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ON CERTAIN PROPERTY OF THE NORMS
BY MODULARS

By

Tetsuya SHIMOGAKI

Let \( R \) be a universally continuous semi-ordered linear space. A functional \( m(a)(a \in R) \) is said to be a modular\(^1\) on \( R \) if it satisfies the following modular conditions:

1. \( 0 \leq m(a) \leq \infty \) for all \( a \in R \);
2. if \( m(\xi a) = 0 \) for all \( \xi > 0 \), then \( a = 0 \);
3. for any \( a \in R \) there exists \( \alpha > 0 \) such that \( m(\alpha a) < \infty \);
4. for every \( a \in R \), \( m(\xi a) \) is a convex function of \( \xi \);
5. \( |a| \leq |b| \) implies \( m(a) \leq m(b) \);
6. \( a \wedge b \neq 0 \) implies \( m(a + b) = m(a) + m(b) \);
7. \( 0 \leq a_\lambda \uparrow a \) implies \( m(a) = \sup_{\lambda \in A} m(a_\lambda) \).

In \( R \), we define functionals \( ||a||, ||a|| (a \in R) \) as follows

\[
||a|| = \inf_{\xi > 0} \frac{1 + m(\xi a)}{\xi}, \quad ||a|| = \inf_{m(\xi a) < 1} \frac{1}{|\xi|}.
\]

Then it is easily seen that both \( ||a|| \) and \( ||a|| \) are norms on \( R \) and \( ||a|| \leq ||a|| \leq 2||a|| \) for all \( a \in R \). \( ||a|| \) is said to be the first norm by \( m \) and \( ||a|| \) is said to be the second norm by \( m \). Let \( \overline{R}^m \) be the modular conjugate space of \( R \) and \( \overline{m} \) be the conjugate modular of \( m \) then we can introduce the norms by \( \overline{m} \) as above. It is known that if \( R \) is semi-regular, the first norm by the conjugate modular \( \overline{m} \) is the conjugate norm of the second norm by \( m \) and the second norm by the conjugate modular \( \overline{m} \) is the conjugate norm of the first norm by \( m \). Since \( ||a|| \) and \( ||a|| \) are semi-continuous by (7), they are reflexive norms (cf. [7]).

If a modular \( m \) is of \( L_\nu \)-type, i.e., \( m(\xi x) = \xi^\nu m(x) \) for all \( x \in R, \xi \geq 0 \),

---

1) We owe the notations and the terminologies using here to the book: H. NAKANO [3].
2) The conjugate modular \( \overline{m} \) is defined as \( m(\bar{a}) = \sup_{x \in R} \{ \bar{a}(x) - m(x) \} \) for every \( \bar{a} \in \overline{R}^m \), where \( \overline{R}^m \) is the space of the modular bounded universally continuous linear functionals on \( R \).
then we have $\frac{||x||}{||x||} = p^{\frac{1}{p}}q^{\frac{1}{q}}$ for all $0 \neq x \in R$, where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (In the case of $p = 1$, we have $\frac{||x||}{||x||} = 1$). The converse of this is studied by S. YAMAMURO [5] and I. AMEMIYA [1]. They proved that if the ratios of two norms are constant for all $0 \neq x \in R$, it is of $L_p$-type essentially. So in the general case, the ratios of two norms are not constant.

A modular $m$ is said to be bounded if there exist real numbers $1 < p_1 \leq p_\wedge < \infty$, such that

$$\xi^{p_1}m(x) \leq m(\xi x) \leq \xi^{p_\wedge}m(x)$$

for all $\xi \geq 1$ and $x \in R$. In [6], S. YAMAMURO obtained that if a modular $m$ on $R$ is bounded then we have

$$||x|| \geq r ||x||$$

for all $x \in R$, where $r > 1$ is a fixed constant.

In this paper we investigate the case when the two norms by a modular $m$ satisfy

$$\inf_{0 \neq x \in R} \frac{||x||}{||x||} = r > 1$$

(In this case we say that the norms have property (\*) throughout this paper).

As showed above, a bounded modular $m$ has that property (\*), but the converse of this is not true in general.

In §1 we prove that if the norms by a modular $m$ satisfy the property (\*) then it is uniformly finite and uniformly increasing, provided that $R$ has no atomic element (Theorem 1.1). And we obtain conversely that if a modular $m$ is uniformly finite and uniformly increasing then the norms by $m$ have the property (\*) (Theorem 1.4). Thus, we can see that if $R$ has no atomic element, then the property (\*) is equivalent to uniform finiteness and uniform increasingness of modular $m$. Theorem 1.2 shows that uniform simpleness of a modular $m$ implies uniform finiteness, in the case when $R$ has no atomic element. Finally some special cases, where the property (\*) is equivalent to boundedness of modular are discussed.

In §2 we define uniform $p$-properties, that is, uniformly $p$-finite, $p$-increasing, $p$-simple and $p$-monotone modulars, to determine the degrees of uniform finiteness, increasingness and etc.. Theorems 2.1 and 2.2
show that there exist the conjugate relations between uniformly $p$-finite modular and uniformly $q$-increasing modular, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, Theorems 2.3 and 2.4 show the similar relations between uniformly $p$-simple modular and uniformly $q$-monotone modular. In the case when $R$ has no atomic element, we have more precisely than in §1, that if a modular $m$ is uniformly $p$-simple it is uniformly $p$-finite (Theorem 2.5). There is a modular which is uniformly finite but not uniformly $p$-finite for any $1 \leq p < \infty$.

In §3 we prove that if the norms by modular $m$ have the property (*) then $r$ (which appears in (*)) determines the degrees of uniform finiteness and uniform increasingness of $m$. Truely, in the case when $R$ has no atomic element, we obtain that if the norms by a modular $m$ have the property (*), $m$ is uniformly $p$-increasing and uniformly $q$-finite, where $p, q$ are positive numbers such that $r = p^\frac{1}{p} q^\frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \leq q$ (Theorem 3.1). The converse of this is not true in general. We show an example of this fact at the end of this paper.

§1. Let $R$ be a modulared semi-ordered linear space with a modular $m$.

A modular $m$ is said to be uniformly finite, if

$$\sup_{m(x) \leq 1} m(\xi x) < \infty$$

for all $\xi > 0$.

A modular $m$ is said to be uniformly increasing, if

$$\lim_{\xi \to \infty} \inf_{m(x) \geq 1} \frac{m(\xi x)}{\xi} = +\infty$$

In [4; Theorems 5.2, 5.3] it is shown that if a modular $m$ is uniformly finite, then the conjugate modular $\overline{m}$ of $m$ is uniformly increasing and if a modular $m$ is uniformly increasing then the conjugate modular $\overline{m}$ is uniformly finite.

Now we shall prove the following

Theorem 1.1. Suppose $R$ has no atomic element. If the norms by a modular $m$ have the property (*), then $m$ is uniformly finite and uniformly increasing.

Proof. 1). Let $r$ be a number, in the sequel, such that $r = \inf_{0 \neq x \in R} \frac{||x||}{||x||}$. Then we have

$$\inf_{0 \neq x \in R^{m}} \frac{||x||}{||x||} = r$$

\((*')\).
In fact we have for every $\bar{x} \in \overline{R}^m$
$$\|\bar{x}\| = \sup_{\|x\| \leqslant 1} |\bar{x}(x)| = \frac{1}{r} \|\bar{x}\|.$$  
Since the norms $\|x\|, \|\bar{x}\|$ are reflexive, we obtain ($*'$).

2). If $m$ is not uniformly finite, then there exists a number $\xi_0 \geq 1$ such that
$$\sup_{m(x) \leq 1} m(\xi x) < +\infty \quad \text{for all } \xi < \xi_0,$$
$$\sup_{m(x) \leq 1} m(\eta x) = +\infty \quad \text{for all } \eta > \xi_0.$$  
Since $r > 1$, we obtain a number $a$ such that $1 > a > 0$ and $ar - 1 > 0$, and we can find also $\varepsilon > 0$ such that $a(\xi_0 + \epsilon) < \xi_0$.

Then by the definition of $\xi_0$, we can find a sequence of elements $\{x_n\}$ ($n=1,2,\cdots$) such that
$$m(x_n) \leq 1, \ m(a(\xi_0 + \epsilon)x_n) \leq k, \ m((\xi_0 + \epsilon)x_n) \geq 1 \quad (n=1,2,\cdots),$$
where $k$ is a fixed positive number.

Since $R$ has no atomic element, we can obtain also a sequence of projectors $\{[p_n]\}$ ($n=1,2,\cdots$) such that
$$m(a(\xi_0 + \epsilon)[p_n]x_n) \leq \frac{k}{n}, \ m((\xi_0 + \epsilon)[p_n]x_n) \geq 1.$$  
Putting $y_n = (\xi_0 + \epsilon)[p_n]x_n$, we have
$$m(y_n) \geq 1, \ m(ay_n) \leq \frac{k}{n} \quad (n=1,2,\cdots).$$

This implies $\lim_{n \to \infty} \frac{1 + m(ay_n)}{a} = \alpha < r$ and contradicts ($*'$), because on the other hand, we have $\|y_n\| \geq 1$ and $\|y_n\| \leq \frac{1 + m(ay_n)}{a}$ for all $n \geq 1$.

Then by 1) $\overline{m}$ is also uniformly finite, thus $m$ is uniformly increasing. This completes the proof.

In the proof of the theorem above, we have shown that if a modular $m$ is not uniformly finite, then there exists a sequence of elements $y_n$ such that

3) We note here that $\overline{m} (x) = \sup_{x \in \overline{R}^m} \{x(x) - m(x)\} \leq m(x)$ for all $x \in R$ by virtue of the definition of conjugate modular. If $R$ is semi-regular, then modular $m$ is reflexive; i.e. $m(x) = \overline{m} (x) = \sup_{x \in \overline{R}^m} \{x(x) - m(x)\}$ for all $x \in R$ ([3]; §39).
On Certain Property of the Norms by Modulars

For some $\xi>0$. Then the sequence $\{y_n\}$ is conditionally modular convergent to 0, but it is not modular convergent. A modular $m$ is said to be uniformly simple if conditionally modular convergence coincides with modular convergence, i.e., $\lim m(x_n) = 0$ implies $\lim m(\xi x_n) = 0$ for every $\xi \geq 0$.

Thus we have

**Theorem 1.2.** Suppose that $R$ has no atomic element. If a modular $m$ is uniformly simple, then it is uniformly finite.

The conjugate property to uniform simpleness of modular is uniform monotoneness. Therefore we obtain also

**Theorem 1.3.** Suppose that $R$ has no atomic element. If a modular $m$ is uniformly monotone, then it is uniformly increasing.

The converse part of Theorem 1.1 is always true (without the assumption that $R$ has no atomic element). That is, we obtain

**Theorem 1.4.** If a modular $m$ is uniformly finite and uniformly increasing, then the norms by $m$ have the property $(*).$

*Proof.* If the property $(*)$ is not satisfied, then we can find $x_n \geq 0$ such that

$$1 \leq ||x_n|| < 1 + \frac{1}{n}, \quad ||x|| = m(x_n) = 1 \quad (n=1,2,\ldots).$$

And we can find also $\xi_n > 0$ such that

$$1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right)\xi_n$$

for all $n \geq 1$ by the definition of the first norm.

Considering a subsequence of $\{\xi_n\}$, it is sufficient for us to investigate only the following cases.

1) In this case, $\{\xi_n\}$ satisfies $0 < \xi_n \leq 1$ for all $n \geq 1$. If $\xi_n \leq \xi_0 < 1$ for some $\xi_0 < 1$, then we obtain

$$\left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_0} > 1 \quad (n=1,2,\ldots).$$

This is a contradiction.

---

4) A modular $m$ is said to be uniformly monotone, if $\lim_{\xi \to 0} \frac{1}{\xi} \sup_{m(x) \leq 1} m(\xi x) = 0$. 

---

Proof. If the property $(*)$ is not satisfied, then we can find $x_n \geq 0$ such that

$$1 \leq ||x_n|| < 1 + \frac{1}{n}, \quad ||x|| = m(x_n) = 1 \quad (n=1,2,\ldots).$$

And we can find also $\xi_n > 0$ such that

$$1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right)\xi_n$$

for all $n \geq 1$ by the definition of the first norm.

Considering a subsequence of $\{\xi_n\}$, it is sufficient for us to investigate only the following cases.

1) In this case, $\{\xi_n\}$ satisfies $0 < \xi_n \leq 1$ for all $n \geq 1$. If $\xi_n \leq \xi_0 < 1$ for some $\xi_0 < 1$, then we obtain

$$\left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_0} > 1 \quad (n=1,2,\ldots).$$

This is a contradiction.

Now without a loss of a generality, we may
assume that
\[ \xi_n \uparrow 1, \quad 1 - \xi_n < \frac{1}{n} \quad (n=1,2,\cdots). \]
Since we have
\[ m(\xi_n x_n) < \left(1 + \frac{1}{n}\right) \xi_n - 1 \leq \frac{1}{n} \]
and \( m(\xi x) \) is a non-decreasing convex function of \( \xi \geq 0 \), we obtain
\[ m((1+(1-\xi_n))x_n) \geq 1 + \frac{n-1}{n} \quad (n=1,2,\cdots), \]
and furthermore
\[ m((1+n(1-\xi_n))x_n) \geq 1 + (n-1) \quad (n=1,2,\cdots). \]
This implies
\[ \sup_{m(x) \leq 1} m(2x) \geq \sup_{n=1,2,\cdots} \frac{m(\xi_n x_n)}{\xi_n} \geq \frac{1}{n} + 1 \]
which contradicts that \( m \) is uniformly finite.

2). In this case, \( \{\xi_n\} (n=1,2,\cdots) \) satisfies \( 1 \leq \xi_n \) for all \( n \geq 1 \). By definition of \( \{\xi_n\} \), we have
\[ 1 + \frac{1}{n} \geq \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_n} + 1 \quad \text{for all } n \geq 1. \]
This implies \( n \leq \xi_n \) for all \( n \geq 1 \). Therefore we may assume \( \xi_n \uparrow +\infty \) \( (n=1,2,\cdots) \), so we obtain
\[ \lim \inf_{\xi \rightarrow m(x) \geq 1} \frac{m(\xi x)}{\xi} \leq \lim_{n \rightarrow \infty} \frac{m(\xi_n x_n)}{\xi_n} \leq \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1, \]
which contradicts that \( m \) is uniformly increasing. This completes the proof.

In the case when a modular \( m \) on \( R \) is of unique spectra ([3]; §54), the property (*) implies boundedness of \( m \). In fact we have

**Theorem 1.5.** If a modular \( m \) on \( R \) is of unique spectra, then boundedness of \( m \) is equivalent to the property (*).

The proof is easily obtained by simple calculations, so it is omitted.

In the case of the constant modular ([3]; §55), the property (*) does not imply simplicity of \( m \), and even in the case of the simple constant modular it does not generally imply the boundedness of \( m \) (the examples are easily obtained). Only in the particular case, we have
Theorem 1.6. If a modular $m$ on $R$ is constant, monotone complete and $R$ has neither complete constant element nor atomic element, then the property $(\ast)$ is equivalent to boundedness of $m$.

Proof. By Theorem 1.1 $m$ is finite, then $m$ is upper bounded by Theorem 55.10 in [3]. Since $\overline{m}$ is constant and has no complete constant element $[3; \S 55]$, $\overline{m}$ is also upper bounded, that is, $m$ is lower bounded. Thus $m$ is a bounded modular on $R$.

§ 2. In this section we investigate the degrees of uniform properties of modulars.

Set for $\xi \geq 1$

\[ f(\xi) = \sup_{m(\xi x) \leq 1} m(\xi x) \quad \text{and} \quad g(\xi) = \inf_{m(\xi x) \geq 1} m(\xi x), \]

then $f(\xi)$ and $g(\xi)$ are defined in $[1, \infty)$ and non-decreasing functions. In the following, let $p$ be a number such that $1 < p < \infty$.

Definition 2.1. A modular $m$ on $R$ is said to be uniformly $p$-finite if there exist $\gamma > 0$ and $\xi_0 \geq 1$ such that

\[ f(\xi) \leq \gamma \xi^p \quad \text{for all} \quad \xi \geq \xi_0. \]

Definition 2.2. A modular $m$ on $R$ is said to be uniformly $p$-increasing, if there exist $\gamma > 0$ and $\xi_0 \geq 1$ such that

\[ g(\xi) \geq \gamma \xi^p \quad \text{for all} \quad \xi \geq \xi_0. \]

It is easily seen that if $m$ is uniformly $p$-finite, it is also uniformly $p'$-finite for $p \leq p'$, and if $m$ is uniformly $p$-increasing it is also uniformly $p''$-increasing for $1 \leq p'' \leq p$.

In the sequel, we set $q = \frac{p}{p-1}$. Now we have

Theorem 2.1. If a modular $m$ is uniformly $p$-finite, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-increasing.

Proof. We have by the assumption for some $\rho_0 \geq 1$, $\gamma > 0$,

\[ f(\xi) \leq \gamma \xi^p \quad (\xi \geq \rho_0 \geq 1). \]

If $\overline{m}(\overline{x}) \geq 1$, $\overline{x} \in \overline{R}^m$ and $0 < a < 1$, we can find $x_0$ such that $\overline{x}(x_0) > a$, $m(x_0) \leq 1$. For such $x_0$, we have by the definition of conjugate modular

\[ \overline{m}(\lambda x) \geq \lambda m(x) - m(x_0) \geq a\lambda^p - \gamma \rho^p \]

for all $\rho \geq \rho_0$. This implies

\[ \overline{m}(\lambda \overline{x}) \geq \sup_{\rho \geq \rho_0} \{a\lambda^p - \gamma \rho^p\} \]
for all $\bar{x} \in \mathbb{R}^m$ such that $\overline{m}(\bar{x}) \geq 1$.

Then we have for $\lambda \geq \lambda_0 = \frac{rp}{a} \rho_0^{\frac{p}{q}}$, 

$$\overline{m}(\lambda \bar{x}) \geq \frac{rp}{q} \left( \frac{\alpha}{\rho r} \right)^{\frac{q}{p}} \lambda^q.$$ 

Hence the conjugate modular $\overline{m}$ is uniformly $q$-increasing modular by definition.

**Theorem 2.2.** If a modular $m$ is uniformly $p$-increasing, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-finite.

**Proof.** By the assumption we have for some $r$ and $\rho_0$ 

$$m(x) \geq 1 \text{ implies } m(\rho x) \geq r \rho^p \quad \text{ for } \rho \geq \rho_0.$$ 

Set $\lambda_0 = \text{Max} \left( \frac{1}{2} \rho_0^{p-1}, 1 \right)$ and for $\lambda \geq \lambda_0$ we define $\rho = \rho(\lambda)$ such that 

$$\rho(\lambda) = \left( \frac{2}{r} \frac{\alpha}{\rho r} \right)^{\frac{q}{p}} \lambda^q.$$ 

Then we have $\rho \geq \rho_0$. Thus we obtain $\frac{m(\rho x)}{\rho} \geq r \rho^{p-1} = 2\lambda$.

If $\bar{x} \in \mathbb{R}^m$, $\overline{m}(\bar{x}) \leq 1$ and $1 \leq m(x) < +\infty$, then there is $\xi > 0$ such that 

$$m \left( \frac{1}{\xi} x \right) = 1, \quad 0 < \frac{1}{\xi} < 1$$ 

and hence by the definition of the conjugate modular $\overline{m}(\bar{x})$ we obtain 

$$\bar{x} \left( \frac{1}{\xi} x \right) \leq \overline{m}(\bar{x}) + m \left( \frac{1}{\xi} x \right) \leq 2.$$ 

For such $\xi$, if $\xi \geq \rho(\lambda)$, then we have 

$$\lambda \bar{x}(x) - m(x) = \xi \left( \lambda \bar{x} \left( \frac{1}{\xi} x \right) - m \left( \frac{1}{\xi} x \right) \right) \leq 0,$$

and if $0 < \xi \leq \rho(\lambda)$, then we have 

$$\lambda \bar{x}(x) - m(x) \leq \xi \lambda \bar{x} \left( \frac{1}{\xi} x \right) \leq 2 \rho \lambda = 2 \lambda \left( \frac{2}{r} \frac{\alpha}{\rho r} \right)^{\frac{q}{p}}.$$ 

If $\overline{m}(\bar{x}) \leq 1$, $m(x) \leq 1$, we have also 

$$\lambda \bar{x}(x) - m(x) \leq \lambda (\overline{m}(\bar{x}) + m(x)) - m(x) \leq 2\lambda.$$ 

Therefore we obtain consequently 

$$\overline{m}(\lambda \bar{x}) \leq 2 \lambda \rho = \gamma_0 \lambda^q$$ 

for all $\lambda \geq \lambda_0$ where $\gamma_0 = 2^q \left( \frac{1}{r} \right)^{\frac{q}{p}}$. Hence the conjugate modular $\overline{m}$ is uniformly $q$-
finite modular.

As similarly as uniformly $p$-finite modulars, we can define uniformly
$p$-simple and uniformly $p$-monotone modular. In order to define them,
we set for $0 \leq \xi \leq 1$
\[
\varphi(\xi) = \sup_{m(\xi x) \leq 1} m(\xi x), \quad \psi(\xi) = \inf_{m(\xi x) \geq 1} m(\xi x).
\]
Then $\varphi(\xi), \psi(\xi)$ are defined in $[0,1]$ and finite non-decreasing functions.

**Definition 2.3.** A modular $m$ on $R$ is said to be uniformly $p$-simple if there exist $r>0$, and $0<\xi_0\leq 1$, such that
\[
\varphi(\xi) \geq r\xi^p
\]
for all $0 \leq \xi \leq \xi_0$.

**Definition 2.4.** A modular $m$ on $R$ is said to be uniformly $p$-monotone, if there exist $r>0$ and $0<\xi_0\leq 1$, such that
\[
\varphi(\xi) \leq r\xi^p
\]
for all $0 \leq \xi \leq \xi_0$.

It is easily seen that if $m$ is uniformly $p$-simple, it is also uniformly $p'$-simple for $p \leq p'$, and if $m$ is uniformly $p$-monotone, it is also uniformly $p'$-monotone for $1 \leq p'' \leq p$.

Concerning uniformly $p$-simple and uniformly $q$-monotone modulars there exist the conjugate relations, in fact we have

**Theorem 2.3.** If a modular $m$ on $R$ is uniformly $p$-monotone, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-simple.

**Theorem 2.4.** If a modular $m$ on $R$ is uniformly $p$-simple, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-monotone.

The proofs of these theorems are analogous to those of Theorems 4.9, 4.10 in [4] and of Theorems 2.1, 2.2, so it is omitted.

Concerning uniform simpleness and uniform finiteness we proved in Theorem 2.2 that uniform simpleness implies uniform finiteness, provided that $R$ has no atomic element. On uniformly $p$-simple modular we obtain more precisely

**Theorem 2.5.** Let $R$ has no atomic element. If a modular $m$ on $R$ is uniformly $p$-simple, then it is uniformly $p$-finite.

**Proof.** It is known already that $m$ is uniformly finite. If it is not uniformly $p$-finite, then there exists a sequence of real numbers $\xi_n \geq 0 (n=1,2,\cdots)$ such that
\[
+\infty > f(\xi_n) > n\xi_n^p, \quad \xi_n \uparrow +\infty \quad (n=1,2,\cdots).
\]
And by definition of $f(\xi)$, we can choose a sequence of elements $\{x_n\}$ $(n=1,2,\cdots)$ such that
\begin{equation*}
m(\xi_{n}x_{n}) > n\xi_{n}^{p}, \quad m(x_{n}) = 1 \quad (n=1,2,\cdots).
\end{equation*}

Here, we can assume without a loss of generality that

\begin{equation*}
m(\xi_{n}x_{n}) = N_{n}
\end{equation*}

where \( N_{n} \) is a natural number, for every \( n \geq 1 \). Because, if there are \( \eta > 0 \) and \( \xi_{0} \geq 1 \) satisfying \( m(\xi x) \leq \eta \xi^{p} \) for every \( \xi \geq \xi_{0} \) such that \( m(\xi x) \) is a natural number, then we have \( m(\xi x) \leq (\eta + 1)\xi^{p} \) for all \( \xi \geq \xi_{0} \). This shows that \( m \) is uniformly \( p \)-finite.

Then we can find a sequence of projectors \( \{[p_{n}]\} \) \( (n=1,2,\cdots) \) by orthogonal decompositions of \( x_{n} \) \( (n=1,2,\cdots) \) such that

\begin{equation*}
m([p_{n}]\xi_{n}x_{n}) = 1, \quad m([p_{n}]x_{n}) < \frac{1}{n\xi_{n}^{p}} \quad (n=1,2,\cdots),
\end{equation*}

since \( m(\xi_{n}x_{n}) \) is natural number for all \( n \geq 1 \). Set \( y_{n} = [p_{n}]\xi_{n}x_{n} \) and \( \eta_{n} = \frac{1}{\xi_{n}} \) for every \( n \geq 1 \), then we have \( m(y_{n}) = 1 \) and \( m(\eta_{n}y_{n}) < \frac{\eta_{n}^{p}}{n} \). Since \( \lim_{n \to \infty} \eta_{n} = 0 \), we show that \( m \) is not uniformly \( p \)-simple. Thus the proof is completed.

Corresponding to Theorem 2.5 we have

\textbf{Theorem 2.6.} Let \( R \) have no atomic element. If a modular \( m \) on \( R \) is uniformly \( p \)-monotone, then it is uniformly \( p \)-increasing.

It will be conjectured that if a modular \( m \) is uniformly finite, then it is uniformly \( p \)-finite for some \( 1 < p < +\infty \). But the following example shows that it is not true.

\textbf{Example.} Set \( \phi(u) = \left\{ \begin{array}{ll} \frac{1}{2}u & u \leq 2 \\ e^{u-2} & u > 2 \end{array} \right. \)

and consider \textit{Orlicz} sequence space \( l_{\phi} \). Then \( l_{\phi} \) is uniformly finite as easily seen, but not uniformly \( p \)-finite for any \( 1 < p < +\infty \). This example shows at the same time that there exists a modular \( m \) which is uniformly increasing but not uniformly \( p \)-increasing for any \( 1 < p < +\infty \).

I. \textsc{Amemiya} proved in [2] that if a modular \( m \) on \( R \) is monotone complete and finite, then \( m \) is semi-upper bounded, i.e., \( m(2x) < \gamma m(x) \) for every \( x \) such that \( m(x) \geq \epsilon \) for some fixed \( \gamma, \epsilon > 0 \), provided that \( R \) has no atomic element. Applying this result, it is seen that the above conjecture is affirmative, in the case when \( m \) is monotone complete and \( R \) has no atomic element. In fact we have
Theorem 2.7. Suppose that \( R \) has no atomic element and \( m \) is monotone complete. If \( m \) is uniformly finite (finite) then it is uniformly \( p \)-finite for some \( p > 1 \).

§ 3. To any \( r \) such that \( 1 < r \leq 2 \), there exist a unique pair of positive numbers \((p, q)\) satisfying the following

1) \[ r = p^\frac{1}{p} q^\frac{1}{q} \]
2) \[ \frac{1}{p} + \frac{1}{q} = 1 \]
3) \[ 1 \leq p \leq 2 \leq q \]

This correspondence is unique and it is easily seen that if \( r_n \) is convergent increasingly to 2, then the corresponding \( p_n(q_n) \) is also convergent increasingly (decreasingly) to 2.

If the norms of modular \( m \) have the property (\(*\)) we can find a pair of numbers such that \( r = p^\frac{1}{p} q^\frac{1}{q} \). It is already seen that \( m \) is uniformly finite and uniformly increasing, provided that \( R \) has no atomic element. Now we shall show that \((p, q)\) gives the degrees of uniform finiteness and increasingness. In fact we can state

Theorem 3.1. Suppose that \( R \) has no atomic element. If the norms by a modular \( m \) have the property (\(*\)), then \( m \) is uniformly \( p \)-increasing and uniformly \( q \)-finite.

Proof. Set \( a = (\frac{p}{q})^\frac{1}{q} \), then \( ra - 1 = a^q \).

Thus we obtain by assumption,

\( m(x) = 1 \) implies \( m(ax) \geq a^q \).

If \( m(x) = 1 + \frac{m}{n} \) (for natural numbers \( m < n \)), we can decompose orthogonally \( x = x_1 + x_2 + \cdots + x_{n+m} \) such that

\[ m(x_i) = m(x_j) = \frac{1}{n} \quad (i, j = 1, 2, \ldots, n+m). \]

The numbers of \( i \) such that \( m(ax_i) < a^q m(x_i) \) are less than \( n \), because if there exists \((i_1, i_2, \ldots, i_n)\) such that \( m(ax_{i_\nu}) > a^q m(x_{i_\nu}) (\nu = 1, 2, \ldots, n) \), then we have \( m \left( a \sum_{\nu=1}^{n} x_{i_\nu} \right) < a^q m \left( \sum_{\nu=1}^{n} x_{i_\nu} \right) \) and \( m \left( \sum_{\nu=1}^{n} x_{i_\nu} \right) = 1 \). This is a contradiction.

Thus there exists \( \{i_k\} \) \((k = 1, 2, \ldots, n)\) such that \( m(ax_{i_k}) \geq a^q m(x_{i_k}) (k = 1, 2, \ldots, n) \).
1, 2, ⋯, m). Putting \( y = \sum_{k=1}^{m} x_{i_{k}} \) we have \( m(x - y) = 1 \) and

\[
\begin{align*}
m(ay) &\geq a^{q}m(y) \\
m(a(x - y)) &\geq a^{q}m(x - y).
\end{align*}
\]

Hence we obtain \( m(ax) \geq a^{q}m(x) \). Generally, if \( 1 \leq m(x) < 2 \), since \( m(\xi x) \) is continuous function of \( \xi \), we have also

\[
m(ax) \geq a^{q}m(x).
\]

Since \( m(x) \) is finite for all \( x \in R \) and \( R \) has no atomic element, we have for \( x \) such that \( m(x) = 1 \)

\[
m(a\xi x) \geq a^{q}m(\xi x)
\]

for all \( \xi \geq 1 \).

Here, putting \( \beta = \frac{1}{a} > 1 \), we obtain

\[
m(\beta^{n}x) \leq \beta^{q\cdot n}m(x)
\]

for all \( x \) such that \( m(x) = 1 \). From this we have

\[
m(\xi x) \leq \beta^{\xi^{q}} \quad \xi \geq \beta,
\]

which shows that \( m \) is uniformly \( q \)-finite. By Theorem 2.1 and \((\ast)\) we can see \( m \) is uniformly \( p \)-increasing.

**Remark 1.** The converse of the theorem is not true. For example, set

\[
\phi(u) = \begin{cases} u^{\frac{3}{2}} & u \leq 2 \\ \frac{1}{\sqrt{2}} u^{2} & u > 2 \end{cases}
\]

Then the \( OR_{1,ICZ} \) space \( L[0,1] \) is uniformly 2-finite and uniformly 2-increasing, but it is easily seen that there is an element such that \( \frac{|x|}{\|x\|} < 2 \). And for any \( 1 < a < 2 \), we can get the example of modulared space such that \( m \) is uniformly 2-finite and uniformly 2-increasing but the norms by \( m \) do not satisfy \( \inf_{x \neq 0} \frac{|x|}{\|x\|} \geq a \).

**Remark 2.** If \( R \) is a discrete modulared semi-ordered linear space, the property \((\ast)\) dose not imply finiteness of \( m \), and even if in the case where \( m \) is finite, the property \((\ast)\) does not imply uniform finiteness of \( m \). The examples are obtained easily. In this case the equivalent
condition to the property (*) is unknown.

References