ON CERTAIN PROPERTY OF THE NORMS
BY MODULARS

By

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Let $R$ be a universally continuous semi-ordered linear space. A functional $m(a)(a\in R)$ is said to be a modular \(^{1}\) on $R$ if it satisfies the following modular conditions:

1. $0 \leq m(a) \leq \infty$ for all $a \in R$;
2. if $m(\xi a) = 0$ for all $\xi > 0$, then $a = 0$;
3. for any $a \in R$ there exists $a > 0$ such that $m(aa) < \infty$;
4. for every $a \in R$, $m(\xi a)$ is a convex function of $\xi$;
5. $|a| \leq |b|$ implies $m(a) \leq m(b)$;
6. $a \land b = \backslash 0$ implies $m(a + b) = m(a) + m(b)$;
7. $0 \leq a_{\lambda} \uparrow a \in R$ implies $m(a) = \sup_{\lambda \in A} m(a_{\lambda})$.

In $R$, we define functionals $||a||$, $||a||$ ($a \in R$) as follows

$$||a|| = \inf_{\xi > 0} \frac{1 + m(\xi a)}{\xi}, \quad ||a|| = \inf_{m(\xi a) < 1} \frac{1}{|\xi|}.$$ 

Then it is easily seen that both $||a||$ and $||a||$ are norms on $R$ and $||a|| \leq ||a|| \leq 2||a||$ for all $a \in R$. $||a||$ is said to be the first norm by $m$ and $||a||$ is said to be the second norm by $m$. Let $\overline{R}^{m}$ be the modular conjugate space of $R$ and $\overline{m}$ be the conjugate modular of $m^{\circ}$ then we can introduce the norms by $\overline{m}$ as above. It is known that if $R$ is semi-regular, the first norm by the conjugate modular $\overline{m}$ is the conjugate norm of the second norm by $m$ and the second norm by the conjugate modular $\overline{m}$ is the conjugate norm of the first norm by $m$. Since $||a||$ and $||a||$ are semi-continuous by (7), they are reflexive norms (cf. [7]).

If a modular $m$ is of $L_{p}$-type, i.e., $m(\xi x) = \xi^{p}m(x)$ for all $x \in R$, $\xi \geq 0$, $^{1)$ We owe the notations and the terminologies using here to the book: H. NAKANO [3].

$^{2}$ The conjugate modular $\overline{m}$ is defined as $m(\overline{a}) = \sup \{ \overline{a}(x) - m(x) \}$ for every $\overline{a} \in \overline{R}^{m}$, where $\overline{R}^{m}$ is the space of the modular bounded universally continuous linear functionals on $R$. 

then we have \[ \frac{||x||}{||x||} = p^{\frac{1}{p}} q^{\frac{1}{q}} \] for all \( 0 \neq x \in \mathbb{R} \), where \( 1 \leq p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) (In the case of \( p = 1 \), we have \( \frac{||x||}{||x||} = 1 \)). The converse of this is studied by S. YAMAMURO [5] and I. AMEMIYA [1]. They proved that if the ratios of two norms are constant for all \( 0 \neq x \in \mathbb{R} \), it is of \( L_p \)-type essentially. So in the general case, the ratios of two norms are not constant.

A modular \( m \) is said to be bounded if there exist real numbers \( 1 < p_1 \leq p_\wedge < \infty \), such that
\[ \xi^{p_1} m(x) \leq m(\xi x) \leq \xi^{p_\wedge} m(x) \]
for all \( \xi \geq 1 \) and \( x \in \mathbb{R} \). In [6], S. YAMAMURO obtained that if a modular \( m \) on \( \mathbb{R} \) is bounded then we have
\[ ||x|| \geq r \|x\| \]
for all \( x \in \mathbb{R} \), where \( r > 1 \) is a fixed constant.

In this paper we investigate the case when the two norms by a modular \( m \) satisfy
\[ \inf_{0 \neq x \in \mathbb{R}} \frac{||x||}{||x||} = r > 1 \]  
(In this case we say that the norms have property \( (*) \) throughout this paper).

As showed above, a bounded modular \( m \) has that property \( (*) \), but the converse of this is not true in general.

In §1 we prove that if the norms by a modular \( m \) satisfy the property \( (*) \) then it is uniformly finite and uniformly increasing, provided that \( \mathbb{R} \) has no atomic element (Theorem 1.1). And we obtain conversely that if a modular \( m \) is uniformly finite and uniformly increasing then the norms by \( m \) have the property \( (*) \) (Theorem 1.4). Thus, we can see that if \( \mathbb{R} \) has no atomic element, then the property \( (*) \) is equivalent to uniform finiteness and uniform increasingness of modular \( m \). Theorem 1.2 shows that uniform simpleness of a modular \( m \) implies uniform finiteness, in the case when \( \mathbb{R} \) has no atomic element. Finally some special cases, where the property \( (*) \) is equivalent to boundedness of modular are discussed.

In §2 we define uniform \( p \)-properties, that is, uniformly \( p \)-finite, \( p \)-increasing, \( p \)-simple and \( p \)-monotone modulars, to determine the degrees of uniform finiteness, increasingness and etc.. Theorems 2.1 and 2.2
show that there exist the conjugate relations between uniformly $p$-finite modular and uniformly $q$-increasing modular, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, Theorems 2.3 and 2.4 show the similar relations between uniformly $p$-simple modular and uniformly $q$-monotone modular. In the case when $R$ has no atomic element, we have more precisely than in §1, that if a modular $m$ is uniformly $p$-simple it is uniformly $p$-finite (Theorem 2.5). There is a modular which is uniformly finite but not uniformly $p$-finite for any $1 \leq p < \infty$.

In §3 we prove that if the norms by modular $m$ have the property (*) then $r$ (which appears in (*)) determines the degrees of uniform finiteness and uniform increasingness of $m$. Truely, in the case when $R$ has no atomic element, we obtain that if the norms by a modular $m$ have the property (*), $m$ is uniformly $p$-increasing and uniformly $q$-finite, where $p, q$ are positive numbers such that $r = p^\frac{1}{p} q^\frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \leq q$ (Theorem 3.1). The converse of this is not true in general. We show an example of this fact at the end of this paper.

§1. Let $R$ be a modulared semi-ordered linear space with a modular $m$.

A modular $m$ is said to be uniformly finite, if

$$\sup_{m(x) \leq 1} m(\xi x) < \infty \text{ for all } \xi > 0.$$ 

A modular $m$ is said to be uniformly increasing, if

$$\lim_{\xi \to \infty} \inf_{m(x) \geq 1} \frac{m(\xi x)}{\xi} = +\infty.$$ 

In [4; Theorems 5.2, 5.3] it is shown that if a modular $m$ is uniformly finite, then the conjugate modular $\overline{m}$ of $m$ is uniformly increasing and if a modular $m$ is uniformly increasing then the conjugate modular $\overline{m}$ is uniformly finite.

Now we shall prove the following

**Theorem 1.1.** Suppose $R$ has no atomic element. If the norms by a modular $m$ have the property (*), then $m$ is uniformly finite and uniformly increasing.

**Proof.** 1). Let $r$ be a number, in the sequel, such that $r = \inf_{0 < \|x\| \in R} \|\overline{x}\|$. Then we have

$$\inf_{0 < x \in R^m} \frac{\|\overline{x}\|}{\|x\|} = r \quad (\ast').$$
In fact we have for every \( \overline{x} \in \overline{R}^m \)

\[
\|\overline{x}\| = \sup_{|x| \leq 1} |\overline{x}(x)| \leq \sup_{r ||x||} |\overline{x}(x)| = \frac{1}{r} \|\overline{x}\|.
\]

Since the norms \( ||x||, \|x\|^r \) are reflexive, we obtain \( (*)' \).

2). If \( m \) is not uniformly finite, then there exists a number \( \xi_0 \geq 1 \) such that

\[
\sup_{m(x) \leq 1} m(\xi x) \leq +\infty \quad \text{for all } \xi < \xi_0,
\]

\[
\sup_{m(x) \leq 1} m(\eta x) = +\infty \quad \text{for all } \eta > \xi_0.
\]

Since \( r > 1 \), we obtain a number \( a \) such that \( 1 > a > 0 \) and \( ar - 1 > 0 \), and we can find also \( \varepsilon > 0 \) such that \( a(\xi_0 + \varepsilon) < \xi_0 \).

Then by the definition of \( \xi_0 \), we can find a sequence of elements \( \{x_n\} \) (\( n=1,2,\cdots \)) such that

\[
m(x_n) \leq 1, \quad m(a(\xi_0 + \varepsilon)x_n) \leq k, \quad m((\xi_0 + \varepsilon)x_n) \geq n \quad (n=1,2,\cdots),
\]

where \( k \) is a fixed positive number.

Since \( R \) has no atomic element, we can obtain also a sequence of projectors \( \{[p_n]\} \) (\( n=1,2,\cdots \)) such that

\[
m(a(\xi_0 + \varepsilon)[p_n]x_n) \leq \frac{k}{n}, \quad m((\xi_0 + \varepsilon)[p_n]x_n) \geq 1.
\]

Putting \( y_n = (\xi_0 + \varepsilon)[p_n]x_n \), we have

\[
m(y_n) \geq 1, \quad m(ay_n) \leq \frac{k}{n} \quad (n=1,2,\cdots).
\]

This implies \( \lim_{n \to \infty} \frac{1+m(ay_n)}{a} = \frac{1}{a} < r \) and contradicts \( (*) \), because on the other hand, we have \( \|y_n\| \geq 1 \) and \( \|y_n\| \leq \frac{1+m(ay_n)}{a} \) for all \( n \geq 1 \).

Then by 1) \( \overline{m} \) is also uniformly finite, thus \( m \) is uniformly increasing. This completes the proof.

In the proof of the theorem above, we have shown that if a modular \( m \) is not uniformly finite, then there exists a sequence of elements \( y_n \) such that

3) We note here that \( \overline{m}(x) = \sup_{x \in \overline{R}^m} \{|x(x) - m(x)|\} \leq m(x) \) for all \( x \in R \) by virtue of the definition of conjugate modular. If \( R \) is semi-regular, then modular \( m \) is reflexive; i.e. \( m(x) = \overline{m}(x) = \sup_{x \in \overline{R}^m} \{|x(x) - m(x)|\} \) for all \( x \in R \) ([3]; §39).
On Certain Property of the Norms by Modulars

$m(y_n) \geq 1$, \( \lim_{n \to \infty} m(\xi y_n) = 0 \quad (n=1,2,\cdots) \)

for some \( \xi > 0 \). Then the sequence \( \{y_n\} (n=1,2,\cdots) \) is conditionally modular convergent to 0, but it is not modular convergent. A modular \( m \) is said to be uniformly simple if conditionally modular convergence coincides with modular convergence, i.e., \( \lim m(x_n) = 0 \) implies \( \lim m(\xi x_n) = 0 \) for every \( \xi \geq 0 \).

Thus we have

**Theorem 1.2.** Suppose that \( R \) has no atomic element. If a modular \( m \) is uniformly simple, then it is uniformly finite.

The conjugate property to uniform simpleness of modular is uniform monotoneness. Therefore we obtain also

**Theorem 1.3.** Suppose that \( R \) has no atomic element. If a modular \( m \) is uniformly monotone, then it is uniformly increasing.

The converse part of Theorem 1.1 is always true (without the assumption that \( R \) has no atomic element). That is, we obtain

**Theorem 1.4.** If a modular \( m \) is uniformly finite and uniformly increasing, then the norms by \( m \) have the property \((*)\).

**Proof.** If the property \((*)\) is not satisfied, then we can find \( x_n \geq 0 \) \( (n=1,2,\cdots) \) such that

\[
1 \leq ||x_n|| < 1 + \frac{1}{n}, \quad ||x|| = m(x_n) = 1 \quad (n=1,2,\cdots).
\]

And we can find also \( \xi_n > 0 \) such that

\[
1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right)\xi_n
\]

for all \( n \geq 1 \) by the definition of the first norm.

Considering a subsequence of \( \{\xi_n\} \), it is sufficient for us to investigate only the following cases.

1) In this case, \( \{\xi_n\} \) satisfies \( 0 < \xi_n \leq 1 \) for all \( n \geq 1 \). If \( \xi_n \leq \xi_0 < 1 \) \( (n=1,2,\cdots) \) for some \( \xi_0 < 1 \), then we obtain

\[
\left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_0} > 1 \quad (n=1,2,\cdots).
\]

This is a contradiction. Now without a loss of a generality, we may

4) A modular \( m \) is said to be uniformly monotone, if \( \lim_{\xi \to 0} \sup_{m(\xi x) < 1} \frac{1}{\xi} \cdot m(\xi x) = 0 \).
assume that
\[ \xi_n \uparrow 1, \quad 1 - \xi_n < \frac{1}{n} \quad (n=1,2,\cdots). \]

Since we have
\[ m(\xi_n x_n) < \left( 1 + \frac{1}{n} \right) \xi_n - 1 \leq \frac{1}{n} \]
and \( m(\xi x) \) is a non-decreasing convex function of \( \xi \geq 0 \), we obtain
\[ m((1+(1-\xi_n)) x_n) \geq 1 + \frac{n-1}{n} \quad (n=1,2,\cdots), \]
and furthermore
\[ m((1+n(1-\xi_n)) x_n) \geq 1 + (n-1) \quad (n=1,2,\cdots). \]
This implies
\[ \sup_{m(x) \leq 1} m(2x) \geq \sup_{n=1,2,\ldots} m(2x_n) \geq \sup_{n=1,2,\ldots} (1+(n-1)) = +\infty, \]
which contradicts that \( m \) is uniformly finite.

2). In this case, \( \{\xi_n\} \ (n=1,2,\cdots) \) satisfies \( 1 \leq \xi_n \) for all \( n \geq 1 \). By definition of \( \{\xi_n\} \), we have
\[ 1 + \frac{1}{n} \geq \frac{1+m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_n} + 1 \quad \text{for all } n \geq 1. \]
This implies \( n \leq \xi_n \) for all \( n \geq 1 \). Therefore we may assume \( \xi_n \uparrow +\infty \ (n=1,2,\cdots) \), so we obtain
\[ \lim \inf_{\xi \rightarrow +\infty \atop \xi \in \{\xi_n\}, \xi_n \uparrow 1} \frac{m(\xi x)}{\xi} \leq \lim_{n \rightarrow \infty} \frac{m(\xi_n x_n)}{\xi_n} \leq \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = 1, \]
which contradicts that \( m \) is uniformly increasing. This completes the proof.

In the case when a modular \( m \) on \( R \) is of unique spectra ([3]; §54), the property (*) implies boundedness of \( m \). In fact we have

**Theorem 1.5.** If a modular \( m \) on \( R \) is of unique spectra, then boundedness of \( m \) is equivalent to the property (*).

The proof is easily obtained by simple calculations, so it is omitted.

In the case of the constant modular ([3]; §55), the property (*) does not imply simplicity of \( m \), and even in the case of the simple constant modular it does not generally imply the boundedness of \( m \) (the examples are easily obtained). Only in the particular case, we have
Theorem 1.6. If a modular $m$ on $R$ is constant, monotone complete and $R$ has neither complete constant element nor atomic element, then the property $(\ast)$ is equivalent to boundedness of $m$.

Proof. By Theorem 1.1 $m$ is finite, then $m$ is upper bounded by Theorem 55.10 in [3]. Since $\overline{m}$ is constant and has no complete constant element [3; §55], $\overline{m}$ is also upper bounded, that is, $m$ is lower bounded. Thus $m$ is a bounded modular on $R$.

§ 2. In this section we investigate the degrees of uniform properties of modulars.

Set for $\xi \geq 1$

$$f(\xi) = \sup_{m(\xi x) \leq 1} m(\xi x) \quad \text{and} \quad g(\xi) = \inf_{m(\xi x) \geq 1} m(\xi x),$$

then $f(\xi)$ and $g(\xi)$ are defined in $[1, \infty)$ and non-decreasing functions. In the following, let $p$ be a number such that $1 < p < \infty$.

Definition 2.1. A modular $m$ on $R$ is said to be uniformly $p$-finite if there exist $r > 0$ and $\xi_0 \geq 1$ such that

$$f(\xi) \leq r \xi^p \quad \text{for all} \quad \xi \geq \xi_0.$$

Definition 2.2. A modular $m$ on $R$ is said to be uniformly $p$-increasing, if there exist $r > 0$ and $\xi_0 \geq 1$ such that

$$g(\xi) \geq r \xi^p \quad \text{for all} \quad \xi \geq \xi_0.$$

It is easily seen that if $m$ is uniformly $p$-finite, it is also uniformly $p'$-finite for $p \leq p'$, and if $m$ is uniformly $p$-increasing it is also uniformly $p'$-increasing for $1 \leq p' \leq p$.

In the sequel, we set $q = \frac{p}{p-1}$. Now we have

Theorem 2.1. If a modular $m$ is uniformly $p$-finite, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-increasing.

Proof. We have by the assumption for some $\rho_0 \geq 1$, $r > 0$,

$$f(\xi) \leq r \xi^p \quad (\xi \geq \rho_0 \geq 1).$$

If $\overline{m}(\overline{x}) \geq 1$, $\overline{x} \in \overline{R}^m$ and $0 < a < 1$, we can find $x_0$ such that $\overline{x}(x_0) > a$, $m(x_0) \leq 1$. For such $x_0$, we have by the definition of conjugate modular

$$\overline{m}(\lambda \overline{x}) \geq \lambda \overline{x}(\rho x_0) - m(\rho x_0) \geq a \lambda^p - \gamma \rho^p$$

for all $\rho \geq \rho_0$. This implies

$$\overline{m}(\lambda \overline{x}) \geq \sup_{\rho \geq \rho_0} \{a \lambda^p - \gamma \rho^p\}$$
for all $x \in \mathbb{R}^m$ such that $m(x) \geq 1$.

Then we have for $\lambda \geq \lambda_0 = \frac{rp}{\alpha} \rho_o ^{\frac{p}{q}}$,

\[ m(\lambda x) \geq \frac{rp}{q} \left( \frac{\alpha}{p\gamma} \right)^q \lambda^q. \]

Hence the conjugate modular $\bar{m}$ is uniformly $q$-increasing modular by definition.

**Theorem 2.2.** If a modular $m$ is uniformly $p$-increasing, then the conjugate modular $\bar{m}$ of $m$ is uniformly $q$-finite.

**Proof.** By the assumption we have for some $r$ and $\rho_o$

\[ m(x) \geq 1 \implies m(p x) \geq rp^q \quad \text{for } \rho \geq \rho_o. \]

Set $\lambda_0 = \text{Max} \left( \frac{r}{2} \rho_o ^{p-1}, 1 \right)$ and for $\lambda \geq \lambda_0$ we define $\rho = \rho(\lambda)$ such that

\[ \rho(\lambda) = \left( \frac{2}{r} \lambda \right)^{\frac{q}{p}}. \]

Then we have $\rho \geq \rho_o$. Thus we obtain $m(p x) / \rho \geq rp^{p-1} = 2\lambda$.

If $x \in \mathbb{R}^m$, $m(x) \leq 1$ and $1 \leq m(x) < +\infty$, then there is $\xi > 0$ such that

\[ m \left( \frac{1}{\xi} x \right) = 1, \quad 0 < \frac{1}{\xi} < 1 \]

and hence by the definition of the conjugate modular $\bar{m}(\bar{x})$ we obtain

\[ \bar{x} \left( \frac{1}{\xi} x \right) \leq \bar{m}(\bar{x}) + m \left( \frac{1}{\xi} x \right) \leq 2. \]

For such $\xi$, if $\xi \geq \rho(\lambda)$, then we have

\[ \lambda \bar{x}(x) - m(x) = \xi \left\{ \lambda \bar{x} \left( \frac{1}{\xi} x \right) - \frac{1}{\xi} m \left( \frac{1}{\xi} x \right) \right\} \leq 0, \]

and if $0 < \xi \leq \rho(\lambda)$, then we have

\[ \lambda \bar{x}(x) - m(x) \leq \xi \lambda \bar{x} \left( \frac{1}{\xi} x \right) \leq 2p \lambda = 2 \lambda \left( \frac{2}{r} \lambda \right)^{\frac{q}{p}}. \]

If $\bar{m}(\bar{x}) \leq 1$, $m(x) \leq 1$, we have also

\[ \lambda \bar{x}(x) - m(x) \leq \lambda (\bar{m}(\bar{x}) + m(x)) - m(x) \leq 2\lambda. \]

Therefore we obtain consequently

\[ \bar{m} (\lambda \bar{x}) \leq 2 \lambda \rho = r_o \lambda^q \quad \text{for all } \lambda \geq \lambda_0 \]

where $r_o = 2^q \left( \frac{1}{r} \right)^{\frac{q}{p}}$. Hence the conjugate modular $\bar{m}$ is uniformly $q$-
finite modular.

As similarly as uniformly $p$-finite modulars, we can define uniformly $p$-simple and uniformly $p$- monotone modular. In order to define them, we set for $0 \leq \xi \leq 1$

$$\varphi(\xi) = \sup_{m(\xi) \leq 1} m(\xi x), \quad \psi(\xi) = \inf_{m(\xi) \geq 1} m(\xi x).$$

Then $\varphi(\xi), \psi(\xi)$ are defined in $[0,1]$ and finite non-decreasing functions.

**Definition 2.3.** A modular $m$ on $R$ is said to be uniformly $p$-simple if there exist $r > 0$, and $0 < \xi_0 \leq 1$, such that

$$\varphi(\xi) \geq r \xi^p$$

for all $0 \leq \xi \leq \xi_0$.

**Definition 2.4.** A modular $m$ on $R$ is said to be uniformly $p$-monotone, if there exist $r > 0$ and $0 < \xi_0 \leq 1$, such that

$$\psi(\xi) \leq \gamma \xi^p$$

for all $0 \leq \xi \leq \xi_0$.

It is easily seen that if $m$ is uniformly $p$-simple, it is also uniformly $p'$-simple for $p \leq p'$, and if $m$ is uniformly $p$-monotone, it is also uniformly $p''$-monotone for $1 \leq p'' \leq p$.

Concerning uniformly $p$-simple and uniformly $q$-monotone modulars there exist the conjugate relations, in fact we have

**Theorem 2.3.** If a modular $m$ on $R$ is uniformly $p$-monotone, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-simple.

**Theorem 2.4.** If a modular $m$ on $R$ is uniformly $p$-simple, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-monotone.

The proofs of these theorems are analogous to those of Theorems 4.9, 4.10 in [4] and of Theorems 2.1, 2.2, so it is omitted.

Concerning uniform simpleness and uniform finiteness we proved in Theorem 2.2 that uniform simpleness implies uniform finiteness, provided that $R$ has no atomic element. On uniformly $p$-simple modular we obtain more precisely

**Theorem 2.5.** Let $R$ has no atomic element. If a modular $m$ on $R$ is uniformly $p$-simple, then it is uniformly $p$-finite.

**Proof.** It is known already that $m$ is uniformly finite. If it is not uniformly $p$-finite, then there exists a sequence of real numbers $\xi_n \geq 0 (n=1,2,\cdots)$ such that

$$+ \infty > f(\xi_n) > n\xi_n^p, \quad \xi_n \uparrow + \infty \quad (n=1,2,\cdots).$$

And by definition of $f(\xi)$, we can choose a sequence of elements $\{x_n\}$ $\quad (n=1,2,\cdots)$ such that
Here, we can assume without a loss of generality that
\[ m(\xi_n x_n) = N_n \]
where \( N_n \) is a natural number, for every \( n \geq 1 \). Because, if there are \( r > 0 \) and \( \xi_0 \geq 1 \) satisfying \( m(\xi x) \leq r\xi^p \) for every \( \xi \geq \xi_0 \) such that \( m(\xi x) \) is a natural number, then we have \( m(\xi x) \leq (r+1)\xi^p \) for all \( \xi \geq \xi_0 \). This shows that \( m \) is uniformly \( p \)-finite.

Then we can find a sequence of projectors \( \{[p_n]\} (n=1,2,\cdots) \) by orthogonal decompositions of \( x_n (n=1,2,\cdots) \) such that
\[ m([p_n] \xi_n x_n) = 1, \quad m([p_n] x_n) < \frac{1}{n\xi_n^p} \quad (n=1,2,\cdots), \]
since \( m(\xi_n x_n) \) is natural number for all \( n \geq 1 \). Set \( y_n = [p_n] \xi_n x_n \) and \( \eta_n = \frac{1}{\xi_n} \) for every \( n \geq 1 \), then we have \( m(y_n) = 1 \) and \( m(\eta_n y_n) < \frac{\eta_n^p}{n} \). Since \( \lim_{n \to \infty} \eta_n = 0 \), we show that \( m \) is not uniformly \( p \)-simple. Thus the proof is completed.

Corresponding to Theorem 2.5 we have

**Theorem 2.6.** Let \( R \) have no atomic element. If a modular \( m \) on \( R \) is uniformly \( p \)-monotone, then it is uniformly \( p \)-increasing.

It will be conjectured that if a modular \( m \) is uniformly finite, then it is uniformly \( p \)-finite for some \( 1 < p < +\infty \). But the following example shows that it is not true.

**Example.** Set \( \phi(u) = \begin{cases} \frac{1}{2} u & u \leq 2 \\ e^{u-2} & u > 2 \end{cases} \)
and consider ORLICZ sequence space \( l_\phi \). Then \( l_\phi \) is uniformly finite as easily seen, but not uniformly \( p \)-finite for any \( 1 < p < +\infty \). This example shows at the same time that there exists a modular \( m \) which is uniformly increasing but not uniformly \( p \)-increasing for any \( 1 < p < +\infty \).

I. AMEMIYA proved in [2] that if a modular \( m \) on \( R \) is monotone complete and finite, then \( m \) is semi-upper bounded, i.e., \( m(2x) \leq r m(x) \) for every \( x \) such that \( m(x) \geq \epsilon \) for some fixed \( r, \epsilon > 0 \), provided that \( R \) has no atomic element. Applying this result, it is seen that the above conjecture is affirmative, in the case when \( m \) is monotone complete and \( R \) has no atomic element. In fact we have
Theorem 2.7. Suppose that $R$ has no atomic element and $m$ is monotone complete. If $m$ is uniformly finite (finite) then it is uniformly $p$-finite for some $p>1$.

§ 3. To any $r$ such that $1< r \leq 2$, there exist a unique pair of positive numbers $(p, q)$ satisfying the following

1) $r = p^{\frac{1}{p}} q^{\frac{1}{q}}$
2) $\frac{1}{p} + \frac{1}{q} = 1$
3) $1 \leq p < 2 \leq q$.

This correspondence is unique and it is easily seen that if $r_n$ is convergent increasingly to 2, then the corresponding $p_n (q_n)$ is also convergent increasingly (decreasingly) to 2.

If the norms of modular $m$ have the property (*) we can find a pair of numbers such that $r = p^{\frac{1}{p}} q^{\frac{1}{q}}$. It is already seen that $m$ is uniformly finite and uniformly increasing, provided that $R$ has no atomic element. Now we shall show that $(p, q)$ gives the degrees of uniform finiteness and increasingness. In fact we can state

Theorem 3.1. Suppose that $R$ has no atomic element. If the norms by a modular $m$ have the property (*), then $m$ is uniformly $p$-increasing and uniformly $q$-finite.

Proof. Set $a = \left( \frac{p}{q} \right)^{\frac{1}{q}}$, then $ra - 1 = a^q$.

Thus we obtain by assumption,

$m(x) = 1$ implies $m(ax) \geq a^q$.

If $m(x) = 1 + \frac{m}{n}$ (for natural numbers $m < n$), we can decompose orthogonally $x = x_1 + x_2 + \cdots + x_{n+m}$ such that

$m(x_i) = m(x_j) = \frac{1}{n}$ \hspace{1cm} (i, j = 1, 2, \ldots, n+m).

The numbers of $i$ such that $m(ax_i) < a^q m(x_i)$ are less than $n$, because if there exists $(i_1, i_2, \ldots, i_n)$ such that $m(ax_{i_\nu}) > a^q m(x_{i_\nu})$ ($\nu = 1, 2, \ldots, n$), then we have $m(a \sum_{\nu=1}^{n} x_{i_\nu}) < a^q m \left( \sum_{\nu=1}^{n} x_{i_\nu} \right)$ and $m \left( \sum_{\nu=1}^{n} x_{i_\nu} \right) = 1$. This is a contradiction.

Thus there exists \{i_k\} ($k = 1, 2, \ldots, m$) such that $m(ax_k) \geq a^q m(x_k)$ ($k =
Putting $y = \sum_{k=1}^{m} x_{i_k}$ we have $m(x-y) = 1$ and
\[ m(ay) \geq a^q m(y) \]
\[ m(\alpha(x-y)) \geq a^q m(x-y). \]
Hence we obtain $m(ax) \geq a^q m(x)$. Generally, if $1 \leq m(x) < 2$, since $m(\xi x)$ is continuous function of $\xi$, we have also
\[ m(ax) \geq a^q m(x). \]
Since $m(x)$ is finite for all $x \in R$ and $R$ has no atomic element, we have for $x$ such that $m(x) = 1$
\[ m(\alpha x) \geq a^q m(x) \quad \text{for all } \xi \geq 1. \]
Here, putting $\beta = \frac{1}{a} > 1$, we obtain
\[ m(\beta^n x) \leq \beta^{q\cdot n} m(x) \quad (n=1,2,\cdots) \]
for all $x$ such that $m(x) = 1$. From this we have
\[ m(\xi x) \leq \beta^q \xi \quad \xi \geq \beta, \]
which shows that $m$ is uniformly $q$-finite. By Theorem 2.1 and (*) we can see $m$ is uniformly $p$-increasing.

Remark 1. The converse of the theorem is not true. For example, set
\[ \phi(u) = \begin{cases} u^{\frac{3}{2}} & u \leq 2 \\ \frac{1}{\sqrt{2}} u^2 & u > 2. \end{cases} \]
Then the $O_{R}LICZ$ space $L[0,1]$ is uniformly 2-finite and uniformly 2-increasing, but it is easily seen that there is an element such that $\frac{||x||}{||x||} < 2$. And for any $1 < \alpha < 2$, we can get the example of modulared space such that $m$ is uniformly 2-finite and uniformly 2-increasing but the norms by $m$ do not satisfy $\inf_{x \neq 0} \frac{||x||}{||x||} \geq \alpha$.

Remark 2. If $R$ is a discrete modulared semi-ordered linear space, the property (*) dose not imply finiteness of $m$, and even if in the case where $m$ is finite, the property (*) does not imply uniform finiteness of $m$. The examples are obtained easily. In this case the equivalent
condition to the property (*) is unknown.

References


