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ON CERTAIN PROPERTY OF THE NORMS
BY MODULARS

By

Tetsuya SHIMOGAKI

Let $R$ be a universally continuous semi-ordered linear space. A functional $m(a)(a \in R)$ is said to be a modular on $R$ if it satisfies the following modular conditions:

1) $0 \leq m(a) \leq \infty$ for all $a \in R$;
2) if $m(\xi a) = 0$ for all $\xi > 0$, then $a = 0$;
3) for any $a \in R$ there exists $a > 0$ such that $m(aa) < \infty$;
4) for every $a \in R$, $m(\xi a)$ is a convex function of $\xi$;
5) $|a| \leq |b|$ implies $m(a) \leq m(b)$;
6) $a \wedge b \neq 0$ implies $m(a + b) = m(a) + m(b)$;
7) $0 \leq a_{\lambda} \uparrow a$ implies $m(a) = \sup_{\lambda \in A} m(a_{\lambda})$.

In $R$, we define functionals $\|a\|, ||a|| (a \in R)$ as follows

$$\|a\| = \inf_{\xi > 0} \frac{1 + m(\xi a)}{\xi}, \quad ||a|| = \inf_{m(\xi a) \leq 1} \frac{1}{|\xi|}.$$ 

Then it is easily seen that both $\|a\|$ and $||a||$ are norms on $R$ and $\|a\| \leq ||a|| \leq 2\|a\|$ for all $a \in R$. $\|a\|$ is said to be the first norm by $m$ and $||a||$ is said to be the second norm by $m$. Let $\bar{R}^m$ be the modular conjugate space of $R$ and $\bar{m}$ be the conjugate modular of $m^{p}$, then we can introduce the norms by $\bar{m}$ as above. It is known that if $R$ is semi-regular, the first norm by the conjugate modular $\bar{m}$ is the conjugate norm of the second norm by $m$ and the second norm by the conjugate modular $\bar{m}$ is the conjugate norm of the first norm by $m$. Since $\|a\|$ and $||a||$ are semi-continuous by (7), they are reflexive norms (cf. [7]).

If a modular $m$ is of $L_{p}$-type, i.e., $m(\xi x) = \xi^{p}m(x)$ for all $x \in R$, $\xi \geq 0$,

1) We owe the notations and the terminologies using here to the book: H. NAKANO [3].
2) The conjugate modular $\bar{m}$ is defined as $m(\overline{a}) = \sup \{\overline{a}(x) - m(x)\}$ for every $\overline{a} \in \bar{R}^m$, where $\bar{R}^m$ is the space of the modular bounded universally continuous linear functionals on $R$. 
then we have $\frac{||x||}{||x||} = p^\frac{1}{p} q^\frac{1}{q}$ for all $0 \neq x \in \mathbb{R}$, where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (In the case of $p=1$, we have $\frac{||x||}{||x||} = 1$). The converse of this is studied by S. Yamamura [5] and I. Amemiya [1]. They proved that if the ratios of two norms are constant for all $0 \neq x \in \mathbb{R}$, it is of $L_p$-type essentially. So in the general case, the ratios of two norms are not constant.

A modular $m$ is said to be bounded if there exist real numbers $1 < p_1 \leq p_\wedge < \infty$, such that

$$\xi^{p_1}m(x) \leq m(\xi x) \leq \xi^{l_2}m(x)$$

for all $\xi \geq 1$ and $x \in \mathbb{R}$. In [6], S. Yamamura obtained that if a modular $m$ on $\mathbb{R}$ is bounded then we have

$$||x|| \geq r ||x||$$

for all $x \in \mathbb{R}$, where $r > 1$ is a fixed constant.

In this paper we investigate the case when the two norms by a modular $m$ satisfy

$$\inf_{0 \neq x \in \mathbb{R}} \frac{||x||}{||x||} = r > 1 \tag{*}$$

(In this case we say that the norms have property (*) throughout this paper).

As showed above, a bounded modular $m$ has that property (*), but the converse of this is not true in general.

In §1 we prove that if the norms by a modular $m$ satisfy the property (*) then it is uniformly finite and uniformly increasing, provided that $\mathbb{R}$ has no atomic element (Theorem 1.1). And we obtain conversely that if a modular $m$ is uniformly finite and uniformly increasing then the norms by $m$ have the property (*) (Theorem 1.4). Thus, we can see that if $\mathbb{R}$ has no atomic element, then the property (*) is equivalent to uniform finiteness and uniform increasingness of modular $m$. Theorem 1.2 shows that uniform simpleness of a modular $m$ implies uniform finiteness, in the case when $\mathbb{R}$ has no atomic element. Finally some special cases, where the property (*) is equivalent to boundedness of modular are discussed.

In §2 we define uniform $p$-properties, that is, uniformly $p$-finite, $p$-increasing, $p$-simple and $p$-monotone modulars, to determine the degrees of uniform finiteness, increasingness and etc. Theorems 2.1 and 2.2
show that there exist the conjugate relations between uniformly \( p \)-finite modular and uniformly \( q \)-increasing modular, where \( 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1 \). On the other hand, Theorems 2.3 and 2.4 show the similar relations between uniformly \( p \)-simple modular and uniformly \( q \)-monotone modular. In the case when \( R \) has no atomic element, we have more precisely than in §1, that if a modular \( m \) is uniformly \( p \)-simple it is uniformly \( p \)-finite (Theorem 2.5). There is a modular which is uniformly finite but not uniformly \( p \)-finite for any \( 1 \leq p < \infty \).

In §3 we prove that if the norms by modular \( m \) have the property (\*) then \( r \) (which appears in (\*)) determines the degrees of uniform finiteness and uniform increasingness of \( m \). Truely, in the case when \( R \) has no atomic element, we obtain that if the norms by a modular \( m \) have the property (\*), \( m \) is uniformly \( p \)-increasing and uniformly \( q \)-finite, where \( p, q \) are positive numbers such that \( r = p^\frac{1}{p}q^\frac{1}{q} \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p \leq q \) (Theorem 3.1). The converse of this is not true in general. We show an example of this fact at the end of this paper.

§1. Let \( R \) be a modulared semi-ordered linear space with a modular \( m \).

A modular \( m \) is said to be uniformly finite, if
\[
\sup_{m(x) \leq 1} m(\xi x) < \infty \quad \text{for all } \xi > 0.
\]

A modular \( m \) is said to be uniformly increasing, if
\[
\lim_{\xi \to \infty} \inf_{m(x) \geq 1} m(\xi x) = +\infty.
\]

In [4; Theorems 5.2, 5.3] it is shown that if a modular \( m \) is uniformly finite, then the conjugate modular \( \overline{m} \) of \( m \) is uniformly increasing and if a modular \( m \) is uniformly increasing then the conjugate modular \( \overline{m} \) is uniformly finite.

Now we shall prove the following

**Theorem 1.1.** Suppose \( R \) has no atomic element. If the norms by a modular \( m \) have the property (\*), then \( m \) is uniformly finite and uniformly increasing.

**Proof.** 1). Let \( r \) be a number, in the sequel, such that \( r = \inf_{0 \neq x \in R} \frac{||x||}{||x||} \).

Then we have
\[
\inf_{0 \neq x \in R} \frac{||x||}{||x||} = r \quad \text{(**').}
\]
In fact we have for every $\bar{x} \in \overline{R}^m$

$$\|\bar{x}\| = \sup_{|x| \leq 1} |\bar{x}(x)| \leq \sup_{r(\|x\|)} |\bar{x}(x)| = \frac{1}{r} \|\bar{x}\|.$$ 

Since the norms $|x|$, $\|x\|$ are reflexive, we obtain (*').

2). If $m$ is not uniformly finite, then there exists a number $\xi_0 \geq 1$ such that

$$\sup_{m(x) \leq 1} m(\xi x) < \infty$$

for all $\xi < \xi_0$.

$$\sup_{m(x) \leq 1} m(\gamma x) = \infty$$

for all $\gamma > \xi_0$.

Since $r > 1$, we obtain a number $a$ such that $1 > a > 0$ and $ar - 1 > 0$, and we can find also $\varepsilon > 0$ such that $a(\xi_0 + \varepsilon) < \xi_0$.

Then by the definition of $\xi_0$, we can find a sequence of elements $\{x_n\}$ $(n=1,2,\cdots)$ such that

$$m(x_n) \leq 1,$$ $$m(a(\xi_0 + \varepsilon)x_n) \leq k,$$ $$m((\xi_0 + \varepsilon)x_n) \geq n$$

$(n=1,2,\cdots)$, where $k$ is a fixed positive number.

Since $R$ has no atomic element, we can obtain also a sequence of projectors $\{[p_n]\}$ $(n=1,2,\cdots)$ such that

$$m(a(\xi_0 + \varepsilon)[p_n]x_n) \leq \frac{k}{n},$$ $$m((\xi_0 + \varepsilon)[p_n]x_n) \geq 1.$$ 

Putting $y_n = (\xi_0 + \varepsilon)[p_n]x_n$, we have

$$m(y_n) \geq 1,$$ $$m(ay_n) \leq \frac{k}{n}$$

$(n=1,2,\cdots)$.

This implies $\lim_{n \to \infty} \frac{1 + m(ay_1)}{a} = \frac{1}{a} < \gamma$ and contradicts (*'), because on the other hand, we have $\|y_n\| \geq 1$ and $\|y_n\| \leq \frac{1 + m(ay_n)}{a}$ for all $n \geq 1$.

Then by 1) $\bar{m}$ is also uniformly finite, thus $m$ is uniformly increasing. This completes the proof.

In the proof of the theorem above, we have shown that if a modular $m$ is not uniformly finite, then there exists a sequence of elements $y_n$ such that

3) We note here that $\bar{m}(x) = \sup_{x \in R^m} \{r(x) - m(x)\} \leq m(x)$ for all $x \in R$ by virtue of the definition of conjugate modular. If $R$ is semi-regular, then modular $m$ is reflexive; i.e. $m(x) = \bar{m}(x) = \sup_{x \in R^m} \{r(x) - m(x)\}$ for all $x \in R$ ([3]; §39).
\[ m(y_n) \geqq 1, \quad \lim_{n \to \infty} m(\xi y_n) = 0 \quad (n=1, 2, \cdots) \]

for some \( \xi > 0 \). Then the sequence \( \{y_n\} (n=1, 2, \cdots) \) is conditionally modular convergent to 0, but it is not modular convergent. A modular \( m \) is said to be uniformly simple if conditionally modular convergence coincides with modular convergence, i.e., \( \lim_{n \to \infty} m(x_n) = 0 \) implies \( \lim_{n \to \infty} m(\xi x_n) = 0 \) for every \( \xi \geqq 0 \).

Thus we have

**Theorem 1.2.** Suppose that \( R \) has no atomic element. If a modular \( m \) is uniformly simple, then it is uniformly finite.

The conjugate property to uniform simpleness of modular is uniform monotoneness. Therefore we obtain also

**Theorem 1.3.** Suppose that \( R \) has no atomic element. If a modular \( m \) is uniformly monotone, then it is uniformly increasing.

The converse part of Theorem 1.1 is always true (without the assumption that \( R \) has no atomic element). That is, we obtain

**Theorem 1.4.** If a modular \( m \) is uniformly finite and uniformly increasing, then the norms by \( m \) have the property (\( * \)).

**Proof.** If the property (\( * \)) is not satisfied, then we can find \( x_n \geqq 0 \) \((n=1, 2, \cdots)\) such that

\[ 1 \leqq ||x|| < 1 + \frac{1}{n}, \quad ||x|| = m(x_n) = 1 \quad (n=1, 2, \cdots). \]

And we can find also \( \xi_n > 0 \) such that

\[ 1 + m(\xi_n x_n) < \left(1 + \frac{1}{n}\right)\xi_n \]

for all \( n \geqq 1 \) by the definition of the first norm.

Considering a subsequence of \( \{\xi_n\} \), it is sufficient for us to investigate only the following cases.

1) In this case, \( \{\xi_n\} \) satisfies \( 0 < \xi_n \leqq 1 \) for all \( n \geqq 1 \). If \( \xi_n \leqq \xi_0 < 1 \) \((n=1, 2, \cdots)\) for some \( \xi_0 < 1 \), then we obtain

\[ \left(1 + \frac{1}{n}\right) > \frac{1 + m(\xi_n x_n)}{\xi_n} \geqq \frac{1}{\xi_0} > 1 \quad (n=1, 2, \cdots). \]

This is a contradiction. Now without a loss of a generality, we may

---

4) A modular \( m \) is said to be uniformly monotone, if \( \lim_{\xi \to 0} \sup_{m(\xi x) \leqq 1} m(\xi x) = 0 \).
assume that
\[ \xi_n \uparrow 1, \quad 1 - \xi_n < \frac{1}{n} \quad (n=1,2,\cdots). \]
Since we have
\[ m(\xi_n x_n) < (1 + \frac{1}{n})\xi_n - 1 \leq \frac{1}{n} \]
and \( m(\xi x) \) is a non-decreasing convex function of \( \xi \geq 0 \), we obtain
\[ m((1+(1-\xi_n))x_n) \geq 1 + \frac{n-1}{n} \quad (n=1,2,\cdots), \]
and furthermore
\[ m((1+n(1-\xi_n))x_n) \geq 1 + (n-1) \quad (n=1,2,\cdots). \]
This implies
\[ \sup_{m(x) \leq 1} \inf_{m(\xi x) \leq 1} \frac{m(\xi x)}{\xi} \leq \lim_{n \to \infty} \frac{m(\xi_n x_n)}{\xi_n} \leq \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1, \]
which contradicts that \( m \) is uniformly finite.

2). In this case, \( \{\xi_n\} \ (n=1,2,\cdots) \) satisfies \( 1 \leq \xi_n \) for all \( n \geq 1 \). By definition of \( \{\xi_n\} \), we have
\[ 1 + \frac{1}{n} \geq \frac{1 + m(\xi_n x_n)}{\xi_n} \geq \frac{1}{\xi_n} + 1 \quad \text{for all } n \geq 1. \]
This implies \( n \leq \xi_n \) for all \( n \geq 1 \). Therefore we may assume \( \xi_n \uparrow +\infty \) \((n=1,2,\cdots)\), so we obtain
\[ \lim_{\xi \to m(x) \leq 1} \frac{m(\xi x)}{\xi} \leq \lim_{n \to \infty} \frac{m(\xi_n x_n)}{\xi_n} \leq \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1, \]
which contradicts that \( m \) is uniformly increasing. This completes the proof.

In the case when a modular \( m \) on \( R \) is of unique spectra ([3]; §54), the property (*) implies boundedness of \( m \). In fact we have

**Theorem 1.5.** If a modular \( m \) on \( R \) is of unique spectra, then boundedness of \( m \) is equivalent to the property (*).

The proof is easily obtained by simple calculations, so it is omitted. In the case of the constant modular ([3]; §55), the property (*) dose not imply simpleness of \( m \), and even in the case of the simple constant modular it dose not generally imply the boundedness of \( m \) (the examples are easily obtained). Only in the particular case, we have
Theorem 1.6. If a modular $m$ on $R$ is constant, monotone complete and $R$ has neither complete constant element nor atomic element, then the property $(\ast)$ is equivalent to boundedness of $m$.

Proof. By Theorem 1.1 $m$ is finite, then $m$ is upper bounded by Theorem 55.10 in [3]. Since $\bar{m}$ is constant and has no complete constant element [3; §55], $\bar{m}$ is also upper bounded, that is, $m$ is lower bounded. Thus $m$ is a bounded modular on $R$.

§ 2. In this section we investigate the degrees of uniform properties of modulars.

Set for $\xi \geq 1$

$$ f(\xi) = \sup_{m(\xi x) \leq 1} m(\xi x) \quad \text{and} \quad g(\xi) = \inf_{m(\xi x) \geq 1} m(\xi x), $$

then $f(\xi)$ and $g(\xi)$ are defined in $[1, \infty)$ and non-decreasing functions. In the following, let $p$ be a number such that $1 < p < \infty$.

Definition 2.1. A modular $m$ on $R$ is said to be uniformly $p$-finite if there exist $r > 0$ and $\xi_0 \geq 1$ such that

$$ f(\xi) \leq r \xi^p \quad \text{for all} \quad \xi \geq \xi_0. $$

Definition 2.2. A modular $m$ on $R$ is said to be uniformly $p$-increasing, if there exist $r > 0$ and $\xi \geq 1$ such that

$$ g(\xi) \geq r \xi^p \quad \text{for all} \quad \xi \geq \xi_0. $$

It is easily seen that if $m$ is uniformly $p$-finite, it is also uniformly $p'$-finite for $p \leq p'$, and if $m$ is uniformly $p$-increasing it is also uniformly $p''$-increasing for $1 \leq p'' \leq p$.

In the sequel, we set $q = \frac{p}{p-1}$. Now we have

Theorem 2.1. If a modular $m$ is uniformly $p$-finite, then the conjugate modular $\bar{m}$ of $m$ is uniformly $q$-increasing.

Proof. We have by the assumption for some $\rho_0 \geq 1, r > 0$,

$$ f(\xi) \leq r \xi^p \quad (\xi \geq \rho_0 \geq 1). $$

If $\bar{m}(\bar{x}) \geq 1, \bar{x} \in \overline{R}^m$ and $0 < a < 1$, we can find $x_0$ such that $\bar{x}(x_0) > a, m(x_0) \leq 1$. For such $x_0$, we have by the definition of conjugate modular

$$ \bar{m}(\lambda \bar{x}) \geq \lambda \bar{x}(\rho x_0) - m(\rho x_0) \geq a \lambda \rho - \gamma \rho^p $$

for all $\rho \geq \rho_o$. This implies

$$ \bar{m}(\lambda \bar{x}) \geq \sup_{\rho \geq \rho_o} \{a \lambda \rho - \gamma \rho^p\} $$
for all $\bar{x} \in \overline{R}^m$ such that $\overline{m}(\bar{x}) \geq 1$.

Then we have for $\lambda \geq \lambda_0 = \frac{r \rho}{\alpha} \frac{\rho_0^p}{\rho}$,

$$\overline{m}(\lambda \bar{x}) \geq \frac{r \rho}{q} \left( \frac{\alpha}{\rho_0^{\frac{p}{q}}} \right)^q \lambda^q.$$ 

Hence the conjugate modular $\overline{m}$ is uniformly $q$-increasing modular by definition.

**Theorem 2.2.** If a modular $m$ is uniformly $p$-increasing, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-finite.

**Proof.** By the assumption we have for some $\gamma$ and $\rho_0$,

$$m(x) \geq 1 \text{ implies } m(\rho x) \geq \gamma \rho^p \quad \text{ for } \rho \geq \rho_0.$$ 

Set $\lambda_0 = \text{Max} \left( \frac{r}{2} \rho_0^{q-1}, 1 \right)$ and for $\lambda \geq \lambda_0$ we define $\rho = \rho(\lambda)$ such that

$$\rho(\lambda) = \left( \frac{2}{r} \lambda \right)^\frac{q}{p}.$$ 

Then we have $\rho \geq \rho_0$. Thus we obtain $\frac{m(\rho x)}{\rho} \geq \gamma \rho^{p-1} = 2\lambda$.

If $\bar{x} \in \overline{R}^m$, $\overline{m}(\bar{x}) \leq 1$ and $1 \leq m(x) < +\infty$, then there is $\xi > 0$ such that

$$m \left( \frac{1}{\xi} x \right) = 1, \quad 0 < \frac{1}{\xi} < 1$$ 

and hence by the definition of the conjugate modular $\overline{m}(\bar{x})$ we obtain

$$\bar{x} \left( \frac{1}{\xi} x \right) \leq \overline{m}(\bar{x}) + m \left( \frac{1}{\xi} x \right) \leq 2.$$ 

For such $\xi$, if $\xi \geq \rho(\lambda)$, then we have

$$\lambda \bar{x}(x) - m(x) = \xi \left\{ \lambda \bar{x} \left( \frac{1}{\xi} x \right) - m \left( \frac{1}{\xi} x \right) \right\} \leq 0,$$

and if $0 < \xi \leq \rho(\lambda)$, then we have

$$\lambda \bar{x}(x) - m(x) \leq \xi \lambda \bar{x} \left( \frac{1}{\xi} x \right) \leq 2\rho \lambda = 2\lambda \left( \frac{2}{r} \lambda \right)^\frac{q}{p}.$$ 

If $\overline{m}(\bar{x}) \leq 1$, $m(x) \leq 1$, we have also

$$\lambda \bar{x}(x) - m(x) \leq \lambda (\overline{m}(\bar{x}) + m(x)) - m(x) \leq 2\lambda.$$ 

Therefore we obtain consequently

$$\overline{m}(\lambda \bar{x}) \leq 2\lambda \rho = \gamma_0 \lambda^q \quad \text{ for all } \lambda \geq \lambda_0$$ 

where $\gamma_0 = 2^q \left( \frac{1}{r} \right)^\frac{q}{p}$. Hence the conjugate modular $\overline{m}$ is uniformly $q$-
finite modular.

As similarly as uniformly $p$-finite modulars, we can define uniformly $p$-simple and uniformly $p$-monotone modular. In order to define them, we set for $0 \leq \xi \leq 1$

$$
\varphi(\xi) = \sup_{m(\xi x) \leq 1} m(\xi), \quad \psi(\xi) = \inf_{m(\xi x) \geq 1} m(\xi).
$$

Then $\varphi(\xi), \psi(\xi)$ are defined in $[0, 1]$ and finite non-decreasing functions.

**Definition 2.3.** A modular $m$ on $R$ is said to be **uniformly** $p$-simple if there exist $\gamma > 0$, and $0 < \xi_0 \leq 1$, such that

$$
\varphi(\xi) \geq \gamma \xi^p
$$

for all $0 \leq \xi \leq \xi_0$.

**Definition 2.4.** A modular $m$ on $R$ is said to be **uniformly** $p$-monotone, if there exist $\gamma > 0$ and $0 < \xi_0 \leq 1$, such that

$$
\varphi(\xi) \leq \gamma \xi^p
$$

for all $0 \leq \xi \leq \xi_0$.

It is easily seen that if $m$ is uniformly $p$-simple, it is also uniformly $p'$-simple for $p \leq p'$, and if $m$ is uniformly $p$-monotone, it is also uniformly $p'$-monotone for $1 \leq p'' \leq p$.

Concerning uniformly $p$-simple and uniformly $q$-monotone modulars there exist the conjugate relations, in fact we have

**Theorem 2.3.** If a modular $m$ on $R$ is uniformly $p$-monotone, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-simple.

**Theorem 2.4.** If a modular $m$ on $R$ is uniformly $p$-simple, then the conjugate modular $\overline{m}$ of $m$ is uniformly $q$-monotone.

The proofs of these theorems are analogous to those of Theorems 4.9, 4.10 in [4] and of Theorems 2.1, 2.2, so it is omitted.

Concerning uniform simpleness and uniform finiteness we proved in Theorem 2.2 that uniform simpleness implies uniform finiteness, provided that $R$ has no atomic element. On uniformly $p$-simple modular we obtain more precisely

**Theorem 2.5.** Let $R$ has no atomic element. If a modular $m$ on $R$ is uniformly $p$-simple, then it is uniformly $p$-finite.

**Proof.** It is known already that $m$ is uniformly finite. If it is not uniformly $p$-finite, then there exists a sequence of real numbers $\xi_n \geq 0$ $(n = 1, 2, \cdots)$ such that

$$
+ \infty > f(\xi_n) > \xi_n^p, \quad \xi_n \uparrow + \infty \quad (n = 1, 2, \cdots).
$$

And by definition of $f(\xi)$, we can choose a sequence of elements $\{x_n\}$ $(n = 1, 2, \cdots)$ such that
Here, we can assume without a loss of generality that

\[ m(\xi_{n}x_{n}) = N_{n} \]

where \( N_{n} \) is a natural number, for every \( n \geq 1 \). Because, if there are \( r > 0 \) and \( \xi_{0} \geq 1 \) satisfying \( m(\xi x) \leq r \xi^{p} \) for every \( \xi \geq \xi_{0} \) such that \( m(\xi x) \) is a natural number, then we have \( m(\xi x) \leq (r + 1) \xi^{p} \) for all \( \xi \geq \xi_{0} \). This shows that \( m \) is uniformly \( p \)-finite.

Then we can find a sequence of projectors \( \{[p_{n}]\} \) (\( n = 1, 2, \cdots \)) by orthogonal decompositions of \( x_{n} \) (\( n = 1, 2, \cdots \)) such that

\[ m([p_{n}] \xi_{n}x_{n}) = 1, \quad m([p_{n}] x_{n}) < \frac{1}{n \xi_{n}^{p}} \quad (n = 1, 2, \cdots), \]

since \( m(\xi_{n}x_{n}) \) is natural number for all \( n \geq 1 \). Set \( y_{n} = [p_{n}] \xi_{n}x_{n} \) and \( \eta_{n} = \frac{1}{\xi_{n}} \) for every \( n \geq 1 \), then we have \( m(y_{n}) = 1 \) and \( m(\eta_{n}y_{n}) < \frac{\eta_{n}^{p}}{n} \). Since \( \lim_{n \to \infty} \eta_{n} = 0 \), we show that \( m \) is not uniformly \( p \)-simple. Thus the proof is completed.

Corresponding to Theorem 2.5 we have

**Theorem 2.6.** Let \( R \) have no atomic element. If a modular \( m \) on \( R \) is uniformly \( p \)-monotone, then it is uniformly \( p \)-increasing.

It will be conjectured that if a modular \( m \) is uniformly finite, then it is uniformly \( p \)-finite for some \( 1 < p < +\infty \). But the following example shows that it is not true.

**Example.** Set \( \phi(u) = \begin{cases} \frac{1}{2} u & u \leq 2 \\ e^{u-2} & u > 2 \end{cases} \)

and consider Orlicz sequence space \( l_{\phi} \). Then \( l_{\phi} \) is uniformly finite as easily seen, but not uniformly \( p \)-finite for any \( 1 < p < +\infty \). This example shows at the same time that there exists a modular \( m \) which is uniformly increasing but not uniformly \( p \)-increasing for any \( 1 < p < +\infty \).

I. Amešiya proved in [2] that if a modular \( m \) on \( R \) is monotone complete and finite, then \( m \) is semi-upper bounded, i.e., \( m(2x) < r m(x) \) for every \( x \) such that \( m(x) \geq \epsilon \) for some fixed \( r, \epsilon > 0 \), provided that \( R \) has no atomic element. Applying this result, it is seen that the above conjecture is affirmative, in the case when \( m \) is monotone complete and \( R \) has no atomic element. In fact we have
Theorem 2.7. Suppose that \( R \) has no atomic element and \( m \) is monotone complete. If \( m \) is uniformly finite (finite) then it is uniformly \( p \)-finite for some \( p > 1 \).

§ 3. To any \( r \) such that \( 1 < r \leq 2 \), there exist a unique pair of positive numbers \((p, q)\) satisfying the following

1) \( r = p \frac{1}{p} q \frac{1}{q} \)
2) \( \frac{1}{p} + \frac{1}{q} = 1 \)
3) \( 1 \leq p \leq 2 \leq q \).

This correspondence is unique and it is easily seen that if \( r_n \) is convergent increasingly to 2, then the corresponding \( p_n (q_n) \) is also convergent increasingly (decreasingly) to 2.

If the norms of modular \( m \) have the property (*) we can find a pair of numbers such that \( r = p \frac{1}{p} q \frac{1}{q} \). It is already seen that \( m \) is uniformly finite and uniformly increasing, provided that \( R \) has no atomic element. Now we shall show that \( (p, q) \) gives the degrees of uniform finiteness and increasingness. In fact we can state

Theorem 3.1. Suppose that \( R \) has no atomic element. If the norms by a modular \( m \) have the property (*), then \( m \) is uniformly \( p \)-increasing and uniformly \( q \)-finite.

Proof. Set \( a = (\frac{p}{q})^{\frac{1}{q}} \), then \( \gamma a - 1 = a^q \).

Thus we obtain by assumption,

\[
m(x) = 1 \implies m(ax) \geq a^q.
\]

If \( m(x) = 1 + \frac{m}{n} \) (for natural numbers \( m < n \)), we can decompose orthogonally \( x = x_1 + x_2 \cdots + x_{n+m} \) such that

\[
m(x_i) = m(x_j) = \frac{1}{n} \quad (i, j = 1, 2, \cdots, n+m).
\]

The numbers of \( i \) such that \( m(ax_i) < a^q m(x_i) \) are less than \( n \), because if there exists \( (i_1, i_2, \cdots, i_n) \) such that \( m(ax_\nu) > a^q m(x_\nu) (\nu = 1, 2, \cdots, n) \), then we have \( m\left(a \sum_{\nu=1}^{n} x_\nu\right) < a^q m\left(\sum_{\nu=1}^{n} x_\nu\right) \) and \( m\left(\sum_{\nu=1}^{n} x_\nu\right) = 1 \). This is a contradiction.

Thus there exists \( \{i_k\} \) \((k = 1, 2, \cdots, m)\) such that \( m(ax_k) \geq a^q m(x_k) \);
1, 2, \ldots, m). Putting $y=\sum_{k=1}^{m}x_{i_k}$ we have $m(x-y)=1$ and

$$m(ay) \geq a^q m(y)$$
$$m(a(x-y)) \geq a^q m(x-y).$$

Hence we obtain $m(ax) \geq a^q m(x)$. Generally, if $1 \leq m(x) < 2$, since $m(\xi x)$ is continuous function of $\xi$, we have also

$$m(ax) \geq a^q m(x).$$

Since $m(x)$ is finite for all $x \in R$ and $R$ has no atomic element, we have for $x$ such that $m(x)=1$

$$m(a\xi x) \geq a^q m(\xi x) \quad \text{for all} \quad \xi \geq 1.$$  

Here, putting $\beta = \frac{1}{a} > 1$, we obtain

$$m(\beta^n x) \leq \beta^{q \cdot n} m(x) \quad (n = 1, 2, \cdots)$$

for all $x$ such that $m(x)=1$. From this we have

$$m(\xi x) \leq \beta^{q \xi} \quad \xi \geq \beta,$$

which shows that $m$ is uniformly $q$-finite. By Theorem 2.1 and (*) we can see $m$ is uniformly $p$-increasing.

**Remark 1.** The converse of the theorem is not true. For example, set

$$\phi(u) = \begin{cases} u^{\frac{3}{2}} & u \leq 2 \\ \frac{1}{\sqrt{2}} u^2 & u > 2 \end{cases}.$$  

Then the $\text{OHI}_{IICZ}$ space $L[0,1]$ is uniformly $2$-finite and uniformly $2$-increasing, but it is easily seen that there is an element such that $\frac{||x||}{||x||} < 2$. And for any $1 < a < 2$, we can get the example of modulared space such that $m$ is uniformly $2$-finite and uniformly $2$-increasing but the norms by $m$ do not satisfy $\inf_{x \neq 0} \frac{||x||}{||x||} \geq a$.

**Remark 2.** If $R$ is a discrete modulared semi-ordered linear space, the property (*) dose not imply finiteness of $m$, and even if in the case where $m$ is finite, the property (*) does not imply uniform finiteness of $m$. The examples are obtained easily. In this case the equivalent
condition to the property (*) is unknown.

References