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POSITIVE LINEAR OPERATORS IN SEMI-ORDERED LINEAR SPACES

By

Tsuyoshi ANDÔ

Since in 1907 O. PERRON [9] discovered a remarkable spectral property of positive matrices and shortly later G. FROBENIUS [1], [2] and R. JENTZSCH [4] investigated and generalized it further, many authors have considered special properties of positive linear operators. Especially M. KREIN and M. A. RUTMAN [5] considered with success a generalization to Banach spaces with a cone. They obtained particularly important results, when the space is lattice ordered or the cone has an interior point. In this paper, we consider spectral properties of positive compact (=completely continuous) linear operators on a universally continuous Banach space (= conditionally complete Banach lattice). Our main aim is to generalize the results of G. FROBENIUS [2] to infinite dimensional spaces.

In §1 preliminary definitions are summarized. In §2 the fundamental theorem on the maximum positive spectrum is proved (Theorem 2.1). In §3 we define completely positive linear operators. The operators of this class play a similar rôle as strongly positive operators in [5]. In §4 we obtain under some additional conditions a necessary and sufficient condition for that a positive compact linear operator is quasi-nilpotent (Theorem 4.7). In §5 the proper values with maximum modulus of a positive compact linear operator are determined (Theorem 5.2).

§1. Preliminaries. We recall briefly definitions from the theory of semi-ordered linear spaces and linear operators. A lattice ordered linear space (with real scalar) $R$ is said to be universally continuous, if for any $a_{\lambda} \geqq 0 (\lambda \in \Lambda)$ there exists $\cap_{\lambda \in \Lambda} a_{\lambda}$. A linear manifold $N$ of $R$ is said to be normal if there exists a positive linear projection $[N]$ of $R$ onto $N$ such that $|x-[N]x| \wedge |y| = 0$ for $x \in R$ and $y \in N$. The projection

*) The author of the present paper expresses his thanks to Dr. S. YAMAMURO at Institute for Advanced Study, Princeton, who kindly communicated the main results of [5], which was not available to him.
onto the normal manifold generated by $a$ is denoted by $[a]$. Normal manifolds and order-projections correspond to each other in one-to-one way.

When we consider spectral problems, it is convenient to define a complex extension $\hat{R}$ of $R$, whose elements consist of all pair of elements of $R$, $(a, b) = a + i b$, the absolute value of $a + i b$ is defined by

$$|a + i b| = \bigcup_{0 \leq \theta \leq 2\pi} |a \cos \theta + b \sin \theta|.$$  

When $R$ is normed, the norm, in this paper, satisfies the following additional condition:

$$|a| \leq |b| \implies ||a|| \leq ||b||,$$

The norm on $\hat{R}$ is defined, when $R$ is normed, by

$$||a + i b|| = ||a + i b||.$$

A norm on $R$ is said to be continuous if $a_\lambda \downarrow \lambda \in \Lambda 0$ implies $||a_\lambda|| \downarrow \lambda \in \Lambda 0$. A bounded linear functional $\tilde{a}$ is said to be universally continuous, if $a_\lambda \downarrow \lambda \in \Lambda 0$ implies $\inf_{\lambda \in \Lambda} |\tilde{a}(a_\lambda)| = 0$. $\widehat{R}$ and $\overline{R}$ denote the space of all bounded linear functionals on $R$ and that of all universally continuous linear functionals respectively. For the other notations and definitions, we refer to [6].

For a bounded linear operator $A$ on a complex Banach space $R$ into itself, $\sigma(A)$ denotes the set of all spectra of $A$, and $\rho(A)$ the resolvent set. If $(\lambda I - A)x = 0$ has a non-trivial solution in $R$, $\lambda$ is said to be a proper value and its solutions are proper elements. We put

$$r(A) = \sup_{\xi \in \sigma(A)} |\xi|$$

$$R(\lambda) = (\lambda I - A)^{-1} \quad \text{for } \lambda \in \rho(A).$$

It is well known (cf. [3]) that

$$r(A) = \lim_{\nu \to \infty} ||A^\nu||^{\frac{1}{\nu}}$$

$A$ is said to be quasi-nilpotent, if $r(A) = 0$.

A bounded linear operator $A$ is said to be compact if the unit sphere is mapped by $A$ into a compact set. We assume in this paper the results of F. Riesz ([10], chap. IV and V) concerning the spectral properties of compact linear operators. If $A$ is a compact linear operator, every non-zero spectrum is a proper value and the corresponding proper manifold is each finite dimensional. $\sigma(A)$ constitutes a totally discon-
nected set with the only possible limiting point 0. For a non-zero complex number \( \lambda \) we can define a bounded linear projection operator \( E(\lambda) \) relative to \( A \) by

\[
E(\lambda) = \frac{1}{2\pi i} \int \frac{1}{\zeta - \lambda} R(\zeta) d\zeta
\]

where the integration is formed along a Jordan curve surrounding \( \lambda \), whose boundary and interior intersect \( \sigma(A) \) in \( \lambda \) alone. The index \( \mu(\lambda) \) of \( \lambda \) is the smallest integer \( n \) satisfying \( (\lambda I-A)^n E(\lambda) = 0 \). For other definitions in operator theories, we refer to [3].

Examples of universally continuous Banach spaces are: \( L_p, l_p \) (\( 1 \leq p \leq \infty \)), and more generally modulared spaces studied in [6].

As we develop in the following a theory of positive compact linear operators in a universally continuous Banach space, it has immediate applications to the theory of integral equations or linear equations with infinite unknowns in the spaces mentioned above.

§ 2. The maximum positive spectrum. Throughout the paper \( R \) denotes a universally continuous semi-ordered Banach space and \( A \) a linear operator on \( R \) into itself, if the contrary is not mentioned.

Though some of results are known under weaker conditions (see [5]), we prove them for the sake of completeness, since under our conditions proofs are sometimes simple.

\( A \) is said to be positive, if \( a \geq 0 \) implies \( Aa \geq 0 \).

**Lemma 2.1.** A positive linear operator is necessarily bounded.

**Proof.** For \( \tilde{a} \in \overline{R} \), putting \( \tilde{b}(x) = \tilde{a}(Ax) \), \( \tilde{b} \) is an \( (o) \)-bounded linear functional on \( R \), so by Theorem 31.3 in [6], is norm-bounded. This means that the image of the unit sphere by \( A \) is weakly bounded. The assertion follows from the known theorem on weakly bounded sets.

**Theorem 2.1.** If \( A \) is positive, \( r(A) \) is in \( \sigma(A) \).

**Proof.** Suppose \( r=r(A) \) is not in \( \sigma(A) \). Then \( R(\lambda) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\lambda^{\nu+1}} \) for \( \lambda > r \) and \( \sup_{\lambda > r} ||R(\lambda)|| < \infty \). Since, considering \( \tilde{R} \), by (1.3)

\[
||R(\lambda e^{i\theta})|| \leq ||R(\lambda)|| \quad \text{for} \quad \lambda > r, \ 0 \leq \theta < 2\pi \text{ and } x > 0,
\]

hence \( \sup_{\lambda > r} ||R(\lambda e^{i\theta})|| < \infty \), this implies \( re^{i\theta} \notin \sigma(A), 0 \leq \theta < 2\pi \), contradicting the assumption (cf. [3]).

**Corollary 2.1.1.** If \( A \) is positive, \( R(\lambda) \geq 0 \) if and only if \( \lambda > r \).

**Proof.** If \( R(\lambda) \geq 0 \), \( \lambda \) is apparently real and by the resolvent equation
$R(\lambda)-R(\mu)=\frac{R(\lambda)R(\mu)}{t^{l}-\lambda}\geqq 0$ for $\mu>\text{Max}(\lambda, r)$. Hence as in Theorem 2.1 $\lambda>r(A)$. The converse part is obvious.

Corollary 2.1.2. Let $A$ be positive. If for a $\lambda>0$, there exists $x\geqq 0$, such that $\lambda x\geqq Ax$, then $\lambda\leq r(A)$.

Proof. If $\lambda/\sigma(A)$, $\lambda<\mu(r)$ by Theorem 2.1. If $\lambda/\rho(A)$, from the hypothesis $R(\lambda)$ is not positive, the assertion follows from Corollary 2.1.

Concerning the indice on the circle of radius $r=r(A)$, we obtain

Theorem 2.2. If $A$ is positive compact with $r=r(A)>0$,

$$\mu(\lambda)\leq \mu(r)$$

for $|\lambda|=r$

Proof. By the Laurent resolution ([3], p. 109) $\sup_{0<\delta<\epsilon}e^{\mu(r)}||R(r+\epsilon)||<\infty$ for a small $\delta$. Since from (1.3) $e^{\mu(r)}||R(e^{i\beta}+\epsilon e^{i\gamma})||\leq e^{\mu(r)}||R(r+\epsilon)||$, we obtain $\mu(\lambda)\leq \mu(r)$.

Theorem 2.3. If $A$ is positive compact $r=r(A)>0$, $A$ has a positive proper element corresponding to the proper value $r$.

Proof. We know that $\lim_{\epsilon\rightarrow 0}e^{\mu(r)}R(r+\epsilon)=(A-rI)^{\mu(r)-1}E(r)$ ([3], p. 109). Since $R(r+\epsilon)\geqq 0$, $(A-rI)^{\mu(r)-1}E(r)\geqq 0$, there exists $x>0$ with $y=(A-rI)^{\mu(r)-1}E(r)x>0$. We obtain that $Ay=ry$ and $y>0$.

§ 3. Completely positive linear operators. A linear operator $A$ is said to be universally continuous, if $a_{\lambda}\downarrow \lambda \in 0$ implies $\bigcap_{\lambda\in A}Aa_{\lambda}=0$.

Lemma 3.1. If $A$ is positive, compact and universally continuous, the range of the conjugate operator $A^{*}$ is contained in $\overline{R}$.

Proof. Since $A$ is positive and compact, for any $a_{\lambda}\downarrow \lambda \in 0$, $\{Aa_{\lambda}\}_{\lambda\in A}$ has a limiting point and it must be equal to 0. So $\lim_{\lambda}Aa_{\lambda}=0$. This implies that $A^{*}\tilde{a}$ is universally continuous for every $\tilde{a}\in \overline{R}$.

An element $a$ of $\overline{R}$ is said to be complete, if $[a]=I$. A bounded linear functional $\tilde{a}$ is said to be complete if $|\tilde{a}|(a)=0$ implies $\tilde{a}=0$. We remark that if $a$ is complete, $\tilde{a}\in \overline{R}$ $|\tilde{a}|(a)=0$ implies $\tilde{a}=0$.

Theorem 3.1. Let $A$ be universally continuous, positive and compact with $r=r(A)>0$. If $A$ has a positive complete proper element, then $\mu(r)=1$ and the proper element corresponds to $r$.

Proof. Let $Aa=\lambda a>0$ and $[a]=I$. If $r\neq \lambda$, $\tilde{a}(a)=0$ for the positive proper element of $A^{*}$ corresponding to $r$, which exists by Theorem 2.3 and is universally continuous by Lemma 3.1. This implies $\tilde{a}=0$. So
\[ \lambda \text{ must be equal to } r. \] 
Suppose that \( \mu(r) \geq 2. \) There exists a positive linear functional \( \tilde{b} = (A^* - rI)^{\mu(r) - 1}E^*(r)x \) as in Theorem 2.3. But by Lemma 3.1 \( \tilde{b}(a) = 0 \) and \( \tilde{b} \in \text{dom} \), this implies \( \tilde{b} = 0 \), contradicting the assumption.

The special property of the proper manifold corresponding to is contained in:

**Theorem 3.2.** If \( A \) is positive compact and \( A^* \) has a positive complete proper element \( \tilde{a} \), the proper manifold corresponding to \( r \) of \( A \) is a linear lattice manifold.

**Proof.** \( \tilde{a} \) corresponds to \( r \) by Theorem 3.1. If \( Aa = ra \), \( Aa^+ \geq ra^+ \), so we obtain \( \tilde{a}(Aa^+ - ra^+) = 0. \) Since \( \tilde{a} \) is complete, this implies that \( Aa = ra \). Hence the proper manifold corresponding to \( r \) is a linear lattice manifold.

**Corollary 3.2.** Under the same assumption as Theorem 3.2, if \( a \in \text{dom} \) is a proper element corresponding to \( \xi \) with \( |\xi| = r \), \( A|a| = r|a| \).

**Proof.** By the definition (1.1), \( Aa = \xi a \) we have \( A \cdot |a| \geq r|a| \). The assertion follows as above.

As a special class of positive linear operators, we define: a positive universally continuous linear operator \( A \) is said to be completely positive if

\[ (3.1) \quad \bigcup_{\nu=1}^{\infty} [A^\nu x] = I \quad \text{for every } x > 0 \]

This means that for a positive \( x > 0 \), \( a \cap A^\nu x = 0 (\nu = 1, 2, \cdots) \) implies \( a = 0. \)

**Lemma 3.2.** If \( A \) is positive and universally continuous, for \( a > 0 \), putting \( \bigcup_{\nu=1}^\infty [A^\nu a] = [N] \), we obtain an invariant normal manifold, that is,

\[ (3.2) \quad A[N] = [N]A[N] \]

**Proof.** Since for \( x > 0 \), \( [N]x = \bigcup_{\nu=1}^\infty (x \cap A(\cdots + A^\nu a) \cap A^\nu a) \) and \( A \) is universally continuous,

\[ A[N]x = \bigcup_{\nu=1}^\infty A(x \cap A(\cdots + A^\nu a)) \] so we obtain \( [A[N]x] \leq \bigcup_{\nu=1}^\infty [Aa + \cdots + A^\nu a] = [N] \).

Complete positiveness corresponds to "Unzerlegbarkeit" in [2], as is seen in the following:

**Theorem 3.3.** A positive universally continuous linear operator is completely positive if and only if it has no non-trivial invariant normal manifold.

This is an immediate consequence of Lemma 3.2.
Lemma 3.3. If $A$ is compact and completely positive, every positive proper element of $A$ (and $A^*$) is complete.

Proof. If $a$ is a proper element of $A$, $[a]=\{A^\nu a\} (\nu=1,2,\cdots)$ and so $[a]=I$. If $\tilde{a}$ is a proper element of $A^*$, $\tilde{a}(a)=0$ implies $\tilde{a}(A^\nu a)=0 (\nu=1,2,\cdots)$. Since $\tilde{a}$ is in $\overline{R}$ by Lemma 3.1, $a=0$.

Theorem 3.4. If $A$ is compact and completely positive with $r=r(A)>0$, the multiplicity of the proper value $r$ is equal to 1 (cf. §5 later).

Proof. By Theorem 3.2 and Lemma 3.3, the proper space corresponding to $r$ is a linear lattice manifold. Since $A$ is completely positive, all positive proper elements are complete, so the multiplicity must be equal to 1.

Next we consider the distribution of proper values corresponding to positive proper elements.

For a bounded linear operator $A$ on a complex Banach space $R$ into itself, the spectral radius of $x$, $r(x,A)$ or $r(x)$ (if there is no confusion), is defined by

$$\tag{3.2} r(x,A) \equiv r(x) = \lim_{\nu \to \infty} \|A^\nu x\|^{\frac{1}{\nu}}$$

This functional satisfies the following properties:

1) $0 \leq r(x) \leq r(A)$
2) $\sup_{x \in R} r(x) = r(A)$
3) $r(ax) = r(x)$ for $a \neq 0$
4) $r(x+y) \leq \text{Max} \{r(x), r(y)\}$
5) $r(Ax) = r(x)$

We remark that $r(x)$ is nothing but the maximum modulus of singularities of analytic continuation of $R(\lambda)x (\lambda \in \rho(A))$.

Since the set of all spectra of a compact operator is a totally disconnected set with the only possible limiting point 0, the functional $r(x)$ is rather convenient.

Lemma 3.4. If $A$ is compact, the functional $r(x)$ satisfies the following:

a) $r(x) = \text{Max} \{|\lambda| : E(\lambda)x \neq 0\}$ for $x \neq 0$

b) $r(x)$ is lower semi-continuous,

c) the range of $r(x)$ coincides with the set $\{|\lambda| ; \lambda \in \sigma(A)\}$. 
Proof. a) is an immediate consequence of remarks stated above. b) follows from a). c) is evident, since every non-zero spectrum is a proper value.

For a positive compact linear operator, we obtain:

Theorem 3.5. If $A$ is positive, universally continuous and compact, the set \{r(x); r(x) > 0, x > 0\} coincides with the set of all non-zero proper values corresponding to positive proper elements.

Proof. For $\lambda > 0$, the set $S_{\lambda} = \{x; r(|x|) \leq \lambda\}$ is a closed linear manifold by Lemma 3.4. If $0 \leq x_\rho \uparrow_0 x (x_\rho \in S_{\lambda}, x \in S_{\lambda}),$ because $E(\xi)$ is universally continuous, hence $S_{\lambda}$ is normal. If there exists $x \in S_{\lambda}$ such that $r(|x|) = \lambda$, by formula (5) $A[S_{\lambda}] = [S_{\lambda}]A[S_{\lambda}]$ and $r(A[S_{\lambda}]) = \lambda$. Hence by Theorem 2.1 there exists $0 \leq a_\lambda \in S_{\lambda}$ such that $Aa_\lambda = \lambda a_\lambda$. Conversely if $Aa = \rho a > 0$, $r(a) = \rho$.

Theorem 3.6. Let $A$ be positive, universally continuous and compact, with $r = r(A) > 0$, $A$ is completely positive if and only if

a) $A$ has a unique positive proper element (up to scalar) which is complete,

b) $r(x) > 0$ for every $x > 0$.

Proof. If $A$ is completely positive, then by Theorems 3.2 and 3.4 the positive proper element is unique and complete. Since $A^*$ has a positive complete proper element $\tilde{a}$ corresponding to $r$, for any positive $a > 0$, $r(a) \geq \lim_{\nu \to \infty} |\tilde{a}(A^\nu a)|^{\frac{1}{\nu}} = r$. Conversely suppose that $A$ satisfies a) and b). If a normal manifold $N$ is invariant relative to $A$, by b) $r(A[N]) > 0$ and by Theorem 2.3 there exists a positive proper element in $N$ which is not complete.

We may replace the condition b) by the condition

b') $A^*$ has a complete positive proper element.

Lemma 3.5. For a compact linear operator $A$ on a Banach space $R$

$$\sup_{|\lambda| = r} \mu(\lambda) \leq 1 \; \text{if and only if} \; \|A^\nu\| \leq M \cdot r^\nu \quad (\nu = 1, 2, \ldots)$$

for some $M$, where $r = r(A)$. And in this case

$$E(r) = \lim_{\nu \to \infty} \frac{1}{\nu} \sum_{k=1}^{\nu} \left( \frac{A}{r} \right)^k$$

(3.3)

The proof is well known (cf. [10] and [5]).

If the maximum spectrum is not simple, the following decomposition holds:
Theorem 3.7. Let $A$ be positive, universally continuous and compact. If $A$ and $A^*$ both have positive complete proper elements, there exists a decomposition of identity such that

\[ \sum_{\nu=1}^{n} [a_{\nu}] = I, \quad [a_{\nu}] [a_{\mu}] = 0 \quad (\nu \neq \mu) \]

and $A$ is completely positive on $[a_{\nu}] R (\nu=1,2,\ldots,n)$.

Proof. By Lemma 3.5 and Theorem 3.2 the positive projection $E(r)$ is written in a form $E(r)x = \sum_{\nu=1}^{n} \tilde{a}_{\nu}(x)a_{\nu}$ such that

\[ Aa_{\nu} = ra_{\nu} > 0, \quad A^{*}\tilde{a}_{\nu} = r\tilde{a}_{\nu} > 0, \quad \tilde{a}_{\nu}(a_{\mu}) = \delta_{\nu\mu} \quad (\nu, \mu=1,2,\ldots,n). \]

Since $\bigcup_{\nu=1}^{n} [a_{\nu}] = \bigcup_{\nu=1}^{n} [\tilde{a}_{\nu}]^{R} = I$ and $A[a_{\nu}] = [a_{\nu}]A[a_{\nu}]$, $A^{*}[\tilde{a}_{\nu}] = [\tilde{a}_{\nu}]A^{*}[\tilde{a}_{\nu}]$ $(\nu=1,2,\ldots,n)$, $A$ is completely positive on $[a_{\nu}] R (\nu=1,2,\ldots,n)$.

We define a somewhat weaker condition than complete positiveness: a positive linear operator $A$ is said to be naturally decomposable, if there exists a decomposition of identity $[N_{\rho}](\rho \in \Lambda)$ and $[M]$ such that

\[ I = \sum_{\rho \in \Lambda} [N_{\rho}] + [M], \quad [N_{\rho}][N_{\rho'}] = 0 \quad (\rho \neq \rho'), \quad [N_{\rho}][M] = 0 \]

\[ A[N_{\rho}] = [N_{\rho}]A \quad \text{and} \quad A[M] = [M]A, \]

\[ r(A[N_{\rho}]) > 0 \quad (\rho \in \Lambda) \quad \text{and} \quad r(A[M]) = 0, \]

and $A$ acts as a completely positive operator on $[N_{\rho}] R (\rho \in \Lambda)$.

Theorem 3.8 Let $A$ be positive, universally continuous and compact. $A$ is naturally decomposable if and only if

\[ \bigcup [a] = \bigcup [\tilde{a}]^{R} \]

where a) (and $\tilde{a}$) varies in all positive proper elements of $A$ (and of $A^{*}$ respectively).

Proof. If $A$ is naturally decomposable with the decomposition (3.5), then by Lemma 3.3. $\bigcup [a] = \bigcup [\tilde{a}]^{R} = \bigcup_{\rho \in \Lambda} [N_{\rho}]$. Conversely, let $\bigcup [a] = \bigcup [\tilde{a}]^{R}$. We arrange the proper values of $A$ corresponding to positive proper elements in the descending order $\lambda_{1} > \lambda_{2} > \cdots$. Since each corresponding proper manifold is finite dimensional, there exist maximum projectors $[p_{\nu}]$ corresponding to $\lambda_{\nu} (\nu=1,2,\ldots)$. Analogously we obtain $\tilde{q}_{\nu}$ relative to $A^{*}$ corresponding to $\mu_{1} > \mu_{2} > \cdots$. By hypothesis $\bigcup_{\nu} [p_{\nu}] = \bigcup_{\nu} [\tilde{q}_{\nu}]^{R} = \bigcup_{\rho \in \Lambda} [N_{\rho}]$ and $\tilde{q}_{\nu}(p_{\nu}) = 0$ if $\lambda_{\nu} \neq \mu_{\nu}$. We obtain $[p_{\nu}] = [\tilde{q}_{\nu}]^{R} (\nu=1,2,\ldots)$ and $[p_{\nu}]A = A[p_{\nu}]$. 


Since \((1 - \bigcup_{\nu=1}^{\infty} [p_{\nu}])A\) is quasi-nilpotent by Theorem 2.3, we obtain the assertion by Theorem 3.7.

Next we consider a characterization of natural decomposability analogous to Theorem 3.6. For a projection \([N]\), we put

\[
r([N]) = \sup_{[N]x=x} r(x) \quad \text{and} \quad r^*([N]) = \sup_{[N]x=x} r^*(x)
\]

where

\[
r^*(x) = \lim_{\nu \to \infty} \|A^{*\nu}x\|^\frac{1}{\nu}
\]

**Theorem 3.9.** Let \(A\) be positive, universally continuous and compact, and \(R\) semi-regular. \(A\) is naturally decomposable, if and only if it satisfies

a) \[r([N]) = r^*([N]) \quad \text{for every projection \([N]\),}
\]

b) \[\sup_{\nu=1, \ldots} \frac{\|A^\nu x\|}{r(x)^\nu} < \infty \quad \text{for } x > 0 \text{ with } r(x) > 0.
\]

**Proof.** If \(A\) is naturally decomposable with the decomposition (3.5), it is easy to see that for a projection \([N]\) and \(x > 0\)

\[
r([N]) = \sup_{[N]x=x} r(A[N]) = r^*([N])
\]

\[
r(x) = \sup_{[N]x=x} r(A[N])
\]

b) follows from Lemma 3.5. Conversely, suppose that a) and b) are satisfied. Put \(S_{\lambda} = \{x; r(|x|) \leq \lambda\}\) and \(\overline{S}_{\lambda} = \{\overline{a}; \overline{a} \in \overline{R}, r^*(|\overline{a}|) \leq \lambda\}\). By Lemma 3.4 and a) we have \([S_{\lambda}] = [\overline{S}_{\lambda}]^R\), hence \(A[S_{\lambda}] = [S_{\lambda}]A\). If \(a\) and \(b\) are positive proper elements corresponding to different positive proper values \(\lambda_1\) and \(\lambda_2\) respectively. If \([p] = [a][b] \neq 0\), as in the proof of Theorem 3.6,

\[
r([p]) \leq \min \{r([a]), r([b])\} = \min \{\lambda_1, \lambda_2\}
\]

and

\[
r^*([p]) \geq \max \{r([a]), r([b])\} = \max \{\lambda_1, \lambda_2\}
\]

contradicting a). So \(a \wedge b = 0\). Let \(\lambda_1 > \lambda_2 > \ldots\) be proper values of \(A\) corresponding to positive proper elements in the descending order. Putting \([S_{\nu}] = [S_{\lambda_1}] - [S_{\lambda_2}]\), we have \(A[S_{\nu}] = [S_{\nu}]A\ (\nu = 1, 2, 3, \ldots)\). By Lemma 3.5 and b) we can prove that \(E(\lambda_{\nu})x > 0\) and \(r(x) = \lambda_{\nu}\) for \(0 < x \in [S_{\nu}]R\). Hence there exists \(0 < \overline{a} \in \overline{R}\) such that

\[
A^*\overline{a} = \lambda_{\nu}\overline{a} \quad \text{and} \quad [\overline{a}]^R = [S_{\nu}]
\]

Again using a), we obtain that there exists \(0 < a \in R\) such that \(Aa = \lambda_{\nu}a\).
and \([a]=[S_{\nu}].\) Now Theorem 3.7 is applicable.

§ 4. Quasi-nilpotent operators. In this § we consider relations between \(A\) and its restriction to normal manifolds.

Lemma 4.1. Let \(A\) be a compact linear operator on a Banach space \(R.\) If \(F\) is a bounded linear operator such that \(F^{2}=F\) and \(AF=FAF,\) then

\[
\sigma(A) = \sigma(AF) \cap \sigma((I-F)A(I-F)) ,
\]

\[
r(A) = \max \{r(AF), r((I-F)A(I-F))\}
\]

Proof. Let \(0 \neq \lambda \in \rho(AF) \cap \rho((I-F)A(I-F))\). If \((\lambda I-A)x=0, (I-F)(\lambda I-F)A(I-F))x=0\) hence \((I-F)x=0\) and similarly \(Fx=0\), so, \(x=0\), hence \(\lambda \in \rho(A)\). Conversely, if \(\lambda \in \rho(A), (\lambda I-A)^{*}F^* = \rho((I-F)A(I-F))\). Similarly \(\lambda \in \rho(A^*)\) is one-to-one on \(FR\), hence by Riesz’s theorem \(\lambda \in \rho(AF)\). Similarly \(\lambda \in \rho(A^*)\) is one-to-one on \(FR\), hence by Riesz’s theorem \(\lambda \in \rho(AF)\).

For positive linear operators \(A\) and \(B\) such that \(A \geq B\), it is evident that \(r(A) \geq r(B)\). In particular, for any projection \([N]\), \(r([N]A[N]) \leq r(A)\).

Theorem 4.1. Let \(A\) be positive, universally continuous and compact with \(r=r(A)>0\). \(A\) is completely positive, if and only if for a projection \([N]\) \(r([N]A[N]) = r(A)\) implies \([N] = I\).

Proof. Suppose first that \(A\) is completely positive. If \(r([N]A[N]) = r\), by Theorem 2.3 there exists \(a>0\) such that \([N]A[N]a = ra\). Since \(Aa \geq ra\), as in the proof of Theorem 3.2, we obtain \(Aa = ra\). Complete positiveness implies \([N] = [a] = I\). Next suppose that \(A\) is not completely positive. There exists by Theorem 3.3 a non-trivial normal manifold \([N]\) such that \(A[N] = [a] = I\). Lemma 4.1 shows that \(\max \{r(A[N]), r((I-[N])A(I-[N]))\} = r\).

Between a positive proper value distinct from \(r=r(A)\) and \(r([N]A[N])\) the following relation holds:

Theorem 4.2. Let \(A\) be positive and compact, and \(\lambda\) a positive proper value distinct from \(r=r(A)\). For any non-zero projection \([N]\) there exists a non-zero projection \([M]\) such that \([N] \geq [M]\) and \(r((I-[M])A(1-[M])) \geq \lambda\).

Proof. There exists \(a \neq 0\) such that \(AAa = \lambda a\), so \(AAa \geq \lambda a\) and \(AAa \leq \lambda a\). By Corollary 2.1.2, \(r([a]^+)A[a^+] \geq \lambda\) (and \(r([a^-]A[a^-]) \geq \lambda\)) if \([a^+] \neq 0\) and \([a^-] \neq 0\). If \([N][a]=0\), we put \([M]=[N]\). If \([N][a] \neq 0\) and \(a^-=0\), we have \(AAa = \lambda a>0\) and \(r([a]A[a]) = \lambda\). Since by Lemma 4.1 Max \(\{r(A[a]), r((I-[a])A(1-[a]))\} = r\), we put \([M]=[N][a]\). If \([N][a^+] \neq 0\) and \([a^-] \neq 0\), we put \([M]=[N][a^+]\). The other case is treated similarly.

Corollary 4.2. Under the same conditions as in Theorem 4.2, if \(p is
a discrete element of $R$, $r((1-[p])A(1-[p]))\leq\lambda$.


Lemma 4.2. If $A_{\lambda} (\lambda \in \Lambda)$, $(\Lambda$ being a directed set), are compact linear operators defined on a Banach space, such that $\lim_{\lambda} ||A_{\lambda} - A|| = 0$, then

\begin{align}
\lim_{\lambda} \sigma(A_{\lambda}) &= \sigma(A) \quad \text{in the sense of metric}, \\
\lim_{\lambda} r(A_{\lambda}) &= r(A).
\end{align}

The proof is found in [8].

Lemma 4.3. Let $F_{\lambda} (\lambda \in \Lambda)$, $(\Lambda$ being a directed set), be bounded linear operators on a Banach space $R$ such that $F_{\lambda}^{2} = F_{\lambda} (\lambda \in \Lambda)$, $F^{2} = F \sup_{\lambda \in \Lambda} ||F_{\lambda}|| < \infty$ and $\lim_{\lambda} F_{\lambda} x = F x (x \in R)$. If $A$ is a compact linear operator on $R$, then $\lim_{\lambda \in \Lambda} r(F_{\lambda} A F_{\lambda}) = r(F A F)$.

Proof. Since the image of the unit sphere by a compact linear operator is relatively compact, it is easy to see that $\lim_{\lambda \in \Lambda} ||F_{\lambda} A F_{\lambda}|| = 0$. By Lemma 4.2, $\lim_{\lambda} r(F_{\lambda} A F_{\lambda}) = r(F A F)$. But since

\[ r(F A F) = \lim_{\nu \to \infty} \| (F A F)^{\nu} \|^{\frac{1}{\nu}} = \lim_{\nu \to \infty} \| (F A F)^{\nu - 1} A \|^{\frac{1}{\nu}} < r(F A F)^{2} \]

and $\lim_{\lambda} F_{\lambda} x = F x (x \in R)$, we obtain

\[ r(F_{\lambda} A F_{\lambda}) = r(F A F). \]

Theorem 4.3. Let $R$ have no discrete element and be of continuous norm. If $A$ is positive compact, there exist $p_{\lambda} \in R$ ($0 < \lambda \leq r(A)$), such that

\begin{align}
[p_{\lambda}] &\leq [p_{\nu}] \quad (0 < \lambda \leq \nu) \\
r([p_{\lambda}] A [p_{\lambda}]) &= \lambda \quad (0 < \lambda \leq r(A))
\end{align}

Proof. We choose, Zorn’s lemma, a maximal linearly ordered family of projections $[q_{\rho}]$. The assumptions on $R$ and Lemma 4.3 imply that $\sup_{\xi < \rho} r([q_{\xi}] A [q_{\xi}]) = \inf_{\xi > \rho} r([q_{\xi}] A [q_{\xi}])$. We can choose $[p_{\lambda}]$ from $[q_{\rho}]$ with $r([q_{\rho}] A [q_{\rho}]) = \lambda$, the remaining part is easily proved.

In the operator theory, it is important to study conditions assuring non-quasi-nilpotentness. Naturally a problem arises whether complete positiveness implies non-quasi-nilpotentness. We have been able to solve this problem only under some additional conditions.

A bounded linear operator $A$ is said to be totally continuous, if for
any positive $a \in R$ and positive $\overline{a} \in \overline{R}$

\[(4.6) \quad [\overline{p}] \overline{x}(A[\overline{p}]x) = \int \Phi(q, \mathfrak{p})[\overline{p}] \times \overline{\mathcal{D}}R \text{ for } [\overline{p}] \leq [\overline{a}], \]

for a fixed Borel function $\Phi(q, \mathfrak{p})$ on the product space $\mathfrak{E} \times \mathfrak{E}$ of the proper space of $R$ (see [7] §5). H. NAKANO [7] proved that $A$ is totally continuous if and only if $|a_\nu| \leq a$ ($\nu = 1, 2, \cdots$) $\lim_{\nu \rightarrow \infty} a_\nu = 0$ (star-convergence) implies $\lim_{\nu \rightarrow \infty} Aa_\nu = 0$. (It is easy to prove that here star-convergence may be replaced by weak-convergence).

**Lemma 4.4** For a bounded linear operator $A$ on a Banach space, put $B = \sum_{\nu=1}^{\infty} \frac{A^\nu}{\lambda^{\nu+1}}$ for some $\lambda > r(A)$. If $B$ has a non-zero spectrum, $A$ has one also. If $A$ is positive and compact, $B$ is so.

**Proof.** Since $R(\lambda) = \frac{1}{\lambda} I + B$, by the spectral mapping theorem (cf. [3] p. 122) $\sigma(A) = \left\{ \lambda - \frac{1}{\lambda + \xi} ; \xi \in \sigma(B) \right\}$, if $\sigma(B)$ contains non-zero number, $\sigma(A)$ does also.

**Lemma 4.5.** Let $R$ be reflexive as a Banach space. If $A_\nu$, $A$ ($\nu = 1, 2, \cdots$) are positive compact such that $A_1 \leq A_2 \leq A_3 \leq \cdots$ and $\lim_{\nu \rightarrow \infty} A_\nu a = Aa$ $(a \in R)$, then $\lim_{\nu \rightarrow \infty} ||A_\nu - A|| = 0$.

**Proof.** Considering $A$ as a continuous function $\overline{a}(Aa)$ on $S \times \overline{S}$, where $S$ and $\overline{S}$ are the positive unit spheres of $R$ and $\overline{R}$ respectively, topologized by weak topologies. Since $S \times \overline{S}$ is compact, the assertion follows from the well-known theorem of Dini and the definition of norms of operators.

**Theorem 4.5.** Let $R$ be reflexive as a Banach space. If $A$ is compact, totally continuous and completely positive, then $A$ is not quasi-nilpotent.

**Proof.** By Lemma 4.4 considering the compact positive operator $B$, we may assume that $[Ax] = I$ for every $x > 0$. Further we may assume that for some $a > 0$ and $\overline{a} > 0$ $[a] = [\overline{a}]^{=} = I$ and $\overline{a}(a) = 1$. Suppose that $A$ is represented in a form (4.6). $A^2$ satisfies the same conditions as $A$ and the corresponding function may be given by

$\Psi(q, p) = \int \Phi(q, \mathfrak{s}) \Phi(\mathfrak{s}, p) \overline{a}(d\mathfrak{sa})$
If $\Psi(q, p)$ vanishes on a set of positive measure, from the measure theory, there exists a measurable subset $\mathcal{N}\subset\mathcal{G}$ such that

a) the measure of $\mathcal{N}$ is positive, (the measure being defined by $\overline{a}(\{p\}a))$.

b) $\text{meas}(\mathcal{B}_p) > 0$ for $p\in\mathcal{N}$, where

$$\mathcal{B}_p = \{q; \int \varphi(q, \mathcal{S}) \varphi(\mathcal{S}, p) \overline{a}(d\mathcal{S}a) = 0\}$$

c) $\varphi(q, p)$ is measurable with respect to $J$ for $\mathcal{H}\in\mathcal{H}$.

If $\varphi(\mathcal{S}, p) > 0$ on a set of positive measure $\mathcal{C}$ for some $p\in\mathcal{N}$ $\varphi(q, \mathcal{S})=0$ almost everywhere on $\mathcal{G}\times\mathcal{N}$, contradicting the assumption that $[Ax] = I$ for every $x>0$. Thus $\varphi(p, q)=0$ almost everywhere on $\mathcal{C}\times\mathcal{N}$, also contradicting the assumption. Hence $\varphi(q, p)>0$ almost everywhere. It is known [7] that $\overline{a}\otimes a = \bigcap_{\nu=1}^{\infty} \{A^{2}\nu \leftrightarrow \overline{a}\otimes a\}$ and $r(\overline{a}\otimes a)=1$. Lemmas 4.5 and 4.2 imply $r(A^{2})= r(A)^{2}>0$.

Next theorem is proved in [11], but we give a somewhat different proof.

**Theorem 4.6.** Let $F_{\lambda} (0 \leq \lambda \leq 1)$ be bounded linear operators defined on a Banach space $R$ such that $F_{0}=0$, $F_{1}=I$, $F_{\lambda} F_{\mu}=F_{\min(\lambda, \mu)}$ and $\lim F_{\rho} x = \lim F_{\rho} x = F_{\lambda} x (x\in R)$. If $A$ is a compact linear operator on $R$ such that $AF_{\lambda}=F_{\lambda} AF_{\lambda} (0 \leq \lambda \leq 1)$, then $A$ is quasi-nilpotent.

**Proof.** Since by Lemma 4.1 $\max \{r(F_{\lambda} A F_{\lambda}), r((I-F_{\lambda}) A (I-F_{\lambda}))\} = r(A)$, there exists, by induction, sequences $\lambda_{\nu} \uparrow \lambda, \alpha$ and $\mu_{\nu} \downarrow \beta$ such that $r(F_{\nu} - F_{\lambda} \nu) A (F_{\nu} - F_{\lambda} \nu) = r(A), (\nu=1, 2, \cdots)$. But by Lemma 4.3 $r(A) = r(F_{\lambda}) A (F_{\nu} - F_{\lambda} \nu)$. Continuing this method, we obtain $r(A)=0$.

Combining Theorems 4.5 and 4.6, we obtain a generalization of criteria of Volterra-type ([10] p. 147).

**Theorem 4.7.** Let $R$ be reflexive as a Banach space and have no discrete element, and $A$ be positive, compact and totally continuous. Then $A$ is quasi-nilpotent if and only if there exist projections $[N_{\lambda}] (0 \leq \lambda \leq 1)$, such that

$$[N_{0}] = 0, [N_{1}] = \bigcup_{x\in R} [A x] \quad \cap [N_{\rho}] = \bigcup_{\rho \leq \lambda} [N_{\rho}] = [N_{\lambda}]$$

$$A[N_{\lambda}] = [N_{\lambda}] A[N_{\lambda}] \quad (0 \leq \lambda \leq 1).$$

**Proof.** Since by Theorem 4.5 any non-trivial invariant normal manifold contains the other non-trivial one, the proof proceeds as in Theorem 4.3.

§ 5. **Proper values with maximum modulus.** Here the distribution
of proper values with maximum modulus of a positive compact linear operator is considered.

Lemma 5.1. If $A$ is compact and completely positive with $r=r(A)>0$, the proper values with maximum modulus are all simple and are the solutions of the equation

\[(5.1) \quad \xi^k - r^k = 0 \quad \text{for some } k.\]

Proof. Since by Theorem 3.1 $\mu(r)=1$, the assertion follows from Theorem 8.1 of [5].

Lemma 5.2. For a compact linear operator $A$ on a Banach space $R$, the following conditions are equivalent to each other

1) $\lim_{\nu \to \infty} A^\nu x = E(1)x$ (for every $x \in R$)
2) $\lim_{\nu \to \infty} A^\nu x = E(1)x$ (for every $x \in R$)
3) $\lim_{\nu \to \infty} ||A^\nu - E(1)|| = 0$
4) $r(A) \leq 1$ and 1 is the only possible proper value with modulus 1.

The proof is similar to that of Lemma 3.5.

Theorem 5.1. Let $A$ be compact and completely positive with $r(A)=1$. The following conditions are equivalent:

1) 1 is the unique proper value with maximum modulus,
2) $A^\nu x$ exists and is complete for every $x > 0$,
3) $A^\nu$ ($\nu = 1, 2, \ldots$) are all completely positive.

Proof. By Theorem 2.1 and Lemma 5.2, 1) and 2) are equivalent. Let 2) be satisfied. Since $\lim_{\nu \to \infty} A^\nu x = E(1)x$, 3) follows from 2) by the definition. Finally if $A^\nu$ ($\nu = 1, 2, \ldots$) are all completely positive and $\lambda$ is a proper value with $|\lambda|=1$ distinct from 1, then by Lemma 5.1 there exists a positive integer $k$ such that $\lambda^k=1$, so 1 is a proper value of $A^k$ of multiplicity greater than 1, contradicting the assumption by Theorem 3.4.

Lemma 5.3. Let $R$ be reflexive as a Banach space and $A$ be compact. If $[N_A]_{\lambda \in A} \subseteq \sigma(A)$ for all $N_A$ and $\sigma([N_A]_{\lambda \in A}) \subseteq \sigma(A)$, then $\sigma([N] A[N]) \subseteq \sigma(A)$.

Proof. The reflexivity implies that that the norms of $R$ and $\bar{R}$ are both continuous. Since $\lim_{\nu \to \infty} ||[N_A]_{\lambda \in A} - [N] A[N]|| = 0$, by Lemma 4.2 $\sigma([N] A[N]) = \lim_{\lambda \in A} \sigma([N_A]_{\lambda \in A}) = \sigma(A)$.

Theorem 5.2. Let $R$ be reflexive as a Banach space. If $A$ is positive
compact, the proper values with maximum modulus coincides with the solutions of the equation

\[
\prod_{i=1}^{n} (\xi^{k_i} - r^{k_i}) = 0
\]

where \( k_i(i=1,2,\ldots,n) \) are some positive integers.

Proof. Reflexivity of \( R \) implies universal continuity of \( A \). Let \( \lambda \) be a proper value with maximum modulus. Considering, by Zorn's lemma, a maximal linearly ordered family \([N_\rho]_{\rho \in \Lambda} \) such that \( \lambda \in \sigma([N_\rho] A[N_\rho]) \subseteq \sigma(A) \). Putting \( [N] = \cap [N_\rho] \), by Lemma 5.3, we obtain \( \lambda \in \sigma([N] A[N]) \subseteq \sigma(A) \). Since \( r([N] A[N]) \leq r(A) \), \( \lambda \) is a proper value of \([N] A[N] \) with maximum modulus. By Lemma 4.1, the maximal hypothesis implies that there is no \( 0 < [M] < [N] \) with \( [N] A[M] = [M] A[M] \), that is, \([N] A[N] \) is completely positive. Hence by Lemma 5.1 \( \lambda \) is a solution of an equation \( \xi^k = r^k \) for some \( k \) and all its solutions are contained in \( \sigma([N] A[N]) \subseteq \sigma(A) \). Since \( A \) has only a finite number of proper values on the circle with radius \( r \), this completes the proof.

References