MODULARS ON SEMI-ORDERED LINEAR SPACES II[1)]
APPROXIMATELY ADDITIVE MODULARS

By

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The additive modulars on universally continuous semi-ordered linear spaces were defined firstly by H. NAKANO and discussed in his book [8]; 1950. These spaces are more general and more extensive than the so-called ORLICZ spaces which were defined by W. ORLICZ in 1931.

Recently he, jointed with Y. MIYAKAWA gave the definitions of more general modulars on universally continuous semi-ordered linear spaces [11]. Above all they defined two kinds of semi-additive modulars which may be considered as slight generalizations of additive modulars. One of them is the upper semi-additive modular and the other is the lower semi-additive modular, and it will be proved that a modular which is upper semi-additive and lower semi-additive at the same time, is an additive modular. But there exist many examples which are upper or lower semi-additive, but not additive modulars. For examples, a norm of a universally continuous linear space can be considered a linear and lower semi-additive modular if the norm is semi-continuous, and in many cases it is not an additive modular except the so-called L-type norm. Furthermore, for a semi-continuous norm $\|x\|$ of a universally continuous semi-ordered linear space R, if we define $m(x)$ as

$$\begin{cases} m(x) = 0 & \text{if} \quad \|x\| \leqq 1 \\ m(x) = +\infty & \text{otherwise}, \end{cases}$$

then we have an upper semi-additive modular $m(x)$ on R, and it is not generally additive except the so-called M-type norm.

The purpose of this paper is to investigate the theory of some classes of semi-additive modulars which are near to additive cases, and discuss the relations with additive modulars. They are the upper semi-additive modulars for which there exists the greatest among the additive ones smaller than the given modulars, or the lower semi-

1) This paper continues to the paper: Y. MIYAKAWA and H. NAKANO [11].

additive modulars for which there exists the least among the additive ones greater than the given modulars.

In §1 and §2, we shall discuss the semi-additive modulars and define the approximately additive modulars. Furthermore we shall show that any spaces having semi-additive modulars can be decomposed into the approximately additive part and the singular part.

The last part of §2 and §3 will be devoted to find some conditions that semi-additive modulars are approximately additive ones or not. In §4 and §5, we shall investigate the equivalence problems and the uniform properties such as uniformly simpleness, between approximately additive modulars and additive ones. The Bi-modulars which were defined in [11] will be discussed in §6, along the line of the present methode. Another approximate methode to additive modulars will be considered in §7.

Throughout the paper we use the notations and terminologies same as [8] and [11].

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§1. Semi-additive modulars. Let R be a universally continuous linear space; in other words conditionally complete vector lattice. We say that a functional $m(x)$ $x\in R$, whose value takes real or $+\infty$, is an *additive modular* on R if it satisfies the following conditions.

1) $0\leqq m(x)\leqq +\infty$.
2) If $m(\xi x)=0$ for every real ξ, then $x=0$.
3) For every element $x\in R$ there exists $\alpha\neq 0$ such that
$$m(\alpha x)<+\infty.$$
4) $m(\alpha x+\beta y)\leqq \alpha m(x)+\beta m(y)$ if $\alpha+\beta=1$, $\alpha,\beta\geqq 0$ for every x, y in R.
5) $|x|\leqq |y|$ implies $m(x)\leqq m(y)$.
6) If $0\leqq x_\lambda \uparrow_{\lambda\in\Lambda} x$ (that is for any $\lambda_1,\lambda_2\in\Lambda$ there exists $\lambda_3\in\Lambda$ such that $x_{\lambda_1}\smile x_{\lambda_2}\leqq x_{\lambda_3}$ and $\bigcup_{\lambda\in\Lambda} x_\lambda=x$), then we have
$$m(x)=\sup_{\lambda\in\Lambda} m(x_\lambda).$$
7) If $x\frown y=0$, then $m(x+y)=m(x)+m(y)$.

The condition 7) is equivalent to the following two conditions.

7′) If $x,y\geqq 0$, then we have $m(x+y)\geqq m(x)+m(y)$.

7″) If $x,y\geqq 0$, then we have $m(x\smile y)\leqq m(x)+m(y)$.

Definition 1.1. *If a functional $m(x)$ $x \in R$ satisfies* 1), 2), 3), 4), 5), 6), *and* 7′) *instead of* 7) *we say that $m(x)$ is an upper semi-additive modular.*

If a functional $m(x)$ $x \in R$ satisfies 1), 2), 3), 4), 5), 6), 7″) *instead of* 7), *and the following condition* 2′), *we say that $m(x)$ is a lower semi-additive modular.*

2′) $m(0) = 0$.

If R has an upper semi-additive modular m, then $m(x)$ $(x \in R)$ has the condition 2′) from the condition 7′) and 1): $m(0)+m(0) \leqq m(0)$ and $m(0) \geqq 0$.

Let $\bar{R}$ be the totallity of the universally continuous linear functionals of R; namely $\bar{R} \ni \bar{x}$ means that $x_\lambda \downarrow_{\lambda \in \Lambda} 0$ implies $\inf_{\lambda \in \Lambda} |\bar{x}(x_\lambda)| = 0$. Then $\bar{R}$ is a universally continuous semi-ordered linear space by the usual order. $\bar{R}$ is said to be the *conjugate space* of R.

In the sequel throughout this paper, it may be supposed that R is semi-regular; which means that if $\bar{x}(x)=0$ for every $\bar{x} \in \bar{R}$, then we have $x=0$, roughly speaking R has sufficiently many universally continuous linear functionals.

An element $\bar{x}$ of $\bar{R}$ is said to be modular bounded by m if there exist positive numbers α and β such that $\alpha \bar{x}(x) \leqq \beta + m(x)$ for every x in R. The totallity of such elements constitutes a complete semi-normal manifold (in other word, dense ideal) in $\bar{R}$, (cf. [8]) and is denoted by $\bar{R}^m$. $\bar{R}^m$ is said to be the *modular conjugate space* of R by m.

In $\bar{R}^m$, we define the functional $\bar{m}(\bar{x})$ of $\bar{x} \in \bar{R}^m$ such that

$$(*) \qquad \bar{m}(\bar{x}) = \sup_{x \in R} \left\{ \bar{x}(x) - m(x) \right\}$$

where $m(x)$ is an upper or lower semi-additive modular. $\bar{m}(\bar{x})$ will be called the *conjugate modular* of m.

The main results of [11], obtained by M. Miyakawa and H. Nakano are the following theorems.

Theorem 1.1. *If $m(x)$ is an upper semi-additive modular on R, then $\bar{m}(\bar{x})$ is a lower semi-additive modular on $\bar{R}^m$. Similary if $m(x)$ is a lower semi-additive modular on R, then $\bar{m}(\bar{x})$ is an upper semi-additive modular on $\bar{R}^m$.*

From this theorem the conjugate modular of an additive modular is also an additive one.

Theorem 1.2. (*Reflexivity theorem*)

$$m(x) = \sup_{x \in \bar{R}^m} \left\{ \bar{x}(x) - \bar{m}(\bar{x}) \right\} .$$

Namely $m(x)$ can be considered as the conjugate modular of $\bar{m}$ which is a modular on $\bar{R}^m$, because R is contained in the modular conjugate space of $\bar{R}^m$.

Let N be a *normal manifold* of R, this means that $N=N^{\perp\perp}$ where $N^{\perp}=\{x \mid |x|\frown|y|=0$ for every $y\in N\}$. For such N, every element $x\in R$ can be written uniquely as $x=x_1+x_2$, $x_1\in N$, $x_2\in N^{\perp}$. The operator $[N]x=x_1$ is called the *projection operator* by the normal manifold N. For any element $p\in R$, $[p]$ means the projection operator by the normal manifold $\{p\}^{\perp\perp}$ (which is called the normal manifold generated by p). $[p]$ is called the *projector* by p. The totaliy of projection operators constitutes a Boolean lattice $\mathfrak{B}$ with the usual meet, join, and complement; where $[R]$ is the unit and $[0]$ is 0. Furthermore $\mathfrak{B}$ is a complete lattice because R is universally continuous, and the set of projectors is an ideal in $\mathfrak{B}$.

Now let m be a lower semi-additive modular on R. We can define the functional m_1 such that

$$m_1(x) = \sup_{\sum_{i=1}^{n}[p_i]=[x]} \sum_{i=1}^{n} m([p_i]x) \qquad \text{for} \quad x\in R\ ,$$

where $\sum_{i=1}^{n}[p_i]=[x]$ means the orthogonal decomposition of $[x]$. Then $m_1(x)$ satisfies modular conditions 1), 2), 4), 5), 6); these conditions are deduced immediately from the lower semi-additive modular conditions except 6).

Relative to 6): if $0\leqq x_\lambda \uparrow_{\lambda\in\Lambda} x$ and $\sum_{i=1}^{n}[p_i]x=x\ (i=1,\cdots,n)$, then

$$\sum_{i=1}^{n} m([p_i]x) \leqq \sup_{\lambda\in\Lambda}\sum_{i=1}^{n} m([p_i]x_\lambda) \leqq \sup_{\lambda\in\Lambda}\left(\sum_{i=1}^{n} m_1([p_i]x_\lambda)\right)$$
$$\leqq \sup_{\lambda\in\Lambda} m_1(x_\lambda)\ ,$$

and so we have $m_1(x)=\sup_{\lambda\in\Lambda} m_1(x_\lambda)$, because the inverse inequality is immediately followed.

Moreover $m_1(x)$ satisfies 7):

$$\text{if} \quad x\frown y=0 \quad \text{then} \quad m_1(x)+m_1(y)=m_1(x+y)\ .$$

Now S is the set of element x such that $m_1(\alpha x)<+\infty$ for some $\alpha>0$. Then S is a semi-normal manifold in R, and m_1 is an additive modular on S. Set $M=S^{\perp\perp}$ and $N=S^{\perp}$. S is a complete semi-normal manifold of M.

Supposing that there exists a functional n satisfying 1), 2), 4), 5), 6), 7), except 3) and $n(x) \geqq m(x)$ for every $x \in R$, we see immediately $n(x) \geqq m_1(x)$ for every $x \in R$. Therefore we may say: S is the existence part of the additive modular greater than m.

By these considerations we get the following theorem.

Theorem 1.3. *Let m be a lower semi-additive modular on R. Then we can decompose R into the two mutually orthogonal normal manifold M and N (denoting $R = M \oplus N$) where M contains a complete semi-normal manifold S on which there exists an additive modular m_1 greater than m, and on any semi-normal manifold in N there is no additive modular greater than m. We can select m_1 as the least among the additive modulars greater than m on S.*

REMARK. In the above theorem S may be coincide with M for some times, but in generally different.

In generally, if a functional n on R satisfies the conditions 1), 2′), 3), 4), 5), 6), 7), then the functional $\bar{n}(\bar{x})$, $\bar{x} \in \bar{R}$ defined by the formula (*), satisfies the condition 1), 2), 4), 5), 6), 7); and the converse of this fact also follows. If $n(x) = 0$ for every $x \in R$, then $\bar{n}(\bar{x}) = +\infty$ for every $0 \neq \bar{x} \in \bar{R}$. The converse of this fact may be deduced. Furthermore we see the generalization of Theorem 1.2:

Corollary of Theorem 1.2. *If $n(x)$ satisfies the additive modular conditions except 2) or 3), then we have*

$$n(x) = \sup_{\bar{x} \in \bar{R}} \left\{ \bar{x}(x) - \bar{n}(\bar{x}) \right\}$$

where $\bar{R}$ may be replaced by any complete semi-normal manifold of $\bar{R}$.

Now we shall consider the case, where R has an upper semi-additive modular m on R. The conjugate modular $\bar{m}$ of m on $\bar{R}^m$ is the lower semi-additive modular by Theorem 1.1 and then by Theorem 1.3. $\bar{R}^m$ can be decomposed into $\bar{M}$ and $\bar{N}$: $\bar{R}^m = \bar{M} \oplus \bar{N}$, and in $\bar{M}$ there exists a complete semi-normal manifold $\bar{S}$ on which the additive modular $\bar{m}_1$ exists. $\bar{m}_1$ may be the least functional greater than $\bar{m}$, satisfying the additive modular conditions except 3).

Denoting by M the totality of elements $x \in R$ such that $\bar{x}(|x|) = 0$ for every $\bar{x} \in \bar{N}$ and $\bar{x} \geqq 0$, and by N the totality of elements $x \in R$ such that $\bar{x}(|x|) = 0$ for every $\bar{x} \in \bar{M}$ and $\bar{x} \geqq 0$,
then we have

$$R = M \oplus N .$$

The element of M can be considered as the element of the conjugate

space of $\bar{S}$. Furthermore the modular conjugate space of $\bar{S}$ by $\bar{m}_1$ contains M. Because, the functional $\bar{\bar{m}}_1(x)$, $x \in R$ defined by the conjugate formula of (*) may satisfy the additive modular conditions except 2) and if $x \in M$ and $\bar{\bar{m}}_1(\alpha x)=0$ for every number α, then by Corollary of Theorem 1.2 ($: \bar{\bar{m}}_1(\bar{x})=\bar{m}_1(\bar{x})$), we can find that $\bar{m}_1(\bar{x})$ may take the value $+\infty$ for every $\bar{x}$ such that $\bar{x}(y)=0$ for every $y \geqq 0$ and $y \in \{x\}^{\perp}$ and so $x=0$ by the definitions of M and $\bar{S}$.

If there exists a functional n satisfying 1), 3), 4), 5), 6), 7), and less than m, then for the conjugate $\bar{n}$, $\bar{m}$, it follows that

$$\bar{n}(\bar{x}) \geqq \bar{m}_1(\bar{x}) \qquad \text{for every} \quad \bar{x} \in \bar{R}^m ,$$

by Theorem 1.3. Hence by Corollary of Theorem 1.2, we have

$$n(x) = \bar{\bar{n}}(x) \leqq \bar{\bar{m}}_1(x) \qquad \text{for every} \quad x \in R ,$$

therefore we obtain the following theorem.

Theorem 1.4. *Let m be an upper semi-additive modular on R. Then we can find a functional m_1 less than m such that R can be decomposed as $R=M \oplus N$ where m_1 is additive in M and 0 in N. We can choose m_1 as the greatest modular on M among the additive ones less than m.*

REMARK. The formulation of m_1 in Theorem 1.4 is not clear as Theorem 1.3, but in some cases we get the strict formula. We call a modular m on R to be *finite* if $m(x) < +\infty$ for every $x \in R$. In finite cases, we have:

Theorem 1.5. *If R has an upper semi-additive modular m, and m is finite, then*

$$m_1(x) = \inf_{[x]=\sum_{i=1}^{n}[p_i]} \sum_{i=1}^{n} m([p_i]x) \qquad \text{for every} \quad x \in R .$$

Proof. We put for every element $x \in R$,

$$m_2(x) = \inf_{[x]=\sum_{i=1}^{n}[p_i]} \sum_{i=1}^{n} m([p_i]x)$$

then $m_2(x)$ satisfies 1), 3), 4), 5), of the modular conditions except 2), 6), by the definition of m_2.

We will prove 6). For every positive elements x and y, we have

$$x = (1-\varepsilon)\, y + \varepsilon \left(y + \frac{1}{\varepsilon}(x-y) \right) \qquad \text{for} \quad 1 > \varepsilon > 0 ,$$

and hence by 4) of m,

$$m(x) \leqq m(y) + \varepsilon m\left(y + \frac{1}{\varepsilon}(x-y)\right)$$

namely,

$$m(x) - m(y) \leqq \varepsilon m\left(y + \frac{1}{\varepsilon}(x-y)\right)$$

$$= \varepsilon m\left\{\frac{1}{2}2y + \frac{1}{2}\frac{2}{\varepsilon}(x-y)\right\}$$

$$\leqq \frac{\varepsilon}{2} m(2y) + \frac{\varepsilon}{2} m\left(\frac{2}{\varepsilon}(x-y)\right) \cdots\cdots\cdots\cdots (1)$$

Let $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$. For any decomposition $[x] = \sum_{i=1}^{n} [p_i]$ $(i = 1, 2, \cdots, n)$,

$$m([p_i]x) - m([p_i]x_\lambda) \leqq \frac{\varepsilon}{2} m(2[p_i]x) + \frac{\varepsilon}{2} m\left(\frac{2}{\varepsilon}[p_i](x - x_\lambda)\right)$$

$$(i = 1, 2, \cdots, n).$$

Summing up 1 to n, we have by virtue of 7′) and above (1)

$$\sum_{i=1}^{n} m([p_i]x) - \sum_{i=1}^{n} m([p_i]x_\lambda)$$

$$\leqq \frac{\varepsilon}{2} \sum_{i=1}^{n} m(2[p_i]x) + \frac{\varepsilon}{2} \sum_{i=1}^{n} m\left(\frac{2}{\varepsilon}([p_i](x - x_\lambda))\right)$$

$$\leqq \frac{\varepsilon}{2} m(2x) + \frac{\varepsilon}{2} m\left(\frac{2}{\varepsilon}(x - x_\lambda)\right),$$

and hence

$$m_2(x) - m_2(x_\lambda) \leqq \frac{\varepsilon}{2} m(2x) + \frac{\varepsilon}{2} m\left(\frac{2}{\varepsilon}(x - x_\lambda)\right).$$

By virtue of 6) of the modular conditions, for any $\varepsilon > 0$ we have:

$$m_2(x) - \sup_{\lambda \in \Lambda} m_2(x_\lambda) \leqq \frac{\varepsilon}{2} m(2x),$$

and ε being arbitrary, we obtain

$$m_2(x) = \sup_{\lambda \in \Lambda} m_2(x_\lambda)$$

which is the condition 6) of m_2.

Furthermore m_2 satisfies 7).

Let $n(x)$, $x \in R$ satisfy 1), 3), 4), 5), 6), 7), and be less than m. Then by the definition of m_2, we see immediately,

$$m_2(x) \geqq n(x), \quad \text{hence} \quad m_2(x) = m_1(x). \qquad \text{Q. E. D.}$$

§ 2. Approximately additive modulars. Let R have a lower or

upper semi-additive modular m. By Theorem 1.3 and 1.4, we can decompose R into the mutually orthogonal normal manifolds M and N such that $R=M\oplus N$, where some complete semi-normal manifold of M has an additive modular related to the given semi-additive modular, and so we will consider espatially M, in other words we will consider the case: $N=\{0\}$.

Definition 2.1. *If we have $R=M$, namely $N=\{0\}$ in Theorem 1.3, then the lower semi-additive modular m is said to be lower approximately additive. Similarly we can define an upper approximately additive modular.*

In this section, we shall investigate the modulars which are approximately additive or not. We say an element x of R to be *atomic* if $[x]>0$, and $[x]\geqq[y]$ implies $[y]=0$ or $[y]=[x]$, therefore $[x]R$ is an one-dimensional space. If the set of atomic elements of R are complete in R, then R is called *discrete*. For example (l_1) is discrete.

Theorem 2.1. *If R is discrete, then any semi-additive modular on R is an approximately additive one.*

Proof. If x is an atomic element, then $m=m_1$ in $[x]R$, because $[x]R$ can not be decomposed any more. Hence $R=M$, because the atomic elements are complete in R.

In the case where R is not discrete, we can obtain the examples of modulars which are not approximately additive on any semi-normal manifold of R.

If the modulars m on R satisfy the property such that $m(x)=0$ implies $x=0$, then m is said to be *simple*. (cf. [8])

Theorem 2.2. *If R has no atomic elements, and m is a lower semi-additive modular on R, and furthermore we suppose the following conditions:*

(1) *m is simple,*

(2) *For any elements $x, y\in R$ such that $m(x)=m(y)$,*

and $x\frown y=0$, there exists a definite positive number $1>\delta>0$ such that

$$m(x+y)\leqq(1-\delta)\left\{m(x)+m(y)\right\},$$

then m is not approximately additive on any semi-normal manifold of R.

Proof. We shall show $m_1(x)=+\infty$ for every $x\neq 0$. Suppose that $m_1(x)<+\infty$ and $x>0$, then we will prove that there exist y and z such that $m(y)=m(z)$, $x=y+z$, $y\frown z=0$.

For any $\varepsilon>0$, there exists orthogonal decomposition of $[x]$ such

that $\sum_{i=1}^{n_\varepsilon}[p_i]=[x]$ and $m([p_i]x)\leqq\varepsilon$ $(i=1,2,\cdots,n_\varepsilon)$, because R has no atomic element and $m_1(x)<+\infty$. For this decomposition we can select $[p_{i_k}]$ $(k=1,2,\cdots,s)$ and $[p_{j_l}]$ $(l=1,2,\cdots,t)$ $t+s=n_\varepsilon$ from $\{[p_i], i=1,2,\cdots,n_\varepsilon\}$ such that

$$m([p_{i_1}]x+\cdots+[p_{i_s}]x)\leqq m([p_{j_1}]x+\cdots+[p_{j_t}]x)$$

and

$$m([p_{i_1}]x+\cdots+[p_{i_s}]x+[p_{j_1}]x)\geqq m([p_{j_2}]x+\cdots+[p_{j_t}]x)$$

where $[p_{j_k}]$ $(k=1,2,\cdots,s)$, $[p_{j_l}]$ $(l=1,2,\cdots,t)$ are different from each others. We put

$$\begin{aligned}[p_{i_1}]+\cdots+[p_{i_s}] &= [p''_\varepsilon]\,.\\ [p_{j_1}]+\cdots+[p_{j_t}] &= [p'_\varepsilon]\,.\\ [p_{j_1}] &= [p_\varepsilon]\,.\end{aligned}$$

Assuming $\varepsilon_1\geqq\varepsilon_2\geqq\cdots$, and $\lim_{n\to\infty}\varepsilon_n=0$, we can suppose $[p''_{\varepsilon_1}]\leqq[p''_{\varepsilon_2}]\leqq\cdots$ and $[p'_{\varepsilon_1}]\geqq[p'_{\varepsilon_2}]\geqq\cdots$ and $[p_{\varepsilon_n}]\downarrow_n 0$, since $m_1(x)<+\infty$.

Putting $[p''_{\varepsilon_n}]\uparrow_n[p'']$ and $[p'_{\varepsilon_n}]\downarrow_n[p']$, we can conclude $m([p']x)=m([p'']x)$, and $[p']+[p'']=[x]$ from the construction of $[p']$ and $[p'']$, and the semi-continuity of m.

Hence

$$m(x)=m([p']x+[p'']x)\leqq(1-\delta)\left\{m([p']x)+m([p'']x)\right\}\,,$$

therefore for every integer n,

$$\left(\frac{1}{1-\delta}\right)^n m(x)\leqq\sup_{[x]=\sum_{i=1}^{l}[p_i]}\sum_{i=1}^{l}m([p_i]x)=m_1(x)$$

which is $m_1(x)=+\infty$, because $m(x)\neq 0$ from the hypothesis (1). This prove the theorem. Q.E.D.

A norm of a universally continuous semi-ordered linear space can be considered a lower semi-additive modular. If R is L_p-space $(p>1)$ and we consider the L_p-norm as the lower semi-additive modular, then this modular is not approximately additive on arbitrary semi-normal manifold. But more general spaces have this property. L_p-space $(p>1)$ is uniformly convex by the sense of Clerkson and Nakano (cf. [10]), and if a norm of a universally continuous linear space is uniformly convex, and moreover this space has no atomic elements, then the hypothesis of Theorem 2.2 may be satisfied by considering the norm

as a lower semi-additive modular. Hence we have the Corollary of Theorem 2.2:

Corollary of Theorem 2.2. *If a universally continuous semi-ordered linear space has a semi-continuous norm, and this norm is uniformly convex, and furthermore has no atomic element, then the lower semi-additive modular defined by this norm is not approximately additive on any semi-normal manifold in this space.*

As the similar theorem on upper semi-additive modulars, we have:

Theorem 2.3. *If an upper semi-additive modular m on R having no atomic elements, satisfies the following conditions:*

(1) m *is finite,*

(2) $m(x+y) \geqq (1+\delta)\{m(x)+m(y)\}$ *if* $x \frown y=0$ *and* $m(x)=m(y)$, *(δ is a definite positive number independent from the choice of x and y),*

then m is not an approximately additive modular on any semi-normal manifold in R.

Proof. We shall prove that $m_1(x)=0$ for every element x of R. For any element $x \in R$, we consider $\mathfrak{M}$: the totality of projectors such that $m([p]x)=0$ and $[p] \leqq [x]$. If we put $\bigcup_{[p]\in\mathfrak{M}} [p]=[p']$, then $m_1([p']x)=0$ by the definition of m_1, because of the semi-continuity of m and Theorem 1.5. If $m(x-[p']x)=0$, then we have $m_1(x)=0$.

Supposing that $m(x-[p']x)>0$, for any $\varepsilon>0$ we can decompose $x-[p']x=y$ such that $y=\sum_{i=1}^{n_\varepsilon}[q_i]y$ $(i=1, 2, \cdots, n_\varepsilon)$ and $m([q_i]y) \leqq \varepsilon$ $(i=1, 2, \cdots, n_\varepsilon)$. As like as the preceeding theorem, there exist projectors $[q'_\varepsilon]=[q_{i_1}]+\cdots+[q_{i_s}]$, $[q''_\varepsilon]=[q_{j_1}]+\cdots+[q_{j_t}]$ and $[q_{j_1}]=[q_\varepsilon]$, $[y]=[q_{i_1}]+\cdots+[q_{i_s}]+[q_{j_1}]+\cdots+[q_{j_t}]$ such that

$$m(([q'_\varepsilon]+[q_\varepsilon])y) \geqq m(([q''_\varepsilon]-[q_\varepsilon])y),$$
$$m([q'_\varepsilon]y) \leqq m([q''_\varepsilon]y),$$
$$m([q_\varepsilon]y) \leqq \varepsilon .$$

Hence, if $\varepsilon \to 0$, we can select $[q'_\varepsilon]$ and $[q''_\varepsilon]$ such that $[q'_\varepsilon]$ is increasing and $[q''_\varepsilon]$ is decreasing, and furthermore we can suppose that $[q_\varepsilon]$ is decreasing.

As we can find projector $[q']$, $[q'']$ such that

$$[q'_\varepsilon] \uparrow_\varepsilon [q'], \quad \text{and} \quad [q''_\varepsilon] \downarrow_\varepsilon [q''],$$

it follows that $[q']+[q'']=[y]$, and $[q_\varepsilon]\downarrow_\varepsilon 0$ from the choice of y. Furthermore we conclude $m([q]y)=m([q']y)$, and so by the hypothesis of this theorem $m(y)\geqq(1+\delta)\ \{m([q]y)+m([q'])y)\}$. For every integer $n\geqq 2$, we have

$$\left(\frac{1}{1+\delta}\right)^n m(y)\geqq m_1(y),\quad \text{therefore}\quad m_1(y)=0\ .$$

Which shows $m_1(x)=0$. Q. E. D.

The conditions of the above two theorems can be changed into slight different forms. In the following we state the theorem of this type.

Theorem 2.4. *If a lower semi-additive modular m on R has the following conditions:*

(1) *if $m(x)=m(y)$ and $x\frown y=0$, then there exists a positive number $\eta<1$ such that for some suitably choosen for x and y.*

$$\frac{1}{\eta}m(\eta(x+y))\leqq(1-\delta)\left\{m(x)+m(y)\right\}\ ,\qquad 1>\delta>0$$

where δ is independent from the choice of x and y.

(2) $\inf\limits_{\xi>0}\dfrac{m(\xi x)}{\xi}>0$ *for every element $x\in R$ different from* 0.

(3) *R has no atomic elements.*

then m is not an approximately additive modular on any semi-normal manifold of R.

In the upper semi-additive case:

Theorem 2.5. *Let m be an upper semi-additive modular on R. If m satisfies the following conditions,*

(1) *for any elements x and y such that $m(x)=m(y)$ and $x\frown y=0$, there exists a definite positive number $\delta>0$ such that*

$$\frac{1}{\eta}m(\eta(x+y))\geqq(1+\delta)\left\{m(x)+m(y)\right\}$$

for $\eta>1$ which is suitably choosen for x and y.

(2) $\sup\limits_{\xi>0}\dfrac{m(\xi x)}{\xi}<+\infty$ *for every element $x\in R$.*

(3) *R has no atomic elements,*

then m is not an approximately additive modular on any semi-normal manifold of R.

Proofs of the above two theorems are similar to that of Theorem

2.2 and 2.4 and so they are omitted.

§ 3. Ascending modulars. A lower or upper semi-additive modular on R is said to be *linear* if $m(\xi x)=\xi m(x)$ for every element x in R and every positive number ξ. If m is a lower semi-additive modular on R, then it will be considered as a norm on R by the modular condition 4). In the case of upper semi-additive modulars, we obtain the following.

Theorem 3.1. *If m is an upper semi-additive and linear modular on R, then m is an additive one.*

Proof. By the modular condition 4) we can find that

$$m(x+y) \leqq \frac{1}{2} m(2x) + \frac{1}{2} m(2y) = m(x) + m(y) .$$

Combining the modular condition 7′), we have the additive property:

$$x \frown y = 0 \quad \text{implies} \quad m(x+y) = m(x) + m(y) .$$

If a semi-additive modular $m(x)$, $x \in R$ takes only 0 or $+\infty$ as its value, then m is called a *singular* modular. The relation between the linear modular and the singular one is conjugate. (cf. [8]).

Theorem 3.2. *If a modular m is linear, then the conjugate modular $\overline{m}$ of m is singular. Conversely, if a modular m is singular, then the conjugate modular $\overline{m}$ of m is linear.*

Let R have a semi-continuous norm $\|x\|$, $x \in R$, then the norm may be considered as a linear lower semi-additive modular, and the conjugate modular $\overline{m}$ of m on $\overline{R}^m$ satisfies

$$(\overset{*}{*}) \begin{cases} \overline{m}(\overline{x}) = 0 & \text{if} \quad \|\overline{x}\| \leqq 1 \\ \overline{m}(\overline{x}) = +\infty & \text{if} \quad \|\overline{x}\| > 1 \end{cases}$$

where $\overline{x}$ is in the norm conjugate space of R by this norm, and $\|\overline{x}\| = \sup_{\|x\| \leqq 1} |\overline{x}(x)|$.

REMARK. The conjugate form of Corollary of Theorem 2.2 may be explained as follows: *If a norm of a universally continuous semi-ordered linear space (semi-continuous norm) is uniformly even and this space has no atomic element, then the modular defined by the formula* $(\overset{*}{*})$ *is not approximately additive,* because the conjugate norm of a uniformly even norm is uniformly convex. (cf. [10]).

We say that a modular m on R is *ascending* if

$$\inf_{\xi > 0} \frac{m(\xi x)}{\xi} > 0 \quad \text{for} \quad x \in R \text{ different from } 0 .$$

Hence linear modulars are ascending ones.

Theorem 3.3. *If an upper semi-additive modular m on R is ascending, then m is approximately additive.*

Proof. Putting $m'(x)=\inf_{\xi>0}\frac{m(\xi x)}{\xi}$, we shall prove that $m'(x)$ is a linear and upper semi-additive modular. $m'(x)$ satisfies the conditions 1), 2), 3), 4), 5), and 7′) of modulars by the definition of m'. As to 6),

if $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x$, then we have

$$m(\xi_0(x-x_\lambda))\downarrow_{\lambda\in\Lambda}0 \quad \text{for some number} \quad \xi_0>0,$$

therefore we have $\sup_{\lambda\in\Lambda}m'(x_\lambda)=m'(x)$ from the relation: $m'(x-x_\lambda)+m'(x_\lambda)\geqq m'(x)$. Hence, $m'(x)$ $x\in R$ is a linear and upper semi-additive modular. By Theorem 3.1. $m'(x)$ $x\in R$ is an additive modular on R. Hence R has the modular m_1 which is greater than m' by Theorem 2.4; therefore m is an approximately additive modular on R. Q.E.D.

Let m be an arbitrary upper semi-additive modular on R, and $m'(x)=\inf_{\xi>0}\frac{m(\xi x)}{\xi}$ as in the proof of the above theorem. If we set $N=\{x|m'(x)=0\}$, then N is a normal manifold of R. Because, if $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}a$; namely $a-a_\lambda\downarrow_{\lambda\in\Lambda}0$, then we have $m'(a-a_\lambda)\leqq m(\xi_0(a-a_\lambda))\downarrow_{\lambda\in\Lambda}0$ for some ξ_0. By the inequality: $m'(a)\leqq m'(a_\lambda)+m'(a-a_\lambda)$, we have $m'(a)=0$ if $m'(a_\lambda)=0$, which deduce the following theorem.

Theorem 3.4. *If m is an upper semi-additive modular on R, then R can be written as follows: $R=N\oplus N^\perp$ where m is ascending on N and m is not ascending for every normal manifold in $N^\perp$.*

We say that an element x of R is a *zero-element* by the modular m, if $m(x)=0$. If the set of zero elements in R is complete in R, then R is said to be *semi-singular* by the modular m.

When R has a lower semi-additive modular m, and N is the normal manifold of R which is generated by the set of zero-elements in R, we can write $R=N\oplus N^\perp$ and $m(x)=0$ implies $x=0$ for $x\in N^\perp$. The relation between the ascending modular and the semi-singular one is similar to the case of the additive modular.

Theorem 3.5. *If m is a lower semi-additive modular on R, and semi-singular, then m is an approximately additive one.*

The proof of this theorem will be found by the next theorem and Theorem 3.3. But we will show an immediate proof: if $m(x)=0$, then

$m_1(x) = \sup_{\sum_{i=1}^{n}[p_i]=[x]} \sum_{i=1}^{n} m([p_i]x) = 0$. Such zero-elements are complete in R, and so we see that M in Theorem 1.3 equals to R.

Theorem 3.6. *Let R have a lower semi-additive and semi-singular modular m. The conjugate modular $\bar{m}$ of m which is an upper semi-additive modular on $\bar{R}^m$, is ascending.*

Proof. For any positive element $\bar{x} \in \bar{R}^m$, we have $\bar{m}(\bar{x}) = \sup_{x \in R} \{\bar{x}(x) - m(x)\}$. Since the zero-elements are complete in R, we can find an element $x' > 0$, such that $\bar{x}(x') \neq 0$ and $m(x') = 0$. Hence

$$\frac{\bar{m}(\xi\bar{x})}{\xi} = \frac{1}{\xi} \sup_{x \in R} \left\{\xi\bar{x}(x) - m(x)\right\} \geqq \bar{x}(x') > 0$$

for every number ξ, which shows that $\bar{m}$ is an ascending modular on $\bar{R}^m$.

Theorem 3.7. *Let m be an ascending and upper semi-additive modular on R. Then the conjugate modular $\bar{m}$ on $\bar{R}^m$ is a semi-singular and lower semi-additive one.*

Proof. Let $m'(x)$ be as in Theorem 3.3, then $m'(x)$ is a linear and additive modular on R. If we put

$$\begin{aligned} \bar{m}(\bar{x}) &= \sup_{x \in R} \left\{\bar{x}(x) - m(x)\right\} \\ &\leqq \sup_{x \in R} \left\{\bar{x}(x) - m'(x)\right\} = \bar{m}'(\bar{x}) \qquad \text{for } \bar{x} \in \bar{R}^{m'}, \end{aligned}$$

then $\bar{m}'(\bar{x})$ is a singular and additive modular on $\bar{R}^{m'}$ by Theorem 3.2. Since $\bar{m}(\bar{x}) \leqq \bar{m}'(\bar{x})$, $\bar{x} \in \bar{R}^{m'} \subset \bar{R}^m$, and $\bar{R}^{m'}$ is a complete semi-normal manifold of $\bar{R}^m$, we see that $\bar{m}(\bar{x})$ is semi-singular.

By the above two theorems, we can state that an upper semi-additive modular is ascending if and only if the conjugate modular by this modular is semi-singular, and conversely a lower semi-additive modular is semi-singular if and only if the conjugate modular is ascending.

§ **4.** Essentially additive modulars. Let m be a lower or upper approximately additive modular on R. Namely there exists an additive modular m_1 which was given by Theorem 1.3 and 1.4. Before we investigate the relation between m and m_1, we shall give a rough speaking of the properties of the modular norms which were given by Nakano. Naturally, these properties follow in the case of general modulars, and whose special cases are semi-additive modulars.

If we put

$$(1) \qquad |||x||| = \inf_{m(\xi x)\leqq 1} \frac{1}{\xi} \qquad \text{for} \quad x\in R\,,$$

then $|||x|||$ is a norm on R and will be called the *second norm* of m. If we put

$$(2) \qquad ||x|| = \sup_{\bar{m}(\bar{x})\leqq 1} |\bar{x}(x)|$$

then we have a norm on R and it will be called the *first norm* of m. Another form of this norm is:

$$(3) \qquad ||x|| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi}$$

which was given by I. Amemiya. Between the first norm and the second norm, we have the relation:

$$|||x||| \leqq ||x|| \leqq 2|||x|||\,.$$

For the first and the second norm of the conjugate modular $\bar{m}$ on $\bar{R}^m$, we have

$$(4) \qquad ||x|| = \sup_{|||\bar{x}|||\leqq 1} |\bar{x}(x)|\,, \qquad |||x||| = \sup_{||\bar{x}||\leqq 1} |\bar{x}(x)|\,,$$

$$(5) \qquad ||\bar{x}|| = \sup_{|||x|||\leqq 1} |\bar{x}(x)|\,, \qquad |||\bar{x}||| = \sup_{||x||\leqq 1} |\bar{x}(x)|\,.$$

Let m be a general modular on a universally continuous semi-ordered linear space R. We call m to be *monotone complete* if $0\leqq x_\lambda \uparrow_{\lambda\in\Lambda}$ and $m(x_\lambda)$, $\lambda\in\Lambda$ are bounded, then there exists the upper bound of x_λ: $\bigcup_{\lambda\in\Lambda} x_\lambda = x\in R$.

A modular m on R is monotone comlete if and only if the first or second norm of the modular m is monotone complete. (cf. [11]). The modular conjugate space $\bar{R}^m$ of R by m is always monotone complete by the conjugate modular $\bar{m}$. If a semi-regular space R has a modular (a lower or upper semi-additive), then R is imbeded in the space which has a monotone complete modular, because of Theorem 1.2. Supposing that n and n' are both lower or upper modulars on R and $n(x)\geqq n'(x)$ for every $x\in R$. If n' is monotone complete, then n is monotone complete by the definition. The additive modular m_1 on S which is defined in Theorem 2.2 is monotone complete if m is a monotone complete modular on R.

$\bar{x}\in\bar{R}$ is modular bounded if and only if $\bar{x}$ is bounded by the first or second norm as the functional on R.

A sequence $\{x_\nu\}$ $(\nu=1, 2, \cdots)$, $x_\nu\in R$ is said to be *modular convergent* to x if $\lim_{\nu\to\infty} m(\xi(x_\nu - x))=0$ for every $\xi>0$.

The modular m is said to be *complete* if

$$\lim_{\mu,\nu\to\infty} m(\xi(x_\nu - x_\mu)) = 0 \qquad \text{for every } \xi>0$$

implies the modular convergence of the sequence $x_\nu \in R\,(\nu=1,2,\cdots)$. A modular m is complete if and only if the first or second norm of m is complete as the norm of R.

The following theorem was proved by NAKANO. (cf. [8]).

Theorem 4.1. *If m is a monotone complete modular on R, then m is complete.*

Proof. If m is a monotone complete modular on R, then the first or second norm by m is monotone complete. If the norm is monotone complete, then the norm is complete. (cf. [1]). Hence m is complete because the modular norm is complete.

Now we shall come back to the approximately additive modulars.

Let m be a lower approximately additive modular on R. Then there exists a complete semi-normal manifold S of R and an additive modular m_1 such that $m_1(x)\geqq m(x)$, $x\in S$ as already shown by Theorem 1.3.. Hence we obtain the first norm and the second norm of the additive modular m_1 on S such as (1), (2) above and denoted by $\|x\|_1$ and $|||x|||_1$. For any element $x\in S$, we have the relations:

$$\|x\| \leqq \|x\|_1, \quad x\in S; \qquad |||x||| \leqq |||x|||_1, \quad x\in S.$$

In the case of an upper approximately additive modular m on R, we see by Theorem 1.4, $m_1(x)\leqq m(x)$ for every $x\in R$, and hence

$$\|x\|_1 \leqq \|x\|, \quad x\in R; \qquad |||x|||_1 \leqq |||x|||, \quad x\in R.$$

In the lower or upper semi-additive case, if the norm $\|x\|$ and $\|x\|_1$ are equivalent: that is $\|x\|\leqq\alpha\|x\|_1$ and $\|x\|_1\leqq\alpha'\|x\|$ for some number α, α', then we say that the approximately additive modular is *essentially additive*, which is equivalent to the fact that both second norms are equivalent. (If the modular is lower semi-additive, then we say *lower essentially additive* and if the modular is upper semi-additive, then we say *upper essentially additive*).

Theorem 4.2. *In order that an approximately additive modular m on R is essentially additive, it is necessary and sufficient that $\bar{R}^m$ coincide with $\bar{R}^{m_1}$.*

Proof. The necessary condition is evident from the definition of the essentially additive modulars.

Conversely, suppose that $\bar{R}^m=\bar{R}^{m_1}$. The modular conjugate spaces

$\bar{R}^m$ and $\bar{R}^{m_1}$ by m and m_1 are monotone complete and have the semi-additive modulars. Hence, by Theorem 4.1 of this section, we see that the first norms and second norms of $\bar{m}$ and $\bar{m}_1$ defined by m and m_1 are complete. By the theorem of NAKANO (cf. [8]) or the BANACH's theorem, we see that these norms are equivalent each others. By the formula (4) of this section the first norms or second norms of m and m_1 are equivalent to each others.

In the case of lower approximately additive modulars, we see:

Theorem 4.3. *If m is a monotone complete and lower approximately additive modular on R, then S in Theorem 1.3 coincide with R if and only if m is essentially additive.*

In the case of upper approximately additive modulars, we have:

Theorem 4.4. *Supposing that R has a monotone complete upper approximately additive modular m, m is essentially additive if and only if m_1 is monotone complete in R.*

Theorem 4.3 and 4.4 are the immediate consequences from Theorem 4.2 and the following lemma which is due to NAKANO.

Lemma 4.1. *If a modular m on R is monotone complete, then any element of $\bar{R}$ is modular bounded: that is $\bar{R}^m=\bar{R}$.*

REMARK. This lemma follows even if m is complete. (cf. [8]).

Proof. By virtue of the monotone completeness of m, $x_\nu \geqq 0$ and $m(x_\nu)\leqq 1$ $(\nu=1,2,\cdots)$ imply that $\sum_{\nu=1}^{n} \frac{1}{2^\nu} x^\nu$ is order convergent to x $\left(: x = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} x_\nu\right)$. Let some $\bar{x}\in\bar{R}$ be not modular bounded. Then there exists x_ν for every ν, $\nu=1,2,\cdots$ such that $\bar{x}(x_\nu)\geqq 2^\nu$ and $m(x_\nu)\leqq 1$. Hence, for $x=\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} x_\nu$ we obtain $\bar{x}(x)\geqq k$ $(k=1,2,\cdots)$: $\bar{x}(x)=+\infty$ which is contradict. Q.E.D. (This proof is same as [8].)

In the following we will explain the another conditions for an approximately additive modular to be essentially additive in the case where the modular is upper semi-additive and finite. For this purpose we prove the following lemma.

Lemma 4.2. *Let m be an upper semi-additive and finite modular on R. Then $y_\lambda \downarrow_{\lambda\in\Lambda} 0$ implies $m(y_\lambda)\downarrow_{\lambda\in\Lambda} 0$.*

Proof. Let some $y_{\lambda_0}(\lambda_0\in\Lambda)$ be fixed. For any $y_\lambda \leqq y_{\lambda_0}$, it follows by the modular condition 7′) that $m(y_\lambda)+m(y_{\lambda_0}-y_\lambda)\leqq m(y_{\lambda_0})$. Hence, by the semi-continuity of m, namely 6), we have

$$m(y_{\lambda_0}-y_\lambda)\uparrow_{\lambda\in\Lambda} m(y_{\lambda_0})$$

which implies

$$m(y_\lambda)\downarrow_{\lambda\in\Lambda} 0\ .$$

Theorem 4.5. *If a finite modular m on R is monotone complete and upper approximately additive, then m is essentially additive if and only if*

$$\sum_{\nu=1}^{\infty} m(x_\nu)<+\infty \quad \textit{implies the existence of} \quad \sum_{\nu=1}^{\infty} x_\nu \qquad \text{(in order sense)}$$

for every positive mutually orthogonal sequence $x_\nu\in R$ $(\nu=1,2,\cdots)$.

Proof. Let m be an essentially additive modular on R. By Theorem 4.4 we see that m_1 is monotone complete. Hence, from the relations $\sum_{\nu=1}^{\infty} m_1(x_\nu)\leqq \sum_{\nu=1}^{\infty} m(x_\nu)<+\infty$ and $\sum_{\nu=1}^{n} m_1(x_\nu)=m_1\left(\sum_{\nu=1}^{n} x_\nu\right)$ we have $\bigcup_{\nu=1}^{\infty} x_\nu = \sum_{\nu=1}^{\infty} x_\nu=x\in R$, which shows the necessity of this theorem.

Sufficient condition: Firstly we consider the Boolean ring of projection operator of R. As R is universally continuous, this Boolean ring $\mathfrak{B}$ is complete as a lattice. Now let $\mathfrak{P}$ be the set of all dual maximal ideal of $\mathfrak{B}$. Then $\mathfrak{P}$ is compact by the usually introduced topology. (Consider the family of dual maximal ideals including some fixed projection operator. In this topology such families are open bases).

We shall prove that there are mutually orthogonal normal manifolds which cover $R: R=N_1\oplus N_2\oplus\cdots\oplus N_n$ with the following properties: there exist positive numbers $\varepsilon_i>0$ $(i=1,2,\cdots n,)$ such that

$$m_1(x)\leqq\varepsilon_i \quad \text{implies} \quad m(x)\leqq 1 \qquad \text{for every } x\in N_i\ (i=1,2,\cdots,n).$$

For any element $\mathfrak{p}$ of $\mathfrak{P}$, if we find the normal manifold N whose projection operator is in $\mathfrak{p}$ and which satisfies that $m_1(x)\leqq\varepsilon$ implies $m(x)\leqq 1$ for $x\in N$, then we can select finite numbers of normal manifolds N_i $(i=1,2,\cdots,k)$ which generate R and satisfy the above condition, by the compactness of $\mathfrak{P}$. Taking the intersection or difference of N_i $(i=1,2,\cdots,k)$ each other, we can find the mutually orthogonal manifolds as above.

Supposing that for some $\mathfrak{p}_0\in\mathfrak{P}$, there does not exist a projection operator in $\mathfrak{p}_0$ which satisfies above condition. If $\mathfrak{p}_0$ has an atomic projection operator $[N]$, then m and m_1 coincide with each others in $[N]R$, since $[N]R$ is one-dimentional. Hence $\mathfrak{p}_0$ has no atomic projection operator.

Firstly we can find a positive element x_1 in R such that

$$m_1(x_1) < \frac{1}{2}, \qquad m(x_1) > 1 .$$

Since $\mathfrak{p}_0$ is not atomic, we have $\bigcap_{[N]\in\mathfrak{p}_0}[N]=0$, and hence $0\neq[x_1]-[x_1][N]\bar{\in}\mathfrak{p}_0$ for some $[N]\in\mathfrak{p}_0$. Hence we can take x_1 as $[x_1]\bar{\in}\mathfrak{p}_0$ without changing the above inequality, since m and m_1 are semi-continuous. For any natural number ν, by induction, we can find mutually orthogonal positive elements x_ν in R such that

$$m_1(x_\nu) < \frac{1}{2^\nu}, \qquad m(x_\nu) > 1$$

and

$$x_\nu \frown x_\mu = 0 \quad \text{if} \quad \nu \neq \mu \qquad\qquad (\nu, \mu = 1, 2, \cdots)$$

and furthermore $[N_\nu]\bar{\in}\mathfrak{p}_0$, where $[N_\nu]$ is a projection operator generated by $x_1, x_2, x_3, \cdots, x_{\nu-1}, x_\nu$.

On virtue of the finiteness of m, we see that for any x_ν, there exist mutually orthogonal projectors $[p_{\nu,i}]$ $(i=1, 2, \cdots, i_\nu)$ such that

$$\sum_{i=1}^{i_\nu}[p_{\nu,i}] = [x_\nu] \quad \text{and} \quad \sum_{i=1}^{i_\nu} m([p_{\nu,i}]x_\nu) \leqq \frac{1}{2^\nu}$$

by Theorem 1.5.

Hence we have $\sum_{\nu=1}^{\infty}\sum_{i=1}^{i_\nu} m([p_{\nu,i}]x_\nu) < +\infty$ which implies the existence of $\sum_{i,\nu}[p_{\nu,i}]x_\nu$ which means the order convergence of $\sum_{\nu=1}^{n} x_\nu$ to some element $y\in R$. Therefore there exists $\sum_{\nu=\mu}^{\infty} x_\nu = y_\mu$ for every $\mu=1, 2, \cdots$. Then we have $y_\mu \downarrow_\mu 0$. By Lemma 4.2 we see that

$$m(y_\mu) \downarrow_\mu 0 \quad \text{and} \quad m(x_\nu) \leqq m(y_\nu) ,$$

which contradict the assumption.

Hence we can find mutually orthogonal manifolds N_i $(i=1, 2, \cdots, n_0)$ such that $R=N_1\oplus N_2\oplus\cdots\oplus N_{n_0}$ and $m_1(x)\leqq\varepsilon_i$ implies $m(x)\leqq 1$ for $x\in N_i$ $(i=1, 2, \cdots, n_0)$; here ε_i may be choosen as $0<\varepsilon_i<1$.

Because $|||x_i|||_1\leqq\varepsilon_i$ implies $m_1(x_i)\leqq\varepsilon_i$ for $x_i\in N_i$, we see that

$$|||x_i|||_1 \leqq \varepsilon_i \quad \text{implies} \quad |||x_i||| \leqq 1 .$$

Therefore $|||x|||_1 \leqq \operatorname*{Min}_{\leqq i\leqq n_0}\{\varepsilon_i\}$ implies $|||x|||\leqq n_0$, for $x\in R$, which prove the theorem.

Another sufficient conditions for essentially additive modulars are as follows.

Theorem 4.6. *Let m be an upper semi-additive modular on R. If m satisfies the following condition:*

(*1) *there exists a definite positive number δ such that*

$$\frac{m(x)}{|||x|||} \geqq \delta$$

for every element $x \in R$ different from 0,
then m is not only an approximately additive but also an essentially additive modular on R.

REMARK. Naturally a linear upper semi-additive modular satisfies the condition (*1). We may say that the condition (*1) is near to the linear condition.

Before we prove this theorem, we shall consider the conjugate condition of (*1):

(*2) *there exists a positive number $\varepsilon > 0$ such that*

$$|||x||| \leqq \varepsilon \quad \textit{implies} \quad m(x) = 0 .$$

Theorem 4.7. *Let m be an upper semi-additive modular on R satisfying the condition (*1). Then the conjugate modular $\bar{m}$ of m satisfies the condition (*2).*

Proof. By the definition of the conjugate modular, we have

$$\bar{m}(\bar{a}) = \sup_{a \in R} \{\bar{a}(a) - m(a)\}$$

$$\leqq \sup_{R \ni a \neq 0} |||a||| \left\{ ||\bar{a}|| - \frac{m(a)}{|||a|||} \right\}$$

Hence, we find that

$$\bar{m}(\bar{a}) = 0 \qquad \text{if} \quad ||\bar{a}|| < \delta .$$

Because the second norm and the second norm of m are equivalent, the condition (*2) for $\bar{m}$ follows.

The conjugate form of this theorem is:

Theorem 4.8. *Let m be a lower semi-additive modular satisfying the condition (*2). Then the conjugate modular $\bar{m}$ of m satisfies the condition (*1).*

Proof. For an element $\bar{a} > 0$, $\bar{a} \in \bar{R}$, we have

$$\frac{\bar{m}(\bar{a})}{|||\bar{a}|||} = \sup_{a \in R} \left\{ \frac{\bar{a}}{|||\bar{a}|||}(a) - \frac{m(a)}{|||\bar{a}|||} \right\}$$

$$\geqq \sup_{||a|| \leqq \varepsilon} \left\{ \frac{\bar{a}}{|||\bar{a}|||}(a) - \frac{m(a)}{|||\bar{a}|||} \right\}$$

$$= \sup_{\|a\| \leq \varepsilon} \left\{ \frac{\bar{a}}{|||\bar{a}|||}(a) \right\} = \varepsilon > 0 ,$$

because $\sup_{\|a\| \leq \varepsilon} |\bar{a}(a)| = \varepsilon |||\bar{a}|||$ and $\bar{a} > 0$, therefore Theorem 4.8 follows.

If we prove the following theorem, we have the proof of Theorem 4.6 by Theorem 4.7 and 4.8.

Theorem 4.9. *Let m be a lower semi-additive on R satisfying the condition* (*2). *Then m is not only an approximately additive but also an essentially additive modular on R.*

Proof. If m satisfies the condition of (*2), then $\bar{\bar{m}}$ on $\bar{\bar{R}}^{\bar{m}} \supset R$, being monotone complete, satisfies also. Hence we can suppose that m is monotone complete.

For any element $a \in R$, we can find a positive number ξ such that $|||\xi a||| < \varepsilon < 1$, which implies $m(\xi a) = 0$. Hence

$$m_1(\xi a) = \sup_{[a] = \sum_{i=1}^{n} [p_i]} \sum_{i=1}^{n} m(\xi [p_i] a) = 0 ;$$

which shows $R = S$ in Theorem 1.3. By Theorem 4.3, we find the proof of this theorem.

A singular and lower semi-additive modular satisfies the condition (*2). We may say that this condition is near to the singular modular condition.

§ 5. Uniformly structures of modulars. Uniform structures concerning modulars and modular norms were investigated by NAKANO. (cf. [10]). He defined various kinds of uniform properties such as uniformly simpleness and studied the relations between each others. In this section we will study the relations of m and m_1 which is defined by an approximately additive modular m, concerning with these properties. And we will generalize the theorems in the case of additive modulars to the case where modulars are approximately additive or essentially additive. For this purpose we will explain the definitions of uniform structures in slight different forms from the NAKANO's ones.

We call a modular m to be *uniformly simple* on R, *if*

$$\lim_{\nu \to \infty} m(x_\nu) = 0 \quad \textit{implies} \quad \lim_{\nu \to \infty} |||x_\nu||| = 0 .$$

Here we may change the second norm by the first norm.

This definition is equivalent to: for any $\xi > 0$,

$$\inf_{0 \neq x \in R} \omega(\xi \mid x) > 0 \quad \text{where} \quad \omega(\xi \mid x) = m\left(\xi \frac{x}{|||x|||}\right)$$

which was given by NAKANO.

Similarly we call a modular m to be *uniformly monotone* if

$$\lim_{\nu\to\infty} |||x_\nu||| = 0 \quad \textit{and} \quad x_\nu \neq 0 \qquad \nu = 1, 2, \cdots$$

implies

$$\lim_{\nu\to\infty} \frac{m(x_\nu)}{|||x_\nu|||} = 0 .$$

Here we may change the second norm by the first norm.

This definition is equivalent to:

$$\lim_{\xi\to 0} \sup_{0\neq x\in R} \frac{1}{\xi}\, \omega(\xi \mid x) = 0 .$$

Uniformly simpleness and uniformly monotonity are the dual properties to each others.

If a modular m on R is uniformly simple, then the conjugate modular $\bar{m}$ of m on $\bar{R}^m$ is uniformly monotone. And conversely, if m is a uniformly monotone modular on R, then the conjugate modular $\bar{m}$ of m on $\bar{R}^m$ is uniformly simple.

This fact was proved in [10], but we show the simple skech of these proofs as follows.

Let $\varepsilon>0$ and $\delta>0$ be such that

$$|||x||| \geqq \varepsilon \quad \text{implies} \quad m(x) \geqq \delta .$$

We have

$$\bar{x}(x) - m(x) \leqq \frac{|||x|||}{\varepsilon}\, \bar{x}\left(\frac{\varepsilon x}{|||x|||}\right) - \frac{|||x|||}{\varepsilon}\, m\left(\frac{\varepsilon x}{|||x|||}\right)$$

$$\leqq \frac{|||x|||}{\varepsilon}(\varepsilon\, ||\bar{x}|| - \delta) \leqq 0 ,$$

$$\text{if} \quad ||\bar{x}|| \leqq \frac{\delta}{\varepsilon} \quad \text{and} \quad |||x||| \geqq \varepsilon ,$$

therefore

$$\bar{m}(\bar{x}) = \sup_{x\in R} \left\{\bar{x}(x) - m(x)\right\} \leqq ||\bar{x}||\, \varepsilon , \qquad \text{for} \quad ||\bar{x}|| \leqq \frac{\delta}{\varepsilon} .$$

Since ε may be arbitrary, we see that $\bar{m}$ is uniformly monotone.

Conversely, let m be uniformly monotone. For any number $\varepsilon'>0$ there exists $\delta>0$ such that

$$|||x||| \leqq \delta \quad \text{implies} \quad m(x) \leqq \varepsilon'\, |||x||| .$$

Then

$$\begin{aligned}\bar{m}(\bar{x}) &= \sup_{x \in R} \left\{\bar{x}(x) - m(x)\right\} \\ &\geqq \sup_{|||x||| \leqq \delta} \left\{\bar{x}(x) - m(x)\right\} \\ &\geqq \sup_{|||x||| \leqq \delta} \{||\bar{x}|| \cdot |||x||| - \varepsilon' |||x|||\} \\ &= (||\bar{x}|| - \varepsilon')\,\delta\,.\end{aligned}$$

If we choose ε' suitably for the given number $\varepsilon > 0$,

$$||\bar{x}|| \geqq \varepsilon \quad \text{implies} \quad \bar{m}(\bar{x}) \geqq (\varepsilon - \varepsilon')\,\delta > 0\,,$$

which shows that $\bar{m}$ is uniformly simple.

Let m be a modular on R, we call m to be *uniformly finite* if *for every natural number n there exists a positive number n' such that*

$$|||x||| \leqq n \quad \textit{implies} \quad m(x) \leqq n'\,,$$

in other words:

$$\sup_{0 \neq x \in R} \omega(\xi \mid x) < +\infty \qquad \text{for every} \quad \xi > 0\,.$$

We will call a modular m to be *uniformly increasing* if *for any natural number n there exists a positive number n' such that*

$$|||x||| \geqq n' \quad \textit{implies} \quad m(x) \geqq n\,|||x|||\,,$$

in other words:

$$\lim_{\xi \to \infty} \frac{1}{\xi} \inf_{0 \neq x \in R} \omega(\xi \mid x) = +\infty\,.$$

In these definitions, we may change the second norm by the first norm.

We will show that *if m is uniformly finite, then $\bar{m}$ is uniformly increasing.* For any $l > 0$ there exists a positive number $l' > 0$ such that

$$||x|| \leqq l \quad \text{implies} \quad m(x) \leqq l'\,.$$

Then

$$\begin{aligned}\bar{m}(\bar{x}) &= \sup_{x \in R} \left\{\bar{x}(x) - m(x)\right\} \\ &\geqq \sup_{||x|| \leqq l} \left\{|||\bar{x}||| \cdot ||x|| - l'\right\} \\ &\geqq |||\bar{x}|||\, l - l'\,.\end{aligned}$$

Making $|||\bar{x}|||$ sufficiently large, we see easily $\bar{m}$ is uniformly increasing.

Conversely, *if m is uniformly increasing*, then for any positive number l there exists a positive number l' such that

$$\|x\| \geqq l' \quad \text{implies} \quad m(x) \geqq l\,\|x\| .$$

Then we can find that:

$$\bar{x}(x) - m(x) \leqq |||\bar{x}||| \cdot \|x\| - l\,\|x\| \leqq 0$$

if $\|x\| \geqq l'$, $|||\bar{x}||| \leqq l$.

Therefore

$$\bar{m}(\bar{x}) = \sup_{x \in R} \left\{ \bar{x}(x) - m(x) \right\} \leqq |||\bar{x}|||\, l' , \qquad \text{if} \quad |||\bar{x}||| < l ,$$

which shows *$\bar{m}$ is uniformly finite.*

If an additive modular m on R is uniformly simple and furthermore R has no atomic elements, then m is uniformly finite. In an approximately additive case, this theorem may be written as follows.

Theorem 5.1. *Let m be an upper approximately additive modular on R and uniformly simple. If R has no atomic elements, then for any positive number l there exists a positive number l' such that*

$$|||x||| \leqq l \quad \textit{implies} \quad m_1(x) \leqq l' .$$

Hence m_1 is finite.

Proof. Suppose that we can select $a_\nu \in R\,(\nu = 1, 2, \cdots)$ such that

$$|||a_\nu||| \leqq \varepsilon < 1 \quad \text{and} \quad m_1(la_\nu) \geqq 2^\nu \cdot \nu \qquad \text{for some number} \quad l > 0 .$$

Since R has no atomic elements, for every a_ν $(\nu = 1, 2, \cdots)$ we can find orthogonal projectors $[p_{\nu,i}]\,(i = 1, 2, \cdots, \nu)$ such that

$$\sum_{i=1}^{\nu} [p_{\nu,i}]a_\nu = a_\nu \qquad \text{and} \qquad m([p_{\nu,i}]a_\nu) \leqq \frac{1}{\nu} .$$

Hence we have

$$m_1(l[p_{\nu,i_0}]a_\nu) \geqq 2^\nu \qquad \text{for some} \quad i_0,\ 1 \leqq i_0 \leqq \nu .$$

Let $[p_{\nu,i_0}] = [p_\nu]$, namely

$$m(l[p_\nu]a_\nu) \geqq m_1(l[p_\nu]a_\nu) \geqq 2^\nu \quad \text{and} \quad m([p_\nu]a_\nu) \leqq \frac{1}{\nu} .$$

The sequence $[p_\nu]a_\nu$ is convergent to 0 by the norm, because $\lim_{\nu \to \infty} m([p_\nu]a_\nu) = 0$ and m is uniformly simple. But this contradicts to $|||l[p_\nu]a_\nu||| \geqq |||l[p_\nu]a_\nu|||_1 \geqq 1$.

The conjugate form of this theorem is as follows:

Theorem 5.2. *Let m be a lower approximately additive and uniformly increasing modular on R. If R has no atomic elements, then for any positive number l, there exists a positive number l' such that*

$$|||a||| \geqq l' \quad \textit{implies} \quad \frac{m_1(a)}{|||a|||} \geqq l\,.$$

Hence for every element $x \in R$ different from 0, we have

$$\lim_{\xi \to \infty} \frac{m_1(\xi x)}{\xi} = +\infty\,,$$

where m_1 is defined in Theorem 1.3.

KALIGUNA's theorem which is generalized by S. YAMAMURO in the cases of additive modulars is also applicable to the lower semi-additive cases.

Theorem 5.3. *A lower semi-additive modular m on R is uniformly simple if m is simple and monotone complete and furthermore the modular norms of m are continuous.* ($y_\lambda \downarrow_{\lambda \in \Lambda} 0$ *implies* $\inf_{\lambda \in \Lambda} |||y_\lambda||| = 0$.)

Proof. Suppose that there exist $x_\nu > 0$ $(\nu = 1, 2, \cdots)$ such that

$$\lim_{\nu \to \infty} m(x_\nu) = 0 \text{ and } |||x_\nu||| \geqq \delta > 0 \quad \text{for some positive number } \delta.$$

We can choose x_ν as $m(x_\nu) \leqq \dfrac{1}{2^\nu}$.

Then

$$x_\nu \cup x_{\nu+1} \cup \cdots = y_\nu$$

exists by the lower additivity: the modular condition 7″) and the monotone completeness of m.

We have

$$m(y_\nu) \leqq \frac{1}{2^{\nu-1}} \qquad (\nu = 1, 2, \cdots)\,,$$

therefore the sequence y_ν is order convergent to 0, since m is simple.

$$|||y_\nu||| \geqq |||x_\nu||| \quad \text{implies} \quad \lim_{\nu \to \infty} |||x_\nu||| = 0\,.$$

But this is a contradiction, and hence m is uniformly simple.

In the case of an upper essentially additive modulars we may prove the following.

Theorem 5.4. *Let m be an upper essentially additive modular on R. If m is monotone complete, simple and modular norms are continuous, then m is uniformly simple.*

Before we prove this theorem, we need the following lemma.

Lemma 5.1. *Let m be an upper essentially additive modular on R. If m is simple, then we have for every element $\bar{x}\in\bar{R}^m$*

$$\inf_{\xi>0}\frac{\bar{m}(\xi\bar{x})}{\xi}=0\,.$$

Proof of Lemma: We put

$$\bar{M}=\left\{\bar{x}\,|\,\bar{m}'(\bar{x})=\inf_{\xi>0}\frac{\bar{m}(\xi\bar{x})}{\xi}=0\right\}.$$

We shall prove that $\bar{M}$ is a normal manifold of $\bar{R}^m$. For this purpose we need only prove

$$\bar{a}_\lambda\in\bar{M}\,(\lambda\in\Lambda),\ \bar{a}_\lambda\uparrow_{\lambda\in\Lambda}\bar{a}\ \text{ implies }\ \bar{a}\in\bar{M}\,.$$

A functional of $\bar{R}^{m_1}$:

$$\bar{m}_1'(\bar{a})=\inf_{\xi>0}\frac{\bar{m}_1(\xi\bar{a})}{\xi}\,,\quad \bar{a}\in\bar{R}^{m_1}$$

can be considered a bounded linear functional on the positive elements in $\bar{R}^{m_1}$. Since m is an essentially additive modular we have by Theorem 4.2 $\bar{R}^m=\bar{R}^{m_1}$.

From

$$\bar{m}'(\bar{a})\leqq\bar{m}'(\bar{a}-\bar{a}_\lambda)+\bar{m}'(\bar{a}_\lambda),\quad \bar{m}'(\bar{a}-\bar{a}_\lambda)\leqq\bar{m}_1'(\bar{a}-\bar{a}_\lambda)\downarrow_{\lambda\in\Lambda}0\,,$$

we have

$$\sup_{\lambda\in\Lambda}\bar{m}'(\bar{a}_\lambda)=\bar{m}'(\bar{a})=0\,,$$

which shows that $\bar{a}\in\bar{M}$.

Let $\bar{R}^m$ be decomposed such as: $\bar{R}^m=\bar{M}\oplus\bar{N}$, where in $\bar{N}$ we have a linear modular $\bar{m}'$.

For an element $x\in[\bar{N}]^R R$[1], we have

$$\begin{aligned} m(x)&=\sup_{\bar{x}\in\bar{N}}\left\{\bar{x}(x)-\bar{m}(\bar{x})\right\}\\ &\leqq\sup_{\bar{x}\in\bar{N}}\left\{\bar{x}(x)-\bar{m}'(\bar{x})\right\}=m_2(x)\,.\end{aligned}$$

$\bar{m}'(\bar{x})$ is linear and hence $m_2(x)$ is a singular modular on a complete semi-normal manifold of $[\bar{N}]^R R$, therefore $m(x)$ is not simple in any semi-normal manifold of $[\bar{N}]^R R$: that is $[\bar{N}]=0$. Namely $\bar{M}=\bar{R}^m$. Q.E.D.

1) $[\bar{N}]^R R=\{x\,|\,|x|\frown|y|=0,\ \text{for}\ \bar{x}(|y|)=0,\ \bar{x}\in\bar{N}\}$ Cf. [8].

Proof of Theorem 5.4. Suppose that m is not uniformly simple. Then we can find a sequence of positive elements a_ν $(\nu=1,2,\cdots)$ in R such that $m(a_\nu)\leqq\frac{1}{2^\nu}$ and $|||a_\nu|||\geqq\delta>0$ for some positive number $\delta>0$. From the monotone completeness of m, we obtain $\bar{R}^m=\bar{R}$, and hence by the definition of the conjugate modular we have:

$$m(a_\nu)+\bar{m}(\xi\bar{a})\geqq\xi\,|\bar{a}|\,(a_\nu)$$

for any element $\bar{a}\in\bar{R}$ and $\xi\geqq0$. Let $\bar{a}$ be fixed. By the above Lemma 5.1, we see that for any $\varepsilon>0$ there exists a positive number $\xi_0>0$ such that

$$\frac{\bar{m}(\xi_0\bar{a})}{\xi_0}\leqq\frac{\varepsilon}{2},$$

therefore choosing ν_0 suitably, we obtain $|\bar{a}|\,(a_\nu)\leqq\varepsilon$ for $\nu\geqq\nu_0$, that is

$\lim_{\nu\to\infty}|\bar{a}|\,(a_\nu)=0$: $|\omega|$-$\lim_{\nu\to\infty}a_\nu=0$ in the notation of NAKANO [8]. By Theorem 27.10 in [8] we find that s-ind-$\lim_{\nu\to\infty}a_\nu=0$, since R is supper universally continuous. Then we can select a_{ν_i} $(i=1,2,\cdots)$ such as

$$\text{ind-}\lim_{i\to\infty}a_{\nu_i}=0\,.$$

Because m_1 is monotone complete and $m_1(a_{\nu_i}\cup a_{\nu_{i+1}}\cup\cdots\cup a_{\nu_{i+n}})\leqq m_1(a_{\nu_i})+\cdots+m_1(a_{\nu_{i+n}})\leqq\frac{1}{2^{i-1}}$ for every i, we can find the order limit y_i of $y_{i,n}=a_{\nu_i}\cup\cdots\cup a_{\nu_{i+n}}$ $(i=1,2,\cdots)$. Hence, we have $y_i\downarrow_i y$ for some y, that is $\overline{\lim_{\nu\to\infty}}\,a_\nu=y$. On the other hand, ind-$\lim_{\nu\to\infty}a_\nu=0$ means that there exists a sequence of projectors $[p_i]\uparrow_i[y]$ such that

$$\lim_{\nu\to\infty}[p_i]a_\nu=0=[p_i]y\,,$$

therefore $y=0$ that is $y_i\downarrow_i0$. By the continuity of the modular norm we have $|||y_i|||\downarrow_i0$: namely $|||a_i|||\downarrow_i0$ which is contradict the assumption. Q.E.D.

In an essentially additive modular m, the uniform structures may be transfered to m_1.

Theorem 5.5. *If R is discrete, and an upper essentially additive modular m on R, and m is uniformly simple, then m_1 is uniformly simple.*

Proof. We can suppose that m is monotone complete. m_1 is also monotone complete by the assumption. m_1 is simple because R is discrete.

We shall prove that the modular norms of m are continuous. For any $y_\lambda \downarrow_{\lambda\in\Lambda} 0$, there exist a positive number $\xi_0>0$ and $\lambda_0\in\Lambda$ such that $m(\xi_0 y_{\lambda_0})<+\infty$. Hence we have by the modular condition 6) and 7′)

$$\inf_{\lambda\in\Lambda} m(\xi_0 y_\lambda) = 0 .$$

Since m is uniformly simple, we obtain $\inf_{\lambda\in\Lambda} |||\xi_0 y_\lambda|||=0$ which is $\inf_{\lambda\in\Lambda} |||y_\lambda|||=0$. Because we have $|||x|||_1 \leqq |||x|||$ for every element $x\in R$, the modular norm of m_1 is continuous.

If the additive modular m_1 is considered a lower semi-additive one, m_1 satisfies the assumption of Theorem 5.3.

Hence m_1 is uniformly simple.

The converse of this theorem is:

Theorem 5.6. *If m is an upper essentially additive modular on R and m_1 is uniformly simple, then m is uniformly simple.*

Proof. Let $\lim_{\nu\to\infty} m(x_\nu)=0$, then $\lim_{\nu\to\infty} m_1(x_\nu)=0$. Because m_1 is uniformly simple, we obtain that $\lim_{\nu\to\infty} |||x_\nu|||_1=0$. From the essentially additive property of m we have $\lim_{\nu\to\infty} |||x_\nu|||=0$, which shows that m is uniformly simple.

The dual forms of above theorems are:

Theorem 5.7. *If R is discrete and a lower essentially additive modular m on R is uniformly monotone, then m_1 is uniformly monotone.*

Theorem 5.8. *If m is a lower essentially additive modular on R, and m_1 is uniformly monotone, then m is uniformly monotone.*

In the following we shall explain the theorems of the same kinds without proof.

Theorem 5.9. *Let m be an upper essentially additive modular on R.*
If m is uniformly finite, then m_1 is uniformly finite;
if m_1 is uniformly increasing, then m is uniformly increasing;
if m is uniformly monotone, then m_1 is uniformly monotone;
if m_1 is uniformly monotone, then m is uniformly monotone.

In the case of lower semi-additive modulars we see the dual forms of above theorems.

Theorem 5.10. *Let m be a lower essentially additive modular on R.*
If m is uniformly simple, then m_1 is uniformly simple;
if m_1 is uniformly simple, then m is uniformly simple;
if m_1 is uniformly finite, then m is uniformly finite;
if m is uniformly increasing, then m_1 is uniformly increasing.

REMARK 1. On the upper additive cases, the uniformly finiteness of m_1 does not imply the uniformly finiteness of m. And m_1 is not uniformly increasing for some times, even if m is uniformly increasing. In the lower semi-additive cases we have the similar remarks.

REMARK 2. We say that a modular m on R is *upper bounded* if there exists a positive number γ such that $m(2x)\leqq\gamma m(x)$ for every $x\in R$. Similarly a modular m on R is called *lower bounded* if there exists a number $\gamma>2$ such that $m(2x)\geqq\gamma m(x)$ for every $x\in R$. If m is a modular and upper bounded, then the conjugate modular $\bar{m}$ is lower bounded. Conversely if m is lower bounded, then $\bar{m}$ is upper bounded, by the same argument in [8]. Furthermore if m is an approximately additive modular on R, and m is upper or lower bounded, then m_1 is also upper or lower bounded by the construction of m_1.

If a modular m on R is upper bounded, then m is uniformly simple and uniformly finite, and if a modular m on R is lower bounded then m is uniformly monotone and uniformly increasing. Hence, in these cases, uniform properties of m may be transfered to m_1.

§ 6. Bi-modulars. *Bi-modulars* are the functionals of two variable and were defined by M. Miyakawa and H. Nakano [11]. Let X, Y be universally continuous semi-ordered linear spaces. Then the functional $M(x, y)$ of two variables $x\in X$ and $y\in Y$ is said to be a *bi-modular* of X and Y, *if the following conditions are satisfied.*

1) *$M(x, y)$ is an additive modular on X if $y\neq 0$ is fixed.*
2) $M(x, |y_1|+|y_2|)=M(x, y_1)+M(x, y_2)$.
3) *For any number $\alpha>0$, $M(x, \alpha y)=\alpha M(x, y)$.*
4) *For any element $y\in Y$, there exists $\alpha\neq 0$ such that*

$$M(\alpha x, y)<+\infty \qquad \textit{for every} \quad x\in X.$$

If we have $M(x, y)<+\infty$ for every $x\in M$ and $y\in Y$, we say $M(x, y)$ is *finite.*

From $M(x, y)$ we can define a norm modular of M if Y has a norm and a *double modular* of M if Y has a complete additive modular. The *double modular $M_D(x)$ of M is defined as follows*:

$$M_D(x)=\sup_{y\in Y}\left\{M(x, y)-m_s(y)\right\}$$

if there exists an complete additive modular $m_s(y)$ on Y.

This modular is upper semi-additive on X.

In this section we shall consider the condition that the double modular

is an approximately additive modular. In the following we suppose that M is finite.

Let Y be one-dimensional space. Then the modular on $Y=\{\xi\}$ may be considered a convex function of real variable: $m_s(\xi)=\varphi(\xi)$, $\xi\in Y$.

Hence, we have

$$M_D(x) = \sup_{\xi} \left\{M(x,\xi)-m_s(\xi)\right\}$$
$$= \sup_{\xi} \left\{\xi M(x,1)-\varphi(\xi)\right\} = \bar{\varphi}\left\{M(x,1)\right\}$$

where $\bar{\varphi}$ has the YOUNG-relation with φ. (that is $\bar{\varphi}(\eta)=\sup_{\xi}\{\eta\xi-\varphi(\xi)\}$).

In the case where X has no atomic elements, any element $x\in X$ can be decomposed as $\sum_{i=1}^{n} x_i=x$ for every natural number n such that

$$\frac{1}{n} M(x,1) = M(x_i, 1)\,,$$

$$x_i = [p_i]x \quad \text{and} \quad \sum_{i=1}^{n}[p_i]=[x] \quad (i=1,2,\cdots,n)\,.$$

Hence

$$\sum_{i=1}^{n} \varphi\left\{M(x_i, 1)\right\}=n\,\bar{\varphi}\left\{\frac{1}{n} M(x,1)\right\} \text{ convergent to } \bar{\varphi}'(0)\,M(x,1) \text{ as } n\to\infty,$$

where $\bar{\varphi}'(0)$ is the r-differential coefficient of $\bar{\varphi}(\xi)$ at 0. Therefore we have

$$M_{D,1}(x) = \inf_{\sum_{i=1}^{n}[p_i]=[x]} \sum_{i=1}^{n} M_D([p_i]x)=\bar{\varphi}'(0)\cdot M(x,1)$$

which is defined in Theorem 1.4.

$\bar{\varphi}'(0)>0$ if and only if $\bar{\varphi}(\xi)$ is ascending as the modular: that is $\varphi(\xi)$ is semi-singular as the modular by Theorem 3.7, which is the proof of the following theorem.

Theorem 6.1. *Let X have no atomic elements and Y be an one-dimensional modulared linear space. Then the double modular $M_D(x)$ of finite bi-modular M is upper approximately additive if and only if Y is semi-singular as a modulared space.*

Let Y be a discrete space and $m_s(y)$, $y\in Y$ be an additive modular on Y. Then the element y of Y may be written as

$$y = \{\xi_\lambda\}_{\lambda\in\Lambda} \quad \text{and} \quad m_s(y) = \sum_{\lambda\in\Lambda} \varphi_\lambda(\xi_\lambda)$$

where $\varphi_\lambda(\lambda\in\Lambda)$ are the convex functions of real variable.

By the same argument as above, if X has no atomic elements, we see:

$$M_D(x) = \sum_{\lambda \in \Lambda} \bar{\varphi}_\lambda \left\{ M(x, 1_\lambda) \right\}$$

where $\bar{\varphi}_\lambda (\lambda \in \Lambda)$ has the YOUNG-relation with φ_λ, and 1_λ are the element of Y such that $\xi_\lambda = 1$ and otherwise 0.

Hence we have:

Theorem 6.2. *If X has no atomic elements and Y is discrete. Then we have*

$$M_{D,1}(x) = \sum_\lambda \bar{\varphi}'_\lambda(0) \cdot M(x, 1_\lambda) .$$

Hence M_D is approximately additive if and only if at least one of $\varphi_\lambda (\lambda \in \Lambda)$ is semi-singular.

In the case where Y is not discrete, if m_s is semi-singular, we have the following theorem.

Theorem 6.3. *If m_s is a semi-singular modular on Y, then the double modular $M_D(x)$ is approximately additive.*

Proof. For any $0 \neq y_0 \in Y$ such that $m_s(\alpha y_0) = 0$ for some $\alpha > 0$, we have

$$\begin{aligned} M_D(x) &= \sup_{y \in Y} \left\{ M(x, y) - m_s(y) \right\} \\ &\geqq \sup_{\xi > 0} \left\{ M(x, \xi y_0) - m_s(\xi y_0) \right\} = \bar{\varphi}_{y_0} \left\{ M(x, y_0) \right\} \end{aligned}$$

where $\bar{\varphi}_{y_0}(\eta)$ is the YOUNG-relation with $m_s(\xi y_0) = \varphi_{y_0}(\xi)$. Therefore we have

$$M_{D,1}(x) \geqq \bar{\varphi}'_{y_0}(0) \cdot M(x, y_0)$$

by Theorem 6.1 which means that M_D is approximately additive.

In the above three theorems we see that the modular norm of M_D is stronger than the modular norm of $M(x, y)$ for a fixed $y \in Y$, where $m_s(\xi y)$ is semi-singular modular on reals ξ.

If Y has no atomic elements and m_s is not a semi-singular modular on Y (: that is simple), we have the examples of double modulars which are approximately additive or not.

We will consider the following case.

Theorem 6.4. *Let X have no atomic elements and an complete additive modular $m_s(y)$ on Y be simple. If we have $\{ f_x(y), x \in X \}$ are finite dimensional in the space of bounded linear functional on Y, then M_D is not approximately*

additive modular, where $f_x(y) = M(x, y^+) - M(x, y^-)$, $x \in X$.

Proof. We need only prove the case that $\{f_x(y), x \in X\}$ is one-dimensional.

Let $M(x_0, y_0) \neq 0$ for some $x_0 \in X$, $y_0 \in Y$. For any $x \in X$, $y \in Y$, we can find a number λ such that

$$\lambda M(x_0, y_0) = M(x, y_0), \quad \lambda M(x_0, y) = M(x, y),$$

therefore

$$M(x, y) = M(x, y_0) \frac{M(x, y)}{M(x_0, y_0)}.$$

Putting

$$\frac{M(x_0 y^+)}{M(x_0, y_0)} - \frac{M(x_0, y^-)}{M(x_0, y_0)} = f(y) \quad \text{and} \quad M(x, y_0) = m(x),$$

we have

$$M(x, y) = m(x) \{f(y^+) + f(y^-)\}$$

Because X has no atomic elements and f can be considered a modular bounded linear functional by the completeness of m_s, we see that M_D is not approximately additive by the same argument as above.

REMARK. If X is discrete, then a double modular M_D of M is always approximately additive by Theorem 2.1.

§ 7. Infinitely linear modulars. This section is devoted to the approximate methode different from the preceeding ones.

Let m be an upper semi-additive modular on R. We say that an element x of R is *infinitely linear* if

$$\lim_{\xi \to \infty} \frac{m(\xi x)}{\xi} < +\infty.$$

The totality of infinitely linear elements constitutes a semi-normal manifold of R because of the modular condition 4) and 5). If this semi-normal manifold M is complete in R, we say that R *is infinely linear* by a modular m. In M, if we put

$$n(x) = \lim_{\xi \to \infty} \frac{m(\xi x)}{\xi} \quad \text{for} \quad x \in M,$$

then $n(x)$ satisfies 1), 2), 3), 4), 5), 6), 7′) of the modular conditions. Furthermore we see that n is linear. By Theorem 3.1 we see that n is a linear additive modular on M. We shall investigate the relation

between an infinitely linear modular m and its conjugate modular.

Theorem 7.1. *If m is an upper semi-additive modular on R, and R is infinitely linear by m, then $\bar{m}$ on $\bar{R}^m$ is infinite: this means that for every element $0 \neq \bar{x} \in \bar{R}^m$, there exists some number $\alpha > 0$ such that $\bar{m}(\alpha\bar{x}) = +\infty$.*

Proof. We see that: (as same as [8])

$$\bar{m}(\alpha\bar{x}) \geqq \sup_{\xi \geqq 0} \{\alpha\bar{x}(\xi a) - m(\xi a)\}$$

$$= \sup_{\xi \geqq 0} \xi \left\{\alpha\bar{x}(a) - \frac{m(\xi a)}{\xi}\right\}.$$

If α is sufficiently large and $\bar{x}(a) \neq 0$ for some $a \in M$, then the right hand of the above inequality equals to $+\infty$. This prove the theorem.

The conjugate form of this theorem is:

Theorem 7.2. *If m is a lower semi-additive modular on R, and R is infinite by this modular m, then the conjugate modular $\bar{m}$ on $\bar{R}^m$ is infinitely linear.*

Proof. Let $\bar{x}$ be an arbitrary element of $\bar{R}^m$. The set $N = \{x \mid \bar{x}(|x|) = 0, x \in R\}$ is a normal manifold of R and $N^\perp \neq \{0\}$ if $\bar{x} \neq 0$. For any element $x \in N^\perp$, $\bar{x}(|x|) = 0$ implies $x = 0$, and hence $N^\perp$ is supper-universally continuous, therefore $[x]R\,(x \in N^\perp)$ is totally continuous.

We shall show that for any element $x > 0$, $x \in N^\perp$, we can find some positive number $\nu_0 > 0$ and $0 < [p_0] \leqq [x]$ such that we have

$$m(\nu_0[p]x) = +\infty \qquad \text{for every } 0 < [p] \leqq [p_0].$$

Suppose that we can not find such $[p_0]$ for every ν; $(\nu = 1, 2, \cdots)$.

Let ν be fixed. $(\nu = 1, 2, \cdots)$.

For any $[p] > 0$, there exist $0 < [p'] \leqq [p]$ such that

$$m(\nu[p']x) < +\infty .$$

If $m(\nu[p'_i]x) < +\infty$ $(i = 1, \cdots, s)$, then $m\{\nu([p'_1] \cup \cdots \cup [p'_s])x\} < +\infty$ from the lower semi-additive condition 7″).

Hence for the fixed ν, we can find the sequence of projectors $[p_i^\nu]$ $(i = 1, 2, \cdots)$ such that

$$m(\nu[p_i^\nu]x) < +\infty \qquad (i = 1, 2, \cdots)$$

and

$$[p_i^\nu] \uparrow_i [x] \qquad (\nu = 1, 2, \cdots).$$

From the totally continuity of $[x]R$, we can find $0 < [p_0] \leqq [x]$ such that

$m(\nu[p_0]x) < +\infty$ for every $\nu = 1, 2, \cdots$ which means that $0 \neq [p_0]x$ is

a finite element in R. This contradict the assumption.

If we have $m(p)<+\infty$, for some $p\in R$, then $[p_0]p\leqq\nu_0[p_0]x$. Because, if we have not

$$[p_0]p\leqq\nu_0[p_0]x\,,\quad\text{that is}\quad 0\neq[([p_0]p-\nu_0[p_0]x)^+]\leqq[p_0]\,,$$

then for any $[p_1]$ such that

$$0\neq[p_1]\leqq[([p_0]p-\nu_0[p_0]x)^+]$$

we obtain $[p_1]p\geqq\nu_0[p_1]x$, therefore it follows that $m([p_1]p)=+\infty$ from the modular condition 5), which is contradict.

We find that:

$$\begin{aligned}\bar{m}(\xi\bar{x}[p_0]) &= \sup_{x_1\in R}\left\{\xi\bar{x}([p_0]x_1)-m(x_1)\right\}\\ &\leqq \xi\bar{x}(\nu_0([p_0]x))<+\infty\,,\end{aligned}$$

therefore $\dfrac{\bar{m}(\xi\bar{x}[p_0])}{\xi}<+\infty$, namely $\bar{x}[p_0]\neq 0$ is infinitely linear element in $\bar{R}^m$. This means that $\bar{R}^m$ is infinitely linear by the modular $\bar{m}$.

From these theorems we see that if an upper semi-additive modular is infinitely linear, then there exists a linear modular greater than this modular and if a lower semi-additive modular is infinite, then there exists a singular modular less than this modular.

In the first case, if the modular m is monotone complete then the modular $n(x)=\lim\limits_{\xi\to\infty}\dfrac{m(\xi x)}{\xi}$ defined in the complete semi-normal manifold of the space is monotone complete. And furthermore if all the element of the space is infinetely linear, then the modular norms defined by m and n are equivalent to each others. Another argument similar to the approximately additive cases also follows. In order to avoid the repeated process, we omitt the further arguments for the present topics.

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