ON CONJUGATELY SIMILAR TRANSFORMATIONS

By

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Introduction. H. Nakano in this book [1] defined the modulared semi-ordered linear space \( R(m) \), that is, \( R \) is a universally continuous\(^1\) semi-ordered linear space where a functional \( m(a) \) \((a \in R)\) is defined such as the following seven properties are satisfied:

1) \( 0 \leq m(a) \leq +\infty \) for all \( a \in R \);
2) if \( m(\xi a) = 0 \) for all \( \xi \geq 0 \), then \( a = 0 \);
3) for any \( a \in R \) there exists \( \alpha > 0 \) such that \( m(\alpha a) < +\infty \);

\((M)\) 4) for any \( a \in R \), \( m(\xi a) \) is a convex function of \( \xi \);
5) \( |a| \leq |b| \) implies \( m(a) \leq m(b) \);
6) \( a \cap b = 0 \) implies \( m(a + b) = m(a) + m(b) \);
7) \( 0 \leq a_{\lambda} \uparrow_{\lambda \in \Lambda} a \) implies \( \sup_{\lambda \in \Lambda} m(a_{\lambda}) = m(a) \).

This functional \( m(a) \) \((a \in R)\) is called a modular on \( R \). The well-known space \( L_p([0,1]) \) \((p \geq 1)\) is one of examples of the modulared semi-ordered linear space, putting \( m_p(a) = \int_{0}^{1} \frac{1}{p} |a(t)|^p \) \( dt \) \((p \geq 1)\).

Let \( R \) be a universally continuous semi-ordered linear space and \( \overline{R} \) be the conjugate space of \( R \), that is, the space of all universally continuous\(^2\) linear functionals on \( R \). Especially when \( R \) is a modulared semi-ordered linear space by modular \( m(a) \) \((a \in R)\), a functional \( \overline{a} \in \overline{R} \) is said to be modular bounded if \( \sup_{m(a) \leq 1} |(a, \overline{a})| < +\infty \). The space of all modular bounded functionals \( \overline{R}^m \) is a universally continuous semi-ordered linear space. When we put for \( \overline{a} \in \overline{R} \)
\begin{equation}
\overline{m}(\overline{a}) = \sup_{a \in R} \{(a, \overline{a}) - m(a)\} \ (\overline{a} \in \overline{R}),
\end{equation}

1) A semi-ordered linear space \( R \) is said to be universally continuous if for any system \( a_{\lambda} \geq 0 \) \((\lambda \in \Lambda)\) there exists an element \( \bigcap_{\lambda \in \Lambda} a_{\lambda} \) in \( R \) ([1], p. 17).
2) For any \( \lambda_1, \lambda_2 \in \Lambda \) there exists \( \lambda_3 \in \Lambda \) such that \( a_{\lambda_1} \cup a_{\lambda_2} \leq a_{\lambda_3} \) and \( \bigcup_{\lambda \in \Lambda} a_{\lambda} = a \).
3) A linear functional \( \overline{a} \), \((a, \overline{a}) \) \((a \in R)\), is said to be universally continuous, if for any \( a_{\lambda} \downarrow_{\lambda \in \Lambda} 0 \) we have \( \inf_{\lambda \in \Lambda} |(a_{\lambda}, \overline{a})| = 0 \) ([1], p. 81).
\( \bar{a} \) is modular bounded if and only if we can find \( \alpha > 0 \) such that \( m(\alpha \bar{a}) < + \infty \) ([1], p. 169). And \( \bar{m}(\bar{a}) (\bar{a} \in \bar{R}^m) \) is a modular on \( \bar{R}^m \). Namely \( \bar{R}^m(\bar{m}) \) is also a modulared semi-ordered linear space. This \( \bar{R}^m(\bar{m}) \) is called a modular conjugate space of \( R(m) \) and \( \bar{m}(\bar{a}) (\bar{a} \in \bar{R}^m) \) is called a conjugate modular of \( m(\alpha) (\alpha \in R) \). If we put \( R(m) = L_p([0,1]) (m_p) (p > 1) \), then we have \( \bar{R} = \bar{R}^m = L_q([0,1]) \) and \( \bar{m}_p = m_q \left( \frac{1}{p} + \frac{1}{q} = 1 \right) \).

In [1] the concept of a conjugately similar transformation on \( R \) was introduced as one method to construct a modular on the universally continuous semi-ordered linear space \( R \) and it was tried to represent the modular as an integration of a conjugately similar transformation. A conjugately similar transformation \( T \) is a mapping from \( R^+ \), the positive cone of \( R \), to \( \bar{R}^+ \) satisfying the following two condition:  

1) if \( a \geq b \geq 0 \), then \( Ta \geq Tb \geq 0 \);  
2) for any \( a \in R^+ \) and a normal manifold \( N \) in \( R \) we have \( (Ta)[N] \)

\( = T([N]a), \) where \( [N] \) is the projection operator \( \delta \) of \( N \).

If \( R \) is reflexive \( (R = \bar{R}) \) and there exists a conjugately similar transformation \( T \) on \( R \) such that \( T \) is an onto- and one to one-mapping, \( R \) is called a conjugately similar space by \( T \). For instance \( L_p([0,1]) (p > 1) \) is a conjugately similar space by a conjugately similar transformation \( T \left| a \right| = \left| a \right|^{p} \in L_q([0,1]) \) \( (a \in L_p([0,1])) \).

Main results in [1] concerning the relation between modulars and conjugately similar transformations are the following two: I and II.

I. If we put \( m_T(a) = \int_{0}^{1} (|a|, T \xi |a|) d \xi \) \( (a \in R) \) for any conjugately similar transformation \( T \), then we have a finite modular \( m_T \) on \( R \).

II. If \( R \) is a conjugately similar space by \( T \), then  

1) \( m_T \) and \( m_{\bar{T}} \) are normal \( \delta \) and monotone complete \( \delta \);
2) $T^{-1}$ is also a conjugately similar transformation;
3) $\overline{R}^{m_{T}}=\bar{R}$ and $\overline{m}_{T}=m_{T^{-1}}$;
4) $m_{T}(|a|)+m_{T^{-1}}(T|a|)=(|a|, T|a|) (a \in R)$.

Conversely for the modulared semi-ordered linear space $R(m)$, if $m$ and $\overline{m}$ are normal and monotone complete, then there exists a conjugately similar transformation $T$ such that $R$ is a conjugately similar space by $T$ and $m=m_{T}$ and $\overline{m}=m_{T^{-1}}$.

The purpose of this paper is a generalization of the above results to the most general case. We shall discuss in §1 a generalized conjugately similar transformation and a representation of the modular as the integration of a generalized conjugately similar transformation. In §2 we shall generalize the concept of an inverse transformation of a conjugately similar transformation and study the relation between a conjugate modular and a generalized inverse transformation of a conjugately similar transformation. In §3 we shall state the classification of several known types of modulars in other words, that is, according to the types of conjugately similar transformations. In §4 we shall treat new types of modulars and their conjugate types.

Throughout this paper we shall use notations and terminologies according to H. Nakano’s book [1].

Before entering into the details I wish to express my gratitude to Professor Nakano for his kind encouragement and advice.

§1. Modulars and Conjugately similar transformations. Let $R$ be a universally continuous semi-ordered linear space and $\bar{R}$ be its conjugate space.

**Definition.** A mapping $T$ from a subset $M$ of $R^{+}$ into $\bar{R}^{+}$ is called a *conjugately similar transformation* on $R$ and $M$ is called the domain of $T$, if the following conditions are satisfied

1) if $M \ni a \geqq b \geqq 0$, then $b \in M$;
2) for any $a, b \in M$ we have $a \Leftrightarrow b \in M$;
3) for any $a \in R^{+}$ there exists $\alpha > 0$ such that $\alpha a \in M$;
4) if $M \ni \xi a > 0$ for all $\xi > 0$, then there exists $\xi_{0} > 0$ such that
   \begin{equation}
   T\xi_{0}a > 0;
   \end{equation}
5) $M \ni a \geqq b \geqq 0$ implies $Ta \geqq Tb \geqq 0$;
6) for any $a \in M$ and normal manifold $N$ in $R$ we have $T([N]a) = (Ta)[N]$;
7) If $M \ni a, (\lambda \in \Lambda), a_{\lambda} \uparrow_{\lambda \in \Lambda} a$ and $\sup_{\lambda \in \Lambda} (a_{\lambda}, Ta_{\lambda}) < +\infty$, then we have $\alpha a \in M$ for all $0 \leq \alpha < 1$.

Evidently the above definition is a generalization of (2). We shall use the notation $(T, M)$ to show a conjugately similar transformation having a domain $M$.

From the definition we can see easily the following lemma.

Lemma 1. i) If $(a, Ta) = 0$ for some $a \in M$, then we have $Ta = 0$. Especially $T0 = 0$.

ii) For any $a, b \in M$ such that $a \wedge b = 0$ we obtain $a + b \in M$ and $T(a + b) = Ta + Tb$.

iii) For $a, b \in M$ we have $T(a \vee b) = Ta \vee Tb$.

For instance, if $a, b \in M$ and $a \wedge b = 0$, then from 2) of (C) $a + b \in M$ and from 6) of (C)

$$(x, T(a + b)) = ([a + b], x, T(a + b)) = ([a], x, T(a + b)) + ([b], x, T(a + b))$$

$$(x, T[a](a + b)) + (x, T[b](a + b)) = (x, Ta + Tb)$$

for all $x \in R$. Therefore $T(a + b) = Ta + Tb$.

Theorem 1. For a conjugately transformation $T$ and its domain $M$

$$m_{T}(a) = \int_{0}^{1} ([a], Tz[a]) dz$$

is a modular on $R$, where we put $([a], Tz[a]) = +\infty$ for $z[a] \notin M$.

Proof. 1) of (M) is evident. Lemma 1 and 4) of (C) imply 2) and 6) of (M). 3) of (C) implies 3) of (M). 5) of (C) implies 4) and 5) of (M). We prove only 7) of (M).

At first, for any $[p_{\lambda}] \uparrow_{\lambda \in \Lambda} [a]$ we see $\sup_{\lambda \in \Lambda} m_{T}([p_{\lambda}]a) = m_{T}(a)$. Because from 3) of (C) there exists $\xi_{0}, 0 < \xi_{0} \leq +\infty$ such that

$$\begin{cases} \xi_{1} |a| \in M \text{ for all } 0 \leq \xi_{1} < \xi_{0}, \\
\xi_{2} |a| \notin M \text{ for all } \xi_{2} \geq \xi_{0}. \end{cases}$$

For $\xi_{1}, 0 \leq \xi_{1} < \xi_{0}$ we have $([p_{\lambda}]a, Tz_{\lambda}[p_{\lambda}]a) = ([p_{\lambda}]a, Tz_{\lambda}a)$ from 6) of (C). Therefore $\sup_{\lambda \in \Lambda} ([p_{\lambda}]a, Tz_{\lambda}[p_{\lambda}]a) = ([a], Tz_{\lambda}a)$ for all $0 \leq \xi_{1} < \xi_{0}$. If for some $\xi_{2} > \xi_{0}$ we have $([p_{\lambda}]a, Tz_{\lambda}[p_{\lambda}]a) < +\infty$, then from 7) of (C) we obtain $\alpha z_{\lambda}a \in M$ for all $\alpha, 0 \leq \alpha < 1$. This contradicts the property of $z_{0}$. Therefore we have $\sup_{\lambda \in \Lambda} ([p_{\lambda}]a, Tz_{\lambda}[p_{\lambda}]a) = +\infty = ([a], Tz_{\lambda}a)$ for all $\xi_{2}, +\infty > \xi_{2} > \xi_{0}$. Hence we have

$$\sup_{\lambda \in \Lambda} ([p_{\lambda}]a, Tz[p_{\lambda}]a) = ([a], Tz[a])$$

for all $\xi, \xi \neq \xi_{0}$. 

\[m_{T}(a) = \int_{0}^{1} ([a], Tz[a]) dz\]
Therefore
\[
\sup_{i \in \mathcal{A}} m_T([p_i]a) = \sup_{i \in \mathcal{A}} \int_0^1 ([p_i]|a|, T_\xi[p_i]|a|) d\xi = \int_0^1 \sup_{i \in \mathcal{A}} ([p_i]|a|, T_\xi[p_i]|a|) d\xi
= \int_0^1 (|a|, T_\xi|a|) d\xi = m_T(a).
\]

Next for any \(0 \leq a_i \uparrow \lambda \in \mathcal{A} a\) and \(0 \leq \alpha < 1\) we put \(p_i = (a_i - \alpha a)^+\), then we have \([p_i] \uparrow \lambda \in \mathcal{A} [a]\) and \(a \geq a_i \geq [p_i] a_i \geq \alpha [p_i] a\). Therefore \(m_T(a) \geq m_T(a_i) \geq m_T(\alpha [p_i] a)\) and this implies \(m_T(a) \geq \sup_{\lambda \in \mathcal{A}} m_T(a_i) \geq \sup_{\lambda \in \mathcal{A}} m_T(\alpha [p_i] a) = m_T(\alpha a)\)
for all \(0 \leq \alpha < 1\). On the other hand \(\sup_{0 \leq \alpha < 1} m_T(\alpha a) = \sup_{0 \leq \alpha < 1} \int_0^\alpha (a, T_\xi a) d\xi = \int_0^1 (a, T_\xi a) d\xi = m_T(a)\).

For \((T, M)\) satisfying 1)~7) of (C), we put
\[
M_+ = [a; \alpha a \in M\text{ for some } \alpha > 1\text{ depending on } a],
\]
\[
M_- = [a; \alpha a \in M\text{ for all } 0 \leq \alpha < 1\text{ and } \bigcup_{0 \leq \alpha < 1} T\alpha a \text{ exists}],
\]
\[
T_+ a = \bigcap_{\alpha > 1} T\alpha a \text{ for } a \in M_+,
\]
\[
T_- a = \bigcup_{0 \leq \alpha < 1} T\alpha a \text{ for } a \in M_-.
\]
Evidently we have \(M_+ \subset M \subset M_-\) and \(T_- a \leq T a \leq T_+ a (a \in M_+)\).
We can see easily \((T_+, M_+)\) and \((T_-, M_-)\) have the following properties stronger than (C).

1) if \(M_+ \ni a \geq b \geq 0\), then \(b \in M_+\),
2) for any \(a, b \in M_+\) we have \(a \sim b \in M_+\);
3) for any \(a \in \mathbb{R}^+\) there exists \(\alpha > 0\) such that \(\alpha a \in M_+\);
4) if \(M_+ \ni a \geq 0\) for all \(\xi > 0\), then there exists \(\xi_0 > 0\) such that \(T_+ a \geq 0\);
5) \(M_+ \ni a \geq b \geq 0\) implies \(T_+ a \geq T_+ b \geq 0\);
6) for any \(a \in M_+\) and normal manifold \(N\) in \(\mathbb{R}\) we have \(T_+([N]a) = (T_+ a)[N]\);
7) (+) if \(M_+ \ni a_i (i \in \mathcal{A}), a_i \uparrow i \in \mathcal{A} a\) and \(\sup_{i \in \mathcal{A}} (a_i, T_+ a_i) < +\infty\), then we have \(\alpha a \in M_+ \text{ for all } 0 \leq \alpha < 1\);
    (–) if \(M_- \ni a_i (i \in \mathcal{A}), a_i \uparrow i \in \mathcal{A} a\) and \(\sup_{i \in \mathcal{A}} (a_i, T_- a_i) < +\infty\), then we have \(a \in M_-\);
8) (+) for any \(a \in M_+\) there exists \(\alpha > 1\) such that \(\alpha a \in M_+\) and we have \(T_+ a = \bigcap_{\alpha \geq 1} T_+ \alpha a\);
    (–) for any \(a \in M_-\) we have \(T_- a = \bigcup_{0 \leq \alpha < 1} T_- \alpha a\).

From the above \((C_+)\) we can prove
8') (+) for \(a \in M(\lambda \in \Lambda)\) and \(a \downarrow_{\Lambda} a \geq 0\) we have \(T \oplus [a] \downarrow_{\Lambda} T \oplus a\);
(-) for \(a \in M(\lambda \in \Lambda)\) and \(a \downarrow_{\Lambda} a \in M\) we have \(T \ominus a \downarrow_{\Lambda} T \ominus a\).

(+) We put \([a]_{\lambda} = ([a]_{\lambda} - \alpha a)\) for \(\alpha > 1\) such as \(\alpha a \in M\), we have from the above 5) and 6) \(T \oplus [a] \geq T \ominus [p] \geq T \oplus (1 - [p])a a \leq [p] T \ominus a + T \ominus a a\).

Lemma 2. \((T \oplus, M_{\oplus})\) and \((T \ominus, M_{\ominus})\) are also conjugately similar transformations on \(R\) and we have \(m_{T} = m_{\tau_{\oplus}} = m_{\tau_{\ominus}}\).

It is clear from \((C)\) that \((T \oplus, M_{\oplus})\) and \((T \ominus, M_{\ominus})\) are conjugately similar transformations on \(R\). From the definition of \(T \oplus\) and \(T \ominus\) we have for all \(\xi \geq 0\) \((|a|, T \oplus \xi |a|) = \inf_{\epsilon > 0} (|a|, T \eta |a|)\) and \((|a|, T \ominus \xi |a|) = \sup_{0 \leq \epsilon < \xi} (|a|, T \xi |a|)\)

\((\xi \geq 0)\). As \((|a|, T \xi |a|)\) is a monotone function of \(\xi \geq 0\), \((|a|, T \xi |a|)\) is equal to \((|a|, T \xi |a|)\) for almost everywhere \(\xi \geq 0\). Therefore \(m_{T}(a) = m_{\tau_{\oplus}}(a) = m_{\tau_{\ominus}}(a)\) \((a \in R)\).

If we put for a modular \(m\) on \(R\)

\[
D_{\oplus} m(a) = \begin{cases} 
\inf_{\epsilon > 1} \frac{m(\xi a) - m(a)}{\xi - 1} & \text{for } m(a) < +\infty \\
+\infty & \text{for } m(a) = +\infty
\end{cases}
\]

(4)

\[
D_{\ominus} m(a) = \begin{cases} 
\sup_{0 < \xi < \epsilon} \frac{m(a) - m(\xi a)}{1 - \xi} & \text{for } m(a) < +\infty \\
+\infty & \text{for } m(a) = +\infty
\end{cases}
\]

then we can see easily

\[
M_{\oplus} = \{|a|; D_{\oplus} m_{T}(a) < +\infty\},
\]

\((a, T \ominus a) = D_{\ominus} m_{T}(a)\) for \(a \in M_{\oplus}\).

Next, in the following (Theorem 2 and 3) we shall prove the converse of Theorem 1, that is, for any modular semi-ordered linear space \(R(m)\) there exists a conjugately similar transformation \((T, M)\) such that \(m = m_{T}\). Theorem 2 is fundamental. However we shall assume some knowledge about the spectral theory of a semi-ordered linear space ([1], Chap. II.).

At first we state the properties of \(D_{\oplus} m(a)\) \((a \in R)\) ([4], §1).
Let \( D_{\pm}m(\xi a) \) be the derivative at \( \xi \) of a function \( m(\xi a) \) from the right (left) side. We see easily

\[
\xi D_{\pm}m(\xi a) = D_{\pm}m(\xi a) \quad (\xi \geq 0, a \in R),
\]

(6)

\[
m(a) = \int_0^1 D_{\pm}m(\xi a) d\xi = \int_0^1 \frac{D_{\pm}m(\xi a)}{\xi} d\xi \quad (a \in R).
\]

From the convexity of \( m(\xi a) \) \( D_{\pm}m(\xi a) \) is a right (left) continuous increasing function and \( D_{\pm}m(\xi a) \leq D_{\mp}m(\xi a) \).

The property that characterizes the functionals \( D_{\pm}m(a) \) \( (a \in R) \) is the following.

**Lemma 3.**

1) \( 0 \leq D_{\pm}m(a) \leq +\infty \) for all \( a \in R \);

2) if \( D_{\pm}m(\xi a) = 0 \) for all \( \xi \geq 0 \), then \( a = 0 \);

3) for any \( a \in R \) there exists \( a > 0 \) such that \( D_{\pm}m(\alpha a) < +\infty \);

4) (\( + \)) \( \frac{D_{\pm}m(\xi a)}{\xi} \) is a right continuous increasing function of \( \xi > 0 \);

\( (D_{\pm}) \)

(\( - \)) \( \frac{D_{\mp}m(\xi a)}{\xi} \) is a left continuous decreasing function of \( \xi > 0 \);

5) \( |a| \leq |b| \) implies \( D_{\pm}m(\alpha a) \leq D_{\pm}m(\alpha b) \);

6) \( a \cap b = 0 \) implies \( D_{\pm}m(a+b) = D_{\pm}m(a) + D_{\pm}m(b) \);

7) (\( + \)) if \( a_0 \geq a_1 \downarrow_{\lambda \in A} a \geq 0 \) and \( D_{\pm}m(a_0) < +\infty \), then \( \inf_{\lambda \in A} D_{\pm}m(a_1) = D_{\pm}m(a) \);

(\( - \)) if \( 0 \leq a_0 \uparrow_{\lambda \in A} a \), then \( \sup_{\lambda \in A} D_{\pm}m(a_1) = D_{\pm}m(a) \).

1)\( \sim \)4) are all evident and (\( M \)) 6) implies 6).

The proof of 5): We may put \( 0 \leq a \leq b \), \( b = [a]b \) and \( D_{\pm}m(b) < +\infty \), from the spectral theory we can find \( b_n \geq a \) \( (n = 1, 2, \ldots) \) such that \( b_n \uparrow_{n \to \infty} b \)

and \( b_n = \sum_{v=1}^{\infty} e_{v,n}[\nu_n]a, \xi_{v,n} \geq 1 \) \( (n = 1, 2, \ldots; v = 1, 2, \ldots, \kappa_n) \). For any \( \varepsilon > 0 \)

\[
m((1+\varepsilon)b_n) = m(\sum_{v=1}^{\infty} e_{v,n}[\nu_n]a) \geq \sum_{v=1}^{\infty} D_{\pm}m(\xi_{v,n}[\nu_n]a) \geq D_{\pm}m[a]
\]

on the other hand from (\( M \)) 7) \( \lim_{n \to \infty} \frac{m((1+\varepsilon)b_n) - m(b_n)}{\varepsilon} = m((1+\varepsilon)b) - m(b) \), therefore \( \frac{m((1+\varepsilon)b) - m(b)}{\varepsilon} \geq D_{\pm}m(a) \), hence \( D_{\pm}m(b) \geq D_{\pm}m(a) \).

Similarily we have \( D_{\pm}m(b) \geq D_{\pm}m(a) \).

The proof of 7) (\( + \)): If \( a = 0 \), then we have for all \( \xi > 1 \)

\[
0 \leq \inf_{\lambda \in A} D_{\pm}m(a_1)
\]
\[ \inf_{\lambda \in \Lambda} \frac{m(\xi a_{\lambda}) - m(a_{\lambda})}{\xi - 1} \leq \frac{m(\xi a_{\lambda})}{\xi - 1} = \frac{m(a_{\lambda})}{\xi - 1} = 0, \]

because we can prove \( \inf_{\lambda \in \Lambda} m(\xi a_{\lambda}) = 0 \) from \((M)\, 7)\) and the assumption \( m(\xi a_{\lambda}) < +\infty \) ([1], p. 155). For a general case, putting \( p_{\lambda} = ([a] a_{\lambda} - \alpha a)^{+} \) for \( \alpha > 1 \), we have \( a_{\lambda} = [p_{\lambda}] a_{\lambda} + (1 - [p_{\lambda}]) a_{\lambda} \leq [p_{\lambda}] a + (1 - [p_{\lambda}]) \alpha a \). Therefore from 5) and 6) \( D_{+} m(a_{\lambda}) \leq D_{+} m([p_{\lambda}] a_{\lambda}) \leq D_{+} m((1 - [p_{\lambda}]) a_{\lambda}) \leq D_{+} m((1 - [a]) a_{\lambda}) \). As \( \inf_{\lambda \in \Lambda} m([p_{\lambda}] a_{\lambda}) = 0 \) and \( \inf_{\lambda \in \Lambda} m(1 - [a]) a_{\lambda} = 0 \), we have \( D_{+} m(1 - [a]) a_{\lambda} = 0 \). Therefore \( D_{+} m(a) \leq \inf_{\lambda \in \Lambda} D_{+} m(a) \leq D_{+} m(\alpha a) \) for all \( \alpha > 1 \). From 4) \((\lambda)\) we have \( D_{+} m(a) = \inf_{\lambda \in \Lambda} D_{+} m(a_{\lambda}) \).

The proof of 7) \((-)\): With the similar technique for the case of \( D_{-} m(a) < +\infty \) we can obtain \( \sup_{\lambda \in \Lambda} D_{-} m(a_{\lambda}) = D_{-} m(a) \). At first if \( a_{\lambda} = [p_{\lambda}] a \) and \( [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [a] \), then from 6) and 7) of \((M)\) we have \( \sup_{\lambda \in \Lambda} m([p_{\lambda}] a) - m(\xi [p_{\lambda}] a) = 1 - \xi \)

\[ \sup_{0 \leq \xi < 1} \frac{m([p_{\lambda}] a) - m(\xi [p_{\lambda}] a)}{1 - \xi} = \sup_{0 \leq \xi < 1} \frac{m([p_{\lambda}] a) - m(\xi [p_{\lambda}] a)}{1 - \xi} = \sup_{0 \leq \xi < 1} \frac{m([p_{\lambda}] a) - m(\xi [p_{\lambda}] a)}{1 - \xi} = \frac{m(a) - m(\xi a)}{1 - \xi} \]

for \( 0 \leq \xi < 1 \) and therefore we have \( \sup_{\lambda \in \Lambda} D_{-} m([p_{\lambda}] a) = \sup_{\lambda \in \Lambda} D_{-} m(a) \). Next, putting \( p_{\lambda} = (a_{\lambda} - \alpha a)^{+} \) for \( 0 \leq \alpha < 1 \), we have \( \sup_{\lambda \in \Lambda} m(a_{\lambda}) \leq \sup_{\lambda \in \Lambda} m(a_{\lambda}) \leq \sup_{\lambda \in \Lambda} D_{-} m(a_{\lambda}) \leq \sup_{\lambda \in \Lambda} D_{-} m(a_{\lambda}) \).

If \( D_{-} m(a) = +\infty \) and \( \sup_{\lambda \in \Lambda} D_{-} m(a_{\lambda}) = \gamma < +\infty \), then we have a contradiction. Because: There exists some \( \xi_{0} \), \( 0 < \xi_{0} \leq 1 \) such that \( D_{-} m(\xi a) = +\infty \) for \( \xi > \xi_{0} \) and \( D_{-} m(\xi a) < +\infty \) for \( 0 \leq \xi < \xi_{0} \). And \( D_{-} m(\xi_{0} a) = \sup_{\lambda \in \Lambda} D_{-} m(\xi_{0} a) = \sup_{\lambda \in \Lambda} D_{-} m(\xi_{0} a) = \sup_{\lambda \in \Lambda} D_{-} m(\xi_{0} a) = \sup_{\lambda \in \Lambda} D_{-} m(\xi_{0} a) = +\infty \). Therefore \( m(\xi_{0} a) < +\infty \) and \( m(a_{\lambda}) \leq (1 - \xi_{0}) a_{\lambda} \). Hence we have \( m(a) = \sup_{\lambda \in \Lambda} m(a_{\lambda}) \leq (1 - \xi_{0}) + m(\xi_{0} a) < +\infty \). This implies \( D_{-} m(\xi a) < +\infty \) for all \( 1 > \xi \geq \xi_{0} \). This contradicts the property of \( \xi_{0} \).

Remark. For a functional \( f(a) (a \in R) \) there exists a modular \( m \) on \( R \) such that \( f(a) = D_{-} m(a) (a \in R) \) if and only if \( f \) satisfies \((D_{-})\).
Theorem 2. Let \( a \in R^+ \) and \( \mu([p]) \) be a universally additive finite measure on the Boolean ring of projectors satisfying
\[
D_-m([p]a) \leq \mu([p]) \leq D_+m([p]a) \quad \text{for all} \quad [p],
\]
then we can find uniquely \( \overline{a} \in \overline{R}^m \) such that \( \overline{a}[a] = \overline{a} \) and \( \mu([p]) - (\overline{a}[p]a, \overline{a}) \) for all \( [p] \).

Proof. The expression (7) implies \( \mu([p]) = 0 \) for \( [p][a] = 0 \). Further from the universal additivity of \( \mu \) we can find a projector \( [a_1] \leq [a] \) such that \( \mu([p]) = 0 \) for \( [p][a_1] = 0 \) and \( \mu([p]) > 0 \) for \( 0 \neq [p] \leq [a_1] \). For \( [a_1] \) we have also \( m([p]a) = 0 \) for \( [p][a_1] = 0 \), because the first inequality implies \( m([p]a) \leq \mu([p]) \).

The condition (7) means \( \frac{m(\xi[p]a) - m([p]a)}{\xi - 1} \geq \mu([p]) \) for all \( \xi > 1 \) and \( \frac{1 - \xi}{\xi - 1} \leq \mu([p]) \) for all \( \xi < 1 \). Therefore we have \( \mu([p]) \)
\[
1 - m([p]a) + m(\xi[p]a) \geq |\xi| \mu([p]) \quad \text{for all} \quad \xi \quad \text{and} \quad [p].
\]
Hence we obtain
\[
1 - \frac{m([p]a)}{\mu([p])} + \frac{m(\xi[p]a)}{\mu([p])} \geq |\xi| \quad \text{for all} \quad \xi \quad \text{and} \quad 0 \neq [p] \leq [a_1].
\]
We consider the derivative of \( m([p]x) \) by \( \mu([p]) \), that is, for any \( x \in R \) and maximal ideal \( \mathfrak{p} \) consisting of projectors such that \( \mathfrak{p} \ni [a_1] \) we can define
\[
\rho(x, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{m([p]x)}{\mu([p])} \quad (x \in R, \mathfrak{p} \ni [a_1]).
\]
As the inverse expression of this we have
\[
m([p]x) = \int_{[p]} \rho(x, \mathfrak{p}) \mu(d\mathfrak{p}) \quad \text{for all} \quad [p] \leq [a_1].
\]
Tending \([p]\) to \( \mathfrak{p} \) in (8), we obtain for any maximal ideal \( \mathfrak{p} \ni [a_1] \)
\[
1 - \rho(a, \mathfrak{p}) + \rho(x, \mathfrak{p}) \geq |(x, \mathfrak{p})|.
\]
Further we can prove for any \( x \in R \) and maximal ideal \( \mathfrak{p} \ni [a_1] \)
\[
1 - \rho(a, \mathfrak{p}) + \rho(x, \mathfrak{p}) \geq \left| \left( \frac{x}{a}, \mathfrak{p} \right) \right|,
\]
where \( \left( \frac{x}{a}, \mathfrak{p} \right) \) means a relative spectrum of \( x \) by \( a \) at \( \mathfrak{p} \) ([1], p. 34).

Because, if \( \left| \left( \frac{x}{a}, \mathfrak{p}_0 \right) \right| > 0 \) for some \( \mathfrak{p}_0 \ni [a_1] \), then for any \( \xi, 0 \leq \xi < \left| \left( \frac{x}{a}, \mathfrak{p}_0 \right) \right| \),

\[\text{If} \quad [p][q] = 0, \quad \text{then} \quad \mu([p] \cup [q]) = \mu([p]) + \mu([q]). \quad \text{And if} \quad [p_1] \uparrow_{\lambda \in A} [p], \quad \text{then} \quad \sup_{\lambda \in A} \mu([p_1]) = \mu([p]). \]
there exists \([ p_0 ]\) such that \(\forall x \leq [ a ]\) and \([ p ] x \geq \xi [ p ] a\) for all \([ p ]\). Therefore we have \(\rho(x, [ p_0 ]) \geq \rho(\xi a, [ p_0 ])\), hence from (11) \(1 - \rho(\xi a, [ p_0 ]) + \rho(x, [ p_0 ]) \geq |\xi|\), therefore \(1 - \rho(\xi a, [ p_0 ]) + \rho(x, [ p_0 ]) \geq |\xi|\).

Next the expression (12) shows that \((\frac{x}{a}, \mathfrak{p}_0)\) is integrable by \(\mu\) on \([ a ]\).

Because, for \(x \in \mathbb{R}\) we can find \(\alpha > 0\) such that \(m(\alpha x) < +\infty\), from (12) we have \(\frac{1}{\alpha} \{1 - \rho(\xi a, [ p_0 ]) + \rho(\alpha x, \mathfrak{p}_0)\} \geq |(\frac{x}{a}, \mathfrak{p}_0)|\), and considering (10), the left side is integrable by \(\mu\) on \([ a ]\). Therefore

\[
\int_{[a_1]} |(\frac{x}{a}, \mathfrak{p}_0)| \mu(d\mathfrak{p}) \leq \frac{1}{\alpha} \{\mu([a_1]) - m([a_1]a) + m(\alpha [a_1]x)\} < +\infty.
\]

If we put \(L(x) = \int_{[a_1]} (\frac{x}{a}, \mathfrak{p}_0) \mu(d\mathfrak{p}) (x \in \mathbb{R})\), evidently \(L\) is a positive linear functional and we have

\[
L([p]a) = \mu([p][a_1]) = \mu([p]) \quad \text{for all \([p]\), and}
\]

\[
L([p]x) \leq \frac{1}{\alpha} \{\mu([p][a_1]) - m([p][a_1]a) + m(\alpha [p][a_1]x)\}.
\]

\(L\) is universally continuous. Because for any \([p_1] \downarrow_{\lambda \in \Lambda} 0\) we have \(0 \leq \inf_{\lambda \in \Lambda} \{L([p_1]x) \leq \frac{1}{\alpha} \{\mu([p_1][a_1]) - m([p_1][a_1]a) + m(\alpha [p_1][a_1]x)\} = 0\) from the universal additivity of \(\mu\) and the modular condition 7) of \((M)\). Therefore \(L = \overline{a} \in \overline{R}_{+}\). We see easily \(\overline{a}[a] = \overline{a} \geq 0\) and \(([p]a, \overline{a}) = \mu([p])\) for all \([p]\) and for any \(x \in \mathbb{R}\) \((x, \overline{a}) \leq \mu([a_1]) - m([a_1]a) + m([a_1]x) \leq \mu([a]) + m(\alpha m(x))\), considering \(\mu([a_1]) = \mu([a])\) and \(m([a_1]a) = m(a)\). Hence from the inequality \((x, \overline{a}) - m(\alpha m(x))\) we have \(\overline{m}(\overline{a}) = (a, \overline{a}) - m(\alpha m(x)) < +\infty\) and \(\overline{a} \in \overline{R}_{+}\).

Such \(\overline{a}\) is unique, because if \(([p]a, \overline{a}) = ([p]a, \overline{b})\) for all \([p]\), then from the spectral theory we have \(([a]x, \overline{a}) = ([a]x, \overline{b})\) for all \(x \in \mathbb{R}\), that is, \(\overline{a} = \overline{a}[a] = \overline{b}[a] = \overline{b}\).

**Lemma 4.** For any \(a \in \mathbb{R}^{+}\) and \(\overline{a} \in \overline{R}^{+}\) we have \(D_{-}m([p]a) \leq ([p]a, \overline{a}) \leq D_{+}m([p]a)\) for all \([p]\) if and only if \(\overline{a}[a] \in \overline{R}^{+}\) and \(m(a + \overline{m}(\overline{a}[a])) = (a, \overline{a})\).

Because, let be \(D_{-}m([p]a) \leq ([p]a, \overline{a}) \leq D_{+}m([p]a)\) for every \([p]\), then we have proved in the previous theorem \(\overline{m}(\overline{a}[a]) = (a, \overline{a}) - m(a)\) and \(\overline{a}[a] \in \overline{R}^{+}\). Conversely, let be \(m(a + \overline{m}(\overline{a}[a])) = (a, \overline{a})\), then we have \(m([p]a) + \overline{m}(\overline{a}[a][p]) = ([p]a, \overline{a})\) for all \([p]\) ([1], p. 178). On the other hand for \(\xi > 1\) we have \(m(\xi [p]a) + \overline{m}(\overline{a}[a][p]) \geq \xi ([p]a, \overline{a})\) from (1). Therefore
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\[ m(\xi[p]a) - m([p]a) \geq \frac{(\xi > 1)}{\xi - 1} \]

hence \( D_+ m([p]a) \geq ([p]a, \overline{a})(\xi > 1) \).

We put

\[ M_+ = \{ |a|; D_+ m(a) < +\infty \} \]

\[ M_- = \{ |a|; D_- m(a) < +\infty \} \]

Then \((D_+ 7)\) show that for \( a \in M_+ \) \( D_+ m([p]a) \) are the universally additive finite measure on the Boolean ring of all \([p] \). Therefore on account of theorem 2 we find a mapping \( T_+ (T_-) \) from the domain \( M_+ (M_-) \) into \( \mathbb{R}^+ \) such that

\[ ([p]a, T_+ a) + D_+ m([p]a) \]

\[ ([p]a, T_- a) + D_- m([p]a) \]

\((T_+, M_+)\) satisfies \((C_+ )\) and \((T_-, M_-)\) satisfies \((C_-)\).

The proof of \((C_-) 5)\) is obtained from \((D_-) 7)\). And \((C_+) 7)\) is proved easily, because, if \( M_+ \ni a \uparrow_{\lambda \in A} \) and \( \sup_{\lambda \in A} (a \lambda, T_+ a \lambda) < +\infty \), then we have \( \sup_{\lambda \in A} (a \lambda, T_- a \lambda) \leq \sup_{\lambda \in A} (a \lambda, T_- a \lambda) < +\infty \),

Therefore \( T_- (T_-) \) is proved.

Theorem 3. For an arbitrary modulated semi-ordered linear space \( R(m) \) \((T_+, M_+)\) and \((T_-, M_-)\) are conjugately similar transformations on
$R$ and $m=m_{T^+} = m_{T^-}$ and we have $m(a)+\overline{m}(T_a)=\langle a, T_a \rangle$ for $a \in M$. 

We shall consider the conjugately similar transformation on the modular function space ([1], appendix I) which is a concrete representation of a modulared semi-ordered linear space.

Let $\Omega(\mathfrak{B}, \mu)$ be a measure space, that is, $\Omega$ be a abstract space, $\mathfrak{B}$ be a totally additive class of subsets in $\Omega$ and $\mu(B) (B \in \mathfrak{B})$ be a finite measure on $\mathfrak{B}$. Let $\Phi(\xi, \omega)$ be a function on $[0, +\infty) \times \Omega$ satisfying following conditions

1) When $\xi$ is fixed, $\Phi(\xi, \omega)$ is a $\mathfrak{B}$-measurable function;
2) When $\omega$ is fixed, $\Phi(\xi, \omega)$ is a convex non-decreasing left continuous function of $\xi \geq 0$;
3) $\lim_{\xi \to +\infty} \Phi(\xi, \omega) = +\infty$ for every $\omega \in \Omega$;
4) for any $\omega \in \Omega$ there exists $\alpha_\omega > 0$ such that $\Phi(\alpha_\omega, \omega) < +\infty$.

We shall denote by $R_\Phi$ the class of all measurable functions $a(\omega) (\omega \in \Omega)$ such that for some $\alpha > 0$ we have $\int_{\Omega} \Phi(\alpha |a(\omega)|, \omega) \, d\mu < +\infty$. Putting $m_\Phi(a) = \int_{\Omega} \Phi(|a(\omega)|, \omega) \, d\mu$ for all $a \in R_\Phi$, $R_\Phi(m_\Phi)$ is a modulared semi-ordered linear space.

Let $\overline{\Phi}(\xi, \omega)$ be a complementary convex function of $\Phi(\xi, \omega)$ in the sense of Young for every fixed $\omega \in \Omega$. $\overline{\Phi}(\xi, \omega)$ satisfies the same conditions as $\Phi(\xi, \omega)$.

Therefore for $\overline{\Phi}(\xi, \omega)$ we obtain a modulared semi-ordered linear space $R_{\overline{\Phi}}(m_{\overline{\Phi}})$ as we have obtained $R_\Phi(m_\Phi)$ for $\Phi(\xi, \omega)$.

For $\overline{a} \in R_{\overline{\Phi}}$, putting $(a, \overline{a}) = \int_{\Omega} a(\omega) \overline{a}(\omega) \, d\mu$ ($a \in R_\Phi$), we obtain a universally continuous linear functional on $R_\Phi$ and we can prove $R_{\overline{\Phi}}(m_{\overline{\Phi}})$ is the modular conjugate space of $R_\Phi(m_\Phi)$.

Since $\Phi(\xi, \omega)$ is a convex function of $\xi \geq 0$, we denote by $\varphi(\xi, \omega)$ the left derivative at $\xi \geq 0$ of $\Phi(\xi, \omega)$. Then $\varphi(\xi, \omega)$ satisfies also the same conditions as $\Phi(\xi, \omega)$ except the convexity about $\xi \geq 0$ and 3). We see easily $D_m\Phi(a) = \int_{\Omega} |a(\omega)| \varphi(|a(\omega)|, \omega) \, d\mu$ for all $a \in R_\Phi$. Therefore we have $M_- = \{ a; a \geq 0 \text{ and } \int_{\Omega} a(\omega) \varphi(a(\omega), \omega) \, d\mu < +\infty \}$. By Young's inequality we have for $a \in M_- \Phi(a(\omega), \omega) + \overline{\Phi}(\varphi(a(\omega), \omega), \omega) - a(\omega)\varphi(a(\omega), \omega)$, hence $\int_{\Omega} \Phi(a(\omega), \omega) \, d\mu$
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\[ + \int_{\Omega} \Phi(a(\omega), \omega) d\mu = \int_{\Omega} a(\omega) \varphi(a(\omega), \omega) d\mu < +\infty. \]

Therefore \( \varphi(a(\omega), \omega) \in R_{\Phi} \).

Furthermore, since for any \( b \in R_{\Phi} \) we have

\[ D_{-} m_{\Phi}(b) = \int_{\Omega} \chi_{b}(\omega) a(\omega) \varphi(a(\omega), \omega) d\mu = \int_{\Omega} (b)(\omega) \varphi(a(\omega), \omega) d\mu, \]

where \( \chi_{b} \) is a characteristic function of the set \( \{ \omega; b(\omega) \neq 0 \} \), we see \( (T_{-} a)(\omega) = \varphi(a(\omega), \omega) \) for \( a \in M_{-} \).

Especially, if \( \Phi(\xi, \omega) = \frac{1}{p} \xi^{p} \) for some \( p \geq 1 \), that is, \( R_{\Phi} \) is \( L_{p} \)-space, then \( M_{-} = R_{\Phi}^{+} \) and \( T_{-} a = a^{p-1} \) for \( a \in R_{\Phi}^{+} \).

§ 2. Conjugate modulars. Let \( R(m) \) be a modulared semi-ordered linear space and \( \overline{R^{m}(m)} \) be its modular conjugate space. On account of the results in § 1 we can find conjugately similar transformations \( (T, M) \) on \( R \) and \( (\overline{T}, \overline{M}) \) on \( \overline{R^{m}} \) such that \( m = m_{T} \) and \( \overline{m} = m_{\overline{T}} \). Especially if \( R(m) \) is on dimensional, then \( \overline{T} \) is a non-decreasing function and \( \overline{T} \) is the inverse function of \( T \). In this section this relation between \( T \) and \( \overline{T} \) will be generalizated to the general case and at this point of view we shall construct directly \( (\overline{T}, \overline{M}) \) from \( (T, M) \). However, we shall assume that \( R(m) \) is monotone complete.

For a conjugately similar transformation \( (T, M) \) on \( R_{T} m_{T} \) is monotone complete if and only if the following condition is satisfied:

\[ (16) \text{ If } M \ni a_{i} \uparrow_{i \in A} \text{ and } \sup_{i \in A} (a_{i}, T a_{i}) < +\infty, \text{ then there exists } a \text{ in } R \text{ such that } a = \bigcup_{i \in A} a_{i}. \]

Because, from the expression \( m_{T}(a_{i}) = \int_{0}^{a_{i}} T(\xi a_{i}) d\xi \) we have easily \( \left( \frac{1}{2} a_{i}, T \frac{1}{2} a_{i} \right) \leq m(a_{i}) \leq (a_{i}, T a_{i}) \). \( (T, M) \) satisfying (16) is called also monotone completes.

Through this section we shall assume \( R \) is semi-regular\(^{11} \) and \( (T, M) \) is monotone complete and satisfies \( (C_{-}) \). According to the assumption that \( R \) is semi-regular \( R \) can be embedded isomorphically into \( \overline{R} \) and \( R \) is a semi-normal manifold of \( \overline{R} \) (Nakano's theorem about reflexivity). And from the assumption that \( (T, M) \) is monotone complete we have \( \overline{R^{m_{T}}} = \overline{R} \) ([1], p. 173).

\[ 11) \text{ For any } a > 0, a \in R \text{ we can find some } \overline{a} \in \overline{R} \text{ such as } (a, \overline{a}) \neq 0 ([1], \text{p. 92}). \]
We introduce a notation $\overline{a}\succ\overline{b}$ for $\overline{a}, \overline{b} \in \overline{R}^+$ having the following meaning:

(17) For all $[N]$ we have $\overline{a}[N] > \overline{b}[N]$ or $\overline{a}[N] = \overline{b}[N] = 0$.

We see easily for $\overline{a}, \overline{b}, \overline{c} \in \overline{R}^+$

1) if $\overline{a}\succ\overline{b}$, then $\overline{a}[N] > \overline{b}[N]$ for all $[N]$,

2) if $\overline{a}\succ\overline{b}$ or $\overline{a}\succ\overline{c}$, then $\overline{a}\succ\overline{c}$,

3) if $\overline{a}\succ\overline{b}$ and $\overline{a}\succ\overline{c}$, then $\overline{a}\succ\overline{b}\succ\overline{c}$,

(18) 4) if $\overline{b}\succ\overline{a}$ and $\overline{c}\succ\overline{a}$, then $\overline{b}\succ\overline{a}$.

5) we have always $\overline{a}\geq 0$ and $\alpha\overline{a}\succ\overline{a}$ for all $\alpha > 1$,

6) if $\overline{a}\succ\overline{b}$, then $[\overline{a}-\overline{b}]^R = [\overline{a}]^R$,

7) if $\overline{c} = (\overline{a}-\overline{b})^+$, then $[\overline{c}]^R > [\overline{b}]^R$.

For instance, the proof of 3): If $\overline{a}[N] = (\overline{b}\overline{c})[N]$, putting $(\overline{b}\overline{c})^+ = \overline{d}$, we have $(\overline{b}\overline{c})[\overline{d}]^R = \overline{b}[\overline{d}]^R$ and $(\overline{b}\overline{c})(1 - [\overline{d}]^R) = \overline{c}(1 - [\overline{d}]^R)$, therefore $\overline{a}[N][\overline{d}]^R = \overline{b}[N][\overline{d}]^R = 0$ and $\overline{a}[N](1 - [\overline{d}]^R) = \overline{c}[N](1 - [\overline{d}]^R) = 0$, hence $\overline{a}[N] = (\overline{b}\overline{c})[N] = 0$. Similarly we can see 4). The proof of 6): Since $0\leq\overline{a} - \overline{b} \leq \overline{a}$, we have $[\overline{a} - \overline{b}]^R \leq [\overline{a}]^R$. If we put $[N] = [\overline{a}]^R - [\overline{a} - \overline{b}]^R$, then $[N][\overline{a}]^R = [N]$ and $[N][\overline{a} - \overline{b}]^R = 0$. $[N][\overline{a} - \overline{b}]^R = 0$ implies $\overline{a}[N] = \overline{b}[N]$, therefore $\overline{a}[N] = 0$ from the assumption. Hence $[\overline{a}]^R[N] = [N] = 0$, that is, $[\overline{a}]^R = [\overline{a} - \overline{b}]^R$. The proof of 7): Evidently $\overline{a}[\overline{c}]^R \geq \overline{b}[\overline{c}]^R$. If $\overline{a}[\overline{c}]^R[N] = \overline{b}[\overline{c}]^R[N]$, then $(\overline{a} - \overline{b})[\overline{c}]^R[N] = \overline{c}[N] = 0$, therefore $[\overline{c}]^R[N] = 0$ and $\overline{a}[\overline{c}]^R[N] = \overline{b}[\overline{c}]^R[N] = 0$.

For $(T, M)$ we define a mapping $\overline{T}$ from the domain $\overrightarrow{M} \subset \overrightarrow{R}^+$ into $R^+$: $\overrightarrow{M} \ni \overrightarrow{a}$ if and only if $\overrightarrow{a} \geq 0$ and $[[\overrightarrow{a}]^Ra; \overrightarrow{a} \succ Ta]$ is bounded in $R$,

(19) $\overrightarrow{T}a = \bigcup_{\overrightarrow{a} \succ Ta} [\overrightarrow{a}]^Ra$.

Lemma 5.

i) $T(M) \subset \overrightarrow{M}$ and $\overrightarrow{T}a \leq \overrightarrow{a} \forall \overrightarrow{a} \in M$.

ii) $\overrightarrow{T}(\overrightarrow{M}) \subset M$ and $\overrightarrow{T}a \leq \overrightarrow{a} \forall \overrightarrow{a} \in \overrightarrow{M}$.

i): If $Ta \succ Tb$, then we see $[Ta]^R a \geq [Ta]^R b$. Because, if $[Ta]^R a \not\geq [Ta]^R b$, then there exists $[N]$ such as $[Ta]^R[N] a < [Ta]^R[N] b$, therefore $(Ta)[N] = T([Ta]^R[N] a) \leq T([Ta]^R[N] b) \leq T([N] b) = (Tb)[N]$ and hence $(Ta)[N] = (Tb)[N] = 0$, therefore $[Ta]^R[N] = 0$ and hence

12) If we put $N = \{a; (|a|, \overrightarrow{a}) = 0\}$ for a fixed $\overrightarrow{a}$, then $N$ is a normal manifold of $R$. We denote $[\overrightarrow{a}]^R = [N] (11)$.
\[[Ta]^r[N]a=[Ta]^r[N]b=0\], this is a contradiction. Therefore \(Ta \in \overline{M}\) and \(\overline{T}a = \bigcup_{\overline{a} \succ Ta} [\overline{a}]^R a\) for all \(a \in \overline{M}\).

ii): As \(\overline{T}a = \bigcup_{\overline{a} > Ta} [\overline{a}]^R a\) for \(\overline{a} \in \overline{M}\), we have sup \(\bigcup_{\overline{a} > Ta} [\overline{a}]^R a\) \(< + \infty\), therefore from (16) \(\bigcup_{\overline{a}>Ta} [\overline{a}]^R a \in M\) and \(T\overline{T}a = \bigcup_{\overline{a} > Ta} T[\overline{a}]^R a \leq [\overline{a}]^R (\overline{a})^R\) of \((C_\sim)\).

**Lemma 6.** We have \((a, \overline{a}) \leq (a, Ta) + (\overline{T}a, \overline{a})\) for \(a \in M\) and \(\overline{a} \in \overline{M}\).

Because: if we put \(\overline{b} = (Ta - \overline{a})^+\), then \(\overline{a} \leq (Ta)[\overline{b}]^R\), hence we have \([\overline{b}]^R a, \overline{a}) = (a, \overline{a}[\overline{b}]^R) \leq (a, (Ta)[\overline{b}]^R)\). On the other hand we can decompose \(1 - [\overline{b}]^R\) into \([N_1]\) and \([N_2]\) such that \(1 - [\overline{b}]^R = [N_1] + [N_2]\). We have \((Ta)[N_1] = \overline{a}[N_1]\) and \((Ta)[N_2] < \overline{a}[N_2] \leq \overline{a}\), and we have \(\overline{T}a \geq [N_2][\overline{a}]^R a\). Therefore \((N_1 \alpha a, \overline{a}) = (a, \overline{a}[N_1]) = (a, (Ta)[N_1])\) and \((N_2 \alpha a, \overline{a}) = ([\overline{a}]^R[N_2] a, \overline{a}) \leq (\overline{T}a, a)\). And we have \((Ta)[\overline{b}]^R + (a, (Ta)[N_1]) + (\overline{T}a, \overline{a}) \leq (a, Ta) + (\overline{T}a, \overline{a})\).

**Theorem 4.** \((T, \overline{M})\) is a monotone complete conjugately similar transformation on \(\overline{R}\) and satisfies \((C_\sim)\).

**Proof.** 1), if \(\overline{M} \ni \overline{a} \geq \overline{b} > 0\), then evidently \(\overline{b} \in \overline{M}\).

2), if \(\overline{M} \ni \overline{a}, \overline{b}\), then for any \(a \in M\) such as \(\overline{a} > b > Ta\) we have \(\overline{a}[(\overline{a} - \overline{b})^+]^R = (\overline{a} - \overline{b}^+) [(\overline{a} - \overline{b})^+]^R \leq (Ta)[(\overline{a} - \overline{b})^+]^R\), therefore we have \([\overline{a}]^R (\overline{a} - \overline{b})^+]^R a \leq \overline{T}a\). Similarly we have \([\overline{b}]^R [(\overline{b} - a)^+]^R a \leq \overline{T}b\). Hence \((\overline{a}^R \leq [\overline{b}]^R) a = [\overline{a} \leq \overline{b}]^R a \leq \overline{T}a \leq \overline{b}\), therefore \(\overline{a} \leq \overline{b} \in \overline{M}\).

3), for any \(\overline{a} \in \overline{R}^+\) there exists \(a > 0\) such that \(a \overline{a} \in \overline{M}\). Because: We can find \(a > 0\) such that \(D_+(\overline{m}(a \overline{a}) < + \infty\). If \(a \overline{a} > Ta\), then we have \(D_+(\overline{m}(Ta) \leq D_+(\overline{m}(a \overline{a}))\). On the other hand \(m(Ta) = (a, Ta)\) from theorem 3 and hence in the same way in lemma 6 we obtain \((a, Ta) \leq D_+(\overline{m}(a \overline{a}))\). Therefore \(\sup (a, Ta) \leq D_+(\overline{m}(a \overline{a})) < + \infty\). On account of (16) there exists \(\bigcup_{\overline{a} > Ta} [\overline{a}]^R a\), therefore \(a \overline{a} \in \overline{M}\).

4), if \(\overline{M} \ni \xi \overline{a}\) and \(\overline{T} \xi \overline{a} = 0\) for all \(\xi \geq 0\), then \(\overline{a} = 0\). Because: From lemma 5 we have \((a, \xi \overline{a}) \leq (a, Ta)\) for all \(a \in \overline{M}\) and \(\xi \geq 0\), therefore \((a, \overline{a}) = 0\) for all \(a \in \overline{M}\). Hence \(\overline{a} = 0\).

5), if \(\overline{M} \ni \overline{a} \geq \overline{b} > 0\), then evidently \(\overline{T}a \geq \overline{T}b \geq 0\).

6), for any \(\overline{a} \in \overline{M}\) and \([N]\) we have \(\overline{T}(\overline{a}[N]) = [N](\overline{T}a)\). Because: If \(\overline{a} \succ Ta\), then \(\overline{a}[N] > (Ta)[N] = T([N]a)\), therefore \(\overline{T}(\overline{a}[N]) \geq [N](\overline{T}a)\).
Conversely if $\overline{a}[N] \succ Tb$, then $\overline{a} \succ Tb$ and $[\overline{a}]^{R}b \leq \overline{Ta}$ and $[N][\overline{a}]^{R}b \leq [N](\overline{Ta})$, therefore $\overline{T}(\overline{a}[N]) \leq [N](\overline{Ta})$.

7), if $\overline{M} \ni \overline{a}_{1} \uparrow_{i \in A}$ and sup$_{i \in A} (\overline{Ta}_{i}, \overline{a}_{i}) < +\infty$, then we find $\overline{a} \in \overline{M}$ such that $\overline{a} = \bigcup_{i \in A} \overline{a}_{i}$. Because: For any $a \in R^{+}$ there exists $\alpha > 0$ such that $\alpha a \in M$ and from lemma 5 we have $(\alpha a, \overline{a}_{i}) \leq (\alpha a, T\alpha a) + (\overline{Ta}_{i}, \overline{a}_{i})$, therefore sup$_{i \in A} (a, \overline{a}_{i}) < +\infty$ for all $a \in R^{+}$, hence there exists $\overline{a} \in \overline{R}$ such that $\bigcup_{i \in A} \overline{a}_{i} \in A \overline{a}$. We may put $\overline{a} > 0$ and prove $\overline{a} \in \overline{M}$. From lemma 6 $(\overline{Ta}_{i}, T\overline{Ta}_{i}) \leq (\overline{Ta}_{i}, \overline{a}_{i})$, hence $(\overline{Ta}_{i}, T\overline{Ta}_{i}) < +\infty$, on account of (16) we have $\overline{Ta}_{i} \uparrow_{i \in A} a \in M$. If $\overline{a} \succ Tb$ and $[\overline{a}]^{R}b \leq a$, then there exists $[N]$ such that $0 = [N] \leq [\overline{a}]^{R}[b]$ and $[N][b] \succ [N]a$. As $\overline{a}[N] \succ T([N]b)$ and $\overline{a}_{i}[N] \uparrow_{i \in A} \overline{a}[N]$ there exist $\overline{a}_{i0}$ and $[N]$ such that $0 = [N]_{0} \leq [N]$ and $0 = \overline{a}_{i0}[N]_{0} \succ T([N]_{0}b)$, therefore $\overline{T}(\overline{a}_{i0}[N]_{0}) \geq [N]_{0} [\overline{a}_{i0}]^{R}b > [N]_{0} [\overline{a}_{i0}]^{R}a \geq [N]_{0} [\overline{a}_{i0}]^{R}T\overline{a}_{i0} = \overline{T}(\overline{a}_{i0}[N])$, and hence $\overline{T}(\overline{a}_{i0}[N]) = [N]_{0} [\overline{a}_{i0}]^{R}b = 0$ and $\overline{a}_{i0}[N]_{0} = 0$. This is a contradiction. From the above if $\overline{a} \succ Tb$, then $[\overline{a}]^{R}b \leq a$, therefore $\overline{a} \in \overline{M}$.

8), if $0 \leq \overline{a}_{i} \uparrow_{i \in A} \overline{a}$ and $\overline{a} \in \overline{M}$, then $\overline{Ta}_{i} \uparrow_{i \in A} \overline{Ta}$. Because: Evidently $\bigcup_{i \in A} \overline{Ta}_{i} \leq \overline{Ta}$. Let be $\overline{a} \succ Ta$, putting $\overline{b}_{i} = (\overline{a}_{i} - Ta)^{R}$, we have $\overline{a}_{i} \geq \overline{a}_{i}[\overline{b}_{i}]^{R} \succ (T[\overline{b}_{i}]^{R}a)$ from (18) 7). Therefore $\overline{Ta}_{i} \geq [\overline{b}_{i}]^{R}a$ and $\bigcup_{i \in A} \overline{Ta}_{i} \geq \bigcup_{i \in A} [\overline{b}_{i}]^{R}a$.

On the other hand from $\overline{b}_{i} \uparrow_{i \in A} \overline{a} - Ta$ we have $\bigcup_{i \in A} [\overline{b}_{i}]^{R}a = [\overline{a} - Ta]^{R} = [\overline{a}]^{R}$, therefore $\bigcup_{i \in A} [\overline{b}_{i}]^{R}a = [\overline{a}]^{R}a \leq \bigcup_{i \in A} \overline{Ta}_{i}$, hence $\overline{Ta} \leq \bigcup_{i \in A} \overline{Ta}_{i}$. Q.E.D.

In the above theorem it is more desirable to show 3) directly from the property of $(T, M)$. However, we did not succeed in it and we used the property of $\overline{m}_{x}$.

Lemma 7.

i) $m_{x}(a) + \overline{m}_{x}(Ta) = (a, Ta)$ for all $a \in M$.

ii) $m_{x}(\overline{Ta}) + m_{x}(\overline{a}) = (\overline{Ta}, \overline{a})$ for all $a \in \overline{M}$.

i) was proved in lemma 4. The proof of ii): At first we prove $T([p](\overline{Ta})) \leq \overline{a}[p] \leq T_{x}([p](\overline{Ta}))$ for all $[p]$ such that $[p] \leq [\overline{Ta}]$ and $[p](\overline{Ta}) \in M_{x}$. $T([p](\overline{Ta})) = T(\overline{T}(\overline{a}[p])) \leq \overline{a}[p]$ was proved in lemma 6. For $\alpha > 1$ we have $\overline{a}[p] \leq \overline{T}(\alpha [p](\overline{Ta}))$. Because, if $\overline{a}[p] \leq \overline{T}(\alpha [p](\overline{Ta}))$, then from (18) 7) there exists $[p_{0}]$ such that $0 = [p_{0}] \leq \overline{a}[p]$ and $\overline{a}[p_{0}] > T(\alpha [p_{0}](\overline{Ta}))$, therefore $\overline{T}(\overline{a}[p_{0}]) \geq \alpha [p_{0}](\overline{Ta}) = \alpha \overline{T}(\overline{a}[p_{0}])$ and hence $\overline{T}(\overline{a}[p_{0}])$
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$[p_0](\overline{T}\overline{a})=0$ and hence $[p_0]=0$, this is a contradiction. Therefore $\overline{a}[p] \leq T(\alpha[p](\overline{T}\overline{a}))$ for all $\alpha>1$ and $\overline{a}[p] \leq \bigcap T(\alpha[p](\overline{T}\overline{a})) = T, ([p](\overline{T}\overline{a}))$. From the above for all $[p]$ we obtain $D, m_r([p](\overline{T}\overline{a}))=(\overline{T}\overline{a}), T([p](\overline{T}\overline{a})) \leq ([p] \overline{T}\overline{a}, \overline{a}[p]) \leq ([p](\overline{T}\overline{a}), T, ([p](\overline{T}\overline{a}))) = D, m_r([p](\overline{T}\overline{a}))$. On account of lemma 4 we have $m_r(\overline{T}\overline{a})+\overline{m}_r(\overline{a})=(\overline{T}\overline{a}, \overline{a})$.

**Theorem 5.** The modular $m_\overline{r}$ on $\overline{R}$ generated by $(\overline{T}, \overline{M})$ is the conjugate modular $\overline{m}_r$ of $m_r$, that is, $\overline{m}_r=m_\overline{r}$.

**Proof.** For $\overline{a} \in \overline{M}$ we have $\overline{m}_r(\overline{a})=m_\overline{r}(\overline{a})$; Because from the above lemma 7 $m_r(\overline{T}\overline{a})+\overline{m}_r(\overline{a})=(\overline{T}\overline{a}, \overline{a})$ and from the definition of $\overline{m}_r$ ((2)) we have for all $\xi \geq 0$ $m_r(\overline{T}\overline{a})+\overline{m}_r(\xi\overline{a}) \geq(\overline{T}\overline{a}, \xi\overline{a})$. Therefore $m_r(\xi\overline{a})-\overline{m}_r(\overline{a}) \geq(\xi-1)(\overline{T}\overline{a}, \overline{a})$ for all $\xi \geq 0$ and we see $D, m_r(\overline{a}) \leq(\overline{T}\overline{a}, \overline{a}) \leq D, \overline{m}_r(\overline{a})$, hence $D, \xi m_r(\xi\overline{a}) \leq(\overline{T}\xi\overline{a}, \overline{a}) \leq D, \xi^2 \overline{m}_r(\xi\overline{a})$ for all $0 \leq \xi \leq 1$ and $m_\overline{r}(\overline{a})=\int(\overline{T}\xi\overline{a}, \overline{a})d\xi$.

Next if $m_\overline{r}(\overline{a})<+\infty$, then $\alpha\overline{a} \in \overline{M}$ for all $0 \leq \alpha < 1$, therefore $m_\overline{r}(\overline{a})=\sup_0 \overline{m}_r(\alpha\overline{a})= \sup \overline{m}_r(\alpha\overline{a})=\overline{m}_r(\overline{a})$. Remembering the proof of 3) in theorem 4 we see $\overline{a} \in \overline{M}$ for all $D, \overline{m}_r(\overline{a})<+\infty$, therefore for $D, \overline{m}_r(\overline{a})<+\infty$ we see $\overline{m}_r(\overline{a})=m_\overline{r}(\overline{a})$. If $\overline{m}_r(\overline{a})<+\infty$, then $D, \overline{m}_r(\alpha\overline{a})<+\infty$ for all $0 \leq \alpha < 1$ and hence $\overline{m}_r(\alpha\overline{a})=m_\overline{r}(\alpha\overline{a})$ for all $0 \leq \alpha < 1$, therefore $\overline{m}_r(\overline{a})=\sup \overline{m}_r(\alpha\overline{a})= \sup m_\overline{r}(\alpha\overline{a})=m_\overline{r}(\overline{a})$. $\overline{m}_r=m_\overline{r}$ has been proved. Q.E.D.

Essentially theorem 5 has been proved independently from the results of theorem 4. And it is ease to prove theorem 4 from theorem 5. However, it seems to be interesting for us to show theorem 4 independently from theorem 5.

**Theorem 6.** We have $\overline{M}=M$ and $\overline{T}=T$, that is,

i) $M \ni a$ if and only if $a \geq 0$ and $[a][a] \ni a \succ T\overline{a}$ is order bounded in $\overline{R}$,

ii) $Ta= \bigcup \overline{a}[a]$ for $a \in M$.

**Proof.** This fact is evident from theorem 5 and Nakano's theorem about the reflexivity of a modular ([1], p. 175). But in the following we shall prove this theorem directly without using the reflexivity of a modular.

i): We put $\overline{M}$ the totality of $a$ such that $a \geq 0$ and $[a][a] \ni a \succ T\overline{a}$ is order bounded in $\overline{R}$ and $\overline{T}a= \bigcup \overline{a}[a]$. If $a \in M$ and $a \succ T\overline{a}$, then we can
see \( Ta \equiv \overline{a}[a] \) and hence \( a \in \overline{M} \) namely \( M \subseteq \overline{M} \). Because, if \( Ta \equiv \overline{a}[a] \), then there exists \( [p] \) such that \( 0 \equiv [p] \leq [a] \) and \( T[p]a < \overline{a}[p] \), therefore \( \overline{T}([p]) \leq [p]a \rightarrow \overline{T}a[p] \) and hence \( [p]a = 0 \) namely \( [p] = 0 \), this is a contradiction. Next for \( a \in \overline{M} \) and \( [p]a \in M \) we have \( ([p]a, T[p]a) \leq ([p]a, \overline{T}[p]a) \) \( Ta = \overline{T}a \) for all \( a \in M \) will be proved after in ii)), therefore \( \sup ([p]a, T[p]a) < +\infty \). From (16), putting \( [N] = \bigcup [p] \), we have \( [N]a \in M \). Further we can proved \( [N]a = a \) and therefore \( a \in M \) namely \( M \subseteq \overline{M} \). Because, let be \( a_0 = a - [N]a > 0 \), as \( [a_0]M \) \( x \) implies \( x \leq a \), \( [a_0]M \) is upper bounded. If we put \( b = \bigcup x \), we see easily \( b < a_0 \). On account of the assumption that \( R \) is semi-regular there exists \( \overline{a} \in \overline{R}^+ \) such as \( (a_0, \overline{a}) > 0 \) and \( [a]\overline{R}^+ \leq [a_0] = [a] - [N][a] \). Therefore for any \( \xi \geq 0 \), if \( \xi \overline{a} > Tx \), then \( [a]\overline{R}^+ \leq [a_0]x \leq b \), hence \( \overline{M} \not\exists \xi \overline{a} \) and \( T\xi \overline{a} \leq b < a_0 \) for all \( \xi \geq 0 \). Therefore \( a_0 \notin \overline{M} \). This contradicts \( a_0 \leq a \in \overline{M} \).

ii): For any \( a \in M \) we have \( a \in \overline{M} \) from the first part of i) and evidently \( Ta \geq \overline{T}a \) from the definition of \( \overline{T} \). Conversely for \( 0 \leq \alpha < 1 \) we have \( \overline{T}Ta \leq \alpha a < a \), therefore \( Ta = \bigcup_{0 \leq \alpha < 1} \overline{T}aa \leq \overline{T}a \) and hence \( Ta \equiv \bigcup_{0 \leq \alpha < 1} \overline{T}aa \leq \overline{T}a \). Q.E.D.

3. Types of conjugately similar transformations. Through this section \( (T, M) \) is a conjugately similar transformation on \( R \) satisfying \( (C_\_). \) And we assume that the following definitions about types of modulars and classifications of modulars are known. We shall state in the following the relation between types of a modular \( m_r \) and types of \( (T, M) \), where \( m_r(a) = \int_0^1 (|a|, T\xi|a|)d\xi \) for all \( a \in R \), putting \( (|a|, T\xi|a|) = +\infty \) for \( \xi|a| \not\in \overline{M} \).

1) \( m_r \) is singular\(^{13} \) \( \equiv T(M) = \{0\} \).

If \( m_r \) is singular, then \( D_+ m_r(a) = +\infty \) implies \( D_+ m_r([p]a) = 0 \) for all \( [p] \), and hence \( ([p]a, T[a]) = 0 \) for all \( [p] \), that is, \( T[a] = 0 \). Conversely if \( T(M) = \{0\} \), then \( m_r(a) = \int_0^1 (|a|, T\xi|a|)d\xi = 0 \) or \( +\infty \) for all \( a \in R \).

a) \( m_r \) is semi-simple\(^{14} \) \( \equiv \) for any \( \alpha > 0 \) we can find \( \alpha > 0 \) and \( [p] \) such as \( Ta \equiv \overline{a}[p]a > 0 \).

If \( m_r \) is semi-simple, then for any \( \alpha > 0 \) there exist \( \alpha > 0 \) and \( [p] \) such that \( 0 < m_r(\alpha[p]a) = +\infty \) and \( \alpha[p]a \) is domestic ( \( m_r(\xi\alpha[p]a) = +\infty \))

\(^{13} \) For any \( a \in R(m) \) \( m(a) = 0 \) or \( +\infty \) ([1], p. 157).

\(^{14} \) For any \( \alpha > 0 \) there exist \( \xi > 0 \) and \( [p] \) such that \( 0 < m(\xi[p]a) < +\infty \) ([1], p. 156).
for some $\xi > 1$, therefore we have $\alpha[p]a \in M$ and $T\alpha[a] > 0$. Conversely if $m_T$ is not semi-simple, there exists $a > 0$ such that $[a]R$ is singular, therefore for any $\alpha > 0$ and $\alpha[p]a \in M$ implies $T\alpha[p]a = 0$.

We see easily $\alpha$ is a simple domestic element if and only if $|a| \in M$, and $T[p]|a| > 0$ for all $0 \equiv [p] \leq [a]$.

3) $m_T$ is linear if $M = R^+$ and $T\xi a = Ta$ for all $\xi > 0$ and $a \in R^+$.

If $a$ is a positive linear element: $m_T(\xi a) = \xi m_T(a)$ for all $\xi \geq 0$, then we have $D_\xi m_T(\xi[p]a) = D_\xi m_T([p]a) < +\infty$ for all $\xi > 0$ and $[p]$, and hence $([p]a, T\xi a) = ([p]a, Ta)$ for all $\xi > 0$ and $[p]$, hence $T\xi a = Ta$ for all $\xi > 0$. Conversely if $T\xi a = Ta$ for all $\xi > 0$, then $m_T(\xi a) = m_T(a)$ for all $\xi \geq 0$.

Since $m_T$ is non-linear if and only if there exists no linear element except $0$,

4) $m_T$ is non-linear if $M \ni \xi a$ for all $\xi \geq 0$ and $T\xi a = Ta$, then $a = 0$.

Since $m_T(a) = 0$ if and only if $T|a| = 0$,

5) $m_T$ is simple if $Ta = 0$ implies $a = 0$.

6) $m_T$ is semi-singular if $\{a; Ta = 0\}$ is complete in $R$.

7) $m_T$ is monotone if $\bigcap_{\alpha > 0} T\alpha a = 0$ for all $a \geq 0$.

For any $a \in M$ we have $(aa, \bigcap_{a > 0} Taa) \leq m_T(aa) \leq (aa, Taa) (0 \leq a \leq 1)$, therefore $\lim_{a \to 0} \frac{m_T(aa)}{a} = (a, \bigcap_{a > 0} Taa)$. Hence $\lim_{a \to 0} \frac{m_T(aa)}{a} = 0$ if and only if $\bigcap_{a > 0} Taa = 0$.

We have also,

8) $m_T$ is assending if $\bigcap_{a > 0} Taa > 0$ for all $a > 0$.

$a$ is a finite element if and only if $M \ni \xi |a|$ for all $\xi \geq 0$. Therefore

9) $m_T$ is finite if $M = R^+$.

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15) $m(\xi a) = \xi m(a)$ for all $a \in R(m)$ and $\xi \geq 0$ ([1], p. 183).

16) If $m(\xi a) = \xi m(a)$ for all $\xi \geq 0$, then $a = 0$, ([1], p. 183).

17) $m(a) = 0$ if and only if $a = 0$ ([1], p. 187)

18) The set of zero units ([1], p. 125) is complete ([1], p. 187).

19) $\inf_{\xi > 0} \frac{m(\xi a)}{\xi} = 0$ for all $a \in R(m)$. ([1], p. 189).

20) $\inf_{\xi > 0} \frac{m(\xi a)}{\xi} > 0$ for all $a > 0$ ([1], p. 188).
10) $m_T$ is almost finite$^{21}$ $\equiv \{a; M \ni \xi a$ for all $\xi \geq 0\}$ is complete in $R$.
11) $m_T$ is infinite$^{22}$ $\equiv \{a; \text{for any } a > 0 \text{ there exists } \alpha > 0 \text{ such as } \alpha a \notin M.\}$

Since $\left(\frac{1}{2} |a|, \frac{1}{2} |a|, T\frac{1}{2} |a|\right) \leq m_T(\xi a) \leq (|a|, |a|, T|a|)$, \(\lim_{\xi \to +\infty} \frac{m_T(\xi a)}{\xi} = \sup_{\xi \geq 0} (|a|, T\xi |a|).\) Therefore $a$ is an infinitely linear element if and only if $\sup_{\xi \geq 0} (|a|, T\xi |a|) < +\infty$.

12) $m_T$ is infinitely linear$^{23}$ $\equiv \{a; \sup_{\xi \geq 0} (a, T\xi a) < +\infty\}$ is complete in $R$.

13) $m_T$ is increasing$^{24}$ (not infinitely linear) $\equiv \sup_{\xi \geq 0} (a, T\xi a) = +\infty$ for $a > 0$.

$m_T$ is called strictly convex if for any $a > 0$ $m_T(\xi a)$ is a strictly convex function of $\xi \geq 0$.

Evidently we see

14) $m_T$ is strictly convex $\equiv T\xi a > T\xi b$ for $0 < a \in M$ and $0 \leq \xi_1 < \xi_2 \leq 1$.

14') $T$ is one to one mapping on $M$ if and only if $m_T$ is strictly convex.

Because, if $m_T$ is strictly convex, for any $a, b \in M$ and $a \neq b$ there exist $[p]$ and $0 \leq \alpha < 1$ such that $[p]a > \alpha[p]a \geq [p]b$ (or $[p]a \leq \alpha[p]b < [p]b$), therefore $T[p]a > T\alpha [p]a \geq T[p]b$ and hence $T[p]a \neq T[p]b$ and then $T a \neq T b$.

15) $m_T$ is concave type$^{25}$ $\equiv M = R^+$ and $T(\alpha \xi_1 + \beta \xi_2)a \leq \alpha T\xi_1 a + \beta T\xi_2 a$ for any $a \geq 0$, $\xi_1 \geq \xi_2 \geq 0$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

We see $m_T$ is concave type if and only if for any $a \geq 0$ $D_{-\xi}m_T(\xi a)$ is a finite concave function of $\xi$ on $0 < \xi < +\infty$, therefore if and only if $M = R^+$ and $T(\alpha \xi_1 + \beta \xi_2)a \geq \alpha T\xi_1 a + \beta T\xi_2 a$ for any $a \geq 0$, $\xi_1 \geq \xi_2 > 0$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$. Further from this we can prove $T(\alpha \xi_1 + \beta \xi_2)a \geq \alpha T\xi_1 a + \beta T\xi_2 a$ for $\xi_1 \geq \xi_2 \geq 0$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

16) $m_T$ is convex type$^{26}$ $\equiv T(\alpha \xi_1 + \beta \xi_2)a \leq \alpha T\xi_1 a + \beta T\xi_2 a$ for any $a \in M$, $1 \geq \xi_1 \geq \xi_2 \geq 0$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

We see $m_T$ is convex type if and only if $m_T$ is monotone and

21) The set of finite elements $\{m(\xi a) < +\infty$ for all $\xi \geq 0\}$ is complete in $R$ ([1], p. 194).
22) There exists no finite element except zero ([1], p. 197).
23) The set of infinitely linear elements $\left(\sup_{\xi} \frac{m(\xi a)}{\xi} < +\infty\right)$ is complete in $R$ ([1], p. 200).
24) $\sup_{\xi > 0} \frac{m(\xi a)}{\xi} = +\infty$ for all $a > 0$ ([1], p. 200).
25) $D_{+\xi}m(\xi a)$ is a concave function of $\xi$ on $[0, +\infty)$ for all $a \in R$ ([1], p. 224).
26) For all $a \in R \inf_{\xi > 0} \frac{m(\xi a)}{\xi} = 0$ and $D_{+\xi}m(\xi a)$ is a convex function of $\xi$ on $[0, +\infty)$. 

$D_{-}\xi m_{T}(\xi a)$ is a convex function of $\xi$ on $0<\xi\leq 1$, therefore if and only if $\bigcap_{\xi>0}T\xi a=0$ and $T(\alpha\xi_{1}+\beta\xi_{2})a\leq \alpha T\xi_{1}a+\beta T\xi_{2}a$ for any $a\in M$ and $0<\xi_{2}\leq \xi_{1}\leq 1$ and $\alpha+\beta=1$, $\alpha, \beta\geq 0$. These two conditions are equivalent to $T(\alpha\xi_{1}+\beta\xi_{2})a\leq \alpha T\xi_{1}a+\beta T\xi_{2}a$ for any $a\in M$ and $0<\xi_{2}\leq \xi_{1}\leq 1$ and $\alpha+\beta=1$, $\alpha, \beta\geq 0$.

17) $m_{T}$ is upper bounded\(^{27}\) $\equiv$ there exist $\gamma_{1}, \gamma_{2}>1$ such that for all $a\in R$ we have $\gamma_{1}a\in M$ and $T\gamma_{1}a\leq \gamma_{2}Ta$.

If $m_{T}$ is upper bounded, then from the definition there exists $\gamma>2$ such that $m_{T}(\frac{1}{2}4a)\leq \gamma m_{T}(a)$ for all $a\in R$. Therefore $D_{-}m_{T}(\frac{1}{2}4a)\leq m_{T}(4a)$ $\leq \gamma m_{T}(a)$ for all $a\in R$.

$\frac{2\alpha}{\gamma} Ta\geq T\frac{1}{2\alpha} a$ for all $a\in M$. Conversely, if $T\gamma_{1}a\leq \gamma_{2}Ta$ for all $a\in M$, then $M=R^{+}$, and $T\gamma_{1}a\leq \gamma_{2}Ta$ implies $(\gamma_{1}a, T\xi\gamma_{1}a)\leq \gamma_{1}\gamma_{2}(a, T\xi a)$ for all $\xi\geq 0$ and $a\in R^{+}$, therefore $m_{T}(\gamma_{1}a)\geq \frac{1}{\gamma_{1}\gamma_{2}} m_{T}(a)$ for all $a\in R$.

18) $m_{T}$ is lower bounded\(^{28}\) $\equiv$ there exist $0<\gamma_{1}, \gamma_{2}<1$ such that $m_{T}(\gamma_{1}a)\geq \gamma m_{T}(a)$ for all $a\in R$.

Since $m_{T}(\alpha^{\nu}a)\geq \gamma m_{T}(a)$ for all $\nu=1, 2, \cdots$, we can put $\frac{\alpha}{\gamma}<\frac{1}{2}$. Therefore $\frac{2\alpha}{\gamma} Ta\geq T\frac{1}{2\alpha} a$ for all $a\in M$. Conversely, if $T\gamma_{1}a\leq \gamma_{2}Ta$ for all $a\in M$, then we have $m_{T}(\gamma_{1}a)\leq \gamma_{1}\gamma_{2}m_{T}(a)$ for all $a\in R$, that is, $m_{T}(\gamma_{1}a)\geq \frac{1}{\gamma_{1}\gamma_{2}} m_{T}(a)$ for all $a\in R$.

§ 4. Some types of modulars. In this section we shall define some new types of a modular and decide these conjugate types.

4.1. We shall state some conditions about bounded modulars.

Definition. A modular is said to be $d$-upper bounded, if there exists a number $p>1$ such that $D_{-}m(\xi a)\leq \xi^{p}D_{-}m(a)$ for all $a\in R$ and $\xi\geq 1$.

Easily we see

\(^{27}\) There exist $\alpha, \gamma>1$ such that $m(ax)\leq \gamma m(x)$ for all $x\in R$ ([1], p. 214).

\(^{28}\) There exist $\gamma, \alpha>1$ such that $m(ax)\geq \gamma m(x)$ for all $x\in R$ ([1], p. 215).
1) $m_T$ is d-upper bounded $\equiv M = R^+$ and there exists $\alpha > 0$ such that $T^\xi a \leqq \xi^a T a$ for all $a \in R^+$ and $\xi \geqq 1$.

If $m(a)$ ($a \in R$) is d-upper bounded, then evidently $m(a)$ ($a \in R$) is upper bounded. However, generally the converse is not true. For example, let $R$ be one dimensional and $m(\xi) = \xi$ ($0 \leqq \xi \leqq 1$) and $m(\xi) = 2\xi - 1$ ($\xi \geqq 1$), then $m$ is upper bounded, but not d-upper bounded.

**Lemma 8.** If $m$ is d-upper bounded: $D_-m(\xi a) \leqq \xi^p D_-m(a)$ for all $a \in R$ and $\xi \geqq 1, \alpha > 0$ such that $T^\xi a \leqq \xi^\alpha T a$ for all $a \in M$ and $0 \leqq \xi \leqq 1$. If $m(a)(a \in R)$ is d-upper bounded, then evidently $m(a)(a \in R)$ is upper bounded. However, generally the converse is not true. For example, let $R$ be one dimensional and $m(\xi) = \xi (0 \leqq \xi \leqq 1)$ and $m(\xi) = 2\xi - 1 (\xi \geqq 1)$, then $m$ is upper bounded, but not d-upper bounded.

**Lemma 8.** If $m$ is d-upper bounded: $D_-m(\xi a) \leqq \xi^p D_-m(a)$ for all $a \in R$ and $\xi \geqq 1$, then we have for any $0 \leqq y \leqq x$ and $\xi \geqq 1$

$$m(\xi x) + \xi^p m(y) \leqq \xi^p m(x) + m(\xi y).$$

(20)

From the assumption easily we have $m(\xi a) \leqq \xi^p m(a)$ for all $a \in R$ and $\xi \geqq 1$, therefore $m$ is finite. If $y = \eta x$ for some $0 \leqq \eta \leqq 1$, then $m(\xi x) - m(\xi y) = \int_0^1 D_-m(tx) dt = \int_0^1 \xi^p \frac{D_-m(tx)}{t} dt \leqq \int_0^1 \xi^p \frac{D_-m(tx)}{t} dt = \xi^p \{m(x) - m(\xi y)\}$.

Next for any $0 \leqq y \leqq x$ we can find $0 \leqq y_n \leqq x$ such that $y_n \uparrow_{n \to \infty} y$ and $y_n = \sum_{\nu=1}^{\kappa_n} \xi_{\nu,n}[p_{\nu,n}] x$ and $0 \leqq \xi_{\nu,n} \leqq 1 (\nu = 1, 2, \cdots, \kappa_n; n = 1, 2, \cdots)$. From the above we obtain $m(\xi x) + \xi^p m(y_n) \leqq \xi^p m(x) + m(\xi y_n)$ ($n = 1, 2, \cdots$), therefore $m(\xi x) + \xi^p m(y) = \lim_{n \to \infty} \{m(\xi x) + \xi^p m(y_n)\} \leqq \lim_{n \to \infty} \{\xi^p m(x) + m(\xi y_n)\}$.

**Definition.** A modular is said to be d-lower bounded, if there exists a number $p > 1$ such that $D_-m(\xi a) \geqq \xi^p D_-m(a)$ for all $a \in R$ and $\xi \geqq 1$.

2) $m_T$ is d-lower bounded $\equiv$ there exists $\alpha > 0$ such that $T^\xi a \leqq \xi^\alpha T a$ for all $a \in M$ and $0 \leqq \xi \leqq 1$.

Similarly in lemma 8 we obtain

**Lemma 9.** If $m$ is d-lower bounded, then for any $0 \leqq y \leqq x$ and $\xi \geqq 1$ we have

$$m(\xi x) + \xi^p m(y) \geqq \xi^p m(x) + m(\xi y).$$

(21)

**Theorem 7.** If a modular $m(a)$ ($a \in R$) is d-upper bounded, then its conjugate modular $\overline{m}(\overline{a})$ ($\overline{a} \in \overline{R}^m$) is d-lower bounded. And if $m(a)$ ($a \in R$) is d-lower bounded, then $\overline{m}(\overline{a})$ ($\overline{a} \in \overline{R}^m$) is d-upper bounded.

**Proof.** Let $m(a)$ ($a \in R$) be d-upper bounded. Then there exists $p > 1$ such that $D_-m(\xi a) \leqq \xi^p D_-m(a)$ for all $a \in R$ and $\xi \geqq 1$. For any $0 \leqq \overline{a} \in \overline{R}^m$, $x, y \in R$, $\xi \geqq 1$ and $0 \leqq \eta \leqq 1$ we can prove

$$\xi^p (x, \overline{a}) - \xi^p m(x) + \xi^p \eta(y, \overline{a}) - m(\xi \eta) \leqq \overline{m}(\xi^p \overline{a}) + \xi^p \overline{m}(\eta \overline{a}).$$

Because: For $x, y \geqq 0$ we can find $\lceil N \rceil$ such that $\lceil N \rceil y \leqq \lceil N \rceil x$ and $\lceil N^\perp \rceil x \leqq \lceil N^\perp \rceil y$. Therefore from lemma 8 we have
On Conjugately Similar Transformations

\[ \xi^{\lambda}m([N]x) + m(\xi[N]y) \geqq \xi^{\lambda}m([N]y) + m(\xi[N]x). \]

And, since \([N^\perp]x \leqq [N^\perp]y\) and \(0 \leqq \eta \leqq 1\) imply \([N^\perp]x + \eta[N^\perp]y \leqq \eta[N^\perp]x + [N^\perp]y\), we have

\[ \xi^{\lambda}([N^\perp]x, \overline{a}) + \xi^{\lambda}\eta([N^\perp]y, \overline{a}) \leqq \xi^{\lambda}\eta([N^\perp]x, \overline{a}) + \xi^{\lambda}([N^\perp]y, \overline{a}). \]

Hence we obtain

\[ \xi^{\lambda}(x, \overline{a}) - \xi^{\lambda}m(x) + \xi^{\lambda}\eta(y, \overline{a}) - m(\xi y) \leqq \xi^{\lambda}([N]x + [N^\perp]y, \overline{a}) - m(\xi x + [N^\perp]y) \]

\[ \leqq m(\xi^{\lambda-1}\overline{a}) + \xi^{\lambda}m(\eta\overline{a}). \]

Therefore

\[ m(\xi\overline{a}) + m(\xi^{\lambda-1}\eta\overline{a}) = \sup_{x, \overline{a} \in \overline{R}^m} \{ \xi^{\lambda}(x, \overline{a}) - \xi^{\lambda}m(x) + \xi^{\lambda-1}\eta(y, \overline{a}) - m(\xi y) \} \]

\[ = \sup_{x, \overline{a} \in \overline{R}^m} \{ \xi^{\lambda}(x, \overline{a}) - \xi^{\lambda}m(x) + \xi^{\lambda-1}\eta(y, \overline{a}) - m(\xi y) \} \]

\[ \leqq m(\xi^{\lambda-1}\overline{a}) + \xi^{\lambda}m(\eta\overline{a}). \]

This inequality implies for any \(\xi \geqq 1, \overline{a} \in \overline{R}^m\) and \(0 \leqq \eta \leqq 1\), putting \(q = \frac{p}{p - 1}\),

\[ \xi^{\lambda}m(\overline{a}) + m(\xi^{\lambda-1}\eta\overline{a}) \leqq m(\xi\overline{a}) + \xi^{\lambda}m(\eta\overline{a}). \]

Therefore \(D_\lambda m(\overline{a}) \geqq \xi^{\lambda}D_\lambda m(\overline{a})\) for all \(\overline{a} \in \overline{R}^m\) and \(\xi \geqq 1\). In the same way we can prove the dual relation.

Q.E.D.

4.2. We shall state some types which are related to continuous modulars and totally discontinuous modulars.

**Definition.** An element \(a \geqq 0\) is said a \(d\)-discontinuous unit, if \(D_\lambda m(a) < + \infty\), and if \(D_\lambda m(x) < + \infty\) implies \([a]x \leqq a\).

By definition we have

1) if \(a\) is a \(d\)-discontinuous unit, then \(a\) is a discontinuous unit,\(^{29}\)

2) for a \(d\)-discontinuous unit \(a\), \([N]a\) is also a \(d\)-discontinuous unit,

3) for \(d\)-discontinuous units \(a_1\) and \(a_2\) we have \([a_1]a_2 = [a_2]a_1 = a_1 \cap a_2\),

4) for any system \(\lambda \in I, a_\lambda \uparrow_{\lambda \in I}\) of discontinuous units, if \(a_\lambda \uparrow_{\lambda \in I} a\) and \(D_\lambda m(a) < + \infty\), then \(a\) is also a \(d\)-discontinuous unit.

1): Evidently \(m(a) \leqq D_\lambda m(a) < + \infty\), and if \(m(x) < + \infty\), then \(D_\lambda m(ax) < + \infty\) for all \(0 \leqq \alpha < 1\), therefore \([a]ax \leqq a\) for all \(0 \leqq \alpha < 1\) and \([a]x \leqq a\).

2) is evident. 3): By definition \([a_1]a_2 \leqq a_1\), hence \([a_1]a_2 \leqq a_1 \cap a_2\). However, \(a_1 \cap a_2 \leqq [a_1]a_2\). Therefore \([a_1]a_2 = a_1 \cap a_2\). 4): If \(D_\lambda m(x) < + \infty\), then \([a_\lambda]x \leqq a_\lambda (\lambda \in I)\), therefore \([a]x = \bigcup_{\lambda \in I}[a_\lambda]x \leqq \bigcup_{\lambda \in I}a_\lambda = a\).

\(^{29}\) \(a \geqq 0\) is called a discontinuous unit when \(m(a) < + \infty\) and \(m(x) < + \infty\) implies \([a]x \leqq a\) ([1], p. 191).
Definition. A modulated semi-ordered linear space $R(m)$ is said to be totally $d$-discontinuous, if the set of all $d$-discontinuous units is complete in $R$. And $R(m)$ is said to be $d$-continuous, if there exists no $d$-discontinuous unit except 0.

Obviously for any $R(m)$ there exists uniquely a normal manifold $R_1$ such that $R_1(m)$ is totally $d$-discontinuous and $R_1^+(m)$ is $d$-continuous.

**Theorem 8.** In order that $R(m)$ is $d$-continuous, it is necessary and sufficient that $D_{-}m(a)=\sup_{D_{-}m(x)<+\infty}D_{-}m(x)$ for all $a\in R^+$.

**Proof.** Necessity: Let $R(m)$ be $d$-continuous and for some $a>0$ such as $0 \leq x \leq a$ and $D_{-}m(x)<+\infty$, we have $x_i \uparrow_{i \in I} a \leq a$. As $D_{-}m(b)= \sup_{D_{-}m(x)<+\infty}D_{-}m(x)$, we see $b < a$. If we put $[a-b]b=d$, then we can see $d>0$ and $d$ is a $d$-discontinuous unit. Because, we can find $\alpha>0$ such as $D_{-}m(\alpha a)<+\infty$, therefore $\alpha a \leq b$ and $0 < \alpha [a-b]a \leq [a-b]b=d$, further if $D_{-}m(x)<+\infty$, then $x \cap a \leq b$, and hence $(x-b) \cap (a-b) \leq 0$ and $(x-b)^+ \cap (a-b)=0$, therefore $[a-b]x \leq [a-b]b=d$. This contradicts that $R(m)$ is $d$-continuous.

Sufficiency: Let $a$ be $a>0$ and $d$-discontinuous unit, then obviously $D_{-}m(aa)=+\infty$ for all $a>1$. However, if $2a \geq x \geq 0$ and $D_{-}m(x)<+\infty$, then $x=\lceil a \rceil x \leq a$. Therefore $\sup_{0 \leq x \leq 2a}D_{-}m(x)=D_{-}m(a)<+\infty=D_{-}m(2a)$.

Therefore the sufficiency is clear. Q.E.D.

Evidently the property to be totally $d$-discontinuous is weaker than to be singular\(^{13}\) and stronger than to be totally discontinuous.\(^{30}\) And the property to be $d$-continuous is weaker than to be continuous\(^{31}\) and stronger than to be semi-simple.\(^{14}\)

In the following we shall decide the conjugate type of a totally $d$-discontinuous $R(m)$.

**Definition.** An element $a \in R$ is called a semi-linear element, if there exist positive numbers $\xi_0$ and $\gamma_0$ such that $m(\xi_0 a)<+\infty$ and $m(\xi a)=(\xi-\xi_0)\gamma_0+m(\xi_0 a)$ for all $\xi \geq \xi_0$.

By definition easily we have

1) if $a$ is a semi-linear element, then $a$ is an asymptotically linear

\(^{30}\) The set of all discontinuous units is complete in $R$ ([1], p. 193).

\(^{31}\) There exists no discontinuous unit except zero ([1], p. 193).
element.\textsuperscript{32}

2) for semi-linear elements $a_1$ and $a_2$ such as $|a_1| \sim |a_2| = 0$ $a_1 + a_2$ is also a semi-linear element.

3) for semi-linear element $a \,[N]a$ is also a semi-linear element

1) and 2) are evident. 3): As $r_{/0} = \lim_{\xi \rightarrow +\infty} \frac{m(\xi a)}{\xi} \xi* + \infty$ we have $\eta_{0} = \lim_{\xi \star +\infty} \frac{m(\xi a)}{\xi} + \lim_{\xi \rightarrow +\infty} \frac{m(\xi a)}{\xi} = \eta_{1} + \eta_{2}$ from the additivity of a modular, and by the convexity of $m(\xi x)$ we have $m(\xi a) \leq \eta_{1} + m(\xi a)$ for all $\xi \geq \xi_{0}$. Therefore $m(\xi a) = \eta_{1}$ for all $\xi \geq \xi_{0}$.

Definition. A modulared semi-ordered linear space $R(m)$ is called semi-linear, if the set of all semi-linear elements is complete in $R$. And $R(m)$ is called non-semi-linear, if there exists no semi-linear element except 0.

Obviously for any $R(m)$ there exists uniquely a normal manifold $R_1$ such that $R_1(m)$ is semi-linear and $R_1(m)$ is non-semi-linear.

The property to be semi-linear is weaker than to be linear\textsuperscript{15} and stronger than to be asymptotically linear.\textsuperscript{33} And the property to be non-semi-linear is weaker than to be increasing\textsuperscript{24} and stronger than to be non-linear.\textsuperscript{16}

Theorem 9. If $R(m)$ is totally d-discontinuous, then its modular conjugate space $\overline{R}^{m}(\overline{m})$ is semi-linear.

Proof. We can represent $R$ as direct sum of two normal manifolds $R_1$ and $R_2$ such that $R_1(m)$ is singular and $R_2(m)$ is semi-simple. As $\overline{R}^{m} = \overline{R}_1^{m} \oplus \overline{R}_2^{m}$ and $\overline{R}_1^{m}(\overline{m})$ is linear, we may assume further $R(m)$ is semi-simple.

Let $a > 0$ be d-discontinuous unit, on account of theorem 2 we find $\vec{a} \in \overline{R}^{m}$ such that $\overline{m}(\vec{a}) + m(a) = (a, \vec{a})$ and $([p]a, \vec{a}) = D_{-}m([p]a)$ for all $[p]$. And for any $\xi \geq 1$ we see $D_{-}m([p]a) \leq (\xi \vec{a}) \leq D_{+}m([p]a)$ for all $[p]$, because $a$ is a d-discontinuous unit and we see $D_{+}m([p]a) = +\infty$ for $[p]a \neq 0$. Therefore from lemma 4 we have $\overline{m}(\vec{a}) + m(a) = (a, \vec{a})$ for all $\xi \geq 1$ and hence $\overline{m}(\vec{a}) = (a, \vec{a}) - m(a) = (\xi - 1)(a, \vec{a}) + m(\vec{a})$ for all $\xi \geq 1$. Therefore $\vec{a}$ is a semi-linear element. On the other hand, if $[p]a > 0$, then $D_{+}m([p]a) > 0$. Because: If $D_{+}m([p]a) = 0$ and $[p]a > 0$, then we

\textsuperscript{32} a is called asymptotically linear when $\sup_{\xi > 0} \frac{m(\xi a)}{\xi} = \tau < +\infty$ and $\sup \{\xi \tau - m(\xi a)\} < +\infty$.

\textsuperscript{33} The set of all asymptoically linear elements is complete in $R$ ([1], p. 208).
have \( D \cdot m(\xi[q]a) = 0 \) or \(+\infty\) for all \([q] \leq [p]\) and \(\xi \geq 0\), therefore \([p]a\) is singular. This contradicts \( R(m) \) is semi-simple. As \( \int \left( \frac{x}{a}, \psi \right) \) (theorem 2), we see \([a]^{R} = [a]\). Therefore the set of all semi-linear elements in \( \overline{R}^{m}(\overline{m}) \) is complete. Q.E.D.

**Theorem 10.** If \( R(m) \) is semi-linear, then \( \overline{R}^{m}(\overline{m}) \) is totally \( d\)-discontinuous.

**Proof.** Let \( a > 0 \) be semi-linear and \( m(\xi a) = (\xi - 1) \xi + m(a) \) for all \( \xi \geq 1 \) and \( m(a) < +\infty \). Evidently \( \eta = D \cdot m(a) \) and \( m([p]a) = (\xi - 1) D \cdot m([p]a) + m([p]a) \) for all \([p]\) and \(\xi \geq 1\). Therefore \( D \cdot m([p]a) = 0 \) implies \([p]a = 0\), for \( D \cdot m([p]a) = 0 \) implies \( m(\xi [p]a) = m([p]a) \) for all \( \xi \geq 1\), namely \([p]a = 0\).

From theorem 2 we can find \( 0 \leq \overline{a} \in \overline{R}^{m} \) such that \( \overline{m}(\overline{a}) + m(a) = (a, \overline{a}) \) and \( (\overline{a}^{R} a, \overline{a}) + m([p]a) = 0 \) implies \([p]a = 0\). From the equality \( \overline{m}(a) + m(a) = (a, \overline{a}) \) we have \( D \cdot m(\overline{a}) \leq (a, \overline{a}) < +\infty \). And for any \( \xi > 1 \) and \( \overline{a}[p] > 0 \) we see

\[
\overline{m}(\xi \overline{a}[p]) = \sup_{\rho \geq 1} \{ \rho(\xi - 1) D \cdot m([p]a) \} + m([p]a) - m([p]a) \geq \sup_{\rho \geq 1} \{ \rho(\xi - 1) m([p]a) \} = +\infty,
\]

because \( \overline{a}[p] > 0 \) implies \([\overline{a}]^{R}[p] > 0\), therefore \([a][p] > 0\) namely \([p]a > 0\) and \( D \cdot m([p]a) > 0\). Therefore \( \overline{a} \) is a \( d\)-discontinuous unit: If \( D \cdot \overline{m}(\overline{a}) < +\infty \) and \( \overline{a}[p] \leq \overline{a} \), then we can find \( \xi > 1 \) and \([p] \) such that \( \overline{a}[p] > 0 \), hence \( +\infty > D \cdot \overline{m}(\overline{a}) \geq D \cdot \overline{m}(\overline{a}[p]) \geq \overline{m}(\xi \overline{a}[p]) = +\infty \), this is a contradiction. As \([\overline{a}]^{R} = [a],\) the set of all \( d\)-discontinuous units in \( \overline{R}^{m} \) is complete. Q.E.D.

**Finally we state one theorem concerning a \( d\)-continuous modular.**

If we put \( F = \{ a; a \geq 0, m(a) < +\infty \} \), then evidently we see \( M, \subseteq M \subseteq F, \) and \( M, \subseteq F \), namely \( m(a) < +\infty \) implies \( D \cdot m(a) < +\infty \). T. Andō named that type domestic and he proved the interesting theorem: A modular is domestic if and only if a modular is continuous and its modular norm\(^{34}\) is continuous.\(^{35}\) In this result the most interesting point is that \( M, = F \) implies the continuity of a modular norm. Recently further he showed

\[34] \text{||| } a \text{|||} = \inf_{m(\xi a) \leq [\xi]} \frac{1}{|\xi|} (a \in R) ([2], p. 212).
\[35] \text{We have } \lim_{\nu \rightarrow \infty} ||| a_{\nu} \text{|||} = 0 \text{ for any } a_{\nu} \downarrow_{\nu \approx 1}^{\infty} 0 ([1], p. 127)\]
the weaker condition than $M_+=F$ implies the continuity of a modular norm. By the similar method we can see $M_+=M_-$ implies the continuity of a modular norm. Therefore in the following we shall prove

**Theorem 11.** Let $R(m)$ be a modulared semi-ordered linear space. Then $M_+=M_-$ if and only if the modular norm is continuous and the modular is d-continuous.

At first we remark that the modular norm $\|a\|$ (a ∈ R) is continuous if and only if for any system $[p_n]_{n=1}^{\infty}, 0$ and a ∈ R we have $\inf_{n\geq 1} D_m([p_n]a) = 0$. Because; if the norm $\|a\|$ (a ∈ R) is continuous then for $[p_n]_{n=1}^{\infty}, 0$ and a ∈ R we can find $\nu_n (n=1,2, \cdots)$ such that $\|2n[p_n]a\|\leq 1$, hence $D_m([p_n]a)\leq m(2[p_n]a)\leq m(2n[p_n]a)\leq \frac{1}{n}$ for every $n=1,2, \cdots$, therefore $\inf_{n\geq 1} D_m([p_n]a) = 0$. Conversely, if we have $\inf_{n\geq 1} D_m([p_n]a) = 0$ for every $[p_n]_{n=1}^{\infty}, 0$ and a ∈ R, then we find $\nu_n (n=1,2, \cdots)$ such that $m(n[p_n]a) \leq D_m(n[p_n]a) \leq 1$, hence $\|p_n]a\|\leq \frac{1}{n}$ for every $n=1,2, \cdots$, therefore $\inf_{n\geq 1} D_m([p_n]a) = 0$. This implies the continuity of the norm $\|a\|$ (a ∈ R) (11), p. 128).

Next we assume the following lemma that was proved by Andō ([5]):

**Lemma.** Let $R(m)$ be a modulared semi-ordered linear space. If the modular norm is not continuous, then we can find a closed subspace $S$ of $R$ satisfying following conditions: $S(m)$ is a monotone complete modulared semi-ordered linear space and there exist normal manifolds $N_\nu$ of $S (\nu=1,2, \cdots)$ such that $N_\nu (\nu=1,2, \cdots)$ are orthogonal each other and the modular norm is not continuous on all $N_\nu (\nu=1,2, \cdots)$.

**Proof of theorem 10.** Let $R(m)$ be d-continuous and its modular norm is continuous. For a ∈ $M_-$ let $[p_\lambda] (\lambda \in A)$ be all projectors such that $[p_\lambda]a \in M_+$, we see easily $[p_\lambda]_{\lambda \in A}$, If we put $b= \bigcup_{\lambda \in A} [p_\lambda]a$, then we can find $\lambda_n \in A (n=1,2, \cdots)$ $[p_{\lambda_n}]a \uparrow_{n=1}^{\infty} b$ (11), p. 128), therefore $[p_{\lambda_n}] \uparrow_{n=1}^{\infty} [b]$. We have $[a]=[b]$, because, if $[b]<[a]$, then for every $[q]$ such that $0<[q]\leq [a]-[b]$ we have $[q]a \in M_+$, therefore $([a]-[b])a$ is a d-discontinuous unit and non-zero. This fact contradicts the assumption. We put $[q_n]= [a]-[p_{\lambda_n}]$, then we have $[q_n] \uparrow_{n=0}^{\infty} 0$, hence from the continuity of the modular norm we have $\inf_{n\geq 1} D_m(2[q_n]a) = 0$. Therefore we can find $n_0$ such that $D_m(2[q_n]a) < +\infty$, hence $[q_n]a \in M_+$ and $a = [q_n]a + [p_{\lambda_n}]a \in M_+$. Thus we have $M_+=M_-$.

Conversely, let be $M_+=M_-$. Evidently if $a$ is non-zero d-discontinuous
unit, then \( a \in M_- \) and \( a \not\in M_+ \), therefore \( M_+ = M_- \) implies that the modular is \( d \)-continuous. Next we see the continuity of the modular norm. If the norm is not continuous, then without loss of generality from the above lemma we can assume that \( R(m) \) is monotone complete and there exist normal manifolds \( N_\nu \ (\nu = 1, 2, \cdots) \) of \( R \) such that the norm is not continuous on every \( N_\nu \ (\nu = 1, 2, \cdots) \). From the property that the norm is not continuous on \( N_1 \) we can find \( [p_{1,\mu}] \downarrow_{\mu=1}^{\infty} 0 \) and \( 0 \leq a_1 \in N_1 \) such that \( D_- m([p_{1,\mu}] a_1) = +\infty \) for every \( \mu = 1, 2, \cdots \). We put \( \xi \) the infimum of \( \xi \geq 0 \) such that \( \inf_{\mu \geq 1} D_- m(\xi [p_{1,\mu}] a_1) = +\infty \). Then evidently \( 0 < \xi \leq 1 \) and we have \( \inf_{\mu \geq 1} D_- m(\xi [p_{1,\mu}] a_1) = +\infty \), because, if \( D_- m(\xi_1 [p_{1,\mu_0}] a_1) < +\infty \) for some \( \mu_0 \), then \( \xi_1 [p_{1,\mu_0}] a_1 \in M_- = M_+ \), hence we find \( \xi' > \xi_1 \) such that \( \xi' [p_{1,\mu_0}] a_1 \in M_- \), therefore \( \inf_{\mu \geq 1} D_- m(\xi' [p_{1,\mu}] a_1) = 0 \). This implies \( \xi' \leq \xi_1 \), it is a contradiction. Thus we can find \( [p_{\nu,\mu}] \downarrow_{\mu=1}^{\infty} 0 \) and \( 0 < a_\nu \in N_\nu \ (\nu = 1, 2, \cdots) \) such that \( [N_\nu] \geq [p_{\nu,\mu}] \) and \( D_- m([p_{\nu,\mu}] a_\nu) = +\infty \) for every \( \mu = 1, 2, \cdots \). From the property that the norm is not continuous on \( N_1 \) we can find \( \mu_{\nu_0} \) such that \( \alpha \alpha_{\nu_0} > 1 \), therefore \( D_- m(\alpha [p_{\nu_0,\mu_{\nu_0}}] a_{\nu_0}) = +\infty \), hence \( \alpha \not\in M_+ \). This is a contradiction.

Q.E.D.

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