ON SEMI-LOWER BOUNDED MODULARS

By

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W. Orlicz and Z. Birnbaum proved in [7] that an Orlicz space $L_{\phi}(G)$ is finite if and only if the function $\Phi$ satisfies the following condition: for some $\gamma>0$ and $t_0>0$, $\Phi(2t) \leq \gamma \Phi(t)$ for every $t \geq t_0$. (In case of $\text{mes}(G) = +\infty$, $\Phi(2t) \leq \gamma \Phi(t)$ for all $t \geq 0$.)

This fact was generalized for arbitrary monotone complete modulars on non-atomic space by I. Amemiya in [1], that is, suppose that $R$ is a universally continuous semi-ordered linear space and has no atomic element, then every monotone complete finite modular on $R$ is semi-upper bounded. T. Shimogaki showed in [8] a new simple proof of this Amemiya's Theorem. In this paper we investigate the properties of the conjugate modular of a semi-upper bounded modular, i.e. the semi-lower bounded modular. Throughout this paper we use the terminologies and notations used in [5].

In §1 we give corollaries of Amemiya's Theorem and a theorem relate to Amemiya's Theorem. In §2 we investigate the relations between a modular or the modular norms and semi-lower bounded modular. In §3 we express the properties of a semi-upper and semi-lower bounded modular.

§1. Let $R$ be a universally continuous semi-ordered linear space and $m$ be a modular on $R^1$. A modular $m$ is said to be "finite", if $m(x) < +\infty$ for every $x \in R$. A modular $m$ is said to be "monotone complete", if for $0 \leq a_i \uparrow_{i \in A}$, \( \sup_{i \in A} m(a_i) < +\infty \) there exists $a \in R$ for which $a_i \uparrow_{i \in A} a$.

And a modular $m$ is said to be "semi-upper bounded", if for every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon) > 0$ such that $m(x) \geq \varepsilon$ implies $m(2x) \leq \gamma m(x)$.

In [1] I. Amemiya proved:

Theorem 1.1. Suppose that $R$ has no atomic element, then every monotone complete, finite modular on $R$ is semi-upper bounded.

We say a modular $m$ on $R$ to be "domestic", if for any $a \in \{a : m(a) < +\infty, a \in \mathbb{R}\}$ there exists $\xi = \xi(a) > 1$ such that $m(\xi a) < +\infty$. On $R$, we define the two functionals $\|a\|$, $\|\|a\||$ ($a \in R$) as follows:

1) For the definition of the modular see H. Nakano [5].
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\[ ||a|| = \inf_{\xi > 0} \frac{1 + m(\xi a)}{\xi}, \quad |||a||| = \inf_{m(\xi a) \leq 1} \frac{1}{|\xi|}. \]

Then it is easily seen that both \( ||a|| \) and \( |||a||| \) are norms on \( R \) and satisfy always \( |||a||| \leq ||a|| \leq 2|||a||| \) for all \( a \in R \) (cf. [6]). The norms \( ||a|| \) and \( |||a||| \) are called the first norm and the second (or modular) norm by \( m \) respectively.

**Remark 1.1.** (i) If a modular \( m \) on \( R \) is finite, then \( m \) is domestic; (ii) if \( m \) is domestic, then \( \inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) = 1 \); (iii) \( \inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) > 0 \) implies \( ||\cdot||| \) is continuous; (iv) if \( ||\cdot||| \) is continuous, then \( m \) is finite, when \( R \) has no atomic element.

Because, (i) is trivial. (iii) and (iv) is well known\(^2\). Therefore we have only to prove (ii). If \( m\left(\frac{x}{|||x|||}\right) < 1 \) for some \( x \in R \), there exists \( \varepsilon > 0 \) by domesticness such that

\[ 1 < m\left((1+\varepsilon)\frac{x}{|||x|||}\right) < +\infty. \]

Thus there exists \( \gamma < 1 \), for which \( m\left(\gamma(1+\varepsilon)\frac{x}{|||x|||}\right) = 1 \). Therefore we obtain \( \gamma(1+\varepsilon) = \left\|\gamma(1+\varepsilon)\frac{x}{|||x|||}\right\| = 1 \), and hence \( m\left(\frac{x}{|||x|||}\right) = 1 \), contradicting \( m\left(\frac{a}{|||x|||}\right) < 1 \).

A modular norm \( ||x||| (x \in R) \) is said to be "finitely monotone" (cf. [9]), if for every \( \varepsilon > 0 \), there exists an integer \( n_0 = n_0(\varepsilon) \) such that \( x = \oplus \sum_{i=1}^{n} x_i, \quad |||x||| \leq 1, \quad ||x_i||| \geq \varepsilon \ (i = 1, 2, \ldots, n) \) implies \( n \leq n_0 \). A modular \( m \) is said to be "uniformly finite", if

\[ \sup_{m(x) \leq 1} m(\xi x) < +\infty \quad \text{for all } \xi \geq 0. \]

In [9, Theorems 1.1, 2.1 and 2.2], it is shown that if a norm on \( R \) is uniformly monotone\(^3\), then it is finitely monotone; if a modular \( m \) is uniformly finite, then the modular norm by \( m \) is finitely monotone; if the modular norm by \( m \) is finitely monotone, then \( m \) is uniformly finite when \( R \) has no atomic element; if a norm is finitely monotone, then the every norms which is equivalent to it is also finitely monotone.

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2) T. Andō obtained (iii). For (iv) see [1].

3) A norm on \( R \) is said to be uniformly monotone, if for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( a \cap b = 0, \quad ||a|| = 1, \quad ||b|| \geq \varepsilon \) implies \( ||a+b|| \geq 1 + \delta \) (cf. [4]).
\( \overline{R}^m \) denotes the totality of all universally continuous linear functionals\(^4\) on \( R \) which are bounded under the modular norm \( |||\cdot||| \) by \( m \). On \( \overline{R}^m \) the conjugate modular of \( m(x) \) is defined as follows
\[
\overline{m}(\overline{a}) = \sup_{x \in R} \{ \overline{a}(x) - m(x) \} \quad \text{for every } \overline{a} \in \overline{R}^m.
\]
\( \overline{m}(\overline{a}) \) satisfies the modular conditions and is monotone complete (cf. [5, §38]).

It has been known that if \( R \) is semi-regular\(^5\), the first norm by the conjugate modular \( \overline{m} \) is the conjugate norm of the second norm by \( m \) and the second norm by the conjugate modular \( \overline{m} \) is the conjugate norm of the first norm by \( m \).

**Lemma 1** ([5, Theorem 39.4]). If \( R \) is semi-regular, then \( R \) is isometric\(^6\) to a complete semi-normal manifold of the conjugate space \( \overline{R}^m \) of \( \overline{R}^m \) by the correspondence
\[
R \ni a \rightarrow a^{R^{-m}} \in \overline{R}^m, \quad a^{R^{-m}}(\overline{x}) = \overline{a}(a) \quad \text{for } \overline{x} \in \overline{R}^m.
\]

**Corollary 1 of Theorem 1.1.** Suppose that \( R \) has no atomic element. If the modular norm \( |||\cdot||| \) by \( m \) is finitely monotone, then \( m \) is semi-upper bounded.

**Proof.** Since \( m \) is uniformly finite by assumption, \( \overline{m} \) is uniformly finite on \( \overline{R}^m \) ([5, Theorems 48.4, 48.5]). Since \( \overline{m} \) is monotone complete and \( \overline{R}^m \) has no atomic element, we obtain by Theorem 1.1 \( \overline{m} \) is semi-upper bounded on \( \overline{R}^m \). Therefore \( m \) is semi-upper bounded by Lemma 1. Q.E.D.

**Remark 1.2.** If a modular \( m \) is semi-upper bounded and semi-simple, then \( m \) is uniformly finite.

Because, if for some \( \gamma > 1 \) we have \( m(2x) \leq \gamma m(x) \) for every \( x \) such that \( m(x) \geq 1 \), then we have obviously \( m(2^\nu x) \leq \gamma^\nu m(x) \) \((\nu = 1, 2, \cdots)\) for every \( x \) such that \( m(x) \geq 1 \). Since \( m \) is finite by assumption, we obtain
\[
\sup_{m(x) \leq 1} m(2^\nu x) \leq \sup_{1 \leq m(x) \leq 2} m(2^\nu x) \\
\leq \sup_{1 \leq m(x) \leq 2} \gamma^\nu m(x) \leq 2\gamma^\nu < +\infty \quad (\nu = 1, 2, \cdots).
\]

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4) A linear functional \( L \) on \( R \) is said to be universally continuous, if for any \( a_1 |_{\subseteq 0} \) we have \( \inf_{1 \leq a_1} | L(a_1) | = 0 \).

5) \( R \) is said to be semi-regular, if \( \overline{a}[p] = 0 \) for all \( \overline{a} \in \overline{R}^m \) implies \( p = 0 \). For \( p \in R \), \([p]\) denotes the projection operator defined by \([p]x = \bigcup_{x' \sim p} x' \) for all \( x \geq 0 \).

6) A modulared space \( \hat{R} \) with a modular \( m \) is said to be isometric to a modulared space \( \hat{R} \) with a modular \( \overline{m} \) by a correspondence \( R \ni a \rightarrow a^{R^{-m}} \in \hat{R} \), if \( R \) is isomorphic to \( \hat{R} \) by this correspondence and \( m(a) = \overline{m}(a^{\hat{R}}) \) for all \( a \in R \).
Thus, $m$ is uniformly finite.

A norm on $R$ is said to be "monotone", if $0 \leq a < b$ implies $||a|| < ||b||$. A norm on $R$ is said to be "universally monotone complete", if for $0 \leq a_i \uparrow_{i \in A}$, sup $||a_i|| < +\infty$ there exists $a \in R$ such that $a_i \uparrow_{i \in A} a$; if $A = \{1,2, \cdots\}$ we say to be "monotone complete".

**Corollary 2 of Theorem 1.1.** If the modular norm $|||\cdot|||$ by $m$ is monotone and monotone complete, then $m$ is uniformly simple\(^7\), and $m$ is semi-upper bounded when $R$ has no atomic element.

**Proof.** (i) If the modular norm $|||\cdot|||$ by $m$ is monotone, than $|||\cdot|||$ is continuous.

Because, if $\inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) < 1$, there exists $a \in R$ such that $|||a|||=1$ and $m(a)<1$, therefore we can suppose $[a]<1$ without difficulty, and hence there exists $0 < b \in R$ such that $a \uparrow b = 0$, $m(a+b) \leq 1$. Thus, we obtain obviously $|||a+b|||=|||a|||=1$, which is contradicting $|||\cdot|||$ is monotone. Consequently we obtain $\inf_{0 \neq x \in R} \left(\frac{x}{|||x|||}\right) = 1$, and hence $|||\cdot|||$ is continuous by Remark 1.1.

(ii) If the modular norm $|||\cdot|||$ by $m$ is monotone, then $m$ is simple\(^8\). Because, if $m$ is not simple there exists $a \in R$ such that $0 < a$ and $m(a) = 0$, then $m(a+b) = m(b) \leq 1$ for any $0 < b$, $a \uparrow b = 0$ and $|||b|||=1$. Thus we have $|||a+b|||=|||b|||=1$, contradicting assumption that $|||\cdot|||$ is monotone. Thus $m$ is simple.

If the modular norm $|||\cdot|||$ by $m$ is continuous and monotone complete, then $m$ is monotone complete (cf. [5, Theorems 30.20, 40.7]). Thus we obtain $m$ is monotone complete, simple and $|||\cdot|||$ is continuous by (i) and (ii). Therefore $m$ is uniformly simple (cf. [11, Theorem 2.1]).

If $R$ has no atomic element, then uniformly simple modular $m$ is uniformly finite ([10, Theorem 1.2]), and hence we obtain $m$ is semi-upper bounded by Corollary 1 of Theorem 1.1.

**Theorem 1.2.** Suppose that $R$ has no atomic element. Each of the following conditions implies that $m$ is semi-upper bounded

\[(1): \quad \inf_{0 \neq x \in R} m\left(\frac{x}{|||x|||}\right) > 0 \quad \text{for some } 0 < a < 1,\]

\[7\) A modular $m$ is said to be uniformly simple, if $\inf_{\xi > 0} m(\xi x) > 0$ for all $\xi > 0$, that is, $\lim_{\nu \to \infty} m(a_\nu) = 0$ implies $\lim_{\nu \to \infty} |||a_\nu||| = 0$.

\[8\) A modular $m$ on $R$ is said to be simple, if $m(a) = 0$ implies $a = 0$.\]
\[
(2): \quad \sup_{0 \neq x \in R} m(\frac{\alpha}{|||x|||} x) > 0 \quad \text{for some } \alpha \geq 1.
\]

**Proof.** (1): We prove first that the condition:
\[
\inf_{0 \neq x \in R} m(\frac{1-\varepsilon}{|||x|||} x) = \xi > 0 \quad \text{for some } 1 > \varepsilon > 0
\]
implies the condition:
\[
\inf_{0 \neq \overline{x} \in \overline{R}^m} m(\frac{1-\varepsilon'}{|||\overline{x}|||} \overline{x}) > 0 \quad \text{for some } \varepsilon > \varepsilon' > 0.
\]

For \( \overline{x} \in \overline{R}^m \) with \( |||\overline{x}||| = 1 \) there exists \( x_\lambda \in R \) (\( \lambda \in \Lambda \)) such that \( x_\lambda \uparrow_{\lambda \in \Lambda} \overline{x} \) (cf. [5, Theorem 5.34]), because \( R \) is a complete semi-normal manifold of \( \overline{R}^m \) by Lemma 1. Since the modular norm is semi-continuous and reflexive (cf. [3]), we obtain \( |||x_\lambda||| \uparrow_{\lambda \in \Lambda} \overline{x} \), and hence we have
\[
(1-\frac{\varepsilon}{2}) |||x_\lambda||| \uparrow_{\lambda \in \Lambda} \left(1-\frac{\varepsilon}{2}\right).
\]
Consequently there exists \( \lambda_0 \) such that \( (1-\frac{\varepsilon}{2}) |||x_\lambda||| \geq 1-\varepsilon \) for \( \lambda \geq \lambda_0 \).

If \( \inf_{0 \neq x \in R} m(\frac{1-\varepsilon}{|||x|||} x) = \xi > 0 \), we obtain easily \( m(x) \geq \xi \) for every \( x \) such that \( |||x||| \geq 1-\varepsilon \), thus we have obviously \( m\left((1-\frac{\varepsilon}{2})x_\lambda\right) \geq \xi \) for \( \lambda \geq \lambda_0 \).

Therefore we have
\[
\inf_{0 \neq \overline{x} \in \overline{R}^m} m\left(\frac{1-\varepsilon}{|||\overline{x}|||} \overline{x}\right) > 0.
\]

Therefore, we obtain \( |||\overline{a}||| \) (\( \overline{a} \in \overline{R}^m \)) is continuous by Remark 1.1, and, since \( \overline{R}^m \) is non-atomic, \( \overline{m} \) is finite on \( \overline{R}^m \) by Remark 1.1. As \( \overline{m} \) is monotone complete, we obtain \( \overline{m} \) is semi-upper bounded by Theorem 1.1, and hence we obtain finally that \( m \) is semi-upper bounded by Lemma 1.

The proof for the condition (2) is similar.

**Q.E.D.**

§2. Let \( R \) be a modulared semi-ordered linear space with a modular \( m \) and be semi-regular. In this section, our aim is to consider the relations between properties of a modular or the modular norms and its semi-lower boundedness.

A modular \( m \) on \( R \) is said to be "semi-lower bounded" if for every \( \varepsilon > 0 \), there exist \( 1 < a = a(\varepsilon) < \gamma(\varepsilon) = \gamma \) such that \( m(x) \geq \varepsilon \) implies \( m(ax) \geq \gamma m(x) \).

**Theorem 2.1.** If a modular \( m \) is semi-upper bounded and semi-simple, then the conjugate modular \( \overline{m} \) of \( m \) is semi-lower bounded.
Proof. Since the case $\overline{m}(\overline{a})=+\infty$ is trivial, we can assume that $\overline{m}(\overline{a})<+\infty$. For every $\epsilon>0$ there exists $\gamma=\gamma(\epsilon)>0$ such that $m(x)\geq\frac{\epsilon}{3}$ implies $m(2x)\leq\gamma m(x)$, by assumption. Then we have definition

$$\overline{m}\left(\overline{\frac{\gamma}{2}a}\right)=\sup_{x\in R}\left\{ \overline{\frac{\gamma}{2}a(2x)}-m(2x) \right\} \geq \sup_{m(x)\geq\frac{\epsilon}{3}}\left\{ \overline{\frac{\gamma}{2}a(2x)}-m(2x) \right\}
\geq \gamma \sup_{m(x)\geq\frac{\epsilon}{3}}\{\overline{a(x)}-m(x)\} \ (\overline{a}\in\overline{R}^m).$$

For every $0\leq\overline{a}\in\overline{R}^m$ such that $\epsilon\leq\overline{m}(\overline{a})<+\infty$, we have to consider the case

$$\overline{m}(\overline{a})=\sup_{m(x)<\frac{\epsilon}{3}}\{\overline{a(x)}-m(x)\}.$$ 

For any $\delta>0$ there exists $x\in R$ such that $m(x)<\frac{\epsilon}{3}$ and $\overline{a(x)}-m(x)\geq\overline{m}(\overline{a})-\delta$. Since $m$ is uniformly finite by Remark 1.2 there exists $\beta=\beta(a)>1$ such that $m(\beta x)=\frac{\epsilon}{3}$.

Therefore we obtain

$$\overline{a}(\beta x)-m(\beta x)\geq\overline{a}(x)-m(x)-m(\beta x)\geq\overline{m}(\overline{a})-\delta-\frac{\epsilon}{3}.$$ 

Thus we have

$$\gamma \sup_{m(x)\geq\frac{\epsilon}{3}}\{\overline{a(x)}-m(x)\} \geq \gamma\left(\overline{m}(\overline{a})-\frac{\epsilon}{3}\right) \geq \gamma\left(\overline{m}(\overline{a})-\frac{\overline{m}(\overline{a})}{3}\right) = \frac{2}{3}\gamma\overline{m}(\overline{a}),$$

and hence $\overline{m}\left(\overline{\frac{\gamma}{2}a}\right)\geq\frac{2}{3}\gamma\overline{m}(\overline{a})$ for every $\overline{a}$ such that $\overline{m}(\overline{a})\geq\epsilon$. Q.E.D.

Theorem 2.2. If a modular $m$ is semi-lower bounded, then $\overline{m}$ is semi-upper bounded.

Proof. If for every $\epsilon>0$ there exist $\gamma>\alpha>1$ such that $m(x)\geq\epsilon$ implies $m(\alpha x)\geq\gamma m(x)$, then we have by definition

$$\overline{m}\left(\overline{\frac{\gamma}{\alpha}a}\right)=\sup_{x\in R}\left\{ \overline{\frac{\gamma}{\alpha}a(ax)}-m(ax) \right\} \leq \gamma \sup_{m(x)\geq\epsilon}\{\overline{a(x)}-m(x)\} + \sup_{m(x)<\epsilon}\{\gamma\overline{a(x)}-m(ax)\}
\leq \gamma\overline{m}(\overline{a})+\gamma \sup_{m(x)<\epsilon}\{\overline{a(x)}-m(x)\} \leq \gamma\overline{m}(\overline{a})+\gamma\overline{m}(\overline{a})+\epsilon = \gamma(2\overline{m}(\overline{a})+\epsilon),$$

since by definition $|\overline{a(x)}|\leq\overline{m}(\overline{a})+m(x)$ for $\overline{a}\in\overline{R}^m$, $x\in R$.

Thus we have $\overline{m}\left(\overline{\frac{\gamma}{\alpha}a}\right)\leq3\gamma\overline{m}(\overline{a})$ for every $\overline{a}$ such that $\overline{m}(\overline{a})\geq\epsilon$. Q.E.D.
The "conjugate" of "uniformly finite" is "uniformly increasing", i.e.
\[
\lim_{\xi \to \infty} \inf_{m(x) \geq 1} \frac{m(\xi x)}{\xi} = +\infty \quad (\text{cf. } [5, \S 48]).
\]

Theorem 2.3. If a modular \( m \) is semi-lower bounded, then \( m \) is uniformly increasing.

Proof. By assumption there exist \( 1 < \alpha < \gamma \) such that \( m(x) \geq 1 \) implies \( m(\alpha^\nu x) \geq \gamma^\nu m(x) (\nu = 1, 2, \cdots) \).
Therefore we obtain \( \frac{1}{\alpha^\nu} m(\alpha^\nu x) \geq (\frac{\gamma}{\alpha})^\nu m(x) (\nu = 1, 2, \cdots) \) for every \( x \) such that \( m(x) \geq 1 \), and consequently \( m \) is uniformly increasing. Q.E.D.

Since the "conjugate" of "finitely monotone" is "finitely flat", i.e.
\[
|||\cdot||| \text{ by } m \text{ is finitely flat, then } m \text{ is uniformly increasing.}
\]

Theorem 2.4. Suppose that \( R \) has no atomic element. If the modular norm \( |||\cdot||| \) by \( m \) is finitely flat, then \( m \) is semi-lower bounded.

Remark 2.1. If a modular \( m \) is uniformly increasing, then the modular norm is finitely flat. The converse of this is valid, if we suppose that \( R \) has no atomic element (cf. [9]).

A norm \( ||\cdot|| \) on \( R \) is said to be "flat", if for any \( a \neq 0 \), \( a \cap b = 0 \) we have
\[
\lim_{\xi \to 0} \frac{||a + \xi b|| - ||a||}{\xi} = 0.
\]

The "conjugate" of "uniformly simple" is "uniformly monotone", i.e.
\[
\lim_{\xi \to 0} \frac{1}{\xi} \sup_{m(x) \leq 1} m(\xi x) = 0 \quad (\text{cf. } [5, \S 48]).
\]

Theorem 2.5. If the first norm \( ||\cdot|| \) by \( m \) is flat and the first norm \( ||\cdot|| \) by conjugate modular \( \overline{m} \) of \( m \) is continuous, then \( m \) is uniformly monotone, and \( m \) is semi-lower bounded when \( R \) has no atomic element.

Proof. Using Banach's theorem (cf. [6, \S 44]) and reflexivity of the norm \( ||\cdot|| \), we can prove that flatness of \( ||\cdot|| \) implies monotony of \( ||\cdot|| \). Thus we have \( \overline{m} \) is simple by (ii) in proof of Corollary 2 of Theorem 1.1. Since \( ||\overline{a}|| \) is continuous by assumption and \( \overline{m} \) is monotone complete, we obtain \( \overline{m} \) is uniformly simple ([11, Theorem 2.1]). Thus \( m \) is uniformly monotone.
On the other hand, if $m$ is uniformly monotone then $m$ is uniformly increasing when $R$ has no atomic element ([10, Theorem 1.3]). By Theorem 2.4 and Remark 2.1 the proof is completed.

A manifold $K$ of $R$ is said to be "equi-continuous", if for any $\overline{a}_{\nu_{0}}\downarrow_{\nu=1}^{\infty} 0$, $\overline{a}_{\nu}\in \overline{R}^{m}$ and $\varepsilon>0$ there exists $\nu_{0}$ for which we have $\overline{a}_{\nu_{0}}(x)\leq \varepsilon$ for all $x\in K$.

Theorem 2.6. If a modular $m$ is semi-lower bounded, then a manifold $K=\{x:m(x)\leq 1, x\in R\}$ is equi-continuous. The converse of this is true, if we suppose that $R$ has no atomic element.

Proof. If $m$ is semi-lower bounded, $m$ is uniformly increasing by Theorem 2.3. Then we have $\overline{m}$ is uniformly finite, and hence the conjugate norm of the modular norm by $m$ is continuous by Remark 1.1. Therefore we obtain for any $\varepsilon>0$ and $\overline{R}^{m}\ni \overline{a}_{\nu_{0}}\downarrow_{\nu=1}^{\infty} 0$ there exists $\nu_{0}$ such that $\overline{a}_{\nu_{0}}(x)\leq \varepsilon$ for all $x\in K$ ([5, Theorem 31.12]). That is, $K$ is equi-continuous.

Conversely we suppose that $R$ has no atomic element and the manifold $K=\{x:m(x)\leq 1\}$ is equi-continuous. Since we have obviously by definition $\{x:\|x\|\leq 1\}=\{x:m(x)\leq 1\}$, the first norm by $\overline{m}$ is continuous ([5, Theorem 31.12]). Thus we obtain $\overline{m}$ is monotone complete and finite, because $\overline{R}^{m}$ is non-atomic by assumption. Thus we have $\overline{m}$ is semi-upper bounded by Theorem 1.1, therefore we obtain by Theorem 2.1 and Lemma 1 $m$ is semi-lower bounded.

A manifold $K$ of $R$ is said to be "weakly bounded", if

$$\sup_{x\in K} |\overline{a}(x)| < +\infty \text{ for all } \overline{a}\in \overline{R}^{m}.$$  

Theorem 2.7. If a modular $m$ is semi-lower bounded, then every weakly bounded manifold is equi-continuous. The converse of this is truth, if we suppose that $R$ has no atomic element.

Proof. If $m$ is semi-lower bounded, the conjugate norm of a norm by $m$ is continuous. Consequently every manifold $K$ for which $\sup_{x\in \overline{K}} \|x\| < +\infty$ is equi-continuous ([5, Theorem 33.10]). Therefore we have $\sup_{x\in \overline{K}} |\overline{a}(x)| \leq \sup_{x\in \overline{K}} \|\overline{a}\| \cdot \|x\|$ for all $\overline{a}\in \overline{R}^{m}$, and hence $K$ is weakly bounded by definition.

Conversely we suppose that $R$ has no atomic element. Since the norm $\|\cdot\|$ is reflexive (cf. [3]), if a manifold $K$ is weakly bounded, then $K$ is norm bounded, i.e. $\sup_{x\in \overline{K}} \|x\| < +\infty$ ([5, Theorem 32.6]), and equi-continuous by assumption. Then the first norm by the conjugate modular $\overline{m}$ of $m$ is continuous ([5, Theorem 33.10]). Thus we have obviously our conclusion by the method applied to Theorem 2.6. Q.E.D.
Theorem 2.7. Suppose that \( R \) has no atomic element. Each of the following conditions implies \( m \) is semi-lower bounded

\[
\inf_{0 \neq x \in R} \frac{1}{\gamma} m\left( \frac{\gamma}{||x||} x \right) \geq 1 + \delta \quad \text{for some } \gamma, \delta > 0,
\]

\[
\sup_{0 \neq x \in R} m\left( \frac{x}{||x||} \right) < 1.
\]

Proof. (1) For every \( \bar{a} \in \bar{R}^m \) with \( ||\bar{a}|| = 1 \), we have
\[
(1 + \delta)\bar{a}(\xi a) - m(\xi a) \leq \xi(1 + \delta) - \xi(1 + \delta) = 0
\]
for every \( a \in R, ||a|| = 1 \) and \( \xi \geq \gamma \).

Thus we have
\[
\bar{m}((1 + \delta)\bar{a}) = \sup_{||x|| \leq \gamma} \{(1 + \delta)\bar{a}(x) - m(x)\} \leq \gamma(1 + \delta).
\]

Suppose that \( \bar{R}^m \ni \bar{a}_\nu \downarrow_{\nu=1}^\infty 0 \) and \( \inf_{\nu \geq 1} ||\bar{a}_\nu|| = \alpha > 0 \),
then there exist \( \epsilon_0 > 0, \nu_0 \) such that
\[
\frac{||\bar{a}_\nu||}{\alpha - \epsilon_0} \leq 1 + \delta \quad \text{for every } \nu \geq \nu_0.
\]

Since we have
\[
1 + \bar{m}\left( \frac{\bar{a}_\nu}{\alpha - \epsilon_0} \right) \geq \frac{||\bar{a}_\nu||}{\alpha - \epsilon_0} \geq \frac{\alpha}{\alpha - \epsilon_0}
\]
for every \( \nu \geq \nu_0 \).

we obtain
\[
1 + \lim_{\nu \to \infty} \bar{m}\left( \frac{\bar{a}_\nu}{\alpha - \epsilon_0} \right) \geq \frac{\alpha}{\alpha - \epsilon_0} > 1.
\]

Since
\[
\lim_{\nu \to \infty} \bar{m}\left( \frac{\bar{a}_\nu}{\alpha - \epsilon_0} \right) = 0,
\]
this is a contradiction.

Therefore \( ||\bar{a}|| \) is continuous. Thus we have our conclusion by the method applied to Theorem 2.6.

The proof for the condition (2) is similar. Q.E.D.

§3. Let \( R \) be a modulared semi-ordered linear space with a semi-simple modular \( m \). In this section, we express the properties of a semi-upper and semi-lower bounded modulars.

If a modular \( m \) is semi-upper and semi-lower bounded, then \( m \) is said to be "semi-bounded".

Lemma 3.1. Suppose that \( R \) has no atomic element. If the norms by a modular \( m \) have the property:
\[
\inf_{0 \neq x \in R} \frac{||x||}{|||x|||} = \gamma, \quad \text{where } \gamma > 1 \text{ is a fixed constant, then } m \text{ is semi-bounded.}
\]

Proof. We have \( m \) is uniformly finite and uniformly increasing by the assumption (cf. [10, Theorem 1.1]). Therefore we obtain our conclu-
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Lemma 3.2. If a modular \( m \) is semi-bounded, then the norms by \( m \) have the property:
\[
\inf_{0 \neq x \in R} \frac{||x||}{|||x|||} = \gamma
\]
for some \( \gamma > 1 \).

Proof. Since \( m \) is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3, we have our conclusion (cf. [10, Theorem 1.4]).

Q.E.D.

From these Lemmata, we obtain the following theorem.

Theorem 3.1. Suppose that \( R \) has no atomic element. A modular is semi-bounded, if and only if the norms by the modular have the property:
\[
\inf_{0 \neq x \in R} \frac{||x||}{|||x|||} = \gamma
\]
for some \( \gamma > 1 \).

In the case when a modular \( m \) on \( R \) is of unique spectra (cf. [5, §54]), semi-boundedness of \( m \) implies boundedness\(^9\) of \( m \). In fact we have

Theorem 3.2. If a modular \( m \) on \( R \) is of unique spectra\(^{10}\), then semi-boundedness of \( m \) is equivalent to boundedness of \( m \).

Proof. If \( m \) is semi-bounded, then \( m \) is uniformly finite and uniformly increasing by Remark 1.2 and Theorem 2.3. Therefore \( m \) has the upper exponent\(^{10}\) \( \rho_u \) and the lower exponent\(^{10}\) \( \rho_l \) such that \( 1 \leq \rho_l \leq \rho_u < +\infty \) (cf. [5, Theorems 54.8, 54.10]). Thus \( m \) is bounded ([5, Theorems 54.4, 54.5]).

Q.E.D.

A modular \( m \) of unique spectra is uniformly convex\(^{10}\) (or uniformly even\(^{10}\)) if and only if \( 1 < \rho_l \leq \rho_u < +\infty \) for the upper exponent \( \rho_u \) and the lower exponent \( \rho_l \) (cf. [5, §50, §54]). Therefore we obtain also:

Theorem 3.3. A modular \( m \) of unique spectra is uniformly convex (or uniformly even), if and only if \( m \) is semi-bounded.

Theorem 3.4. Suppose that \( R \) has no atomic element. If a modular \( m \) is uniformly convex (or uniformly even), then \( m \) is semi-bounded.

Proof. Let \( m \) be uniformly convex. Then \( m \) is uniformly simple ([5, Theorem 50.1]). Since \( R \) is non-atomic by assumption, \( m \) and \( \overline{m} \) are uniformly finite ([10, Theorem 1.2]), and hence \( m \) and \( \overline{m} \) are semi-upper

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9) A modular \( m \) on \( R \) is said to be upper bounded, if there exist \( \omega, \gamma > 1 \), for which we have \( m(\omega x) \leq \gamma m(x) \) for all \( x \in R \); and \( m \) is said to be lower bounded, if there exist \( \gamma > \omega > 1 \) such that \( m(\omega x) \geq \gamma m(x) \) for all \( x \in R \); if a modular \( m \) is upper and lower bounded, then \( m \) is said to be bounded.

10) For the definitions see [5].
bounded by Corollary 1 of Theorem 1.1. Thus $m$ is semi-bounded.

Let $m$ be uniformly even. Then $m$ is uniformly finite and uniformly monotone ([5, Theorems 51.1, 51.2]), and hence $m$ is semi-bounded by Corollary 1 of Theorem 1.1 and Theorem 2.4. Q.E.D.

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References


