ON GROUPS OF ROTATIONS IN MINKOWSKI SPACE II

By

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§ 1. **Introduction.** The present paper is a continuation of the one with the same title "On groups of rotations in Minkowski space I." [7][1] Let M_{n+1} be an $n+1$-dimensional Minkowski space whose indicatrix I_n is defined by $F(X^1, X^2, \cdots, X^{n+1})=1$. As I_n is an n-dimensional hypersurface in M_{n+1}, we may represent I_n by $n+1$ equations involving n parameters as

$$X^i=X^i(u^\alpha). \quad (i=1, 2, \cdots, n+1;\ \alpha=1, 2, \cdots, n)^{2)}$$

Putting $A_{ijk}(X)=F(X)C_{ijk}$, where $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial X^k}$ and $g_{ij}=\frac{1}{2}\partial^2 F/\partial X^i \partial X^j$, A_{ijk} is a symmetric covariant tensor in M_{n+1}. If we denote by $g_{\alpha\beta}$ and $A_{\alpha\beta\gamma}$ the induced components of g_{ij} and A_{ijk} respectively, i.e.

$$g_{\alpha\beta}=g_{ij}X^i_\alpha X^j_\beta, \quad A_{\alpha\beta\gamma}=A_{ijk}X^i_\alpha X^j_\beta X^k_\gamma \quad (X^i_\alpha=\partial X^i/\partial u^\alpha),$$

I_n may be considered as an n-dimensional compact Riemannian space with the fundamental metric tensor $g_{\alpha\beta}$, and the Riemannian space is characterized by existence of a symmetric covariant tensor $A_{\alpha\beta\gamma}$. Moreover, it is remarkable that in I_n we have

$$R_{\alpha\beta\gamma\delta}=S_{\alpha\beta\gamma\delta}+(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}), \tag{1.1}$$

where $R_{\alpha\beta\gamma\delta}=g_{\alpha\varepsilon}R^{\varepsilon}_{.\beta\gamma\delta}$ and

$$R^{\alpha}_{.\beta\gamma\delta}=\frac{\partial\left\{{\alpha \atop \beta\delta}\right\}}{\partial u^\gamma}-\frac{\partial\left\{{\alpha \atop \beta\gamma}\right\}}{\partial u^\delta}+\left\{{\varepsilon \atop \beta\delta}\right\}\left\{{\alpha \atop \varepsilon\gamma}\right\}-\left\{{\varepsilon \atop \beta\gamma}\right\}\left\{{\alpha \atop \varepsilon\delta}\right\},$$

$$S_{\alpha\beta\gamma\delta}=A^{\cdot\cdot\sigma}_{\alpha\delta}A_{\sigma\gamma\beta}-A^{\cdot\cdot\sigma}_{\alpha\gamma}A_{\sigma\delta\beta}.$$

Also, $A_{\alpha\beta\gamma}$ satisfies the relation $A_{\alpha\beta\gamma;\delta}=A_{\alpha\beta\delta;\gamma}$ [6].[3]

When a centro-affine transformation in M_{n+1}, with its centre at origin, preserves I_n, the transformation is called a *rotation* in M_{n+1}.

1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout the present paper, the Greek indices $\alpha, \beta, \gamma, \cdots$ are supposed to run over the range $1, 2, \cdots, n$.

3) Semi-colon is used to represent the covariant differentiation with respect to the Christoffel symbols made by $g_{\alpha\beta}$.

By this transformation every point on I_n is transformed into a point on the same. When an infinitesimal point transformation on I_n coincides with the transformation of such a class, we call it an *infinitesimal rotation on* I_n [7]. In the previous paper, the present author derived the fundamental equation of an infinitesimal rotation on I_n, i.e.

$$\mathfrak{L}g_{\alpha\beta}=\eta_{\alpha;\beta}+\eta_{\beta;\alpha}=2\eta^{\gamma}A_{\alpha\beta\gamma}\,, \tag{1.2}$$

and discussed about the integrability conditions of the transformation, where $\mathfrak{L}$ denotes Lie differential with respect to an infinitesimal transformation $\overline{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}(u)\delta t$.

The purpose of the present paper is to study the properties of infinitesimal rotations on I_n and develop the structures of r-parameter Lie groups of the transformations, mainly in connection with the properties of the symmetric tensor $A_{\alpha\beta\gamma}$.

In § 2 we shall state a few remarks on the connections between properties of I_n and those of the tensor $A_{\alpha\beta\gamma}$. § 3 contains theorems concerning with properties of r-parameter groups of infinitesimal rotations on I_n. In § 4 we shall discuss about the special cases in which the infinitesimal rotation on I_n coincides with a motion or transformation of other classes. § 5 devoted to give the order of the groups when the infinitesimal rotations on I_n preserve the tensor $A_{\alpha\beta\gamma}$.

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§ 2. Preliminary remarks on the structures of I_n. Let us denote by $\|A_{\alpha\beta\gamma}\|$ the matrix with elements $A_{\alpha\beta\gamma}$, where γ denotes the columns, and α and β the rows. If we consider about the rank of the matrix $\|A_{\alpha\beta\gamma}\|$, the followings are exceptional cases in our discussions.

Case I. Rank $\|A_{\alpha\beta\gamma}\|=0$. In this case, at every point on I_n, $A_{\alpha\beta\gamma}=0$ for every α, β and γ. Therefore, as we have $A_{ijk}=A_{\alpha\beta\gamma}X_i^{\alpha}X_j^{\beta}X_k^{\gamma}$ [6], it follows that $A_{ijk}=0$ and this means that the considered Minkowski space is essentially a Euclidean space.

Case II. Rank $\|A_{\alpha\beta\gamma}\|=1$. In this case, we have

$$\det.\begin{vmatrix} A_{\alpha\beta\tau} & A_{\alpha\beta\omega} \\ A_{\gamma\delta\tau} & A_{\gamma\delta\omega} \end{vmatrix}=0\,. \quad (\alpha,\beta,\gamma,\delta,\tau,\omega=1,2,\cdots,n) \tag{2.1}$$

On the other hand, by means of the symmetric properties of $A_{\alpha\beta\gamma}$, it

is readily seen that

$$S_{\alpha\beta\gamma\delta}=A_{\alpha\delta}^{\cdot\cdot\sigma}A_{\sigma\gamma\beta}-A_{\alpha\gamma}^{\cdot\cdot\sigma}A_{\sigma\delta\beta}=g^{\sigma\tau}(A_{\alpha\tau\delta}A_{\beta\sigma\gamma}-A_{\alpha\tau\gamma}A_{\beta\sigma\delta})\,. \tag{2.2}$$

Comparing (2.1) and (2.2), we find that $S_{\alpha\beta\gamma\delta}=0$. This also implies that the considered Minkowski space is essentially a Euclidean space [6].

According to the definition, as a rotation in M_{n+1} leaves invariant the indicatrix I_n, the transformation preserves the metric of M_{n+1}. However, as we see from the fundamental equation (1.2), in general an infinitesimal rotation on I_n does not preserve the metric of I_n, which is induced from the one of M_{n+1}. If an infinitesimal rotation $\overline{u}^\alpha=u^\alpha+\eta^\alpha(u)\delta t$ coincides with a motion in I_n, the generating vector η^α should satisfy the equation $\eta^\gamma A_{\alpha\beta\gamma}=0$. Therefore if an infinitesimal rotation coincides with a motion in I_n, the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than the maximum value n, and otherwise the transformation must be an identity transformation.

Theorem 2.1. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is equal to the maximum value n, an infinitesimal rotation on I_n does not coincide with a motion in I_n, except an identity transformation.*

§ 3. Groups of infinitesimal rotations on I_n. Let us denote by G_r the r-parameter group of infinitesimal rotations on I_n and by $X_a f\equiv\eta^\alpha_{(a)}\dfrac{\partial f}{\partial u^\alpha}$ $(a=1,2,\cdots,r)$[4] its r linearly independent symbols. Then by means of the fundamental equations (1.2), we get

$$\mathfrak{L}_a g_{\alpha\beta}=\eta_{(a)\alpha;\beta}+\eta_{(a)\beta;\alpha}=2\eta^\gamma_{(a)}A_{\alpha\beta\gamma}\,, \tag{3.1}$$

where $\mathfrak{L}_a$ denotes Lie differentiation with respect to an infinitesimal transformation generated by the vector $\eta^\alpha_{(a)}$.

On the other hand, from the theory of Lie derivatives, we have

$$\mathfrak{L}_a\eta^\alpha_{(b)}=c_{ab}^{c}\eta^\alpha_{(c)}\,, \tag{3.2}$$

$$(\mathfrak{L}_a,\mathfrak{L}_b)T^{\cdot\lambda}_{\cdot\mu\nu}\equiv(\mathfrak{L}_a\mathfrak{L}_b-\mathfrak{L}_b\mathfrak{L}_a)T^{\cdot\lambda}_{\cdot\mu\nu}=c_{ab}^{c}\mathfrak{L}_c T^{\cdot\lambda}_{\cdot\mu\nu}\,, \tag{3.3}$$

where $T^{\cdot\lambda}_{\cdot\mu\nu}$ is an arbitrary tensor and c_{ab}^{c} is a structural constant of G_r and satisfies

$$c_{ab}^{e}=-c_{ba}^{e}\,,\qquad c_{ab}^{e}c_{ec}^{f}+c_{bc}^{e}c_{ea}^{f}+c_{ca}^{e}c_{eb}^{f}=0\,.$$

Applying the formula (3.3) to the fundamental tensor $g_{\alpha\beta}$, we get

$$(\mathfrak{L}_b,\mathfrak{L}_c)g_{\alpha\beta}=c_{bc}^{a}\mathfrak{L}_a g_{\alpha\beta}\,. \tag{3.4}$$

4) Throughout the present paper the Latin indices $a, b, c, \cdots$ take the values $1, 2, \cdots, r$.

Substituting (3.1) and (3.2) into the left hand side of (3.4), it follows that

$$(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta}=2\mathfrak{L}_b(\eta^r_{(c)}A_{\alpha\beta\gamma})-2\mathfrak{L}_c(\eta^r_{(b)}A_{\alpha\beta\gamma})$$
$$=4c^a_{bc}\eta^r_{(a)}A_{\alpha\beta\gamma}+2(\eta^r_{(c)}\mathfrak{L}_bA_{\alpha\beta\gamma}-\eta^r_{(b)}\mathfrak{L}_cA_{\alpha\beta\gamma}).$$

Since we have $c^a_{bc}\mathfrak{L}_a g_{\alpha\beta}=2c^a_{bc}\eta^r_{(a)}A_{\alpha\beta\gamma}$, in consequence of the above equation, (3.4) gives rise to the relation

$$2c^a_{bc}\eta^r_{(a)}A_{\alpha\beta\gamma}+2(\eta^r_{(c)}\mathfrak{L}_bA_{\alpha\beta\gamma}-\eta^r_{(b)}\mathfrak{L}_cA_{\alpha\beta\gamma})=0. \tag{3.5}$$

If X_af $(a=1, 2, \cdots, r)$ are r generators of G_r, it is satisfied that

$$(X_b, X_c)f\equiv X_bX_cf-X_cX_bf=c^a_{bc}X_af.$$

Now, consider the symbols $X_{bc}f$ $(b, c=1, 2, \cdots, r)$ defined by $X_{bc}f=(X_b, X_c)f$, then each of them determines a subgroup G_1 of G_r and its generating vector is $c^a_{bc}\eta^a_{(a)}$. Since c^a_{bc} is constant, it is evident that

$$\mathfrak{L}_{bc}g_{\alpha\beta}=c^a_{bc}\eta_{(a)\alpha;\beta}+c^a_{bc}\eta_{(a)\beta;\alpha}=2c^a_{bc}\eta^r_{(a)}A_{\alpha\beta\gamma}, \tag{3.6}$$

where $\mathfrak{L}_{bc}$ denotes Lie differentiation with respect to an infinitesimal transformation determined by a vector $c^a_{bc}\eta^a_{(a)}$. By means of (3.5) and (3.6), if each transformation of G_r leaves invariant the tensor $A_{\alpha\beta\gamma}$, namely $\mathfrak{L}_aA_{\alpha\beta\gamma}=0$ $(a=1, 2, \cdots, r)$, we must have $\mathfrak{L}_{bc}g_{\alpha\beta}=0$. Hence we have the following

Theorem 3.1. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and each one-parameter group G_1, determined by r generators X_af, leaves invariant the covariant tensor $A_{\alpha\beta\gamma}$, then $X_{bc}f$ are symbols of one-parameter groups of motions on I_n.*

It is well known that, as it follows that

$$(X_{ab}, X_{cd})f=(c^e_{ab}X_e,\ c^f_{cd}X_f)f=c^e_{ab}c^h_{cd}X_{eh}f,$$

symbols $X_{ab}f$ determine a group which is either G_r itself, or an invariant subgroup of G_r called the derived group of G_r. The order of the derived group is determined by the rank of the matrix $\|c^e_{ab}\|$ where e denotes the columns, and a and b the rows. Specially if the rank of the matrix $\|c^e_{ab}\|$ is equal to r, the derived group coincides with the given group G_r. Then, in consequence of Theorem 3.1 we have

Theorem 3.2. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the covariant tensor $A_{\alpha\beta\gamma}$, the derived group is a subgroup of motions on I_n, and G_r itself is a group of motions if the rank of the matrix $\|c^e_{ab}\|$ is equal to r.*

Since Lie derivatives of $g_{\alpha\beta}$ do not vanish for an infinitesimal rotation on I_n, relations $\mathfrak{L}A_{\alpha\beta\gamma}=0$, $\mathfrak{L}A^{\alpha}_{\cdot\beta\gamma}=0$, $\mathfrak{L}A^{\alpha\beta}_{\cdot\cdot\gamma}=0$ and $\mathfrak{L}A^{\alpha\beta\gamma}=0$ are not equivalent each other. However, in consequence of direct calculations, we can obtain the following analogous results concerning the mixed type of tensors $A^{\alpha}_{\cdot\beta\gamma}$, $A^{\alpha\beta}_{\cdot\cdot\gamma}$ and the contravariant tensor $A^{\alpha\beta\gamma}$.

Theorem 3.3. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the tensor $A^{\alpha}_{\cdot\beta\gamma}$, the derived group is a subgroup of motions on I_n, and itself is a group of motions if rank the of the matrix $\|c^{\varepsilon}_{ab}\|$ is equal to r.*

Proof. As r vectors $\eta^{\alpha}_{(a)}$ $(a=1, 2, \cdots, r)$ generate infinitesimal rotations on I_n, we have the following equation which is equivalent to the fundamental equation (3.1):

$$\mathfrak{L}_a g_{\alpha\beta} = 2\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.7}$$

Therefore, we may put the right hand side of (3.4) as follows:

$$c^{a}_{bc}\mathfrak{L}_a g_{\alpha\beta} = 2c^{a}_{bc}\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.8}$$

On the other hand, if we assume that $\mathfrak{L}_a A^{\alpha}_{\cdot\beta\gamma}=0$ $(a=1, 2, \cdots, r)$, by means of the symmetric property of $A_{\alpha\beta\gamma}$ and (3.1), the left member of (3.4) is reduced as follows:

$$\begin{aligned}(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta} &= \mathfrak{L}_b(2\eta_{(c)\gamma}A^{\gamma}_{\cdot\alpha\beta}) - \mathfrak{L}_c(2\eta_{(b)\gamma}A^{\gamma}_{\cdot\alpha\beta}) \\ &= 2(\eta_{(c)\gamma;\varepsilon}\eta^{\varepsilon}_{(b)} - \eta_{(c)\varepsilon;\gamma}\eta^{\varepsilon}_{(b)} + \eta^{\varepsilon}_{(c)}\eta_{(b)\varepsilon;\gamma} - \eta^{\varepsilon}_{(c)}\eta_{(b)\gamma;c})A^{\gamma}_{\cdot\alpha\beta} \\ &= 4(\eta_{(c)\gamma;\varepsilon}\eta^{\varepsilon}_{(b)} - \eta^{\varepsilon}_{(c)}\eta_{(b)\gamma;\varepsilon})A^{\gamma}_{\cdot\alpha\beta}\,.\end{aligned} \tag{3.9}$$

However, as r vectors $\eta^{\alpha}_{(a)}$ $(a=1, 2, \cdots, r)$ satisfy the Maurer-Cartan equation

$$\eta^{\varepsilon}_{(b)}\frac{\partial\eta^{\gamma}_{(c)}}{\partial u^{\varepsilon}} - \eta^{\varepsilon}_{(c)}\frac{\partial\gamma^{\gamma}_{(b)}}{\partial u^{\varepsilon}} = c^{a}_{bc}\eta^{\gamma}_{(a)}\,,$$

we obtain from (3.9)

$$(\mathfrak{L}_b, \mathfrak{L}_c)g_{\alpha\beta} = 4c^{a}_{bc}\eta_{(a)\gamma}A^{\gamma}_{\cdot\alpha\beta}\,. \tag{3.10}$$

Making use of (3.4), (3.8) and (3.10), it follows that

$$\mathfrak{L}_{bc}g_{\alpha\beta} = 2c^{a}_{bc}\eta^{\gamma}_{(a)}A_{\alpha\beta\gamma} = 0\,.$$

The last equation implies the result of Theorem 3.3.

Theorem 3.4. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the tensor $A^{\alpha\beta}_{\cdot\cdot\gamma}$, the derived group is a subgroup of motions on I_n, and itself is a group of motions if the rank of the matrix $\|c^{\varepsilon}_{ab}\|$ is equal to r.*

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\delta}$ and summing for α, we have

$$g^{\alpha\delta}(\mathfrak{L}_a g_{\alpha\beta}) = 2\eta_{(a)\gamma} A^{\gamma\delta}_{\cdot\cdot\beta}. \tag{3.11}$$

Because of our assumption $\mathfrak{L}_a A^{\gamma\delta}_{\cdot\cdot\beta}=0$ $(a=1, 2, \cdots, r)$, from (3.11) it follows that

$$\mathfrak{L}_b g^{\alpha\delta}\mathfrak{L}_a g_{\alpha\beta} - \mathfrak{L}_a g^{\alpha\delta}\mathfrak{L}_b g_{\alpha\beta} + g^{\alpha\delta}(\mathfrak{L}_b, \mathfrak{L}_a) g_{\alpha\beta} = 2A^{\gamma\delta}_{\cdot\cdot\beta}(\mathfrak{L}_b \eta_{(a)\gamma} - \mathfrak{L}_a \eta_{(b)\gamma}).$$

On the other hand it is evident that we have from (3.1)

$$\mathfrak{L}_a g^{\alpha\beta} = -2\eta^{\gamma}_{(a)} A^{\alpha\beta}_{\cdot\cdot\gamma}. \tag{3.12}$$

Making use of (3.4), (3.12) and the Maurer-Cartan equation, the above equation gives us

$$4\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)} S_{\alpha\beta\gamma\delta} = 2c^{c}_{ba}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma}. \tag{3.13}$$

However, from the definition of $S_{\alpha\beta\gamma\delta}$, we must have $S_{\alpha\beta\gamma\delta}=-S_{\beta\alpha\gamma\delta}$. Since $A_{\alpha\beta\gamma}$ is a symmetric covariant tensor, comparing both sides of (3.13), we should have

$$\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)} S_{\alpha\beta\gamma\delta}=0 \quad \text{and} \quad c^{c}_{ba}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma}=0.$$

The above equations enable us to obtain the result of Theorem 3.4.

Finally, with regard to the contravariant tensor $A^{\alpha\beta\gamma}$, we have the following

Theorem 3.5. *If the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is less than n and an r-parameter group G_r of infinitesimal rotations on I_n leaves invariant the tensor $A^{\alpha\beta\gamma}$, the derived group is a subgroup of motions on I_n, and itself is a group of motions if the rank of the matrix $\|c^{e}_{ab}\|$ is equal to r.*

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\delta}$ and $g^{\beta\varepsilon}$, and summing with respect to α and β, it follows that

$$g^{\alpha\delta} g^{\beta\varepsilon}\mathfrak{L}_a g_{\alpha\beta} = 2\eta_{(a)\gamma} A^{\gamma\delta\varepsilon}. \tag{3.14}$$

Because of our assumption $\mathfrak{L}_a A^{\gamma\delta\varepsilon}=0$, we have from (3.14)

$$\begin{aligned}(\mathfrak{L}_b g^{\alpha\delta}) g^{\beta\varepsilon}(\mathfrak{L}_a g_{\alpha\beta}) - (\mathfrak{L}_a g^{\alpha\delta}) g^{\beta\varepsilon}(\mathfrak{L}_b g_{\alpha\beta}) + g^{\alpha\delta}(\mathfrak{L}_b g^{\beta\varepsilon})(\mathfrak{L}_a g_{\alpha\beta}) - g^{\alpha\delta}(\mathfrak{L}_a g^{\beta\varepsilon})(\mathfrak{L}_b g_{\alpha\beta}) \\ = 2A^{\gamma\delta\varepsilon}(\mathfrak{L}_b \eta_{(a)\gamma} - \mathfrak{L}_a \eta_{(b)\gamma}) - g^{\alpha\delta} g^{\beta\varepsilon}(\mathfrak{L}_b, \mathfrak{L}_a) g_{\alpha\beta}.\end{aligned}$$

Substituting (3.12) into the above equation, and making use of (3.4) and the Maurer-Cartan equation, we find that

$$4\eta^{\gamma}_{(a)}\eta^{\delta}_{(b)}(S^{\alpha\beta}_{\cdot\cdot\gamma\delta} + S^{\beta\alpha}_{\cdot\cdot\gamma\delta}) = 2c^{c}_{ba}\eta^{\gamma}_{(c)} A^{\alpha\beta}_{\cdot\cdot\gamma}. \tag{3.15}$$

Since $S^{\alpha\beta}_{\cdot\cdot\gamma\delta}=-S^{\beta\alpha}_{\cdot\cdot\gamma\delta}$ holds good, (3.15) gives rise to the relation $c^{c}_{ba}\eta^{\gamma}_{(c)} A_{\alpha\beta\gamma}=0$. This implies the result of Theorem 3.5.

§ 4. Special calsses of infinitesimal rotations on I_n. Since A_α is a gradient vector [6], $A_{\alpha;\beta}=A_{\beta;\alpha}$ holds good, where $A_\alpha=g^{\beta\gamma}A_{\alpha\beta\gamma}$. Then it follows that

$$\mathfrak{L}A_\alpha=A_{\alpha;\beta}\eta^\beta+A_\beta\eta^\beta_{;\alpha}=(A_\beta\eta^\beta)_{;\alpha}\,. \tag{4.1}$$

On the other hand, if η^α is the generating vector of an infinitesimal rotation on I_n, by means of the fundamental relation (1.2), we get $\eta^\delta_{;\delta}=A_\delta\eta^\delta$. As I_n is a compact Riemannian space, if it is an orientable, in consequence of the theorem of Green, we find from (4.1) and the last relation that

$$\int_{I_n}\eta^\delta_{;\delta}d\sigma=\int_{I_n}A_\delta\eta^\delta d\sigma=0\,. \tag{4.2}$$

If the transformation preserves the vector A_α, that is to say, $\mathfrak{L}A_\alpha=0$, by means of (4.1) it should be satisfied that $(A_\beta\eta^\beta)_{;\alpha}=0$, i.e. $A_\beta\eta^\beta=\text{const.}$ Hence, from (4.2) we get $A_\delta\eta^\delta=0$.

Conversely, from (4.1) it is evident that if $A_\delta\eta^\delta=0$ holds good, we have $\mathfrak{L}A_\delta=0$. Consequently we have the following theorem:

Theorem 4.1. *In order that an infinitesimal rotation on I_n preserves the covariant vector A_α, it is necessary and sufficient that the generating vector of the transformation be orthogonal to the vector A^α.*

Let us suppose that an infinitesimal rotation generated by a vector η^α is a motion on I_n. Then, we should have

$$\mathfrak{L}g_{\alpha\beta}=2\eta^\gamma A_{\alpha\beta\gamma}=0\,. \tag{4.3}$$

In this case the vector η^α satisfies the relation $\eta^\gamma A_\gamma=0$. In consequence of Theorem 4.1, this implies that $\mathfrak{L}A_\delta=0$ holds good. Therefore we have

Theorem 4.2. *If an infinitesimal rotation on I_n is a motion on I_n, the generating vector is orthogonal to the vector A^α and the transformation preserves the vector A^α.*

According to the definition of Lie derivatives, we have

$$\mathfrak{L}A_{\alpha\beta\gamma}=A_{\alpha\beta\gamma;\delta}\eta^\delta+A_{\delta\beta\gamma}\eta^\delta_{;\alpha}+A_{\alpha\delta\gamma}\eta^\delta_{;\beta}+A_{\alpha\beta\delta}\eta^\delta_{;\gamma}\,.$$

However, by means of the fundamental equation (1.2), we can obtain

$$A_{\delta\beta\gamma}\eta^\delta_{;\alpha}=\frac{1}{2}(\mathfrak{L}g_{\beta\gamma})_{;\alpha}-\eta^\delta A_{\delta\beta\gamma;\alpha}\,. \tag{4.4}$$

Making use of the relation $A_{\alpha\beta\gamma;\delta}=A_{\alpha\beta\delta;\gamma}$ (cf. § 1) and (4.4) we find that

$$\mathfrak{L}A_{\alpha\beta\gamma}=-2A_{\alpha\beta\gamma;\delta}\eta^\delta+\frac{1}{2}[(\mathfrak{L}g_{\beta\gamma})_{;\alpha}+(\mathfrak{L}g_{\alpha\gamma})_{;\beta}+(\mathfrak{L}g_{\alpha\beta})_{;\gamma}]\,.$$

Therefore, if $A_{\alpha\beta\gamma;\delta}=0$ holds good in I_n, $\mathfrak{L}g_{\alpha\beta}=0$ gives rise to the relation $\mathfrak{L}A_{\alpha\beta\gamma}=0$.

On the other hand, considering I_n as an affine hypersurface in an $n+1$-dimensional affine space and applying the theory of affine differential geometry, as A. Kawaguchi has shown in [6], it follows that

$$\mathfrak{A}_{\alpha\beta\gamma}=-\left\{A_{\alpha\beta\gamma}-\frac{1}{n+2}(g_{\alpha\beta}A_\gamma+g_{\beta\gamma}A_\alpha+g_{\gamma\alpha}A_\beta)\right\}, \tag{4.5}$$

$$\mathfrak{A}_{\alpha\beta\gamma}{}^{\gamma}=\mathfrak{A}_{\alpha\beta\gamma;}{}^{\gamma}-\frac{1}{2}(2-n)\mathfrak{A}_{\alpha\beta\gamma}\sigma^\gamma, \tag{4.6}$$

where $\mathfrak{A}_{\alpha\beta\gamma}{}^{\gamma}\equiv\mathfrak{A}_{\alpha\beta\gamma\delta}g^{\gamma\delta}$, and $\mathfrak{A}_{\alpha\beta\gamma\delta}$ denotes covariant derivative of $\mathfrak{A}_{\alpha\beta\gamma}$ with respect to $\mathfrak{G}_{\alpha\beta}$.[5] According to the result obtained by A. Kawaguchi, if $\mathfrak{A}_{\alpha\beta\gamma}{}^{\gamma}=0$ holds good at every point on I_n, then I_n $(n\geqq 2)$ is a hyperquadric. Therefore, in consequence of (4.5) and (4.6), if $A_{\alpha\beta\gamma;\delta}=0$ holds good in I_2, then the considered Minkowski space M_3 is equivalent with a Euclidean space. Thus we have

Theorem 4.3. *When $A_{\alpha\beta\gamma;\delta}=0$ is satisfied in I_n $(n\geqq 3)$, if an infinitesimal rotation is a motion in I_n, the transformation leaves invarian the tensor $A_{\alpha\beta\gamma}$.*

In the previous paper [7], the present author has shown that the system of linear partial differential equations determining the group of infinitesimal rotations on I_n is completely integrable and then the maximum order of the group of infinitesimal rotations on I_n is equal to $\frac{1}{2}n(n+1)$.

If an infinitesimal rotation on I_n coincides with a motion on I_n, the generating vector η^α should satisfy the condition $\eta^\gamma A_{\alpha\beta\gamma}=0$. Now, let us suppose that the rank of the matrix $\|A_{\alpha\beta\gamma}\|$ is equal to p, then there exist p independent conditions $\eta^\gamma A_{\alpha\beta\gamma}=0$. Therefore, the order of a subgroup of motions does not exceed $\frac{1}{2}n(n+1)-p$. However, as was stated in §2, since $p\geqq 2$ we can see that the order of the subgroup of motions is less than $\frac{1}{2}n(n+1)-1$.

On the other hand, according to the theorem proved by K. Yano [10], in an n-dimensional Riemannian space for $n\neq 4$, there exists no group of motions of order r such that

$$\frac{1}{2}n(n-1)+1<r<\frac{1}{2}n(n+1)$$

5) $\mathfrak{G}_{\alpha\beta}$ denotes the affine fundamental quantity of the second order, and $\mathfrak{A}_{\alpha\beta\gamma}$ the fundamental tensor of the third order defined on I_n. Cf. W. Blaschke [1].

and if the Riemannian space admits a group of motions of order $\frac{1}{2}n(n-1)+1$, then the group is transitive.

If we call our attention to the fact that a group of infinitesimal rotations on I_n is intransitive [6], we have the following

Theorem 4.4. *In I_n $(n \neq 4)$, the order of the subgroup of motions, contained in the group of infinitesimal rotations on I_n, does not exceed $\frac{1}{2}n(n-1)$.*

Remembering the fact that the indicatrix I_n is a compact Riemannian space, in consequence of the theorems concerning affine motion or homothetic transformation in a compact Riemannian space [2], [8], we get

Lemma 1. *If an infinitesimal rotation on I_n is an affine motion, it is necessarily a motion on I_n.*

Lemma 2. *If an infinitesimal rotation on I_n is a homothetic transformation, it is necessarily a motion on I_n.*

Now, we shall consider the case when an infinitesimal rotation on I_n is a conformal transformation on I_n. Then according to the theory of Lie derivatives, it should be satisfied that [9]

$$\mathfrak{L} g_{\alpha\beta} = 2\eta^{\gamma} A_{\alpha\beta\gamma} = 2\phi g_{\alpha\beta}, \tag{4.7}$$

where ϕ is a scalar function. Contracting $g^{\alpha\beta}$ to the second and third terms of (4.7), we have $\eta^{\gamma} A_{\gamma} = n\phi$. Then as $A_{\gamma;\delta} = A_{\delta;\gamma}$ it follows that

$$\mathfrak{L} A_{\delta} = A_{\delta;\gamma}\eta^{\gamma} + A_{\gamma}\eta^{\gamma}_{;\delta} = n\phi_{;\delta}.$$

The last relation shows us that if the transformation leaves invariant the covariant vector A_{δ}, the scalar function ϕ becomes constant, and from (4.7) the transformation is a homothetic one. Then, by means of Lemma 2, we have

Theorem 4.5. *When an infinitesimal rotation on I_n is a conformal transformation, if the transformation preserves the covariant vector A_{δ}, it is necessarily a motion.*

Next, we shall consider the case when an infinitesimal rotation on I_n is a projective transformation on I_n. In this case it must be satisfied that [9]

$$\mathfrak{L}\left\{{\alpha \atop \beta\gamma}\right\} = \delta^{\alpha}_{\beta}\varphi_{\gamma} + \delta^{\alpha}_{\gamma}\varphi_{\beta}, \tag{4.8}$$

where φ_{α} be an arbitrary covariant vector.

On the other hand, if the vector η^{α} is the generating vector of an infinitesimal rotation on I_n, by means of (1.2), we have from the symmetric properties of $A_{\alpha\beta\gamma}$ and the relation $A_{\alpha\beta\gamma;\delta} = A_{\alpha\beta\delta;\gamma}$,

$$\mathfrak{L}\left\{{\alpha \atop \beta\gamma}\right\}=\frac{1}{2}g^{\alpha\delta}[(\mathfrak{L}g_{\delta\beta})_{;\gamma}+(\mathfrak{L}g_{\delta\gamma})_{;\beta}-(\mathfrak{L}g_{\beta\gamma})_{;\delta}]$$
$$=A^{\alpha}_{\cdot\beta\gamma;\delta}\eta^{\delta}+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,. \tag{4.9}$$

Comparing (4.8) with (4.9) we find that

$$\delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}=A^{\alpha}_{\cdot\beta\gamma;\delta}\eta^{\delta}+\eta^{\delta}_{;\lambda}(\delta^{\lambda}_{\gamma}A^{\alpha}_{\cdot\beta\delta}+\delta^{\lambda}_{\beta}A^{\alpha}_{\cdot\gamma\delta}-g^{\alpha\lambda}A_{\beta\gamma\delta})\,. \tag{4.10}$$

Therefore, if we contract α and β in (4.10), it follows that $(n+1)\varphi_{\gamma}=\mathfrak{L}A_{\gamma}$. The last relation gives us that if the transformation leaves invariant A_{γ}, it becomes that $\varphi_{\gamma}=0$, and from (4.8) we have $\mathfrak{L}\left\{{\alpha \atop \beta\gamma}\right\}=0$. Therefore the transformation becomes an affine motion. Then, by means of Lemma 1, we have

Theorem 4.6. *When an infinitesimal rotation on I_n is a projective transformation, if the transformation preserves the covariant vector A_{δ}, it is necessarily a motion.*

If an n-dimensional Riemannian space I_n is a space of constant curvature, I_n is equivalent with a hypersphere in an $n+1$-dimensional Euclidean space and the considered Minkowski space M_{n+1} is essentially a Euclidean space [6]. Although a space of constant curvature is an Einstein space, there exists an Einstein space which is not a space of constant curvature. Then, we shall consider the case when I_n is an Einstein space.

If an infinitesimal rotation $\overline{u}^{\alpha}=u^{\alpha}+\eta^{\alpha}(u)\delta t$ is a motion or a conformal transformation, as η^{α} satisfies (4.3) or (4.7) respectively, it follows that $\eta^{\alpha}S_{\alpha\beta\gamma\delta}=0$. Then, we have from (1.1)

$$\eta^{\alpha}R_{\alpha\beta\gamma\delta}=\eta^{\alpha}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma})\,. \tag{4.11}$$

Multiplying (4.11) by $g^{\beta\gamma}$ and summing for β and γ, we get $\eta^{\alpha}\{R_{\alpha\gamma}-(1-n)g_{\alpha\gamma}\}=0$. If I_n is an Einstein space, substituting the relation $R_{\alpha\gamma}=\frac{R}{n}g_{\alpha\gamma}$ into the last equation, we get $\eta_{\gamma}\{R-n(1-n)\}=0$. Hence, at any point, if $R\neq n(1-n)$, then η_{γ} must be zero at the same point. Thus we have

Theorem 4.7. *If I_n is an Einstein space and $R\neq n(1-n)$ is satisfied at every point on I_n, an infinitesimal rotation does not coincide with a motion or a conformal transformation with respect to the metric on I_n.*

K. Yano and T. Nagano [12] has shown the following theorem concerning the transformation in a compact Einstein space:

Theorem I. *In a compact Einstein space with negative scalar curvature, there does not exist an infinitesimal projective transformation.*

Also, T. Sumitomo [8] has proved the following theorem concerning the conformal transformation in a compact Riemannian space:

Theorem II. *In a compact Riemannian manifold M with non-pasitive constant scalar curvature: $R=const.\leqq 0$, $C_0(M)$ coincides with $I_0(M)$.*[6]

In consequence of Theorem I and Theorem II, we have

Theorem 4.8. *If I_n is an Einstein space with $R=n(1-n)$, an infinitesimal rotation on I_n does not coincide with a conformal and a projective transformation in I_n.*

§ 5. Order of groups of infinitesimal rotations on I_n preserving $A_{\alpha\beta\gamma}$.

In the following, we shall consider the groups of infinitesimal rotations on I_n such that each element of the groups preserves one of the tensors $A_{\alpha\beta\gamma}$, $A^{\alpha}_{\cdot\beta\gamma}$, $A^{\alpha\beta}_{\cdot\cdot\gamma}$ and $A^{\alpha\beta\gamma}$.[7]

Case I. $\mathfrak{L}A^{\alpha}_{\cdot\beta\gamma}=0$. In this case we have

$$A^{\cdot\cdot\alpha}_{\beta\gamma;\delta}\eta^{\delta}+A^{\cdot\cdot\alpha}_{\delta\gamma}\eta^{\delta}_{;\beta}+A^{\cdot\cdot\alpha}_{\beta\delta}\eta^{\delta}_{;\gamma}-A^{\cdot\cdot\delta}_{\beta\gamma}\eta^{\alpha}_{;\delta}=0\,.$$

Putting

$$T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}=\delta^{\varepsilon}_{\beta}A^{\cdot\cdot\alpha}_{\delta\gamma}+\delta^{\varepsilon}_{\gamma}A^{\cdot\cdot\alpha}_{\beta\delta}-\delta^{\alpha}_{\delta}A^{\cdot\cdot\varepsilon}_{\beta\gamma}\,, \tag{5.1}$$

and we denote by $\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}\|$ the matrix of elements $T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}$, where $\binom{\varepsilon}{\delta}$ denotes the columns and $\binom{\alpha}{\beta\gamma}$ the rows.

Lemma 1. *If the rank of the matrix $\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\gamma}\|$ is less than n, it follows that $A^{\cdot\cdot\alpha_1}_{\alpha_3\alpha_2}=0$.*

Proof. From (5.1) we can easily see that $T\binom{\alpha_1}{\alpha_j}\binom{\alpha_i}{\alpha_3\alpha_2}=-\delta^{\alpha_i}_{\alpha_j}A^{\cdot\cdot\alpha_1}_{\alpha_3\alpha_2}$,[8] where $\alpha_i\neq\alpha_j$ for $i\neq j$. Then we can select from $\|T\binom{\varepsilon}{\delta}\binom{\alpha}{\beta\delta}\|$ the following n-rowed square matrix:

$$\begin{array}{c|cccc}
 & \binom{\alpha_1}{\alpha_2}_{\alpha_3} & \binom{\alpha_2}{\alpha_2}_{\alpha_3} & \cdots\cdots & \binom{\alpha_n}{\alpha_2}_{\alpha_3} \\
\hline
\binom{\alpha_1}{\alpha_1} & -A^{\cdot\cdot\alpha_1}_{\alpha_3\alpha_2} & 0 & \cdots\cdots & 0 \\
\binom{\alpha_1}{\alpha_2} & 0 & -A^{\cdot\cdot\alpha_1}_{\alpha_3\alpha_2} & & \vdots \\
\vdots & \vdots & & * & \vdots \\
 & & & * & \\
 & & & * & \\
\binom{\alpha_1}{\alpha_n} & 0 & 0 & \cdots\cdots & -A^{\cdot\cdot\alpha_1}_{\alpha_3\alpha_2}
\end{array}\,.$$

6) $I_0(M)$ denotes the connected component of the identical transformations in $I(M)$, where $I(M)$ is the totality of isometric transformations of M. $C_0(M)$ denotes the connected component of the conformal transformations in $C(M)$, where $C(M)$ is the totality of conformal transformations of M.

7) In the theory of groups in an asymmetric affine connection space, I. P. Egorov [3] proved that "If the rank of matrix $\|T\binom{\rho}{\sigma}\binom{\kappa}{\mu\lambda}\|\equiv\|\delta^{\sigma}_{\mu}S^{\cdot\cdot\kappa}_{\mu\sigma}+\delta^{\rho}_{\lambda}S^{\cdot\cdot\kappa}_{\mu\sigma}-\delta^{\kappa}_{\sigma}S^{\cdot\cdot\rho}_{\mu\lambda}\|$ is less than n, then $S^{\cdot\cdot\kappa}_{\mu\lambda}=0$.", where $S^{\cdot\cdot\kappa}_{\mu\lambda}=\frac{1}{2}(\Gamma^{\kappa}_{\mu\lambda}-\Gamma^{\kappa}_{\lambda\mu})$ and $\Gamma^{\kappa}_{\mu\lambda}$ are parameters of an affine connection. The methods of our proof are analogous as those of I. P. Egorov.

8) Indices α_i, α_j take the values $1, 2, \cdots, n$.

Therefore, if the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $(-A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1})^n=0$ and this implies $A_{\alpha_3\alpha_2}^{\cdot\cdot\alpha_1}=0$. Q.E.D.

Lemma 2. *If the rank of the matrix* $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2}=0$.[9]

Proof. Noting that $\alpha_1 \neq \alpha_2$, from the symmetric properties of $A_{\beta\gamma}^{\cdot\cdot\alpha}$, we have from (5.1) that $T(^{\alpha_i}_{\alpha_1})(^{\alpha_2}_{\alpha_j\alpha_2})=\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2}+\delta^{\alpha_i}_{\alpha_2}A_{\alpha_1\alpha_j}^{\cdot\cdot\alpha_2}$. Then, if we suppose that $\alpha_1<\alpha_2$, we can select from $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|ccccccccc}
 & (^{\alpha_2}_{1\ \alpha_2}) & (^{\alpha_2}_{2\ \alpha_2}) & \cdots & (^{\alpha_2}_{\alpha_1\ \alpha_2}) & \cdots & (^{\alpha_2}_{\alpha_2\ \alpha_2}) & \cdots & (^{\alpha_2}_{n\ \alpha_2}) \\
\hline
(^{1}_{\alpha_1}) & A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
(^{2}_{\alpha_1}) & 0 & A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
(^{\alpha_1}_{\alpha_1}) & \vdots & & & A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2} & & & & \vdots \\
\vdots & 0 & & & & * & & & \vdots \\
(^{\alpha_2}_{\alpha_1}) & A_{1\alpha_1}^{\cdot\cdot\alpha_2} & A_{2\alpha_1}^{\cdot\cdot\alpha_2} & * & A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_2} & * & 2A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2} & * & A_{n\alpha_1}^{\cdot\cdot\alpha_2} \\
\vdots & \vdots & & & & & & * & 0 \\
(^{n}_{\alpha_1}) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2}
\end{array}
$$

Therefore, by virtue of Lemma 1, if the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $2(A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2})^n=0$ and this implies $A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2}=0$. By the same way we can prove $A_{\alpha_1\alpha_2}^{\cdot\cdot\alpha_2}=0$ for the case $\alpha_1>\alpha_2$. Q.E.D.

Lemma 3. *If the rank of the matrix* $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1}=0$.

Proof. Noting that $\alpha_1 \neq \alpha_2$, from the symmetric properties of $A_{\beta\gamma}^{\cdot\cdot\alpha}$, we have from (5.1) that $T(^{\alpha_1}_{\alpha_j})(^{\alpha_i}_{\alpha_2\alpha_2})=-\delta^{\alpha_i}_{\alpha_j}A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1}$. Then, if we suppose that $\alpha_1<\alpha_2$, we can select from $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|ccccccccc}
 & (^{\ 1}_{\alpha_2\ \alpha_2}) & (^{\ 2}_{\alpha_2\ \alpha_2}) & \cdots & (^{\alpha_1}_{\alpha_2\ \alpha_2}) & \cdots & (^{\alpha_2}_{\alpha_2\ \alpha_2}) & \cdots & (^{\ n}_{\alpha_2\ \alpha_2}) \\
\hline
(^{\alpha_1}_{1}) & -A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
(^{\alpha_1}_{2}) & 0 & -A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
(^{\alpha_1}_{\alpha_1}) & \vdots & & & -A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1} & & & & \vdots \\
\vdots & \vdots & & & & * & & & \vdots \\
(^{\alpha_1}_{\alpha_2}) & \vdots & & & & & -A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1} & & \vdots \\
\vdots & \vdots & & & & & & * & 0 \\
(^{\alpha_1}_{n}) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1}
\end{array}
$$

9) In the following, we do not sum up with respect to the indices α_i $(i=1, 2, \cdots, n)$.

Therefore, if the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $(-A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1})^n=0$ and this implies $A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1}=0$. By the same way, we can prove $A_{\alpha_2\alpha_2}^{\cdot\cdot\alpha_1}=0$ for the case $\alpha_1>\alpha_2$. Q.E.D.

Lemma 4. *If the rank of the matrix* $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1}=0$.

Proof. From (5.1), it follows that $T(^{\alpha_1}_{\alpha_i})(^{\alpha_j}_{\alpha_1\alpha_1})=2A_{\alpha_1\alpha_j}^{\cdot\cdot\alpha_j}-\delta^{\alpha_j}_{\alpha_i}A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1}$. Then we can select from $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|cccccc}
 & (^{1}_{\alpha_1\alpha_1}) & (^{2}_{\alpha_1\alpha_1}) & \cdots\cdots & (^{\alpha_1}_{\alpha_1\alpha_1}) & \cdots\cdots & (^{n}_{\alpha_1\alpha_1}) \\
\hline
(^{\alpha_1}_{1}) & -A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1} & 0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & 0 \\
(^{\alpha_1}_{2}) & 0 & A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1} & & & & \vdots \\
\vdots & \vdots & & * & & & \vdots \\
(^{\alpha_1}_{\alpha_1}) & \vdots & & & A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1} & & \vdots \\
\vdots & \vdots & & & & * & 0 \\
(^{\alpha_1}_{n}) & 0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & 0 & -A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1}
\end{array}
$$

Therefore, if the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ is less than n, we must have $-(-A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1})^n=0$ and this implies $A_{\alpha_1\alpha_1}^{\cdot\cdot\alpha_1}=0$. Q.E.D.

Summarizing the results of above Lemmas, we see that if the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha}_{\beta\gamma}) \|$ is less than n, it follows that $A_{\beta\gamma}^{\cdot\cdot\alpha}=0$ for every α, β and γ. Consequently, as the maximum order of groups of the infinitesimal rotations on I_n is equal to $\frac{1}{2}n(n+1)$ [7], it follows that

Theorem 5.1. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A_{\beta\gamma}^{\cdot\cdot\alpha}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case II. $\mathfrak{L}A^{\alpha\beta}_{\cdot\cdot\gamma}=0$. In this case we have

$$A^{\alpha\beta}_{\cdot\cdot\gamma;\delta}\eta^{\delta}-A^{\varepsilon\beta}_{\cdot\cdot\gamma}\eta^{\alpha}_{;\varepsilon}-A^{\alpha\varepsilon}_{\cdot\cdot\gamma}\eta^{\beta}_{;\varepsilon}+A^{\alpha\beta}_{\cdot\cdot\delta}\eta^{\delta}_{;\gamma}=0\,.$$

We put $T(^{\varepsilon}_{\delta})(^{\alpha\beta}_{\gamma})=\delta^{\alpha}_{\delta}A^{\varepsilon\beta}_{\cdot\cdot\gamma}+\delta^{\beta}_{\delta}A^{\alpha\varepsilon}_{\cdot\cdot\gamma}-\delta^{\varepsilon}_{\beta}A^{\alpha\beta}_{\cdot\cdot\delta}$, and consider the rank of the matrix $\| T(^{\varepsilon}_{\delta})(^{\alpha\beta}_{\gamma}) \|$. Making use of the analogous way as those in Case I, we obtain

Lemma 5. *If the rank of the matrix* $\| T(^{\varepsilon}_{\delta})(^{\alpha\beta}_{\gamma}) \|$ *is less than* n, *it follows that* $A^{\alpha\beta}_{\cdot\cdot\gamma}=0$.

Consequently, we obtain the following

Theorem 5.2. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A^{\alpha\beta}_{\cdot\cdot\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case III. $\mathfrak{L}A_{\alpha\beta\gamma}=0$. In this case we have

$$A_{\alpha\beta\gamma;\delta}\eta^{\delta}+A_{\delta\beta\gamma}\eta^{\delta}_{;\alpha}+A_{\alpha\delta\gamma}\eta^{\delta}_{;\beta}+A_{\alpha\beta\delta}\eta^{\delta}_{;\gamma}=0\,.$$

We put

$$T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\equiv\delta^{\lambda}_{\alpha}A_{\delta\beta\gamma}+\delta^{\lambda}_{\beta}A_{\alpha\delta\gamma}+\delta^{\lambda}_{\gamma}A_{\alpha\beta\delta} \tag{5.2}$$

and denote by $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the matrix of elements $T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})$, where $\binom{\lambda}{\delta}$ denotes the columns and $(_{\alpha\beta\gamma})$ the rows.

Lemma 6. *If the rank of the matrix* $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1\alpha_1}=0$.

Proof. Making use of the symmetric properties of $A_{\alpha\beta\gamma}$, we have from (5.2) that $T\binom{\alpha_i}{\alpha_1}(_{\alpha_1\alpha_1\alpha_j})=2\delta^{\alpha_i}_{\alpha_1}A_{\alpha_1\alpha_1\alpha_j}+\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_1\alpha_1}$. Then, we can select from $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the following n-rowed square matrix:

$$\begin{array}{c|cccccc}
 & (_{\alpha_1\alpha_1 1}) & (_{\alpha_1\alpha_1 2}) & \cdots\cdots & (_{\alpha_1\alpha_1\alpha_1}) & \cdots\cdots & (_{\alpha_1\alpha_1 n}) \\
\hline
\binom{1}{\alpha_1} & A_{\alpha_1\alpha_1\alpha_1} & 0 & \cdots\cdots & & \cdots\cdots & 0 \\
\binom{2}{\alpha_1} & 0 & A_{\alpha_1\alpha_1\alpha_1} & & & & \vdots \\
\vdots & \vdots & & * & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & * & * & * & 3A_{\alpha_1\alpha_1\alpha_1} & * & \vdots \\
\vdots & \vdots & & & & * & \vdots \\
\binom{n}{\alpha_1} & 0 & \cdots\cdots & & & 0 & A_{\alpha_1\alpha_1\alpha_1}
\end{array}$$

Therefore, if the rank of the matrix $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ is less than n, we must have $3(A_{\alpha_1\alpha_1\alpha_1})^n=0$, i.e. $A_{\alpha_1\alpha_1\alpha_1}=0$. Q.E.D.

Lemma 7. *If the rank of the matrix* $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_1\alpha_2}=0$.

Proof. From (5.2), by means of the symmetric properties of $A_{\alpha\beta\gamma}$, we have $T\binom{\alpha_i}{\alpha_1}(_{\alpha_j\alpha_2\alpha_2})=\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_2\alpha_2}+2\delta^{\alpha_i}_{\alpha_2}A_{\alpha_1\alpha_2\alpha_j}$. Then, if we suppose that $\alpha_2<\alpha_2$, we can select from $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ the following n-rowed square matrix:

$$\begin{array}{c|ccccccccc}
 & (_{1\alpha_2\alpha_2}) & (_{2\alpha_2\alpha_2}) & \cdots\cdots & (_{\alpha_1\alpha_2\alpha_2}) & \cdots\cdots & (_{\alpha_2\alpha_2\alpha_2}) & \cdots\cdots & (_{n\alpha_2\alpha_2}) \\
\hline
\binom{1}{\alpha_1} & A_{\alpha_1\alpha_2\alpha_2} & 0 & \cdots\cdots & & & & & 0 \\
\binom{2}{\alpha_1} & 0 & A_{\alpha_1\alpha_2\alpha_2} & & & & & & \vdots \\
\vdots & \vdots & & * & & & & & \vdots \\
\binom{\alpha_1}{\alpha_1} & \vdots & & & A_{\alpha_1\alpha_2\alpha_2} & & & & \vdots \\
\vdots & \vdots & & & & * & & & \vdots \\
\binom{\alpha_2}{\alpha_1} & * & * & * & * & * & 3A_{\alpha_1\alpha_2\alpha_2} & * & * \\
\vdots & \vdots & & & & & & * & 0 \\
\binom{n}{\alpha_1} & 0 & \cdots\cdots & & & & & 0 & A_{\alpha_1\alpha_2\alpha_2}
\end{array}$$

Therefore, if the rank of the matrix $\|T\binom{\lambda}{\delta}(_{\alpha\beta\gamma})\|$ is less than n, we must have $3(A_{\alpha_1\alpha_2\alpha_2})^n=0$, i.e. $A_{\alpha_1\alpha_2\alpha_2}=0$. By the same way, we can prove $A_{\alpha_1\alpha_2\alpha_2}=0$ for the case $\alpha_1<\alpha_2$. Q.E.D.

Lemma 8. *If the rank of the matrix* $\| T^{\lambda}_{\delta}({}_{\alpha\beta\gamma}) \|$ *is less than* n, *it follows that* $A_{\alpha_1\alpha_2\alpha_3}=0$.

Proof. In consequence of the symmetric property of $A_{\alpha\beta\gamma}$, we may assume that $\alpha_1<\alpha_2<\alpha_3$ without loss of generality. Thus, we have from (5.2) that $T^{\alpha_i}_{\alpha_1}({}_{\alpha_j\alpha_2\alpha_3})=\delta^{\alpha_i}_{\alpha_j}A_{\alpha_1\alpha_2\alpha_3}+\delta^{\alpha_i}_{\alpha_2}A_{\alpha_j\alpha_1\alpha_3}+\delta^{i}_{3}A_{\alpha_j\alpha_2\alpha_1}$. Then we can select from $\| T^{\lambda}_{\delta}({}_{\alpha\beta\gamma}) \|$ the following n-rowed square matrix:

$$
\begin{array}{c|ccccccccc}
 & ({}_{1\alpha_1\alpha_2}) & ({}_{2\alpha_2\alpha_3}) & \cdots\cdots & ({}_{\alpha_1\alpha_2\alpha_3}) & \cdots\cdots & ({}_{\alpha_2\alpha_2\alpha_3}) & \cdots\cdots & ({}_{\alpha_3\alpha_2\alpha_3}) & \cdots\cdots & ({}_{n\alpha_2\alpha_3}) \\
\hline
({}^{1}_{\alpha_1}) & A_{\alpha_1\alpha_2\alpha_3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
({}^{2}_{\alpha_1}) & * & A_{\alpha_1\alpha_2\alpha_3} & & & & & & & & \vdots \\
\vdots & * & & * & & & & & & & \vdots \\
({}^{\alpha_1}_{\alpha_1}) & * & & & A_{\alpha_1\alpha_2\alpha_3} & & & & & & \vdots \\
\vdots & * & & & & * & & & & & \vdots \\
({}^{\alpha_2}_{\alpha_3}) & * & & & & & 2A_{\alpha_1\alpha_2\alpha_3} & & & & \vdots \\
\vdots & * & & & & & & * & & & \vdots \\
({}^{\alpha_3}_{\alpha_1}) & * & & & & & & & 2A_{\alpha_1\alpha_2\alpha_3} & & \vdots \\
\vdots & * & & & & & & & & * & 0 \\
({}^{n}_{\alpha_1}) & * & * & * & * & * & * & * & * & * & A_{\alpha_1\alpha_2\alpha_3}
\end{array}
$$

Therefore, if the rank of the matrix $\| T^{\lambda}_{\delta}({}_{\alpha\beta\gamma}) \|$ is less than n, we must have $4(A_{\alpha_1\alpha_2\alpha_3})^n=0$, i.e. $A_{\alpha_1\alpha_2\alpha_3}=0$. Q.E.D.

Summarizing the results of the above Lemmas, we see that if the rank of the matrix $\| T^{\lambda}_{\delta}({}_{\alpha\beta\gamma}) \|$ is less than n, then it follows that $A_{\alpha\beta\gamma}=0$ for every α, β and γ. Consequently we have

Theorem 5.3. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A_{\alpha\beta\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

Case IV. $\mathfrak{L}A^{\alpha\beta\gamma}=0$. In this case we can easily obtain the following theorem by means of the analogous way as in Case III:

Theorem 5.4. *The order of the group of infinitesimal rotations on* I_n, *preserving the tensor* $A^{\alpha\beta\gamma}$, *does not exceed* $\frac{1}{2}n(n-1)$.

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