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ON GROUPS OF ROTATIONS IN MINKOWSKI
SPACE II

By

Tamao NAGAI

§ 1. Introduction. The present paper is a continuation of the one
with the same title "On groups of rotations in Minkowski space I." [7]
Let $M_{n+1}$ be an $n+1$-dimensional Minkowski space whose indicatrix $I_n$

is defined by $F(X^1, X^2, \cdots, X^{n+1})=1$. As $I_n$ is an $n$-dimensional hypersurface

in $M_{n+1}$, we may represent $I_n$ by $n+1$ equations involving $n$ parameters as

$$X^i = X^i(u^\alpha). \quad (i=1, 2, \cdots, n+1; \alpha=1, 2, \cdots, n)$$

Putting $A_{ijk}(X) = F(X)C_{ijk}$, where $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial X^k}$ and $g_{ij} = \frac{1}{2} \frac{\partial^2 F}{\partial X^i \partial X^j}$,

$A_{ijk}$ is a symmetric covariant tensor in $M_{n+1}$. If we denote by $g_{a\beta}$ and $A_{a\beta\gamma}$ the induced components of $g_{ij}$ and $A_{ijk}$ respectively, i.e.

$$g_{a\beta} = g_{ij}X^i_a X^\beta_j, \quad A_{a\beta\gamma} = A_{ijk}X^i_a X^j_b X^k_c \quad (X^i_a = \partial X^i/\partial u^\alpha),$$

$I_n$ may be considered as an $n$-dimensional compact Riemannian space with the fundamental metric tensor $g_{a\beta}$, and the Riemannian space is characterized by existence of a symmetric covariant tensor $A_{a\beta\gamma}$. Moreover, it is remarkable that in $I_n$ we have

$$(1.1) \quad R_{a\beta\gamma\delta} = S_{a\beta\gamma\delta} + (g_{a\epsilon}g_{\beta\delta} - g_{a\delta}g_{\beta\gamma}),$$

where $R_{a\beta\gamma\delta} = g_{a\epsilon}R_{\epsilon\beta\gamma\delta}$ and

$$R_{a\beta\gamma\delta} = \frac{\partial^2 \{a\beta\gamma\delta\}}{\partial u^\epsilon} - \frac{\partial \{a\beta\gamma\delta\}}{\partial u^\epsilon} + \left\{a\beta\gamma\delta\right\} \left\{a\beta\gamma\delta\right\} - \left\{a\beta\gamma\delta\right\} \left\{a\beta\gamma\delta\right\}.$$

Also, $A_{a\beta\gamma}$ satisfies the relation $A_{a\beta\gamma;\delta} = A_{a\beta;\gamma}A_{\gamma\delta} - A_{a\gamma;\delta}A_{\beta\gamma}$ [6].

When a centro-affine transformation in $M_{n+1}$, with its centre at

origin, preserves $I_n$, the transformation is called a rotation in $M_{n+1}$.

1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout the present paper, the Greek indices $\alpha, \beta, \gamma, \cdots$ are supposed to run over the range $1, 2, \cdots, n$.

3) Semi-colon is used to represent the covariant differentiation with respect to the Christoffel symbols made by $g_{a\beta}$. 

By this transformation every point on $I_n$ is transformed into a point on the same. When an infinitesimal point transformation on $I_n$ coincides with the transformation of such a class, we call it an infinitesimal rotation on $I_n$ [7]. In the previous paper, the present author derived the fundamental equation of an infinitesimal rotation on $I_n$, i.e.

$$\mathcal{L}g_{\alpha\beta} = \eta_{\alpha;\beta} + \eta_{\beta;\alpha} - 2\eta^\gamma A_{\alpha\beta\gamma},$$

and discussed about the integrability conditions of the transformation, where $\mathcal{L}$ denotes Lie differential with respect to an infinitesimal transformation $\bar{u} = u^\alpha + \eta^\alpha(u)\delta t$.

The purpose of the present paper is to study the properties of infinitesimal rotations on $I_n$ and develop the structures of $r$-parameter Lie groups of the transformations, mainly in connection with the properties of the symmetric tensor $A_{\alpha\beta\gamma}$.

In § 2 we shall state a few remarks on the connections between properties of $I_n$ and those of the tensor $A_{\alpha\beta\gamma}$. § 3 contains theorems concerning with properties of $r$-parameter groups of infinitesimal rotations on $I_n$. In § 4 we shall discuss about the special cases in which the infinitesimal rotation on $I_n$ coincides with a motion or transformation of other classes. § 5 devoted to give the order of the groups when the infinitesimal rotations on $I_n$ preserve the tensor $A_{\alpha\beta\gamma}$.

The present author wishes to express his sincere thanks to Dr. A. Kawaguchi and Dr. Y. Katsurada for their constant guidances and criticisms, and also thanks to Mr. T. Sumitomo who gave the author many valuable suggestions.

§ 2. Preliminary remarks on the structures of $I_n$. Let us denote by $||A_{\alpha\beta\gamma}||$ the matrix with elements $A_{\alpha\beta\gamma}$, where $\gamma$ denotes the columns, and $\alpha$ and $\beta$ the rows. If we consider about the rank of the matrix $||A_{\alpha\beta\gamma}||$, the followings are exceptional cases in our discussions.

Case I. Rank $||A_{\alpha\beta\gamma}|| = 0$. In this case, at every point on $I_n$, $A_{\alpha\beta\gamma} = 0$ for every $\alpha, \beta$ and $\gamma$. Therefore, as we have $A_{ijk} = A_{\alpha\beta\gamma}X_i^\alpha X_j^\beta X_k^\gamma$ [6], it follows that $A_{ijk} = 0$ and this means that the considered Minkowski space is essentially a Euclidean space.

Case II. Rank $||A_{\alpha\beta\gamma}|| = 1$. In this case, we have

$$\det \begin{vmatrix} A_{\alpha\beta\gamma} & A_{\alpha\beta\omega} \\ A_{\gamma\delta\tau} & A_{\tau\delta\omega} \end{vmatrix} = 0. \quad (\alpha, \beta, \gamma, \delta, \tau, \omega = 1, 2, \cdots, n)$$

On the other hand, by means of the symmetric properties of $A_{\alpha\beta\gamma}$, it
is readily seen that
\[(2.2) \quad S_{a\beta\gamma\delta} = A_{a\delta}^{\sigma}A_{\sigma\gamma\beta} - A_{a\gamma}^{\sigma}A_{\sigma\delta\beta} = g^{\sigma\tau}(A_{a\tau\delta}A_{\beta\sigma\gamma} - A_{a\tau\gamma}A_{\beta\sigma\delta}).\]

Comparing (2.1) and (2.2), we find that $S_{a\beta\gamma\delta} = 0$. This also implies that the considered Minkowski space is essentially a Euclidean space \([6]\).

According to the definition, as a rotation in $M_{n+1}$ leaves invariant the indicatrix $I_n$, the transformation preserves the metric of $M_{n+1}$. However, as we see from the fundamental equation (1.2), in general an infinitesimal rotation on $I_n$ does not preserve the metric of $I_n$, which is induced from the one of $M_{n+1}$. If an infinitesimal rotation $\overline{u}^\alpha = u^\alpha + \gamma^\alpha(u)\delta t$ coincides with a motion in $I_n$, the generating vector $\gamma^\alpha$ should satisfy the equation $\gamma^\alpha A_{a\beta\gamma}=0$. Therefore if an infinitesimal rotation coincides with a motion in $I_n$, the rank of the matrix $||A_{a\beta\gamma}||$ is less than the maximum value $n$, and otherwise the transformation must be an identity transformation.

**Theorem 2.1.** If the rank of the matrix $||A_{a\beta\gamma}||$ is equal to the maximum value $n$, an infinitesimal rotation on $I_n$ does not coincide with a motion in $I_n$, except an identity transformation.

§ 3. Groups of infinitesimal rotations on $I_n$. Let us denote by $G_r$ the $r$-parameter group of infinitesimal rotations on $I_n$ and by $X_a f \equiv \eta_{(a)}^\alpha \frac{\partial f}{\partial u^\alpha}$ $(a=1, 2, \cdots, r)^4$ its $r$ linearly independent symbols. Then by means of the fundamental equations (1.2), we get
\[(3.1) \quad \mathfrak{L}_a g_{\alpha\beta} = \eta_{(a)\alpha,\beta} + \eta_{(a)\beta,\alpha} - 2\eta_{(a)\gamma}A_{a\beta\gamma},\]
where $\mathfrak{L}_a$ denotes Lie differentiation with respect to an infinitesimal transformation generated by the vector $\eta_{(a)}^\alpha$.

On the other hand, from the theory of Lie derivatives, we have
\[(3.2) \quad \mathfrak{L}_a \mathfrak{L}_b g_{\alpha\beta} = c_{ab}^\gamma \mathfrak{L}_\gamma g_{\alpha\beta},\]
\[(3.3) \quad (\mathfrak{L}_b, \mathfrak{L}_c) T_{\mu\nu}^\lambda \equiv (\mathfrak{L}_b \mathfrak{L}_c - \mathfrak{L}_c \mathfrak{L}_b) T_{\mu\nu}^\lambda = c_{bc}^e \mathfrak{L}_e T_{\mu\nu}^\lambda,\]
where $T_{\mu\nu}$ is an arbitrary tensor and $c_{ab}^\gamma$ is a structural constant of $G_r$ and satisfies
\[c_{ab}^\gamma = -c_{ba}^\gamma, \quad c_{ab}^\gamma c_{\beta\gamma}^\delta + c_{bc}^\gamma c_{\gamma\delta}^\alpha + c_{ca}^\gamma c_{\alpha\beta}^\delta = 0.\]

Applying the formula (3.3) to the fundamental tensor $g_{\alpha\beta}$, we get
\[(3.4) \quad (\mathfrak{L}_b, \mathfrak{L}_c) g_{\alpha\beta} = c_{bc}^\gamma \mathfrak{L}_\gamma g_{\alpha\beta}.\]

4) Throughout the present paper the Latin indices $a, b, c, \cdots$ take the values $1, 2, \cdots, r$. 

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Substituting (3.1) and (3.2) into the left hand side of (3.4), it follows that

\[
(Q_{b} Q_{c})g_{ab} = 2Q_{b}(\eta_{(a)}^{c} A_{ab}) - 2Q_{c}(\eta_{(b)}^{c} A_{ab}) \\
= 4c^{a}_{bc}\gamma_{(a)}^{c} A_{ab} + 2(\gamma_{(a)}^{c} Q_{b} A_{ab} - \gamma_{(b)}^{c} Q_{c} A_{ab}) .
\]

Since we have \(c_{bc}^{a} = 2c_{bc}^{a} A_{ab},\) in consequence of the above equation, (3.4) gives rise to the relation

\[
(3.5) \quad 2c^{a}_{bc}\gamma_{(a)}^{c} A_{ab} + 2(\gamma_{(a)}^{c} Q_{b} A_{ab} - \gamma_{(b)}^{c} Q_{c} A_{ab}) = 0 .
\]

If \(X_{a}f\) \((a = 1, 2, \cdots, r)\) are \(r\) generators of \(G_{r},\) it is satisfied that

\[
(X_{a}, X_{c})f = X_{c}X_{a}f - X_{a}X_{c}f = c_{bc}^{a}X_{a}f .
\]

Now, consider the symbols \(X_{bc}f\) \((b, c = 1, 2, \cdots, r)\) defined by \(X_{bc}f = (X_{b}, X_{c})f,\) then each of them determines a subgroup \(G_{b}\) of \(G_{r}\) and its generating vector is \(c_{bc}^{a}.\) Since \(c_{bc}^{a}\) is constant, it is evident that

\[
(3.6) \quad Q_{bc}g_{ab} = c_{bc}^{a}\gamma_{(a)}^{c} A_{ab} + c_{bc}^{a}\gamma_{(b)}^{c} A_{ab} = 2c_{bc}^{a}\gamma_{(a)}^{c} A_{ab} ,
\]

where \(Q_{bc}\) denotes Lie differentiation with respect to an infinitesimal transformation determined by a vector \(c_{bc}^{a}.\) By means of (3.5) and (3.6), if each transformation of \(G_{r}\) leaves invariant the tensor \(A_{ab},\) namely \(Q_{a} A_{ab} = 0 \) \((a = 1, 2, \cdots, r),\) we must have \(Q_{bc}g_{ab} = 0.\) Hence we have the following

**Theorem 3.1.** If the rank of the matrix \(\|A_{ab}\|\) is less than \(n\) and each one-parameter group \(G_{i},\) determined by \(r\) generators \(X_{a}f,\) leaves invariant the covariant tensor \(A_{ab},\) then \(X_{bc}f\) are symbols of one-parameter groups of motions on \(I_{a}.

It is well known that, as it follows that

\[
(X_{ab}, X_{cd})f = (c_{ab}^{e} X_{e}, c_{cd}^{e} X_{e})f = c_{ab}^{e} c_{cd}^{e} X_{e}f ,
\]

symbols \(X_{ab}f\) determine a group which is either \(G_{r}\) itself, or an invariant subgroup of \(G_{r}\) called the derived group of \(G_{r}.\) The order of the derived group is determined by the rank of the matrix \(\|c_{ab}\|\) where \(e\) denotes the columns, and \(a\) and \(b\) the rows. Specially if the rank of the matrix \(\|c_{ab}\|\) is equal to \(r,\) the derived group coincides with the given group \(G_{r}.\) Then, in consequence of Theorem 3.1 we have

**Theorem 3.2.** If the rank of the matrix \(\|A_{ab}\|\) is less than \(n\) and an \(r\)-parameter group \(G_{r}\) of infinitesimal rotations on \(I_{a}\) leaves invariant the covariant tensor \(A_{ab},\) the derived group is a subgroup of motions on \(I_{a},\) and \(G_{r}\) itself is a group of motions if the rank of the matrix \(\|c_{ab}\|\) is equal to \(r.\)
Since Lie derivatives of $g_{a\beta}$ do not vanish for an infinitesimal rotation on $I_n$, relations $\mathcal{L}A^a_{a\beta}=0, \mathcal{L}A^a_{\beta\gamma}=0, \mathcal{L}A^a_{\alpha\beta}=0$ and $\mathcal{L}A^{a\beta}_{\alpha\gamma}=0$ are not equivalent each other. However, in consequence of direct calculations, we can obtain the following analogous results concerning the mixed type of tensors $A^a_{a\beta}, A^a_{\beta\gamma}$ and the contravariant tensor $A^{a\beta}_{\alpha\gamma}$.

**Theorem 3.3.** If the rank of the matrix $||A^a_{a\beta}||$ is less than $n$ and an $r$-parameter group $G_r$ of infinitesimal rotations on $I_n$ leaves invariant the tensor $A^a_{a\beta}$, the derived group is a subgroup of motions on $I_n$, and itself is a group of motions if the rank of the matrix $||c_{ab}^e||$ is equal to $r$.

**Proof.** As $r$ vectors $\eta^a_{(a)}$ ($a=1,2,\cdots,r$) generate infinitesimal rotations on $I_n$, we have the following equation which is equivalent to the fundamental equation (3.1):

$$\mathcal{L}_a g_{a\beta} = 2\eta^a_{(a)\gamma} A^a_{a\beta}.$$

Therefore, we may put the right hand side of (3.4) as follows:

$$c_{bc}^a \mathcal{L}_a g_{a\beta} = 2c_{bc}^a \eta^a_{(a)\gamma} A^a_{a\beta}.$$

On the other hand, if we assume that $\mathcal{L}_a A^a_{a\beta}=0$ ($a=1,2,\cdots,r$), by means of the symmetric property of $A^a_{a\beta}$ and (3.1), the left member of (3.4) is reduced as follows:

$$(\mathcal{L}_b, \mathcal{L}_c) g_{a\beta} = 2(2\eta^a_{(a)\gamma} A^a_{a\beta}) - 2(2\eta^a_{(a)\gamma} A^a_{a\beta})$$

$$= 2(\eta^a_{(a)\gamma} A^a_{a\beta} - \eta^a_{(a)\gamma} A^a_{a\beta})$$

$$= 4(\eta^a_{(a)\gamma} A^a_{a\beta} - \eta^a_{(a)\gamma} A^a_{a\beta}).$$

However, as $r$ vectors $\eta^a_{(a)}$ ($a=1,2,\cdots,r$) satisfy the Maurer-Cartan equation

$$\eta^a_{(a)} \frac{\partial \eta^a_{(a)}}{\partial u^\xi} = \eta^a_{(a)} \frac{\partial \eta^a_{(a)}}{\partial u^\xi} = c_{bc}^a \eta^a_{(a)},$$

we obtain from (3.9)

$$(\mathcal{L}_b, \mathcal{L}_c) g_{a\beta} = 4c_{bc}^a \eta^a_{(a)\gamma} A^a_{a\beta}.$$

Making use of (3.4), (3.8) and (3.10), it follows that

$$\mathcal{L}_b g_{a\beta} = 2c_{bc}^a \eta^a_{(a)\gamma} A^a_{a\beta} = 0.$$

The last equation implies the result of Theorem 3.3.

**Theorem 3.4.** If the rank of the matrix $||A^a_{a\beta}||$ is less than $n$ and an $r$-parameter group $G_r$ of infinitesimal rotations on $I_n$ leaves invariant the tensor $A^a_{a\beta}$, the derived group is a subgroup of motions on $I_n$, and itself is a group of motions if the rank of the matrix $||c_{ab}^e||$ is equal to $r$. 

On Groups of Rotations in Minkowski Space

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\beta}$ and summing for $\alpha$, we have

\begin{equation}
(3.11) \quad g^{\alpha\beta}(\mathcal{L}_a g_{\alpha\beta}) = 2\eta_{(a)\gamma}A_{\beta}^{\gamma\delta}.
\end{equation}

Because of our assumption $\mathcal{L}_a A^{\gamma\delta} = 0$ ($a=1, 2, \ldots, r$), from (3.11) it follows that

\begin{equation}
(3.11) \quad g^{\alpha\beta}(\mathcal{L}_a g_{\alpha\beta}) = 2\eta_{(a)\gamma}A_{\beta}^{\gamma\delta}.
\end{equation}

On the other hand it is evident that we have from (3.1)

\begin{equation}
(3.12) \quad \mathcal{L}_a g^{\alpha\beta} = -2\eta_{(a)}^{\gamma}A_{\gamma}^{\alpha\beta}.
\end{equation}

Making use of (3.4), (3.12) and the Maurer-Cartan equation, the above equation gives us

\begin{equation}
(3.13) \quad 4\eta_{(a)\gamma}^{\delta}(S_{\alpha\beta\gamma\delta} + S_{\alpha\beta\gamma\delta}) = 2c_{ba}^{\sigma}\eta_{(c)}^{\gamma}A_{\gamma}^{\alpha\beta}.
\end{equation}

However, from the definition of $S_{\alpha\beta\gamma\delta}$, we must have $S_{\alpha\beta\gamma\delta} = -S_{\beta\alpha\gamma\delta}$. Since $A_{\alpha\beta\gamma}$ is a symmetric covariant tensor, comparing both sides of (3.13), we should have

\begin{equation}
\eta_{(a)\gamma}^{\delta}S_{\alpha\beta\gamma\delta} = 0 \quad \text{and} \quad c_{ba}^{\sigma}\eta_{(c)}^{\gamma}A_{\gamma}^{\alpha\beta} = 0.
\end{equation}

The above equations enable us to obtain the result of Theorem 3.4.

Finally, with regard to the contravariant tensor $A^{\alpha\beta\gamma}$, we have the following

Theorem 3.5. If the rank of the matrix $||A_{\alpha\beta\gamma}||$ is less than $n$ and an r-parameter group $G_r$ of infinitesimal rotations on $I_n$ leaves invariant the tensor $A_{\alpha\beta\gamma}$, the derived group is a subgroup of motions on $I_n$, and itself is a group of motions if the rank of the matrix $||c_{\alpha\beta\gamma}||$ is equal to $r$.

Proof. Multiplying the fundamental equation (3.1) by $g^{\alpha\beta}$ and $g^{\beta\gamma}$, and summing with respect to $\alpha$ and $\beta$, it follows that

\begin{equation}
(3.14) \quad g^{\alpha\beta}g^{\beta\gamma}(\mathcal{L}_a g_{\alpha\beta}) = 2\eta_{(a)\gamma}A_{\gamma}^{\alpha\beta}.
\end{equation}

Because of our assumption $\mathcal{L}_a A^{\gamma\delta} = 0$, we have from (3.14)

\begin{align*}
&(\mathcal{L}_b g^{\alpha\beta})(\mathcal{L}_a g_{\alpha\beta}) - (\mathcal{L}_a g^{\alpha\beta})(\mathcal{L}_b g_{\alpha\beta}) + g^{\alpha\beta}(\mathcal{L}_b g^{\beta\gamma})(\mathcal{L}_a g_{\gamma\delta}) - g^{\alpha\beta}(\mathcal{L}_a g^{\beta\gamma})(\mathcal{L}_b g_{\gamma\delta}) \\
&= 2A_{\gamma}^{\alpha\beta}(\mathcal{L}_b \eta_{(a)\gamma} - \mathcal{L}_a \eta_{(b)\gamma}) - g^{\alpha\beta}g^{\beta\gamma}(\mathcal{L}_b, \mathcal{L}_a) g_{\alpha\beta}.
\end{align*}

Substituting (3.12) into the above equation, and making use of (3.4) and the Maurer-Cartan equation, we find that

\begin{equation}
(3.15) \quad 4\eta_{(a)\gamma}^{\delta}(S_{\alpha\beta\gamma\delta} + S_{\alpha\beta\gamma\delta}) = 2c_{ba}^{\sigma}\eta_{(c)}^{\gamma}A_{\gamma}^{\alpha\beta}.
\end{equation}

Since $S_{\alpha\beta\gamma\delta} = -S_{\beta\alpha\gamma\delta}$ holds good, (3.15) gives rise to the relation $c_{ba}^{\sigma}\eta_{(c)}^{\gamma}A_{\gamma}^{\alpha\beta} = 0$. This implies the result of Theorem 3.5.
§ 4. Special classes of infinitesimal rotations on \( I_n \). Since \( A_\alpha \) is a gradient vector \([6]\), \( A_{\alpha;\beta} = A_{\beta;\alpha} \) holds good, where \( A_\alpha = g^{\beta\gamma}A_{\alpha\beta\gamma} \). Then it follows that

\[
\mathfrak{L}A_\alpha = A_{\alpha;\beta}\eta^\beta + A_{\beta}\eta_{;\alpha}^\beta = (A_{\beta}\eta^\beta)_{;\alpha}.
\]

On the other hand, if \( \eta^\alpha \) is the generating vector of an infinitesimal rotation on \( I_n \), by means of the fundamental relation (1.2), we get \( \eta_{;\beta}^\alpha = A_\beta\eta^\alpha \). As \( I_n \) is a compact Riemannian space, if it is an orientable, in consequence of the theorem of Green, we find from (4.1) and the last relation that

\[
\int_{I_n} \eta_{;\delta}^\delta d\sigma = \int_{I_n} A_\delta \eta^\delta d\sigma = 0.
\]

If the transformation preserves the vector \( A_\alpha \), that is to say, \( \mathfrak{L}A_\alpha = 0 \), by means of (4.1) it should be satisfied that \( (A_\beta\eta^\beta)_{;\alpha} = 0 \), i.e. \( A_\beta\eta^\beta = \text{const} \). Hence, from (4.2) we get \( A_\beta\eta^\beta = 0 \).

Conversely, from (4.1) it is evident that if \( A_\beta\eta^\beta = 0 \) holds good, we have \( \mathfrak{L}A_\alpha = 0 \). Consequently we have the following theorem:

**Theorem 4.1.** In order that an infinitesimal rotation on \( I_n \) preserves the covariant vector \( A_\alpha \), it is necessary and sufficient that the generating vector of the transformation be orthogonal to the vector \( A^\alpha \).

Let us suppose that an infinitesimal rotation generated by a vector \( \eta^\alpha \) is a motion on \( I_n \). Then, we should have

\[
\mathfrak{L}g_{\alpha\beta} = 2\eta^\gamma A_{\beta\gamma} = 0.
\]

In this case the vector \( \eta^\alpha \) satisfies the relation \( \eta^\alpha A_\alpha = 0 \). In consequence of Theorem 4.1, this implies that \( \mathfrak{L}A_\alpha = 0 \) holds good. Therefore we have

**Theorem 4.2.** If an infinitesimal rotation on \( I_n \) is a motion on \( I_n \), the generating vector is orthogonal to the vector \( A^\alpha \) and the transformation preserves the vector \( A^\alpha \).

According to the definition of Lie derivatives, we have

\[
\mathfrak{L}A_{\alpha\beta\gamma} = A_{\alpha\beta\gamma;\delta} \eta^\delta + A_{\delta\beta\gamma} \eta_{;\alpha}^\delta + A_{\alpha\delta\gamma} \eta_{;\beta}^\delta + A_{\alpha\beta\delta} \eta_{;\gamma}^\delta.
\]

However, by means of the fundamental equation (1.2), we can obtain

\[
A_{\alpha\beta;\gamma;\delta} = \frac{1}{2} (\mathfrak{L}g_{\beta\gamma})_{;\alpha} - \gamma^\delta A_{\alpha\beta;\gamma;\delta}.
\]

Making use of the relation \( A_{\alpha\beta;\gamma;\delta} = A_{\alpha\beta;\gamma;\delta} \) (cf. § 1) and (4.4) we find that

\[
\mathfrak{L}A_{\alpha\beta\gamma} = -2A_{\alpha\beta;\gamma} \eta^\delta + \frac{1}{2} [(\mathfrak{L}g_{\beta\gamma})_{;\alpha} + (\mathfrak{L}g_{\alpha\gamma})_{;\beta} + (\mathfrak{L}g_{\alpha\beta})_{;\gamma}].
\]
Therefore, if $A_{\alpha\beta\gamma \delta} = 0$ holds good in $I_n$, $\mathfrak{g}_{\alpha\beta} = 0$ gives rise to the relation $\mathfrak{g}A_{\alpha\beta} = 0$.

On the other hand, considering $I_n$ as an affine hypersurface in an $n+1$-dimensional affine space and applying the theory of affine differential geometry, as A. Kawaguchi has shown in [6], it follows that

\begin{equation}
\mathfrak{U}_{\alpha\beta\gamma} = -\left\{A_{\alpha\beta\gamma} - \frac{1}{n+2}(g_{\alpha\beta}A_{\gamma} + g_{\beta\gamma}A_{\alpha} + g_{\gamma\alpha}A_{\beta})\right\},
\end{equation}

\begin{equation}
\mathfrak{U}_{\alpha\beta\gamma}^\gamma = \mathfrak{U}_{\alpha\beta\gamma}^{-\gamma} - \frac{1}{2}(2-n)\mathfrak{U}_{\alpha\beta\gamma} \sigma^\gamma,
\end{equation}

where $\mathfrak{U}_{\alpha\beta\gamma}^\gamma \equiv \mathfrak{A}_{\alpha\beta\gamma} g^\gamma$, and $\mathfrak{U}_{\alpha\beta\gamma}^\gamma$ denotes covariant derivative of $\mathfrak{U}_{\alpha\beta\gamma}$ with respect to $\mathfrak{G}_{\alpha\beta}$. According to the result obtained by A. Kawaguchi, if $\mathfrak{U}_{\alpha\beta\gamma}^\gamma = 0$ holds good at every point on $I_n$, then $I_n$ ($n \geq 2$) is a hyperquadric. Therefore, in consequence of (4.5) and (4.6), if $A_{\alpha\beta\gamma \delta} = 0$ holds good in $I_2$, then the considered Minkowski space $M_n$ is equivalent with a Euclidean space. Thus we have

**Theorem 4.3.** When $A_{\alpha\beta\gamma \delta} = 0$ is satisfied in $I_n$ ($n \geq 3$), if an infinitesimal rotation is a motion in $I_n$, the transformation leaves invariant the tensor $A_{\alpha\beta\gamma}$.

In the previous paper [7], the present author has shown that the system of linear partial differential equations determining the group of infinitesimal rotations on $I_n$ is completely integrable and then the maximum order of the group of infinitesimal rotations on $I_n$ is equal to $\frac{1}{2}n(n+1)$.

If an infinitesimal rotation on $I_n$ coincides with a motion on $I_n$, the generating vector $\gamma^\alpha$ should satisfy the condition $\gamma^\alpha A_{\alpha\beta\gamma} = 0$. Now, let us suppose that the rank of the matrix $||A_{\alpha\beta\gamma}||$ is equal to $p$, then there exist $p$ independent conditions $\gamma^\alpha A_{\alpha\beta\gamma} = 0$. Therefore, the order of a subgroup of motions does not exceed $\frac{1}{2}n(n+1) - p$. However, as was stated in §2, since $p \geq 2$ we can see that the order of the subgroup of motions is less than $\frac{1}{2}n(n+1) - 1$.

On the other hand, according to the theorem proved by K. Yano [10], in an $n$-dimensional Riemannian space for $n \neq 4$, there exists no group of motions of order $r$ such that

\[ \frac{1}{2} n(n-1) + 1 < r < \frac{1}{2} n(n+1) \]

5) $\mathfrak{g}_{\alpha\beta}$ denotes the affine fundamental quantity of the second order, and $\mathfrak{U}_{\alpha\beta\gamma}$ the fundamental tensor of the third order defined on $I_n$. Cf. W. Blaschke [1].
and if the Riemannian space admits a group of motions of order $\frac{1}{2}n(n-1)+1$, then the group is transitive.

If we call our attention to the fact that a group of infinitesimal rotations on $I_n$ is intransitive [6], we have the following

**Theorem 4.4.** In $I_n$ $(n \neq 4)$, the order of the subgroup of motions, contained in the group of infinitesimal rotations on $I_n$, does not exceed $\frac{1}{2}n(n-1)$.

Remembering the fact that the indicatrix $I_n$ is a compact Riemannian space, in consequence of the theorems concerning affine motion or homothetic transformation in a compact Riemannian space [2], [8], we get

**Lemma 1.** If an infinitesimal rotation on $I_n$ is an affine motion, it is necessarily a motion on $I_n$.

**Lemma 2.** If an infinitesimal rotation on $I_n$ is a homothetic transformation, it is necessarily a motion on $I_n$.

Now, we shall consider the case when an infinitesimal rotation on $I_n$ is a conformal transformation on $I_n$. Then according to the theory of Lie derivatives, it should be satisfied that [9]

\[(4.7) \quad \mathfrak{L}^\gamma g_{\alpha\beta} = 2\gamma^\gamma A_{\alpha\beta\gamma} = 2\phi g_{\alpha\beta},\]

where $\phi$ is a scalar function. Contracting $g^{\alpha\beta}$ to the second and third terms of (4.7), we have $\gamma^\gamma A_\gamma = n\phi$. Then as $A_{\gamma\delta} = A_{\delta\gamma}$ it follows that

\[\mathfrak{L} A_\delta = A_{\delta\gamma} \gamma^\gamma + A_\gamma \gamma^\gamma_\delta = n\phi_\delta.\]

The last relation shows us that if the transformation leaves invariant the covariant vector $A_\delta$, the scalar function $\phi$ becomes constant, and from (4.7) the transformation is a homothetic one. Then, by means of Lemma 2, we have

**Theorem 4.5.** When an infinitesimal rotation on $I_n$ is a conformal transformation, if the transformation preserves the covariant vector $A_\delta$, it is necessarily a motion.

Next, we shall consider the case when an infinitesimal rotation on $I_n$ is a projective transformation on $I_n$. In this case it must be satisfied that [9]

\[(4.8) \quad \mathfrak{L} \varphi_\alpha = \partial^\gamma_\delta \varphi_\gamma + \delta^\gamma_\delta \varphi_\gamma,\]

where $\varphi_\alpha$ be an arbitrary covariant vector.

On the other hand, if the vector $\gamma^\alpha$ is the generating vector of an infinitesimal rotation on $I_n$, by means of (1.2), we have from the symmetric properties of $A_{\alpha\beta\gamma}$ and the relation $A_{\alpha\beta\gamma;\delta} = A_{\alpha\beta\delta;\gamma,}$
\[ \Omega[\beta^2] = \frac{1}{2} g^{a} \left[ (\Omega g_{d^2})_{,r} + (\Omega g_{d^2})_{,e} - (\Omega g_{d^2})_{,r} \right] \]
\[ = A^a_{\beta^2,\gamma^2} + \gamma^2_r (\delta^2_{\alpha^2} A^a_{\beta^2,\gamma^2} + \delta^2_{\beta^2} A^a_{\gamma^2,\delta^2} - g^{a} A^a_{\beta^2,\gamma^2}). \]

Comparing (4.8) with (4.9) we find that
\[ \delta^2_{\alpha^2} \varphi_r + \delta^2_{\beta^2} \varphi_r = A^a_{\beta^2,\gamma^2} + \gamma^2_r (\delta^2_{\alpha^2} A^a_{\beta^2,\gamma^2} + \delta^2_{\beta^2} A^a_{\gamma^2,\delta^2} - g^{a} A^a_{\beta^2,\gamma^2}). \]

Therefore, if we contract \( \alpha \) and \( \beta \) in (4.10), it follows that \( (n+1) \varphi_r = \Omega A_r. \)

The last relation gives us that if the transformation leaves invariant \( A_r \), it becomes that \( \varphi_r = 0 \), and from (4.8) we have \( \Omega[\beta^2] = 0. \) Therefore the transformation becomes an affine motion. Then, by means of Lemma 1, we have

**Theorem 4.6.** When an infinitesimal rotation on \( I_n \) is a projective transformation, if the transformation preserves the covariant vector \( A_a \), it is necessarily a motion.

If an \( n \)-dimensional Riemannian space \( I_n \) is a space of constant curvature, \( I_n \) is equivalent with a hypersphere in an \( n+1 \)-dimensional Euclidean space and the considered Minkowski space \( M_{n+1} \) is essentially a Euclidean space \([6]\). Although a space of constant curvature is an Einstein space, there exists an Einstein space which is not a space of constant curvature. Then, we shall consider the case when \( I_n \) is an Einstein space.

If an infinitesimal rotation \( \Omega^* = u^* + \gamma^*(u) dt \) is a motion or a conformal transformation, as \( \gamma^* \) satisfies (4.3) or (4.7) respectively, it follows that \( \gamma^* S_{\alpha\beta\gamma} = 0. \) Then, we have from (1.1)
\[ \gamma^* R_{\alpha\beta\gamma} = \gamma^*(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}). \]

Multiplying (4.11) by \( g^{\alpha\delta} \) and summing for \( \beta \) and \( \gamma \), we get \( \gamma^* \{ R_{\alpha\gamma} - (1-n) g_{\alpha\gamma} \} = 0. \) If \( I_n \) is an Einstein space, substituting the relation \( R_{\alpha\gamma} = \frac{\mathfrak{R}}{\eta} g_{\alpha\gamma} \) into the last equation, we get \( \gamma^* \{ R - n(1-n) \} = 0. \) Hence, at any point, if \( R \equiv n(1-n) \), then \( \gamma^* \) must be zero at the same point. Thus we have

**Theorem 4.7.** If \( I_n \) is an Einstein space and \( R \equiv n(1-n) \) is satisfied at every point on \( I_n \), an infinitesimal rotation does not coincide with a motion or a conformal transformation with respect to the metric on \( I_n \).

K. Yano and T. Nagano \([12]\) has shown the following theorem concerning the transformation in a compact Einstein space:

**Theorem I.** In a compact Einstein space, with negative scalar curvature, there does not exist an infinitesimal projective transformation.

Also, T. Sumitomo \([8]\) has proved the following theorem concerning the conformal transformation in a compact Riemannian space:
Theorem II. In a compact Riemannian manifold $M$ with non-positive constant scalar curvature: $R=\text{const.}\leqq 0$, $C_0(M)$ coincides with $I_0(M)$.

In consequence of Theorem I and Theorem II, we have

Theorem 4.8. If $I_n$ is an Einstein space with $R=n(1-n)$, an infinitesimal rotation on $I_n$ does not coincide with a conformal and a projective transformation in $I_n$.

§ 5. Order of groups of infinitesimal rotations on $I_n$ preserving $A_{\alpha\beta\gamma}$.
In the following, we shall consider the groups of infinitesimal rotations on $I_n$ such that each element of the groups preserves one of the tensors $A_{\alpha\beta\gamma}$, $A_{\alpha\beta}^{\gamma}$, $A_{\alpha}^{\beta\gamma}$ and $A^{\alpha\beta\gamma}$.

Case I. $\mathfrak{L}A_{\alpha\beta\gamma}=0$. In this case we have

$$A_{\alpha\beta\gamma}\eta^\gamma\partial_{\beta}+A_{\alpha\beta}^{\gamma}\eta_{\gamma}\partial_{\beta}+A_{\alpha}^{\beta\gamma}\eta_{\gamma}\partial_{\beta}-A_{\alpha}^{\beta}\eta_{\gamma}\partial_{\beta}=0.$$ 

Putting

$$T(\alpha\beta\gamma)(\alpha\beta\gamma)=\delta^{\alpha}_{\beta}A_{\alpha\beta}^{\gamma}$$
and we denote by $||T(\alpha\beta\gamma)||$ the matrix of elements $T(\alpha\beta\gamma)$, where $\alpha$ denotes the columns and $\beta\gamma$ the rows.

Lemma 1. If the rank of the matrix $||T(\alpha\beta\gamma)||$ is less than $n$, it follows that $A_{\alpha\beta\gamma}=0$.

Proof. From (5.1) we can easily see that $T(\alpha\beta\gamma)(\alpha\beta\gamma)=\delta_{\alpha}^{\beta}A_{\alpha\beta}^{\gamma}$, where $\alpha\neq\beta$ for $i\neq j$. Then we can select from $||T(\alpha\beta\gamma)||$ the following $n$-rowed square matrix:

$$\begin{bmatrix} (\alpha \beta \gamma) \\ (\alpha \gamma) \\ (\beta \gamma) \\ (\gamma) \\ \vdots \\ (\alpha_1) \end{bmatrix} \begin{bmatrix} A_{\alpha\beta\gamma} \\ 0 \\ \cdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} (\alpha_2) \\ 0 \\ \cdots \\ -A_{\alpha_2\alpha_2} \end{bmatrix}$$

6) $I_0(M)$ denotes the connected component of the identical transformations in $I(M)$, where $I(M)$ is the totality of isometric transformations of $M$. $C_0(M)$ denotes the connected component of the conformal transformations in $C(M)$, where $C(M)$ is the totality of conformal transformations of $M$.

7) In the theory of groups in an asymmetric affine connection space, I. P. Egorov [3] proved that "If the rank of matrix $||T(\alpha\beta\gamma)||=||\frac{1}{2}(\delta_{\alpha}\partial_{\beta}\eta^{\gamma}\partial_{\beta}+\delta_{\alpha}\partial_{\beta}\eta^{\gamma}\partial_{\beta}-\delta_{\alpha}\partial_{\beta}\eta^{\gamma}\partial_{\beta})||$ is less than $n$, then $S_{\alpha\beta\gamma}=0."$, where $S_{\alpha\beta\gamma}^{\gamma}=\frac{1}{2}(\Gamma_{\alpha\beta\gamma}^{\gamma}-\Gamma_{\alpha\beta}^{\gamma}\partial_{\gamma})$ and $\Gamma_{\alpha\beta\gamma}$ are parameters of an affine connection.

The methods of our proof are analogous as those of I. P. Egorov.

8) Indices $\alpha_i, \alpha_j$ take the values 1, 2, …, $n$. 

21x367, 283x367, 330x367, 383x367, 430x367, 477x367, 524x367, 571x367, 618x367, 665x367, 712x367, 759x367
Therefore, if the rank of the matrix $||T(\epsilon)|_{(\beta)}||$ is less than $n$, we must have $(-A^{\cdot \cdot a_{2}})^{n}=0$ and this implies $A^{\cdot \cdot a_{2}}=0$. Q.E.D.

Lemma 2. If the rank of the matrix $||T(\epsilon)|_{(\beta)}||$ is less than $n$, it follows that $A^{\cdot \cdot a_{2}}=0$.

Proof. Noting that $\alpha_{1} \neq \alpha_{2}$, from the symmetric properties of $A^{\cdot \cdot a_{2}}$, we have from (5.1) that $T(\epsilon)\delta_{(\alpha_{1}\alpha_{2})}^{(a_{2})}=\delta_{a_{2}}^{\cdot \cdot a_{2}}A^{\cdot \cdot a_{2}}+\delta_{a_{2}}^{\cdot \cdot a_{2}}A^{\cdot \cdot a_{2}}$. Then, if we suppose that $\alpha_{1}<\alpha_{2}$, we can select from $||T(\epsilon)|_{(\beta)}||$ the following $n$-rowed square matrix:

\[
\begin{pmatrix}
(1_{a_{1}}) & (2_{a_{2}}) & \cdots & (n\cdot a_{2}) & \cdots & (n\cdot a_{2}) & \cdots & (n\cdot a_{2})
\end{pmatrix}
\]

\[
(1_{a_{1}}) \quad A^{\cdot \cdot a_{2}} \quad 0 \quad \cdots \quad \cdots \quad 0
\]

\[
(2_{a_{1}}) \quad 0 \quad A^{\cdot \cdot a_{2}} \quad \cdots \quad \cdots \quad \cdots
\]

\[
(\cdots) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
(n_{a_{1}}) \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

Therefore, by virtue of Lemma 1, if the rank of the matrix $||T(\epsilon)|_{(\beta)}||$ is less than $n$, we must have $2(A^{\cdot \cdot a_{2}})^{n}=0$ and this implies $A^{\cdot \cdot a_{2}}=0$. By the same way we can prove $A^{\cdot \cdot a_{2}}=0$ for the case $\alpha_{1} \geq \alpha_{2}$. Q.E.D.

Lemma 3. If the rank of the matrix $||T(\epsilon)|_{(\beta)}||$ is less than $n$, it follows that $A^{\cdot \cdot a_{2}}=0$.

Proof. Noting that $\alpha_{1} \neq \alpha_{2}$, from the symmetric properties of $A^{\cdot \cdot a_{2}}$, we have from (5.1) that $T(\epsilon)\delta_{(\alpha_{1}\alpha_{2})}^{(a_{2})}=-\delta_{a_{2}}^{\cdot \cdot a_{2}}A^{\cdot \cdot a_{2}}$. Then, if we suppose that $\alpha_{1}<\alpha_{2}$, we can select from $||T(\epsilon)|_{(\beta)}||$ the following $n$-rowed square matrix:

\[
\begin{pmatrix}
(1_{a_{1}}) & (2_{a_{2}}) & \cdots & (n\cdot a_{2}) & \cdots & (n\cdot a_{2}) & \cdots & (n\cdot a_{2})
\end{pmatrix}
\]

\[
(1_{a_{1}}) \quad -A^{\cdot \cdot a_{2}} \quad 0 \quad \cdots \quad \cdots \quad 0
\]

\[
(2_{a_{1}}) \quad 0 \quad -A^{\cdot \cdot a_{1}} \quad \cdots \quad \cdots \quad \cdots
\]

\[
(\cdots) \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
(n_{a_{1}}) \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

9) In the following, we do not sum up with respect to the indices $\alpha_{i}$ ($i=1, 2, \cdots, n$).
Therefore, if the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ is less than $n$, we must have $(-A_{\alpha_1 \beta \gamma})^n = 0$ and this implies $A_{\alpha_1 \beta \gamma} = 0$. By the same way, we can prove $A_{\alpha_1 \beta \gamma} = 0$ for the case $\alpha_1 > \alpha_2$. Q.E.D.

**Lemma 4.** If the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ is less than $n$, it follows that $A_{\beta \alpha \gamma} = 0$.

**Proof.** From (5.1), it follows that $T(\mathfrak{a}(\alpha_1 \beta \gamma)) = 2A_{\alpha_1 \alpha_2}^\beta - \delta_{\alpha_1}^\beta A_{\alpha_2 \alpha_1}^\gamma$. Then we can select from $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ the following $n$-rowed square matrix:

\[
\begin{pmatrix}
(a_1 \alpha_1) & (a_1 \alpha_2) & \cdots & (a_1 \alpha_1) & \cdots & (a_1 \alpha_1) \\
(a_1 \alpha_1) & 0 & \cdots & 0 & \cdots & 0 \\
(a_1 \alpha_2) & & \ddots & \ddots & \cdots & \ddots \\
(a_1 \alpha_1) & & & 0 & \cdots & \ddots \\
(a_1 \alpha_2) & & & & \ddots & \ddots \\
(a_1 \alpha_1) & & & & & 0 \\
(a_1 \alpha_1) & & & & & \cdots \\
\end{pmatrix}
\]

Therefore, if the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ is less than $n$, we must have $(-A_{\alpha_1 \beta \gamma})^n = 0$ and this implies $A_{\alpha_1 \beta \gamma} = 0$. Q.E.D.

Summarizing the results of above Lemmas, we see that if the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ is less than $n$, it follows that $A_{\beta \gamma \alpha} = 0$ for every $\alpha, \beta$ and $\gamma$. Consequently, as the maximum order of groups of the infinitesimal rotations on $I_n$ is equal to $\frac{1}{2}n(n+1)$ [7], it follows that

**Theorem 5.1.** The order of the group of infinitesimal rotations on $I_n$, preserving the tensor $A_{\beta \gamma \alpha}$, does not exceed $\frac{1}{2}n(n-1)$.

**Case II.** $\mathfrak{L}A_{\gamma \beta \alpha} = 0$. In this case we have

\[A_{\alpha \beta \gamma}^\delta - A_{\alpha \beta \gamma}^\delta + A_{\alpha \gamma \beta}^\delta - A_{\alpha \gamma \beta}^\delta + A_{\beta \gamma \alpha}^\delta - A_{\beta \gamma \alpha}^\delta = 0 .\]

We put $T(\mathfrak{a}(\alpha_1 \beta \gamma)) = \delta_{\alpha_1}^\gamma A_{\beta \gamma \alpha}^\delta + \delta_{\beta}^\gamma A_{\alpha \gamma \beta}^\delta - \delta_{\gamma}^\beta A_{\alpha \beta \gamma}^\delta$ and consider the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$. Making use of the analogous way as those in Case I, we obtain

**Lemma 5.** If the rank of the matrix $||T(\mathfrak{a}(\alpha_1 \beta \gamma)||$ is less than $n$, it follows that $A_{\beta \gamma \alpha} = 0$.

Consequently, we obtain the following

**Theorem 5.2.** The order of the group of infinitesimal rotations on $I_n$, preserving the tensor $A_{\beta \gamma \alpha}$, does not exceed $\frac{1}{2}n(n-1)$.

**Case III.** $\mathfrak{L}A_{\beta \gamma \alpha} = 0$. In this case we have

\[A_{\beta \gamma \alpha}^\delta + A_{\beta \gamma \alpha}^\delta + A_{\alpha \beta \gamma}^\delta + A_{\gamma \beta \alpha}^\delta + A_{\alpha \gamma \beta}^\delta = 0 .\]

We put
(5.2) \[ T(i)(\alpha_{\beta}) = \delta_{\alpha}^{i}A_{\beta \gamma} + \delta_{\beta}^{i}A_{\alpha \gamma} + \delta_{\gamma}^{i}A_{\alpha \beta} \]
and denote by \[ ||T(i)(\alpha_{\beta})|| \] the matrix of elements \[ T(\alpha_{\beta})(\alpha_{\gamma}) \], where \( (i) \) denotes the columns and \( (\alpha_{\beta}) \) the rows.

**Lemma 6.** If the rank of the matrix \[ ||T(i)(\alpha_{\beta})|| \] is less than \( n \), it follows that \( A_{\alpha_{1}\alpha_{1}\alpha_{1}} = 0 \).

**Proof.** Making use of the symmetric properties of \( A_{\alpha_{\beta}} \), we have from (5.2) that \( T(\alpha_{1})(\alpha_{1}\alpha_{2}) = 2\delta_{\alpha_{1}}^{\alpha_{1}}A_{\alpha_{1}\alpha_{1}\alpha_{2}} + \delta_{\alpha_{2}}^{\alpha_{1}}A_{\alpha_{1}\alpha_{2}\alpha_{2}} \). Then, we can select from \[ ||T(i)(\alpha_{\beta})|| \] the following \( n \)-rowed square matrix:

\[
\begin{array}{cccccc}
(\alpha_{1}) & (\alpha_{1} \alpha_{2}) & \cdots & (\alpha_{1} \alpha_{n}) & \cdots & (\alpha_{1} \alpha_{n}) \\
(\alpha_{2}) & A_{\alpha_{1} \alpha_{2}} & 0 & \cdots & \cdots & 0 \\
(\alpha_{3}) & 0 & A_{\alpha_{1} \alpha_{2}} & \cdots & \cdots & \cdots \\
(\alpha_{4}) & \vdots & \vdots & \ddots & \cdots & \cdots \\
(\alpha_{5}) & \vdots & \vdots & \cdots & \ddots & \cdots \\
(\alpha_{n}) & 0 & \cdots & \cdots & \cdots & A_{\alpha_{1} \alpha_{n}} \\
\end{array}
\]

Therefore, if the rank of the matrix \[ ||T(i)(\alpha_{\beta})|| \] is less than \( n \), we must have \( 3(A_{\alpha_{1} \alpha_{2} \alpha_{2}})^{n} = 0 \), i.e. \( A_{\alpha_{1} \alpha_{2} \alpha_{2}} = 0 \). Q.E.D.

**Lemma 7.** If the rank of the matrix \[ ||T(i)(\alpha_{\beta})|| \] is less than \( n \), it follows that \( A_{\alpha_{1} \alpha_{1} \alpha_{1}} = 0 \).

**Proof.** From (5.2), by means of the symmetric properties of \( A_{\alpha_{\beta}} \), we have \( T(\alpha_{1})(\alpha_{2} \alpha_{2}) = \delta_{\alpha_{1}}^{\alpha_{1}}A_{\alpha_{1} \alpha_{2} \alpha_{2}} + 2\delta_{\alpha_{2}}^{\alpha_{1}}A_{\alpha_{1} \alpha_{2} \alpha_{2}} \). Then, if we suppose that \( \alpha_{2} < \alpha_{1} \), we can select from \[ ||T(i)(\alpha_{\beta})|| \] the following \( n \)-rowed square matrix:

\[
\begin{array}{cccccc}
(\alpha_{1}) & (\alpha_{1} \alpha_{2}) & \cdots & (\alpha_{1} \alpha_{2}) & \cdots & (\alpha_{1} \alpha_{2}) \\
(\alpha_{2}) & A_{\alpha_{1} \alpha_{2} \alpha_{2}} & 0 & \cdots & \cdots & 0 \\
(\alpha_{3}) & 0 & A_{\alpha_{1} \alpha_{2} \alpha_{2}} & \cdots & \cdots & \cdots \\
(\alpha_{4}) & \vdots & \vdots & \ddots & \cdots & \cdots \\
(\alpha_{5}) & \vdots & \vdots & \cdots & \ddots & \cdots \\
(\alpha_{n}) & 0 & \cdots & \cdots & \cdots & A_{\alpha_{1} \alpha_{2} \alpha_{2}} \\
\end{array}
\]

Therefore, if the rank of the matrix \[ ||T(i)(\alpha_{\beta})|| \] is less than \( n \), we must have \( 3(A_{\alpha_{1} \alpha_{2} \alpha_{2}})^{n} = 0 \), i.e. \( A_{\alpha_{1} \alpha_{2} \alpha_{2}} = 0 \). By the same way, we can prove \( A_{\alpha_{1} \alpha_{2} \alpha_{2}} = 0 \) for the case \( \alpha_{1} < \alpha_{2} \). Q.E.D.
Lemma 8. If the rank of the matrix \( ||T(\lambda\delta)(\alpha\beta\gamma)|| \) is less than \( n \), it follows that \( A_{\alpha_1\alpha_2\alpha_3} = 0 \).

Proof. In consequence of the symmetric property of \( A_{\alpha\beta\gamma} \), we may assume that \( \alpha_1 < \alpha_2 < \alpha_3 \) without loss of generality. Thus, we have from (5.2) that \( T(\alpha_1)(\alpha_2\alpha_3) = \delta_{\alpha_1}^{\alpha_2}A_{\alpha_1\alpha_2\alpha_3} + \delta_{\alpha_2}^{\alpha_3}A_{\alpha_2\alpha_3\alpha_1} + \delta_{\alpha_3}^{\alpha_1}A_{\alpha_3\alpha_1\alpha_2} \). Then we can select from \( ||T(\lambda\delta)(\alpha\beta\gamma)|| \) the following \( n \)-rowed square matrix:

\[
\begin{pmatrix}
(\alpha_1\alpha_2) & (\alpha_2\alpha_3) & \cdots & (\alpha_3\alpha_4) & \cdots & (\alpha_4\alpha_5) & \cdots & (\alpha_5\alpha_6) \\
(1,\alpha_1) & A_{\alpha_1\alpha_2\alpha_3} & 0 & \cdots & (\alpha_1\alpha_3) & \cdots & (\alpha_3\alpha_4) & \cdots & (\alpha_4\alpha_5) \\
(2,\alpha_1) & * & A_{\alpha_1\alpha_2\alpha_3} & \cdots & * & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_2) & * & * & A_{\alpha_1\alpha_2\alpha_3} & \cdots & * & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_3) & * & * & * & \cdots & 2A_{\alpha_1\alpha_3\alpha_2} & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_4) & * & * & * & * & \cdots & 2A_{\alpha_1\alpha_4\alpha_3} & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_5) & * & * & * & * & * & \cdots & * & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_6) & * & * & * & * & * & * & \cdots & * & \cdots & * & \cdots & * \\
(\alpha_1,\alpha_7) & * & * & * & * & * & * & * & \cdots & * & \cdots & * & \cdots & * \\
\end{pmatrix}
\]

Therefore, if the rank of the matrix \( ||T(\lambda\delta)(\alpha\beta\gamma)|| \) is less than \( n \), we must have \( 4(A_{\alpha_1\alpha_2\alpha_3})^n = 0 \), i.e. \( A_{\alpha_1\alpha_2\alpha_3} = 0 \). Q.E.D.

Summarizing the results of the above Lemmas, we see that if the rank of the matrix \( ||T(\lambda\delta)(\alpha\beta\gamma)|| \) is less than \( n \), then it follows that \( A_{\alpha\beta\gamma} = 0 \) for every \( \alpha, \beta \) and \( \gamma \). Consequently we have

**Theorem 5.3.** The order of the group of infinitesimal rotations on \( I_n \), preserving the tensor \( A_{\alpha\beta\gamma} \), does not exceed \( \frac{1}{2}n(n-1) \).

**Case IV.** \( 2A_{\alpha\beta\gamma} = 0 \). In this case we can easily obtain the following theorem by means of the analogous way as in Case III:

**Theorem 5.4.** The order of the group of infinitesimal rotations on \( I_n \), preserving the tensor \( A_{\alpha\beta\gamma} \), does not exceed \( \frac{1}{2}n(n-1) \).

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**References**


