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ON THE RATIOS OF THE NORMS DEFINED
BY MODULARS

By

Tetsuya SHIMOGAKI

§ 1. Let $R$ be a modulared semi-ordered linear space and $m(x)$ ($x \in R$) be a modular\(^1\) on $R$. Since $0 \leq m(\xi x)$ is a non-trivial convex function of real number $\xi \geq 0$ for every $0 \neq x \in R$, we can define two kinds of norms by the modular $m$ as follows:

$$
\|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\phi) \leq 1} \frac{1}{|\xi|} \quad (x \in R).
$$

The former of them is said to be the first norm by $m$ and the latter to be the second (or modular) norm by $m$.

Let $\overline{R}^{m}$ be the modular conjugate space of $R$ and $\overline{m}$ be the conjugate modular\(^2\) of $m$. Then we can also define the norms on $\overline{R}^{m}$ by $\overline{m}$ as above. It is well-known [4; § 40] that if $R$ is semi-regular\(^3\) the first norm by the conjugate modular $\overline{m}$ is the conjugate one of the second norm by $m$ and the second norm by $\overline{m}$ is the conjugate one of the first norm by $m$. Since $\|\cdot\|$ and $|||\cdot|||$ are semi-continuous, they are reflexive [3]. We have always $|||x||| \leq \|x\| \leq 2|||x|||$ for all $x \in R$, that is, $1 \leq \frac{\|x\|}{|||x|||}$ $\leq 2$ for all $0 \neq x \in R$.

When the ratios of these two norms are equal to a constant number, i.e. $\frac{\|x\|}{|||x|||} = \gamma$ holds for each $0 \neq x \in R$, S. Yamamuro [8] and I. Amemiya [1] succeeded in showing that the modular $m$ is of $L^p$-type essentially, i.e. $m(\xi x) = \xi^n m(x)$ for all $x \in R$ and $\xi \geq 0$, where $1 \leq p$.

In the earlier paper [7] the author investigated the case that the

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1) For the definition of a modular see [4]. The notations and terminologies used here are the same as in [4 or 7].

2) $\overline{R}^{m}$ is the totality of all linear functionals $\overline{a}$ on $R$ such that $\inf_{t \in A} |\overline{a}(x_t)| = 0$ for every $x_t |_{t \in A}$ and $\sup_{m(x) \leq 1} |\overline{a}(x)| < +\infty$. The conjugate modular $\overline{m}$ of $m$ on $\overline{R}^{m}$ is defined as $\overline{m}(\overline{a}) = \sup_{x \in R} \{\overline{a}(x) - m(x)\}$ ($\overline{a} \in \overline{R}^{m}$).

3) $R$ is said to be semi-regular, if $\overline{a}(x) = 0$ for all $\overline{a} \in \overline{R}^{m}$ implies $x = 0$. 

ratios satisfy the condition:
\[(1.2) \quad \inf_{0 \neq x \in R} \frac{\|x\|}{\|x\downarrow\|} > 1,\]
and proved that it is equivalent to uniform finiteness of both $m$ and $\overline{m}$, provided that $R$ is non-atomic.

In an Orlicz space $L_{\Phi}^*(G)$, which is one of the concrete examples of modulared semi-ordered linear spaces, the similar results concerning the ratios were found independently by D. V. Salekhov in [6] under more restricted circumstances.

In this paper we shall consider the following conditions on the ratios of the norms by a modular $m$:
\[(1.3) \quad \frac{\|x\|}{\|x\downarrow\|} < 2 \quad \text{for all } 0 \neq x \in R;\]
\[(1.4) \quad \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\downarrow\|} < 2,\]
and study their relations to the properties of the modular $m$. We shall show in §2 that if the condition (1.3) is satisfied, then either $m(\xi x) < \xi^2 m(x)$ (for all $\xi > 1$ and $x \in R$ with $m(x) \geq 1$), or $m(\xi x) > \xi^2 m(x)$ (for all $\xi > 1$ and $x \in R$ with $+\infty > m(x) \geq 1$) holds, provided that $R$ is non-atomic. And as for (1.4) we shall show in §3 that (1.4) implies that either $m(\xi x) \leq \xi^p m(x)$ for all $\xi \geq 1$ and $x \in R$ with $m(x) \geq 1$ or $m(\xi x) \geq \xi^{p'} m(x)$ for all $\xi \geq 1$ and $x \in R$ with $m(x) \geq 1$ holds, where $p, p'$ are real numbers with $1 \leq p < 2 < p' \leq +\infty$, provided that $R$ is non-atomic.

The difference between the conditions (1.2) and (1.4) exists in the point of their topological properties, that is, the former of them remains valid for any modular $m'$ equivalent 5) to the original one except a finite dimensional space, but the later does not hold in general. Thus we can not obtain the explicit conditions equivalent to (1.4) with respect to the modular $m$ in general case. For a modular of unique spectra, however, we shall estimate $\sup_{0 \neq x \in R} \|x\|$ and $\inf_{0 \neq x \in R} \|x\|$ exactly in §4 by applying the results obtained in §§2 and 3.

Throughout this paper we denote by $R$ a modulared semi-ordered linear space and by $m$ a modular on $R$. For any $p \in R$ we denote by $[p]$

4) For the definition of Orlicz space $L_{\Phi}^*(G)$ see [2] or [9].

5) Two modulars $m$ and $m'$ on $R$ are called equivalent, if their norms are equivalent to each other.
a projection operator defined by $p: [p]x = \bigcup_{n=1}^{\infty}(n|p| \cap x)$ for all $0 \leq x \in R$. $R$ is called to be non-atomic, if any $0 \neq a \in R$ is decomposed into $a=b+c$ such that $|b|\cap|c|=0$, $b \not\subseteq 0$ and $c \not\subseteq 0$. Since $m(x+y)=m(x)+m(y)$ for any $x,y \in R$ with $|x|\cap|y|=0$, $a \in R$ with $m(a)<+\infty$ can be decomposed into $a=[p]a+(1-[p])a$ for some $p \in R$ such that $m([p]a)=m((1-[p])a)$, if $R$ is non-atomic. Here we note that $m(\xi x)$ is a continuous function of $\xi \in [0, \eta]$ for each $x \in R$, if $m(\eta x)<+\infty$, because $m(\xi x)$ is a positive convex function of $\xi \geq 0$ for each $x \in R$.

§ 2. We put for every $x \in R$ with $m(x)<+\infty$

(2.1) \[ \pi_{+}(x)=\inf_{\epsilon>0} \frac{1}{\epsilon} \{ m((1+\epsilon)x)-m(x) \} \]

and

(2.2) \[ \pi_{-}(x)=\sup_{\epsilon>0} \frac{1}{\epsilon} \{ m(x)-m((1-\epsilon)x) \} , \]

and for $x \in R$ with $m(x)=+\infty$ we put

(2.3) \[ \pi_{+}(x)=\pi_{-}(x)=+\infty . \]

Then it follows from the definitions that $0 \leq \pi_{-}(x) \leq \pi_{+}(x)$ for all $x \in R$ and both $\pi_{+}(\xi x)$ and $\pi_{-}(\xi x)$ are non-decreasing functions of $\xi \geq 0$ for every $x \in R$ and are orthogonally additive, that is, $\pi_{+}(x+y)=\pi_{+}(x)+\pi_{+}(y)$ if $x \perp y$, $x, y \in R$. Furthermore $\pi_{-}(\xi x)$ is a right-hand continuous function of $\xi \geq 0$ for every $x \in R$, since $m(\xi x)$ is a convex function of $\xi \geq 0$. In fact, we have for each $\xi_{0} \geq 0$

\[ \lim_{\xi \downarrow \xi_{0}} \pi_{+}(\xi x) = \inf_{\xi>\xi_{0}} \pi_{+}(\xi x) = \inf_{\xi>\xi_{0}} \inf_{\epsilon>0} \frac{1}{\epsilon} \{ m((1+\epsilon)\xi x)-m(\xi x) \} = \pi_{+}(\xi_{0} x), \]

if $m(\alpha \xi_{0} x)<+\infty$ for some $\alpha>1$. If $m(\alpha \xi_{0} x)=+\infty$ for all $\alpha>1$ or $m(\xi_{0} x)=+\infty$, we have $\pi_{+}(\xi x)=+\infty$ and

\[ \lim_{\xi \downarrow \xi_{0}} \pi_{+}(\xi x) = \pi_{+}(\xi_{0} x) . \]

Similarly $\pi_{-}(\xi x)$ is a left-hand continuous function of $\xi$ for every $x \in R$, that is, \[ \lim_{\xi \uparrow \xi_{0}} \pi_{-}(\xi x) = \pi_{-}(\xi_{0} x) \]

for each $\xi_{0} \geq 0$.

We put also

\[ S=\{ x : x \in R, \| x \|=1 \} , \]

\[ S_{m}=\{ x : x \in R, m(x)=1 \} \quad \text{and} \quad S_{c}=\{ x : x \in S, m(x)<1 \} . \]

From the definition of the second norm it is clear that $S=S_{m} \cup S_{c}$, $S_{m} \cap S_{c} = \phi$ and $m(\xi x)=+\infty$ for all $x \in S_{c}$ and $\xi>1$.

6) Two elements $x, y$ are called mutually orthogonal, if $|x|\cap|y|=0$ and then we write $x \perp y$. For a subset $A$ of $R$, $A^{\perp}$ denotes the set of all $x \in R$ with $x \perp y$ for all $y \in A$. 

\[ S_{m}=\{ x : x \in R, m(x)=1 \} \quad \text{and} \quad S_{c}=\{ x : x \in S, m(x)<1 \} . \]
Lemma 1. We have $||x||=2$ for $x \in S$ if and only if $x \in S_m$ and $\pi_-(x) \leq 2 \leq \pi_+(x)$.

Proof. If $||x||=2$ for $x \in S$, then we have by the formula (1.1) $2 \xi \leq 1 + m(\xi x)$ for every $\xi > 0$. This implies $m(x) = 1$ and $\frac{1 - m(\xi x)}{1 - \xi} \leq 2 \leq \frac{m(\eta x) - 1}{\eta - 1}$ for every $0 < \xi < 1 < \eta$. It follows therefore $\pi_-(x) \leq 2 \leq \pi_+(x)$.

Conversely $\pi_-(x) \leq 2 \leq \pi_+(x)$ implies $\frac{m(x) - m(\xi x)}{1 - \xi} \leq 2 \leq \frac{m(\eta x) - m(x)}{\eta - 1}$ for every $0 < \xi < 1 < \eta$, which yields $2 \xi \leq 1 + m(\xi x)$ for all $\xi > 0$ in virtue of $m(x) = 1$. Hence we have $||x||=2$. Q.E.D.

Lemma 2. We have $\pi_-(x)>2$ for $x \in S_m$, if and only if $||x||<2$ and $||x||=\inf_{1<\xi<1} \frac{1+m(\xi x)}{\xi}$.

Proof. If $\pi_-(x)>2$ for $x \in S_m$, then we have for some $0 < \xi < 1$ $2 < \frac{1 - m(\xi x)}{1 - \xi}$, which implies $2 > \frac{1 + m(\xi x)}{\xi}$ and $||x||<2$. Since $\pi_+(x) \geq \pi_-(x)$, we obtain $\frac{m(\eta x) - 1}{\eta - 1} \geq \pi_-(x) > 2$ for all $\eta > 1$. It follows from above that $\frac{1 + m(\eta x)}{\eta} > 2$ for $\eta > 1$ and $||x||=\inf_{1<\xi<1} \frac{1+m(\xi x)}{\xi}$. Conversely let $||x||<2$, $||x||=\inf_{1<\xi<1} \frac{1+m(\xi x)}{\xi}$ and $x \in S_m$, then there exists $\xi_0$ ($0 < \xi_0 < 1$) such that $2 > \frac{1 + m(\xi_0 x)}{\xi_0}$. This implies $2 < \frac{1 - m(\xi_0 x)}{1 - \xi_0}$ and $\pi_-(x) > 2$. Q.E.D.

Lemma 3. We have $\pi_+(x)<2$ for $x \in S$ if and only if $||x||<2$ and $||x||=\inf_{1<\xi} \frac{1+m(\xi x)}{\xi}$.

Proof. If $x \in S$ and $\pi_+(x)<2$, then we have $m(\xi_0 x) < +\infty$ for some $1 < \xi_0$ by the definition of $\pi_+(x)$. Thus we have $x \in S_m$. The remainder of the proof can be obtained by the similar way as above. Q.E.D.

Now we put

$$S_* = \left\{ x : x \in S, \ ||x||=\inf_{1<\xi<1} \frac{1+m(\xi x)}{\xi} \right\}$$

and

$$S^* = \left\{ x : x \in S, \ ||x||=\inf_{1<\xi} \frac{1+m(\xi x)}{\xi} \right\}.$$  

It is clear by Lemmata 1–3 that $S_* \cap S^* = S$, $S_* \subset S_*$ and that $x \in S_* \cap S^*$ implies $||x||=2$. 

The following lemma plays essential rôle in this paper.

**Lemma 4.** If there exist mutually orthogonal elements $x, y \in R$ with $x \in S_*$ and $y \in S^*$, then there exists an element $z \in S$ such that $\|z\| = 2$.

**Proof.** If $\|x\| = 2$ (or $\|y\| = 2$) holds, then the above assertion is clearly true. Hence we suppose $\|x\| < 2$ and $\|y\| < 2$. We put for every positive number $\alpha$ with $0 \leq \alpha \leq 1$

\begin{equation}
\varphi(\alpha) = \sup_{\|ax + \beta y\| = 1} \beta.
\end{equation}

Since $y \in S^*$ implies $m(y) = 1$, it is easily seen that $\varphi(\alpha)$ is a continuous function of $\alpha$ ($0 \leq \alpha \leq 1$) and if $\alpha$ runs decreasingly from 1 to 0, $\varphi(\alpha)$ does increasingly from $\xi_0$ to 1, where $\xi_0 = \sup_{m(\xi y) = 1 - m(x)} \xi$.

Now we put $z_\alpha = \alpha x + \varphi(\alpha)y$. It is clear that $z_\alpha \in S_m$ and

\begin{equation}
\pi_-(z_\alpha) = \pi_-(ax) + \pi_-(\varphi(\alpha)y)
\end{equation}

for all $0 \leq \alpha \leq 1$.

By Lemma 3, $\|y\| < 2$ and $y \in S^*$ imply

\begin{equation}
\pi_-(z_\alpha) \leq \pi_+(z_\alpha) = \pi_+(y) < 2.
\end{equation}

And if $x \in S_m$, then we have by Lemma 2

\begin{equation}
\pi_-(z_\alpha) = \pi_-(\alpha x) + \pi_-(\varphi(\alpha)y) \geq \pi_-(\alpha x) + \pi_-(\varphi(\alpha)y).
\end{equation}

Since $\pi_-(\xi x)$ is left-hand continuous, we obtain

\[2 \geq \lim_{n \to \infty} \pi_-(z_{\alpha_n}) \geq \lim_{n \to \infty} \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n)y) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n)y),\]

which implies $\pi_-(z_{\alpha_0}) \leq 2$. On the other hand, since $\pi_-(z_{\alpha_n}) \leq 2$ implies $\alpha_0 < 1$, we can find a sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that $1 \geq \alpha'_n \geq \alpha_n$ implies $0 \leq \varphi(\alpha') \leq \varphi(\alpha) \leq 1$

\[\pi_-(z_{\alpha_n}) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n)y) \geq \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n)y).\]

Since $\pi_-(\xi x)$ is right-hand continuous, we obtain

\[2 \leq \lim_{n \to \infty} \pi_+(z_{\alpha_n}) = \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_n)y) \leq \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_n)y),\]

which yields $\pi_+(z_{\alpha_0}) \geq 2$. Therefore we have $\pi_-(z_{\alpha_0}) \leq 2 \leq \pi_+(z_{\alpha_0})$ and $a$
fortiori $||z_0||=2$ by Lemma 1.

We denote by $S_{x,y}$ the totality of all elements $z \in S$ such that $z = \alpha x + \beta y$ for some $0 \leq \alpha, \beta$.

**Lemma 5.** If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S^*$ and $z \in S_*$, then there exists $z_0 \in S_{x,y}$ such that $||z_0||=2$.

**Proof.** Here we may also assume without loss of generality that $\pi_+(x) < 2$, $\pi_+(y) < 2$ and $\pi_-(z) > 2$. We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha = \alpha x + \varphi(\alpha)y$ for $0 \leq \alpha \leq 1$. It follows from $x, y \in S^*$ that $z_\alpha$ belongs to $S_m$ for each $0 \leq \alpha \leq 1$. Since there exists some $1 > \alpha' > 0$ such that $\pi_-(z) = \pi_-(z_{\alpha'}) > 2$ we can put $\alpha_0 = \inf \alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1 \geq \alpha_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) \geq 2$ ($n=1, 2, \ldots$). Since

$$2 \leq \pi_+(z_{\alpha_n}) = \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_n)y) \leq \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_0)y)$$

implies

$$2 \leq \lim_{n \to \infty} \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_0)y) = \pi_+(\alpha_0 x) + \pi_+(\varphi(\alpha_0)y),$$

we have $2 \leq \pi_+(z_{\alpha_0})$. On the other hand, $\pi_+(z_{\alpha_0}) \geq 2$ implies $\alpha_0 > 0$ and hence we can find also a sequence of positive numbers $\{\alpha'_n\}_{n=1}^\infty$ such that $0 \leq \alpha'_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha'_n}) < 2$ ($n=1, 2, \ldots$). Since

$$2 > \pi_-(z_{\alpha'_n}) = \pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha'_n)y) \geq \pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha_0)y)$$

implies

$$2 \geq \lim_{n \to \infty} \pi_-(\alpha'_n x) + \pi_-(\varphi(\alpha_0)y) = \pi_-(\alpha_0 x) + \pi_-(\varphi(\alpha_0)y),$$

we have $2 \geq \pi_-(z_{\alpha_0})$. Therefore we obtain $\pi_-(z_{\alpha_0}) \leq 2 \leq \pi_+(z_{\alpha_0})$, which implies $||z_0||=2$. Q.E.D.

Here we note that if there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $z \in S^*$ then $m(x+y) > 1$. Hence applying the similar method as in the proof of Lemma 5, we have

**Lemma 6.** If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S_*$ and $z \in S^*$, then there exists $z_0 \in S_{x,y}$ with $||z_0||=2$.

Collecting the results of the above Lemmata, we have

**Theorem 2.1.** In order that the condition (1.3) is satisfied, that is,

$$\frac{||x||}{||z||} < 2 \text{ for all } 0 \neq x \in R,$$

it is necessary and sufficient that either

$$\pi_+(x) < 2 \quad \text{for all } x \in S$$

or
\[(2.9') \quad \pi_-(x) > 2 \quad \text{for all } x \in S_m \]

holds.

**Proof. Necessity.** When \( R \) is one-dimensional, the assertion comes directly from Lemmata 2 and 3. Thus we may assume that the dimension of \( R \) is greater than two. Now let \( R = N_1 \oplus N_2 \), where \( N_i \) \((i=1, 2)\) are normal manifolds and \( N_1^\perp = N_2 \). For an element \( x_0 \in N_1 \cap S \) the condition (1.3) implies either \( x_0 \in S_* \) or \( x_0 \in S^* \).

First let \( x_0 \in S_* \). Then Lemma 4 and the condition (1.3) imply \( N_2 \cap S \subseteq S_* \), which implies also \( N_1 \cap S \subseteq S_* \) by Lemma 4. Therefore we obtain \( S \subseteq S_* \) by Lemma 6. Thus we can see that (2.9') holds good in virtue of Lemma 2.

Secondly let \( x_0 \in S^* \), then we have by the same manner \( N_2 \cap S = S^* \) and \( N_1 \cap S = S^* \). This implies that (2.8) holds good in virtue of Lemmata 3 and 5. Q.E.D.

**Sufficiency.** Since \( x \in S_* \) implies \( 1 + m(x) < 2 \), we have \( ||x|| < 2 \) for all \( x \in S_* \). Thus we can see that Lemmata 2 and 3 assure that (2.8) (or (2.9')) implies the condition (1.3). Q.E.D.

A modular \( m \) on \( R \) is said to be **finite** if \( m(x) < +\infty \) for all \( x \in R \). Since we have \( S = S_m \), in case \( R \) is finite, we have immediately from Theorem 2.1

**Corollary 1.** Let a modular \( m \) be finite. In order that the condition (1.3) holds, it is necessary and sufficient that either the condition (2.8) or

\[(2.9) \quad \pi_-(x) > 2 \quad \text{for all } x \in S \]

holds.

In order that we discuss the condition (1.3) more precisely in case of non-atomic \( R \), we need to prove

**Lemma 7.** If \( R \) non-atomic and \( S = S^* \), then the modular \( m \) is finite.

**Proof.** If there exists \( 0 \leq x \in R \) with \( m(x) = +\infty \), we put \( a_0 = \inf_{m(\eta x) = +\infty} \xi \) and \( x_0 = a_0 x \). It follows \( m(\gamma x_0) = +\infty \) for all \( \gamma > 1 \) and \( m(\xi x_0) < +\infty \) for all \( 0 \leq \xi < 1 \). When \( m(x_0) < +\infty \) holds, we can find an element \( p \in R \) such that \( m(\eta[p]x_0) = +\infty \) for all \( \eta > 1 \) and \( m([p]x_0) \leq 1 \), since \( R \) has no atomic elements. For such \([p]x_0 \) we have \([p]x_0 \in S_* \), which is inconsistent with \( S^* = S \). Now let \( m(x_0) = +\infty \) hold. Since \( R \) is non-atomic, we can find \( p \in R \) with \( m(\frac{2}{3}[p]x_0) \leq \frac{1}{4} \) and \( \lim_{\xi \uparrow 1} m(\xi[p_0]x_0) = +\infty \). Now we put
\[ \alpha = \frac{1}{\left\| \frac{2}{3} \mathbf{p} \mathbf{x}_0 \right\|} \] and 
\[ y = \alpha \frac{2}{3} \mathbf{p} \mathbf{x}_0. \] Then we have 
\[ 1 < \alpha < \frac{3}{2} \] and 
\[ y \in S_m. \] Hence
\[ 1 - \frac{1}{4} \frac{1}{3} > 2. \]
This contradicts the assumption: \( S = S^* \) by Lemmata 1 and 2. Therefore we have proved that 
\( m(x) < +\infty \) for all \( x \in R \).

**Q.E.D.**

**Theorem 2.2.** Let \( R \) be non-atomic and the condition (1.3) be satisfied, then the modular \( m \) satisfies one of the following conditions:

\[ m(\xi x) < \xi^2 m(x) \quad \text{for all } \xi > 1 \text{ and } x \in R \text{ with } m(x) \geq 1; \]

\[ m(\xi x) > \xi^2 m(x) \quad \text{for all } \xi > 1 \text{ and } +\infty > m(x) \geq 1. \]

**Proof.** In virtue of the foregoing theorem we know that one of the conditions (2.8) or (2.9') is true. First we suppose that (2.8) holds.

Then Lemma 7 together with Lemma 3 implies that \( m \) is finite. If 
\[ m(x) = N + \frac{m}{n} \] (where \( N, m \) and \( n \) are natural numbers with \( m \leq n \)), we can decompose orthogonally \( x \) into 
\[ x = \sum_{i=1}^{N-1} x_i + \sum_{j=1}^{n+m} y_j \] such that 
\[ m(x_i) = 1 \quad (i = 1, 2, \ldots, N-1) \] and 
\[ m(y_j) = \frac{1}{n} \quad (j = 1, 2, \ldots, n+m). \]

The number of \( j \) satisfying \( \pi_+(y_j) \geq 2m(y_j) \) is less than \( n \), because if there exist \( j_1, j_2, \ldots, j_n \) with 
\[ \pi_+(y_j_k) \geq 2m(y_j_k) \quad (k = 1, 2, \ldots, n), \] we have 
\[ \sum_{k=1}^{n} y_j_k \in S_m \] and 
\[ \pi_+ \left( \sum_{k=1}^{n} y_j_k \right) = \sum_{k=1}^{n} \pi_+(y_j_k) \geq 2m(y_j_k) = 2m \left( \sum_{k=1}^{n} y_j_k \right) = 2, \] which is inconsistent with (2.8).

Hence we can find \( \{ j_p \} \) (\( 1 \leq p \leq m \)) such that 
\[ \pi_+(y_{j_p}) < 2m(y_{j_p}) \quad (p = 1, 2, \ldots, m). \]
Put \( y_0 = \sum_{p=1}^{m} y_{j_p} \), we obtain 
\[ m \left( \sum_{j=1}^{m+n} y_j - y_0 \right) = 1 \] and 
\[ \pi_+ \left( \sum_{j=1}^{m+n} y_j - y_0 \right) < 2. \]
Therefore we have
\[ \pi_+(x) = \sum_{i=1}^{N-1} \pi_+(x_i) + \pi_+ \left( \sum_{j=1}^{m+n} y_j - y_0 \right) + \pi_+(y_0) < 2(N-1) + 2 + 2m(y_0) = 2 \left( N + \frac{m}{n} \right) = 2m(x), \]

hence \( \pi_+(x) < 2m(x) \). In general, if \( 1 < m(x) \), we can find \( \{ \alpha_n \}_{n=1}^{\infty} \) with 
\[ \alpha_n + n+1 = 1 \] and \( m(\alpha_n x) \) is a rational number for each \( n \geq 1 \). It follows
\[ \pi_+(x) = \lim_{n \to \infty} \pi_+(\alpha_n x) \leq \lim_{n \to \infty} 2m(\alpha_n x) = 2m(x), \]
hence

\[ (2.12) \quad \pi_+(x) \leq 2m(x) \quad \text{for all } x \in R \text{ with } 1 \leq m(x). \]

Now we put

\[ (2.13) \quad m'_\xi(x) = \lim_{\xi \to 0} \frac{m((\xi + \epsilon)x) - m(\xi x)}{\epsilon} \]

for each \( x \in R \) and \( \xi \geq 0 \). It is clear that \( \xi \cdot m'_\xi(x) = \pi_+(\xi x) \) for all \( \xi > 0 \) and \( x \in R \). (2.12) implies

\[ (2.14) \quad \frac{m'_\xi(x)}{m(\xi x)} \leq \frac{2}{\xi} \quad (\xi > 1) \]

for every \( x \in R \) with \( m(x) \geq 1 \). Integrating both sides of (2.14) from 1 to \( \eta > 1 \) with respect to \( \xi \), we have

\[ (2.15) \quad \log \frac{m(\eta x)}{m(x)} \leq 2 \log \xi \quad (\eta > 1). \]

In formula (2.15), however, the equal sign does not hold in any case. Indeed, since as is shown above, the set of all \( \xi \) satisfying \( \xi m'_\xi(x) = \pi_+(\xi x) < 2m(\xi x) \) is dense in \([1, +\infty)\) and \( m(\xi x) \) is a continuous function of \( \xi \), there exists an interval \( (\xi_0, \eta_0) \subseteq (1, \eta) \) such that

\[ \xi m'_\xi(x) = \pi_+(\xi x) < 2m(\xi x) \]

holds for all \( \xi \in (\xi_0, \eta_0) \). Therefore we have

\[ m(\eta x) < \eta^2 m(x) \]

for all \( \eta > 1 \) and \( x \in R \) with \( m(x) \geq 1 \).

By the quite same manner we can prove that the condition (2.11) is satisfied, if we assume that the condition (2.9') is true. Q.E.D.

§ 3. Here we consider the case that the norms defined by a modular \( m \) satisfy (1.4), that is, \( \sup_{0 \neq x \in R} \frac{||x||}{|||x|||} < 2 \). From the results proved in § 2 we have

Theorem 3.1. If the condition (1.4) is satisfied, then either

\[ (3.1) \quad \sup_{x \in S} \pi_+(x) < 2 \]

or

\[ (3.2) \quad \inf_{x \in S_m} \pi_-(x) > 2 \]

holds.

Proof. In virtue of Theorem 2.1, we can see that either (2.8) or (2.9') holds. First let (2.8) be true and set \( \gamma = \sup_{0 \neq x \in R} \frac{||x||}{|||x|||} \). Then for
each $x \in S$ and $\varepsilon > 0$, there exists $\xi > 1$ such that \( \frac{1 + m(\xi x)}{\xi} < \gamma + \varepsilon \) by Lemma 3. From this

\[
\pi_+(x) \leq \frac{m(\xi x) - 1}{\xi - 1} < \gamma + \varepsilon
\]

follows, if $\gamma + \varepsilon < 2$. Hence we have $\gamma \geq \pi_+(x)$ for all $x \in S$.

On the other hand, we can prove by the same way\(^7\) that (2.9') together with (1.4) implies (3.2).

**Q.E.D.**

**Remark 1.** The converse of Theorem 3.1 does not remain true in general. It is easily verified that there exists a modular which does not fulfill (1.4) but satisfies (3.1) (or (3.2)).

**Remark 2.** As is seen in the proof of Theorem 3.1, it is clear that

\[
\sup_{x \in S} \pi_+(x) \leq \gamma \quad \text{or} \quad \inf_{x \in S} \pi_-(x) \geq \frac{\gamma}{\gamma - 1} \quad \text{holds respectively, where} \quad \gamma = \sup_{0 \neq x \in R} \frac{||x||}{||x||} < 2.
\]

As for non-atomic $R$, corresponding to Theorem 2.2, we have

**Theorem 3.2.** Let $R$ be non-atomic and the condition (1.4) be satisfied, then either

\[
\begin{align*}
(3.3) & \quad m(\xi x) \leq \xi^p m(x) \quad \text{for all} \quad \xi \geq 1 \quad \text{and} \quad x \in R \quad \text{with} \quad m(x) \geq 1; \\
(3.4) & \quad m(\xi x) \geq \xi^{p'} m(x) \quad \text{for all} \quad \xi \geq 1 \quad \text{and} \quad x \in R \quad \text{with} \quad m(x) \geq 1,
\end{align*}
\]

where $p$ and $p'$ are real numbers with $1 \leq p < 2 < p' \leq +\infty^8$.

**Proof.** In virtue of the preceding theorem we need only to verify implications: (3.1)$\rightarrow$(3.3) and (3.2)$\rightarrow$(3.4). And these implications can be ascertained by the same manner as in the proof of Theorem 2.2. Here we may choose $p$, $p'$ as $p = \gamma$ and $p' = \frac{\gamma}{\gamma - 1}$ respectively, where

\[
\gamma = \sup_{0 \neq x \in R} \frac{||x||}{||x||}. \quad \text{Q.E.D.}
\]

**Remark 3.** As is easily verified by calculating $||x||$ of $x \in S$, the condition (3.3) is the sufficient one for (1.4) at the same time. On the other hand, (3.4) is not such a one in general.

\[\S 4.\] At last we deal with a modular of unique spectra \([4; \S 54]\) and estimate exactly $\sup_{0 \neq x \in R} \frac{||x||}{||x||}$ and $\inf_{0 \neq x \in R} \frac{||x||}{||x||}$ as applications of the

\[7)\quad \text{We note that} \quad 2 - \gamma \xi \geq \frac{\gamma}{\gamma - 1} (1-\xi) \quad \text{holds if} \quad 1 < \gamma < 2 \quad \text{and} \quad 1 < \gamma \xi.
\]

\[8)\quad \text{When} \quad p' = +\infty, \quad \text{we put} \quad \xi^\infty = \infty \quad \text{if} \quad \xi > 1.
\]
On the Ratios of the Norms defined by Modulars

preceding theorems and those of [7]. An element $0 \leq s \in R$ is said to be simple, if $m(s) < +\infty$ and $m([p]s) = 0$ implies $[p]s = 0$. And a modular $m$ is said to be of unique spectra if $m(\xi s) = \int [\xi^p]m(dp)\xi^p$ for all $\xi \geq 0$ and simple elements $s \in R$. Function spaces $L^{p(t)}$ (where $p(t)$ is a measurable function with $p(t) \geq 1$) are the examples of modulated spaces whose modulared are of unique spectra, where the modulared are defined as $m(\varphi) = \int_0^1 |\varphi(t)|^{p(t)}dt$ and $m(\mathfrak{x}) = \sum_{\nu=1}^\infty |\xi_{\nu}|^{p_{\nu}}$ respectively. When $m$ is of unique spectra, we denote by $\rho_u$, $\rho_l$ the upper exponent of $m$: $\rho_u = \sup_{\mathfrak{p} \in \in} \rho(\mathfrak{p})$ and the lower exponent of $m$: $\rho_l = \inf_{\mathfrak{p} \in \in} \rho(\mathfrak{p})$ respectively. There exist normal manifolds $N_1$, $N_2$ such that $R = N_1 \oplus N_2$ and $\rho(\mathfrak{p})$ is finite for all $\mathfrak{p} \in U_{[N_1]}$ and $\rho(\mathfrak{p}) = +\infty$ for all $\mathfrak{p} \in U_{[N_2]}$, that is, $m$ is singular in $N_2$. For any $0 \neq x \in N_2$ we have $||x|| = |||x|||$ and $S = S_*$. Therefore we obtain

**Theorem 4.1.** If a modular $m$ is of unique spectra, then we have

(with the conventions $\frac{1}{\infty} = 0$ and $\infty^0 = 1$)

\[
\sup_{0 \neq x \in R} \frac{||x||}{|||x|||} \begin{cases} = 2, & \text{if } \rho_l \leq 2 \leq \rho_u, \\ = \frac{1}{\rho_u} q_u \frac{1}{q_u}, & \text{if } \rho_u < 2, \\ = \rho_l \frac{1}{q_l} q_l, & \text{if } \rho_l > 2, \end{cases}
\]

where $q_u$ and $q_l$ are real numbers with $\frac{1}{\rho_u} + \frac{1}{q_u} = 1$ and $\frac{1}{\rho_l} + \frac{1}{q_l} = 1$.

**Proof.** As $\frac{||x||}{|||x|||} = 1$ for all $0 \neq x \in N_2$, we may consider only the

9) $\mathfrak{p}$ is a point of representation space $\in$ of $R$, i.e. the maximal ideal of normal manifolds of $R$. For $N$, we denote by $U_{[N]}$ the totality of all $\mathfrak{p} \in \in$ with $N \in \mathfrak{p}$. $\rho(\mathfrak{p})$ is a continuous function on $\in$ with $\rho(\mathfrak{p}) \geq 1$. 
ratios of the norms in $N_i$. When $\gamma \leq \rho(p) \leq \gamma'$ for all $p \in U_{[p]} \subseteq U_{[N_i]}$, then we have for all $x \in R \xi m([p]x) \leq m(\xi [p]x) \leq \xi' m([p]x)$ ($\xi \geq 1$) and $\gamma' m([p]x) \geq m(\gamma[p]x) \geq \gamma'' m([p]x)$ ($0 \leq \gamma \leq 1$). From this and Lemma 4 we have $\sup_{x \in R} \frac{||x||}{|||x|||} = 2$, if $\rho_i \leq 2 \leq \rho_u$. Since $\sup_{x \in S} \pi_+(x) < 2$ if and only if $\rho_u < 2$ and $\inf_{x \in S} \pi_-(x) > 2$ if and only if $\rho_i > 2$ for $m$, we have by Lemmata 2 and 3 that $||x|| \leq \rho_u^{1/\rho_u} q_u^{1/\rho_u} (x \in S)$ and $||x|| \leq \rho_i^{1/\rho_i} q_i^{1/\rho_i}$ respectively according to (3.1) and (3.2). Therefore we complete the proof.

Q.E.D.

Similarly we can conclude by Theorem 3.1 in [7]

**Theorem 4.2.** If a modular $m$ is of unique spectra, then we have

$$\inf_{0 \neq x \in R} \frac{||x||}{|||x|||} = \min \left\{ \rho_i^{1/\rho_i} q_i^{1/\rho_i}, \rho_u^{1/\rho_u} q_u^{1/\rho_u} \right\}.$$