### Instructions for use

**Title**
ON THE RATIOS OF THE NORMS DEFINED BY MODULARS

**Author(s)**
Shimogaki, Tetsuya

**Citation**
Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 15(1-2): 001-012

**Issue Date**
1960

**Doc URL**
http://hdl.handle.net/2115/56009

**Type**
bulletin (article)

**File Information**
JFSHIU_15_N1-2_001-012.pdf

*Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP*
ON THE RATIOS OF THE NORMS DEFINED
BY MODULARS

By

Tetsuya SHIMOGAKI

§ 1. Let $R$ be a modulared semi-ordered linear space and $m(x)$ $(x \in R)$ be a modular\(^1\) on $R$. Since $0 \leq m(\xi x)$ is a non-trivial convex function of real number $\xi \geq 0$ for every $0 \neq x \in R$, we can define two kinds of norms by the modular $m$ as follows:

\[
||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\phi) \leq 1} \frac{1}{|\xi|} (x \in R).
\]

The former of them is said to be the first norm by $m$ and the latter to be the second (or modular) norm by $m$.

Let $\overline{R}^m$ be the modular conjugate space of $R$ and $\overline{m}$ be the conjugate modular\(^2\) of $m$. Then we can also define the norms on $\overline{R}^m$ by $\overline{m}$ as above. It is well-known [4; § 40] that if $R$ is semi-regular\(^3\) the first norm by the conjugate modular $\overline{m}$ is the conjugate one of the second norm by $m$ and the second norm by $\overline{m}$ is the conjugate one of the first norm by $m$. Since $||\cdot||$ and $|||\cdot|||$ are semi-continuous, they are reflexive [3]. We have always $|||x||| \leq ||x|| \leq 2 |||x|||$ for all $x \in R$, that is, $1 \leq \frac{||x||}{|||x|||}$ $\leq 2$ for all $0 \neq x \in R$.

When the ratios of these two norms are equal to a constant number, i.e. $\frac{||x||}{|||x|||} = \gamma$ holds for each $0 \neq x \in R$, S. Yamamuro [8] and I. Amemiya [1] succeeded in showing that the modular $m$ is of $L^p$-type essentially, i.e. $m(\xi x) = \xi^p m(x)$ for all $x \in R$ and $\xi \geq 0$, where $1 \leq p$.

In the earlier paper [7] the author investigated the case that the

---

1) For the definition of a modular see [4]. The notations and terminologies used here are the same as in [4 or 7].

2) $\overline{R}^m$ is the totality of all linear functionals $\overline{a}$ on $R$ such that $\inf_{t \in A} |\overline{a}(x_t)| = 0$ for every $x_t$, $t \in A$ and $\sup_{m(\xi) \leq 1} |\overline{a}(x)| < +\infty$. The conjugate modular $\overline{m}$ of $m$ on $\overline{R}^m$ is defined as $\overline{m}(\overline{a}) = \sup_{x \in R} \{\overline{a}(x) - m(x)\}$ ($\overline{a} \in \overline{R}^m$).

3) $R$ is said to be semi-regular, if $\overline{a}(x) = 0$ for all $\overline{a} \in \overline{R}^m$ implies $x = 0$. 

---
ratios satisfy the condition:

\[(1.2) \inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|} > 1,\]

and proved that it is equivalent to uniform finiteness of both \(m\) and \(\overline{m}\), provided that \(R\) is non-atomic.

In an Orlicz space \(L_{\Phi}^{*}(G)^4\), which is one of the concrete examples of modulared semi-ordered linear spaces, the similar results concerning the ratios were found independently by D. V. Salekhov in [6] under more restricted circumstances.

In this paper we shall consider the following conditions on the ratios of the norms by a modular \(m\):

\[(1.3) \frac{\|x\|}{\|x\|} < 2 \quad \text{for all } 0 \neq x \in R;\]

or

\[(1.4) \sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|} < 2,\]

and study their relations to the properties of the modular \(m\). We shall show in § 2 that if the condition (1.3) is satisfied, then either \(m(\xi x) < \xi^2 m(x)\) (for all \(\xi > 1\) and \(x \in R\) with \(m(x) \geq 1\)), or \(m(\xi x) > \xi^2 m(x)\) (for all \(\xi > 1\) and \(x \in R\) with \(+\infty > m(x) \geq 1\)) holds, provided that \(R\) is non-atomic. And as for (1.4) we shall show in § 3 that (1.4) implies that either \(m(\xi x) \leq \xi^p m(x)\) for all \(\xi \geq 1\) and \(x \in R\) with \(m(x) \geq 1\) or \(m(\xi x) \geq \xi^{p'} m(x)\) for all \(\xi \geq 1\) and \(x \in R\) with \(m(x) \geq 1\) holds, where \(p, p'\) are real numbers with \(1 \leq p < 2 < p' \leq +\infty\), provided that \(R\) is non-atomic.

The difference between the conditions (1.2) and (1.4) exists in the point of their topological properties, that is, the former of them remains valid for any modular \(m'\) equivalent to the original one except a finite dimensional space, but the later dose not hold in general. Thus we can not obtain the explicit conditions equivalent to (1.4) with respect to the modular \(m\) in general case. For a modular of unique spectra, however, we shall estimate \(\sup_{0 \neq x \in R} \frac{\|x\|}{\|x\|}\) and \(\inf_{0 \neq x \in R} \frac{\|x\|}{\|x\|}\) exactly in § 4 by applying the results obtained in §§ 2 and 3.

Throughout this paper we denote by \(R\) a modulared semi-ordered linear space and by \(m\) a modular on \(R\). For any \(p \in R\) we denote by \([p]\)

\[4)\] For the definition of Orlicz space \(L_{\Phi}^{*}(G)\) see [2] or [9].

\[5)\] Two modulars \(m\) and \(m'\) on \(R\) are called equivalent, if their norms are equivalent to each other.
On the Ratios of the Norms defined by Modulars

A projection operator defined by $p : [p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x)$ for all $0 \leq x \in R$. $R$ is called to be non-atomic, if any $0 \neq a \in R$ is decomposed into $a = b + c$ such that $|b| = |c| = 0$, $b \neq 0$ and $c \neq 0$. Since $m(x+y) = m(x) + m(y)$ for any $x, y \in R$ with $|x| = |y| = 0$, $a \in R$ with $m(a) < +\infty$ can be decomposed into $a = [p]a + (1 - [p])a$ for some $p \in R$ such that $m([p]a) = m((1 - [p])a)$, if $R$ is non-atomic. Here we note that $m(\xi x)$ is a continuous function of $\xi \in [0, \eta]$ for each $x \in R$, if $m(\eta x) < +\infty$, because $m(\xi x)$ is a positive convex function of $\xi \geq 0$ for each $x \in R$.

§ 2. We put for every $x \in R$ with $m(x) < +\infty$

(2.1) $\pi_{+}(x) = \inf_{\epsilon > 0} \frac{1}{\epsilon} \{m((1 + \epsilon)x) - m(x)\}$

and

(2.2) $\pi_{-}(x) = \sup_{\epsilon > 0} \frac{1}{\epsilon} \{m(x) - m((1 - \epsilon)x)\}$,

and for $x \in R$ with $m(x) = +\infty$ we put

(2.3) $\pi_{+}(x) = \pi_{-}(x) = +\infty$.

Then it follows from the definitions that $0 \leq \pi_{-}(x) \leq \pi_{+}(x)$ for all $x \in R$ and both $\pi_{+}(\xi x)$ and $\pi_{-}(\xi x)$ are non-decreasing functions of $\xi \geq 0$ for every $x \in R$ and are orthogonally additive, that is, $\pi_{+}(x+y) = \pi_{+}(x) + \pi_{+}(y)$ if $x \perp y \pound$, $x, y \in R$. Furthermore $\pi_{+}(\xi x)$ is a right-hand continuous function of $\xi \geq 0$ for every $x \in R$, since $m(\xi x)$ is a convex function of $\xi \geq 0$. In fact, we have for each $\xi_0 \geq 0$ $\lim_{\xi \downarrow \xi_0} \pi_{+}(\xi x) = \inf_{\xi_0 > \xi > 0} \frac{1}{\xi} \{m((1 + \epsilon)\xi x) - m(\xi_0 x)\} = \pi_{+}(\xi_0 x)$, if $m(\alpha \xi_0 x) < +\infty$ for some $\alpha > 1$. If $m(\alpha \xi_0 x) = +\infty$ for all $\alpha > 1$ or $m(\xi_0 x) = +\infty$, we have $\pi_{+}(\xi_0 x) = +\infty$ and $\lim_{\xi \downarrow \xi_0} \pi_{+}(\xi x) = \pi_{+}(\xi_0 x)$. Similarly $\pi_{-}(\xi x)$ is a left-hand continuous function of $\xi$ for every $x \in R$, that is, $\lim_{\xi \downarrow \xi_0} \pi_{-}(\xi x) = \pi_{-}(\xi_0 x)$ for each $\xi_0 \geq 0$.

We put also

$S = \{x : x \in R, \|x\| = 1\}$,

$S_m = \{x : x \in R, m(x) = 1\}$ and $S_c = \{x : x \in S, m(x) < 1\}$.

From the definition of the second norm it is clear that $S = S_m \wedge S_c$, $S_m \wedge S_c = \emptyset$ and $m(\xi x) = +\infty$ for all $x \in S_c$ and $\xi > 1$.

6) Two elements $x, y$ are called mutually orthogonal, if $|x| \cap |y| = 0$ and then we write $x \perp y$. For a subset $A$ of $R$, $A^\perp$ denotes the set of all $x \in R$ with $x \perp y$ for all $y \notin A$. 
Lemma 1. We have $||x||=2$ for $x \in S$ if and only if $x \in S_m$ and $\pi_-(x) \leq 2 \leq \pi_+(x)$.

Proof. If $||x||=2$ for $x \in S$, then we have by the formula (1.1) $2 \xi \leq 1 + m(\xi x)$ for every $\xi > 0$. This implies $m(x) = 1$ and $\frac{1 - m(\xi x)}{1 - \xi} \leq 2 \leq \frac{m(\eta x) - 1}{\eta - 1}$ for every $0 < \xi < 1 < \eta$. It follows therefore $\pi_-(x) \leq 2 \leq \pi_+(x)$.

Conversely $\pi_-(x) \leq 2 \leq \pi_+(x)$ implies $\frac{m(x) - m(\xi x)}{1 - \xi} \leq 2 \leq \frac{m(\eta x) - m(x)}{\eta - 1}$ for every $0 < \xi < 1 < \eta$, which yields $2 \xi \leq 1 + m(\xi x)$ for all $\xi > 0$ in virtue of $m(x) = 1$. Hence we have $||x||=2$. Q.E.D.

Lemma 2. We have $\pi_-(x) > 2$ for $x \in S_m$, if and only if $||x|| < 2$ and $||x|| = \inf_{\frac{1}{\xi} < \xi < 1} \frac{1 + m(\xi x)}{\xi}$.

Proof. If $\pi_-(x) > 2$ for $x \in S_m$, then we have for some $0 < \xi < 1$ $2 < \frac{1 - m(\xi x)}{1 - \xi}$, which implies $2 \leq \frac{1 + m(\xi x)}{\xi}$ and $||x|| < 2$. Since $\pi_+(x) \geq \pi_-(x) > 2$, we obtain $\frac{m(\eta x) - 1}{\eta - 1} \geq \pi_-(x) > 2$ for all $\eta > 1$. It follows from above that $\frac{1 + m(\eta x)}{\eta} > 2$ for $\eta > 1$ and $||x|| = \inf_{\frac{1}{\xi} < \xi < 1} \frac{1 + m(\xi x)}{\xi}$. Conversely let $||x|| < 2$, $||x|| = \inf_{\frac{1}{\xi} < \xi < 1} \frac{1 + m(\xi x)}{\xi}$ and $x \in S_m$, then there exists $\xi_0$ ($0 < \xi_0 < 1$) such that $2 > \frac{1 + m(\xi_0 x)}{1 - \xi_0}$. This implies $2 < \frac{1 - m(\xi_0 x)}{1 - \xi_0}$ and $\pi_-(x) > 2$. Q.E.D.

Lemma 3. We have $\pi_+(x) < 2$ for $x \in S$ if and only if $||x|| < 2$ and $||x|| = \inf_{1 < \xi} \frac{1 + m(\xi x)}{\xi}$.

Proof. If $x \in S$ and $\pi_+(x) < 2$, then we have $m(\xi_0 x) < +\infty$ for some $1 < \xi_0$ by the definition of $\pi_+(x)$. Thus we have $x \in S_m$. The remainder of the proof can be obtained by the similar way as above. Q.E.D.

Now we put

$$S_*=\{x: x \in S, \ ||x|| = \inf_{\frac{1}{\xi} < \xi < 1} \frac{1 + m(\xi x)}{\xi}\}$$

and

$$S^*=\{x: x \in S, \ ||x|| = \inf_{1 < \xi} \frac{1 + m(\xi x)}{\xi}\}.$$ 

It is clear by Lemmata 1–3 that $S_* \cup S^*=S$, $S \subset S_*$ and that $x \in S_* \cap S^*$ implies $||x||=2$. 

The following lemma plays essential rôle in this paper.

**Lemma 4.** If there exist mutually orthogonal elements \(x, y \in R\) with \(x \in S_*\) and \(y \in S^*\), then there exists an element \(z \in S\) such that \(||z||=2\).

**Proof.** If \(||x||=2\) (or \(||y||=2\)) holds, then the above assertion is clearly true. Hence we suppose \(||x||<2\) and \(||y||<2\). We put for every positive number \(\alpha\) with \(0 \leq \alpha \leq 1\)

\[
\varphi(\alpha) = \sup_{||ax+\beta y||=1} \beta.
\]

Since \(y \in S^*\) implies \(m(y)=1\), it is easily seen that \(\varphi(\alpha)\) is a continuous function of \(\alpha\) \((0 \leq \alpha \leq 1)\) and if \(\alpha\) runs decreasingly from 1 to 0, \(\varphi(\alpha)\) does increasingly from \(\xi_0\) to 1, where \(\xi_0 = \sup_{m(\xi y)=1-m(x)} \xi\).

Now we put \(z_\alpha = \alpha x + \varphi(\alpha)y\). It is clear that \(z_\alpha \in S_m\) and

\[
\pi_+(z_\alpha) = \pi_+(\alpha x) + \pi_+(\varphi(\alpha)y) \quad \text{for all } 0 \leq \alpha \leq 1.
\]

By Lemma 3, \(||y||<2\) and \(y \in S^*\) imply

\[
\pi_-(z_0) \leq \pi_+(z_0) = \pi_+(y) < 2.
\]

And if \(x \in S_m\), then we have by Lemma 2

\[
\pi_-(z_1) = \pi_-(x) + \pi_-(\xi_0 y) \geq \pi_-(x) > 2.
\]

On the other hand, \(x \in S_c\) implies \(\pi_+(x)=\pi_+(z_1)=+\infty\), because \(m(\xi x)=+\infty\) for all \(\xi>1\). Thus we have \(||z_1||=2\) by Lemma 1, if \(\pi_-(z_1)\leq 2\). Therefore, we may also suppose that both (2.6) and (2.7) hold good.

In virtue of (2.6) we can put \(\alpha_0 = \sup_{\pi_+(z_\alpha) \leq 2} \alpha\). For such \(\alpha_0 \geq 0\) there exists a sequence of positive numbers such that \(0 \leq \alpha_n \downarrow_{n=1}^{\infty} \alpha_0\) and \(\pi_-(z_{\alpha_n}) \leq \pi_+(z_{\alpha_n}) < 2\). We have by (2.5) and by the fact that \(1 \geq \alpha' \geq \alpha \geq 0\) implies \(0 \leq \varphi(\alpha') \leq \varphi(\alpha) \leq 1\)

\[
\pi_-(z_{\alpha_n}) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n y)) \geq \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0 y)).
\]

Since \(\pi_-(z_{\alpha_n})\) is left-hand continuous, we obtain

\[
2 \geq \lim_{n \to \infty} \pi_-(z_{\alpha_n}) \geq \lim_{n \to \infty} \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_n y)) = \pi_-(\alpha_n x) + \pi_-(\varphi(\alpha_0 y)),
\]

which implies \(\pi_-(z_{\alpha_n}) \leq 2\). On the other hand, since \(\pi_-(z_{\alpha_n}) \leq 2\) implies \(\alpha_0 < 1\), we can find a sequence \(\{\alpha_n\}_{n=1}^{\infty}\) such that \(1 \geq \alpha_n \downarrow_{n=1}^{\infty} \alpha_0\) and \(\pi_+(z_{\alpha_n}) \geq 2\). We have also by (2.5)

\[
2 \leq \lim_{n \to \infty} \pi_+(z_{\alpha_n}) = \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_n y)) \leq \pi_+(\alpha_\alpha x) + \pi_+(\varphi(\alpha_0 y)).
\]

Since \(\pi_+(z_{\alpha_n})\) is right-hand continuous, we obtain

\[
2 \leq \lim_{n \to \infty} \pi_+(z_{\alpha_n}) \leq \lim_{n \to \infty} \pi_+(\alpha_n x) + \pi_+(\varphi(\alpha_0 y)) = \pi_+(\alpha_0 x) + \pi_+(\varphi(\alpha_0 y)),
\]

which yields \(\pi_+(z_{\alpha_n}) \geq 2\). Therefore we have \(\pi_-(z_{\alpha_n}) \leq 2 \leq \pi_+(z_{\alpha_n})\) and \(z_{\alpha_n}\)
We denote by $S_{x,y}$ the totality of all elements $z \in S$ such that $z = \alpha x + \beta y$ for some $0 \leq \alpha, \beta$.

**Lemma 5.** If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S^*$ and $z \in S_*$, then there exists $z_0 \in S_{x,y}$ such that $||z_0|| = 2$.

**Proof.** Here we may also assume without loss of generality that $\pi_+(x) < 2$, $\pi_+(y) < 2$ and $\pi_-(z) > 2$. We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha = \alpha x + \varphi(\alpha)y$ for $0 \leq \alpha \leq 1$. It follows from $x, y \in S^*$ and $z \in S_*$, that $\pi_-(z) = \pi_-(z_0) > 2$ we can put $\alpha_0 = \inf \alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1 \geq \alpha_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) > 2$.

We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha = \alpha x + \varphi(\alpha)y$ for $0 \leq \alpha \leq 1$. It follows from $x, y \in S^*$ and $z \in S_*$, that $\pi_-(z) = \pi_-(z_0) > 2$ we can put $\alpha_0 = \inf \alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1 \geq \alpha_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) > 2$.

We also define $\varphi(\alpha)$ by the formula (2.4) and put $z_\alpha = \alpha x + \varphi(\alpha)y$ for $0 \leq \alpha \leq 1$. It follows from $x, y \in S^*$ and $z \in S_*$, that $\pi_-(z) = \pi_-(z_0) > 2$ we can put $\alpha_0 = \inf \alpha$. Then we can find a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $1 \geq \alpha_n \downarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha_n}) > 2$.

Since $2 \leq \pi_+(z_\alpha) = \pi_+(\alpha x) + \pi_+(\varphi(\alpha)y) \leq \pi_+(\alpha x) + \pi_+(\varphi(\alpha)y)$ implies

$$2 \leq \lim_{\alpha \rightarrow \infty} \pi_+(\alpha x) + \pi_+(\varphi(\alpha)y) = \pi_+(\alpha x) + \pi_+(\varphi(\alpha)y)$$

we have $2 \leq \pi_+(z_0)$. On the other hand, $\pi_+(z_\alpha) \geq 2$ implies $\alpha_0 > 0$ and hence we can find also a sequence of positive numbers $\{\alpha'_n\}_{n=1}^\infty$ such that $0 \leq \alpha'_n \rightarrow_{n=1}^\infty \alpha_0$ and $\pi_-(z_{\alpha'_n}) < 2$ ($n = 1, 2, \cdots$). Since

$$2 > \pi_-(z_{\alpha'_n}) = \pi_-(\alpha' x) + \pi_-(\varphi(\alpha')y) \geq \pi_-(\alpha' x) + \pi_-(\varphi(\alpha')y)$$

implies

$$2 \geq \lim_{\alpha \rightarrow \infty} \pi_-(\alpha' x) + \pi_-(\varphi(\alpha')y) = \pi_-(\alpha' x) + \pi_-(\varphi(\alpha')y)$$

we have $2 \geq \pi_-(z_0)$. Therefore we obtain $\pi_-(z_0) \leq 2 \leq \pi_+(z_0)$, which implies $||z_0|| = 2$. Q.E.D.

Here we note that if there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $z \in S^*$ then $m(x+y) > 1$. Hence applying the similar method as in the proof of Lemma 5, we have

**Lemma 6.** If there exist mutually orthogonal elements $x, y \in S$ and $z \in S_{x,y}$ such that $x, y \in S_*$ and $z \in S^*$, then there exists $z_0 \in S_{x,y}$ with $||z_0|| = 2$.

Collecting the results of the above Lemmata, we have

**Theorem 2.1.** In order that the condition (1.3) is satisfied, that is,

$$\frac{||x||}{||x||} < 2$$

for all $0 \neq x \in R$, it is necessary and sufficient that either

$$(2.8) \quad \pi_+(x) < 2 \quad \text{for all } x \in S$$

or
On the Ratios of the Norms defined by Modulars

\[ (2.9') \quad \pi_{-}(x) > 2 \quad \text{for all } x \in S_m \]

holds.

**Proof. Necessity.** When \( R \) is one-dimensional, the assertion comes directly from Lemmata 2 and 3. Thus we may assume that the dimension of \( R \) is greater than two. Now let \( R = N_1 \oplus N_2 \), where \( N_i \) (\( i = 1, 2 \)) are normal manifolds and \( N_1^\perp = N_2 \). For an element \( x_0 \in N_1 \cap S \) the condition (1.3) implies either \( x_0 \in S_* \) or \( x_0 \in S^* \).

First let \( x_0 \in S_* \). Then Lemma 4 and the condition (1.3) imply \( N_2 \cap S \subseteq S_* \), which implies also \( N_1 \cap S \subseteq S_* \) by Lemma 4. Therefore we obtain \( S \subseteq S_* \) by Lemma 6. Thus we can see that (2.9') holds good in virtue of Lemma 2.

Secondly let \( x_0 \in S^* \), then we have by the same manner \( N_1 \cap S = S_* \) and \( N_2 \cap S = S^* \). This implies that (2.8) holds good in virtue of Lemmata 3 and 5. Q.E.D.

**Sufficiency.** Since \( x \in S_c \) implies \( 1 + m(x) < 2 \), we have \( ||x|| < 2 \) for all \( x \in S_c \). Thus we can see that Lemmata 2 and 3 assure that (2.8) holds good in virtue of Lemmata 2 and 3. Q.E.D.

A modular \( m \) on \( R \) is said to be finite if \( m(x) < +\infty \) for all \( x \in R \).

Since we have \( S = S_m \), in case \( R \) is finite, we have immediately from Theorem 2.1

**Corollary 1.** Let a modular \( m \) be finite. In order that the condition (1.3) holds, it is necessary and sufficient that either the condition (2.8) or

\[ (2.9) \quad \pi_{-}(x) > 2 \quad \text{for all } x \in S \]

holds.

In order that we discuss the condition (1.3) more precisely in case of non-atomic \( R \), we need to prove

**Lemma 7.** If \( R \) non-atomic and \( S = S^* \), then the modular \( m \) is finite.

**Proof.** If there exists \( 0 \leq x \in R \) with \( m(x) = +\infty \), we put \( a_0 = \inf_{m([x]) = +\infty} \xi \) and \( x_0 = \alpha_0 x \). It follows \( m(\gamma x_0) = +\infty \) for all \( \eta > 1 \) and \( m(\xi x_0) < +\infty \) for all \( 0 \leq \xi < 1 \). When \( m(x_0) = +\infty \) holds, we can find an element \( p \in R \) such that \( m(\eta[p]x_0) = +\infty \) for all \( \eta > 1 \) and \( m([p]x_0) \leq 1 \), since \( R \) has no atomic elements. For such \( [p]x_0 \) we have \( [p]x_0 \in S_* \), which is inconsistent with \( S^* = S \). Now let \( m(x_0) = +\infty \) hold. Since \( R \) is non-atomic, we can find \( p \in R \) with \( m\left(\frac{2}{3}[p]x_0\right) \leq \frac{1}{4} \) and \( \lim_{t \uparrow 1} m(\xi[p_0]x_0) = +\infty \). Now we put
\[ \alpha = \frac{1}{\left\| \frac{2}{3} [p] x_0 \right\|} \text{ and } y = \alpha \frac{2}{3} [p] x_0. \] Then we have \( 1 < \alpha < \frac{3}{2} \) and \( y \in S_m \), hence

\[ \pi_-(y) = \pi_-(\alpha \frac{2}{3} [p] x_0) \geq \frac{m(\alpha \frac{2}{3} [p] x_0) - m(\frac{2}{3} [p] x_0)}{\alpha - 1} \geq \frac{1 - \frac{1}{4}}{\frac{1}{3}} > 2. \]

This contradicts the assumption: \( S = S^* \) by Lemmata 1 and 2. Therefore we have proved that \( m(x) < +\infty \) for all \( x \in R \). Q.E.D.

**Thoreme 2.2.** Let \( R \) be non-atomic and the condition (1.3) be satisfied, then the modular \( m \) satisfies one of the following conditions:

(2.10) \( m(\xi x) < \xi^2 m(x) \) for all \( \xi > 1 \) and \( x \in R \) with \( m(x) \geq 1 \);

(2.11) \( m(\xi x) > \xi^2 m(x) \) for all \( \xi > 1 \) and \( +\infty > m(x) \geq 1 \).

**Proof.** In virtue of the foregoing theorem we know that one of the conditions (2.8) or (2.9') is true. First we suppose that (2.8) holds. Then Lemma 7 together with Lemma 3 implies that \( m \) is finite. If \( m(x) = N + \frac{m}{n} \) (where \( N, m \) and \( n \) are natural numbers with \( m \leq n \)), we can decompose orthogonally \( x \) into \( x = \sum_{i=1}^{N-1} x_i + \sum_{j=1}^{n+m} y_j \) such that \( m(x_i) = 1 \) (\( i = 1, 2, \ldots, N-1 \)) and \( m(y_j) = \frac{1}{n} \) (\( j = 1, 2, \ldots, n+m \)). The number of \( j \) satisfying \( \pi_+(y_j) \geq 2m(y_j) \) is less than \( n \), because if there exist \( j_1, j_2, \ldots, j_n \) with \( \pi_+(y_{j_k}) \geq 2m(y_{j_k}) \) (\( k = 1, 2, \ldots, n \)), we have \( \sum_{k=1}^{n} y_{j_k} \in S_m \) and \( \pi_+(\sum_{k=1}^{n} y_{j_k}) = \sum_{k=1}^{n} \pi_+(y_{j_k}) \geq \sum_{k=1}^{n} 2m(y_{j_k}) = 2m(\sum_{k=1}^{n} y_{j_k}) = 2 \), which is inconsistent with (2.8). Hence we can find \( \{j_p\} \) (\( 1 \leq p \leq m \)) such that \( \pi_+(y_{j_p}) < 2m(y_{j_p}) \) (\( p = 1, 2, \ldots, m \)). Putting \( y_0 = \sum_{p=1}^{m} y_{j_p} \), we obtain \( m(\sum_{j=1}^{m+n} y_j - y_0) = 1 \) and \( \pi_+(\sum_{j=1}^{m+n} y_j - y_0) < 2 \). Therefore we have

\[ \pi_+(x) = \sum_{i=1}^{N-1} \pi_+(x_i) + \pi_+(\sum_{j=1}^{m+n} y_j - y_0) + \pi_+(y_0) < 2(N-1) + 2 + 2m(y_0) = 2 \left( N + \frac{m}{n} \right) = 2m(x), \]

hence \( \pi_+(x) < 2m(x) \). In general, if \( 1 < m(x) \), we can find \( \{\alpha_n\}_{n=1}^{\infty} \) with \( \alpha_n \downarrow_{n=1}^{\infty} 1 \) and \( m(\alpha_n x) \) is a rational number for each \( n \geq 1 \). It follows

\[ \pi_+(x) = \lim_{n \to \infty} \pi_+(\alpha_n x) \leq \lim_{n \to \infty} 2m(\alpha_n x) = 2m(x), \]
On the Ratios of the Norms defined by Modulars

hence

\[(2.12) \quad \pi_+(x) \leq 2m(x) \quad \text{for all } x \in R \text{ with } 1 \leq m(x).\]

Now we put

\[(2.13) \quad m'_\ell(\xi) = \lim_{\varepsilon \to 0} \frac{m((\xi + \varepsilon)x) - m(\xi x)}{\varepsilon} \]

for each $x \in R$ and $\xi \geq 0$. It is clear that $\xi \cdot m'_\ell(\xi) = \pi_+(\xi x)$ for all $\xi > 0$ and $x \in R$. (2.12) implies

\[(2.14) \quad \frac{m'_\ell(\xi)}{m(\xi x)} \leq \frac{2}{\xi} \quad (\xi > 1)\]

for every $x \in R$ with $m(x) \geq 1$. Integrating both sides of (2.14) from 1 to $\eta > 1$ with respect to $\xi$, we have

\[(2.15) \quad \log \frac{m(\eta x)}{m(x)} \leq 2 \log \xi \quad (\eta > 1).\]

In formula (2.15), however, the equal sign does not hold in any case. Indeed, since as is shown above, the set of all $\xi$ satisfying $\xi m'_\ell(\xi) = \pi_+(\xi x) < 2m(\xi x)$ is dense in $[1, +\infty)$ and $m(\xi x)$ is a continuous function of $\xi$, there exists an interval $(\xi_0, \eta_0) \subseteq (1, \eta)$ such that

$\xi m'_\ell(\xi) = \pi_+(\xi x) < 2m(\xi x)$

holds for all $\xi \in (\xi_0, \eta_0)$. Therefore we have

$m(\eta x) < \eta^2 m(x)$

for all $\eta > 1$ and $x \in R$ with $m(x) \geq 1$.

By the quite same manner we can prove that the condition (2.11) is satisfied, if we assume that the condition (2.9') is true. Q.E.D.

§ 3. Here we consider the case that the norms defined by a modular $m$ satisfy (1.4), that is, $\sup_{0 \neq x \in R} \frac{||x||}{|||x|||} < 2$. From the results proved in § 2 we have

**Theorem 3.1.** If the condition (1.4) is satisfied, then either

\[(3.1) \quad \sup_{x \in S} \pi_+(x) < 2 \]

or

\[(3.2) \quad \inf_{x \in S} \pi_-(x) > 2 \]

holds.

**Proof.** In virtue of Theorem 2.1, we can see that either (2.8) or (2.9') holds. First let (2.8) be true and set $\gamma = \sup_{0 \neq x \in R} \frac{||x||}{|||x|||}$. Then for
each \( x \in S \) and \( \varepsilon > 0 \), there exists \( \xi > 1 \) such that \( \frac{1+m(\xi x)}{\xi} < \gamma + \varepsilon \) by Lemma 3. From this

\[
\pi_+(x) \leq \frac{m(\xi x) - 1}{\xi - 1} < \gamma + \varepsilon
\]

follows, if \( \gamma + \varepsilon < 2 \). Hence we have \( \gamma \geq \pi_+(x) \) for all \( x \in S \).

On the other hand, we can prove by the same way\(^7\) that (2.9') together with (1.4) implies (3.2).

Q.E.D.

Remark 1. The converse of Theorem 3.1 does not remain true in general. It is easily verified that there exists a modular which does not fulfill (1.4) but satisfies (3.1) (or (3.2)).

Remark 2. As is seen in the proof of Theorem 3.1, it is clear that

\[
\sup_{x \in S} \pi_+(x) \leq \gamma \quad \text{or} \quad \inf_{x \in S_{m}} \pi_-(x) \geq \frac{\gamma}{\gamma - 1}
\]

holds respectively, where \( \gamma = \sup_{0 \neq x \in R} \frac{||x||}{|||x|||} \) and \( \xi > 1 \).

As for non-atomic \( R \), corresponding to Theorem 2.2, we have

Theorem 3.2. Let \( R \) be non-atomic and the condition (1.4) be satisfied, then either

\[
\begin{align*}
(3.3) & \quad m(\xi x) \leq \xi^p m(x) \quad \text{for all } \xi \geq 1 \text{ and } x \in R \text{ with } m(x) \geq 1; \\
(3.4) & \quad m(\xi x) \geq \xi^{p'} m(x) \quad \text{for all } \xi \geq 1 \text{ and } x \in R \text{ with } m(x) \geq 1,
\end{align*}
\]

where \( p \) and \( p' \) are real numbers with \( 1 \leq p < 2 < p' \leq +\infty \)\(^8\).

Proof. In virtue of the preceding theorem we need only to verify implications: (3.1) \( \Rightarrow \) (3.3) and (3.2) \( \Rightarrow \) (3.4). And these implications can be ascertained by the same manner as in the proof of Theorem 2.2. Here we may choose \( p, p' \) as \( p = \gamma \) and \( p' = \frac{\gamma}{\gamma - 1} \) respectively, where

\[
\gamma = \sup_{0 \neq x \in R} \frac{||x||}{|||x|||}.
\]

Q.E.D.

Remark 3. As is easily verified by calculating \( ||x|| \) of \( x \in S \), the condition (3.3) is the sufficient one for (1.4) at the same time. On the other hand, (3.4) is not such a one in general.

\( \S \) 4. At last we deal with a modular of unique spectra \([4; \S 54]\) and estimate exactly \( \sup_{0 \neq x \in R} \frac{||x||}{|||x|||} \) and \( \inf_{0 \neq x \in R} \frac{||x||}{|||x|||} \) as applications of the

---

7) We note that \( 2 - \gamma \xi \geq \frac{\gamma}{\gamma - 1} (1 - \xi) \) holds if \( 1 < \gamma < 2 \) and \( 1 < \gamma \xi \).

8) When \( p' = +\infty \), we put \( \xi^\infty = \infty \) if \( \xi > 1 \).
On the Ratios of the Norms defined by Modulars

preceding theorems and those of [7]. An element $0 \leq s \in R$ is said to be simple, if $m(s) < +\infty$ and $m([p]s) = 0$ implies $[p]s = 0$. And a modular $m$ is said to be of unique spectra if $m(\xi s) = \int_{[\xi]}^{\xi^{(p)}} m(dps)^{\rho}$ for all $\xi \geq 0$ and simple elements $s \in R$. Function spaces $L^{p(t)}$ (where $p(t)$ is a measurable function with $p(t) \geq 1$ $(0 \leq t \leq 1)$): the totality of all measurable functions $\varphi(t)$ such that

$$\int_{0}^{1} |\alpha \varphi(t)|^{p(t)} dt < +\infty$$

for some $\alpha > 0$.

and sequence spaces $l^{p_{\nu}}$ (where $p_{\nu} \geq 1$ $(\nu \geq 1)$): the totality of all sequences $\mathfrak{x} = (\xi_{\nu})_{(\nu \geq 1)}$ such that

$$\sum_{\nu=1}^{\infty} |\alpha \xi_{\nu}|^{p_{\nu}} < +\infty$$

for some $\alpha > 0$.

are the examples of modulared spaces whose modulars are of unique spectra, where the modulars are defined as $m(\varphi) = \int_{0}^{1} |\varphi(t)|^{p(t)} dt$ and $m(x) = \sum_{\nu=1}^{\infty} |\xi_{\nu}|^{p_{\nu}}$ respectively. When $m$ is of unique spectra, we denote by $\rho_{u}, \rho_{l}$ the upper exponent of $m$ and the lower exponent of $m$: $\rho_{u} = \sup_{\mathfrak{p} \in \infty} \rho(\mathfrak{p})$ and $\rho_{l} = \inf_{\mathfrak{p} \in \infty} \rho(\mathfrak{p})$ respectively. There exist normal manifolds $N_{1}, N_{2}$ such that $R = N_{1} \oplus N_{2}$ and $\rho(p)$ is finite for all $\mathfrak{p} \in U_{[N_{2}]}$ and $\rho(\mathfrak{p}) = +\infty$ for all $\mathfrak{p} \in U_{[N_{1},]}$ that is, $m$ is singular in $N_{2}$. For any $0 \neq x \in N_{2}$ we have $\|x\| = \|\|x\||$ and $S = S_{\ast}$. Therefore we obtain

**Theorem 4.1.** If a modular $m$ is of unique spectra, then we have

(11) $\sup_{0 \neq x \in R} \frac{\|x\|}{\|\|x\||} \left\{ \begin{array}{ll} = 2, & \text{if } \rho_{l} \leq 2 \leq \rho_{u}, \\ = \frac{1}{\rho_{u}} q_{u}^{\frac{1}{\rho_{u}}} q_{u}, & \text{if } \rho_{u} < 2, \\ = \frac{1}{\rho_{l}} q_{l}^{\frac{1}{\rho_{l}}} q_{l}, & \text{if } \rho_{l} > 2, \end{array} \right.$

where $q_{u}$ and $q_{l}$ are real numbers with $\frac{1}{\rho_{u}} + \frac{1}{q_{u}} = 1$ and $\frac{1}{\rho_{l}} + \frac{1}{q_{l}} = 1$.

**Proof.** As $\frac{\|x\|}{\|\|x\||} = 1$ for all $0 \neq x \in N_{2}$, we may consider only the

9) $\mathfrak{p}$ is a point of representation space $\in$ of $R$, i.e. the maximal ideal of normal manifolds of $R$. For $N$, we denote by $U_{[N]}$ the totality of all $\mathfrak{p} \in \in$ with $N \in \mathfrak{p}$. $\rho(\mathfrak{p})$ is a continuous function on $\in$ with $\rho(\mathfrak{p}) \geq 1$. 


ratios of the norms in $N_1$. When $\gamma \leq \rho(p) \leq \gamma'$ for all $p \in U_{[p]_1} \subseteq U_{[p]_3}$, then we have for all $x \in R$, $\xi' m([p]x) \leq m(\xi [p]x) \leq \xi'' m([p]x)$ ($\xi \geq 1$) and $\gamma' m([p]x) \geq m(\gamma [p]x) \geq \gamma'' m([p]x)$ ($0 \leq \gamma \leq 1$). From this and Lemma 4 we have $\sup_{0 \neq x \in R} \frac{||x||}{||x||} = 2$, if $\rho_t \leq 2 \leq \rho_u$. Since $\sup_{x \in S} \pi_-(x) < 2$ if and only if $\rho_u < 2$ and $\inf_{x \in S} \pi_+(x) > 2$ if and only if $\rho_l > 2$ for $m$, we have by Lemmata 2 and 3 that $||x|| \leq \rho_u^{\frac{1}{\rho_u}} q_u^{\frac{1}{q_u}} (x \in S)$ and $||x|| \leq \rho_l^{\frac{1}{\rho_l}} q_l^{\frac{1}{q_l}}$ respectively according to (3.1) and (3.2). Therefore we complete the proof.

Similarly we can conclude by Theorem 3.1 in [7]

**Theorem 4.2.** If a modular $m$ is of unique spectra, then we have

$$\inf_{0 \neq x \in R} \frac{||x||}{||x||} = \min \left\{ \rho_t^{\frac{1}{\rho_t}} q_t^{\frac{1}{q_t}}, \rho_u^{\frac{1}{\rho_u}} q_u^{\frac{1}{q_u}} \right\}.$$

Department of Mathematics,
Hokkaido University, Sapporo.

**References**


