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CONTINUOUS FILTERING AND ITS SPECTRAL SEQUENCE

By

Hisashi NUMATA

0. A filtering $f$ of a ring $A$ is a integer valued function on $A$ satisfying the following three conditions:

\begin{align}
(0.1) & \quad f(x+y) \geq \min\{f(x), f(y)\}, \quad x, y \in A, \\
(0.2) & \quad f(xy) \geq f(x)+f(y), \\
(0.3) & \quad f(0) = +\infty.
\end{align}

Thus, the notion of filtering can be regarded as a generalization of discrete valuation of a field. For purely algebraic interest, it seems to be natural to consider a continuous filtering as the generalization of continuous valuation.

In this note, we consider a real valued function $F$ on $A$ satisfying the above three conditions. We call $F$ a continuous filtering of $A$, and the ring $A$ is said to be a continuously filtered ring.

Sections 1 and 2 are devoted to describe analogous definitions notations and relations to those of J. Leray [1], and the main parts of this note are sections 3 and 4.

1. A ring $A$ is called a continuously graded ring if

\[ A = \sum_{p \in R} A^{[p]} \] (direct sum, $R$ is the set of reals)

where $\{A^{[p]}\}$ are submodules of $A$ and satisfy

\[ A^{[p]} \cdot A^{[q]} \subset A^{[p+q]} \]

A continuously filtered ring $A$ is called a continuously filtered differential ring if $A$ has a differentiation $(d, a)$ subjected to

\begin{align}
& d^2 = 0, \\
& adx + dax = 0, \quad x, y \in A, \\
& d(xy) = dx \cdot y + ax \cdot dy, \quad (a \text{ is an automorphism of } A), \\
& F(ax) = F(x).
\end{align}

A differentiation $(d, a)$ is called homogeneous of degree $r$ ($r \in R$) if
$(d, a)$ is a differentiation of a continuously graded ring $A$ and
\[
d A^{[p]} \subset A^{[p+r]} \quad \text{for any } p \in R.
\]

If $B$ is an ideal of a continuously filtered ring $A$, then $A/B$ becomes a continuously filtered ring if we define
\[
\bar{F}(\bar{x}) = \sup_{x \in \bar{x}} F(x) \quad \text{for } \bar{x} \in A/B.
\]

1. If $B$ is an ideal of $A$, then $A/B$ becomes a continuously filtered ring if we define
\[
(1.1) \quad \overline{F}(\overline{x})=\sup_{x \in \overline{x}} F(x) \quad \text{for } \overline{x} \in A/B.
\]

2. From now on, $A$ means a continuously filtered differential ring. We set
\[
A^p = \{ x \mid x \in A, F(x) \geq p \} \quad p \in R,
\]
then this is a submodule of $A$, and
\[
A^p \subset A^q \quad \text{if } p \geq q, \quad \bigcup_p A^p = A,
\]
\[
A^p \cdot A^q \subset A^{p+q}.
\]
Define, for $\epsilon > 0$,
\[
G_\epsilon(A) = \sum_p A^p / A^{p+\epsilon}
\]
and define the multiplication by
\[
(x^p \mod A^{p+\epsilon})(x^q \mod A^{q+\epsilon}) = x^p \cdot x^q \mod A^{p+q+\epsilon},
\]
Then $G_\epsilon(A)$ becomes a continuously graded ring, called the $\epsilon$-graded ring of $A$. If we put
\[
C = \text{kernel of } d, \quad D = \text{image of } d,
\]
\[
C^p = A^p \cap C, \quad D^p = A^p \cap D,
\]
\[
C^p = \{ x \mid x \in A^p, \; dx \in A^{p+r} \}, \quad dC^p = D^p + r,
\]
then we have
\[
(2.1) \quad D^p \subset D^{p+r}, \quad \bigcup_{r \in R} D^p = D^p, \quad D^p \subset C^p, \quad C^{p+r} \subset C^p,
\]
\[
(2.2) \quad C^{p+r} = C^p \cap A^{p+r} \subset C^p,
\]
\[
(2.3) \quad D^{p+r} = D^p \cap A^{p+r} \subset D^p,
\]
\[
(2.4) \quad C^p \cdot C^q \subset C^{p+q},
\]
\[
(2.5) \quad C^p \cdot D^q \subset C^{p+q} + D^{p+q}, \quad D^{p+q} \cdot C^p \subset C^{p+q} + D^{p+q},
\]
(2.4) implies that $\sum_{p \in R} C^p$ (direct sum of modules $C^p$) can be considered to be a continuously graded ring, while (2.5) means that
\[
\sum_{p \in R} (C^{p+r} + D^{p+r})
\]
is an ideal of $\sum_{p \in R} C^p$.

We define
\[ H_{r+\epsilon}(A) = \sum_p C^p_{r+\epsilon}/(C^p_{r-\epsilon} + D^p_{r-\epsilon}) \cdot \]

Then \( H_{r+\epsilon}(A) \) has a differentiation \((d_{r,\epsilon}, a_{r,\epsilon})\) of homogeneous of degree \( r \) by

\[
d_{r,\epsilon}h_{r,[p]} = dc^p \mod (C^p_{r-\epsilon} + D^p_{r-\epsilon}) ,
\]

\[
a_{r,\epsilon}h_{r,[p]} = ac^p \mod (C^p_{r-\epsilon} + D^p_{r-\epsilon}) ,
\]

where \( h_{r,[p]} \in H_{r+\epsilon}(A) \) is homogeneous of degree \( p \) and \( c^p \in h_{r,[p]} \). Next, we define the cohomology ring of \( H_{r+\epsilon}(A) \), we use the notation \( H(H_{r+\epsilon}(A)) \).

A parallel argument to that of J. Leray [1] Chap. I, \S 9 shows that

\[
C(H_{r+\epsilon}(A)) = \text{kernel of } d_{r,\epsilon} = \sum_p (C^p_{r+\epsilon} + C^p_{r-\epsilon})/(C^p_{r-\epsilon} + D^p_{r-\epsilon}) ,
\]

\[
D(H_{r+\epsilon}(A)) = \text{image of } d_{r,\epsilon} = \sum_p (C^p_{r-\epsilon} + D^p_{r+\epsilon} + D^p_{r-\epsilon})/(C^p_{r-\epsilon} + D^p_{r-\epsilon})
= \sum_p (C^p_{r-\epsilon} + D^p_{r-\epsilon})/(C^p_{r-\epsilon} + D^p_{r-\epsilon}) ,
\]

whence

\[
H(H_{r+\epsilon}(A)) = \sum_p (C^p_{r-\epsilon} + C^p_{r-\epsilon})/(C^p_{r+\epsilon} + D^p_{r-\epsilon})
= \sum_p C^p_{r+\epsilon}/[C^p_{r+\epsilon} \cap (C^p_{r+\epsilon} + D^p_{r-\epsilon})]
= \sum_p C^p_{r+\epsilon}/(C^p_{r+\epsilon} + D^p_{r-\epsilon}) = H_{r+\epsilon+\epsilon}(A) .
\]

3. In this section, we proceed to define an inverse mapping system of \( \{H_{r+\epsilon}(A)\}_{\epsilon>0} \) and consider the projective limit of this system. Since

\[
C^p_{r+\epsilon} + D^p_{r+\epsilon} \supset C^p_{r+\epsilon} + D^p_{r-\epsilon},
\]

for \( 0<\sigma<\tau \), we can define a natural inverse mapping \( \pi^\tau_{\epsilon} : H_{r+\epsilon}(A) \to H_{r+\epsilon}(A) \).

The projective limit of this system is denoted by

\[
\text{p-lim}_{\epsilon} H_{r+\epsilon}(A) = H_r ,
\]

and we define a differentiation \((d_r, a_r)\) by

\[
d_r h_r = (\cdots, d_{r,\epsilon} h_{r,\epsilon}, \cdots) ,
\]

\[
a_r h_r = (\cdots, a_{r,\epsilon} h_{r,\epsilon}, \cdots)
\]

for

\[
h_r = (\cdots, h_{r,\epsilon}, \cdots) \in H_r (\pi^\tau_{\epsilon} h_{r,\epsilon} = h_{r,\epsilon}) .
\]

It is easy to see that the above definition of \((d_r, a_r)\) has no inconvenience. Also we can define naturally an inverse system of \( \{H_{r+\epsilon}(A)\} \) and the projective limit

\[
\text{p-lim}_{\epsilon} H_{r+\epsilon}(A) ,
\]

because of
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\[ C^{p}_{r+\sigma} \supset C^{p}_{r-\sigma} \quad \text{and} \quad C^{p+\sigma}_{r} + D^{p}_{r} \supset C^{p+\tau}_{r} + D^{p}_{r}. \]

For (3.1) and (3.2), the following relation is true:

\[ H(H_{r}) = H(p\lim \limits_{\sigma} H_{r,\sigma}(A)) = p\lim \limits_{\sigma} H(H_{r,\sigma}(A)) \]

\[ = p\lim \limits_{\sigma} H_{r+\sigma,\sigma}(A). \]

For the proof, a straightforward computation shows that

\[ C(H_{r}) = \text{kernel of } d_{r} = p\lim \limits_{\sigma} C(H_{r,\sigma}(A)) \]

\[ D(H_{r}) = \text{image of } d_{r} = p\lim \limits_{\sigma} D(H_{r,\sigma}(A)), \]

so that we get

\[ H(H_{r}) = C(H_{r})/D(H_{r}) \cong p\lim \limits_{\sigma} \{C(H_{r,\sigma}(A))/D(H_{r,\sigma}(A))\}. \]

(\pi_{\tau}^{\sigma} \text{ induce the natural inverse system of } C(H_{r,\sigma}(A))/D(H_{r,\sigma}(A)).)

4. We define another continuously graded ring

\[ H_{\infty,\sigma}(A) = \sum_{p} C^{p}/(C^{p+\sigma} + D^{p}). \]

Then we have

(4.1) \[ H_{\infty,\sigma}(A) = G_{\sigma}(H(A)), \]

where \( H(A) \) is the cohomology ring of \( A \) with the filtering defined as (1.1). The proof is analogous to that for discrete filtration and is omitted.

Next we consider

\[ I_{r,\sigma} = \sum_{p} \left( \bigcap_{n>0} C^{p}_{r+n\sigma} \right)/(C^{p+\sigma}_{r-\sigma} + D^{p}_{r-\sigma}) \]

and an ideal of \( I_{r,\sigma} \)

\[ J_{r,\sigma} = \sum_{p} (C^{p+\sigma} + D^{p})/(C^{p+\sigma}_{r-\sigma} + D^{p}_{r-\sigma}). \]

Then we have easily

\[ I_{r,\sigma}/J_{r,\sigma} \cong I_{r+\tau,\sigma}/J_{r+\tau,\sigma} \quad \text{for } t > \sigma, \]

therefore we identify all \( I_{r+t,\sigma} \), and denote

\[ \lim \limits_{r\rightarrow\infty} H_{r,\sigma}(A). \]

An analogous relation to (4.1) holds

\[ G_{\sigma}(H(A)) \subset \lim \limits_{r\rightarrow\infty} H_{r,\sigma}(A). \]

Again, if we use the natural inverse system, then we get

\[ p\lim \limits_{\sigma} (G_{\sigma}(H(A))) \subset p\lim \limits_{\sigma} (\lim \limits_{r\rightarrow\infty} H_{r,\sigma}(A)). \]
5. In 3 and 4, we defined two limits of $H_{r,\sigma}(A)$, $p$-lim and $\lim$. These two operations are not commutative, because $\lim (\lim_{r \to \infty} H_{r,\sigma}(A))$ can be always defined, while $\lim_{r \to \infty} (p\text{-}\lim_{\sigma} H_{r,\sigma}(A))$ cannot be defined so far as we use only the natural procedure.

Hokkaido University

References
