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PARTIALLY ORDERED ABELIAN SEMIGROUPS. III
ON THE REVERSIBLE PARTIAL ORDER DEFINED
ON AN ABELIAN SEMIGROUP

By
Osamu NAKADA

In their paper,¹ Ben Dushnik and E. W. Miller introduced the concept of the reversible partial order and expressed the theorem about this concept. In this Part III, I shall show that the same one is held in the partially ordered abelian semigroup by adding the certain condition.

Definition 1. A set $S$ is said to be a partially ordered abelian semigroup (p.o. semigroup), when $S$ is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the homogeneity: $a \geq b$ implies $ac \geq bc$ for any $c$ of $S$.

A partial order which satisfies the condition (III) is called a partial order defined on an abelian semigroup.

Moreover, if a partial order defined on an abelian semigroup $S$ is a linear order, then $S$ is said to be a linearly ordered abelian semigroup (l.o. semigroup). (Definition 1, O.I.)

Definition 2. Let $\mathcal{S} = \{P_a\}$ be any set of partial orders, each defined on the same abelian semigroup $S$. We define the new partial order $P$ on $S$ as follows: For any two elements $a, b$, we put $a \geq b$ in $P$ if and only if $a \geq b$ in every $P_a$ of the set $\mathcal{S}$. Indeed, $P$ is again a partial order defined on $S$. This partial order $P$ is said to be the product of the partial orders $P_a$ or to be realized by the set $\mathcal{S}$ of partial orders $P_a$. (Definition 9, O.I.)

By the dimension of a partial order $P$ defined on an abelian semigroup $S$ is meant the smallest cardinal number $\mathfrak{m}$ such that $P$ is realized by $\mathfrak{m}$ linear orders defined on $S$.

Definition 3. Let $P$ and $Q$ be two partial orders defined on the same

abelian semigroup $S$, and suppose that any two distinct elements of $S$ are comparable in just one of these partial orders; in such a case we shall say that $P$ and $Q$ are conjugate partial orders. A partial order will be called reversible if and only if it has a conjugate. 2)

If $P$ is a partial order defined on $S$, then the partial order obtained from $P$ by inverting the sense of all ordered pairs will be called a dual order, which is denoted by $P^*$.  

**Theorem 1.** Let $P$ and $Q$ are conjugate partial orders defined on an abelian semigroup $S$. Then we can define a linear order $L_1$ on $S$ such that $a>b$ in $L_1$ if and only if $a>b$ in either $P$ or $Q$; denoted by $L_1=P+Q$. Similary $L_2=P+Q^*$ is a linear order defined on $S$.

**Proof.** We shall prove only the transivity of $L_1$.

From $a>b$, $b>c$ in $L_1$, we can consider the following four cases:

(i) $a>b$, $b>c$ in $P$, (ii) $a>b$, $b>c$ in $Q$, (iii) $a>b$ in $P$, $b>c$ in $Q$, (iv) $a>b$ in $Q$, $b>c$ in $P$.

In cases (i) and (ii), $a>c$ in $L_1$ is cleary. In case (iii), if $c>a$ in $P$ or $Q$, then $c>b$ in $P$ or $b>a$ in $Q$ respectively, which is assured, therefore $a>c$ in $P$ or $Q$ and hence in $L_1$. Similary, in case (iv) $a>c$ in $L_1$ is held.

**Theorem 2.** The following two properties of a partial order $P$ defined on an abelian semigroup $S$ are equivalent to each other:

1. $P$ is reversible.
2. The dimension of $P$ is 2.

**Proof.** We shall show first that (1) implies (2). Suppose that the partial order $P$ defined on $S$ is reversible, and let $Q$ be a partial order defined on $S$ conjugate to $P$ and $Q^*$ be the dual order of $Q$. Then by Theorem 1, $L_1=P+Q$ and $L_2=P+Q^*$ are linear extensions of $P$ and it is obvious that $P$ is realized by linear orders $L_1$ and $L_2$.

Next we show that (2) implies (1). Let $L_1$ and $L_2$ be any two linear orders defined on $S$ which together realize $P$. We define the other order $Q$ as follows: $a>b$ in $Q$ if and only if $a$ and $b$ are non-comparable in $P$ and $a>b$ in $L_1$ (likewise $a>b$ in $L_1$ and $b>a$ in $L_2$). Then $a>b$ and $b>a$ in $Q$ are contradictory. If $a>b$ and $b>c$ in $Q$, then we have $a>b>c$ in $L_1$ and $c>b>a$ in $L_2$, hence $a>c$ in $Q$. $a>b$ in $Q$ implies that $ac\geq bc$ in $L_1$ and $bc\geq ac$ in $L_2$, i.e. $ac\geq bc$ in $Q$. Therefore $Q$ is a partial order defined on $S$. Evidently $P$ and $Q$ are conjugate.

**Definition 4.** A linear extension $L$ of a partial order $P$ defined on

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an abelian semigroup $S$ will be called separating if and only if there
exist three elements $a$, $b$ and $c$ in $S$ such that $a>c$ in $P$, and $b$ is not
comparable with either $a$ or $c$ in $P$, while in $L$ we have $a>b>c$.

**Theorem 3.** Let $P$ be a partial order defined on an abelian semi-
group $S$ which satisfies the condition $(E)$. Then the following three
properties of a partial order $P$ are equivalent to each other:

1. $P$ is reversible.
2. The dimension of $P$ is 2.
3. There exists a linear extension of $P$ which is non-separating.

Proof. (1) and (2) are equivalent by Theorem 2.

We show now that (2) implies (3) without the condition $(E)$. Let
$L_1$ and $L_2$ be any two linear orders defined on $S$ which together realize
$P: P = L_1 \times L_2$. If $L_1$ is separating, then there exist three elements $a$, $b$
and $c$ such that $a>c$ in $P$, $a>b>c$ in $L_1$ and $b$ is not comparable with
either $a$ or $c$ in $P$. Hence we have $c>b>a$ in $L_2$ which is impossible.

To show that (3) implies (1) we shall suppose that $L$ is a non-
separating linear extension of $P$. We define the other order $Q$ as follows:
$a>b$ in $Q$ if and only if $a$ and $b$ are non-comparable in $P$ and $a>b$ in $L$.
Then clearly $a>b$ and $b>a$ in $Q$ are contradictory. If $a>b$ and $b>c$ in
$Q$, then we have $a>b>c$ in $L$ and $a$ and $c$ are non-comparable in $P$, for
otherwise $a>c$ in $P$ would imply that $L$ is separating contrary to the
assumption, hence we have $a>c$ in $Q$. $a>b$ in $Q$ implies that $ac\geq bc$ in
$L$. If $ac>bc$ in $P$, then by the condition $(E)$ $a>b$ in $P$ which is impos-
sible. Hence $ac=bc$ or $ac$ and $bc$ are non-comparable in $P$, and hence
$ac\geq bc$ in $Q$. Therefore $Q$ is a partial order defined on $S$. Clearly $P$
and $Q$ are conjugate.

**Definition 5.** Let $S$ be a p.o. semigroup and $P$ be the partial order
defined on $S$. For any element $a$ of $S$, we denote the set of all elements
$x$ such that $x \leq a$ in $P$ by $\bar{a}$. Then the correspondence $a \rightarrow \bar{a}$ is one-to-one.
We put $\bar{a} \supseteq \bar{b}$ if and only if $\bar{b}$ is a subset of $\bar{a}$, likewise $a \geq b$ in $P$, then
the family $\bar{S} = \{ \bar{a} \}$ is become a partially ordered set. Next we define the
product $\bar{a} \cdot \bar{b} = \bar{ab}$, then the family $\bar{S}$ is a commutative semigroup, more-
over $\bar{S}$ become a p.o. semigroup. Clearly $S$ and $\bar{S}$ are order-isomorphic.

More generally, if there exists a one-to-one correspondence between

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4) Condition (E) (order cancellation law):
   \[ ac > bc \text{ in } P \text{ implies } a > b \text{ in } P. \]
5) See Definition 3, O.I.
the elements of the p.o. semigroup $S$ and the family $\Re$ of subsets of the certain set $\mathcal{R}$ (a subset of $\mathcal{R}$ which corresponds with an element $a$ of $\mathcal{R}$, denote by $s(a)$), and $a \geq b$ in $P$ if and only if $s(a) \supseteq s(b)$ (in the sense of set-inclusion), then by the defining the product $s(a) \cdot s(b) = s(ab)$, two p.o. semigroups $S$ and $\Re$ are order-isomorphic.

Any family $\Re$ of the subsets of the set $\mathcal{R}$ which has the above properties will be called a representation of $P$.

**Theorem 4.** Let $P$ be a partial order defined on an abelian semigroup $S$ which satisfies the condition (E). Then the following two properties are equivalent to each other:

1. $P$ is reversible.
2. There exists a representation of $P$ by means of a family $\Re = \{I_a\}$ of closed intervals on some l.o. semigroup $\mathcal{R}$, and let $I_a = [\alpha_1, \alpha_2], I_b = [\beta_1, \beta_2], I_{ac} = [\gamma_1, \gamma_2], I_{bc} = [\delta_1, \delta_2]$, and if $a$ and $b$ are non-comparable in $P$, then $\alpha_1 < \beta_1$ (and $\alpha_2 < \beta_2$) implies $\gamma_1 \leq \delta_1$ (and $\gamma_2 \leq \delta_2$) or its dual.

**Proof.** We shall show (1) implies (4). Let $P$ be reversible, and hence the dimension of $P$ is 2. Let $A$ and $B$ be any two linear orders defined on $S$ which together realize $P$.

Let $S'$ be a l.o. semigroup which is anti-order-isomorphic to the l.o. semigroup $S$ in the linear order $B$, where the set $S'$ is disjoint from $S$, and the linear order defined on $S'$ is denoted by $B'$.

Let $R$ be the union of $S, S'$ and the new element 0 which belongs to neither $S$ nor $S'$.

We define the multiplication in $R$ as follows:

$$0 \cdot 0 = 0,$$

$$x \cdot 0 = 0 \cdot x = 0 \quad \text{for any } x \in S \text{ or } S',$$

$$a \cdot a' = a' \cdot a = 0 \quad \text{for any } a \in S \text{ and } a' \in S',$$

and for any two elements $x$ and $y$ of $S(S')$ the product is the same as in $S(S')$.

Thus $R$ becomes the abelian semigroup under the multiplication introduced above.

Let us now define the order-relation $L$ in $R$ as follows:

$$x > y \text{ in } L \quad (x, y \in S) \quad \text{if and only if } x > y \text{ in } A,$$

$$x > y \text{ in } L \quad (x, y \in S') \quad \text{if and only if } x > y \text{ in } B',$$

and we put

$$a > 0 > a' \text{ in } L \quad (a \in S, a' \in S').$$

Then $R$ becomes a l.o. semigroup.
For each $a$ in $S$ denote by $a'$ the image of $a$ in $S'$, and denote by $I_a$ the closed interval $[a', a]$ of $R$.

We will show that the family $\mathfrak{R} = \{I_a\}$ of all such intervals is a representation of $P$. Suppose first that $a > b$ in $P$. Then $a > b$ in $A$ and $a' < b'$ in $B'$, so that we have $a' < b' < a$ in $L$. This means that $I_b$ is a proper subset of $I_a$.

Let $I_a = [a', a], I_b = [b', b], I_{ac} = [a'c', ac], I_{bc} = [b'c', bc]$. If $a$ and $b$ are non-comparable in $P$, then from $a > b$ ($a' > b'$) in $L$ we have $ac \geq bc$ ($a'c' \geq b'c'$) in $L$ or its dual.

We prove that (4) implies (1). Suppose that $P$ is a partial order which is represented by a family $\mathfrak{R}$ of intervals taken from some l.o. semigroup $R$, whose linear order is denoted by $L$. For each $a$ in $S$, denote by $I_a$ the interval of the family $\mathfrak{R}$ which corresponds to $a$. We notice first that if $a$ and $b$ are distinct elements of $S$ which are not comparable in $P$, then $I_a$ and $I_b$ cannot have the same left (right)-hand end-point.

Suppose that $I_a = [\alpha_1, \alpha_2], I_b = [\beta_1, \beta_2], I_c = [\gamma_1, \gamma_2], \cdots$.

We define a new partial order $Q$ defined on $S$ as follows:

(i) $a$ and $b$ are not comparable in $P$,
(ii) $\alpha_1 < \beta_1$ (and $\alpha_2 < \beta_2$) in $L$.

It is easy to see that $Q$ is the partial order defined on the set $S$. We shall now prove the homogeneity. Let $a > b$ in $Q$ and $I_{ac} = [\lambda_1, \lambda_2], I_{bc} = [\mu_1, \mu_2]$. If $ac$ and $bc$ are distinct and comparable in $P$, then by the condition $(E)$ $a$ and $b$ are comparable in $P$ which is impossible. If $ac$ and $bc$ are non-comparable in $P$, then $\lambda_1 < \mu$, $\lambda_2 < \mu_2$ and hence $ac > bc$ in $Q$.

**Example 1.** Let $S_1$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation

$$a^m b^n = ab^n$$

for any positive integers $m$ and $n$.

By putting the order-relation

$$P: \begin{cases} a^{m+1} > a^m \\ b^n > ab^n \end{cases}$$

for any positive integer $n$

$S_1$ becomes a p.o. semigroup, and the partial order $P$ is reversible. Its conjugate order $Q$ is as follows:

$$Q: \begin{cases} a^m > b^n > b^{n+1} > ab^{n+1} \\ a^m > ab^n > b^{n+1} \end{cases}$$

for any positive integers $m$ and $n$.

The linear orders which together realize $P$ are
$a^{m+1} > a^m > b^n > ab^n > b^{n+1} > ab^{n+1}$
and
$b^{n+1} > ab^{n+1} > b^n > ab^n > a^{m+1} > a^m$
for any positive integers $m$ and $n$.

**Example 2.** Let $S_2$ be an abelian semigroup generated by two elements $a$ and $b$ with the relation
\[ a^m b^n = b^n \]
for any positive integers $m$ and $n$.

By putting the two order-relations $A$ and $B$
\[
A:\quad a^{m+1} > a^m > b^n > b^{n+1}
B:\quad b^{n+1} > b^n > a^{m+1} > a^m
\]
for any positive integers $m$ and $n$,
$S_2$ becomes a l.o. semigroup in the orders $A$ and $B$ respectively. Let $P$ be the partial order which is the product of $A$ and $B$, that is
\[
P:\quad a^{m+1} > a^m.
\]
Then $P$ has the conjugate order $Q$ such that
\[
Q:\quad a^m > b^n > b^{n+1}.
\]

**Example 3.** Let $S'_3$ and $S''_3$ be free abelian semigroups generated by elements $a$ and $b$ respectively. And by defining the order-relations
\[
a^{m+1} > a^m \quad (m \geq 1), \quad b^{n+1} > b^n \quad (n > 1),
\]
$S'_3$ and $S''_3$ becomes a l.o. semigroup and a p.o. semigroup respectively. Let $S_3$ be the direct product of $S'_3$ and $S''_3$. Then $S_3$ becomes a p.o. semigroup by introducing the following order-relation $P$:
\[
a^ib^j > a^m b^n
\]
if and only if
\[
a^i > a^m \quad \text{or} \quad a^i = a^m \quad \text{and} \quad b^j > b^n.
\]

Since $ab^2$ and $ab$ are non-comparable in $P$ in spite of $(ab^2)(ab) = a^2b^3 > a^2b^2 = (ab)(ab)$, $P$ does not satisfy the condition $(E)$.

Now, in $S''_3$ we define the another order-relation:
\[
b^{n+1} > b^n \quad (n \geq 1),
\]
then we get the non-separating linear extension of $P$.
But we cannot realize the partial order $P$ by two linear orders.

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