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**Instructions for use**
A NOTE ON STRICTLY GALOIS EXTENSION
OF PRIMARY RINGS

By

Takesi ONODERA and Hisao TOMINAGA

Let \( R \) be a primary ring with minimum condition (for one-sided ideals). One of the present authors proved in [1] that if \( R \) is strictly Galois with respect to \( \mathfrak{G} \) then \( R \) possesses a \( \mathfrak{G} \)-normal basis element. The purpose of this note is to present a slight generalization of this fact.

In what follows, \( R \) be always a primary ring with minimum condition which is strictly Galois with respect to (an \( F \)-group) \( \mathfrak{G} \) of order \( n \), \( N \mid 1 \) a subring of \( R \) with minimum condition such that \( N\mathfrak{G}=N \) and \( R \) possesses a linearly independent right \( N \)-basis consisting of \( t \) elements. Further, we set \( t=nq+r \), where \( 0 \leq r < n \). Under this situation, our theorem can be stated as follows:

**Theorem.** There exist \( q \) elements \( x_1, \ldots, x_q \in R \) and a \( \mathfrak{G}N_r \)-submodule \( M \) of \( R \) such that

1. \( M \) is \( \mathfrak{G}N_r \)-homomorphic to \( \mathfrak{G}N_r \) and possesses a linearly independent right \( N \)-basis consisting of \( r \) elements,
2. \( R = \sum_{i=1}^{q} \bigoplus_{\sigma \in \mathfrak{G}^\oplus} \sum (x_i \sigma)N \oplus M \).

**Proof.** As is shown in [1], \( \text{Hom}_{\mathfrak{G}}(R, R) = \mathfrak{G}R_r = \sum_{\sigma \in \mathfrak{G}^\oplus} \sigma R_r \), where \( S = J(\mathfrak{G}, R) \). Since \([R:S] = n\), and so, since \( R \) is \( S \)-left regular, \( R \) is \( \text{Hom}_{\mathfrak{G}}(R, R) \)-right regular too. In fact, \( R^{(n)} \) is \( \mathfrak{G}R_r \)-isomorphic to \( \mathfrak{G}R_r \), where \( R^{(n)} \) means the direct sum of \( n \)-copies of \( R \) as \( \mathfrak{G}R_r \)-module. Accordingly, \( R^{(n)} \) is \( \mathfrak{G}N_r \)-isomorphic to \( \mathfrak{G}R_r \) of course. Now let \( R = u_1 N \oplus \cdots \oplus u_{\ell} N \).

Then, we have \( \mathfrak{G}R_r = \mathfrak{G} \sum_{i=1}^{\ell} u_i N, \mathfrak{G} = \sum_{i=1}^{\ell} u_i \mathfrak{G}N_r \). Hence, \( \mathfrak{G}R_r \) is \( \mathfrak{G}N_r \)-isomorphic to \( (\mathfrak{G}N_r)^{(q)} \), and so we have eventually that \( R^{(n)} \) is \( \mathfrak{G}N_r \)-isomorphic to \( (\mathfrak{G}N_r)^{(r)} \). Here let \( \mathfrak{p}_1, \ldots, \mathfrak{p}_s \) be all the non-isomorphic directly indecomposable direct summands of the \( \mathfrak{G}N_r \)-module \( R \) (or \( \mathfrak{G}N_r \) itself). And, in the Remak decompositions of \( \mathfrak{G}N_r \)-modules \( R \) and \( \mathfrak{G}N_r \), the re-

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1) As to notations and terminologies used in this note, we follow [1]. And we will use freely the results cited in [1].

2) \( N \) does not necessarily contain the subring \( S = J(\mathfrak{G}, R) \).
spective numbers of directly indecomposable components which are isomorphic to \( v_i \) will be denoted by \( n_i \) and \( m_i \). Then, our isomorphism mentioned above yields at once \( n_i n = m_i t = m_i (nq + r) \), whence we have \( m_i r = nk_i \) with some non-negative integer \( k_i < m_i \). Consequently, it follows that \( n_i = m_i q + k_i (i = 1, \ldots, s) \). This proves clearly the existence of a \( \mathfrak{G}N_{r} \)-isomorphism \( \varphi \) of \( R \) onto \( (\mathfrak{G}N_{r})^{(q)} \oplus T \), where \( T = \sum_{i=1}^{s} \oplus v_i^{(k_i)} \). Recalling here \( m_i > k_i \), we see that \( T \) is \( \mathfrak{G}N_{r} \)-homomorphic to \( \mathfrak{G}N_{r} \). Now, let \( y_i = (0, \ldots, 0, 1, \ldots, 0) \in (\mathfrak{G}N_{r})^{(q)} \). Then, one will easily verify that \( x_i = \varphi^{-1}[y_i] \) (\( i = 1, \ldots, q \)) and \( M = \varphi^{-1}[T] \) are desired ones.

Our theorem may be considered as a generalization of [1, Theorem 1]. Moreover, in case \( R \) is a division ring we obtain the following which secures the existence of the so-called semi-normal basis.

**Corollary.** Let \( R \) be a division ring which is strictly Galois with respect to \( \mathfrak{G} \) of order \( n \), and \( N \) a division subring of \( R \) with \( N \mathfrak{G} = N \) and \( [R : N]_{r} = t \). If \( t = nq + r \) (\( 0 \leq r < n \)) then there exist some \( x_0, x_1, \ldots, x_q \in R \) such that \( R \cong \sum_{i=1}^{q} \oplus_{\sigma \in \mathfrak{G}} \sum (x_i \sigma)N \oplus \sum_{\tau} (x_0 \tau)N \), where \( \tau \) runs over a suitable subset of \( \mathfrak{G} \) consisting of \( r \) elements.

**References**


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