PARTIALLY ORDERED ABELIAN SEMIGROUPS. IV

ON THE EXTENTION OF THE CERTAIN NORMAL PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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In Part I\(^1\) of this series, I noted that for any two elements \(x\) and \(y\) non-comparable in the strong partial order \(P\) defined on an abelian semigroup \(S\) there exists an extension \(Q\) of \(P\) such \(x>y\) in \(Q\) if and only if \(P\) is normal. In this Part IV, I shall discuss the extension of the partial order under the weak condition than strongness.

Definition 1. A set \(S\) is said to be a partially ordered abelian semigroup (p.o. semigroup), when \(S\) is (I) an abelian semigroup (not necessarily contains the unit element), (II) a partially ordered set, and satisfies (III) the homogeneity: \(a \geq b\) implies \(ac \geq bc\) for any \(c\) of \(S\).

A partial order which satisfies the condition (III) is called a partial order defined on an abelian semigroup.

Moreover, if a partial order defined on an abelian semigroup \(S\) is a linear order, then \(S\) is said to be a linearly ordered abelian semigroup (l.o. semigroup).

We write \(a//b\) in \(P\) for \(a\) and \(b\) are non-comparable in \(P\).

Definition 2. Let \(P\) be a partial order defined on an abelian semigroup \(S\). We consider the following conditions for the partial order \(P\):

- (E): \(ac \geq bc\) in \(P\) implies \(a \geq b\) in \(P\). (order cancellation law)
- (G): Let \(x\) and \(y\) be any two elements non-comparable in \(P\). Then there exists an extension of \(P\) in which \(x>y\).
- (H): If \(a//b\) in \(P\), then \(ua \neq ub\) for any \(u\) in \(S\).
- (K): If \(a//b\) in \(P\), then \(ua//ub\) in \(P\) for any \(u\) in \(S\).
- (L): Let \(a//b\) and \(u//v\) in \(P\) respectively. If \(au \neq bv\), then \(au//bv\) in \(P\).

Strongness: \( ac \geq bc \) in \( P \) implies \( a \geq b \) in \( P \).

Normality: \( a^n \geq b^n \) in \( P \) for some positive integer \( n \) implies \( a \geq b \) in \( P \).

**Theorem 1.** Let \( P \) be a partial order defined on an abelian semigroup \( S \). Then \( P \) satisfies the condition (K) if and only if \( P \) satisfies the conditions (E) and (H).

**Proof.** Clearly the condition (K) implies the condition (H). If \( P \) satisfies the condition (K) and \( ac > bc \) in \( P \), then \( a \) and \( b \) are comparable in \( P \). And hence we have \( a > b \) in \( P \).

Conversely, let \( P \) satisfy the conditions (H) and (E) and let \( a \parallel b \) in \( P \). Then we have \( ua = ub \) for any \( u \) in \( S \) by the condition (H). If \( ac \) and \( bc \) are comparable in \( P \) for some \( c \) in \( S \), say that \( ac > bc \) in \( P \), then we have \( a > b \) in \( P \) by the condition (E), this is impossible.

**Theorem 2.** Let \( P \) be a normal partial order defined on abelian semigroup \( S \) which satisfies the condition (K). If \( a \parallel b \) in \( P \), then \( u'a' \parallel u'b' \) in \( P \) for any \( u \) in \( S \) and any integers \( i (\geq 0) \) and \( j (> 0) \), where if \( i = 0 \), \( u'a' \parallel u'b' \) means that \( a' \parallel b' \).

**Proof.** By the normality, \( a \parallel b \) in \( P \) implies \( a' \parallel b' \) in \( P \) for any positive integer \( j \). And hence we have \( u'a' \parallel u'b' \) in \( P \) by the condition (K).

**Theorem 3.** Let \( P \) be a normal partial order defined on an abelian semigroup \( S \) which satisfies the condition (K) and \( x \) and \( y \) be any two elements non-comparable in \( P \). Then there exists a normal extension \( Q \) of \( P \) such that \( x > y \) in \( Q \).

**Proof.** Let \( P \) be a normal partial order defined on \( S \) and the elements \( x \) and \( y \) are not comparable in \( P \). Let us define a relation \( Q \) as follows:

We put \( a > b \) in \( Q \) if and only if \( a \parallel b \) and there exist two non-negative integers \( n \) and \( m \), such that not both zero and

\[(\S) \quad a^n y^m \geq b^n x^m \quad \text{in} \quad P,\]

where if \( m = 0 \) or \( n = 0 \) \((\S)\) means that \( a^n \geq b^n \) or \( y^m \geq x^m \) in \( P \) respectively.

First we note that \( n \) is never zero, for otherwise we should have \( y^m \geq x^m \) in \( P \), whence by the normality we have \( y \geq x \) in \( P \) against the hypothesis.

(i) We begin with verifying that \( a > b \) and \( b > a \) in \( Q \) are contradictory. Suppose that \( a > b \) and \( b > a \) in \( Q \), namely \( a^n y^m \geq b^n x^m \) and \( b^i y^j \geq a^i x^j \) in \( P \) for some non-negative integers \( n \), \( m \), \( i \), \( j \). By multiplying \( i \) times the first, \( n \) times the second inequality, we obtain \((ab)^{mi+nj}x^mj \geq (ab)^{mi+nj}x^mj \) in \( P \), which contradicts the condition (K). If \( m = j = 0 \), then we have \( a > b \) and \( b > a \) in \( P \), which is impossible.
(ii) We show the transitivity of \( Q \). Assume that \( a > b \) and \( b > c \) in \( Q \), i.e., for some non-negative integers \( n, m, i, j \), \( a^n y^m \geq b^n x^m \) and \( b^j y^i \geq c^i x^j \) in \( P \). By multiplying as in (i) we get \( a^{ni} y^{mi+nj} \geq b^{ni} x^{mi+nj} \) in \( P \). Here \( ni \) is not zero, and \( a = c \) is impossible by the condition (K), so that \( a > c \) in \( Q \). If \( m = j = 0 \), then we have \( a > b \), \( b > c \) in \( P \), and hence \( a > c \) in \( P(Q) \).

(iii) We prove next the homogeneity of \( Q \). Suppose that \( a > b \) in \( Q \). If \( ac \neq bc \), from \( (ac)^n y^m \geq (bc)^n x^m \) in \( P \) we have \( ac > bc \) in \( Q \). Therefore \( a > b \) in \( Q \) implies \( ac \geq bc \) in \( Q \) for any \( c \) of \( S \).

(iv) \( Q \) is an extension of \( P \), for if \( a > b \) in \( P \), then \( ay^0 > bx^0 \) in \( P \), therefore \( a > b \) in \( Q \).

(v) It is clear that \( x > y \) in \( Q \). In fact, \( xy \geq yx \) in \( P \).

(vi) We may prove the normality of \( Q \). Indeed, supposing \( a^n > b^n \) in \( Q \) for some positive integer \( n \), i.e., \( (a^n)^i y^i \geq (b^n)^i x^i \) in \( P \), we see at once that \( a > b \) in \( Q \).

(vii) If \( a \parallel b \) in \( Q \), then \( a \parallel b \) in \( P \), and hence \( ua \equiv ub \) for any \( u \) in \( S \). Therefore, \( Q \) satisfies the condition (H).

**Theorem 4.** Let \( P \) be a partial order defined on an abelian semigroup \( S \) which satisfies the condition (G) and let \( a \parallel b, u \parallel v \) in \( P \). If \( au \neq bv \) and \( av = bu \), then \( au \parallel bv \) in \( P \).

**Proof.** Suppose that \( au \) and \( bv \) are comparable in \( P \), say that \( au > bv \) in \( P \). There exists an extension \( Q \) of \( P \) such that \( v > u \) in \( Q \). Then we have \( bv \geq bu = av \geq au \) in \( Q \), that is, we have \( bv \geq au \) in \( Q \). This contradicts the assumption.

**Theorem 5.** Let \( P \) be a partial order defined on an abelian semigroup \( S \) which satisfies the condition (G) and let \( a \parallel b, u \parallel v \) in \( P \). If \( au \neq bv \) and \( av = bu \), then \( au \parallel bv \) or \( av \parallel bu \) in \( P \).

**Proof.** Suppose that \( au \) and \( bv \) are comparable in \( P \), say that \( au > bv \) in \( P \). If \( bu > av \) in \( P \), then we consider an extension \( Q \) of \( P \) such that \( v > u \) in \( Q \). Then we have \( bv \geq bu > av \geq au \) in \( Q \), that is, \( bv > au \) in \( Q \), this is absurd. If \( av > bu \) in \( P \), then we consider an extension \( Q \) of \( P \) such that \( b > a \) in \( Q \). Then we have \( bv \geq av > bu \geq au \) in \( Q \), that is, \( bv > au \) in \( Q \), which leads the contradiction also. Therefore, \( bu \parallel av \) in \( P \).

**Theorem 6.** Let \( P \) be a normal partial order defined on an abelian semigroup \( S \) which satisfies the condition (K). If \( a \parallel b \) and \( x \parallel y \) in \( P \), then \( a^n y^m > b^n x^m \) or \( a^n y^m \parallel b^n x^m \) in \( P \) \((a^n x^m > b^n y^m \) or \( a^n x^m \parallel b^n y^m \) in \( P \)) for any integers \( m (\geq 0) \) and \( n (> 0) \).

**Proof.** If \( a^n y^m = b^n x^m \) for some positive integers \( m \) and \( n \), then we
have \( a^n x^m \geqq b^n x^m = a^n y^m \geqq b^n y^m \) in \( P \), that is, \( b^n x^m \geqq b^n y^m \) in \( P \) which contradicts the condition (K).

By the existence of the extension \( Q \) of \( P \) such that \( y > x \) in \( Q \), we have \( a^n y^m \geqq b^n x^m \) in \( Q \). Hence, if \( a^n y^m \) and \( b^n x^m \) are comparable in \( P \), then we have \( a^n y^m > b^n x^m \) in \( P \).

**Theorem 7.** Let \( P \) be a normal partial order defined on an abelian semigroup \( S \) which satisfies the conditions (K) and (L) and let \( x \parallel y \) in \( P \). For two distinct elements \( a \) and \( b \), the following two properties are equivalent to each other:

1. \( a > b \) in \( P \) or \( a^n y^m = b^n x^m \)
2. \( a^n y^m \geqq b^n x^m \) in \( P \)

for some integers \( m \) (\( \geqq 0 \)) and \( n \) (\( > 0 \)), where if \( m = 0 \), \( a^n y^m \) and \( b^n x^m \) means that \( a^n \) and \( b^n \) respectively.

**Proof.** (1) implies (2): If \( a > b \) in \( P \), then we can write \( a y^o \geqq b x^o \) in \( P \).

(2) implies (1): If \( a \parallel b \) in \( P \), then by the normality we have \( m > 0 \) and \( a^n \parallel b^n \), \( y^m \parallel x^m \) in \( P \). Therefore, \( a^n y^m = b^n x^m \) by the condition (L).

If \( m = 0 \), then \( a^n \geqq b^n \), and hence \( a > b \) in \( P \).

If \( m > 0 \) and \( b > a \) in \( P \), then \( b^n > a^n \), and hence we have \( b^n x^m \geqq b^n x^m \), \( b^n y^m \geqq a^n y^m \) in \( P \). Therefore, we have \( b^n x^m \geqq a^n y^m \geqq b^n x^m \geqq a^n x^m \) in \( P \), that is, \( b^n y^m \geqq b^n x^m \) in \( P \) which contradicts the condition (K). Therefore, we have \( a > b \) in \( P \).

Moreover, in this case, \( a > b \) in \( P \) if and only if \( a^n y^m > b^n x^m \) in \( P \) for some integers \( m \) (\( \geqq 0 \)) and \( n \) (\( > 0 \)).

**Theorem 8.** Let \( P \) be a normal partial order defined on an abelian semigroup \( S \) which satisfies the conditions (K) and (L) and \( x \) and \( y \) be any two elements non-comparable in \( P \). Then there exists a normal extension \( Q \), which satisfies the condition (K), of \( P \) such that \( x > y \) in \( Q \).

**Proof.** By Theorem 3, there exists the normal extension \( Q \) of \( P \) which satisfies the condition (H) such that \( x > y \) in \( Q \).

The order-relation \( Q \) is as follows:

- \( a > b \) in \( Q \) if and only if \( a > b \) in \( P \), or \( a \parallel b \) in \( P \) and \( a^n y^m = b^n x^m \) for some positive integers \( m \) and \( n \).

- (viii) Suppose that \( ac > bc \) in \( Q \). If \( ac > bc \) in \( P \), then we have \( a > b \) in \( P(Q) \). If \( ac \parallel bc \) in \( P \), then \( a \parallel b \) in \( P \) and \( (ac)^n y^m = (bc)^n x^m \), i.e., \( c^n(a^n y^m) = c^n(b^n x^m) \) for some positive integers \( m \) and \( n \). By the condition (K) of \( P \), \( a^n y^m \) and \( b^n x^m \) are comparable in \( P \). And hence we have \( a^n y^m = b^n x^m \)
by the condition (L) of $P$. Therefore, we have $a \succ b$ in $Q$. Thus $Q$ satisfies the conditions (H) and (E), that is, the condition (K).

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