A GENERALIZATION OF MAZUR-ORLICZ THEOREM
ON FUNCTION SPACES

By

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1. Introduction. Let \( \Omega(B, \mu) \) be a locally finite measure space. By
many investigators various function spaces consisting of locally almost
finite \( B \)-measurable functions on \( \Omega \) have been considered as a generaliza-
tion of the so-called \( L_p \)-spaces on \( \Omega \) \( (1 \leq p \leq +\infty) \). One of them is \( L_{M(u, \omega)} \)-
space (Musielak-Orlicz [3], [4]).

Let \( M(u, \omega) \) be a function on \( [0, +\infty] \times \Omega \) with the following
properties (it will be called (M)-function);
1) \( 0 \leq M(u, \omega) \leq +\infty \) for all \( (u, \omega) \in [0, +\infty] \times \Omega \),
2) \( \lim_{u \to 0} M(u, \omega) = 0 \) for all \( \omega \in \Omega \),
3) \( M(u, \omega) \) is a non-decreasing and left continuous function of \( u \)
(M)
4) \( \lim_{u \to +\infty} M(u, \omega) > 0 \) for all \( \omega \in \Omega \),
5) \( M(u, \omega) \) is locally \( B \)-measurable as a function of \( \omega \) for all
\( u \in [0, +\infty] \).

Using this function \( M(u, \omega) \) we can define a functional \( \rho_M(x) \) on locally
almost finite \( B \)-measurable functions \( x(\omega) \) \( (\omega \in \Omega) \) by the formula

\[
(1) \quad \rho_M(x) = \int_{\Omega} M(|x(\omega)|, \omega) d\mu
\]

If \( L_{M(u, \omega)} \) denotes the set of all \( x(\omega) \) such that \( \rho_M(\alpha x) < +\infty \) for a positive
number \( \alpha = \alpha(x) \) depending on \( x \), \( L_{M(u, \omega)} \) is a vector space.

As special cases, \( L_{M(u, \omega)} \) coincides with four typical spaces respectively:

1) \( \Omega \) is covered by the family of measurable sets of finite measure.
2) Correctly speaking, we shall consider only the functions which are almost finite
real valued and \( B \)-measurable in every measurable set of finite measure. And two functions
\( x(\omega) \) and \( y(\omega) \) are identified if \( x(\omega) = y(\omega) \) except on a set of measure zero in every measurable
set of finite measure.
3) Since \( M(u, \omega) \) can be replaced by \( M(u-0, \omega) \), the left side continuity is not essential
for the definition of the space \( L_{M(u, \omega)} \).
4) It is unnecessary for \( M(u, \omega) \) to be almost finite valued.
5) (M)-2) and 3) imply the measurability of a function \( M(|x(\omega)|, \omega) \). The integration
on \( \Omega \) means the supremum of integrations on every finite measured set.
1) $L_p$-space ($0 < p \leq +\infty$), when $M(u, \omega) = u^p$,
2) $L_{N(u)}$-space (Orlicz [7]), when $M(u, \omega) = N(u)$ and $N(u)$ is a convex function of $u$,
3) $L_{M(u)}$-space (Mazur-Orlicz [2]), when $M(u, \omega) = M(u)$,
4) $L_{N(u, \omega)}$-space (Nakano [5]), when $M(u, \omega) = N(u, \omega)$ and $N(u, \omega)$ is a convex function of $u$ for all $\omega \in \Omega$.

In view of generalization of a constructive method, the relation between above four spaces is shown with the following schema,

$$ L_p (1 \leq p \leq +\infty) \downarrow $$

(2)

$$ L_{N(u)} \rightarrow L_{N(u, \omega)} $$

$$ L_p (0 < p < 1) \rightarrow L_{M(u)} \downarrow \rightarrow L_{M(u, \omega)} $$

In the spaces $L_{N(u)}$ and $L_{N(u, \omega)}$, if we put

$$ \| x \|_N = \inf \{ \varepsilon > 0 ; \rho_N(x/\varepsilon) \leq 1 \} , $$
we have a complete norm (B-norm) on $L_{N(u)}$ and $L_{N(u, \omega)}$ respectively ([1], [5]). In the spaces $L_{M(u)}$ and $L_{M(u, \omega)}$, putting

$$ \| x \|_M = \inf \{ \varepsilon > 0 ; \rho_M(x/\varepsilon) \leq \varepsilon \} , $$
we have a complete quasi-norm (F-norm) on $L_{M(u)}$ and $L_{M(u, \omega)}$ respectively ([2], [3]). We can see easily

$$ \lim_{n \to \infty} \rho_M(\alpha x_n) = 0 \text{ (for all } \alpha \geq 0). $$

Mazur-Orlicz has shown in [2] the following result:

"Given $L_{M(u)}$-space, the necessary and sufficient condition for to exist a convex (M)-function $N(u)$ such as $L_{M(u)} = L_{N(u)}$ is that the linear topology induced by the quasi-norm $\| x \|_M$ is locally convex."

The purpose of this paper is to generalize this result to the problem of the relation between $L_{M(u, \omega)}$ and $L_{N(u, \omega)}$. In §2 we shall define the abstract $L_{M(u, \omega)}$-space, and in §3 the problem will be studied in an abstract form. If $\Omega(B, \mu)$ is non-atomic, we obtain a similar result to the above Mazur-Orlicz theorem (Theorem 2). Although in general it does not hold in an atomic case, under some assumption it can be proved also (Theorem 3).

6) If $p = +\infty$, then we put $u^{+\infty} = 0 (0 \leq u \leq 1)$ and $= +\infty (u > 1)$.
7) H. Nakano calls $L_{N(u, \omega)}$ a modulared function space in [5] (appendix).
8) It has been proved under an additional condition: $M(2u) \leq KM(u)$ for all $u \geq u_0 > 0$ (non-atomic case) or $M(2u) \leq KM(u)$ for all $0 \leq u \leq u_0$ (atomic case).
2. Modulated vector lattice. First of all, we shall define a modulated vector lattice \( R(\rho) \) as the abstraction of \( L_{M(u,\omega)} \)-spaces. Let \( R \) be a conditionally complete\(^9\) vector lattice. A functional on \( R \) with values \( 0 \leq \rho(x) \leq +\infty \) will be called a modular\(^{10}\) \([4],[5],[6]\) when the following conditions are satisfied;

1) \( \rho(\alpha x) = 0 \) for all \( \alpha \geq 0 \) if and only if \( x = 0 \), 
2) \( \inf_{x > 0} \rho(\alpha x) = 0 \) for all \( x \in R \), 
3) \( |x| \leq |y| \) implies \( \rho(x) \leq \rho(y) \), 
4) \( x \leq y = 0 \) implies \( \rho(x+y) = \rho(x) + \rho(y) \), 
5) \( 0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x \)\(^{11}\) implies \( \sup_{\lambda \in \Lambda} \rho(x_{\lambda}) = \rho(x) \), 
6) for any orthogonal system \( x_{\lambda} \geq 0 \) \((\lambda \in \Lambda)\) such as \( \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \) we can find \( x \in R \) and \( x = \sum_{\lambda \in \Lambda} x_{\lambda} \)\(^{12}\) (orthogonal completeness).

Moreover, if \( \rho \) satisfies the following condition (C), \( \rho \) will be called a convex modular;

\[
\rho(\alpha x) \text{ is a convex function of } \alpha \text{ for all } x \in R. 
\]

We shall call \( R \) where a (convex) modular is defined a (convex) modulated vector lattice. A convex modulated vector lattice will be said briefly the Nakano space\(^{13}\). We can see easily that \( L_{M(u,\omega)}(\rho_{N}) \) is a modulated vector lattice and \( L_{N(u,\omega)}(\rho_{N}) \) is the Nakano space.

The \((\rho)\)-condition implies some properties;

\[
(5) \quad \rho(x \leq y) + \rho(x \leq y) = \rho(x) + \rho(y) \text{ for } x, y \geq 0, 
\]

\[
(6) \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \text{ for } x, y \in R, \alpha, \beta \geq 0, \alpha + \beta = 1. 
\]

It has been shown in \([3]\) and \([4]\) that the property (6) defines a ordered\(^{14}\) quasi-norm \( \|x\|_{\rho} \) on \( R \) by the formula

\[
(7) \quad \|x\|_{\rho} = \inf \{\epsilon > 0 ; \rho(x/\epsilon) \leq \epsilon\} \quad (x \in R). 
\]

We can see easily \( \lim_{n \to +\infty} \|x_{n}\|_{\rho} = 0 \) if and only if \( \lim_{n \to +\infty} \rho(\alpha x_{n}) = 0 \) for all \( \alpha \geq 0 \).

\(10\) For the first time the name ‘modular’ was used by H. Nakano, when \((\rho)\)-1\(\sim 5\) and \((\rho)\) were satisfied. The convex modular defined in this paper coincides with the monotone-complete modular in Nakano’s terminology \((\[5]\))\). The orthogonal completeness \((\rho)-6\) implies the monotone completeness (cf. Remark of Lemma 1). The condition \((\rho)\) is stronger than that in \([4]\) and of the quasi-modular in \([8]\).

\(11\) For any \( \lambda_{1}, \lambda_{2} \in \Lambda \) there exists \( \lambda_{3} \in \Lambda \) such as \( x_{\lambda_{1}} \cup x_{\lambda_{2}} \leq x_{\lambda_{3}} \) and \( \cup_{\lambda \in \Lambda} x_{\lambda} = x \).

\(12\) \( \sum_{\lambda \in \Lambda} x_{\lambda} = \bigcup_{\Lambda \subset \Lambda'} \sum_{\lambda \in \Lambda} x_{\lambda} \), where \( \Lambda' \) is a finite subset of \( \Lambda \).

\(13\) In \([5]\) it is called a monotone-complete modulated semi-ordered linear space.

\(14\) \( |x| \leq |y| \) implies \( \|x\|_{\rho} \leq \|y\|_{\rho} \).
In this section we shall prove that \( \| x \|_\rho \) is a complete quasi-norm on \( R \).

**Lemma 1.** The necessary and sufficient condition for a directed system of positive elements \( 0 \leq x_{\lambda} \uparrow_{\lambda \in A} \) to be order-bounded is that the following two conditions are satisfied:

(i) \( \sup_{\lambda \in A} \rho(\alpha x_{\lambda}) < +\infty \) for some \( \alpha > 0 \),

(ii) for any \( p \in R \) (\( p \neq 0 \)) we can find two positive numbers \( \beta_2 > \beta_1 > 0 \) such that \( \sup_{\lambda \in A} (\beta_1 [p] x_{\lambda}) < \rho(\beta_2 p) \).

**Proof.** Supposing \( 0 \leq x_{\lambda} \uparrow_{\lambda \in A} x \), then (\( \rho \)-2) and (3) imply (i). Since 
\[
\sup_{\alpha > 0} \rho(\alpha p) > 0 \quad (p \neq 0) \quad \text{and} \quad \inf_{\alpha > 0} \rho(\alpha [p] x) = 0,
\]
we have easily (\( \rho \)-11).

Sufficiency: First, (ii) implies the fact that for a given \( p > 0 \) we can find \( 0 < [q] \leq [p] \) such that \( [q] x_{\lambda} (\lambda \in \Lambda) \) is order-bounded. Because; in the contrary case, we can obtain the decomposition of \( [p] \), \( [p] = [q_1] \oplus \cdots \oplus [q_n] \), and \( \lambda_i \in \Lambda \) (\( 1 \leq i \leq n \)) such that \( \beta_2 [q_i] p < \beta_1 [q_i] x_{\lambda_i} (1 \leq i \leq n) \), hence \( \beta_2 [p] = \sum_{i=1}^{n} \beta_2 [q_i] p \leq \sum_{i=1}^{n} \beta_1 [q_i] x_{\lambda_i} \leq \beta_1 [p] x_{\lambda_0} \) for some \( \lambda_0 \in \Lambda \). This implies the contradiction:
\[
\rho(\beta_2 p) \leq \rho(\beta_1 [p] x_{\lambda_0}) < \rho(\beta_2 p).
\]
Therefore, if we put \( [p_\gamma] (\gamma \in \Gamma) \) a maximal orthogonal system of projectors such as \( [p_\gamma] x_{\lambda} (\lambda \in \Lambda) \) is order-bounded, then we have \( \sum [p_\gamma] = I \).

Putting \( [p_\gamma] x_{\lambda} \uparrow_{\lambda \in A} y_\gamma \), since \( \rho(\alpha y_\gamma) = \sup_{\lambda \in A} \rho(\alpha [p_\gamma] x_{\lambda}) \), we see \( \sum \rho(\alpha y_\gamma) = \sup_{\lambda \in A} \rho(\alpha [p_\gamma] x_{\lambda}) = \sup_{\lambda \in A} \rho(\alpha x_{\lambda}) < +\infty \) ((i)). Hence the orthogonal completeness ((\( \rho \)-6)) implies the existence \( x \in R \) such as \( x = \sum_{\gamma \in \Gamma} \alpha y_\gamma = \bigcup_{\lambda \in A} \alpha X_{\lambda} \), that is \( x_{\lambda} \uparrow_{\lambda \in A} x/\alpha \).

**Remark.** When \( \sup_{\alpha > 0} \rho(\alpha p) = +\infty \) (\( p \neq 0 \)) is satisfied, (ii) follows from (i).

**Theorem 1.** \( \| x \|_\rho \) (\( x \in R \)) is a complete quasi-norm on \( R(\rho) \).

**Proof.** Let \( x_{\nu} (\nu = 1, 2, \ldots) \) be a Cauchy sequence, and we assume \( \| x_{\nu+1} - x_{\nu} \|_{\rho} \leq 1/2^\nu \) (\( \nu = 1, 2, \ldots \)). Putting \( |x_2 - x_1| + \cdots + |x_n - x_{n-1}| = z_n \) (\( n \geq 2 \)) and \( \sum_{\nu=m}^{n} |x_{\nu+1} - x_{\nu}| = y_{n,m} \) (\( n \geq 1, m \geq n \)), we see \( y_{1,n} = z_n \) and \( \| y_{1,n} \|_{\rho} \leq 1/2^{n-1} \), that

\[ 15) \quad [p] \text{ is a projection operator and defined as follows } [p] x = \bigcup_{v=1}^{\infty} (x \ominus v \ | \ p \ |) \text{ for all } x \geq 0, \]
\[ 16) \quad [q] x \leq [p] x \text{ for all } x \geq 0. \]
\[ 17) \quad [p] x = \sum_{i=1}^{\infty} [q_i] x \text{ for all } x \in R \text{ and } [q_i] [q_j] = 0 \text{ (} i \neq j \text{).} \]
\[ 18) \quad \sum_{\gamma \in \Gamma} [p_\gamma] x = x \text{ for all } x \geq 0. \]
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is, $\rho(2^{n-1}y_{n,m}) \leq 1/2^{n-1}$, and $y_{n,m} \uparrow_{n \geq m \geq 1}$. The non-decreasing sequence $y_{1,m} \uparrow_{m \geq 1}$ satisfies (i) and (ii) in the previous Lemma 1. First, $\sup_{m \geq 1} \rho(y_{1,m}) \leq 1$ follows from $\|y_{1,m}\|_{\rho} \leq 1 (m \geq 1)$. Next for any $p \neq 0$ we can find a positive number $\beta_{2}>0$ and an integer $n \geq 1$ such as $1/2^{n-1} < \rho(\beta_{2}p)$, and further $\beta_{1}>0$ such as $2\beta_{1} < 2^{n-1}$ and $\rho(2\beta_{1}x_{n}) < 1/2^{n-1}$. Hence $\rho(\beta_{1}[p]y_{1,m}) \leq \rho(2\beta_{1}y_{n,m}) \leq 1/2^{n-1} + \rho(2^{n-1}y_{n,m}) \leq 1/2^{n-1} + 1/2^{n-1} = 1/2^{n-2} < \rho(\beta_{2}p)$. By Lemma 1 we can put $\sum_{\nu=n}^{\infty} |x_{\nu+1} - x_{\nu}| = \bigcup_{m \geq n} y_{n,m} = y_{n}$ $(n \geq 1)$. This implies also that the sequence $x_{\nu}(\nu=1,2, \cdots)$ converges to an element $x_{0}$ in order, that is, $0-\lim_{\nu \to \infty} x_{\nu} = x_{0}$. We see $|x_{0} - x_{n}| = |0-\lim_{m \to \infty} \sum_{\nu=n}^{m} (x_{\nu+1} - x_{\nu})| \leq \bigcup_{m \geq n} y_{n,m} = y_{n}$, hence $\|x_{0} - x_{n}\|_{\rho} \leq \|y_{n}\|_{\rho}^{20)} = \sup_{m \geq n} \|y_{n,m}\|_{\rho} \leq 1/2^{n-1}$, that is, $\lim_{n \to \infty} \|x_{0} - x_{n}\|_{\rho} = 0$. Q.E.D.

3. Local convexity of the linear topology in modulared vector lattices.

A. Non-atomic case. Let $R(\rho)$ be a non-atomic modulared vector lattice, we have the following main theorem.

Theorem 2. In a non-atomic modulared vector lattice $R(\rho)$ the following four conditions are equivalent each other;

a) the metric linear topology induced by $\|x\|_{\rho}$ is normable,
b) the metric linear topology induced by $\|x\|_{\rho}$ is locally convex,
c) there exists a convex modular $m(x)$ on $R(\rho)$ ($R$ is the Nakano space),
d) there exists a complete ordered norm $\|\|x\||$ on $R(\rho)$ ($R$ is a Banach lattice).

Proof. (b)$\to$(c). First, we shall prove the following fact:
For any $\varepsilon > 0$ we can find a positive number $\delta = \delta(\varepsilon) > 0$ such that

(*) $\rho(x/\varepsilon) > \varepsilon$ implies $\sum_{i=1}^{l} \rho(n_{i}x_{i}/\delta)/n_{i} > \delta$,

where $\{x_{i} ; 1 \leq i \leq l\}$ is an arbitrary orthogonal decomposition of $x$, $x = \sum_{i=1}^{l} \oplus x_{i}$, and $n_{i}$ $(1 \leq i \leq l)$ are arbitrary positive integers.

19) $\bigcap_{n=1}^{\infty} \bigcup_{\nu \geq n} x_{\nu} = \bigcup_{n=1}^{\infty} \bigcap_{\nu \geq n} x_{\nu}$ and it is denoted by $0-\lim_{n \to \infty} x_{\nu} = x_{0}$.
20) (\rho)-5 implies $\sup_{x_{\nu} \in E} \|x_{\nu}\|_{\rho} = \|x\|_{\rho}$ for all $0 \leq x_{\nu} \in E$.
21) For every $a \in R$, $a \geq 0$ we can find $b, c > 0$ such as $a = b + c$ and $b \cap c = 0$.
22) $x = \sum_{i=1}^{l} x_{i}$ and $|x_{i} \cap x_{j}| = 0$ $(i \neq j)$. 


Because, from the local convexity of $\|x\|_r$ for any $\varepsilon > 0$ we can find a positive number $\delta = \delta(\varepsilon) > 0$ such that $\|x_i\|_r \leq \delta$ ($1 \leq i \leq l$) imply $\|\sum_{i=1}^{l} x_i/l\|_r \leq \varepsilon$, that is,

$$\rho(x_i/\delta) \leq \delta \quad (1 \leq i \leq l) \text{ imply } \rho\left(\sum_{i=1}^{l} x_i/\varepsilon l\right) \leq \varepsilon .$$

Hence, if $\sum_{i=1}^{l} \rho(n_i x_i/\delta)/n_i \leq \delta$, $x = \sum_{i=1}^{l} \oplus x_i$ and $n_i$ ($1 \leq i \leq l$) are positive integers, then in view of the assumption that $R$ is non-atomic, we can find an orthogonal decomposition of $x_i$ such that

$$\left\{ \begin{array}{l} x_i = \sum_{\nu=1}^{n_i} \oplus x_{i,\nu} \quad (1 \leq i \leq l) \\ \rho(n_i x_{i,\nu}/\delta) = \rho(n_i x_i/\delta)/n_i \quad (1 \leq \nu \leq n_i, 1 \leq i \leq l) \end{array} \right).$$

If we put $y_{\nu_1,\nu_2,\ldots,\nu_l} = \sum_{i=1}^{l} \oplus n_i x_{i,\nu_i}$ ($1 \leq \nu_i \leq n_i$), then the total number of elements $y_{\nu_1,\nu_2,\ldots,\nu_l}$ is $n_1 n_2 \cdots n_l$ and the sum of them equals to $n_1 n_2 \cdots n_l x$, because the multiplicity of $n_i x_{i,j}$ in the summation is $n_1 n_2 \cdots n_i$, we have

$$\sum_{1 \leq \nu_i \leq n_i} y_{\nu_1,\nu_2,\ldots,\nu_l} = \sum_{i=1}^{l} \sum_{j=1}^{n_i} n_1 n_2 \cdots n_i x_{i,j} \quad 1 \leq i \leq l$$

$$= \sum_{i=1}^{l} n_1 n_2 \cdots n_i x = n_1 n_2 \cdots n_l x .$$

On the other hand

$$\rho(y_{\nu_1,\nu_2,\ldots,\nu_l}/\delta) = \sum_{i=1}^{l} \rho(n_i x_{i,\nu_i}/\delta) = \sum_{i=1}^{l} \rho(n_i x_i/\delta)/n_i \leq \delta .$$

Therefore from (8), (10) and (11) we see

$$\rho\left(\sum_{1 \leq \nu_i \leq n_i} y_{\nu_1,\nu_2,\ldots,\nu_l}/\varepsilon n_1 n_2 \cdots n_i \right) = \rho(x/\varepsilon) \leq \varepsilon .$$

Thus (*) has been proved.

And the following fact is a direct consequence of (*),

$$\sup_{x \geq 0} \rho(\alpha x) < + \infty \text{ if and only if } x = 0.$$ 

Since $\sup_{x \geq 0} \rho(\alpha x) = \gamma < + \infty$ implies $\inf_{n \geq 1} \rho(n x/\delta)/n = 0$ for all $\delta > 0$, by (*)

$$\rho(x/\varepsilon) \leq \varepsilon$$

for all $\varepsilon > 0$, hence $\|x\|_r = 0$, that is, $x = 0$.

Putting, for $\delta_1 = \delta(1) > 0$,

23) This is a method used oftenly in non-atomic cases. Confer [4] or [5].
The functional $\overline{\rho}(x)$ $(x \in R)$ has the following properties:

1) $\overline{\rho}(x) \leq 1/\delta_1 \cdot \rho(2x/\delta_1)$ for all $x \in R$,
2) $|x| \leq |y|$ implies $\overline{\rho}(x) \leq \overline{\rho}(y)$,
3) $x \perp y = 0$ implies $\overline{\rho}(x + y) = \overline{\rho}(x) + \overline{\rho}(y)$,
4) $x \cap y = 0$ implies $\overline{\rho}(x + y) = \overline{\rho}(x) + \overline{\rho}(y)$,
5) $\overline{\rho}(tx)/t$ $(t > 0)$ is a non-decreasing function of $t > 0$ for all $x \in R$,
6) $\sup_{\lambda \in \Lambda} \overline{\rho}(\lambda x) = \overline{\rho}(x)$.

($\overline{\rho}$-1) is obvious from the definition of $\overline{\rho}$. ($\overline{\rho}$)-2 is a simple consequence of ($\ast$); if $\rho(x) > 1$, $x = \sum_{i=1}^{l} \bigoplus_{1} x_i$ and $\eta_i \geq 1$ $(1 \leq i \leq l)$, then we see $\sum_{i=1}^{l} \rho(2\eta_i x_i/\delta_1) \eta_i \geq \sum_{i=1}^{l} \rho(2n_i x_i/\delta_1) 2\delta_i n_i$ where $n_i, 1 \leq i \leq l$ are positive integers such as $n_i \leq \eta_i < n_i + 1$ $(1 \leq i \leq l)$. ($\overline{\rho}$)-3 and 4 are easily implied from ($\overline{\rho}$)-3 and 4 respectively.

Next we shall check ($\overline{\rho}$)-5; for $t_2 > t_1 > 0$ we have

$$\overline{\rho}(t_1 x)/t_1 = \inf_{\xi \geq 1} \sum_{i=1}^{l} \rho(2t_1 \eta_i x_i/\delta_i) \eta_i / t_1 \delta_i \leq \inf_{\xi \geq 1} \sum_{i=1}^{l} \rho(2\xi_i x_i/\delta_i) / \xi_i \delta_i = \overline{\rho}(t_2 x)/t_2.$$

($\overline{\rho}$)-6 is shown as follows; $\overline{\rho}(x) < +\infty$ if and only if $\rho(2x/\delta_1) < +\infty$. Hence

$$0 \leq \overline{\rho}(x) - \overline{\rho}(\lambda x) = \overline{\rho}(x - \lambda x) \leq \rho(2(x - \lambda x)/\delta_1) \delta_1 < +\infty,$$

and inf $\rho(2(x - \lambda x)/\delta_1) = 0$ is effected by ($\overline{\rho}$)-5.

Next we put $\overline{\rho}(x)$ $(x \in R)$ as follows:

$$\overline{\rho}(x) = \begin{cases} \sup \overline{\rho}(\lambda x), & \text{if there exists } \lambda \in \Lambda \text{ and } [p_\lambda] \uparrow y \in R \text{ and } [x] \\ +\infty, & \text{elsewhere.} \end{cases}$$

We see obviously $\overline{\rho}(x) \leq \overline{\rho}(x)$ $(x \in R)$ and $\overline{\rho}(x) = \overline{\rho}(x)$, if $\overline{\rho}(x) < +\infty$. The functional $\overline{\rho}(x)$ $(x \in R)$ has the same properties as ($\overline{\rho}$) and moreover has the stronger property than 6) of ($\overline{\rho}$):

If $\lambda = \lambda x$, then sup $\overline{\rho}(\lambda x) = \overline{\rho}(x)$.

Now we can construct a convex modular $m(x)$ $(x \in R)$:

$$m(x) = \int_{0}^{x} \overline{\rho}(tx) dt \quad (x \in R).$$

24) $[p_\lambda] y \uparrow y \in R \text{ for all } y \geq 0.$
Evidently we see
(17) \[ \overline{\rho}(x/2) \leq m(x) \leq \rho(x) \quad (x \in R). \]

It is obvious also that this functional \( m(x) \) on \( R \) satisfies (C) from the fact that \( \overline{\rho}(tx)/t \) is a non-decreasing function of \( t > 0 \).

We shall check the modular condition \((\rho)\) about \( m(x) \) on \( R \) from the fact that \( \overline{\rho}(tx)/t \) is a non-decreasing function of \( t > 0 \).

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implies \[ \sup_{\alpha \geq 0} = \rho(\alpha x) = 0, \]

hence from the definition of \( \overline{\rho} \) and \((\overline{\rho})\) we can see \[ \sup_{\alpha \geq 0} \rho(\alpha x) \leq 1, \]

therefore \( x = 0 \) follows from (12).

\[ \overline{\rho}(x) - 1) \]

is evident from (17) and \((\overline{\rho})\) and \((\overline{\rho})\) are almost evident. \((\overline{\rho})\).

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and since \( m(\alpha x) = \int_{0}^{\alpha} \rho(tx)/dt \)

is a left-continuous function of \( \alpha \geq 0 \),

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hence from the definition of \( \overline{\rho} \) and \((\overline{\rho})\) we can see \[ \sup_{\alpha \geq 0} \rho(\alpha x) \leq 1, \]

therefore \( x = 0 \) follows from (12).

\[ \overline{\rho}(x) - 1) \]

is evident from (17) and \((\overline{\rho})\) and \((\overline{\rho})\) are almost evident. \((\overline{\rho})\).

\[ \sup_{\alpha \geq 0} m(\alpha x) = 0 \]
implies \[ \sup_{\alpha \geq 0} = \rho(\alpha x) = 0, \]

and since \( m(\alpha x) = \int_{0}^{\alpha} \rho(tx)/dt \)

is a left-continuous function of \( \alpha \geq 0 \),

therefore \[ \sup_{\alpha \geq 0} \rho(\alpha x) \leq 1, \]

therefore \( x = 0 \) follows from (12).
\[ \sum_{v=1}^{\infty} ||x_{v}||_{1}\leq 1, \text{ from the completeness of } ||x||_{1}, \text{ we can find } x_{0}\in R \text{ and } \lim_{n\to 0} ||x_{0} - \sum_{v=1}^{n} x_{v}||_{1} = 0. \] And
\[ 0 \leq ||x_{0\cap}x_{v} - x_{v}||_{1} = ||x_{0\cap}x_{v} - \left(\sum_{i=1}^{n} x_{i}\right)_{\cap}x_{v}||_{1} \leq ||x_{0} - \sum_{i=1}^{n} x_{i}||_{1} \to 0 (n\to \infty), \]
hence \( x_{0\cap}x_{v} = x_{v} \), that is, \( x_{0} \geq x_{v} \) \((v = 1, 2, \ldots)\). Therefore we have a contradiction:
\[ ||x_{0}/\nu||_{2} \leq ||x_{v}/\nu||_{2} \geq \varepsilon_{0} \text{ and } \lim_{n\to \infty} ||x_{0}/\nu||_{2} \geq \varepsilon_{0} > 0. \]
Thus, given \( \lim_{n\to \infty} ||y_{n}||_{1} = 0 \), for any \( \varepsilon > 0 \) we have \( ||y_{n}/\gamma_{*}||_{1} \leq \delta \), for almost all \( n \), hence \( ||\gamma_{*}y_{n}/\gamma_{*}||_{2} = ||y_{n}||_{2} \leq \varepsilon \) for almost all \( n \), that is, \( \lim_{n\to \infty} ||y_{n}||_{2} = 0 \).

Q.E.D.

Remark. Under the assumption \( \sup_{x>0} \rho(\alpha x) = +\infty \) \((x \neq 0)\), the condition b) in the above Theorem 2 may be replaced with the following.

b') for some \( \varepsilon_{0} > 0 \) \( \{x; ||x||_{\rho} \leq \varepsilon_{0}\} \) contains a convex neighbourhood of 0.

The application to function spaces. The detailed proof will be omitted. Let \( m(x) \) be a convex modular \( L_{M(u, \omega)} \). By Radon-Nikodym's theorem we can find a convex \((M)\)-function \( N(u, \omega) \) and \( m(x) \) can be represented as follows
\[ m(x) = \int_{\Omega} \int_{\Omega} N[|x(\omega)|, \omega]\,d\mu. \]
The orthogonal completeness of \( m \) implies \( L_{N(u, \omega)} = L_{M(u, \omega)} \). Thus by Theorem 2 Mazur-Orlicz's result in \S 1 can be generalized;

\[ \text{Given } L_{M(u, \omega)}-\text{space on non-atomic measure space } \Omega(B, \mu), \text{ the necessary and sufficient condition to exist a convex \((M)\)-function } N(u, \omega) \text{ such as } L_{M(u, \omega)} = L_{N(u, \omega)} \text{ is that the linear topology induced by } ||x||_{M} \text{ on } L_{M(u, \omega)} \text{ is locally convex.} \]

B. Atomic case. In an atomic modulared vector lattice \( R(\rho) \) the above Theorem 2 does not hold in general. The so-called \( S \)-space is a counter example. Putting \( \Omega = \{\omega_{1}, \omega_{2}, \cdots\} \), \( \mu(\omega_{n}) = 1 \) and \( M(u, \omega_{n}) = u/2^{n}(1+u) \) \((n = 1, 2, \cdots)\), then \( L_{M(u, \omega)} \) is \( S \)-space on \( \Omega \). It is easily proved that \( ||x||_{M} \) on \( S(\Omega) \) is locally convex, but not normable.

Now we shall consider the following assumption:
\[ (**): \quad R = \bigoplus_{\nu=1}^{\infty} R_{\nu}, \text{ and } R(\rho) = R_{\nu}(\rho)_{\nu} \geq 1, \]

26) Since \( |x\cap y - x \cap y| = |x - y| \) \(([5])\), we have \( ||x\cap y - x \cap z||_{\rho} \leq ||x - y||_{\rho}. \)

27) Every \( R_{\nu} \) is a normal subspace of \( R \), that is, \( R_{\nu} \) is a linear subspace and if \( R_{\nu} \ni x \) and \( |x| \leq |y| \), then \( y \in R_{\nu} \) and \( \exists x_{1} \in R_{\nu}, (x_{1} \in R_{\nu}, \exists x \) imply \( x \in R_{\nu} \) \(([5])\). \( R = \bigoplus_{\nu=1}^{\infty} R_{\nu} \) means that \( |x|_{\nu} \leq |x|_{\rho} \) \( (\nu \geq 1) \).

28) There exists an isomorphism \( I_{\nu} \) from \( R \) onto \( R_{\nu} \) such as \( \rho(x) = \rho(I_{\nu}x) \) for all \( x \in R. \)
where \( R_{\nu} \)\((\nu=1,2, \cdots)\) are normal subspaces and orthogonal each other.

For instance, if \( \Omega(2^\nu, \mu) \) is an atomic measure space and for any \( \omega_0 \in \Omega \), \( \Omega_{\omega_0} = \{ \omega ; M(u, \omega_0) = M(u, \omega) \mu(\omega) \} \) for all \( u \geq 0 \) is an infinite set, then \( L_{\Omega(\omega_0)} \) satisfies (**) . As a special case, \( L_{\Omega(\omega)} \) on an atomic measure \( \Omega(2^\nu, \mu) \), where \( \mu(\omega) = 1 (\omega \in \Omega) \) and \( \Omega \) is infinite, satisfies (**).

**Theorem 3.** In the modulared vector lattice \( R(\rho) \) satisfying the assumption (**), four conditions in Theorem 2 are equivalent each other. Moreover the condition b) can be replaced by the weaker condition:

b’) for some \( \varepsilon_0 > 0 \) \( \{ x ; \| x \|_\rho \leq \varepsilon_0 \} \) contains a convex neighbourhood of 0.

**Proof.** It is sufficient to show b’)→c). It follows from b’) that there is \( \delta_0 > 0 \) such that

\[
\rho(x_i/\delta_0) \leq \delta_0 \quad (1 \leq i \leq l)
\]

Using the assumption (**), we shall show

\[
(*) \quad \rho(x/\delta_0) \leq \delta_0 \implies \sum_{\nu=1}^{n_i} \rho(x_i/\varepsilon_0 n_i) \leq \varepsilon_0,
\]

where \( \{ x_i ; 1 \leq i \leq l \} \) is an arbitrary orthogonal decomposition of \( x \), \( x = \sum_{i=1}^{l} \oplus x_i \), and \( n_i \) \((1 \leq i \leq l)\) are arbitrary positive integers.

Because; from (**) we can find \( x_{i_{\nu}} \in R_{\nu} \) \((1 \leq \nu \leq n_i, 1 \leq i \leq l)\) such that

\[
(21) \begin{cases}
x_{i_{\nu}} \cdot x_{j_{\mu}} = 0 \quad ((i, \nu) \neq (j, \mu)) \\
\rho(\alpha x_{i_{\nu}}) = \rho(\alpha x_i) \quad \text{for all } \alpha \geq 0 \quad (1 \leq \nu \leq n_i, 1 \leq i \leq l).
\end{cases}
\]

If we put \( y_{\nu_{1},\nu_{2},\ldots,\nu_{l}} = \sum_{i=1}^{l} \oplus x_{i_{\nu_{i}}} \), then the total number of elements \( y_{\nu_{1},\nu_{2},\ldots,\nu_{l}} \) is \( n_1 n_2 \cdots n_l \) and we have

\[
(22) \sum_{1 \leq \nu_1 \leq n_1, 1 \leq \nu_2 \leq n_2, \ldots, 1 \leq \nu_l \leq n_l} y_{\nu_{1},\nu_{2},\ldots,\nu_{l}}/n_1 n_2 \cdots n_l = \sum_{i=1}^{l} \oplus \sum_{\nu=1}^{n_i} \oplus x_{i_{\nu}}/n_i.
\]

On the other hand

\[
(23) \rho(y_{\nu_{1},\nu_{2},\ldots,\nu_{l}}/\delta_0) = \sum_{i=1}^{l} \rho(x_{i_{\nu_{i}}}/\delta_0)
\]

\[= \sum_{i=1}^{l} \rho(x_i/\delta_0) = \rho(x/\delta_0) \leq \delta_0,
\]

therefore (20) and (22) imply

\[
(24) \sum_{i=1}^{l} n_i \rho(x_i/\varepsilon_0 n_i) = \sum_{i=1}^{l} \sum_{\nu=1}^{n_i} \rho(x_{i_{\nu}}/\varepsilon_0 n_i)
\]

\[\leq \rho\left( \sum_{1 \leq \nu_1 \leq n_1, 1 \leq \nu_2 \leq n_2, \ldots, 1 \leq \nu_l \leq n_l} y_{\nu_{1},\nu_{2},\ldots,\nu_{l}}/\varepsilon_0 n_1 n_2 \cdots n_l \right) \leq \varepsilon_0.
\]
Next, we put
\begin{equation}
\tilde{\rho}(x) = \sup_{x=\sum_{i} \xi_{i}x_{i}} \sum_{0<\xi_{i}\leq 1} \rho(\xi_{i}x_{i}/\epsilon_{0})/\xi_{i} (x\in R).
\end{equation}

The functional \(\tilde{\rho}(x) (x\in R)\) has the following properties:

1) \(\rho(x/\epsilon_{0})\leq\tilde{\rho}(x)\) for all \(x\in R\),

2) \(\rho(x/\delta_{0})\leq\delta_{0}\) implies \(\tilde{\rho}(x)\leq 2\epsilon_{0}\),

3) \(|x|\leq|y|\) implies \(\tilde{\rho}(x)\leq\tilde{\rho}(y)\),

4) \(x\cap y=0\) implies \(\tilde{\rho}(x+y)=\tilde{\rho}(x)+\tilde{\rho}(y)\),

5) \(\tilde{\rho}(tx)/t (t>0)\) is a non-decreasing function of \(t>0\) for all \(x\in R\).

6) \([p_{\lambda}]\uparrow_{\lambda\in\Lambda} [x]\) implies \(\sup\tilde{\rho}(\lambda p_{\lambda} x)=\tilde{\rho}(x)\).

\((\tilde{\rho})-2)\) is a direct consequence of \((**)*\), and other properties are obvious from the definition of \(\tilde{\rho}\).

Now we can construct a convex modular \(m(x)\) on \(R\):
\begin{equation}
m(x) = \int_{0}^{1} \tilde{\rho}(tx)/t \, dt (x\in R).
\end{equation}

Evidently we see
\begin{equation}
\rho(x/2\epsilon_{0})\leq\tilde{\rho}(x/2)\leq m(x)\leq\tilde{\rho}(x) (x\in R).
\end{equation}

It is easy to check the convex modular condition: \((\rho)\) and \((C)\). \((C)\) follows from \((\tilde{\rho})-5)\), \((\rho)\)-1) and 6) are implied by \((26)\). \((\rho)\)-3), 4) and 5) are almost obvious. \((\rho)\)-2): It is sufficient to see \(m(\alpha x)<+\infty\) for some \(\alpha=\alpha(x)>0\). For \(x\) we can find \(\alpha>0\) such as \(\rho(\alpha x/\delta_{0})\leq\delta_{0}\), hence from \((\tilde{\rho})-2)\) and \((26)\) we have \(m(\alpha x)\leq\tilde{\rho}(\alpha x)\leq 2\epsilon_{0}<+\infty\).

Finally we remark that in an atomic modulared vector lattice \(R(\rho)\) it can be proved that \(\|x\|_{\rho}\) is normable if and only if there is a convex modular on \(R(\rho)\) (cf. \([8]\)). It will be studied in another paper.

References


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