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SOME REMARKS ON CARTAN-BRAUER-HUA THEOREM¹⁾

By

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Concerning Cartan-Brauer-Hua theorem, T. Nagahara and H. Tominaga proved the following [3, Lemma 3.5]:

Let U be a ring with 1, and B a two-sided simple subring of U containing 1. If A is a division subring of U containing 1 such that B is invariant relative to all inner automorphisms determined by non-zero elements of A , then either $A \subseteq B$ or $A \subseteq V_U(B)$.

In what follows, by making use of the same method as in the proof of this fact, we shall present a slight generalization of [3, Lemma 3.5] (Theorem 1) and an extension of [2, Theorem 7.13.1 (2)] (Theorem 2). And finally, we shall prove that Theorem 2 is still valid for inner automorphisms provided A is a simple ring (Theorem 3). Throughout the present note, a ring will mean a ring with the identity element 1, and a subring one with this identity element.

Our first theorem containing [3, Lemma 3.5] can be stated as follows:

Theorem 1. *Let U be a ring, and A and B a subring of U satisfying minimum condition for right ideals and a two-sided simple subring of U respectively. If for each $a \in A$ and $b \in B$ there exists an element $b_1 \in B$ such that $ab = b_1a$, then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

Proof. To be easily seen from the proof of [3, Lemma 3.5], it suffices to prove that $A = (A \cap B) \cup V_A(B)$. Let a be an arbitrary element of A . If a and 1 are linearly left independent over B , then for each $b \in B$, $ab = b_1a$ and $(a+1)b = b_2(a+1)$ yield $(b_1 - b_2)a + (b - b_2) = 0$, whence it follows $b_1 = b_2 = b$. Consequently, we obtain $a \in V_A(B)$. If, on the other hand, a and 1 are linearly dependent, then there holds $d_1a = d_2$ for some non-zero $d_1 \in B$. In case $d_2 \neq 0$, since B is two-sided simple, we obtain $da = 1$ for some $d \in B$. And so, recalling that A satisfies minimum condition for right ideals, one will readily see that a is a regular element of A . And then, $aB = Ba = B$ will yield at once $a \in B$. In case $d_2 = 0$ too, since $d_1(a+1) = d_1 \neq 0$, we obtain $a+1 \in B$. Thus, in either case, a is contained in B . We have proved therefore $A = (A \cap B) \cup V_A(B)$.

Combining our method with the one employed in the proof of [2,

1) The author wishes to express his gratitude to Prof. G. Azumaya for his kind guidance.

Theorem 7.13.1 (2)], we can obtain the following:

Theorem 2.²⁾ *Let U be a ring and B a two-sided simple subring of U . If B is not of characteristic 2 and A is a subring of U such that B is invariant relative to all inner derivations determined by elements of A , then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

Proof. Let a be an arbitrary element of A . For any element $b \in B$, we set $[b, a] = ba - ab = b_1$, $[[b, a]a] = b_2$, $[b, a^2] = b_3$ where b_1, b_2 and b_3 are in B . Then, one will easily see that $2b_1a = 2(ba^2 - aba) = b_2 + b_3 \in B$. And, if a and 1 are linearly left independent over B , we obtain $b_1 = 0$. This means obviously that $a \in V_A(B)$. On the other hand, if a and 1 are linearly dependent: $b^*a - b^{**} = 0$ with non-zero $b^* \in B$, then noting that B is two-sided simple, it will be easy to see that $a \in A \cap B$. We have proved therefore $A = (A \cap B) \cup V_A(B)$. Now, the rest of the proof is the same with the latter half of the proof of [3, Lemma 3.5].

Finally, we shall present the following:

Theorem 3. *Let U be a ring and B a two-sided simple subring of U . If B is not of characteristic 2, and A a simple subring of U such that B is invariant relative to all inner automorphisms determined by regular elements of A , then either $A \subseteq B$ or $A \subseteq V_U(B)$.*

Proof. Let K be the prime field of A (which is evidently contained in the center of B), and let a be an arbitrary α -biregular element³⁾ of A ($0 \neq \alpha \in K$). If a and 1 are linearly left independent over B , then for an arbitrary $b \in B$, $ab = b^*a$ and $(a - \alpha)b = b^{**}(a - \alpha)$ yield at once $(b^* - b^{**})a + \alpha(b^{**} - b) = 0$, whence it follows $b^* = b^{**} = b$. Hence we obtain $a \in V_A(B)$. On the other hand, if a and 1 are linearly dependent, then it will be easy to see that $a \in B$. Since each element of A is a sum of biregular elements by [1], the fact proved above will show that B is invariant relative to all inner derivations determined by elements of A . Hence, our assertion is a direct consequence of Theorem 2.

References

- [1] A. HATTORI: On invariant subrings, Jap., J. Math., 21 (1951), 121-129.
- [2] N. JACOBSON: Structure of rings, Providence (1956).
- [3] T. NAGAHARA and H. TOMINAGA: On Galois and locally Galois extensions of simple rings, Math. J. Okayama Univ., 10 (1961) to appear.

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2) This theorem is essentially due to Dr. H. Tominaga who kindly permitted us to cite it here. We are indebted to him for his helpful suggestions and advices.

3) Cf. [1].