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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense \([1]\)) and \( \rho \) be a functional which satisfies the following four conditions:

1. \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);
2. \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \); \( 1) \)
3. If \( \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \) for a mutually orthogonal system \( \{x_{\lambda}\}_{\lambda \in \Lambda} \), there exists \( x_{0} \in R \) such that \( x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \) and \( \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) \);
4. \( \lim_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper \([2]\), we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular \( m \) on \( R \) for which every universally continuous linear functional \( f \) is continuous with respect to the norm defined by the modular \( m \) \([2; \text{Theorem 3.1}]\).

Recently in \([6]\) J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

1. \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);
2. \( \rho(-x) = \rho(x) \);
3. \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);
4. \( \alpha_{n} \to 0 \) implies \( \rho(\alpha_{n} x) \to 0 \) for every \( x \in R \);
5. for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( || \cdot ||_{0} \) defined by the formula:

\[ ||x||_{0} = \rho(x) \]

1) \( x \perp y \) means \( |x| \cap |y| = 0 \).
2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma \).
3) For the definition of a modular, see \([3]\).
4) A linear functional \( f \) is called universally continuous, if \( \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \) for any \( a_{\lambda} \downarrow 0 \).
5) \( R \) is called semi-regular, if for any \( x \neq 0, x \in R \), there exists a universally continuous linear functional \( f \) such that \( f(x) = 0 \).

In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\}^6 \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \sim \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0=\{ x : x \in R, \rho^*(\xi x)=0 \) for all \( \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^{\perp} \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^{\perp} \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^{\perp} \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^{\perp} \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \S 83]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [3].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \([ p ] \) is a projector: \([ p ]x = \bigcup_{n=1}^{\infty} (n| p \cap x) \) for all \( x \geq 0 \) and \( 1-[ p ] \) is a projection operator onto the normal manifold \( N = \{ p \}^{\perp} \), that is, \( x = [ p ]x + (1-[ p ])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

\begin{equation}
(2.1) \quad \rho(0) = 0
\end{equation}
\begin{equation}
(2.2) \quad \rho([p]x) \leq \rho(x) \quad \text{for all} \quad p, x \in R
\end{equation}
\begin{equation}
(2.3) \quad \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \quad \text{for any} \quad [p_i] \uparrow_{i \in A} [p].
\end{equation}

In the argument below, we have to use the additional property of $\rho$:

\begin{equation}
(\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if} \quad |x| \leq |y|, \quad x, y \in R,
\end{equation}
which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\begin{equation}
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{equation}

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

\[ \sum_{i \in A} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \quad \text{for all} \quad x \in R. \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of} \quad (\rho.3). \]

For such $x_0$,

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_i|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. $\rho'$ fulfills $(\rho.3)$.

If $\rho'$ does not fulfill $(\rho.4)$, we have for some $x_0 \in R$,

\[ \rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all} \quad n \geq 1. \]

By $(\rho.2)$ and $(\rho.4)$, $x_0$ cannot be written as $x_0 = \sum_{\nu=1}^{k} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq k$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'\left(\frac{1}{\nu} x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'\left(\frac{1}{\nu} x_2\right) = +\infty \) (\( \nu = 1, 2, \ldots \))
and
\[ \rho'\left(\frac{1}{2} x_2'\right) > 2. \]
There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).
Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.
Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).
Because, putting \( [p] = \lceil |x| - |y| \rceil \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \]
\[ \leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta |y|) \]
\[ = \rho([p]|x| + (1-[p])|y|) \]
\[ = \rho([p]x) + \rho((1-[p])y) \]
\[ \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \( (\rho.4) \). Thus, in the above theorems, we cannot replace \( (\rho.4) \) by the weaker condition:

\( (\rho.4'') \) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \( (\rho.1), (\rho.2), (\rho.3) \) and \( (\rho.4'') \), but does not \( (\rho.4) \). For this \( \rho_0 \), we obtain

\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \( [0,1] \) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \operatorname{mes} \{ t : x(t) = \frac{1}{i} \}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \( (A.4) \), namely,

\[ (\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \( (\rho.5) \). Now we put

\[ (2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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$d(x+y)=d(x)+d(y)$ if $x \perp y$.

Hence, putting

(2.6) \[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that $(\rho.1), (\rho.2), (\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all $x \in R$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

(2.7) \[ \inf_{\lambda \in \Lambda} d([p_\lambda]x) = 0 \]

for any $[p_\lambda] \downarrow_{\lambda \in \Lambda} 0$. In fact, if we suppose the contrary, we have

\[ \inf_{\nu \in \Lambda} d([p_\nu]x_0) \geq \alpha > 0 \]

for some $[p_\nu] \downarrow_{\nu \in \Lambda} 0$ and $x_0 \in R$.

Hence,

\[ \rho'(\frac{1}{\nu}[p_\nu]x_0) \geq d([p_\nu]x_0) \geq \alpha \]

for all $\nu \geq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

\[ [p_{\lambda_n}] \supseteq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}[p_{\lambda_n}]x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0) = +\infty, \]

which is inconsistent with $(\rho.4)$. Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n \geq 1} 0$ and \( \inf_{n=1,2,...} d([p_n]x) = 0 \) by (2.7). Since $(1 - [p_n])n |y| \geq (1 - [p_n]) |x|$ and

\[ d(\alpha x) = d(x) \]

for $\alpha > 0$ and $x \in R$, we obtain
As $n$ is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y) \]
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

\section*{Theorem 2.3.} $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies $(\rho.3)$ (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies
\begin{equation}
(\rho.4') \quad \sup_{x \in R} \sup_{0 \leq |y| \leq |x|} \rho'(\xi x) = K < +\infty.
\end{equation}

\section*{Proof.} Let $\rho$ satisfy $(\rho.4)$. We need to prove
\begin{equation}
(2.9) \quad \sup_{x \in R} d(x) = \sup_{x \in R} \sup_{\xi \to 0} \rho'(\xi x) = K' < +\infty,
\end{equation}
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'\left(\frac{1}{\nu}x\right)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\epsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with
\[ \rho(x) \leq \epsilon \quad \text{implies} \quad \rho'\left(\frac{1}{\nu_0}x\right) \leq \gamma. \]

In $N_0$, we have obviously
\[ \sup_{x \in N_0} \sup_{\xi \to 0} \rho'(\xi x) = \gamma_0 < +\infty. \]

If $\epsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by $(\rho.4')$, and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (i = 1, 2, \ldots, n), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho^{'}\left(\frac{1}{\nu_0}\alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho^{'}\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho^{'}(y_j) + \rho^{'}\frac{z}{\nu_0} \\
\leq n\gamma + \sum_{j=1}^{m} \rho^{'}(y_j) + \rho^{'}\frac{z}{\nu_0} \\
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho^{'}(\xi x_0) \leq \rho^{'}\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^* \)

\[
\lim_{\xi \to 0} \rho^{'}(\xi x) \leq \gamma
\]

Therefore, we obtain

\[
\sup_{x \in R} \{\lim_{\xi \to 0} \rho^{'}(\xi x)\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^{*}(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{i_0}) = d(\sum_{i=1}^{k} x_{i_0}) = \lim_{\xi \to 0} \rho^{'}(\xi \sum_{i=1}^{k} x_{i_0}) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho^{'}(x_i) = \sum_{i \in A} \rho^{*}(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^{*}(x_i) = \rho^{*}(x_0) \) by \( (\rho.4) \) and \( (2.7) \). Therefore \( \rho^{*} \) satisfies \( (\rho.3) \).

On the other hand, suppose that \( \rho^{*} \) satisfies \( (\rho.3) \) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{\mu} d(x_i) = d(\sum_{i=1}^{\mu} x_i) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
$$\lim_{t \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu$ (\nu \geq 1) and 
$$\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty.$$ 
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$, 
$$\rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty,$$
which is inconsistent with (\rho.4). Therefore we have 
$$\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.$$ Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set: 
$$R_0 = \{x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1\},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\footnote{A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup x_\lambda \in R$, $0 \leq x_\lambda \in S(\lambda \in \Lambda)$ implies $\cup x_\lambda \in S$.} of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$. Putting 
$$[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]$$
we have 
$$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \text{ and } 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx,$$
which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into 
$$R = R_0 \oplus R_0^\perp.$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda \in R, x_\lambda \in [p]R_0\}$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^0$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

\footnote{This means that $x \in R$ is written by $y + z$, $y \in R_0$ and $z \in R_0^\perp$.}

\footnote{$S(\Lambda)$ is the set of all real functions defined on $\Lambda$.}
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Theorem 3.1. Let \( R \) be a quasi-modular space. Then \( R_0^\perp \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) which is semi-continuous, i.e.

\[
\sup_{i \in A} \| x_i \|_0 = \| x \|_0
\]

for any \( 0 \leq x, x_i \in A \cdot x \).

Proof. In virtue of Theorems 2.1 and 2.2, \( \rho^* \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \). Now we put

\[
(3.1) \quad \| x \|_0 = \inf \left\{ \xi ; \rho^* \left( \frac{1}{\xi} x \right) \leq \xi \right\}.
\]

Then,

i) \( 0 \leq \| x \|_0 = \| -x \|_0 < \infty \) and \( \| x \|_0 = 0 \) is equivalent to \( x = 0 \); follows from \((\rho.1), (\rho.6), (2.1)\) and the definition of \( R_0^\perp \).

ii) \( \| x + y \|_0 \leq \| x \|_0 + \| y \|_0 \) for any \( x, y \in R \); follows also from \((A.3)\) which is deduced from \((\rho.4)\).

iii) \( \lim_{\alpha_n \searrow 0} \| \alpha_n x \|_0 = 0 \) and \( \lim_{\alpha_n \searrow a} \| \alpha x \|_0 = 0 \); is a direct consequence of \((\rho.5)\). At last we shall prove that \( \| \cdot \|_0 \) is semi-continuous. From ii) and iii), it follows that \( \lim \| \alpha x \|_0 = \| \alpha x \|_0 \) for all \( x \in R_0^\perp \) and \( \alpha \geq 0 \). If \( x \in R_0^\perp \) and \( \{ p_i \} \uparrow \{ x \} \), for any positive number \( \xi \) with \( \| \{ p \} x \|_0 \geq \xi \) we have \( \rho^*(\frac{1}{\xi} \{ p \} x) > \xi \), which implies \( \sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi} \{ p_i \} x) > \xi \) and hence \( \sup_{\lambda \in \Lambda} \| \{ p_i \} x \|_0 \geq \xi \). Thus we obtain

\[
\sup_{\lambda \in \Lambda} \| \{ p_i \} x \|_0 = \| \{ p \} x \|_0 \quad \text{if} \quad \{ p_i \} \uparrow \{ x \}.
\]

Let \( 0 \leq x_1 \uparrow_{i \in A} x \). Putting

\[
\{ p_n, i \} = \{ (x_i - (1 - \frac{1}{n}) x) \}
\]

we have

\[
\{ p_n, i \} \uparrow_{i \in A} \{ x \} \quad \text{and} \quad \{ p_n, i \} x_i \geq \{ p_n, i \} \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).
\]

As is shown above, since

\[
\sup_{i \in A} \| \{ p_n, i \} x_i \|_0 \geq \sup_{i \in A} \| \{ p_n, i \} \left( 1 - \frac{1}{n} \right) x \|_0 = \| \left( 1 - \frac{1}{n} \right) x \|_0,
\]

we have

\[
\sup_{i \in A} \| x_i \|_0 \geq \| \left( 1 - \frac{1}{n} \right) x \|_0
\]

and also \( \sup_{i \in A} \| x_i \|_0 \geq \| x \|_0 \). As the converse inequality is obvious by iv), \( \| \cdot \|_0 \) is semi-continuous.

Q.E.D.

Remark 2. By the definition of \((3.1)\), we can see easily that \( \lim \| x_n \|_0 = 0 \) if and only if \( \lim \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \).
In order to prove the completeness of quasi-norm \( \| \cdot \|_0 \), the next Lemma is necessary.

**Lemma 2.** Let \( p_{n, \nu}, x_{\nu} \geq 0 \) and \( a \geq 0 \) \((n, \nu = 1, 2, \cdots)\) be the elements of \( R_0^\perp \) such that

\[
\begin{align*}
(p_{n, \nu})^\uparrow_{\nu=1}^\infty &= ([p_n]a)_{\nu=1}^\infty; \\
(p_{n, \nu})x_{\nu} &\geq n[p_{n, \nu}]a \text{ for all } n, \nu \geq 1.
\end{align*}
\]

Then \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence of \( R_0^\perp \) with respect to \( \| \cdot \|_0 \).

**Proof.** We shall show that there exist a sequence of projectors \( [q_m]^\downarrow_{m=1}^\infty \) \((m \geq 1)\) and sequences of natural numbers \( \nu_m, n_m \) such that

\[
\begin{align*}
\| [q_m]a \|_0 &> \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \cdots) \tag{3.4}
\end{align*}
\]

and

\[
\begin{align*}
n_m[q_m]a &\geq [q_m]x_{\nu_m-1} \quad \text{for } m=2, 3, \cdots \tag{3.5}
\end{align*}
\]

where \( \delta = \| [p_0]a \|_0 \).

In fact, we put \( n_1 = 1 \). Since \( [p_{1, \nu}][p_0] \uparrow_{\nu=1}^\infty [p_0] \) and \( \| \cdot \|_0 \) is semi-continuous, we can find a natural number \( \nu_1 \) such that

\[
\| [p_{1, \nu_1}][p_0]a \|_0 > \frac{\| [p_0]a \|_0}{2} = \frac{\delta}{2}.
\]

We put \( [q_1] = [p_{1, \nu_1}][p_0] \). Now, let us assume that \( [q_m], \nu_m, n_m \) \(( m=1, 2, \cdots, k)\) have been taken such that (3.4) and (3.5) are satisfied.

Since \( [(na-x_{\nu})^+]^\uparrow_{n=1}^\infty [a] \) and \( \| [q_k]a \|_0 > \frac{\delta}{2} \), there exists \( n_{k+1} \) with

\[
\| (n_{k+1}a-x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}.
\]

For such \( n_{k+1} \), there exists also a natural number \( \nu_{k+1} \) such that

\[
\| [p_{n_{k+1}, \nu_{k+1}}](n_{k+1}a-x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of \( \| \cdot \|_0 \). Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}](n_{k+1}a-x_{\nu_k})^+[q_k]a,
\]

because

\[
[q_{k+1}] \leq [q_k], \quad \| [q_{k+1}]a \| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a
\]

by (3.3) and \( [q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k} \) by (3.5).

For the sequence thus obtained, we have for every \( k \geq 3 \)
\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| q_{k+1} (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1} q_{k+1} a - n_{k} q_{k+1} a \|_0 \geq \| q_{k+1} a \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu_{k-1}})^+]\) implies \([q_{k+1}] n_k a \geq [q_{k+1}] x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R_0^+\) is an F-space with \(\| \cdot \|_0\) if and only if \(\rho\) satisfies (\(\rho.4'\)).

**Proof.** If \(\rho\) satisfies (\(\rho.4'\)), \(\rho^*\) is a quasi-modular which fulfills also (\(\rho.5\)) and (\(\rho.6\)) in virtue of Theorem 2.3. Since \(\| x \|_0 (= \inf \{ \xi ; \rho^* (x/\xi) \leq \xi \})\) is a quasi-norm on \(R_0^+\), we need only to verify completeness of \(\| \cdot \|_0\). At first let \(\{x_{\nu}\}_{\nu \geq 1} \subset R_0^+\) be a Cauchy sequence with \(0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots} x_r \). Since \(\rho^*\) satisfies (\(\rho.3\)), there exists \(0 \leq x_0 \in R_0^+\) such that \(x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n.r}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{r=1}^{\infty} [p_{n,r}] = [p_n]\), we obtain

\[ (1 - [p_{n.r}]) x_{\nu} \geq n (1 - [p_{n.r}]) x_0 \]

for all \(n, r \geq 1\) and \([p_n]\).

Since \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty} [p_n] = 0\), that is, \(\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]\). And

\[ (1 - [p_{n,r}]) \geq 1 - [p_{n,r}] \]

implies

\[ n (1 - [p_n]) x_0 \geq (1 - [p_n]) x_r \geq 0. \]

Hence we have

\[ y_n = \bigcup_{r=1}^{\infty} (1 - [p_n]) x_r \in R_0^+, \]

because \(R_0^+\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 \leq +\infty, \]

which implies

\[ \| y_n \|_0 \leq \sup_{\nu \geq 1} \| (1 - [p_n]) x_{\nu} \|_0 \leq \gamma \]

for every \(n \geq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma\). This implies
\[ \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma \]

for all \( n \geq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of \( z = \bigcup_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\[ z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_n])x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-[p_n])x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} . \]

By semi-continuity of \( ||\cdot||_0 \), we have

\[ ||z-x_{\nu}||_0 \leq \sup_{\mu \geq \nu} ||x_{\mu}-x_{\nu}||_0 \]

and furthermore \( \lim_{\nu \to \infty} ||z-x_{\nu}||_0 = 0 \).

Secondly let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that

\[ ||y_{\nu+1}-y_{\nu}||_0 \leq \frac{1}{2^{\nu}} \]

for all \( \nu \geq 1 \).

This implies

\[ ||\sum_{\nu=m}^{n} y_{\nu+1}-y_{\nu}||_0 \leq \sum_{\nu=m}^{n} ||y_{\nu+1}-y_{\nu}||_0 \leq \frac{1}{2^{m-1}} \]

for all \( n > m \geq 1 \).

Putting \( z_n = \sum_{\nu=1}^{n} \ |y_{\nu+1}-y_{\nu} |, \) we have a Cauchy sequence \( \{z_n\}_{n \geq 1} \) with \( 0 \leq z_n \leq z \).

Then by the fact proved just above,

\[ z_0 = \bigcup_{n=1}^{\infty} z_n = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_n])x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-[p_n])x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} . \]

Since \( \sum_{\nu=1}^{\infty} \ |y_{\nu+1}-y_{\nu} | \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \) is also convergent and

\[ ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})-y_n||_0 = ||\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})||_0 \leq ||z_0-z_n||_0 \to 0 . \]

Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that

\[ \lim_{\nu \to \infty} ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})-x_{\nu}||_0 = 0 . \]

Therefore \( ||\cdot||_0 \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( ||\cdot||_0 \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu}x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \( (\rho.4') \). Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ defined by (3.1) and $R$ becomes an $F$-space with $|| \cdot ||_0$ if and only if $\rho$ fulfils $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

$$(4.1) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}$$

and show that $|| \cdot ||_1$ is also a quasi-norm on $L$ and

$$(4.2) \quad ||x||_0 \leq ||x||_1 \leq 2||x||_0$$

for all $x \in L$

hold, where $|| \cdot ||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < +\infty (x \in L)$ and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim ||x_n||_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim ||\alpha_n x||_1 = 0$ for all $\alpha_n \downarrow 0$ and that $\lim ||x_n||_1 = 0$ implies $\lim ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$  

This yields

$$||x+y||_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right)$$

$$\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $||x+y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $|| \cdot ||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$  

10) For the convex modular $m$, we can define two kinds of norms such as

$$(3.1) \quad ||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $|| \cdot ||$ and $|| \cdot \|$ respectively.
This yields (4.2), since we have \( ||x||_{0} \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( ||x||_{0} > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \)(A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( ||\cdot||_{1} \) on \( L \) which is equivalent to \( ||\cdot||_{0} \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
||x||_{1} = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^{*}(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R_{1}^{\perp} \) and \( ||\cdot||_{1} \) is complete if and only if \( \rho \) satisfies (\( \rho.4' \)), where \( \rho^{*} \) and \( R_{0} \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
||x||_{0} \leq ||x||_{1} \leq 2||x||_{0} \quad \text{for all } x \in R_{1}^{\perp}.
\]

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-

modular \( \rho^{*} \) on \( R \) satisfies \( \rho.1 \sim \rho.6 \) except \( \rho.3 \) and \( \rho^{*}(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_{0} = \{0\} \)). A sequence of elements \( \{x_{\nu}\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-\lim_{\nu \to \infty} x_{\nu} = a \), if there exists a sequence of elements \( \{a_{\nu}\}_{\nu \geq 1} \) such that \( |x_{\nu} - a| \leq a_{\nu} \) (\( \nu \geq 1 \)) and \( a_{\nu} \downarrow 0 \). And a sequence of elements \( \{x_{\nu}\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-\lim_{\nu \to \infty} x_{\nu} = a \), if for any subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \), there exists a subsequence \( \{z_{\nu}\}_{\nu \geq 1} \) of \( \{y_{\nu}\}_{\nu \geq 1} \) with \( o-\lim_{\nu \to \infty} z_{\nu} = a \). A quasi-norm \( ||\cdot|| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} ||a_{\nu}|| = 0 \) for any \( a_{\nu} \downarrow 0 \). In the sequel, we write by \( ||\cdot||_{0} \) (or \( ||\cdot||_{1} \)) the quasi-norm defined on \( R \) by \( \rho^{*} \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( ||\cdot||_{0} \) (or \( ||\cdot||_{1} \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
\text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z] R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \(\{[p_n]\}_{n \geqq 1}\) such that \(\rho([p_n]x)=+\infty\) and \([p_n] \downarrow_{n=1}^{\infty}0\). Hence by (3.1) it follows that \(||[p_n]x||_0>1\) for all \(n \geqq 1\), which contradicts the continuity of \(||\cdot||_0\).

**Sufficiency.** Let \(a_{\nu} \downarrow_{\nu=1}^{\infty}0\) and put \([p_{\nu}^\epsilon]=[(a_n-\epsilon a_1)^+\] for any \(\epsilon>0\) and \(n \geqq 1\).

This implies

\[
\rho^*(\xi a_n) \leqq \rho^*([p_{\nu}^\epsilon]a_1) + \rho^*(\xi(1-[p_{\nu}^\epsilon])a_1)
\]

for all \(n \geqq 1\) and \(\xi \geqq 0\). In virtue of (5.1) and \([p_{\nu}^\epsilon] \downarrow_{n=1}^{\infty}0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*([p_{\nu}^\epsilon]a_1)<+\infty\), and hence \(\inf_{n \geqq 1} \rho^*([p_{\nu}^\epsilon]a_1) =0\) by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain

\[
\inf_{n \geqq 1} \rho^*([p_{\nu}^\epsilon]a_1) \leqq \rho^*(\xi a_1).
\]

Since \(\epsilon\) is arbitrary, \(\lim_{n \rightarrow \infty} \rho^*([p_{\nu}^\epsilon]a_1)=0\) follows. Hence we infer that \(\inf_{n \geqq 1} ||a_n||_0=0\) and \(||\cdot||_0\) is continuous in view of Remark 2 in \S 3. Q.E.D.

**Corollary.** \(||\cdot||_0\) is continuous, if

\[
(5.2) \quad \rho^*(a_\nu) \rightarrow 0 \text{ implies } \rho^*(\alpha a_\nu) \rightarrow 0 \text{ for every } \alpha \geqq 0.
\]

From the definition, it is clear that \(s\)-\(\lim_{\nu \rightarrow \infty} x_{\nu} =0\) implies \(\lim_{\nu \rightarrow \infty} ||x_{\nu}||_0=0\), if \(||\cdot||_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \rightarrow \infty} ||x_{\nu}||_0=0\) (or \(\lim_{\nu \rightarrow \infty} ||x_{\nu}||=0\)) implies \(\lim_{\nu \rightarrow \infty} s\)-\(\lim_{\nu \rightarrow \infty} x_{\nu} =0\), if \(||\cdot||_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho.3\)).

If we replace \(\lim_{\nu \rightarrow \infty} ||x_{\nu}||_0=0\) by \(\lim_{\nu \rightarrow \infty} \rho(x_{\nu})=0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x)=0 \text{ implies } x=0.
\]

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(||\cdot||_0\) is complete, \(\rho(a_{\nu}) \rightarrow 0\) implies \(s\)-\(\lim_{\nu \rightarrow \infty} a_{\nu} =0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous, i.e. \(\rho^*(x)=\sup_{y_1 \in A} \rho^*(x y_1)\) for any \(0 \leqq x \uparrow_{i \in A} x\). If

11) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x)=\inf_{y_1 \uparrow_{i \in A} x} \{\sup_{j \in A} \rho^*(y_1)\}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \rightarrow 0\) is equivalent to \(\rho_*(x) \rightarrow 0\).
\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{R} \) in virtue of (\( \rho.3 \)).

Now, since
\[ \rho\left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) = 0. \]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^\nu} \) (\( \nu = 1, 2, \cdots \)). Therefore we have \( s\text{-lim} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^{*} \) satisfies (5.3) and \( ||\cdot||_{0} \) is complete and continuous, then (5.2) holds.

**References**


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