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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(3-4), 202-218</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1961</td>
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<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56023">http://hdl.handle.net/2115/56023</a></td>
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<td>File Information</td>
<td>JFSHIU_15_N3-4_202-218.pdf</td>
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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense \[1\]) and $\rho$ be a functional which satisfies the following four conditions:

\begin{align*}
(\rho.1) \quad & 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all } x \in R; \\
(\rho.2) \quad & \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any } x, y \in R \text{ with } x \perp y; \\
(\rho.3) \quad & \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \quad \text{for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda}, \\
(\rho.4) \quad & \lim_{\xi \to 0} \rho(\xi x) < +\infty \quad \text{for all } x \in R.
\end{align*}

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper \[2\], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^5\) $m$ on $R$ for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) $m$ \[2; \text{Theorem 3.1}\].

Recently in \[6\] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

\begin{align*}
(A.1) \quad & \rho(x) \geq 0 \text{ and } \rho(x) = 0 \text{ if and only if } x = 0; \\
(A.2) \quad & \rho(-x) = \rho(x); \\
(A.3) \quad & \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1; \\
(A.4) \quad & \alpha_{n} \to 0 \text{ implies } \rho(\alpha_{n}x) \to 0 \quad \text{for every } x \in R; \\
(A.5) \quad & \text{for any } x \in L \text{ there exists } \alpha > 0 \text{ such that } \rho(\alpha x) < +\infty.
\end{align*}

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by the formula;

\begin{align*}
1) \quad & x \perp y \text{ means } |x| \cap |y| = 0. \\
2) \quad & \text{A system of elements } \{x_{\lambda}\}_{\lambda \in \Lambda} \text{ is called mutually orthogonal, if } x_{\lambda} \perp x_{\gamma} \text{ for } \lambda \neq \gamma. \\
3) \quad & \text{For the definition of a modular, see \[3\].} \\
4) \quad & \text{A linear functional } f \text{ is called universally continuous, if } \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \text{ for any } a_{\lambda} \downarrow 0. \\
5) \quad & R \text{ is called semi-regular, if for any } x \neq 0, x \in R, \text{ there exists a universally continuous linear functional } f \text{ such that } f(x) > 0.
\end{align*}

This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano \[3 \text{ and } 4\]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.\ explicitly.

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1) $x \perp y$ means $|x| \cap |y| = 0$. 
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$. 
3) For the definition of a modular, see \[3\]. 
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$. 
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano \[3 \text{ and } 4\].
(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( \rho(1) \sim \rho(4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \} \) for all \( \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ \rho^* \sup_{x \in R} \{ \lim_{a \to 0} \rho(ax) \} < +\infty \]  

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S 83 \)]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n | p | \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N=\{p\}^\perp \), that is, \( x = [p]x + (1 - [p])x \).

\[ 6 \] This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

\begin{align}
(2.1) \quad & \rho(0) = 0; \\
(2.2) \quad & \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) \quad & \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \quad \text{for any } [p_i] \uparrow_{i \in A} [p].
\end{align}

In the argument below, we have to use the additional property of \( \rho \):

\( \rho(x) \leq \rho(y) \) if \( |x| \leq |y| \), \( x, y \in R \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

Proof. We put for every \( x \in R \),

\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[ \sum_{i \in A} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \quad \text{for all } x \in R. \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of } (\rho.3). \]

For such \( x_0 \),

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in R \),

\[ \rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n \geq 1. \]

By \((\rho.2) \) and \((\rho.4)\), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{r} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq r \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[x_0 = x_1 + x_1', \quad x_1 \perp x_1',\]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) \((\nu = 1, 2, \cdots)\) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[x_1 = x_2 + x_2', \quad x_2 \perp x_2',\]
where
\[\rho'(\frac{1}{\nu} x_2) = +\infty \quad (\nu = 1, 2, \cdots)\]
and
\[\rho'(\frac{1}{2} x_2') > 2.\]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho'(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[\rho'(\frac{1}{\nu} y_\nu) \geq \nu\]
and
\[0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R\]
and
\[\rho'(\frac{1}{\nu} y_0) \geq \rho'(\frac{1}{\nu} y_\nu) \geq \nu,\]
which contradicts \( \rho' \). Therefore \( \rho' \) has to satisfy \( \rho'(4) \). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \( \rho (5) \), \( \rho \) does also \( \text{A.3) in } \S 1 : \)
\[\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)\]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( \lfloor p \rfloor = \lfloor |x| - |y| \rfloor \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha \lfloor p \rfloor |x| + \alpha (1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|) \]
\[ = \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|) \leq \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor) y) \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[ (\rho.4'') \quad \text{for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty. \]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0^*(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \{ t : x(t) = \frac{1}{i} \}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[ (\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ (2.5) \quad d(x) = \lim_{\xi \to 0} \rho'({\xi}x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x + y) = d(x) + d(y) \quad \text{if} \ x \perp y. \]

Hence, putting

\[(2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad \text{if} \ x \in R. \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[(2.7) \quad \inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0 \]

for any \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in \Lambda\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{2 \in \Lambda}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}(p_{\lambda_n} - p_{\lambda_{n+1}})x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}(p_{\lambda_m} - p_{\lambda_{m+1}})x_0) = +\infty, \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[(2.8) \quad d(x) = d(y), \quad \text{if} \ [x] = [y]. \]

We put \([p_{n}] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then,

\[ [p_{n}] \downarrow_{n=1}^{\infty} 0 \]

and

\[ \inf_{n=1,2,...} d([p_{n}]x) = 0 \]

by \((2.7)\). Since \((1-[p_{n}])n |y| \geq (1-[p_{n}])|x|\)

and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
$d(x) = d([p_n]x) + d((1-[p_n])x)$
$\leq d([p_n]x) + d(n(1-[p_n])y)$
$\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies
$d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y)$,
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
\[
\rho^*(x) = \rho^*([y]x) + \rho^*([x]-[y])x) \\
= \rho([y]x) - d([y]x) + \rho^*([x]-[y])x) \\
\geq \rho^*(y) - d(y) + \rho^*([x]-[y])x) \\
\geq \rho^*(y).
\]
Thus $\rho^*$ satisfies (ρ.5).

Q.E.D.

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (ρ.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies (ρ.4').

**Proof.** Let $\rho$ satisfy (ρ.4). We need to prove
\[
\sup_{x \in K} d(x) = \sup_{x \in K} \{\lim_{\xi \to 0} \rho'(\xi x)\} = K' < +\infty,
\]
where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]
Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_0(x) = \rho'(\frac{1}{\nu} x)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^+$ with
\[
\rho(x) \leq \varepsilon \text{ implies } \rho'(\frac{1}{\nu_0} x) \leq \gamma.
\]
In $N_0$, we have obviously
\[
\sup_{x \in N_0} \{\lim_{\xi \to 0} \rho'(\xi x)\} = \gamma_0 < +\infty.
\]
If $\varepsilon \leq 2K$, for any $x_0 \in N_0^+$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (ρ.4'), and hence there exists always an orthogonal decomposition such that
where $\frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon$ for every $j = 1, 2, \cdots, m$ and $\rho(z) \leq \frac{\varepsilon}{2}$. From above, we get $n \leq \frac{4K}{\varepsilon}$ and $m \leq \frac{2K}{\varepsilon}$. This yields

$$ \rho' \left( \frac{1}{\nu_0} \alpha_0 x_0 \right) \leq n \gamma + \sum_{j=1}^{m} \rho' (y_j) + \rho' \left( \frac{z}{\nu_0} \right) $$

Hence, we obtain

$$ \lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho' \left( \frac{\alpha_0}{\nu_0} x_0 \right) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right) $$

in case of $\varepsilon \leq 2K$. If $2K \leq \varepsilon$, we have immediately for $x \in N_0^\perp$

$$ \lim_{\xi \to 0} \rho'(\xi x) \leq \gamma $$

Therefore, we obtain

$$ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma' $$

where

$$ \gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0 $$

Let $\{x_i\}_{i \in A}$ be an orthogonal system with $\sum_{i \in A} \rho^*(x_i) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in A$, we have

$$ \sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma' $$

which implies $\sum_{i \in A} d(x_i) \leq \gamma'$. It follows that

$$ \sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty $$

which implies $x_0 = \sum_{i \in A} x_i \in R$ and $\sum_{i \in A} \rho^*(x_i) = \rho^*(x_0)$ by (\(\rho.4\)) and (2.7). Therefore $\rho^*$ satisfies (\(\rho.3\)).

On the other hand, suppose that $\rho^*$ satisfies (\(\rho.3\)) and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_i\}_{i \geq 1}$ such that

$$ \sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu $$

Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be an orthogonal system with $\sum_{\lambda \in \Lambda} \rho^*(x_\lambda) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in \Lambda$, we have

$$ \sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma' $$

which implies $\sum_{\lambda \in \Lambda} d(x_\lambda) \leq \gamma'$. It follows that

$$ \sum_{\lambda \in \Lambda} \rho'(x_\lambda) = \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) + \sum_{\lambda \in \Lambda} d(x_\lambda) < +\infty $$

which implies $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in R$ and $\sum_{\lambda \in \Lambda} \rho^*(x_\lambda) = \rho^*(x_0)$ by (\(\rho.4\)) and (2.7). Therefore $\rho^*$ satisfies (\(\rho.3\)).

On the other hand, suppose that $\rho^*$ satisfies (\(\rho.3\)) and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_i\}_{i \geq 1}$ such that

$$ \sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu $$
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
\lim_{t \to 0} \rho^*(\xi x) = 0$, there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu \ (\nu \geq 1)$ and \[ \sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < + \infty. \]
It follows that \[ x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R \] and \[ d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu) \] from ($\rho.3$).
\[ \rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = + \infty, \]
which is inconsistent with ($\rho.4$). Therefore we have 
\[ \sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < + \infty. \] Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \ \text{for all} \ n \geq 1 \}, \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let \[ x = \bigcup_{\lambda \in \Lambda} x_\lambda \] with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$. Putting
\[ [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \]
we have 
\[ [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \] and \[ 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx, \]
which implies \[ \rho^*(n[p_{n,\lambda}]x) = 0 \] and \[ \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0. \] Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.
Therefore, $R$ is orthogonally decomposed into \[ R = R_0 \oplus R_0^\perp. \]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, \[ [p]R_0 \] is universally complete, i.e. for any orthogonal system \[ \{x_\lambda\}_{\lambda \in \Lambda}(x_\lambda \in [p]R_0), \]
there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^p$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if \[ \bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda) \] implies \[ \bigcup_{\lambda \in \Lambda} x_\lambda \in S. \]

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^{\perp}$ becomes a quasi-normed space with a quasi-norm $||.||_{0}$ which is semi-continuous, i.e.
\[
\sup_{i \in I} ||x_{i}||_{0} = ||x||_{0}
\]
for any $0 \leq x_{i} \uparrow_{i \in I} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^{*}$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put
\[
(3.1) \quad ||x||_{0} = \inf \{ \xi ; \rho^{*}(\frac{1}{\xi}x) \leq \xi \}.
\]

Then,

i) $0 \leq ||x||_{0} = ||-x||_{0} < \infty$ and $||x||_{0} = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_{0}^{\perp}$.

ii) $||x+y||_{0} \leq ||x||_{0} + ||y||_{0}$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_{n} \uparrow 0} ||\alpha_{n}x||_{0} = 0$ and $\lim ||\alpha x||_{0} = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||.||_{0}$ is semi-continuous. From ii) and iii), it follows that $\lim ||\alpha x||_{0} = ||\alpha x||_{0}$ for all $x \in R_{0}^{\perp}$ and $\alpha_{0} \geq 0$. If $x \in R_{0}^{\perp}$ and $[p_{\lambda}]_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $||[p]x||_{0} > \xi$ we have $\rho^{*}(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^{*}(\frac{1}{\xi}[p_{\lambda}]x) > \xi$ and hence $\sup ||p_{\lambda}x||_{0} \geq \xi$. Thus we obtain
\[
\sup_{\lambda \in \Lambda} ||p_{\lambda}x||_{0} = ||[p]x||_{0}, \text{ if } [p_{\lambda}]_{\lambda \in \Lambda} [p].
\]

Let $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$. Putting
\[
[p_{n,\lambda}] = \left[ (x_{\lambda} - (1 - \frac{1}{n})x) \right]^{*}
\]
we have
\[
[p_{n,\lambda}]_{\lambda \in \Lambda} [x] \text{ and } [p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}](1 - \frac{1}{n})x \quad (n \geq 1).
\]
As is shown above, since
\[
\sup_{\lambda \in \Lambda} ||[p_{n,\lambda}]x_{\lambda}||_{0} \geq \sup_{\lambda \in \Lambda} ||[p_{n,\lambda}](1 - \frac{1}{n})x||_{0} = \left(1 - \frac{1}{n}\right)||x||_{0},
\]
we have
\[
\sup_{\lambda \in \Lambda} ||x_{\lambda}||_{0} \geq \left(1 - \frac{1}{n}\right)||x||_{0}
\]
and also $\sup ||x_{\lambda}||_{0} \geq ||x||_{0}$. As the converse inequality is obvious by iv), $||.||_{0}$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim ||x_{n}||_{0} = 0$ if and only if $\lim \rho(\xi x_{n}) = 0$ for all $\xi \geq 0$. 

\[ \square \]
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\begin{align}
(p_{n, \nu}) & \uparrow_{\nu=1}^{\infty} (p_n) \quad \text{with} \quad \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0, \tag{3.2} \\
[p_{n, \nu}]x_{\nu} & \geq n[p_{n, \nu}]a \quad \text{for all} \quad n, \nu \geq 1. \tag{3.3}
\end{align}

Then \{x_\nu\}_{\nu \geq 1} is not a Cauchy sequence of $R_0^\perp$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m]_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
\| [q_m]a \|_0 & > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m = 1, 2, \cdots) \tag{3.4} \\
n_m[q_m]a & \geq [q_m]x_{\nu_{m-1}} \quad \text{and} \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots), \tag{3.5}
\end{align}

where $\delta = \| [p_0]a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ \| [p_{1, \nu_1}][p_0]a \|_0 > \frac{\delta}{2} \]

We put $[q_1] = [p_{1, \nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n_1a-x_{\nu_1})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k]a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ \| (n_{k+1}a-x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ \| [p_{n_{k+1}, \nu_{k+1}}](n_{k+1}a-x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}, \]

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}](n_{k+1}a-x_{\nu_k})^+[q_k], \]

because

\[ [q_{k+1}] \subseteq [q_k], \quad [q_{k+1}]a \| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and $[q_{k+1}]a \geq [q_{k+1}]a$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0} \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0} \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a||_{0} \geqq ||[q_{k+1}]a_{0}||_{0} \geqq \frac{\delta}{2}$, since $[q_{k+1}] \leqq [q_{k}] \leqq [(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^\perp$ is an F-space with $|| \cdot ||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leqq x_{0} \in R_{0}^\perp$ such that $x_{0} = \bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}] = [(x_{\nu} - nx_{0})^{+}]$ and $\bigcup_{v=1}^{\infty}[p_{n,v}] = [p_{n}]$, we obtain

(3.6) $[p_{n,v}]x_{\nu} \geqq n[p_{n,v}]x_{0}$ for all $n, \nu \geqq 1$ and $[p_{n}]_{\nu=0}^{\infty} = 0$. Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}] = 0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}] - [p_{n}]) = [x_{0}]$. And

$$(1 - [p_{n,v}]) \geqq (1 - [p_{n}])$$

$(n, \nu \geqq 1)$ implies

$$n(1 - [p_{n}])x_{0} \geqq (1 - [p_{n}])x_{\nu} \geqq 0.$$ 

Hence we have

$$y_{n} = \bigcup_{v=1}^{\infty} (1 - [p_{n}])x_{\nu} \in R_{0}^\perp,$$

because $R_{0}^\perp$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $|| \cdot ||_{0}$

$$\gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_{0} < +\infty,$$

which implies

$$||y_{n}||_{0} = \sup_{\nu \geqq 1} ||(1 - [p_{n}])x_{\nu}||_{0} \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $|| \cdot ||_{0}$. We put $z_{1} = y_{1}$ and $z_{n} = y_{n} - y_{n-1} (n \geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0} = ||y_{n}||_{0} \leqq \gamma$. This implies
\[ \sum_{\nu=1}^{n} \rho^*(\frac{z_{\nu}}{1+\gamma}) = \rho^*(\frac{y_n}{1+\gamma}) \leq \gamma \]

for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of \( z = \bigcup_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\[ z = \bigcup_{n=1}^{\infty} y_n = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1- [p_n]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1- [p_n]) x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} \]

By semi-continuity of \( || \cdot ||_0 \), we have

\[ ||z-x_{\nu}||_0 \leq \sup_{\mu \geq \nu} ||x_{\mu} - x_{\nu}||_0 \]

and furthermore

\[ \lim_{\nu \to \infty} ||z-x_{\nu}||_0 = 0. \]

Secondly let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that

\[ ||y_{\nu+1} - y_{\nu}||_0 \leq \frac{1}{2^{\nu}} \]

for all \( \nu \geq 1 \).

This implies

\[ ||\sum_{\nu=m}^{n} |y_{\nu+1} - y_{\nu}|||_0 \leq \sum_{\nu=m}^{n} ||y_{\nu+1} - y_{\nu}||_0 \leq \frac{1}{2^{m-1}} \]

for all \( n > m \geq 1 \).

Putting \( z_n = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}| \), we have a Cauchy sequence \( \{z_n\}_{n \geq 1} \) with \( 0 \leq z_n \uparrow \infty \).

Then by the fact proved just above,

\[ z_0 = \bigcup_{n=1}^{\infty} z_n = \bigcup_{n=1}^{\infty} |y_{n+1} - y_{n}| \in R_0^\perp \quad \text{and} \quad \lim_{n \to \infty} ||z_0 - z_n||_0 = 0. \]

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and

\[ ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n||_0 = ||\sum_{\nu=m}^{\infty} (y_{\nu+1} - y_{\nu})||_0 \leq ||z_0 - z_n||_0 \to 0. \]

Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that

\[ \lim_{\nu \to \infty} ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu}||_0 = 0. \]

Therefore \( || \cdot ||_0 \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( || \cdot ||_0 \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \( (\rho.4') \).

Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let \( R \) be a quasi-modular space which includes no universally complete normal manifold. Then \( R \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by (3.1) and \( R \) becomes an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) fulfils (ρA').

§4. Another Quasi-norm. Let \( L \) be a modular space in the sense of Musielak and Orlicz (§1). Here we put for \( x \in L \)

\[
(4.1) \quad \| x \|_1 = \inf_{\xi > 0} \left( \frac{1}{\xi} + \rho(\xi x) \right)^{10)}
\]

and show that \( \| \cdot \|_1 \) is also a quasi-norm on \( L \) and

\[
(4.2) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all } x \in L
\]
hold, where \( \| \cdot \|_0 \) is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that \( 0 \leq \| x \|_1 = \| -x \|_1 < +\infty \) (\( x \in L \)) and that \( \| x \|_1 = 0 \) is equivalent to \( x = 0 \). Since \( \alpha_n \downarrow_{n=1}^\infty 0 \) implies \( \lim_{n \to \infty} \rho(\alpha_n x) = 0 \) for each \( x \in L \) and \( \lim \| x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \), we obtain that \( \lim \| \alpha x_n \|_1 = 0 \) for all \( \alpha \downarrow_{n=1}^\infty 0 \) and that \( \lim \| x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \| \alpha x_n \|_1 = 0 \) for all \( \alpha > 0 \). If \( \| x \|_1 < \alpha \) and \( \| y \|_1 < \beta \), there exist \( \xi, \eta > 0 \) such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta} (x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y)\right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore \( \| x + y \|_1 \leq \| x \|_1 + \| y \|_1 \) holds for any \( x, y \in L \) and \( \| \cdot \|_1 \) is a quasi-norm on \( L \). If \( \xi \rho(\xi x) \leq 1 \) for some \( \xi > 0 \) and \( x \in L \), we have \( \rho(\xi x) \leq \frac{1}{\xi} \) and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq 2 \frac{1}{\xi}.
\]

10) For the convex modular \( m \), we can define two kinds of norms such as

\[
\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi x} \| x \|
\]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing \( m(\xi x) \) by \( \xi \rho(\xi x) \) in \( \| \cdot \|_1 \) and \( \| \cdot \| \) respectively.
This yields (4.2), since we have \( \| x \|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \| x \|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \) (A.5) in §1, then the formula (4.1) yields a quasi-norm \( \| \cdot \| \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\| x \|_1 = \inf_{\xi \geq 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}
\]

(4.3)

is a semi-continuous quasi-norm on \( R \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.4' \)), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\]

(4.4)

for all \( x \in R \).

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies (\( \rho.1 \)\( \sim \) (\( \rho.6 \)) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-\lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that

\[ |x_n - a| \leq a_n \quad (n \geq 1) \]  

and \( a_n \downarrow 0 \). And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-\lim_{n \to \infty} x_n = a \), if for any subsequence

\[ \{y_n\}_{n \geq 1} \text{ of } \{x_n\}_{n \geq 1}, \text{ there exists a subsequence } \{z_n\}_{n \geq 1} \text{ of } \{y_n\}_{n \geq 1} \text{ with } o-\lim_{n \to \infty} z_n = a. \]

A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} \| a_n \| = 0 \) for any \( a_n \downarrow 0 \). In the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]_R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
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sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \([p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( ||[p_n]x||_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( ||\cdot||_0 \).

**Sufficiency.** Let \( a_{\nu} \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon] a_1) + \rho^*(\xi \epsilon(1 - [p_n^\epsilon]) a_1)
\]

for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \([p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon] a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_n^\epsilon] a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} ||a_n||_0 = 0 \) and \( ||\cdot||_0 \) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \( ||\cdot||_0 \) is continuous, if

(5.2) \( \rho^*(a) \rightarrow 0 \) implies \( \rho^*(\alpha a) \rightarrow 0 \) for every \( \alpha \geq 0 \).

From the definition, it is clear that s-lim \( x = 0 \) implies \( \lim_{\nu \rightarrow \infty} ||x_{\nu}|| = 0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} ||x_{\nu}|| = 0 \) (or \( \lim_{\nu \rightarrow \infty} ||x_{\nu}|| = 0 \)) implies s-lim \( x = 0 \), if \( ||\cdot||_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu \rightarrow \infty} ||x_{\nu}|| = 0 \) by \( \lim \rho(x_{\nu}) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \( \rho^*(x) = 0 \) implies \( x = 0 \).

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_0 \) is complete, \( \rho(a) \rightarrow 0 \) implies s-lim \( a = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{\lambda \in \Lambda} \rho^*(x_{\lambda}) \) for any \( 0 \leq x_{\lambda} \leq x \) for some \( \Lambda \) and \( x_{\lambda} \). If

11) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf \{ \sup_{y_{\lambda} \in \Lambda} \rho^*(y_{\lambda}) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_*(x) \rightarrow 0 \).
$\rho(a_\nu) \leq \frac{1}{2^\nu}$ \hspace{1cm} (\nu \geq 1),

we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_\nu| \in \mathcal{R}$ in virtue of $(\rho.3)$.

Now, since

$$\rho\left(\bigcup_{\nu \geq \nu}^{\infty} |a_\nu|\right) \leq \sum_{\nu=1}^{\infty} \rho(a_\nu) \leq \frac{1}{2^{\nu-1}}$$

holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_\nu|\right)\right) = 0$ and hence (5.3) implies

$$\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_\nu|\right) = 0.$$

Thus we see that $\{a_\nu\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_\nu\}_{\nu \geq 1}$ with $\rho(b_\nu) \to 0$, we can find a subsequence $\{b'_\nu\}_{\nu \geq 1}$ of $\{b_\nu\}_{\nu \geq 1}$ with $\rho(b'_\nu) \leq \frac{1}{2^\nu}$ ($\nu = 1, 2, \cdots$). Therefore we have $s\text{-lim}_{\nu \to \infty} b'_\nu = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete and continuous, then (5.2) holds.

**References**