ON F-NORMS OF QUASI-MODULAR SPACES

By
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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ \[0 \leq \rho(x) = \rho(-x) \leq +\infty\] for all $x \in R$;

$(\rho.2)$ \[\rho(x+y) = \rho(x) + \rho(y)\] for any $x, y \in R$ with $x \perp y^{1)}$;

$(\rho.3)$ If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}^{2)}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

$(\rho.4)$ \[\lim_{t \to 0} \rho(tx) < +\infty\] for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular$^{3)}$ $m$ on $R$ for which every universally continuous linear functional$^{4)}$ is continuous with respect to the norm defined by the modular$^{5)}$ $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

$(A.1)$ \[\rho(x) \geq 0\] and \[\rho(x) = 0\] if and only if $x = 0$;

$(A.2)$ \[\rho(-x) = \rho(x)\];

$(A.3)$ \[\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)\] for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

$(A.4)$ \[\alpha_{n} \to 0\] implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;

$(A.5)$ for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by the formula;

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1) $x \perp y$ means $| x | \cap | y | = 0$.

2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.

3) For the definition of a modular, see [3].

4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in A} f(a_{\lambda}) = 0$ for any $a_{\lambda} \in A_{\lambda}$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.****
(1.1) 

\[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( \rho(1) \sim \rho(4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \) for all \( \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ \rho(4) \]

\[ \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \S 83]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), 

\[ \lfloor p \rfloor \]

is a projector: \( \lfloor p \rfloor x = \bigcup_{n=1}^{\infty} (n \lfloor p \rfloor \cap x) \) for all \( x \geq 0 \) and \( 1 - \lfloor p \rfloor \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = \lfloor p \rfloor x + (1 - \lfloor p \rfloor) x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

\begin{align*}
(2.1) \quad & \rho(0) = 0; \\
(2.2) \quad & \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) \quad & \rho([p]x) = \sup_{\lambda \in A} \rho([p_{\lambda}]x) \quad \text{for any } [p_{\lambda}] \uparrow_{\lambda \in A} [p].
\end{align*}

In the argument below, we have to use the additional property of $\rho$:

\begin{align*}
(\rho.5) \quad & \rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \quad x, y \in R,
\end{align*}

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\begin{align*}
(2.4) \quad & \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{align*}

It is clear that $\rho'$ satisfies the conditions $(\rho.1)$, $(\rho.2)$ and $(\rho.5)$.

Let $\{x_{i}\}_{i \in A}$ be an orthogonal system such that \(\sum_{i \in A} \rho'(x_{i}) < +\infty\), then

\begin{align*}
\sum_{i \in A} \rho(x_{i}) < +\infty,
\end{align*}

because

\begin{align*}
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\end{align*}

We have

\begin{align*}
x_{0} = \sum_{i \in A} x_{i} \in R
\end{align*}

and

\begin{align*}
\rho(x_{0}) = \sum_{i \in A} \rho(x_{i}) \quad \text{in virtue of } (\rho.3).
\end{align*}

For such $x_{0}$,

\begin{align*}
\rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{i \in A} \rho([x_{i}]y)
\end{align*}

\begin{align*}
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_{0}|} \rho([x_{i}]y) = \sum_{i \in A} \rho'(x_{i})
\end{align*}

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_{0} \in R$,

\begin{align*}
\rho'\left(\frac{1}{n} x_{0}\right) = +\infty \quad \text{for all } n \geq 1.
\end{align*}

By $(\rho.2)$ and $(\rho.4)$, $x_{0}$ can not be written as $x_{0} = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_{0}$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))
and
\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2. \) In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,...} \) such that
\[ \rho(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,...} \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho(\frac{1}{\nu} y_0) \leq \rho(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts \((\rho.4)\). Therefore \( \rho' \) has to satisfy \((\rho.4)\). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \((\rho.5)\), \( \rho \) does also \((A.3)\) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \]
\[ \leq \rho(\alpha \lceil p \rceil |x| + \alpha(1 - \lceil p \rceil) |y| + \beta \lceil p \rceil |x| + (1 - \lceil p \rceil) \beta |y|) \]
\[ = \rho(\lceil p \rceil |x| + (1 - \lceil p \rceil) |y|) \]
\[ = \rho(\lceil p \rceil x) + \rho((1 - \lceil p \rceil)y) \]
\[ \leq \rho(x) + \rho(y). \]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[ (\rho.4'') \quad \text{for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty. \]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^\infty i \text{ mes} \{t : x(t) = \frac{1}{i}\}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4), \) namely,

\[ (\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

Proof. In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ (2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
Hence, putting
\[(2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R),\]
we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since
\[d(x) \leq \rho'(x)\]
and
\[d(\alpha x) = d(x)\]
for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note
\[(2.7) \quad \inf_{\lambda \in \Lambda} d([p_\lambda]x) = 0\]
for any \([p_\lambda] \downarrow_{\lambda \in \Lambda} 0\). In fact, if we suppose the contrary, we have
\[\inf_{\lambda \in \Lambda} d([p_\lambda]x_0) \geq \alpha > 0\]
for some \([p_\lambda] \downarrow_{\lambda \in \Lambda} 0\) and \(x_0 \in R\).

Hence,
\[\rho'\left(\frac{1}{\nu}[p_\lambda]x_0\right) \geq d([p_\lambda]x_0) \geq \alpha\]
for all \(\nu \geq 1\) and \(\lambda \in \Lambda\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in \Lambda}\) such that
\[[p_{\lambda_n}] \geq [p_{\lambda_{n+1}}]\]
and
\[\rho'\left(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0\right) \geq \frac{\alpha}{2}\]
for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies
\[\rho'\left(\frac{1}{n}x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0\right) = +\infty,
\]
which is inconsistent with \((\rho.4)\). Secondly we shall prove
\[(2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y].\]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n \to \infty} 0\) and \(\inf_{n=1, 2, \ldots} d([p_n]x) = 0\) by \((2.7)\). Since \((1 - [p_n])n|y| \geq (1 - [p_n])|x|\) and
\[d(\alpha x) = d(x)\]
for \(\alpha > 0\) and \(x \in R\), we obtain
\[ d(x) = d([p_n]x) + d((1-[p_n])x) \]
\[ \leq d([p_n]x) + d(n(1-[p_n])y) \]
\[ \leq d([p_n]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) = \inf_{n=1,2,...} d([p_n]x) + d(y), \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[ \rho^*(x) = \rho^*(y) + \rho^*(([x]-[y])x) \]
\[ = \rho'(y) - d(y) + \rho^*(([x]-[y])x) \]
\[ \geq \rho^*(y). \]

Thus \( \rho^* \) satisfies (\( \rho.5 \)). Q.E.D.

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies
\[ (\rho.4') \]
\[ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K < +\infty. \]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ (2.9) \quad \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty, \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho('\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \varepsilon, \gamma, \) a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[ \rho(x) \leq \varepsilon \implies \rho'(\frac{1}{\nu_0}x) \leq \gamma. \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty. \]

If \( \varepsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
where $\frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon$ for every $i = 1, 2, \ldots, n$, $y_j$ is an atomic element with $\rho(y_j) > \varepsilon$ for every $j = 1, 2, \ldots, m$ and $\rho(z) \leq \frac{\varepsilon}{2}$. From above, we get $n \leq \frac{4K}{\varepsilon}$ and $m \leq \frac{2K}{\varepsilon}$. This yields

$$\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho(y_j) + \rho\left(\frac{z}{\nu_0}\right) \leq n\gamma + \sum_{j=1}^{m} \rho(y_j) + \rho\left(\frac{z}{\nu_0}\right) \leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \sup_{0 \leq a \leq a_0} \rho(\alpha x) + \gamma.$$ 

Hence, we obtain

$$\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right).$$

in case of $\varepsilon \leq 2K$. If $2K \leq \varepsilon$, we have immediately for $x \in N^0_0$

$$\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.$$ 

Therefore, we obtain

$$\sup_{x \in R} \{\lim_{\xi \to 0} \rho'(\xi x)\} \leq \gamma'$$

where

$$\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.$$

Let $\{x_i\}_{i \in A}$ be an orthogonal system with $\sum_{i \in A} \rho^*(x_i) < +\infty$. Then for arbitrary $\lambda_1, \ldots, \lambda_k \in A$, we have

$$\sum_{i=1}^{k} d(x_i) = d(\sum_{i=1}^{k} x_i) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_i) \leq \gamma'$$

which implies $\sum_{i \in A} d(x_i) \leq \gamma'$. It follows that

$$\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,$$

which implies $x_0 = \sum_{i \in A} x_i \in R$ and $\sum_{i \in A} \rho^*(x_i) = \rho^*(x_0)$ by $(\rho.4)$ and (2.7). Therefore $\rho^*$ satisfies $(\rho.3)$.

On the other hand, suppose that $\rho^*$ satisfies $(\rho.3)$ and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_i\}_{i \geq 1}$ such that

$$\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu$$
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
$$\lim_{\xi \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu$ ($\nu \geq 1$) and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x) < +\infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$,
$$\rho^*(\xi x_0) = \sum_{\nu=1}^{\infty} \rho^*(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty,$$
which is inconsistent with (\rho.4). Therefore we have
$$\sup_{x \in R} \left( \lim_{\epsilon \to 0} \rho(\xi x) \right) \leq \sup_{x \in R} d(x) < +\infty.$$
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geq 1\},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$. Putting \[ [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \]
we have \[ \lbrack p_{n,\lambda} \rbrack \uparrow_{\lambda \in \Lambda} [x] \] and \[ 2n [p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}] nx, \]
which implies $\rho^*(n [p_{n,\lambda}] x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n [p_{n,\lambda}] x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp.\(^8)$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda} \subset [p]R_0$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

\(^7)\) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in R$, $\lambda \in \Lambda$ implies $\{x_\lambda\}_{\lambda \in \Lambda} \subset S$.

\(^8)\) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

\(^9)\) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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**Theorem 3.1.** Let $R$ be a quasi-modular space. Then $R^\perp$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e. 
\[
\sup_{i\in\Lambda} \|x_i\|_0 = \|x\|_0 \text{ for any } 0 \leq x_i \uparrow_{i\in\Lambda} x.
\]

**Proof.** In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put 
\[(3.1) \quad \|x\|_0 = \inf \left\{ \xi ; \rho^*\left(\frac{1}{\xi}x\right) \leq \xi \right\} .
\]

Then,
\[i ) \quad 0 \leq \|x\|_0 = \|-x\|_0 < \infty \text{ and } \|x\|_0 = 0 \text{ is equivalent to } x = 0 ; \text{ follows from } (\rho.1), (\rho.6), (2.1) \text{ and the definition of } R^\perp.
\]
\[ii ) \quad \|x+y\|_0 \leq \|x\|_0 + \|y\|_0 \text{ for any } x, y \in R ; \text{ follows also from (A.3)} .
\]
\[iii ) \quad \lim_{\alpha_{n^{-}} \uparrow 0} \|\alpha_n x\|_0 = 0 \text{ and } \lim_{x_{n} \uparrow 0} \|\alpha x\|_0 = 0 ; \text{ is a direct consequence of (\rho.5). At last we shall prove that } ||\cdot||_0 \text{ is semi-continuous. From ii) and iii), it follows that } \lim_{a \rightarrow \alpha_{0}} \|\alpha x\|_0 = \|\alpha_0 x\|_0 \text{ for all } x \in R^\perp \text{ and } \alpha_0 \geq 0 . \text{ If } x \in R^\perp \text{ and } [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p], \text{ for any positive number } \xi \text{ with } \|[p]x\|_0 > \xi \text{ we have } \rho^*(\frac{1}{\xi}[p]x) > \xi, \text{ which implies } \sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi}[p_{\lambda}]x) > \xi \text{ and hence } \sup_{\lambda \in \Lambda} ||p_{\lambda} x||_0 \geq \xi . \text{ Thus we obtain } \sup_{\lambda \in \Lambda} ||p_{\lambda} x||_0 = ||[p]x||_0 , \text{ if } [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p] .
\]

Let $0 \leq x_{1} \uparrow_{i \in \Lambda} x$. Putting 
\[
[p_{n,1}] = \left[ (x_{1} - (1 - \frac{1}{n})x)^* \right]
\]
we have 
\[
[p_{n,1}] \uparrow_{i \in \Lambda} [x] \text{ and } [p_{n,1}]x_{1} \geq [p_{n,1}](1 - \frac{1}{n})x \quad (n \geq 1).
\]
As is shown above, since 
\[
\sup_{i \in \Lambda} ||[p_{n,1}]x_{1}||_0 = \sup_{i \in \Lambda} \|p_{n,1}\|(1 - \frac{1}{n})x||_0 = \|1 - \frac{1}{n}\|_0 \sup_{i \in \Lambda} ||x_{1}||_0,
\]
we have 
\[
\sup_{i \in \Lambda} ||x_{1}||_0 \geq \|1 - \frac{1}{n}\|_0 \|	ext{.}
\]
and also \( \sup_{i \in \Lambda} ||x_{1}||_0 \geq ||x||_0 \). As the converse inequality is obvious by iv), \( ||\cdot||_0 \) is semi-continuous. Q.E.D.

**Remark 2.** By the definition of (3.1), we can see easily that 
\[
\lim \|x_{n}\|_0 = 0 \text{ if and only if } \lim \rho(\xi x_{n}) = 0 \text{ for all } \xi \geq 0 .
\]
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0$ $(n, \nu=1, 2, \cdots)$ be the elements of $R^\perp_0$ such that

\[(3.2) \quad [p_{n,\nu}] \uparrow_{\nu\rightarrow-1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0;
\]
\[(3.3) \quad [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]a \text{ for all } n, \nu \geq 1.
\]

Then \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence of $R^\perp_0$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[(3.4) \quad ||[q_m]a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \cdots)
\]
and
\[(3.5) \quad n_m[q_m]a \geq [q_m]x_{\nu_m-1}, \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),
\]
where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1=1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[||[p_{1,\nu_1}][p_0]a||_0 > \frac{\delta}{2}.
\]

We put $[q_1]= [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m$ $(m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_1})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[||[p_{n_{k+1}, \nu_{k+1}}][na-x_{\nu_1}][q_k]a||_0 > \frac{\delta}{2}.
\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[||[p_{n_{k+1}, \nu_{k+1}}][n_{k+1}a-x_{\nu_1}]^+[q_k]a||_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put $[q_{k+1}]=[p_{n_{k+1}, \nu_{k+1}}][(n_{k+1}a-x_{\nu_1})^+[q_k]]$, because

\[ [q_{k+1}] \subseteq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_{k+1}}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0}\geqq||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0}\geqq||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0}\geqq||[q_{k+1}]a_{0}||_{0}\geqq\frac{\delta}{2}$, since $[q_{k+1}]\leqq[q_{k}]\leqq[(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a\geqq[q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu\geqq 1}$ is not a Cauchy sequence.

Theorem 3.2. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{\perp}$ is an F-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

Proof. If $\rho$ satisfies $(\rho.4')$, $\rho^{*}$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $x||_{0}(=\inf\{\xi;\rho^{*}(\frac{x}{\xi})\leqq\xi\})$ is a quasi-norm on $R_{0}^{\perp}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu\geqq 1}\subset R_{0}^{\perp}$ be a Cauchy sequence with $0\leqq x_{\nu}\uparrow_{\nu=1,2,\ldots}$. Since $\rho^{*}$ satisfies $(\rho.3)$, there exists $0\leqq x_{0}\in R_{0}^{\perp}$ such that $x_{0}=\bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}]=[(x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{v=1}^{\infty}[p_{n,v}]=[p_{n}]$, we obtain

(3.6) $[p_{n,v}]x_{\nu}\geqq n[p_{n,v}]x_{0}$ for all $n, v\geqq 1$ and $[p_{n}]\downarrow_{n=1}^{\infty}0$. Since $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}]=0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}]-[p_{n}])=[x_{0}]$. And

$(1-[p_{n,v}])\geqq(1-[p_{n}])$ $(n, v\geqq 1)$

implies

$n(1-[p_{n}])x_{0}\geqq(1-[p_{n}])x_{v}\geqq 0$.

Hence we have

$y_{n}=\bigcup_{v=1}^{\infty}(1-[p_{n}])x_{v}\in R_{0}^{\perp}$, because $R_{0}^{\perp}$ is universally continuous. As $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$\gamma=\sup_{\nu\geqq 1}||x_{\nu}||_{0}<+\infty$,

which implies

$||y_{n}||_{0}=\sup_{\nu\geqq 1}||(1-[p_{n}])x_{v}||_{0}\leqq\gamma$

for every $n\geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1}=y_{1}$ and $z_{n}=y_{n}-y_{n-1}$ $(n\geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu\geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0}=||y_{n}||_{0}\leqq\gamma$. This implies
\[ \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma \]

for all \( n \geq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of \( z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from
\[
 z = \bigcup_{n=1}^{\infty} y_n = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} x_{\nu}.
\]

By semi-continuity of \( || \cdot ||_0 \), we have
\[
 ||z - x_{\nu}||_0 \leq \sup_{\nu \geq \mu} ||x_{\nu} - x_{\mu}||_0
\]
and furthermore \( \lim_{\nu \to \infty} ||z - x_{\nu}||_0 = 0 \).

Secondly let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that
\[
 ||y_{\nu+1} - y_{\nu}||_0 \leq \frac{1}{2^\nu} \quad \text{for all } \nu \geq 1.
\]
This implies
\[
 ||\sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu}||_0 \leq \sum_{\nu=m}^{n} ||y_{\nu+1} - y_{\nu}||_0 \leq \frac{1}{2^{n-1}} \quad \text{for all } n > m \geq 1.
\]

Putting \( z_n = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \), we have a Cauchy sequence \( \{z_n\}_{n \geq 1} \) with \( 0 \leq z_n \uparrow_{n=1}^{\infty} \). Then by the fact proved just above, \( z_0 = \bigcup_{n=1}^{\infty} z_n = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_0^\perp \) and \( \lim_{n \to \infty} ||z_0 - z_n||_0 = 0 \).

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and
\[
 ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n||_0 = ||\sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})||_0 \leq ||z_0 - z_n||_0 \to 0.
\]
Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that
\[
 \lim_{\rho \to \infty} ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\rho}||_0 = 0.
\]
Therefore \( || \cdot ||_0 \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( || \cdot ||_0 \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)). Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy (\( \rho.4' \)). Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfills (\rho.A'').

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

$$
(4.1) \quad \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
$$

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

$$
(4.2) \quad \|x\|_0 \leq \|x\|_1 \leq 2 \|x\|_0
$$

for all $x \in L$ hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1). From (A.1), (A.2) and (A.5), it follows that $0 \leq \|x\|_1 = \|-x\|_1 < +\infty$ ($x \in L$) and that $\|x\|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \rightarrow \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \rightarrow \infty} \|x_n\|_1 = 0$ implies $\lim_{n \rightarrow \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \rightarrow \infty} \|\alpha x_n\|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n \rightarrow \infty} \|x_n\|_1 = 0$ implies $\lim_{n \rightarrow \infty} \|\alpha x_n\|_1 = 0$ for all $\alpha > 0$. If $\|x\|_1 < \alpha$ and $\|y\|_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
$$

This yields

$$
\|x+y\| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right)
$$

$$
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
$$

in virtue of (A.3). Therefore $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
$$

10) For the convex modular $m$, we can define two kinds of norms such as

$$
\|x\| = \inf_{\xi > 0} \left\{ \frac{1 + m(\xi x)}{\xi} \right\} \quad \text{and} \quad \|x\| = \inf_{\xi > 0} \\frac{1}{m(\xi x) \leq 1} \|x\|_1
$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|_1$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1) \( \sim \) (A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( \|\cdot\|_1 \) on \( L \) which is equivalent to \( \|\cdot\|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R_0^+ \) and \( \|\cdot\|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
\|x\|_0 \leq \|x\|_1 \leq 2 \|x\|_0 \quad \text{for all } x \in R_0^+.
\]

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \leq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( o\text{-}\lim x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that \( |x_n - a_n| \leq a_n \) \( (n \geq 1) \) and \( a_n \downarrow 0 \). And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( s\text{-}\lim x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( o\text{-}\lim z_n = a \).

A quasi-norm \( \|\cdot\| \) on \( R \) is termed to be continuous, if \( \inf \|a_n\| = 0 \) for any \( a_n \downarrow 0 \). In the sequel, we write by \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \text{ for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \([p_n]\) such that \(\rho([p_n]x)=+\infty\) and \([p_n]\downarrow_{n=1}^{\infty}0\). Hence by (3.1) it follows that \(\| [p_n]x \|_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(\| \cdot \|_0\).

**Sufficiency.** Let \(a_{\nu}\downarrow_{\nu=1}^{\infty}0\) and put \([p_{n}^*] = [(a_n - \varepsilon a_1)^+]\) for any \(\varepsilon > 0\) and \(n \geq 1\). It is easily seen that \([p_{n}^*]\downarrow_{n=1}^{\infty}0\) for any \(\varepsilon > 0\) and \(a_n = [a_1]a_n = [p_{n}^*]a_n + (1 - [p_{n}^*])a_n \leq [p_{n}^*]a_1 + \varepsilon a_1\).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_{n}^*]a_1) + \rho^*(\xi (1 - [p_{n}^*])a_1)
\]

for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p_{n}^*]\downarrow_{n=1}^{\infty}0\), we can find \(n_0\) (depending on \(\xi\) and \(\varepsilon\)) such that \(\rho^*(\xi [p_{n}^*]a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*(\xi [p_{n}^*]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho, 2\)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \varepsilon a_1).
\]

Since \(\varepsilon\) is arbitrary, \(\lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} \| a_n \|_0 = 0\) and \(\| \cdot \|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** \(\| \cdot \|_0\) is continuous, if

(5.2) \(\rho^*(a_\nu) \rightarrow 0\) implies \(\rho^*(\alpha a_\nu) \rightarrow 0\) for every \(\alpha \geq 0\).

From the definition, it is clear that \(s-\lim_{\nu \rightarrow \infty} x_\nu = 0\) implies \(\lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0\), if \(\| \cdot \|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0\) (or \(\lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0\)) implies \(s-\lim_{\nu \rightarrow \infty} x_\nu = 0\), if \(\| \cdot \|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho, 3\))).

If we replace \(\lim_{\nu \rightarrow \infty} \| x_\nu \| = 0\) by \(\lim_{\nu \rightarrow \infty} (x_\nu) = 0\), Theorem 5.2 may fail to be valid in general. By this reason, we must consider the following condition:

(5.3) \(\rho^*(x) = 0\) implies \(x = 0\).

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(\| \cdot \|_0\) is complete, \(\rho(a_\nu) \rightarrow 0\) implies \(s-\lim_{\nu \rightarrow \infty} a_\nu = 0\).

Proof. We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^{11}\) i.e. \(\rho^*(x) = \sup_{\nu \in \Lambda} \rho^*(x_\nu)\) for any \(0 \leq x_{\nu} \uparrow_{\nu \in \Lambda}^i\). If

\(^{11}\) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x) = \inf_{y_\nu \uparrow_{\nu \in \Lambda}^i} \{ \sup_{\nu \in \Lambda} \rho^*(y_\nu) \}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \rightarrow 0\) is equivalent to \(\rho_*(x) \rightarrow 0\).
\[ \rho(a_{\nu}) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of (\( \rho.3 \)).

Now, since
\[ \rho \left( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho \left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu=1}^{\nu} |a_{\mu}| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu=1}^{\nu} |a_{\mu}| \right) = 0. \]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^\nu} \) (\( \nu = 1, 2, \cdots \)). Therefore we have \text{s-lim} b_{\nu} = 0. \quad \text{Q.E.D.} \]

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete and continuous, then (5.2) holds.

**References**


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