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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e., a conditionally complete vector lattice in Birkhoff's sense \[1\]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ \(0 \leq \rho(x) = \rho(-x) \leq +\infty\) for all \(x \in R\);

$(\rho.2)$ \(\rho(x+y) = \rho(x) + \rho(y)\) for any \(x, y \in R\) with \(x \perp y\)\(^{1}\);

$(\rho.3)$ If \(\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty\) for a mutually orthogonal system \(\{x_{\lambda}\}_{\lambda \in \Lambda}\)\(^{2}\), there exists \(x_{0} \in R\) such that \(x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}\) and \(\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})\);

$(\rho.4)$ \(\varlimsup_{\xi \to 0} \rho(\xi x) < +\infty\) for all \(x \in R\).

Then, \(\rho\) is called a quasi-modular and \(R\) is called a quasi-modular space.

In the previous paper \[2\], we have defined a quasi-modular space and proved that if \(R\) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^{3}\) \(m\) on \(R\) for which every universally continuous linear functional\(^{4}\) is continuous with respect to the norm defined by the modular\(^{5}\) \(m\) \[2; Theorem 3.1\].

Recently in \[6\] J. Musielak and W. Orlicz considered a modular \(\rho\) on a linear space \(L\) which satisfies the following conditions:

(A.1) \(\rho(x) \geq 0\) and \(\rho(x) = 0\) if and only if \(x = 0\);

(A.2) \(\rho(-x) = \rho(x)\);

(A.3) \(\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)\) for every \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\);

(A.4) \(\alpha_{n} \to 0\) implies \(\rho(\alpha_{n} x) \to 0\) for every \(x \in R\);

(A.5) for any \(x \in L\) there exists \(\alpha > 0\) such that \(\rho(\alpha x) < +\infty\).

They showed that \(L\) is a quasi-normed space with a quasi-norm \(|\cdot|_{0}\) defined by the formula;

\[1\) \(x \perp y\) means \(|x| \wedge |y| = 0\).

\[2\) A system of elements \(\{x_{\lambda}\}_{\lambda \in \Lambda}\) is called mutually orthogonal, if \(x_{\lambda} \perp x_{\gamma}\) for \(\lambda \neq \gamma\).

\[3\) For the definition of a modular, see \[3\].

\[4\) A linear functional \(f\) is called universally continuous, if \(\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0\) for any \(a_{\lambda} \downarrow \lambda \in \Lambda\).

\[5\) \(R\) is called semi-regular, if for any \(x \neq 0, x \in R\), there exists a universally continuous linear functional \(f\) such that \(f(x) \neq 0\).

\(\rho\) is a generalization of a modular \(m\) in the sense of Nakano \[3\ and \[4\]. In the latter, there is assumed that \(m(\xi x)\) is a convex function of \(\xi \geq 0\) for each \(x \in R\).
On F-Norms of Quasi-Modular Spaces

\[ ||x||_0 = \inf \left\{ \xi : \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an F-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions : (\( \rho.1 \sim \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S \)2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S \)3 that a quasi-norm \( || \cdot ||_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an F-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In \( \S \)4, we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R_0^+ \) which is equivalent to \( || \cdot ||_0 \) such that \( || x ||_0 \leq || x ||_1 \leq 2 || x ||_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S \)83]. At last in \( \S \)5 we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S \)5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n|p | \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N=[p]^1 \), that is, \( x=[p]x+(1-[p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

(2.1) $\rho(0) = 0$;
(2.2) $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
(2.3) $\rho([p]x) = \sup_{i \in A} \rho([p_i]x)$ for any $[p_i]_{i \in A}[p]$.

In the argument below, we have to use the additional property of $\rho$:

$(\rho.5)$ $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

(2.4) $\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y)$.

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

$\sum_{i \in A} \rho(x_i) < +\infty$,

because

$\rho(x) \leq \rho'(x)$ for all $x \in R$.

We have

$x_0 = \sum_{i \in A} x_i \in R$,

and

$\rho(x_0) = \sum_{i \in A} \rho(x_i)$ in virtue of $(\rho.3)$.

For such $x_0$,

$\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)$

$= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_i|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)$

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfill $(\rho.4)$, we have for some $x_0 \in R$,

$\rho'(\frac{1}{n} x_0) = +\infty$ for all $n \geq 1$.

By $(\rho.2)$ and $(\rho.4)$, $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{s_0} \xi_\nu e_\nu$, where $e_\nu$ is an atomic element for each $\nu$ with $1 \leq \nu \leq s_0$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho'\left(\frac{1}{\nu} x_1\right) = +\infty \ (\nu = 1, 2, \ldots) \) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where \( \rho'\left(\frac{1}{\nu} x_2\right) = +\infty \ (\nu = 1, 2, \ldots) \) and
\[ \rho'\left(\frac{1}{2} x'_2\right) > 2. \]
There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that \( \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \) and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).
Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.
Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).
If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).
Because, putting \([p]=[(|x|-|y|)^+]\), we obtain
\[
\begin{align*}
\rho(\alpha x + \beta y) & \leq \rho(|x| + |y|) \\
& \leq \rho(\alpha \lfloor p \rfloor |x| + \alpha(1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|) \\
& = \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|) \\
& = \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor)y) \\
& \leq \rho(x) + \rho(y).
\end{align*}
\]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in \mathbb{R} \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^{\infty} i \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[ \lim_{\xi \to 0} \rho(\xi x) = 0 \] for all \( x \in \mathbb{R} \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. Let \( \rho \) be a quasi-modular on \( \mathbb{R} \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

Proof. In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in \mathbb{R} \) and
For all $x \in R$ and $\alpha > 0$. We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

\begin{equation}
\inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0
\end{equation}

for any $[p_{\lambda}] \downarrow_{i \in A} 0$. In fact, if we suppose the contrary, we have

\[ \inf_{i \in A} d([p_{i}]x_{0}) \geq \alpha > 0 \]

for some $[p_{i}] \downarrow_{i \in A} 0$ and $x_{0} \in R$.

Hence,

\[ \rho'\left(\frac{1}{\nu}[p_{\lambda}]x_{0}\right) \geq d([p_{\lambda}]x_{0}) \geq \alpha \]

for all $\nu \geq 1$ and $\lambda \in A$. Thus we can find a subsequence $\{\lambda_{n}\}_{n \geq 1}$ of $\{\lambda\}_{i \in A}$ such that

\[ [p_{\lambda_{n}}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'\left(\frac{1}{n}([p_{\lambda_{n}}] - [p_{\lambda_{n+1}}])x_{0}\right) \geq \frac{\alpha}{2} \]

for all $n \geq 1$ in virtue of $(\rho.2)$ and (2.3). This implies

\[ \rho'\left(\frac{1}{n}x_{0}\right) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_{m}}] - [p_{\lambda_{m+1}}])x_{0}) = +\infty , \]

which is inconsistent with $(\rho.4)$. Secondly we shall prove

\begin{equation}
(2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y].
\end{equation}

We put $[p_{n}] = [(|x| - n|y|')^+]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_{n}] \downarrow_{n \in N} 0$ and $\inf_{n=1,2,\ldots} d([p_{n}]x) = 0$ by (2.7). Since $(1 - [p_{n}])n |y| \geq (1 - [p_{n}]) |x|$ and

\[ d(\alpha x) = d(x) \]

for $\alpha > 0$ and $x \in R$, we obtain
As $n$ is arbitrary, this implies

$$d(x) \leq \inf_{n=1,2,...} d([p_n]x) + d(y),$$

and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then

$$\rho^*(x) = \rho^*(\{y\}x) + \rho^*(([x] - [y])x)$$

$$\geq \rho'(y) - d(y) + \rho^*(([x] - [y])x)$$

$$\geq \rho^*(y).$$

Thus $\rho^*$ satisfies (ρ.5).

Theorem 2.3. $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (ρ.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

$$(\rho.4') \quad \sup_{x \in R} \{ \lim_{\xi + 0} \rho(\xi x) \} = K < +\infty.$$

**Proof.** Let $\rho$ satisfy (ρ.4). We need to prove

$$(2.9) \quad \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi + 0} \rho'(\xi x) \} = K' < +\infty,$$

where

$$\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'\left(\frac{1}{\nu}x\right)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with

$$\rho(x) \leq \varepsilon \quad \text{implies} \quad \rho'\left(\frac{1}{\nu_0}x\right) \leq \gamma.$$

In $N_0$, we have obviously

$$\sup_{x \in N_0} \{ \lim_{\xi + 0} \rho'(\xi x) \} = \gamma_0 < +\infty.$$

If $\varepsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (ρ.4'), and hence there exists always an orthogonal decomposition such that
where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (\( i = 1, 2, \ldots, n \)), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho\left(y_j\right) + \rho\left(\frac{z}{\nu_0}\right)
\]

\[
\leq n\gamma + \sum_{j=1}^{m} \rho\left(y_j\right) + \rho\left(\frac{z}{\nu_0}\right)
\]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \sum_{0 \leq a \leq a_0} \rho(\alpha x) + \gamma
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho\left(\frac{\xi}{\nu_0} x_0\right) \leq \rho\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^\perp \)

\[
\lim_{\xi \to 0} \rho\left(\xi x\right) \leq \gamma
\]

Therefore, we obtain

\[
\sup_{x \in R} \{\lim_{\xi \to 0} \rho\left(\xi x\right)\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_i) = d\left(\sum_{i=1}^{k} x_i\right) = \lim_{\xi \to 0} \rho\left(\xi \sum_{i=1}^{k} x_i\right) \leq \gamma'
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho^*(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d\left(\sum_{i=1}^{n} x_i\right) \geq \mu
\]
for all \( \mu \geqq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since
\[
\lim_{\xi \to 0} \rho^*(\xi x) = 0,
\]
there exists \( \{\alpha_{\nu}\}_{\nu \geqq 1} \) with \( 0 < \alpha_{\nu} \) \((\nu \geqq 1)\) and
\[
\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu} x) < +\infty.
\]
It follows that \( x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu} x \in R \) and \( d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x) \) from (\( \rho.3 \)). For such \( x_0 \), we have for every \( \xi \geqq 0 \),
\[
\rho'(\xi x_0) \geqq \sum_{\nu=1}^{\infty} d(x_0) = +\infty,
\]
which is inconsistent with (\( \rho.4 \)). Therefore we have
\[
\sup_{x \in R} (\lim_{\xi \to 0} \rho(\xi x)) \leqq \sup_{x \in R} d(x) < +\infty.
\]
Q.E.D.

§3. Quasi-norms. We denote by \( R_0 \) the set:
\[
R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geqq 1 \},
\]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_{\lambda} \) with \( R_0 \ni x_{\lambda} \geqq 0 \) for all \( \lambda \in \Lambda \).

Putting \( [p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+] \), we have \( [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \) and \( 2n[p_{n,\lambda}]x_{\lambda} \geqq \rho^*(nx) = 0 \). Hence, we obtain \( x \in R_0 \), that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp.\(^8\)
\]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0 \), \( [p]R_0 \) is universally complete, i.e. for any orthogonal system \( \{x_{\lambda,\nu} \in \Lambda, x_{\lambda,\nu} \in [p]R_0\} \), there exists \( x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R \). Hence we can also verify without difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \) is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda)^0 \)-space. With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, \ |b| \leqq |a|, \ b \in R \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \bigcup_{\lambda \in \Lambda} x_{\lambda} \in S(\lambda \in \Lambda) \) implies \( \bigcup_{\lambda \in \Lambda} x_{\lambda} \in S \).

8) This means that \( x \in R \) is written by \( x = y + z \), \( y \in R_0 \) and \( z \in R_0^\perp \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
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**Theorem 3.1.** Let \( R \) be a quasi-modular space. Then \( R_{0}^{\perp} \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_{0} \) which is semi-continuous, i.e.,
\[
\sup_{x \in A} \| x \|_{0} = \| x \|_{0}
\]
for any \( 0 \leq x \leq \infty \).

**Proof.** In virtue of Theorems 2.1 and 2.2, \( \rho^{*} \) satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\). Now we put
\[
(3.1) \quad \| x \|_{0} = \inf \left\{ \xi; \rho^{*}\left(\frac{1}{\xi}x\right) \leq \xi \right\}.
\]

Then,
i) \( 0 \leq \| x \|_{0} = \| -x \|_{0} < \infty \) and \( \| x \|_{0} = 0 \) is equivalent to \( x = 0 \); follows from \((\rho.1), (\rho.6), (2.1) \) and the definition of \( R_{0}^{\perp} \).

ii) \( \| x + y \|_{0} \leq \| x \|_{0} + \| y \|_{0} \) for any \( x, y \in R \); follows also from \((A.3)\) which is deduced from \((\rho.4)\).

iii) \( \lim_{a \rightarrow \alpha_{0}} \| \alpha x \|_{0} = 0 \) and \( \lim_{x \in A} \| \alpha x \|_{0} = 0 \); is a direct consequence of \((\rho.5)\). At last we shall prove that \( \| \cdot \|_{0} \) is semi-continuous. From ii) and iii), it follows that \( \lim_{x \in R_{0}^{\perp}} \| \alpha x \|_{0} = \| \alpha_{0} x \|_{0} \) for all \( x \in R_{0}^{\perp} \) and \( \alpha_{0} \geq 0 \). If \( x \in R_{0}^{\perp} \) and \( [p_{\lambda}]_{\lambda \in \Lambda} \geq [p] \), for any positive number \( \xi \) with \( \| [p]x \|_{0} > \xi \) we have \( \rho^{*}\left(\frac{1}{\xi}[p]x\right) > \xi \), which implies \( \sup_{\lambda \in \Lambda} \rho^{*}\left(\frac{1}{\xi}[p_{\lambda}]x\right) > \xi \) and hence \( \sup_{\lambda \in \Lambda} \| p_{\lambda}x \|_{0} \geq \xi \). Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| p_{\lambda}x \|_{0} = \| [p]x \|_{0}, \quad \text{if} \quad [p_{\lambda}]_{\lambda \in \Lambda} \geq [p].
\]

Let \( 0 \leq x_{\lambda} \leq \infty \). Putting
\[
[p_{n,\lambda}] = \left[(x_{\lambda} - \left(1 - \frac{1}{n}\right)x)\right]
\]
we have
\[
[p_{n,\lambda}]_{\lambda \in \Lambda} \geq [p_{n,\lambda}]_{\lambda \in \Lambda} \left(1 - \frac{1}{n}\right)x
\]
\( (n \geq 1) \).

As is shown above, since
\[
\sup_{\lambda \in \Lambda} \| [p_{n,\lambda}]x \|_{0} = \sup_{\lambda \in \Lambda} \| [p_{n,\lambda}]\left(1 - \frac{1}{n}\right)x \|_{0} = \| \left(1 - \frac{1}{n}\right)x \|_{0},
\]
we have
\[
\sup_{\lambda \in \Lambda} \| x \|_{0} \geq \left(1 - \frac{1}{n}\right)x
\]
and also \( \sup_{\lambda \in \Lambda} \| x_{\lambda} \|_{0} \geq \| x \|_{0} \). As the converse inequality is obvious by iv), \( \| \cdot \|_{0} \) is semi-continuous.

**Remark 2.** By the definition of \((3.1)\), we can see easily that
\[
\lim_{n \to \infty} \| x_{n} \|_{0} = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} \rho(\xi x_{n}) = 0 \quad \text{for all} \quad \xi \geq 0.
\]
In order to prove the completeness of quasi-norm $\|\cdot\|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R^\perp_0$ such that

\[(3.2) \quad \lim_{\nu \to -1} p_{n,\nu} = \lim_{\nu \to -1} p_n a = [p_0]a \neq 0; \]

\[(3.3) \quad [p_{n,\nu}]x_{\nu} \geq n [p_{n,\nu}]a \quad \text{for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R^\perp_0$ with respect to $\|\cdot\|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[(3.4) \quad \| [q_m]a \|_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m [q_m]a \quad (m = 1, 2, \cdots) \]

and

\[(3.5) \quad n_m [q_m]a \geq [q_m]x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots), \]

where $\delta = \| [p_0]a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}] [p_0] \uparrow_{\nu = 1}^{\infty} [p_0]$ and $\|\cdot\|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[\| [p_{1,\nu_1}] [p_0]a \|_0 > \frac{\| [p_0]a \|_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,\nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n a - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k]a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[\| (n_{k+1} a - x_{\nu_k})^+ [q_k]a \|_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[\| [p_{n_{k+1},\nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k]a \|_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $\|\cdot\|_0$. Hence we can put

$[q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k]$, because

$[q_{k+1}] \subseteq [q_k]$, $\| [q_{k+1}]a \| > \frac{\delta}{2}$, $[q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}]a$

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0} \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0} \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0} \geqq ||[q_{k+1}]a_{0}||_{0} \geqq \frac{\delta}{2},$$

since $[q_{k+1}] \leqq [q_{k}] \leqq [(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{+}$ is an F-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^{*}$ is a quasi-modular which fulfils also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $||x||_{0} (=\inf \{\xi; \rho^{*}(\frac{x}{\xi}) \leqq \xi\})$ is a quasi-norm on $R_{0}^{+}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^{+}$ be a Cauchy sequence with $0 \leqq x_{\nu} \uparrow_{\nu=1,2,\ldots}$. Since $\rho^{*}$ satisfies $(\rho.3)$, there exists $0 \leqq x_{0} \in R_{0}^{+}$ such that $x_{0} = \bigcup_{\nu=1}^{\infty} x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}] = [(x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{v=1}^{\infty} [p_{n,v}] = [p_{n}]$, we obtain

$$[p_{n,v}]x_{\nu} \geqq n[p_{n,v}]x_{0} \quad \text{for all } n, v \geqq 1 \quad \text{and} \quad [p_{n}]_{v=1}^{\infty} = 0.$$ 

Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty} [p_{n}] = 0$, that is, $\bigcup_{n=1}^{\infty} ([x_{0}] - [p_{n}]) = [x_{0}]$. And

$$(1-[p_{n,v}]) \geqq (1-[p_{n}]) \quad (n, v \geqq 1)$$

implies

$$n(1-[p_{n}])x_{0} \geqq (1-[p_{n}])x_{v} \geqq 0.$$ 

Hence we have

$$y_{n} = \bigcup_{v=1}^{\infty} (1-[p_{n}])x_{v} \in R_{0}^{+},$$

because $R_{0}^{+}$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$$\gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_{0} < +\infty,$$

which implies

$$||y_{n}||_{0} = \sup_{\nu \geqq 1} ||(1-[p_{n}])x_{\nu}||_{0} \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1} = y_{1}$ and $z_{n} = y_{n} - y_{n-1}$ ($n \geqq 2$). It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $||\sum_{v=1}^{n} z_{\nu}||_{0} = ||y_{n}||_{0} \leqq \gamma$. This implies
\[ \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma \]

for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of 
\[ z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-\lfloor p_{\nu} \rfloor) x_{\nu} = \bigcup_{\nu=1}^{\infty} \lfloor x_{\nu} \rfloor x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} . \]

By semi-continuity of \( || \cdot ||_0 \), we have 
\[ || z - x_{\nu} ||_0 \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_0 \]

and furthermore 
\[ \lim_{\nu \to \infty} || z - x_{\nu} ||_0 = 0. \]

Secondly let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^+ \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that 
\[ || y_{\nu+1} - y_{\nu} ||_0 \leq \frac{1}{2^{\nu}} \]

for all \( \nu \geq 1 \).

This implies 
\[ || \sum_{\nu=m}^{n} (y_{\nu+1} - y_{\nu}) ||_0 \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_0 \leq \frac{1}{2^{n-m}} \]

for all \( n > m \geq 1 \).

Putting 
\[ z_n = \sum_{\nu=1}^{n} | y_{\nu+1} - y_{\nu} |, \]
we have a Cauchy sequence \( \{z_n\}_{n \geq 1} \) with \( 0 \leq z_n \uparrow \infty \).

Then by the fact proved just above, 
\[ z_0 = \bigcup_{n=1}^{\infty} z_n = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_0^+ \quad \text{and} \quad \lim_{n \to \infty} || z_0 - z_n ||_0 = 0. \]

Since \( \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and 
\[ || y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n ||_0 = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_0 \leq || z_0 - z_n ||_0 \to 0 . \]

Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that 
\[ \lim_{\nu \to \infty} || y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_0 = 0 . \]

Therefore \( || \cdot ||_0 \) is complete in \( R_0^+ \), that is, \( R_0^+ \) is an F-space with \( || \cdot ||_0 \).

Conversely if \( R_0^+ \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_0^+ \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^+ \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)). Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \( (\rho.4') \). Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0 \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let \( R \) be a quasi-modular space which includes no universally complete normal manifold. Then \( R \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by (3.1) and \( R \) becomes an \( F \)-space with \( \| \cdot \|_0 \) if and only if \( \rho \) fulfills (\( \rho.4' \)).

§4. Another Quasi-norm. Let \( L \) be a modular space in the sense of Musielak and Orlicz (§1). Here we put for \( x \in L \)

\[
\| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10})
\]

and show that \( \| \cdot \|_1 \) is also a quasi-norm on \( L \) and

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\]

hold, where \( \| \cdot \|_0 \) is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that \( 0 \leq \| x \|_1 = \| -x \|_1 < +\infty \) (\( x \in L \)) and that \( \| x \|_1 = 0 \) is equivalent to \( x = 0 \). Since \( \alpha_n\downarrow_{n=1}^{\infty} 0 \) implies \( \lim_{n \to \infty} \rho(\alpha_n x) = 0 \) for each \( x \in L \) and \( \lim \| x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \), we obtain that \( \lim \| \alpha_n x \|_1 = 0 \) for all \( \alpha_n\downarrow_{n=1}^{\infty} 0 \) and that \( \lim \| x_n \|_1 = 0 \) implies \( \lim \| \alpha x_n \|_1 = 0 \) for all \( \alpha > 0 \). If \( \| x \|_1 < \alpha \) and \( \| y \|_1 < \beta \), there exist \( \xi, \eta > 0 \) such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore \( \| x + y \|_1 \leq \| x \|_1 + \| y \|_1 \) holds for any \( x, y \in L \) and \( \| \cdot \|_1 \) is a quasi-norm on \( L \). If \( \xi \rho(\xi x) \leq 1 \) for some \( \xi > 0 \) and \( x \in L \), we have \( \rho(\xi x) \leq \frac{1}{\xi} \) and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular \( m \), we can define two kinds of norms such as

\[
\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \geq 1} \frac{1}{\xi}
\]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing \( m(\xi x) \) by \( \xi \rho(\xi x) \) in \( \| \cdot \| \) and \( \| \cdot \| \) respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above 

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \)(A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( ||\cdot||_1 \) on \( L \) which is equivalent to \( ||\cdot||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^+ \) and \( ||\cdot||_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.4' \)), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
||x||_0 \leq ||x||_1 \leq 2||x||_0 \quad \text{for all } x \in R^+.
\]

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies (\( \rho.1\)\( \sim \)(\( \rho.6 \)) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that

\[
|a_n - a| \leq a_{n+1} \quad (n \geq 1)
\]

and \( a_{\kappa} \downarrow_{\kappa = 1}^{\infty} 0 \). And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-lim_{n \to \infty} x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( o-lim_{n \to \infty} z_n = a \).

A quasi-norm \( ||\cdot|| \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} ||a_n|| = 0 \) for any \( a_{\nu \uparrow_{\nu = 0}^{\infty} 0} \). In the sequel, we write by \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]_R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{[p_n]\}_{n\geqq 1} \) such that \( \rho([p_n]x)=+\infty \) and \( [p_n]x\downarrow_{n=1}^{\infty}0 \). Hence by (3.1) it follows that \( ||[p_n]x||_{0}>1 \) for all \( n\geqq 1 \), which contradicts the continuity of \( ||\cdot||_0 \).

**Sufficiency.** Let \( a_\nu\downarrow_{\nu=1}^{\infty}0 \) and put \( [p_n^\epsilon]=[a_n-\epsilon a_1]^+ \) for any \( \epsilon>0 \) and \( n\geqq 1 \).

This implies

\[
\rho^{*}(\xi x) \leq \rho^{*}(\xi[a_n^\epsilon]a_1) + \rho^{*}(\xi e(1-[p_n^\epsilon]a_1))
\]

for all \( n\geqq 1 \) and \( \xi\geqq 0 \). In virtue of (5.1) and \( [p_n^\epsilon]x\downarrow_{n=1}^{\infty}0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^{*}(\xi[a_n^\epsilon]a_1)<+\infty \), and hence \( \inf_{n\geqq 1} \rho^{*}(\xi[a_n^\epsilon]a_1) =0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)).

Thus we obtain

\[
\inf_{n\geqq 1} \rho^{*}(\xi a_n) \leq \rho^{*}(\xi\epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{\nu\to\infty} \rho^{*}(\xi a_n) =0 \) follows. Hence we infer that \( \inf_{n\geqq 1} ||a_n||_{0}=0 \) and \( ||\cdot||_0 \) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \( ||\cdot||_0 \) is continuous, if

(5.2) \( \rho^{*}(a_\nu)\to 0 \) implies \( \rho^{*}(\alpha a_\nu)\to 0 \) for every \( \alpha\geqq 0 \).

From the definition, it is clear that \( s-\lim_{\nu\to\infty} x_\nu=0 \) implies \( \lim_{\nu\to\infty} ||x_\nu||_{0}=0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** If \( \lim_{\nu\to\infty} ||x_\nu||_{0}=0 \) (or \( \lim_{\nu\to\infty} ||x_\nu||=0 \)) implies \( s-\lim_{\nu\to\infty} x_\nu=0 \), if \( ||\cdot||_0 \) is complete (i.e. \( \rho^{*} \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu\to\infty} ||x_\nu||=0 \) by \( \lim_{\nu\to\infty} \rho(x_\nu)=0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \( \rho^{*}(x)=0 \) implies \( x=0 \).

Truly we obtain

**Theorem 5.3.** If \( \rho^{*} \) satisfies (5.3) and \( ||\cdot||_0 \) is complete, \( \rho(a_\nu)\to 0 \) implies \( s-\lim_{\nu\to\infty} a_\nu=0 \).

**Proof.** We may suppose without loss of generality that \( \rho^{*} \) is semi-continuous,\(^{11}\) i.e. \( \rho^{*}(x)=\sup_{y\downarrow_{\nu\in A}} \rho^{*}(y) \) for any \( 0\leqq x_{\downarrow_{\nu\in A}} \). If

\(^{11}\) If \( \rho^{*} \) is not semi-continuous, putting \( \rho_{*}(x)=\inf_{y_{\downarrow_{\nu\in A}}} \{\sup_{y_{\downarrow_{\nu\in A}}^{x}} \rho^{*}(y)\} \), we obtain a quasi-modular \( \rho_{*} \) which is semi-continuous and \( \rho^{*}(x)\to 0 \) is equivalent to \( \rho_{*}(x)\to 0 \).
\[ \rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_\nu| \in \mathcal{R} \) in virtue of (\( \rho.3 \)).

Now, since
\[
\rho \left( \bigcup_{\nu \geq \nu}^{\infty} |a_\mu| \right) \leq \sum_{\mu \geq \nu}^{\infty} \rho(a_\mu) \leq \frac{1}{2^\nu - 1}
\]
holds for each \( \nu \geq 1 \), \( \rho \left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_\mu| \right) \right) = 0 \) and hence (\( 5.3 \)) implies
\[
\bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_\mu| \right) = 0.
\]
Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \quad (\nu = 1, 2, \ldots) \). Therefore we have \( s-lim b_\nu = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (\( 5.2 \)) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (\( 5.3 \)) and \( || \cdot ||_0 \) is complete and continuous, then (\( 5.2 \)) holds.

**References**


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