ON F-NORMS OF QUASI-MODULAR SPACES

By

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§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

(\( \rho.1 \)) \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);

(\( \rho.2 \)) \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y^{1}) \);

(\( \rho.3 \)) If \( \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \) for a mutually orthogonal system \( \{x_{\lambda}\}_{\lambda \in \Lambda}^{2)} \), there exists \( x_{0} \in R \) such that \( x_{0} = \sum_{\lambda \in \Lambda} x \) and \( \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) \);

(\( \rho.4 \)) \( \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^{3} \) \( m \) on \( R \) for which every universally continuous linear functional\(^{4} \) is continuous with respect to the norm defined by the modular\(^{5} \) \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

(A.1) \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);

(A.2) \( \rho(-x) = \rho(x) \);

(A.3) \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);

(A.4) \( \alpha_{n} \to 0 \) implies \( \rho(\alpha_{n} x) \to 0 \) for every \( x \in R \);

(A.5) for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( \| \cdot \|_{0} \) defined by the formula;

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1) \( x \perp y \) means \( |x| \cap |y| = 0 \).
2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma \).
3) For the definition of a modular, see [3].
4) A linear functional \( f \) is called universally continuous, if \( \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \) for any \( \alpha_{\lambda} \downarrow 0_{\lambda \in \Lambda} \).
5) This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4]. In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
\begin{equation}
\|x\|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \tag{6)}
\end{equation}
and \(\|x_n\|_0 \rightarrow 0\) is equivalent to \(\rho(\alpha x_n) \rightarrow 0\) for all \(\alpha \geq 0\).

In the present paper, we shall deal with a general quasi-modular space \(R\) (i.e., without the assumption that \(R\) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \(R\) and to investigate the condition under which \(R\) is an \(F\)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \(\rho\) on \(R\) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\(\rho.1\)) \sim (\(\rho.4\)) with those of \(\rho\) \([6]\), we can not apply the formula (1.1) directly to \(\rho\) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \(\rho^*\) which satisfies (A.2) \sim (A.5) on an arbitrary quasi-modular space \(R\) in §2 (Theorems 2.1 and 2.2). Since \(R\) may include a normal manifold \(R_0 = \{x : x \in R, \rho^*(\xi x) = 0\ \text{for all } \xi \geq 0\}\) and we can not define a quasi-norm on \(R_0\) in general, we have to exclude \(R_0\) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \(\|\cdot\|_0\) on \(R_0^+\) defined by \(\rho^*\) according to the formula (1.1) is semi-continuous, and in order that \(R_0^+\) is an \(F\)-space with \(\|\cdot\|_0\) (i.e., \(\|\cdot\|_0\) is complete), it is necessary and sufficient that \(\rho\) satisfies
\begin{equation}
\sup_{x \in R} \rho(\alpha x) < +\infty
\end{equation}
(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \(\|\cdot\|_1\) on \(R_0^+\) which is equivalent to \(\|\cdot\|_0\) such that \(\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0\) holds for every \(x \in R_0^+\) (Formulas (4.1) and (4.3)). \(\|\cdot\|_1\) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano \([4; \S 83]\). At last in §5 we shall add shortly the supplementary results concerning the relations between \(\|\cdot\|_0\)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases \([8]\).

Throughout this paper \(R\) denotes a universally continuous semi-ordered linear space and \(\rho\) a quasi-modular defined on \(R\). For any \(p \in R\), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x)\) for all \(x \geq 0\) and \(1 - [p]\) is a projection operator onto the normal manifold \(N = \{p\}^1\), that is, \(x = [p]x + (1 - [p])x\).

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6) This quasi-norm was first considered by S. Mazur and W. Orlicz \([5]\) and discussed by several authors \([6\ or \ 7]\).
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

\begin{align}
(2.1) \quad & \rho(0)=0; \\
(2.2) \quad & \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) \quad & \rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x) \quad \text{for any } [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p].
\end{align}

In the argument below, we have to use the additional property of $\rho$:

\begin{equation}
(\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \ x, y \in R,
\end{equation}

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\begin{equation}
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{equation}

It is clear that $\rho'$ satisfies the conditions $(\rho.1)$, $(\rho.2)$ and $(\rho.5)$.

Let $\{x_{i}\}_{i \in \Lambda}$ be an orthogonal system such that $\sum_{i \in \Lambda} \rho'(x_{i}) < +\infty$, then

\[ \sum_{i \in \Lambda} \rho(x_{i}) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \quad \text{for all } x \in R. \]

We have

\[ x_{0} = \sum_{i \in \Lambda} x_{i} \in R \]

and

\[ \rho(x_{0}) = \sum_{i \in \Lambda} \rho(x_{i}) \quad \text{in virtue of } (\rho.3). \]

For such $x_{0}$,

\[ \rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{i \in \Lambda} \rho([x_{i}]y) \]

\[ = \sum_{i \in \Lambda} \sup_{0 \leq |y| \leq |x_{0}|} \rho([x_{i}]y) = \sum_{i \in \Lambda} \rho'(x_{i}) \]

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_{0} \in R$,

\[ \rho'(1/n x_{0}) = +\infty \quad \text{for all } n \geq 1. \]

By $(\rho.2)$ and $(\rho.4)$, $x_{0}$ can not be written as $x_{0} = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_{0}$ into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu}x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu}x_2) = +\infty \quad (\nu = 1, 2, \ldots) \]

and

\[ \rho'(\frac{1}{2}x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2}y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that

\[ \rho\left(\frac{1}{\nu}y_\nu\right) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]

and

\[ \rho(\frac{1}{\nu}y_0) \geq \rho(\frac{1}{\nu}y_\nu) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)\]
\[\leq \rho(\alpha \lfloor p \rfloor |x| + \alpha (1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|)\]
\[= \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|)\]
\[= \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor) y)\]
\[\leq \rho(x) + \rho(y)\].

**Remark 1.** As is shown above, the existence of \(\rho^*\) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4')\quad \text{for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty.\]

In fact, the next example shows that there exists a functional \(\rho_0\) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4')\), but does not \((\rho.4)\). For this \(\rho_0\), we obtain

\[\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty\]

for all \(x \neq 0\).

**Example.** \(L_1[0,1]\) is the set of measurable functions \(x(t)\) which are defined in \([0,1]\) with

\[\int_0^1 |x(t)| \, dt < +\infty.\]

Putting

\[\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \(\rho\) be a quasi-modular on \(R\). We can find a functional \(\rho^*\) which satisfies \((\rho.1)\)~\((\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \(\rho'\) which satisfies \((\rho.5)\). Now we put

\[(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.\]

It is clear that \(0 \leq d(x) = d(|x|) < +\infty\) for all \(x \in R\) and
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Hence, putting

\[ (2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[ (2.7) \quad \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{n}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \(n \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n=1}^{\infty}\) of \(\{\lambda\}_{\lambda \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}[p_{\lambda_n}]x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}]x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0) = +\infty \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|]^+)\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \(\inf_{\text{any } n_1, 2, ...} d([p_n]x) = 0\) by \((2.7)\). Since \((1 - [p_n])n \geq (1 - [p_n])|x|\) and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
\[ d(x) = d([p_n]x) + d((1 - [p_n])x) \leqq d([p_n]x) + d(n(1 - [p_n])y) \leqq d([p_n]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) \leqq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y), \]
and also \( d(x) \leqq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geqq |y| \), then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x \]
\[ = \rho'(y) - d(y) + \rho^*([x] - [y])x \]
\[ \geqq \rho^*(y). \]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies (\( \rho.4' \))
\[
\sup_{x \in R} \left\{ \lim_{\xi \rightarrow 0} \rho(x) \right\} = K < +\infty.
\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[
\sup_{x \in R} d(x) = \sup_{x \in R} \left\{ \lim_{\xi \rightarrow 0} \rho'(\xi x) \right\} = K' < +\infty,
\]
where
\[
\rho'(x) = \sup_{0 \leqq |y| \leqq |x|} \rho(y).
\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'(1/\nu x) \) for \( \nu \geqq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[
\rho(x) \leqq \epsilon \text{ implies } \rho'(1/\nu_0 x) \leqq \gamma.
\]

In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \left\{ \lim_{\xi \rightarrow 0} \rho'(\xi x) \right\} = \gamma_0 < +\infty.
\]

If \( \epsilon \leqq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leqq 2K \) for all \( 0 \leqq \alpha \leqq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \varepsilon (i = 1, 2, \cdots, n) \), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \\
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \\
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq (\frac{4K+\varepsilon}{\varepsilon}) \gamma + (\frac{4K^2}{\varepsilon})
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N^0_0 \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K+\varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_i) = d(\sum_{i=1}^{k} x_i) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_i) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho^*(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore, \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
\[ \lim_{t \to 0} \rho^*(\xi x) = 0 , \]
there exists $\{ \alpha_\nu \}_{\nu \geq 1}$ with $0 < \alpha_\nu \ (\nu \geq 1)$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty$. It follows that
\[ x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R \]
and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (2.3). For such $x_0$, we have for every $\xi \geq 0$,
\[ \rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty , \]
which is inconsistent with ($\rho.4$). Therefore we have
\[ \sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty . \]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{ x : x \in R , \ \rho^*(nx) = 0 \text{ for all } n \geq 1 \} , \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^7\) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$. Putting
\[ \lbrack p_{n,\lambda} \rbrack = \lbrack (2nx_\lambda - nx)^+ \rbrack , \]
we have
\[ \lbrack p_{n,\lambda} \rbrack \uparrow_{\lambda \in \Lambda} \lbrack x \rbrack \text{ and } 2n \lbrack p_{n,\lambda} \rbrack x_\lambda \geq \lbrack p_{n,\lambda} \rbrack nx , \]
which implies $\rho^*(n \lbrack p_{n,\lambda} \rbrack x) = 0$ and $\sup_{x \in R} \rho^*(n \lbrack p_{n,\lambda} \rbrack x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
\[ R = R_0 \oplus R_0^\perp . \]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $\lbrack p \rbrack R_0$ is universally complete, i.e. for any orthogonal system $\{ x_\lambda \}_{\lambda \in \Lambda} \subseteq \lbrack p \rbrack R_0$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in \lbrack p \rbrack R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{\text{pt}}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

\(^7\) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq \alpha$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$.

\(^8\) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

\(^9\) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ which is semi-continuous, i.e.

$$\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 = \| x \|_0$$

for any $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $\rho(1) \sim (\rho.6)$ except (\rho.3). Now we put

$$(3.1) \quad \| x \|_0 = \inf \{ \xi ; \rho^* \left( \frac{1}{\xi} x \right) \leq \xi \} .$$

Then,

i ) $0 \leq \| x \|_0 = - \| x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from (\rho.1), (\rho.6), (2.1) and the definition of $R_0^\perp$.

ii) $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from (\rho.4).

iii) $\lim_{\alpha_n \to 0^+} \| \alpha x \|_0 = 0$ and $\lim_{\| x \|_0 \to 0} \| \alpha x \|_0 = 0$; is a direct consequence of (\rho.5). At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to 0^+} \| \alpha x \|_0 = \| \alpha_0 x \|_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p, \lambda] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $\| [p] x \|_0 > \xi$ we have $\rho^* \left( \frac{1}{\xi} [p] x \right) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^* \left( \frac{1}{\xi} [p, \lambda] x \right) > \xi$ and hence $\sup_{\lambda \in \Lambda} \| [p, \lambda] x \|_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda \in \Lambda} \| [p, \lambda] x \|_0 = \| [p] x \|_0 , \quad \text{if} \quad [p, \lambda] \uparrow_{\lambda \in \Lambda} [p] .$$

Let $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$. Putting

$$[p_{n, \lambda}] = \left[ (x_{\lambda} - (1 - \frac{1}{n}) x)^+ \right]$$

we have

$$[p_{n, \lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad [p_{n, \lambda}] x_{\lambda} \geq [p_{n, \lambda}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda \in \Lambda} \| [p_{n, \lambda}] x_{\lambda} \|_0 \geq \sup_{\lambda \in \Lambda} \left\| [p_{n, \lambda}] \left( 1 - \frac{1}{n} \right) x \right\|_0 = \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0 ,$$

we have

$$\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 \geq \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0$$

and also $\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim \| x_n \|_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

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In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}$, $x_{\nu} \geq 0$ and $a \geq 0$ ($n, \nu = 1, 2, \cdots$) be the elements of $R_0^-$ such that

\begin{equation}
(p_{n,\nu}) \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0;
\end{equation}

\begin{equation}
[p_n] a \geq n [p_{n,\nu}] a \text{ for all } n, \nu \geq 1.
\end{equation}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^-$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty}$ ($m \geq 1$) and sequences of natural numbers $\nu_m, n_m$ such that

\begin{equation}
||[q_m]a||_0 > \frac{\delta}{2} \text{ and } [q_m] a \geq n_m [q_m] a \quad (m = 1, 2, \cdots)
\end{equation}

and

\begin{equation}
n_m [q_m] a \geq [q_m] x_{\nu_{m-1}} , \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots),
\end{equation}

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[
||[p_{1,\nu_1}][p_0] a||_0 > \frac{||p_0||_0}{2} = \frac{\delta}{2}.
\]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m$ ($m = 1, 2, \cdots, k$) have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a] \text{ and } ||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[
||(n_{k+1}a-x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2}.
\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[
||[p_{n_{k+1}, \nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k] a||_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k],
\]

because

\[
[q_{k+1}] \geq [q_k], \quad ||[q_{k+1}] a|| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}] a
\]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1} [q_{k+1}] a - n_k [q_{k+1}] a \|_0 \geq \| [q_{k+1}] a_0 \|_0 \geq \frac{\delta}{2}, \]

since \[ [q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu_{k-1}})^+] \] implies \[ [q_{k+1}] n_k a \geq [q_{k+1}] x_{\nu_{k-1}} \] by (3.4). It follows from the above that \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence.

**Theorem 3.2.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then \( R_0^\perp \) is an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) satisfies (\( \rho.4' \)).

**Proof.** If \( \rho \) satisfies (\( \rho.4' \)), \( \rho^* \) is a quasi-modular which fulfills also (\( \rho.5 \)) and (\( \rho.6 \)) in virtue of Theorem 2.3. Since \( \rho^* \) satisfies (\( \rho.3 \)), there exists \( 0 \leq x_0 \in R_0^\perp \) such that \( x_{0} = \bigcup_{\nu=1}^\infty x_{\nu} \), as is shown in the proof of Theorem 2.3.

Putting \( [p_{n,\nu}] = [(x_{\nu} - nx_0)^+] \) and \( \bigcup_{\nu=1}^\infty [p_{n,\nu}] = [p_n] \), we obtain

\[ [p_{n,\nu}] \geq n [p_{n,\nu}] x_0 \quad \text{for all } n, \nu \geq 1 \]

and \( \bigcup_{\nu=1}^\infty [p_{n}] = 0 \). Since \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we have in virtue of Lemma 2, \( \bigcap_{n=1}^\infty [p_{n}] = 0 \), that is, \( \bigcup_{n=1}^\infty ([x_{\nu}] - [p_{n}]) = [x_{0}] \). And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_{n}]) \quad (n, \nu \geq 1) \]

implies

\[ n(1 - [p_{n}]) x_0 \geq (1 - [p_{n}]) x_{\nu} \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^\infty (1 - [p_{n}]) x_\nu \in R_0^\perp, \]

because \( R_0^\perp \) is universally continuous. As \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we obtain from the triangle inequality of \( \| \cdot \|_0 \)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_{n}]) x_\nu \|_0 \leq \gamma \]

for every \( n \geq 1 \) by semi-continuity of \( \| \cdot \|_0 \). We put \( z_1 = y_1 \) and \( z_n = y_n - y_{n-1} \) \( (n \geq 2) \). It follows from the definition of \( y_n \) that \( \{z_{\nu}\}_{\nu \geq 1} \) is an orthogonal sequence with \( \| \sum_{\nu=1}^n z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma \). This implies
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\[ \sum_{\nu=1}^{n} \rho^{*} \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^{*} \left( \frac{y_{n}}{1+\gamma} \right) \leq \gamma \]

for all \( n \geq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of \( z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\[ z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \left( 1 - \left[ p_{n} \right] \right) x_{n} = \bigcup_{n=1}^{\infty} \left( 1 - \left[ p_{n} \right] \right) x_{n} = \bigcup_{\nu=1}^{\infty} x_{\nu} \]

By semi-continuity of \( || \cdot ||_{0} \), we have

\[ || z - x_{\nu} ||_{0} \leq \sup_{\nu \geq \nu} || x_{\nu} - x_{\nu} ||_{0} \]

and furthermore \( \lim_{\nu \to \infty} || z - x_{\nu} ||_{0} = 0 \).

Secondly let \{\(x_{\nu}\)\}_{\nu=1}^{\infty} be an arbitrary Cauchy sequence of \( R_{0}^{\perp} \). Then we can find a subsequence \{\(y_{\nu}\)\}_{\nu=1}^{\infty} of \{\(x_{\nu}\)\}_{\nu=1}^{\infty} such that

\[ || y_{\nu+1} - y_{\nu} ||_{0} \leq 2^{-\nu} \]

for all \( \nu \geq 1 \).

This implies

\[ || \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{m-1}} \]

for all \( n \geq m \geq 1 \).

Putting \( z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}| \), we have a Cauchy sequence \{\(z_{n}\)\}_{n=1}^{\infty} with \( 0 \leq z_{n}^{\uparrow} \).

Then by the fact proved just above,

\[ z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{\perp} \] and \( \lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0 \).

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \) is convergent, \( y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and

\[ || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=n+1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0 \]

Since \{\(y_{\nu}\)\}_{\nu=1}^{\infty} is a subsequence of the Cauchy sequence \{\(x_{\nu}\)\}_{\nu=1}^{\infty}, it follows that

\[ \lim_{\nu \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0 \]

Therefore \( || \cdot ||_{0} \) is complete in \( R_{0}^{\perp} \), that is, \( R_{0}^{\perp} \) is an F-space with \( || \cdot ||_{0} \).

Conversely if \( R_{0}^{\perp} \) is an F-space, then for any orthogonal sequence \{\(x_{\nu}\)\}_{\nu=1}^{\infty} \in R_{0}^{\perp}, we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp} \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)). Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \((\rho.4').\) Q.E.D.

Since \( R_{0} \) contains a normal manifold which is universally complete, if \( R_{0}^{\perp} \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfils ($\rho.A'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz ($\S$1). Here we put for $x \in L$

$$||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10}$$

and show that $||\cdot||_1$ is also a quasi-norm on $L$ and

$$||x||_0 \le ||x||_1 \le 2||x||_0$$

hold, where $||\cdot||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \le ||x||_1 = ||-x||_1 < +\infty$ ($x \in L$) and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim ||x_n||_1 = 0$ implies $\lim \rho(\xi x_n) = 0$ for all $\xi \ge 0$, we obtain that $\lim ||\alpha_n x||_1 = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim ||x_n||_1 = 0$ implies $\lim ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ 

This yields

$$||x+y|| \le \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta} (x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y)\right)$$

$$\le \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $||x+y|| \le ||x|| + ||y||$ holds for any $x, y \in L$ and $||\cdot||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \le 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \le \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \le \frac{1}{\xi} + \rho(\xi x) \le \frac{2}{\xi}.$$ 

10) For the convex modular $m$, we can define two kinds of norms such as

$$||x|| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \le 1} \frac{1}{\xi}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm \( \|\cdot\|_1 \) on \( L \) which is equivalent to \( \|\cdot\|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}, \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^+_\rho \) and \( \|\cdot\|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all } x \in R^+_\rho.
\]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{\nu \to \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a_\nu| \leq a_\nu (\nu \geq 1) \) and \( a_\nu \downarrow_\nu 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{\nu \to \infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( \lim_{\nu \to \infty} z_\nu = a \).

A quasi-norm \( \|\cdot\| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} \|a_\nu\| = 0 \) for any \( a_\nu \downarrow_\nu 0 \). In the sequel, we write by \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
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sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( ||[p_n]x||_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( ||\cdot||_0 \).

Sufficiency. Let \( a_{\nu} \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_n^*] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^*] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p_n^*]a_n + (1 - [p_n^*])a_n \leq [p_n^e]a_1 + \epsilon a_1 \).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^*]a_1) + \rho^*(\xi \epsilon (1 - [p_n^*])a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^*] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^*]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1) \).

Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} 1_{a_n} ||_0 = 0 \) and \( ||\cdot||_0 \) is continuous in view of Remark 2 in §3. Q.E.D.

Corollary. \( ||\cdot||_0 \) is continuous, if
\[
(5.2) \quad \rho^*(a_\nu) \rightarrow 0 \implies \rho^*(\alpha a_\nu) \rightarrow 0 \quad \text{for every } \alpha \geq 0.
\]

From the definition, it is clear that \( \text{s-lim} x_\nu = 0 \) implies \( \lim_{v \rightarrow \infty} ||x_\nu||_0 = 0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

\[\text{Theorem 5.2.} \quad \lim_{v \rightarrow \infty} ||x_\nu||_0 = 0 \quad (\text{or } \lim_{v \rightarrow \infty} ||x_\nu|| = 0) \quad \text{implies } \text{s-lim} x_\nu = 0, \text{ if } ||\cdot||_0 \text{ is complete (i.e. } \rho^* \text{ satisfies (5.3)).}
\]

If we replace \( \lim_{v \rightarrow \infty} ||x_\nu|| = 0 \) by \( \lim_{v \rightarrow \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[\rho^*(x) = 0 \quad \text{implies } x = 0. \]

Truly we obtain

\[\text{Theorem 5.3.} \quad \text{If } \rho^* \text{ satisfies (5.3) and } ||\cdot||_0 \text{ is complete, } \rho(a_\nu) \rightarrow 0 \text{ implies } \text{s-lim} a_\nu = 0.
\]

Proof. We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11}\) i.e. \( \rho^*(x) = \sup_{y \in A} \rho^*(y) \) for any \( 0 \leq x \uparrow_{j \in A} x \). If
\[\text{11) If } \rho^* \text{ is not semi-continuous, putting } \rho_*(x) = \inf_{y \uparrow_{j \in A} x} \{ \sup_{j \in A} \rho^*(y_j) \}, \text{ we obtain a quasi-modular } \rho_* \text{ which is semi-continuous and } \rho^*(x_\nu) \rightarrow 0 \text{ is equivalent to } \rho_*(x_\nu) \rightarrow 0.
\]
\[ \rho(a_{\nu}) \leq \frac{1}{2^\nu} \quad (\nu \geq 1) , \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of (\( \rho.3 \)).

Now, since
\[ \rho\left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) = 0 . \]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu = 1, 2, \ldots) \). Therefore we have \( s\text{-lim}_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \|\cdot\|_0 \) is complete and continuous, then (5.2) holds.

**References**


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