ON F-NORMS OF QUASI-MODULAR SPACES

By Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense \[1\]) and $\rho$ be a functional which satisfies the following four conditions:

$\rho.1)$ $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;

$\rho.2)$ $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y^{1)}$;

$\rho.3)$ If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}^{2)}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

$\rho.4)$ $\varlimsup_{\xi \rightarrow 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $^{3)} m$ on $R$ for which every universally continuous linear functional $^{4)} f$ is continuous with respect to the norm defined by the modular $^{5)} m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

$\rho.a)$ $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

$\rho.b)$ $\rho(-x) = \rho(x)$;

$\rho.c)$ $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

$\rho.d)$ $\alpha_{n} \rightarrow 0$ implies $\rho(\alpha_{n} x) \rightarrow 0$ for every $x \in R$;

$\rho.e)$ for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula;

\[ 1) x \perp y \text{ means } |x| \cap |y| = 0. \]

\[ 2) \text{A system of elements } \{x_{\lambda}\}_{\lambda \in \Lambda} \text{ is called mutually orthogonal, if } x_{\lambda} \perp x_{\gamma} \text{ for } \lambda \neq \gamma. \]

\[ 3) \text{For the definition of a modular, see } [3]. \]

\[ 4) \text{A linear functional } f \text{ is called universally continuous, if } \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \text{ for any } a_{\lambda} \downarrow a_{0}. \]

\[ 5) \text{This modular } \rho \text{ is a generalization of a modular } m \text{ in the sense of Nakano } [3 \text{ and } 4]. \]

In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

$\rho.5)$ $\rho(\xi x) < +\infty$ for all $x \in R.
On F-Norms of Quasi-Modular Spaces

$(1.1)$

$$\|x\|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\}$$

and $\|x_n\|_0 \to 0$ is equivalent to $\rho(\alpha x_n) \to 0$ for all $\alpha \geq 0$.

In the present paper, we shall deal with a general quasi-modular space $R$ (i.e. without the assumption that $R$ is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on $R$ and to investigate the condition under which $R$ is an $F$-space with this quasi-norm by making use of the above formula $(1.1)$. Since a quasi-modular $\rho$ on $R$ does not satisfy the conditions $(A.1)$, $(A.2)$, $(A.4)$ and $(A.5)$ in general, as is seen by comparing the conditions: $(\rho.1) \sim (\rho.4)$ with those of $\rho$ $[6]$, we can not apply the formula $(1.1)$ directly to $\rho$ to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular $\rho^*$ which satisfies $(A.2) \sim (A.5)$ on an arbitrary quasi-modular space $R$ in §2 (Theorems 2.1 and 2.2). Since $R$ may include a normal manifold $R_0 = \{x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0\}$ and we can not define a quasi-norm on $R_0$ in general, we have to exclude $R_0$ in order to proceed with the argument further. We shall prove in §3 that a quasi-norm $\|\cdot\|_0$ on $R_0^+$ defined by $\rho^*$ according to the formula $(1.1)$ is semi-continuous, and in order that $R_0^+$ is an $F$-space with $\|\cdot\|_0$ (i.e. $\|\cdot\|_0$ is complete), it is necessary and sufficient that $\rho$ satisfies

$$(\rho.4') \quad \sup_{x \in R} \{ \lim_{\alpha \to 0} \rho(\alpha x) \} < +\infty$$

(Thorem 3.2).

In §4, we shall show that we can define another quasi-norm $\|\cdot\|_1$ on $R_0^+$ which is equivalent to $\|\cdot\|_0$ such that $\|x\|_0 \leq 1 x|\|_1 \leq 2\|x\|_0$ holds for every $x \in R_0^+$ (Formulas $(4.1)$ and $(4.3)$). $\|\cdot\|_1$ has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano $[4; \S 83]$. At last in §5 we shall add shortly the supplementary results concerning the relations between $\|\cdot\|_0$-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases $[8]$.

Throughout this paper $R$ denotes a universally continuous semi-ordered linear space and $\rho$ a quasi-modular defined on $R$. For any $p \in R$, $[p]$ is a projector: $[p]x = \bigcup_{n=1}^{\infty} (n|p |x)$ for all $x \geq 0$ and $1-[p]$ is a projection operator onto the normal manifold $N = \{p\}^1$, that is, $x = [p]x + (1 - [p])x$.  

6) This quasi-norm was first considered by S. Mazur and W. Orlicz $[5]$ and discussed by several authors $[6$ or $7]$. 
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[(2.1) \quad \rho(0)=0; \]
\[(2.2) \quad \rho([p]x)\leq\rho(x) \quad \text{for all } p, x\in R; \]
\[(2.3) \quad \rho([p]x) = \sup_{i\in I} \rho([p_i]x) \quad \text{for any } [p_i]_{i\in I} \uparrow [p]. \]

In the argument below, we have to use the additional property of \( \rho \):
\[(\rho.5) \quad \rho(x)\leq\rho(y) \quad \text{if } |x|\leq|y|, \ x, y\in R, \]
which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x\in R \),
\[(2.4) \quad \rho'(x) = \sup_{0\leq |y|\leq |x|} \rho(y). \]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i\in I} \) be an orthogonal system such that \( \sum_{i\in I} \rho'(x_i) < +\infty \), then
\[\sum_{i\in I} \rho(x_i) < +\infty, \]
because
\[\rho(x)\leq\rho'(x) \quad \text{for all } x\in R. \]

We have
\[x_0 = \sum_{i\in I} x_i \in R \]
and
\[\rho(x_0) = \sum_{i\in I} \rho(x_i) \quad \text{in virtue of } (\rho.3). \]

For such \( x_0 \),
\[\rho'(x_0) = \sup_{0\leq |y|\leq |x_0|} \rho(y) = \sup_{0\leq |y|\leq |x_0|} \sum_{i\in I} \rho([x_i]y) = \sum_{i\in I} \rho([x_i]y) = \sum_{i\in I} \rho'(x_i) \]
holds, i.e., \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in R \),
\[\rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n\geq 1. \]

By \((\rho.2) \) and \((\rho.4) \), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1\leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho\left(\frac{1}{\nu} x_1 \right) = + \infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where
\[ \rho\left(\frac{1}{\nu} x_2 \right) = + \infty \quad (\nu = 1, 2, \ldots) \]
and
\[ \rho\left(\frac{1}{2} x'_2 \right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho\left(\frac{1}{2} y_2 \right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that
\[ \rho\left(\frac{1}{\nu} y_\nu \right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho\left(\frac{1}{\nu} y_0 \right) \geq \rho\left(\frac{1}{\nu} y_\nu \right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).
If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\begin{align*}
\rho(\alpha x + \beta y) &\leq \rho(|x| + |y|) \\
&\leq \rho(\alpha \lfloor p \rfloor |x| + \alpha (1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|) \\
&= \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|) \\
&= \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor) y) \\
&\leq \rho(x) + \rho(y).
\end{align*}

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[
\rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| \, dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[
(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.
\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1)\)~\((\rho.6)\) except \((\rho.3)\).

Proof. In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[
d(x) = \lim_{\xi \to 0} \rho'(\xi x).
\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
$d(x + y) = d(x) + d(y)$ if $x \perp y$.

Hence, putting

$$\rho^*(x) = \rho'(x) - d(x) \quad (x \in R).$$

we can see easily that $(\rho.1)$, $(\rho.2)$, $(\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since

$$d(x) \leq \rho'(x)$$

and

$$d(\alpha x) = d(x)$$

for all $x \in R$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

$$\inf_{\lambda \in A} d([\lambda x]_0) = 0$$

for any $[\lambda x]_{\lambda \in A}$ in $A$. In fact, if we suppose the contrary, we have

$$\inf_{\lambda \in A} d([\lambda x]_0) \geq \alpha > 0$$

for some $[\lambda x]_{\lambda \in A}$ and $x_0 \in R$.

Hence,

$$\rho'(\frac{1}{\nu} [\lambda x]_0) \geq d([\lambda x]_0) \geq \alpha$$

for all $\nu \geq 1$ and $\lambda \in A$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in A}$ such that

$$[\lambda_{n+1}] \leq [\lambda_n]$$

and

$$\rho'(\frac{1}{n} [\lambda_n]_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} [\lambda_m]_0) = +\infty$$

for all $n \geq 1$ in virtue of $(\rho.2)$ and (2.3). This implies

$$\rho'(\frac{1}{n} x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} [\lambda_m]_0) = +\infty,$$

which is inconsistent with $(\rho.4)$. Secondly we shall prove

$$d(x) = d(y), \quad \text{if } [x] = [y].$$

We put $[p_n] = [(|x| - n|y|) +]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n \to \infty} 0$ and $d([p_n] x) = 0$ by (2.7). Since $(1 - [p_n])n |y| \geq |(1 - [p_n])| x |$ and

$$d(\alpha x) = d(x)$$

for $\alpha > 0$ and $x \in R$, we obtain
$d(x) = d([p_n]x) + d((1-[p_n])x)$
$\leq d([p_n]x) + d(n(1-[p_n])y)$
$\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies

$$d(x) \leq \inf_{n=1,2,\ldots} d([p_n]x) + d(y),$$

and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then

$$\rho^*(x) = \rho^*([y]x) + \rho^*([x]-[y])x)$$
$$= \rho^*([y]x) - d([y]x) + \rho^*([x]-[y])x)$$
$$\geq \rho^*(y) - d(y) + \rho^*([x]-[y])x)$$
$$\geq \rho^*(y).$$

Thus $\rho^*$ satisfies $(\rho.5)$.

**Q.E.D.**

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies $(\rho.3)$ (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies $(\rho.4')$

$$\sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K < +\infty.$$

**Proof.** Let $\rho$ satisfy $(\rho.4)$. We need to prove

$$\sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K' < +\infty,$$

where

$$\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'(\frac{1}{\nu} x)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with

$$\rho(x) \leq \varepsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0} x) \leq \gamma.$$ 

In $N_0$, we have obviously

$$\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = \gamma_0 < +\infty.$$

If $\varepsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by $(\rho.4')$, and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) \((i = 1, 2, \cdots, n)\), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq n \gamma + \frac{m}{\nu_0} \rho'(y_j) + \rho'\frac{z}{\nu_0}
\]

\[
\leq 4K \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'\left(\xi x_0\right) \leq \rho'\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \epsilon}{\epsilon}\right) \gamma + \left(\frac{4K^2}{\epsilon}\right)
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_{0}^{+} \)

\[
\lim_{\xi \to 0} \rho'\left(\xi x\right) \leq \gamma
\]

Therefore, we obtain

\[
\sup_{x \in R} \left\{ \lim_{\xi \to 0} \rho'\left(\xi x\right) \right\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0
\]

Let \( \{x_{\lambda}\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^{*}(x_{\lambda}) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\sum_{\nu=1}^{k} d(x_{\nu}) = d\left(\sum_{\nu=1}^{k} x_{\nu}\right) = \lim_{\xi \to 0} \rho'\left(\xi \sum_{\nu=1}^{k} x_{\nu}\right) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_{\lambda}) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho^{*}(x_{\lambda}) = \sum_{\lambda \in A} \rho^{*}(x_{\lambda}) + \sum_{\lambda \in A} d(x_{\lambda}) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_{\lambda} \in R \) and \( \sum_{\lambda \in A} \rho^{*}(x_{\lambda}) = \rho^{*}(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^{*} \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^{*} \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \).

Then we can find an orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \) such that

\[
\sum_{\nu=1}^{n} d(x_{\nu}) = d\left(\sum_{\nu=1}^{n} x_{\nu}\right) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
\[ \lim_{t \to 0} \rho^*(\xi x) = 0, \]
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu (\nu \geq 1)$ and 
\[ \sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty. \]
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$,
\[ \rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty, \]
which is inconsistent with (\rho.4). Therefore we have 
\[ \sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty. \] Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \}, \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\textsuperscript{7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in A} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in A$. Putting 
\[ [p_{n,1}] = [(2nx_\lambda - nx)^+] \]
we have 
\[ [p_{n,1}] \uparrow_{\lambda \in A} [x] \text{ and } 2n[p_{n,1}]x_\lambda \geq [p_{n,1}] nx, \]
which implies $\rho^*(n[p_{n,1}]x) = 0$ and $\lim_{t \to 0} \rho(\xi x) \leq \sup_{x \in R} d(x) < +\infty$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into 
\[ R = R_0 \oplus R_0^\perp. \]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in A}(x_\lambda \in [p]R_0)$, there exists $x_0 = \sum_{\lambda \in A} x_\lambda \in [p] R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(A)^{\text{\circ}}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

\textsuperscript{7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|, b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in A} x_\lambda \in S(\lambda \in A)$ implies $\bigcup_{\lambda \in A} x_\lambda \in S$.

\textsuperscript{8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

\textsuperscript{9) $S(A)$ is the set of all real functions defined on $A$.}
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^{\perp}$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ which is semi-continuous, i.e.,
\[ \sup_{i \in I} \| x_{i} \|_{0} = \| x \|_{0} \]
for any $0 \leq x_{i} \geq x$. 

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^{*}$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put
\[ (3.1) \quad \| x \|_{0} = \inf \left\{ \xi ; \rho^{*}(\frac{1}{\xi} x) \leq \xi \right\} . \]

Then,
\[ i) \quad 0 \leq \| x \|_{0} = \| -x \|_{0} < \infty \quad \text{and} \quad \| x \|_{0} = 0 \]
is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_{0}^{\perp}$. 

\[ ii) \quad \| x + y \|_{0} \leq \| x \|_{0} + \| y \|_{0} \quad \text{for any} \quad x, y \in R ; \]
follows also from (A.3) which is deduced from $(\rho.4)$. 

\[ iii) \quad \lim_{\alpha_{n} \uparrow 0} \| \alpha_{n} x \|_{0} = 0 \quad \text{and} \quad \lim_{\| x_{n} \|_{0} \to 0} \| \alpha x_{n} \|_{0} = 0 ; \]
is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_{0}$ is semi-continuous. From ii) and iii), it follows that $\lim \| \alpha x \|_{0} = \| \alpha_{0} x \|_{0} \quad \text{for all} \quad x \in R_{0}^{\perp} \quad \text{and} \quad \alpha_{0} \geq 0$. If $x \in R_{0}^{\perp}$ and $[p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $\| [p] x \|_{0} > \xi$ we have $\rho^{*}(\frac{1}{\xi} [p] x) > \xi$, which implies $\rho^{*}(\frac{1}{\xi} x) > \xi$ and hence $\lim \| p_{\lambda} x \|_{0} \geq \xi$. Thus we obtain
\[ \sup_{\lambda \in \Lambda} \| p_{\lambda} x \|_{0} = \| [p] x \|_{0} , \quad \text{if} \quad [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p] . \]

Let $0 \leq x_{1} \uparrow_{\lambda \in \Lambda} x$. Putting
\[ [p_{n,\lambda}] = \left[ x_{\lambda} - \left( 1 - \frac{1}{n} \right) x \right] \]
we have
\[ [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad [p_{n,\lambda}] x_{\lambda} \geq [p_{n,\lambda}] \left( 1 - \frac{1}{n} \right) x \quad \text{for} \quad n \geq 1. \]

As is shown above, since
\[ \sup_{\lambda \in \Lambda} [p_{n,\lambda}] x_{\lambda} \|_{0} \geq \sup_{\lambda \in \Lambda} \| [p_{n,\lambda}] \left( 1 - \frac{1}{n} \right) x \|_{0} = \| \left( 1 - \frac{1}{n} \right) x \|_{0} , \]
we have
\[ \sup_{\lambda \in \Lambda} \| x_{\lambda} \|_{0} \geq \| \left( 1 - \frac{1}{n} \right) x \|_{0} \]
and also $\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_{0} \geq \| x \|_{0}$. As the converse inequality is obvious by iv), $\| \cdot \|_{0}$ is semi-continuous.

Remark 2. By the definition of (3.1), we can see easily that
\[ \lim_{n \to \infty} \| x_{n} \|_{0} = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} \rho(\xi x_{n}) = 0 \quad \text{for all} \quad \xi \geq 0. \]
In order to prove the completeness of quasi-norm $\|\cdot\|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0$ $(n, \nu=1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\begin{align}
(3.2) \quad & [p_{n,\nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0; \\
(3.3) \quad & [p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] a \text{ for all } n, \nu \geq 1.
\end{align}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $\|\cdot\|_0$.

**Proof.**
We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
(3.4) \quad & \| [q_m] a \|_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m=1, 2, \cdots) \\
(3.5) \quad & n_m [q_m] a \geq [q_m] x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),
\end{align}

where $\delta = \| [p_0] a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $\|\cdot\|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$$
\| [p_{1,\nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}.
$$

We put $[q_1] = [p_{1,\nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m$ $(m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k] a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$$
\| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
$$

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$$
\| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
$$

in virtue of (3.2) and semi-continuity of $\|\cdot\|_0$. Hence we can put

$$
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k],
$$

because

$$
[q_{k+1}] \leq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
$$

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
On F-Norms of Quasi-Modular Spaces

\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}](x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1}[q_{k+1}]a - n_{k}[q_{k+1}]a \|_0 \geq \| [q_{k+1}]a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_{k}] \leq [(n_{k}a - x_{\nu-1})^+]\) implies \([q_{k+1}]n_{k}a \geq [q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \([x_{\nu}]_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then \( R^0_0 \) is an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) satisfies \((\rho.4')\).

**Proof.** If \( \rho \) satisfies \((\rho.4')\), \( \rho^* \) is a quasi-modular which fulfills also \((\rho.5)\) and \((\rho.6)\) in virtue of Theorem 2.3. Since \( \rho^* \) satisfies \((\rho.3)\), there exists \(0 \leq x_0 \in R^0_0\) such that \(x_0 = \bigcup_{\nu=1}^\infty x_\nu\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,\nu}] = [(x_\nu - nx_0)^+]\) and \(\bigcup_{\nu=1}^\infty [p_{n,\nu}] = [p_n]\), we obtain

\[ (3.6) \quad [p_{n,\nu}]x_\nu \geq n[p_{n,\nu}]x_0 \quad \text{for all } n, \nu \geq 1 \]

and \([p_n] \downarrow_{n=1}^\infty 0\). Since \([x_\nu]_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^\infty [p_n] = 0\), that is, \(\bigcup_{n=1}^\infty ([x_\nu] - [p_n]) = [x_0]\). And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1) \]

implies

\[ n(1 - [p_n])x_0 \geq (1 - [p_n])x_\nu \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^\infty (1 - [p_n])x_\nu \in R^0_0, \]

because \( R^0_0 \) is universally continuous. As \([x_\nu]_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)

\[ \gamma = \sup_{\nu \geq 1} \| x_\nu \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n])x_\nu \|_0 \leq \gamma \]

for every \(n \geq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \([z_\nu]_{\nu \geq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^n z_\nu \|_0 = \| y_n \|_0 \leq \gamma\). This implies
for all \( n \geq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of \( z = \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma \) for all \( n \geq 1 \) by the formula (3.1).

Then \((\rho.3)\) assures the existence of \( z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \).

Truly, it follows from \( z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_n]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-[p_n]) x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} \).

By semi-continuity of \( \| \cdot \|_0 \), we have
\[
\| z - x_{\nu} \|_0 \leq \sup_{\mu \geq \nu} \| x_{\mu} - x_{\nu} \|_0
\]
and furthermore
\[
\lim_{\nu \to \infty} \| z - x_{\nu} \|_0 = 0.
\]

Secondly let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^+ \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that
\[
\| y_{\nu+1} - y_{\nu} \|_0 \leq \frac{1}{2^{m-1}}
\]
for all \( m \geq 1 \).

Putting \( z_n = \sum_{\nu=1}^{n} |y_{\nu+1}-y_{\nu}| \), we have a Cauchy sequence \( \{z_n\}_{n \geq 1} \) with \( 0 \leq z_n \leq \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \in R_0^+ \) and \( \lim_{n \to \infty} \| z_n - z \|_0 = 0 \).

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \) is also convergent and
\[
\| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_0 \|_0 \leq \| \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \|_0 \leq 2^{m-1} \| z_n - z \|_0 \to 0.
\]

Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that
\[
\lim_{\nu \to \infty} \| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_{\nu} \|_0 = 0.
\]

Therefore \( \| \cdot \|_0 \) is complete in \( R_0^+ \), that is, \( R_0^+ \) is an F-space with \( \| \cdot \|_0 \).

Conversely if \( R_0^+ \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_0^+ \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^+ \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in \mathbb{R}} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \( \rho.4' \).

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0 \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfils $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
\|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
\end{equation}

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

\begin{equation}
\|x\|_1 \leq \|x\|_0 \leq 2\|x\|_0
\end{equation}

hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \|x\|_i = \|-x\|_i < +\infty (x \in L)$ and that $\|x\|_i = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \|x_n\|_i = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \|\alpha_n x\|_i = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n \to \infty} \|x_n\|_i = 0$ implies $\lim_{n \to \infty} \|\alpha x_n\|_i = 0$ for all $\alpha > 0$. If $\|x\|_1 < \alpha$ and $\|y\|_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$}

This yields

$$\|x + y\| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta} (x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\xi y)\right)$$

$$\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$}

10) For the convex modular $m$, we can define two kinds of norms such as

$$\|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have $\|x\|_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $\|x\|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)\textasciitilde(A.5) in §1, then the formula (4.1) yields a quasi-norm $\|\cdot\|_1$ on $L$ which is equivalent to $\|\cdot\|_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

(4.3) \[ \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R) \]

is a semi-continuous quasi-norm on $R^+_0$ and $\|\cdot\|_1$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

(4.4) \[ \|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all } x \in R^+_0. \]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies $(\rho.1)\textasciitilde(\rho.6)$ except $(\rho.3)$ and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $\limsup_{n \to \infty} x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a_n| \leq a_{n+1}$ (or $a_{n+1} = 0$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $\liminf_{n \to \infty} x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $\liminf_{n \to \infty} z_n = a$. A quasi-norm $\|\cdot\|$ on $R$ is termed to be continuous, if $\inf_{n \geq 1} \|a_n|| = 0$ for any $a_n \in R$.

In the sequel, we write by $\|\cdot\|_0$ (or $\|\cdot\|_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $\|\cdot\|_0$ (or $\|\cdot\|_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z]^R$ is finite dimensional and $\rho(y) < +\infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector $\{[p_n]\}_{n\geqq 1}$ such that $\rho([p_n]x)=+\infty$ and $[p_n]\downarrow_{n=1}^{\infty}0$. Hence by (3.1) it follows that $\| [p_n]x \|_0>1$ for all $n\geqq 1$, which contradicts the continuity of $\| \cdot \|_0$.

**Sufficiency.** Let $a_\nu\downarrow_{\nu=1}^{\infty}0$ and put $[p_\nu^*]=[(a_n-\varepsilon a_1)^+]$ for any $\varepsilon>0$ and $n\geqq 1$. It is easily seen that $[p_\nu^*]\downarrow_{n=1}^{\infty}0$ for any $\varepsilon>0$ and $a_n=[a_1]a_n=[[p_\nu^*]a_n+(1-[p_\nu^*])a_n]\leqq[p_\nu^*]a_1+\varepsilon a_1$.

This implies

$$\rho^*(\xi a_n)\leqq\rho^*(\xi[p_\nu^*]a_1)+\rho^*(\xi[1-[p_\nu^*])a_1$$

for all $n\geqq 1$ and $\xi\geqq 0$. In virtue of (5.1) and $[p_\nu^*]\downarrow_{n=1}^{\infty}0$, we can find $n_0$ (depending on $\xi$ and $\varepsilon$) such that $\rho^*(\xi[p_\nu^*]a_1)<+\infty$, and hence $\inf_{n\geqq 1}\rho^*(\xi[p_\nu^*]a_1)=0$ by (2.3) in Lemma 1 and (\rho.2). Thus we obtain

$$\inf_{n\geqq 1}\rho^*(\xi a_n)\leqq\rho^*(\xi\varepsilon a_1).$$

Since $\varepsilon$ is arbitrary, $\lim_{n\rightarrow\infty}\rho^*(\xi a_n)=0$ follows. Hence we infer that $\inf_{n\geqq 1}\| a_n \|_0=0$ and $\| \cdot \|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** $\| \cdot \|_0$ is continuous, if

$$\rho^*(a_\nu)\rightarrow 0 \text{ implies } \rho^*(\alpha a_\nu)\rightarrow 0 \text{ for every } \alpha\geqq 0.\tag{5.2}$$

From the definition, it is clear that $s-\lim_{\nu\rightarrow\infty}x_\nu=0$ implies $\lim_{\nu\rightarrow\infty}\| x_\nu \|_0=0$, if $\| \cdot \|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** $\lim_{\nu\rightarrow\infty}\| x_\nu \|_0=0$ (or $\lim_{\nu\rightarrow\infty}\| x_\nu \|_0=0$) implies $s-\lim_{\nu\rightarrow\infty}x_\nu=0$, if $\| \cdot \|_0$ is complete (i.e. $\rho^*$ satisfies (\rho.3)).

If we replace $\lim_{\nu\rightarrow\infty}\| x_\nu \|_0$ by $\lim_{\nu\rightarrow\infty}\rho(x_\nu)=0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

$$\rho^*(x)=0 \text{ implies } x=0.\tag{5.3}$$

Truly we obtain

**Theorem 5.3.** If $\rho^*$ satisfies (5.3) and $\| \cdot \|_0$ is complete, $\rho(a_\nu)\rightarrow 0$ implies $s-\lim_{\nu\rightarrow\infty}a_\nu=0$.

**Proof.** We may suppose without loss of generality that $\rho^*$ is semi-continuous,\footnote{11) If $\rho^*$ is not semi-continuous, putting $\rho_\ast(x)=\inf\{\sup\rho^*(y_i)\}$, we obtain a quasi-modular $\rho_\ast$ which is semi-continuous and $\rho^*(x_\nu)\rightarrow 0$ is equivalent to $\rho_\ast(x_\nu)\rightarrow 0$.} i.e. $\rho^*(x)=\sup_{i\in I}^\rho\rho^*(x_i)$ for any $0\leqq x\downarrow_{i\in I}^\rho x$. If
\[ \rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^\infty |a_\nu| \in \mathcal{R} \) in virtue of \((\rho.3)\).

Now, since
\[
\rho \left( \bigcup_{\nu \geq 1} |a_\mu| \right) = \sum_{\mu \geq \nu} \rho(a_\mu) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho \left( \bigcap_{\nu=1}^\infty \left( \bigcup_{\mu \geq \nu} |a_\mu| \right) \right) = 0 \) and hence \((5.3)\) implies
\[
\bigcap_{\nu=1}^\infty \left( \bigcup_{\mu \geq \nu} |a_\mu| \right) = 0.
\]
Thus we see that \( \{a_\mu\}_{\mu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \quad (\nu = 1, 2, \ldots) \). Therefore we have s-lim_{\nu \to \infty} b_\nu = 0. \quad \text{Q.E.D.}

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies \((5.3)\) and \( \|\cdot\|_0 \) is complete and continuous, then \((5.2)\) holds.

**References**


Mathematical Institute,
Hokkaido University

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