ON $F$-NORMS OF QUASI-MODULAR SPACES

By
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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;
(\rho.2) $\rho(x + y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;
(\rho.3) If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;
(\rho.4) $\varlimsup_{\xi \to 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;
(A.2) $\rho(-x) = \rho(x)$;
(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
(A.4) $\alpha_{n} \to 0$ implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;
(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula:

1) $x \perp y$ means $| x | \cap | y | = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.
   $R$ is called semi-regular, if for any $x \neq 0$, $\exists R$, there exists a universally continuous linear functional $f$ such that $f(x) \leq 0$.
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

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(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \))\( \sim \) (\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S \)2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0=\{ x : x \in R, \rho^*(\xi x)=0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S \)3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[(\rho.4') \quad \sup_{x \in R} \{ \lim_{\alpha \to 0} \rho(\alpha x) \} < +\infty \]

(Theorem 3.2).

In \( \S \)4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \S83]. At last in \( \S \)5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S \)5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p] x = \bigcup_{n=1}^{\infty} (n \upharpoonright p \cap x) \) for all \( x \geq 0 \) and \( 1-\lfloor p \rfloor \) is a projection operator onto the normal manifold \( N=\{ p \}^1 \), that is, \( x=[p]x+(1-\lfloor p \rfloor)x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

\[(2.1) \quad \rho(0)=0;\]
\[(2.2) \quad \rho([p]x)\leq\rho(x) \text{ for all } p, x \in R;\]
\[(2.3) \quad \rho([p]x)=\sup_{i \in A} \rho([p_i]x) \text{ for any } [p_i] \uparrow_{i \in A} [p].\]

In the argument below, we have to use the additional property of $\rho$:

\[(\rho.5) \quad \rho(x)\leq\rho(y) \text{ if } |x|\leq|y|, x, y \in R,\]

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\[(2.4) \quad \rho'(x)=\sup_{0\leq|y|\leq|x|} \rho(y).\]

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i)<+\infty$, then

\[\sum_{i \in A} \rho(x_i)<+\infty,\]

because

\[\rho(x)\leq\rho'(x) \quad \text{ for all } x \in R.\]

We have

\[x_0=\sum_{i \in A} x_i \in R\]

and

\[\rho(x_0)=\sum_{i \in A} \rho(x_i) \quad \text{ in virtue of } (\rho.3).\]

For such $x_0$,

\[\rho'(x_0)=\sup_{0\leq|y|\leq|x_0|} \rho(y)=\sup_{0\leq|y|\leq|x_0|} \sum_{i \in A} \rho([x_i]y)\]

\[=\sum_{i \in A} \sup_{0\leq|y|\leq|x_0|} \rho([x_i]y)=\sum_{i \in A} \rho'(x_i)\]

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_0 \in R$,

\[\rho'(\frac{1}{n}x_0)=+\infty \quad \text{ for all } n \geq 1.\]

By $(\rho.2)$ and $(\rho.4)$, $x_0$ can not be written as $x_0=\sum_{\nu=1}^{\kappa} \xi_\nu e_\nu$, where $e_\nu$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]

where \( \rho'(\frac{1}{\nu}x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]

where

\[ \rho'(\frac{1}{\nu}x_2) = +\infty \quad (\nu = 1, 2, \ldots) \]

and

\[ \rho'(\frac{1}{2}x'_2) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho(\frac{1}{2}y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that

\[ \rho(\frac{1}{\nu}y_\nu) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^\infty y_\nu \in R \]

and

\[ \rho(\frac{1}{\nu}y_0) \geq \rho(\frac{1}{\nu}y_\nu) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in \S 1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [(|x| - |y|)^+] \), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha \lfloor p \rfloor |x| + \alpha (1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|) = \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|) = \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor)y) \leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in \mathbb{R} \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain
\[
\rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]
for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with
\[
\int_0^1 |x(t)| \, dt < +\infty.
\]
Putting
\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},
\]
we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,
\[
(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in \mathbb{R}.
\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( \mathbb{R} \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put
\[
d(x) = \lim_{\xi \to 0} \rho'(\xi x).
\]
It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in \mathbb{R} \) and
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\[ d(x+y) = d(x) + d(y) \]

if \( x \perp y \).

Hence, putting

\[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R) \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \( \rho^* \), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that \((\rho.5)\) is true for \( \rho^* \). First we have to note

\[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0 \]

for any \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\) and \( x_0 \in R \).

Hence,

\[ \rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all \( \nu \geq 1 \) and \( \lambda \in \Lambda \). Thus we can find a subsequence \( \{\lambda_n\}_{n \geq 1} \) of \( \{\lambda\}_{\lambda \in \Lambda} \) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2} \]

for all \( n \geq 1 \) in virtue of \((\rho.2)\) and \((\rho.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x|-n|y|)+]\) for \( x, y \in R \) with \([x] = [y] \) and \( n \geq 1 \). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \( \inf_{n=1,2,...} d([p_n]x) = 0 \) by \((2.7)\). Since \((1-[p_n])n|y| \geq (1-[p_n])|x|\)
and

\[ d(\alpha x) = d(x) \]

for \( \alpha > 0 \) and \( x \in R \), we obtain
$d(x) = d([p_n]x) + d((1-[p_n])x) \\
\leq d([p_n]x) + d(n(1-[p_n])y) \\
\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies

\[ d(x) \leq \inf_{n=1,2} \ldots d([p_n]x) + d(y), \]

and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then

\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x) \\
= \rho'([y]x) - d([y]x) + \rho^*([x] - [y])x) \\
\geq \rho'(y) - d(y) + \rho^*([x] - [y])x) \\
\geq \rho^*(y). \]

Thus $\rho^*$ satisfies $(\rho.5)$.

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies $(\rho.3)$ (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

\[ (\rho.4') \quad \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho'((\xi x) = K < +\infty. \]

**Proof.** Let $\rho$ satisfy $(\rho.4)$. We need to prove

\[ (2.9) \quad \sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho'((\xi x) \leq K' < +\infty, \]

where

\[ \rho'(x) = \sup_{0 \leq y \leq |x|} \rho(y). \]

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'(\frac{1}{\nu}x)$ for $\nu \geq 1$ and $x \in \mathbb{R}$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with

\[ \rho(x) \leq \varepsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma. \]

In $N_0$, we have obviously

\[ \sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'((\xi x) = \gamma_0 < +\infty. \]

If $\varepsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha_0 x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by $(\rho.4')$, and hence there exists always an orthogonal decomposition such that
\[
\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z
\]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) for every \( i = 1, 2, \cdots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho^\prime \left( \frac{1}{\nu_0} \alpha_0 x_0 \right) \leq \sum_{i=1}^{n} \rho^\prime \left( \frac{1}{\nu_0} x_i \right) + \sum_{j=1}^{m} \rho^\prime (y_j) + \rho^\prime \left( \frac{z}{\nu_0} \right)
\]

\[
\leq n \gamma + \sum_{j=1}^{m} \rho^\prime (y_j) + \rho^\prime \left( \frac{z}{\nu_0} \right)
\]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left( \sup_{0 \leq a \leq a_0} \rho(a x) \right) + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho^\prime (\xi x_0) \leq \rho^\prime \left( \frac{\alpha_0}{\nu_0} x_0 \right) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \to 0} \rho^\prime (\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho^\prime (\xi x) \} \leq \gamma,
\]

where

\[
\gamma = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{ x_i \}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{i_\nu}) = d(\sum_{i=1}^{k} x_{i_\nu}) \leq \lim_{\xi \to 0} \rho^\prime (\xi \sum_{i=1}^{k} x_{i_\nu}) \leq \gamma^\prime,
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma^\prime \). It follows that

\[
\sum_{i \in A} \rho^\prime (x_i) \leq \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{ x_i \}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
$$\lim_{\xi \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_{\nu}\}_{\nu \geq 1}$ with $0 < \alpha_{\nu} \leq 1$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu} x_{\nu}) < +\infty$. It follows that 
$$x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R$$
and 
$$d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x_{\nu})$$
from $(\rho.3)$. For such $x_0$, we have for every $\xi \geq 0$,
$$\rho^*(\xi x_{0}) = \sum_{\nu=1}^{\infty} \rho^*(\xi \alpha_{\nu} x_{\nu}) \geq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty,$$
which is inconsistent with $(\rho.4)$. Therefore we have 
$$\sup_{x \in R} (\lim_{\xi \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.$$  Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \ \text{for all} \ n \geq 1 \},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold$^7$ of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let 
$$x = \bigcup_{\lambda \in A} x_{\lambda}$$
with $R_0 \ni x_{\lambda} \geq 0$ for all $\lambda \in A$. We have 
$$\rho^*(nx) = \rho^*(2nx_{\lambda} - nx) \geq \rho^*(nx) = 0$$
for all $n \geq 1$, since
$$\rho^*(nx) = 0.$$
From this, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.
Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp.$$  
In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]_{R_0}$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda} \in R : x_{\lambda} \in [p]_{R_0}\}$, there exists $x_0 = \sum_{\lambda \in A} x_{\lambda} \in [p]_{R_0}$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold $S$ is said to be semi-normal, if $a \in S, \ |b| \leq |a|, b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup_{\lambda \in A} x_{\lambda} \in S(\Lambda)$ implies $\cup_{\lambda \in A} x_{\lambda} \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup_{\lambda \in A} x_{\lambda} \in S(\Lambda)$ implies $\cup_{\lambda \in A} x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^{\perp}$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ which is semi-continuous, i.e. 
\[ \sup_{x \in A} \| x \|_0 = \| x \|_0 \] for any $0 \leq x \uparrow_{x \in A} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^{*}$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

\[ \| x \|_0 = \inf \left\{ \xi ; \rho^{*} \left( \frac{1}{\xi} x \right) \leq \xi \right\} . \]

Then,

i) $0 \leq \| x \|_0 = \| -x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_{0}^{\perp}$.

ii) $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_{n} \uparrow_{\alpha_{n} \in A} 0} \| \alpha_{n} x \|_0 = 0$ and $\lim_{\alpha \uparrow \alpha_{0}} \| \alpha x \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \uparrow \alpha_{0}} \| \alpha x \|_0 = \| \alpha_{0} x \|_0$ for all $x \in R_{0}^{\perp}$ and $\alpha_{0} \geq 0$. If $x \in R_{0}^{\perp}$ and $[p_{\lambda}] \uparrow_{\lambda \in A} [p]$, for any positive number $\xi$ with $\| [p] x \|_0 \geq \xi$ we have $\rho^{*}(\frac{1}{\xi} [p_{\lambda}] x) \geq \xi$, which implies $\sup_{\lambda \in A} \rho^{*}(\frac{1}{\xi} [p_{\lambda}] x) \geq \xi$ and hence $\sup_{\lambda \in A} \| [p_{\lambda}] x \|_0 \geq \xi$. Thus we obtain

$\sup_{\lambda \in A} \| [p_{\lambda}] x \|_0 = \| [p] x \|_0$, if $[p_{\lambda}] \uparrow_{\lambda \in A} [p]$.

Let $0 \leq x_{1} \uparrow_{1 \in A} x$. Putting

$[p_{n,1}] = \left[ (x_{1} - (1 - \frac{1}{n}) x)^{\ast} \right]$ we have

$[p_{n,1}] \uparrow_{1 \in A} [x]$ and $[p_{n,1}] x_{1} \geq [p_{n,1}] \left( 1 - \frac{1}{n} \right) x$ \hspace{1cm} \( n \geq 1 \).

As is shown above, since

$\sup_{1 \in A} \| [p_{n,1}] x_{1} \|_0 \geq \| [p_{n,1}] \left( 1 - \frac{1}{n} \right) x \|_0 = \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0$,

we have

$\sup_{1 \in A} \| x_{1} \|_0 \geq \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0$ and also $\sup_{1 \in A} \| x_{1} \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{\xi \uparrow \infty} \| x_{n} \|_0 = 0$ if and only if $\lim_{\xi \uparrow \infty} \rho(\xi x_{n}) = 0$ for all $\xi \geq 0$. 

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In order to prove the completeness of quasi-norm \( \| \cdot \|_0 \), the next Lemma is necessary.

**Lemma 2.** Let \( p_{n, \nu}, x_{\nu} \geq 0 \) and \( a \geq 0 \) \((n, \nu = 1, 2, \cdots)\) be the elements of \( R_{0}^\perp \) such that

\[
\begin{align*}
& (3.2) \quad [p_{n, \nu}] \uparrow_{n=1}^{\infty} [p_{n}] \text{ with } \bigcap_{n=1}^{\infty} [p_{n}] a = [p_{0}] a = 0; \\
& (3.3) \quad [p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \text{ for all } n, \nu \geq 1.
\end{align*}
\]

Then \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence of \( R_{0}^\perp \) with respect to \( \| \cdot \|_0 \).

**Proof.** We shall show that there exist a sequence of projectors \([q_m] \downarrow_{m=1}^{\infty} (m \geq 1)\) and sequences of natural numbers \( \nu_m, n_m \) such that

\[
\begin{align*}
& (3.4) \quad \| [q_m] a \|_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \\
& (3.5) \quad n_m [q_m] a \geq [q_m] x_{\nu_m} \geq n_{m+1} [q_{m+1}] a \quad (m = 1, 2, \cdots),
\end{align*}
\]

where \( \delta = \| [p_0] a \|_0 \).

In fact, we put \( n_1 = 1 \). Since \([p_{1, \nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0] \) and \( \| \cdot \|_0 \) is semi-continuous, we can find a natural number \( \nu_1 \) such that

\[
\| [p_{1, \nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}.
\]

We put \([q_1] = [p_{1, \nu_1}] [p_0] \). Now, let us assume that \([q_m], \nu_m, n_m \) \((m = 1, 2, \cdots, k)\) have been taken such that (3.4) and (3.5) are satisfied.

Since \([(n a - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a] \) and \( \| [q_k] a \|_0 > \frac{\delta}{2} \), there exists \( n_{k+1} \) with

\[
\| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
\]

For such \( n_{k+1} \), there exists also a natural number \( \nu_{k+1} \) such that

\[
\| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+] [q_k] a \|_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of \( \| \cdot \|_0 \). Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k],
\]

because

\[
[q_{k+1}] \leq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
\]

by (3.3) and \([q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k} \) by (3.5).

For the sequence thus obtained, we have for every \( k \geq 3 \)
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\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \]
\[ \geq \| n_{k+1}[q_{k+1}]a - n_{k}[q_{k+1}]a \|_0 \geq \| [q_{k+1}]a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_k] \leq [(n_{k}a - x_{\nu-1})^+]\) implies \([q_{k+1}]n_{k}a \geq [q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence.

**Theorem 3.2.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) if and only if \( \rho \) satisfies \((\rho.4')\).

**Proof.** If \( \rho \) satisfies \((\rho.4')\), \( \rho^* \) is a quasi-modular which fulfills also \((\rho.5)\) and \((\rho.6)\) in virtue of Theorem 2.3. Since \( x \|_0 (= \inf \{ \xi; \rho^*(x/\xi) \leq \xi \}) \) is a quasi-norm on \( R_0^+ \), we need only to verify completeness of \( \| \cdot \|_0 \). At first let \( \{x_{\nu}\}_{\nu \geq 1} \subset R_0^+ \) be a Cauchy sequence with \( 0 \leq x_{\nu} \uparrow_{\nu=1,2} \ldots \). Since \( \rho^* \) satisfies \((\rho.3)\), there exists \( 0 \leq x_0 \in R_0^+ \) such that \( x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu} \), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,v}] = [(x_{\nu} - nx_0)^+]\) and \( \bigcup_{\nu=1}^{\infty} [p_{n,v}] = [p_{n}] \), we obtain

\[ [p_{n,v}]x_{\nu} \geq n[p_{n,v}]x_0 \]

for all \( n, \nu \geq 1 \) and \([p_{n}] \downarrow_{n=1}^{\infty} 0 \). Since \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we have in virtue of Lemma 2, \( \bigcap_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} [p_{n,v}] = \bigcup_{\nu=1}^{\infty} [p_{n,v}] = [x_0] \), and

\[ (1 - [p_{n,v}]) \geq (1 - [p_{n}]) \]

\((n, \nu \geq 1)\)

implies

\[ n(1 - [p_{n}])x_0 \geq (1 - [p_{n}])x_\nu \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_{n}])x_{\nu} \in R_0^+, \]

because \( R_0^+ \) is universally continuous. As \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we obtain from the triangle inequality of \( \| \cdot \|_0 \)

\[ r = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_{n}])x_{\nu} \|_0 \leq r \]

for every \( n \geq 1 \) by semi-continuity of \( \| \cdot \|_0 \). We put \( z_1 = y_1 \) and \( z_n = y_n - y_{n-1} \)(\( n \geq 2 \)). It follows from the definition of \( y_n \) that \( \{z_{\nu}\}_{\nu \geq 1} \) is an orthogonal sequence with \( \| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq r \). This implies
for all \(n \geq 1\) by the formula (3.1). Then (\(\rho.3\)) assures the existence of \(z = \sum_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}\). This yields \(z = \bigcup_{\nu=1}^{\infty} x_{\nu}\). Truly, it follows from

\[
\sum_{\nu=1}^{n} \rho^{*}\left(\frac{z_{\nu}}{1+\gamma}\right) = \rho^{*}\left(\frac{y_{n}}{1+\gamma}\right) \leq \gamma \quad \text{for all} \quad n \geq 1
\]

by the formula (3.1). Then (\(\rho.3\)) assures the existence of \(z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}\).

This yields \(z = \bigcup_{\nu=1}^{\infty} x_{\nu}\).

By semi-continuity of \(\|\cdot\|_{0}\), we have

\[
\|z - x_{\nu}\|_{0} \leq \sup_{\mu \geq \nu} \|x_{\mu} - x_{\nu}\|_{0}
\]

and furthermore \(\lim_{\nu \to \infty} \|z - x_{\nu}\|_{0} = 0\).

Secondly let \(\{x_{\nu}\}_{\nu \geq 1}\) be an arbitrary Cauchy sequence of \(R_{0}^{+}\). Then we can find a subsequence \(\{y_{\nu}\}_{\nu \geq 1}\) of \(\{x_{\nu}\}_{\nu \geq 1}\) such that

\[
\|y_{\nu+1} - y_{\nu}\|_{0} \leq \frac{1}{2^{\nu}} \quad \text{for all} \quad \nu \geq 1.
\]

This implies

\[
\|\sum_{\nu=m}^{n} (y_{\nu+1} - y_{\nu})\|_{0} \leq \sum_{\nu=m}^{n} \|y_{\nu+1} - y_{\nu}\|_{0} \leq \frac{1}{2^{n-m}}
\]

for all \(n > m \geq 1\).

Putting \(z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|\), we have a Cauchy sequence \(\{z_{n}\}_{n \geq 1}\) with \(0 \leq z_{n} \leq \infty\).

Then by the fact proved just above,

\[
z_{0} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{+} \quad \text{and} \quad \lim_{n \to \infty} \|z_{0} - z_{n}\|_{0} = 0.
\]

Since \(\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|\) is convergent, \(y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})\) is also convergent and

\[
\|y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n}\|_{0} = \|\sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})\|_{0} \leq \|z_{0} - z_{n}\|_{0} \to 0.
\]

Since \(\{y_{\nu}\}_{\nu \geq 1}\) is a subsequence of the Cauchy sequence \(\{x_{\nu}\}_{\nu \geq 1}\), it follows that

\[
\lim_{\nu \to \infty} \|y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu}\|_{0} = 0.
\]

Therefore \(\|\cdot\|_{0}\) is complete in \(R_{0}^{+}\), that is, \(R_{0}^{+}\) is an F-space with \(\|\cdot\|_{0}\).

Conversely if \(R_{0}^{+}\) is an F-space, then for any orthogonal sequence \(\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+}\), we have \(\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}\) for some real numbers \(\alpha_{\nu} > 0\) (for all \(\nu \geq 1\)). Hence we can see that \(\sup_{x \in R} d(x) < +\infty\) by the same way applied in Theorem 2.1. It follows that \(\rho\) must satisfy (\(\rho.4'\)). Q.E.D.

Since \(R_{0}\) contains a normal manifold which is universally complete, if \(R_{0}^{+} = 0\), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills (3.4').

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

$$\| x \|_1 := \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}$$

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

$$\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0$$

hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \lim_{n \to \infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha_n x \|_1 = 0$ for all $\alpha_n \lim_{n \to \infty} 0$ and that $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ 

This yields

$$\| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho(\frac{\xi \eta}{\xi + \eta}(x+y)) = \frac{1}{\xi} + \frac{1}{\eta} + \rho(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$ 

10) For the convex modular $m$, we can define two kinds of norms such as

$$\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have \( ||x||_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( ||x||_0 > \frac{1}{\eta} \). Therefore we can obtain from above

Theorem 4.1. If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \) (A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( || \cdot ||_1 \) on \( L \) which is equivalent to \( || \cdot ||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

Theorem 4.2. If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
(4.3) \quad ||x||_1 = \inf_{t \to 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^+_0 \) and \( || \cdot ||_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
(4.4) \quad ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all } x \in R^+_0.
\]

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-\lim_{\nu \to \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a| \leq a_\nu \) \( (\nu \geq 1) \) and \( a_\nu \downarrow_{\nu = 1}^\infty 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-\lim_{\nu \to +\infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \) there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( o-\lim_{\nu \to +\infty} z_\nu = a \).

A quasi-norm \( || \cdot || \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} ||a_\nu|| = 0 \) for any \( a_\nu \downarrow_{\nu = 1}^\infty 0 \). In the sequel, we write by \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

Theorem 5.1. In order that \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

Proof. Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \([p_n]_{n\geq 1}\) such that \(\rho([p_n]x) = +\infty\) and \([p_n]_{n=1}^{\infty}0\). Hence by (3.1) it follows that \(||[p_n]x||_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(||\cdot||_0\).

**Sufficiency.** Let \(a_{\nu} \downarrow_{\nu=1}^{\infty}0\) and put \([p_n^{\epsilon}] = [(a_n - \epsilon a_1)^+]\) for any \(\epsilon > 0\) and \(n \geq 1\). It is easily seen that \([p_n^{\epsilon}] \downarrow_{n=1}^{\infty}0\) for any \(\epsilon > 0\) and \(a_n = [a_1]a_n = [p_n^{\epsilon}]a_n + (1 - [p_n^{\epsilon}])a_n \leq [p_n^{\epsilon}]a_1 + \epsilon a_1\).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi[p_n^{\epsilon}]a_1) + \rho^*(\xi(1-[p_n^{\epsilon}])a_1)
\]
for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p_n^{\epsilon}]_{n=1}^{\infty}0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*(\xi[p_n^{\epsilon}]a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*(\xi[p_n^{\epsilon}]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho . 2\)). Thus we obtain
\[
\inf_{n \geq 1} \rho^*(\xi a_n) = \rho^*(\xi a_1).
\]

Since \(\epsilon\) is arbitrary, \(\lim_{\nu \to \infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} ||a_n||_0 = 0\) and ||\(\cdot||_0\) is continuous in view of Remark 2 in \(\S 3\). Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** ||\(\cdot||_0\) is continuous, if
\[
(5.2) \quad \rho^*(a_\nu) \to 0 \implies \rho^*(\alpha a_\nu) \to 0 \quad \text{for every } \alpha \geq 0.
\]

From the definition, it is clear that s-lim \(x_\nu = 0\) implies \(\lim_{\nu \to \infty}||x_\nu||_0 = 0\), if ||\(\cdot||_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \to \infty}||x_\nu||_0 = 0\) (or \(\lim_{\nu \to \infty}||x_\nu|| = 0\)) implies s-lim \(x_\nu = 0\), if ||\(\cdot||_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho . 3\)).

If we replace \(\lim_{\nu \to \infty}||x_\nu|| = 0\) by \(\lim_{\nu \to \infty} \rho(x_\nu) = 0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[
(5.3) \quad \rho^*(x) = 0 \implies x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and ||\(\cdot||_0\) is complete, \(\rho(a_\nu) \to 0\) implies s-lim \(a_\nu = 0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^11\) i.e. \(\rho^*(x) = \sup_{\nu \in A} \rho^*(x_\nu)\) for any \(0 \leq x \leq \sum_{i \in A} x_i\). If

\(^{11}\) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x) = \inf_{y_1 \in A} \{\sup_{y \in A} \rho^*(y_1)\}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \to 0\) is equivalent to \(\rho_*(x) \to 0\).
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathbb{R} \) in virtue of \((\rho.3)\).

Now, since
\[
\rho\left( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \right) \right) = 0 \) and hence (5.3) implies
\[
\bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \right) = 0.
\]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^\nu} \) \((\nu = 1, 2, \ldots)\). Therefore we have \( s\text{-}\lim_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_0 \) is complete and continuous, then (5.2) holds.

References


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