<table>
<thead>
<tr>
<th>Section</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>ON F-NORMS OF QUASI-MODULAR SPACES</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Koshi, Shôzô; Shimogaki, Tetsuya</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(3-4), 202-218</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1961</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56023">http://hdl.handle.net/2115/56023</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSHIU_15_N3-4_202-218.pdf</td>
</tr>
</tbody>
</table>

**Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP**
ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;

(\rho.2) $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;

(\rho.3) If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}^{2}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

(\rho.4) $\varlimsup_{x \rightarrow 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $^{3}$ on $R$ for which every universally continuous linear functional$^{4}$ is continuous with respect to the norm defined by the modular$^{5}$ $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

(A.2) $\rho(-x) = \rho(x)$;

(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(A.4) $\alpha_{n} \rightarrow 0$ implies $\rho(\alpha_{n} x) \rightarrow 0$ for every $x \in R$;

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula:

1) $x \perp y$ means $|x| \wedge |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow \in \Lambda 0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

1) $x \perp y$ means $|x| \wedge |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow \in \Lambda 0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

1) $x \perp y$ means $|x| \wedge |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow \in \Lambda 0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.
(1.1) \[ \| x \|_0 = \inf \left\{ \xi; \rho\left(\frac{1}{\xi} x\right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \))~(\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)~(A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x: x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

(\( \rho.4' \)) \[ \sup_{x \in R} \lim_{\alpha \to 0} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0 \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R, [p] \) is a projector: \([p]x = \bigcup \lim_{n \to \infty} (n|p| \cap x)\) for all \( x \geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N = \{p\}^1 \), that is, \( x = [p]x + (1 - [p])x \).

---

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

\[
\begin{align*}
\rho(0) &= 0; \quad (2.1) \\
\rho([p]x) &\leq \rho(x) \quad \text{for all } p, x \in R; \quad (2.2) \\
\rho([p]x) &= \sup_{i \in A} \rho([p_i]x) \quad \text{for any } [p_i] \uparrow_{i \in A} [p]. \quad (2.3)
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\( \rho(x) \leq \rho(y) \) if \( |x| \leq |y| \), \( x, y \in R \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \( \rho(5) \).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \( \rho(5) \) is valid.

Proof. We put for every \( x \in R \),

\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \quad (2.4)
\]

It is clear that \( \rho' \) satisfies the conditions \( \rho(1), \rho(2) \) and \( \rho(5) \).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x)
\]

for all \( x \in R \).

We have

\[
x_0 = \sum_{i \in A} x_i \in R
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i)
\]

in virtue of \( \rho(3) \).

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)
\]

\[
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils \( \rho(3) \).

If \( \rho' \) does not fulfil \( \rho(4) \), we have for some \( x_0 \in R \),

\[
\rho'\left(\frac{1}{n} x_0\right) = +\infty \quad \text{for all } n \geq 1.
\]

By \( \rho(2) \) and \( \rho(4) \), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{s} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \varepsilon \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu}x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu}x_2) = +\infty \quad (\nu = 1, 2, \ldots) \]
and
\[ \rho'(\frac{1}{2}x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_1'| \) such that \( \rho'(\frac{1}{2}y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that
\[ \rho'(\frac{1}{\nu}y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho'(\frac{1}{\nu}y_0) \geq \rho'(\frac{1}{\nu}y_\nu) \geq \nu, \]
which contradicts \( \rho(4) \). Therefore \( \rho' \) has to satisfy \( \rho(4) \). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \( \rho(5) \), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = \lfloor |x| - |y| \rfloor^+\), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta|y|) \]
\[ = \rho([p]|x| + (1-[p])|y|) \]
\[ = \rho([p]x) + \rho((1-[p])y) \leq \rho(x) + \rho(y). \]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition (\( \rho.4 \)). Thus, in the above theorems, we cannot replace (\( \rho.4 \)) by the weaker condition:

(\( \rho.4'' \)) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.3 \)) and (\( \rho.4'' \)), but does not (\( \rho.4 \)). For this \( \rho_0 \), we obtain

\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\( \rho.6 \)) \[ \lim_{\xi \to 0} \rho(\xi x) = 0 \]

for all \( x \in R \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies (\( \rho.1 \))~(\( \rho.6 \)) except (\( \rho.3 \)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies (\( \rho.5 \)). Now we put

(\( 2.5 \)) \[ d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
On $F$-Norms of Quasi-Modular Spaces

Hence, putting

$$d(x+y)=d(x)+d(y) \quad \text{if } x \perp y.$$  \hspace{2cm} (2.6)

we can see easily that $(\rho.1), (\rho.2), (\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since

$$d(x) \leq \rho'(x)$$

and

$$d(\alpha x) = d(x)$$

for all $x \in R$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

$$\inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0$$

for any $[p_{\lambda}] \downarrow_{\lambda \in A} 0$. In fact, if we suppose the contrary, we have

$$\inf_{i \in A} d([p_{i}]x_0) \geq \alpha > 0$$

for some $[p_{i}] \downarrow_{i \in A} 0$ and $x_0 \in R$.

Hence,

$$\rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha$$

for all $\nu \geq 1$ and $\lambda \in A$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in A}$ such that

$$[p_{\lambda_n}] \geq [p_{\lambda_{n+1}}]$$

and

$$\rho'(\frac{1}{n}( [p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2}$$

for all $n \geq 1$ in virtue of $(\rho.2)$ and (2.3). This implies

$$\rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}( [p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty,$$

which is inconsistent with $(\rho.4)$. Secondly we shall prove

$$d(x) = d(y), \quad \text{if } [x] = [y].$$  \hspace{2cm} (2.8)

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n=1} 0$ and $\inf_{n=1,2,...} d([p_n]x) = 0$ by (2.7). Since $(1 - [p_n])n \mid y \mid \geq (1 - [p_n])x$ and

$$d(\alpha x) = d(x)$$

for $\alpha > 0$ and $x \in R$, we obtain
\[d(x) = d([p_{n}]x) + d((1-[p_{n}])x)\]
\[\leqq d([p_{n}]x) + d(n(1-[p_{n}])y)\]
\[\leqq d([p_{n}]x) + d(y)\].

As \(n\) is arbitrary, this implies
\[d(x) \leqq \inf_{n=1, 2, \ldots} d([p_{n}]x) + d(y),\]
and also \(d(x) \leqq d(y)\). Therefore we conclude that (2.8) holds.

If \(|x| \geqq |y|\), then
\[\rho^{*}(x) = \rho^{*}([y]x) + \rho^{*}(([x]-[y])x)\]
\[= \rho'([y]x) - d([y]x) + \rho^{*}(([x]-[y])x)\]
\[\geqq \rho'(y) - d(y) + \rho^{*}(([x]-[y])x)\]
\[\geqq \rho^{*}(y).\]

Thus \(\rho^{*}\) satisfies (\(\rho.5\)).

**Q.E.D.**

**Theorem 2.3.** \(\rho^{*}\) (which is constructed from \(\rho\) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\(\rho.3\)) (that is, \(\rho^{*}\) is a quasi-modular), if and only if \(\rho\) satisfies (\(\rho.4'\))

\[\sup_{x \in \mathbb{R}} \{\lim_{\xi \to 0} \rho'(\xi x)\} = K' < +\infty.\]

**Proof.** Let \(\rho\) satisfy (\(\rho.4\)). We need to prove

\[\sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{\lim_{\xi \to 0} \rho'(\xi x)\} = K' < +\infty,\]

where

\[\rho'(x) = \sup_{0 \leqq |y| \leqq |x|} \rho(y).\]

Since \(\rho'\) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \(n_{0}(x) = \rho(x)\) and \(n_{\nu}(x) = \rho'(\frac{1}{\nu}x)\) for \(\nu \geqq 1\) and \(x \in \mathbb{R}\). Hence we can find positive numbers \(\varepsilon, \gamma, \) a natural number \(\nu_{0}\) and a finite dimensional normal manifold \(N_{0}\) such that \(x \in N_{0}^\perp\) with
\[\rho(x) \leqq \varepsilon\] implies \(\rho'(\frac{1}{\nu_{0}}x) \leqq \gamma.\)

In \(N_{0}\), we have obviously
\[\sup_{x \in N_{0}} \{\lim_{\xi \to 0} \rho'(\xi x)\} = \gamma_{0} < +\infty.\]

If \(\varepsilon \leqq 2K\), for any \(x_{0} \in N_{0}^\perp\), we can find \(\alpha_{0} > 0\) such that \(\rho(\alpha x_{0}) \leqq 2K\) for all \(0 \leqq \alpha \leqq \alpha_{0}\) by (\(\rho.4'\)), and hence there exists always an orthogonal decomposition such that
On $F$-Norms of Quasi-Modular Spaces

\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) for every \( i = 1, 2, \ldots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho'\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho'\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]

\[
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{i_1}) = d(\sum_{i=1}^{k} x_{i_1}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{i_1}) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by \( (\rho.4) \) and \( (2.7) \). Therefore \( \rho^* \) satisfies \( (\rho.3) \).

On the other hand, suppose that \( \rho^* \) satisfies \( (\rho.3) \) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu.
\]
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
\[ \lim_{\xi \to 0} \rho^*(\xi x) = 0, \]
there exists $\{\alpha_\nu\}_{\nu \geqq 1}$ with $0 < \alpha_\nu (\nu \geqq 1)$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty$.
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geqq 0$,
\[ \rho' (\xi x_0) = \sum_{\nu=1}^{\infty} \rho' (\xi \alpha_\nu x_\nu) \geqq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty, \]
which is inconsistent with (\rho.4). Therefore we have
\[ \sup_{x \in R} (\lim_{\xi \to 0} \rho(\xi x)) \leqq \sup_{x \in R} d(x) < +\infty. \]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geqq 1 \}, \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^{7}\) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geqq 0$ for all $\lambda \in \Lambda$.
Putting $[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]$, we have $[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} x$ and $2n [p_{n,\lambda}] x_\lambda \geqq [p_{n,\lambda}] nx$, which implies $\rho^*(n[p_{n,\lambda}] x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}] x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.
Therefore, $R$ is orthogonally decomposed into
\[ R = R_0 \oplus R_0^\perp. \]
In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p] R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda} (x_\lambda \in [p] R_0)$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p] R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{9}\) space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

\(^{7}\) A linear manifold $S$ is said to be semi-normal, if $a \in S, \ |b| \leqq |a|, \ b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$.

\(^{8}\) This means that $x \in R$ is written by $x = y + z, \ y \in R_0$ and $z \in R_0^\perp$.

\(^{9}\) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---

7 A linear manifold $S$ is said to be semi-normal, if $a \in S, \ |b| \leqq |a|, \ b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$. 

8 This means that $x \in R$ is written by $x = y + z, \ y \in R_0$ and $z \in R_0^\perp$. 

9 $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^*_0$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ which is semi-continuous, i.e.
\[
\sup_{\lambda \in \Lambda} || x_{\lambda} ||_0 = || x ||_0
\]
for any $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put
\[
(3.1) \quad || x ||_0 = \inf \{ \xi ; \rho^*(\frac{1}{\xi} x) \leq \xi \}.
\]

Then,
\begin{enumerate}
  \item $0 \leq || x ||_0 = || -x ||_0 < \infty$ and $|| x ||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1), (\rho.6), (2.1)$ and the definition of $R^*_0$.
  \item $|| x + y ||_0 \leq || x ||_0 + || y ||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.
  \item $\lim_{\alpha_n \to 0^+} || \alpha_n x ||_0 = 0$ and $\lim_{\alpha_n \downarrow 0^+} || \alpha x_n ||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim || \alpha x ||_0 = || \alpha_0 x ||_0$ for all $x \in R^*_0$ and $\alpha_0 \geq 0$. If $x \in R^*_0$ and $[p] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $|| [p] x ||_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi} [p_{\lambda}] x) > \xi$ and hence $\sup_{\lambda \in \Lambda} || p_{\lambda} x ||_0 \geq \xi$. Thus we obtain
\[
\sup_{\lambda \in \Lambda} || p_{\lambda} x ||_0 = || [p] x ||_0,
\]
if $[p] \uparrow_{\lambda \in \Lambda} [p]$.

Let $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$. Putting
\[
[p_{n,\lambda}] = \left[ x_{\lambda} - \left( 1 - \frac{1}{n} \right) x \right]
\]
we have
\[
[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] and [p_{n,\lambda}] x_{\lambda} \geq [p_{n,\lambda}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).
\]
As is shown above, since
\[
\sup_{\lambda \in \Lambda} || [p_{n,\lambda}] x_{\lambda} ||_0 \geq \sup_{\lambda \in \Lambda} \left[ p_{n,\lambda} \right] \left( 1 - \frac{1}{n} \right) x = \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1),
\]
we have
\[
\sup_{\lambda \in \Lambda} || x_{\lambda} ||_0 \geq \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1)
\]
and also $\sup_{\lambda \in \Lambda} || x_{\lambda} ||_0 \geq || x ||_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that
\[
\lim_{n \to \infty} || x_n ||_0 = 0 \text{ if and only if } \lim_{n \to \infty} \rho(\xi x_n) = 0 \text{ for all } \xi \geq 0.
\]
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}$, $x_{\nu} \geq 0$ and $a \geq 0$ ($n, \nu = 1, 2, \ldots$) be the elements of $R^+_0$ such that

\begin{align}
(3.2) & \quad [p_{n,\nu}] \uparrow_{\nu=1}^\infty [p_n] \quad \text{with} \quad \cap_{n=1}^\infty [p_n]a = [p_0]a \neq 0; \\
(3.3) & \quad [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]a \quad \text{for all } n, \nu \geq 1.
\end{align}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R^+_0$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
(3.4) & \quad ||[q_m]a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m = 1, 2, \ldots) \\
(3.5) & \quad n_m[q_m]a \geq [q_m]x_{\nu_m} - 1 \quad n_{m+1} > n_m \quad (m = 2, 3, \ldots),
\end{align}

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^\infty [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$$||[p_{1,\nu_1}]a||_0 > \frac{\delta}{2} = \frac{\delta}{2}.$$ 

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \ldots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_i})^+] \uparrow_{n=1}^\infty [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$$||(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}.$$ 

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$$||[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2},$$

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

$$[q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+[q_k],$$

because

$$[q_{k+1}] \leq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a$$

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
$\|x_{\nu_{k+1}} - x_{\nu_{k-1}}\|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0$
\geq \| n_{k+1}[q_{k+1}] a - n_{k}[q_{k+1}] a \|_0 \geq \| [q_{k+1}] a_0 \|_0 \geq \frac{\delta}{2}$,

since $[q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu-1})^+]$ implies $[q_{k+1}] n_{k} a \leq [q_{k+1}] x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^\perp$ is an F-space with $\| \cdot \|_0$ if and only if $\rho$ satisfies ($\rho.4'$).

**Proof.** If $\rho$ satisfies ($\rho.4'$), $\rho^*$ is a quasi-modular which fulfills also ($\rho.5$) and ($\rho.6$) in virtue of Theorem 2.3. Since $\| x \|_0 = \inf \{ \xi ; \rho^*\left(\frac{x}{\xi}\right) \leq \xi \}$ is a quasi-norm on $R_0^\perp$, we need only to verify completeness of $\| \cdot \|_0$. At first let $\{x_{\nu}\}_{\nu \geq 1} \subset R_0^\perp$ be a Cauchy sequence with $0 \leq x_{\nu} \uparrow \nu=1, 2, \ldots$. Since $\rho^*$ satisfies ($\rho.3$), there exists $0 \leq x_0 \in R_0^\perp$ such that $x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = [(x_{\nu} - nx_0)^+]$ and $\bigcup_{\nu=1}^{\infty} [p_{n,\nu}] = [p_n]$, we obtain

$$
[\bigcup_{\nu=1}^{\infty} [p_{n,\nu}]] x_{\nu} \geq n [\bigcup_{\nu=1}^{\infty} [p_{n,\nu}]] x_0
$$

for all $n, \nu \geq 1$ and $[p_n] \downarrow_{n=1}^{\infty} 0$. Since $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty} [p_n] = 0$, that is, $\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]$. And

$$
(1 - [p_{n,\nu}]) \geq (1 - [p_n])
$$

implies

$$
n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_{\nu} \geq 0.
$$

Hence we have

$$
y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_{\nu} \in R_0^\perp,
$$

because $R_0^\perp$ is universally continuous. As $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $\| \cdot \|_0$

$$
\gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty,
$$

which implies

$$
\| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n]) x_{\nu} \|_0 \leq \gamma
$$

for every $n \geq 1$ by semi-continuity of $\| \cdot \|_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ ($n \geq 2$). It follows from the definition of $y_n$ that $\{z_{\nu}\}_{\nu \geq 1}$ is an orthogonal sequence with $\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from

$$z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} (1 - [p_{\nu}])x_{\nu} = \bigcup_{\nu=1}^{\infty} [x_{\nu}]x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.$$

By semi-continuity of $|| \cdot ||_{0}$, we have

$$|| z - x_{\nu} ||_{0} \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_{0}$$

and furthermore

$$\lim_{n \to \infty} || z - x_{n} ||_{0} = 0.$$

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{+}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$.

This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{m-1}}$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} | y_{\nu+1} - y_{\nu} |$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \leq \infty$.

Then by the fact proved just above,

$$z_{0} = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_{0}^{+}$$

and

$$\lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0.$$

Since $\sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} |$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{n})$ is also convergent and

$$|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{n}) - y_{n} ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{n}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0.$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{n \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0.$$

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{+}$, that is, $R_{0}^{+}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{+}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfils $(\rho.A')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
\| x \|_1 = \inf_{\xi > 0} \left( \frac{1}{\xi} + \rho(\xi x) \right)\\
\end{equation}

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

\begin{equation}
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\end{equation}

for all $x \in L$, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\begin{equation}
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\end{equation}

This yields

\begin{align*}
\| x + y \| &\leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta}(x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y) \right) \\
&\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\end{align*}

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\begin{equation}
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\end{equation}

10) For the convex modular $m$, we can define two kinds of norms such as

\begin{align*}
\| x \| &\equiv \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi}\\
\end{align*}

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \) (A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( \| \cdot \|_1 \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^* (\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R \)\( \downarrow \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all } x \in R \downarrow.
\]

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \) and \( \rho^* (\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \leq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that \( |x_n - a| \leq a_n (n \geq 1) \) and \( a_n \to 0 \) as \( n \to \infty \). And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( \lim_{n \to \infty} z_n = a \).

A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} \| a_n \| = 0 \) for any \( \lim_{n \to \infty} a_n = 0 \). In the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]_R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector $\{[p_n]\}_{n \geq 1}$ such that $\rho([p_n]x) = +\infty$ and $[p_n] \downarrow_{n=1}^{\infty} 0$. Hence by (3.1) it follows that $\|[p_n]x\|_0 > 1$ for all $n \geq 1$, which contradicts the continuity of $\|\cdot\|_0$.

**Sufficiency.** Let $a_n \downarrow_{n=1}^{\infty} 0$ and put $[p_n^\epsilon] = [(a_n - \epsilon a_1)^+]$ for any $\epsilon > 0$ and $n \geq 1$. It is easily seen that $[p_n^\epsilon] \downarrow_{n=1}^{\infty} 0$ for any $\epsilon > 0$ and $a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1$.

This implies $\rho^*([a_1]a_n) \leq \rho^*([p_n^\epsilon]a_1) + \rho^*([a_n](1 - [p_n^\epsilon])a_1)$ for all $n \geq 1$ and $\xi \geq 0$. In virtue of (5.1) and $[p_n^\epsilon] \downarrow_{n=1}^{\infty} 0$, we can find $n_0$ (depending on $\xi$ and $\epsilon$) such that $\rho^*([p_n^\epsilon]a_1) < +\infty$, and hence $\inf_{n \geq 1} \rho^*([p_n^\epsilon]a_1) = 0$ by (2.3) in Lemma 1 and (r.2). Thus we obtain

$$\inf_{n \geq 1} \rho^*([a_1]a_n) \leq \rho^*([a_1]a_1).$$

Since $\epsilon$ is arbitrary, $\lim_{n \to \infty} \rho^*([a_1]a_n) = 0$ follows. Hence we infer that $\inf_{n \geq 1} ||a_n||_0 = 0$ and $\|\cdot\|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** $\|\cdot\|_0$ is continuous, if

(5.2) $\rho^*(a_\nu) \to 0$ implies $\rho^*(\alpha a_\nu) \to 0$ for every $\alpha \geq 0$.

From the definition, it is clear that $s\lim_{\nu \to \infty} x_\nu = 0$ implies $\lim_{\nu \to \infty} ||x_\nu||_0 = 0$, if $\|\cdot\|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3]).

**Theorem 5.2.** $\lim_{\nu \to \infty} ||x_\nu||_0 = 0$ (or $\lim_{\nu \to \infty} ||x_\nu|| = 0$) implies $s\lim_{\nu \to \infty} x_\nu = 0$, if $\|\cdot\|_0$ is complete (i.e. $\rho^*$ satisfies (r.3)).

If we replace $\lim_{\nu \to \infty} ||x_\nu|| = 0$ by $\lim_{\nu \to \infty} \rho(x_\nu) = 0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) $\rho^*(x) = 0$ implies $x = 0$.

Truly we obtain

**Theorem 5.3.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete, $\rho(a_\nu) \to 0$ implies $s\lim_{\nu \to \infty} a_\nu = 0$.

**Proof.** We may suppose without loss of generality that $\rho^*$ is semi-continuous, i.e. $\rho^*(x) = \sup_{y \in A} \rho^*(x_y)$ for any $0 \leq x \uparrow_{\nu \in A} x$. If $\rho^*$ is not semi-continuous, putting $\rho_*(x) = \inf_{y \uparrow_{\nu \in A} x} \{\sup_{j \in A} \rho^*(y_j)\}$, we obtain a quasi-modular $\rho_*$ which is semi-continuous and $\rho^*(x) \to 0$ is equivalent to $\rho_*(x) \to 0$. 

11) If $\rho^*$ is not semi-continuous, putting $\rho_*(x) = \inf_{y \uparrow_{\nu \in A} x} \{\sup_{j \in A} \rho^*(y_j)\}$, we obtain a quasi-modular $\rho_*$ which is semi-continuous and $\rho^*(x) \to 0$ is equivalent to $\rho_*(x) \to 0$. 


we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{E}$ in virtue of (\rho.3).

Now, since
\[
\rho\left(\bigcup_{\nu \geq 1} |a_{\nu}|\right) \leq \sum_{\nu \geq 1} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right)\right) = 0$ and hence (5.3) implies
\[
\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu}^{\infty} |a_{\nu}|\right) = 0.
\]
Thus we see that \(\{a_{\nu}\}_{\nu \geq 1}\) is order-convergent to 0.

For any \(\{b_{\nu}\}_{\nu \geq 1}\) with \(\rho(b_{\nu}) \to 0\), we can find a subsequence \(\{b'_{\nu}\}_{\nu \geq 1}\) of \(\{b_{\nu}\}_{\nu \geq 1}\) with \(\rho(b'_{\nu}) \leq \frac{1}{2^{\nu}}\) \((\nu = 1, 2, \ldots)\). Therefore we have $s-$lim $b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\| \cdot \|_0$ is complete and continuous, then (5.2) holds.

References


Mathematical Institute,
Hokkaido University

(Received September 30, 1960)