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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all } x \in R;
(\rho.2) \quad \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any } x, y \in R \text{ with } x \perp y^{1)};
(\rho.3) \quad \text{If } \sum_{i \in A} \rho(x_i) < +\infty \quad \text{for a mutually orthogonal system } \{x_i\}_{i \in A}^{2)},
\quad \there exists \quad x_0 \in R \quad \text{such that } \quad x_0 = \sum_{i \in A} x_i \quad \text{and } \quad \rho(x_0) = \sum_{i \in A} \rho(x_i);
(\rho.4) \quad \varlimsup_{t \to 0} \rho(\xi x) < +\infty \quad \text{for all } x \in R.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular$^{3)}$ $m$ on $R$ for which every universally continuous linear functional$^{4)}$ is continuous with respect to the norm defined by the modular$^{5)}$ $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) \quad \rho(x) \geq 0 \quad \text{and} \quad \rho(x) = 0 \quad \text{if and only if } x = 0;
(A.2) \quad \rho(-x) = \rho(x);
(A.3) \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1;
(A.4) \quad \alpha_n \to 0 \implies \rho(\alpha_n x) \to 0 \quad \text{for every } x \in R;
(A.5) \quad \text{for any } x \in L \quad \text{there exists } \alpha > 0 \quad \text{such that } \rho(\alpha x) < +\infty.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by the formula;

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_i\}_{i \in A}$ is called mutually orthogonal, if $x_\lambda \perp x_\gamma$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{i \in A} f(a_i) = 0$ for any $a_\lambda \downarrow A_0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) = 0$.

This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]
and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \)\( \sim \)(\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim \)(A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x ; x \in R, \rho^*(\xi x) = 0 \) for all \( \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^1 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^1 \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

(\( \rho.4' \)) \[ \sup_{x \in \Xi} \left( \lim_{\alpha \to 0} \rho(\alpha x) \right) < + \infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^1 \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^1 \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( \lfloor p \rfloor \) is a projector: \( \lfloor p \rfloor x = \bigcup_{n=1}^{\infty} (n|p | \cap x) \) for all \( x \geq 0 \) and \( 1 - \lfloor p \rfloor \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = \lfloor p \rfloor x + (1 - \lfloor p \rfloor) x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) & \quad \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \quad \text{for any } [p_i] \uparrow_{i \in A} [p].
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\[\rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \quad x, y \in R,\]

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2)\) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\]

We have

\[
x_0 = \sum_{i \in A} x_i \in R
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of } (\rho.3).
\]

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_i|} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in R \),

\[
\rho'\left(\frac{1}{n} x_0\right) = +\infty \quad \text{for all } n \geq 1.
\]

By \((\rho.2)\) and \((\rho.4)\), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_\nu e_\nu \), where \( e_\nu \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) \((\nu = 1, 2, \cdots)\) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where \( \rho'(\frac{1}{\nu} x_2) = +\infty \) \((\nu = 1, 2, \cdots)\)
and
\[ \rho'(\frac{1}{2} x'_2) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1, 2, \ldots} \) such that
\[ \rho(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1, 2, \ldots} \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho(\frac{1}{\nu} y_0) \geq \rho(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts \((\rho.4)\). Therefore \( \rho' \) has to satisfy \((\rho.4)\). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \((\rho.5)\), \( \rho \) does also \((A.3)\) in \S 1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
$\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)$

$\leq \rho(\alpha[p]|x| + \beta[p]|y| + (1-\alpha[p])\beta|y|)$

$= \rho([p]|x| + \beta(1-[p])|y|)$

$\leq \rho([p]x) + \rho((1-[p])y)$

$= \rho(x) + \rho(y)$

**Remark 1.** As is shown above, the existence of $\rho'$ as a quasi-modular depends essentially on the condition $(\rho.4)$. Thus, in the above theorems, we cannot replace $(\rho.4)$ by the weaker condition:

$(\rho.4'')$ for any $x \in R$, there exists $\alpha \geq 0$ such that $\rho(\alpha x) < +\infty$.

In fact, the next example shows that there exists a functional $\rho_0$ on a universally continuous semi-ordered linear space satisfying $(\rho.1)$, $(\rho.2)$, $(\rho.3)$ and $(\rho.4'')$, but does not $(\rho.4)$. For this $\rho_0$, we obtain

$$\rho_0(x) = \sup_{y : \|y\| \leq \|x\|} \rho_0(y) = +\infty$$

for all $x \neq 0$.

**Example.** $L_1[0,1]$ is the set of measurable functions $x(t)$ which are defined in $[0,1]$ with

$$\int_0^1 |x(t)| \, dt < +\infty.$$ 

Putting

$$\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},$$

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: $(A.4)$, namely,

$$(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0$$

for all $x \in R$.

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an $F$-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let $\rho$ be a quasi-modular on $R$. We can find a functional $\rho^*$ which satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$.

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular $\rho'$ which satisfies $(\rho.5)$. Now we put

$$(2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x).$$

It is clear that $0 \leq d(x) = d(|x|) < +\infty$ for all $x \in R$ and
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\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in \mathbb{R}). \]

we can see easily that (\(\rho.1\)), (\(\rho.2\)), (\(\rho.4\)) and (\(\rho.6\)) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in \mathbb{R}\) and \(\alpha > 0\).

We need to prove that (\(\rho.5\)) is true for \(\rho^*\). First we have to note

\[ \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\nu \in \mathbb{N}} d([p_\nu]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in \mathbb{R}\).

Hence,

\[ \rho'(\frac{1}{\nu}[p_\nu]x_0) \geq d([p_\nu]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{2 \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of (\(\rho.2\)) and (2.3). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty, \]

which is inconsistent with (\(\rho.4\)). Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)]^+\) for \(x, y \in \mathbb{R}\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n \geq 1} 0\) and \(\inf_{n=1, 2, \ldots} d([p_n]x) = 0\) by (2.7). Since \((1-[p_n])n |y| \geq (1-[p_n])|x|\) and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in \mathbb{R}\), we obtain
\[ d(x) = d([p_n]x) + d((1 - [p_n])x) \leq d([p_n]x) + d((1 - [p_n])y) \leq d([p_n]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y), \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x \]
\[ = \rho'([y]x) - d([y]x) + \rho^*([x] - [y])x \]
\[ \geq \rho'(y) - d(y) + \rho^*([x] - [y])x \]
\[ \geq \rho^*(y). \]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies
\[ (\rho.4') \]
\[ \sup_{x \in K} \{ \lim_{\xi \to 0} \rho(\xi x) \} = K < +\infty. \]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ (2.9) \]
\[ \sup_{x \in K} d(x) = \sup_{x \in K} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty, \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'\left(\frac{1}{\nu} x\right) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma, \alpha_0 \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[ \rho(x) \leq \epsilon \quad \text{implies} \quad \rho'\left(\frac{1}{\nu_0} x\right) \leq \gamma. \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty. \]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha_0 x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (i = 1, 2, \cdots, n), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \]

\[
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma .
\]

Hence, we obtain

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^\perp \)

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x) \leq \gamma .
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \rightarrow 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0 .
\]

Let \( \{x_\lambda\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^*(x_\lambda) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\sum_{\nu=1}^{k} d(x_{\lambda_\nu}) - d(\sum_{\nu=1}^{k} x_{\lambda_\nu}) = \lim_{\xi \rightarrow 0} \rho'(\xi \sum_{\nu=1}^{k} x_{\lambda_\nu}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_\lambda) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho'(x_\lambda) = \sum_{\lambda \in A} \rho^*(x_\lambda) + \sum_{\lambda \in A} d(x_\lambda) < +\infty ,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_\lambda \in R \) and \( \sum_{\lambda \in A} \rho^*(x_\lambda) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_\nu\}_{\nu \geq 1} \) such that

\[
\sum_{\nu=1}^{n} d(x_\nu) = d(\sum_{\nu=1}^{n} x_\nu) \geq \mu
\]
for all \( \mu \geq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since 
\[ \lim_{t \to 0} \rho^*(\xi x) = 0, \]
there exists \( \{\alpha_\nu\}_{\nu \geq 1} \) with \( 0 < \alpha_\nu \) (\( \nu \geq 1 \)) and \( \sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty \). It follows that \( x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R \) and \( d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu) \) from (\( \rho.3 \)). For such \( x_0 \), we have for every \( \xi \geq 0 \),
\[ \rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty, \]
which is inconsistent with (\( \rho.4 \)). Therefore we have
\[ \sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty. \]
Q.E.D.

\[ \S 3. \] Quasi-norms. We denote by \( R_0 \) the set:
\[ R_0 = \{x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geq 1\}, \]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_\lambda \) with \( R_0 \ni x_\lambda \geq 0 \) for all \( \lambda \in \Lambda \). Putting
\[ [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \]
we have
\[ [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \]
and \( 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx = \rho^*(nx) = 0 \), which implies \( \rho^*(n[p_{n,\lambda}]x) = 0 \) and \( \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = 0 \). Hence, we obtain \( x \in R_0 \), that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[ R = R_0 \oplus R_0^\perp. \]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0 \), \( [p]R_0 \) is universally complete, i.e. for any orthogonal system \( \{x_\lambda \}_{\lambda \in \Lambda}, x_\lambda \in [p]R_0 \), there exists \( x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R \). Hence we can also verify without difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \) is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda)^{\mathbb{P}} \)-space. With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, \ |b| \leq |a|, b \in R \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \cup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda) \) implies \( \cup_{\lambda \in \Lambda} x_\lambda \in S \).

8) This means that \( x \in R \) is written by \( x = y + z \), \( y \in R_0 \) and \( z \in R_0^\perp \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e. $\sup_{\xi\in\Lambda} ||x_i||_0 = ||x||_0$ for any $0 \leq x_i \leq x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) - (\rho.6)$ except $(\rho.3)$. Now we put

\begin{equation}
||x||_0 = \inf \left\{ \xi : \rho^* \left( \frac{1}{\xi} x \right) \leq \xi \right\}.
\end{equation}

Then,

i) $0 \leq ||x||_0 = ||-x||_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_0^\perp$.

ii) $||x+y||_0 \leq ||x||_0 + ||y||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_{n\to 0}} ||\alpha_n x||_0 = 0$ and $\lim_{\alpha \to \alpha_0} ||\alpha x||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to \alpha_0} ||\alpha x||_0 = ||\alpha_0 x||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_i] \uparrow_{\iota \in \Lambda} [p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^* \left( \frac{1}{\xi} [p] x \right) > \xi$, which implies $\sup_{\iota \in \Lambda} \rho^* \left( \frac{1}{\xi} [p_i] x \right) > \xi$ and hence $\sup_{\iota \in \Lambda} ||[p_i] x||_0 \geq \xi$. Thus we obtain $\sup_{\iota \in \Lambda} ||[p_i] x||_0 = ||[p] x||_0$, if $[p_i] \uparrow_{\iota \in \Lambda} [p]$.

Let $0 \leq x_i \uparrow_{\iota \in \Lambda} x$. Putting $[p_{n,i}] = \left( x_i - \left( 1 - \frac{1}{n} \right) x \right)^*$ we have $[p_{n,i}] \uparrow_{\iota \in \Lambda} [x]$ and $[p_{n,i}] x_i \geq [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \ (n \geq 1)$.

As is shown above, since $\sup_{\iota \in \Lambda} ||[p_{n,i}] x_i||_0 \geq \sup_{\iota \in \Lambda} ||[p_{n,i}] \left( 1 - \frac{1}{n} \right) x||_0 = \left( 1 - \frac{1}{n} \right) x||_0$, we have $\sup_{\iota \in \Lambda} ||x_i||_0 \geq \left( 1 - \frac{1}{n} \right) x||_0$ and also $\sup_{\iota \in \Lambda} ||x_i||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $||\cdot||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{\iota \to \infty} ||x_i||_0 = 0$ if and only if $\lim \rho(\xi x_i) = 0$ for all $\xi \geq 0$.
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

(3.2) \[ [p_{n,\nu}] \uparrow_{\nu=1}^\infty [p_n] \text{ with } \cap_{n=1}^\infty [p_n] a = [p_0] a = 0; \]

(3.3) \[ [p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{n}\}_{n \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

(3.4) \[ ||[q_m] a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \]

and

(3.5) \[ n_m [q_m] a \geq [q_m] x_{\nu_{m-1}} , \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots) , \]

where $\delta = ||[p_0] a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}] [p_0] \uparrow_{\nu=1}^\infty [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}] [p_0] a||_0 \geq \frac{\delta}{2} . \]

We put $[q_1] = [p_{1,\nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na - x_{\nu})^+] \uparrow_{n=1}^\infty [a]$ and $||[q_k] a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ ||(n_{k+1} a - x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2} . \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ ||[p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2} . \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] , \]

because

\[ [q_{k+1}] \leq [q_k] , \quad ||[q_{k+1}] a|| > \frac{\delta}{2} , \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1}[q_{k+1}]a - n_k[q_{k+1}]a \|_0 \geq \| [q_{k+1}]a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu_{k-1}})^+]\) implies \([q_{k+1}]a \leq [q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R^+_{0}\) is an \(F\)-space with \(\| \cdot \|_0\) if and only if \(\rho\) satisfies (\(\rho\.4^\prime\)).

**Proof.** If \(\rho\) satisfies (\(\rho\.4^\prime\)), \(\rho^*\) is a quasi-modular which fulfills also (\(\rho\.5\)) and (\(\rho\.6\)) in virtue of Theorem 2.3. Since \(\| x \|_0 = \inf \{ \xi ; \rho^*(\frac{x}{\xi}) \leq \xi \}\) is a quasi-norm on \(R^+_{0}\), we need only to verify completeness of \(\| \cdot \|_0\). At first let \(\{x_{\nu}\}_{\nu \geq 1} \subset R^+_{0}\) be a Cauchy sequence with \(0 \leq x_{\nu} \uparrow_{\nu=1,2,...}\). Since \(\rho^*\) satisfies (\(\rho\.3\)), there exists \(0 \leq x_0 \in R^+_{0}\) such as is shown in the proof of Theorem 2.3.

Putting \([p_{n,v}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{n=1}^{\infty} [p_{n,v}] = [p_n]\), we obtain
\[
(3.6) \quad [p_{n,v}]x_{\nu} \geq n[p_{n,v}]x_0 \quad \text{for all } n, \nu \geq 1
\]
and \([p_n] \downarrow_{n=0}^{\infty} 0\). Since \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty} [p_n] = 0\), that is, \(\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]\). And
\[
(1 - [p_{n,v}]) \leq (1 - [p_n]) \quad (n, \nu \geq 1)
\]
implies
\[
n(1 - [p_n])x_0 \geq (1 - [p_n])x_v \geq 0.
\]
Hence we have
\[
y_n = \bigcup_{v=1}^{\infty} (1 - [p_n])x_v \in R^+_{0},
\]
because \(R^+_{0}\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)
\[
\gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty,
\]
which implies
\[
\| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n])x_v \|_0 \leq \gamma
\]
for every \(n \geq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma\). This implies
$$\sum_{\nu=1}^{n}\rho^{*}\left(\frac{z_{\nu}}{1+\gamma}\right) = \rho^{*}\left(\frac{y_{n}}{1+\gamma}\right) \leq \gamma$$

for all \(n \geq 1\) by the formula (3.1). Then \((\rho.3)\) assures the existence of \(z = \sum_{\nu=1}^{\infty}z_{\nu} = \bigcup_{\nu=1}^{\infty}y_{\nu}\). This yields \(z = \bigcup_{\nu=1}^{\infty}x_{\nu}\). Truly, it follows from

\[
\exists z_{n} = \sum_{\nu=1}^{n} \left|y_{\nu+1} - y_{\nu}\right|, \text{ we have } ||z_{n}||_{0} \leq \sup_{p \geq n} ||x_{p} - x_{n}||_{0}
\]

and furthermore \(\lim_{n \to \infty} ||z_{n}||_{0} = 0\).

Secondly let \(\{x_{\nu}\}_{\nu \geq 1}\) be an arbitrary Cauchy sequence of \(R_{0}^{+}\). Then we can find a subsequence \(\{y_{\nu}\}_{\nu \geq 1}\) of \(\{x_{\nu}\}_{\nu \geq 1}\) such that

\[
||y_{\nu+1} - y_{\nu}||_{0} \leq \frac{1}{2^{\nu}} \quad \text{for all } \nu \geq 1.
\]

This implies

\[
||\sum_{\nu=1}^{n} (y_{\nu+1} - y_{\nu})||_{0} \leq \sum_{\nu=m}^{n} ||y_{\nu+1} - y_{\nu}||_{0} \leq \frac{1}{2^{m-1}} \quad \text{for all } n > m \geq 1.
\]

Putting \(z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|\), we have a Cauchy sequence \(\{z_{n}\}_{n \geq 1}\) with \(0 \leq z_{n} \leq \infty\). Then by the fact proved just above,

\[
z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{+} \quad \text{and} \quad \lim_{n \to \infty} ||z_{0} - z_{n}||_{0} = 0.
\]

Since \(\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|\) is convergent, \(y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})\) is also convergent and

\[
||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n}||_{0} = ||\sum_{\nu=m}^{\infty} (y_{\nu+1} - y_{\nu})||_{0} \leq ||z_{0} - z_{n}||_{0} \to 0.
\]

Since \(\{y_{\nu}\}_{\nu \geq 1}\) is a subsequence of the Cauchy sequence \(\{x_{\nu}\}_{\nu \geq 1}\), it follows that

\[
\lim_{\nu \to \infty} ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu}||_{0} = 0.
\]

Therefore \(\|\cdot\|_{0}\) is complete in \(R_{0}^{+}\), that is, \(R_{0}^{+}\) is an F-space with \(\|\cdot\|_{0}\).

Conversely if \(R_{0}^{+}\) is an F-space, then for any orthogonal sequence \(\{x_{\nu}\}_{\nu \geq 1} \subset R_{0}^{+}\), we have \(\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}\) for some real numbers \(\alpha_{\nu} > 0\) (for all \(\nu \geq 1\)).

Hence we can see that \(\sup_{x \in R} d(x) < +\infty\) by the same way applied in Theorem 2.1. It follows that \(\rho\) must satisfy \((\rho.4')\). Q.E.D.

Since \(R_{0}\) contains a normal manifold which is universally complete, if \(R_{0}^{+} \\cap 0\), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

(4.1) \[ \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)} \]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

(4.2) \[ \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \]

for all $x \in L$ hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha_n x \|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^\infty 0$ and that $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ \| x + y \|_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (x) + \frac{\xi}{\xi + \eta} (y) \right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{\| m(\xi x) \| \leq 1} \frac{1}{\| \xi \|} \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have $||x||_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $||x||_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $|| \cdot ||_1$ on $L$ which is equivalent to $|| \cdot ||_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$(4.3) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}$$

is a semi-continuous quasi-norm on $R^*_1$ and $|| \cdot ||_1$ is complete if and only if $\rho$ satisfies (\rho.4'), where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$(4.4) \quad ||x||_0 \leq ||x||_1 \leq 2||x||_0$$

for all $x \in R^*_0$.

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies (\rho.1)~(\rho.6) except (\rho.3) and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $\lim_{n \to \infty} x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a_n| \leq a_n (n \geq 1)$ and $a_n \downarrow 0$. And a sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $\ast\lim_{n \to \infty} x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $\lim_{n \to \infty} z_n = a$.

A quasi-norm $|| \cdot ||$ on $R$ is termed to be continuous, if $\inf_{n \geq 1} ||a_n|| = 0$ for any $a_n \downarrow 0$. In the sequel, we write by $|| \cdot ||_0$ (or $|| \cdot ||_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $|| \cdot ||_0$ (or $|| \cdot ||_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

$$(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.$$  

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \(\{[p_n]\}_{n \geq 1}\) such that \(\rho([p_n]x) = +\infty\) and \([p_n] \downarrow_{n=1}^{\infty} 0\). Hence by (3.1) it follows that \(||[p_n]x||_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(||\cdot||_0\).

**Sufficiency.** Let \(a_{\nu} \downarrow_{\nu=1}^{\infty} 0\) and put \([p_n^\epsilon]=([a_n-\epsilon a_1]^+)\) for any \(\epsilon > 0\) and \(n \geq 1\). It is easily seen that \([p_n^\epsilon] \downarrow_{n=1}^{\infty} 0\) for any \(\epsilon > 0\) and

\[
a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1-[p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1.
\]

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi[p_n^\epsilon]a_1) + \rho^*(\xi\epsilon(1-[p_n^\epsilon])a_1)
\]

for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p_n^\epsilon] \downarrow_{n=1}^{\infty} 0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*(\xi[p_n^\epsilon]a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*(\xi[p_n^\epsilon]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi\epsilon a_1).
\]

Since \(\epsilon\) is arbitrary, \(\lim_{n \to \infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} ||a_n||_0 = 0\) and \(||\cdot||_0\) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \(||\cdot||_0\) is continuous, if

\[
(5.2) \quad \rho^*(a_\nu) \to 0 \text{ implies } \rho^*(\alpha a_\nu) \to 0 \quad \text{for every } \alpha \geq 0.
\]

From the definition, it is clear that \(s\)-\(\lim_{\nu \to \infty} x_\nu = 0\) implies \(\lim_{\nu \to \infty} ||x_\nu||_0 = 0\), if \(||\cdot||_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \to \infty} ||x_\nu||_0 = 0\) (or \(\lim_{\nu \to \infty} ||x_\nu|| = 0\)) implies \(s\)-\(\lim_{\nu \to \infty} x_\nu = 0\), if \(||\cdot||_0\) is complete (i.e. \(\rho^*\) satisfies \(\rho(3)\)).

If we replace \(\lim_{\nu \to \infty} ||x_\nu|| = 0\) by \(\lim_{\nu \to \infty} \rho(x_\nu) = 0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x) = 0 \quad \text{implies } x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(||\cdot||_0\) is complete, \(\rho(a_\nu) \to 0\) implies \(s\)-\(\lim_{\nu \to \infty} a_\nu = 0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^{11}\) i.e. \(\rho^*(x) = \sup_{y_\nu \downarrow x} \rho^*(y_\nu)\) for any \(0 \leq x = \sup_{y_\nu \downarrow x} \rho^*(y_\nu)\). If

\[11\] If \(\rho^*\) is not semi-continuous, putting \(\rho_* (x) = \inf_{y_\nu \uparrow x} \sup_j \rho^*(y_\nu)\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \to 0\) is equivalent to \(\rho_* (x) \to 0\).
we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R$ in virtue of \((\rho.3)\).

Now, since
$$\rho\left(\bigcup_{\nu\in \mathbb{N}} |a_{\nu}| \right) \leq \sum_{\nu \geq 1} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}$$
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq 1} |a_{\nu}| \right) \right) = 0$ and hence (5.3) implies
$$\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq 1} |a_{\nu}| \right) = 0.$$ Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \rightarrow 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^{\nu}}$ ($\nu = 1, 2, \cdots$). Therefore we have $s-\lim_{\nu \rightarrow \infty} b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete and continuous, then (5.2) holds.

**References**