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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOYAKI

§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense \([1]\)) and \( \rho \) be a functional which satisfies the following four conditions:

1. \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);
2. \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);
3. If \( \sum_{i \in \Lambda} \rho(x_i) < +\infty \) for a mutually orthogonal system \( \{x_i\}_{i \in \Lambda} \), there exists \( x_0 \in R \) such that \( x_0 = \sum_{i \in \Lambda} x_i \) and \( \rho(x_0) = \sum_{i \in \Lambda} \rho(x_i) \);
4. \( \lim_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper \([2]\), we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular \( m \) on \( R \) for which every universally continuous linear functional is continuous with respect to the norm defined by the modular \( m \) \([2; \text{Theorem } 3.1]\).

Recently in \([6]\) J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

1. \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);
2. \( \rho(-x) = \rho(x) \);
3. \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);
4. \( \alpha_n \to 0 \) implies \( \rho(\alpha_n x) \to 0 \) for every \( x \in R \);
5. for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( || \cdot ||_0 \) defined by the formula:

\[ x \perp y \text{ means } |x| \cap |y| = 0. \]

1. A system of elements \( \{x_i\}_{i \in A} \) is called mutually orthogonal, if \( x_i \perp x_j \) for \( i \neq j \).
2. For the definition of a modular, see \([3]\).
3. A linear functional \( f \) is called universally continuous, if \( \inf_{\lambda \in A} f(a_\lambda) = 0 \) for any \( a_\lambda_{i \in A} \).
4. \( R \) is called semi-regular, if for any \( x \neq 0, x \in R \), there exists a universally continuous linear functional \( f \) such that \( f(x) \neq 0 \).
5. This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano \([3 \text{ and } 4]\).

In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
(1.1) \[ ||x||_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \((\rho.1) \sim (\rho.4)\) with those of \( \rho \) \[6\], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( || \cdot ||_0 \) on \( R_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0 \) is an \( F \)-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \left\{ \lim_{\alpha \to 0} \rho(\alpha x) \right\} < +\infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R_0 \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2||x||_0 \) holds for every \( x \in R_0 \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano \[4; §83\]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases \[8\].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x) \) for all \( x \geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N=\{p\}^1 \), that is, \( x = [p]x + (1-\{p\})x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz \[5\] and discussed by several authors \[6 \text{ or } 7\].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

\begin{align}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho([p]x) \leq \rho(x) \text{ for all } p, x \in R; \\
(2.3) & \quad \rho([p]x) = \sup_{i \in I} \rho([p_i]x) \text{ for any } [p_i]_{i \in I} \uparrow [p].
\end{align}

In the argument below, we have to use the additional property of \( \rho \):

\( (\rho.5) \quad \rho(x) \leq \rho(y) \text{ if } |x| \leq |y|, x, y \in R, \)

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \( (\rho.5) \).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \( (\rho.5) \) is valid.

Proof. We put for every \( x \in R \),

\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

It is clear that \( \rho' \) satisfies the conditions \( (\rho.1), (\rho.2) \) and \( (\rho.5) \).

Let \( \{x_i\}_{i \in I} \) be an orthogonal system such that \( \sum_{i \in I} \rho'(x_i) < +\infty \), then

\[ \sum_{i \in I} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \text{ for all } x \in R. \]

We have

\[ x_0 = \sum_{i \in I} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in I} \rho(x_i) \text{ in virtue of } (\rho.3). \]

For such \( x_0 \),

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in I} \rho([x_i]y) \]

\[ = \sum_{i \in I} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in I} \rho'(x_i) \]

holds, i.e. \( \rho' \) fulfils \( (\rho.3) \).

If \( \rho' \) does not fulfil \( (\rho.4) \), we have for some \( x_0 \in R \),

\[ \rho'(\frac{1}{n}x_0) = +\infty \text{ for all } n \geq 1. \]

By \( (\rho.2) \) and \( (\rho.4) \), \( x_0 \) cannot be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho'\left(\frac{1}{\nu}x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where
\[ \rho'\left(\frac{1}{\nu}x_2\right) = +\infty \] (\( \nu = 1, 2, \ldots \))
and
\[ \rho'\left(\frac{1}{2}x'_2\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho\left(\frac{1}{2}y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho'\left(\frac{1}{\nu}y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho'\left(\frac{1}{\nu}y_0\right) \geq \rho'\left(\frac{1}{\nu}y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)\]
\[\leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta |y|)\]
\[= \rho([p]|x| + (1-[p])|y|)\]
\[= \rho([p]x) + \rho((1-[p])y)\]
\[\leq \rho(x) + \rho(y).\]

**Remark 1.** As is shown above, the existence of \(\rho'\) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \(x \in R\), there exists \(\alpha \geq 0\) such that \(\rho(\alpha x) < +\infty\).

In fact, the next example shows that there exists a functional \(\rho_0\) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \(\rho_0\), we obtain

\[\rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty\]

for all \(x \neq 0\).

**Example.** \(L_1[0,1]\) is the set of measurable functions \(x(t)\) which are defined in \([0,1]\) with

\[\int_0^1 |x(t)| \, dt < +\infty.\]

Putting

\[\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty \mes \left\{ t : x(t) = \frac{1}{i} \right\},\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6)\quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \(\rho\) be a quasi-modular on \(R\). We can find a functional \(\rho^*\) which satisfies \((\rho.1)\sim(\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \(\rho'\) which satisfies \((\rho.5)\). Now we put

\[d(x) = \lim_{\xi \to 0} \rho'(\xi x).\]

It is clear that \(0 \leq d(x) = d(|x|) < +\infty\) for all \(x \in R\) and
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\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting
\[ (2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]
we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since
\[ d(x) \leq \rho'(x) \]
and
\[ d(\alpha x) = d(x) \]
for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note
\[ (2.7) \quad \inf_{\lambda \in A} d([p_{\lambda}]x) = 0 \]
for any \([p_{\lambda}] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have
\[ \inf_{\lambda \in A} d([p_{\lambda}]x_0) \geq \alpha > 0 \]
for some \([p_{\lambda}] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,
\[ \rho'\left(\frac{1}{\nu}[p_{\lambda}]x_0\right) \geq d([p_{\lambda}]x_0) \geq \alpha \]
for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \{\lambda_n\}_{n \geq 1} of \{\lambda\}_{\lambda \in A}
such that
\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]
and
\[ \rho'\left(\frac{1}{n}[p_{\lambda_n}]x_0\right) - \rho'\left(\frac{1}{n}[p_{\lambda_{n+1}}]x_0\right) \geq \frac{\alpha}{2} \]
for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies
\[ \rho'\left(\frac{1}{n}x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0\right) = +\infty, \]
which is inconsistent with \((\rho.4)\). Secondly we shall prove
\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then,
\[ [p_n] \downarrow_{n \geq 1} 0 \quad \text{and} \quad \inf_{n \geq 1} d([p_n]x) = 0 \text{ by } (2.7). \]
Since \((1-[p_n])n \mid y \mid \geq (1-[p_n])|x| \]
and
\[ d(\alpha x) = d(x) \]
for \(\alpha > 0\) and \(x \in R\), we obtain
$d(x) = d([p_n]x) + d(1-[p_n])x)$
$\leq d([p_n]x) + d(n(1-[p_n])y)$
$\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies
$d(x) \leq \inf_{n=1,2,\ldots} d([p_n]x) + d(y)$,
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
$\rho^*(x) = \rho^*([y]x) + \rho^*([x]-[y])x)$
$= \rho^*([y]x) - d([y]x) + \rho^*([x]-[y])x)$
$\geq \rho^*(y) - d(y) + \rho^*([x]-[y])x)$
$\geq \rho^*(y)$.

Thus $\rho^*$ satisfies (\rho.5).

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (\rho.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

$(\rho.4')$

\[ \sup_{x \in R} \{|\lim_{\xi \to 0} \rho^*(\xi x)| = K < +\infty. \]

**Proof.** Let $\rho$ satisfy (\rho.4). We need to prove

(2.9) \[ \sup_{x \in R} d(x) = \sup_{x \in R} \{|\lim_{\xi \to 0} \rho^*(\xi x)| = K' < +\infty, \]

where

$\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y)$.

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $\eta_0(x) = \rho(x)$ and $\eta_\nu(x) = \rho'(\frac{1}{\nu} x)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\gamma, \gamma_0, \gamma_1$ a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N^p$ with

$\rho(x) \leq \epsilon$ implies $\rho'(\frac{1}{\nu_0} x) \leq \gamma$.

In $N_0$, we have obviously

$\sup_{x \in N_0} \{|\lim_{\xi \to 0} \rho'(\xi x)| = \gamma_0 < +\infty. \]

If $\epsilon \leq 2K$, for any $x_0 \in N^+$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (\rho.4'), and hence there exists always an orthogonal decomposition such that

\[ \sup_{x \in N_0} \{|\lim_{\xi \to 0} \rho'(\xi x)| = \gamma_0 < +\infty. \]
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$$\alpha_{0}x_{0}=x_{1}+\cdots+x_{n}+y_{1}+\cdots+y_{m}+z$$

where $\frac{\varepsilon}{2}<\rho(x_{i})\leq \varepsilon$ $(i=1, 2, \cdots, n)$, $y_{j}$ is an atomic element with $\rho(y_{j})>\varepsilon$ for every $j=1, 2, \cdots, m$ and $\rho(z)\leq \frac{\varepsilon}{2}$. From above, we get $n\leq \frac{4K}{\varepsilon}$ and $m\leq \frac{2K}{\varepsilon}$. This yields

$$\rho\left(\frac{1}{\nu_{0}}\alpha_{0}x_{0}\right)\leq \sum_{i=1}^{n}\rho\left(\frac{1}{\nu_{0}}x_{i}\right)\leq \sum_{j=1}^{m}\rho'(y_{j})+\rho'\left(\frac{z}{\nu_{0}}\right)\leq n\gamma+\sum_{j=1}^{m}\rho'(y_{j})+\rho'\left(\frac{z}{\nu_{0}}\right)\leq \frac{4K}{\varepsilon}\gamma+\frac{2K}{\varepsilon}\left\{\sup_{0\leq a\leq a_{0}}\rho(\alpha x)\right\}+\gamma.$$ 

Hence, we obtain

$$\lim_{\xi\to 0}\rho'(\xi x_{0})\leq \rho'\left(\frac{\alpha_{0}}{\nu_{0}}x_{0}\right)\leq \frac{4K+\varepsilon}{\varepsilon}\gamma+\frac{4K^{2}}{\varepsilon}\gamma_{0}.$$ 

in case of $\varepsilon\leq 2K$. If $2K\leq \varepsilon$, we have immediately for $x\in N_{0}^{+}$

$$\lim_{\xi\to 0}\rho'(\xi x)\leq \gamma.$$ 

Therefore, we obtain

$$\sup_{x\in R}\left\{\lim_{\xi\to 0}\rho'(\xi x)\right\}\leq \gamma'$$

where

$$\gamma'=\frac{4K+\varepsilon}{\varepsilon}+\frac{4K^{2}}{\varepsilon}+\gamma_{0}.$$ 

Let $\{x_{\lambda}\}_{\lambda\in A}$ be an orthogonal system with $\sum_{\lambda\in A}\rho^{*}(x_{\lambda})<+\infty$. Then for arbitrary $\lambda_{1}, \cdots, \lambda_{k}\in A$, we have

$$\sum_{\nu=1}^{k}d(x_{\lambda_{\nu}})=d(\sum_{\nu=1}^{k}x_{\lambda_{\nu}})=\lim_{\xi\to 0}\rho'(\xi \sum_{\nu=1}^{k}x_{\lambda_{\nu}})\leq \gamma'',$$

which implies $\sum_{\lambda\in A}d(x_{\lambda})=\gamma'$. It follows that

$$\sum_{\lambda\in A}\rho^{*}(x_{\lambda})=\sum_{\lambda\in A}\rho^{*}(x_{\lambda})+\sum_{\lambda\in A}d(x_{\lambda})<+\infty,$$

which implies $x_{0}=\sum_{\lambda\in A}x_{\lambda}\in R$ and $\sum_{\lambda\in A}\rho^{*}(x_{\lambda})=\rho^{*}(x_{0})$ by (\rho.4) and (2.7). Therefore $\rho^{*}$ satisfies (\rho.3).

On the other hand, suppose that $\rho^{*}$ satisfies (\rho.3) and $\sup_{x\in R}d(x)=+\infty$. Then we can find an orthogonal sequence $\{x_{\nu}\}_{\nu=1}^{n}$ such that

$$\sum_{\nu=1}^{n}d(x_{\nu})=d(\sum_{\nu=1}^{n}x_{\nu})\geq \mu$$
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
$$\lim_{t \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu$ ($\nu \geq 1$) and
$$\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < + \infty.$$ 
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from ($\rho.3$). For such $x_0$, we have for every $\xi \geq 0$,
$$\rho'((\xi x, y) = \sum_{\nu=1}^{\infty} \rho'((\xi \alpha_\nu x_\nu, y) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = + \infty,$$
which is inconsistent with ($\rho.4$). Therefore we have
$$\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < + \infty.$$ 
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{x: x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1\},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\footnote{7) A linear manifold $S$ is said to be semi-normal, if $a \in S, |b| \leq |a|, b \in R$ implies $b \in S$. Since $R$ is univerfally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{x \in \Lambda} x \in R$.} of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$.
Putting
$$[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]$$
we have
$$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \text{ and } 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx,$$
which implies $\rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp.$$ 
In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\nu\}_{\nu \in \Lambda}(x_\nu \in [p]R_0)$, there exists $x_0 = \sum_{\nu \in \Lambda} x_\nu \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration.

Now we can state the theorems which we aim at.

\footnote{8) This means that $x \in R$ is written by $x = y + z, y \in R_0$ and $z \in R_0^\perp$.}
\footnote{9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$.}
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^\perp_0$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.

$$\sup_{x \in A} \| x \|_0 = \| x \|_0$$

for any $0 \leqq x, y \in A.$

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad \| x \|_0 = \inf \left\{ \xi; \rho^*(\frac{1}{\xi} x) \leqq \xi \right\}.$$ 

Then,

i) $0 \leqq \| x \|_0 = \| -x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R^\perp_0$.

ii) $\| x + y \|_0 \leqq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_n \to 0} \| \alpha_n x \|_0 = 0$ and $\lim_{\| x \|_0 \to 0} \| \alpha x \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha_0 \to 0} \| \alpha x \|_0 = \| \alpha_0 x \|_0$ for all $x \in R^\perp_0$ and $\alpha_0 \geqq 0$. If $x \in R^\perp_0$ and $[p_n] \uparrow_{n \to \infty} [p]$, for any positive number $\xi$ with $\| [p] x \|_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{n \to \infty} \rho^*(\frac{1}{\xi} [p_n] x) > \xi$ and hence $\sup_{n \to \infty} \| [p_n] x \|_0 \geqq \xi$. Thus we obtain

$$\sup_{n \to \infty} \| [p_n] x \|_0 = \| [p] x \|_0,$$

if $[p_n] \uparrow_{n \to \infty} [p]$.

Let $0 \leqq x_n \uparrow_{n \to \infty} x$. Putting

$$[p_{n,1}] = \left[ (x_n - (1 - \frac{1}{n}) x) \right],$$

we have

$$[p_{n,1}] \uparrow_{n \to \infty} [x] \quad \text{and} \quad [p_{n,1}] x_n = [p_{n,1}] \left( 1 - \frac{1}{n} \right) x$$

$(n \geqq 1)$. As is shown above, since

$$\sup_{n \to \infty} \| [p_{n,1}] x_n \|_0 \geqq \sup_{n \to \infty} \| [p_{n,1}] \left( 1 - \frac{1}{n} \right) x \|_0 = \left( 1 - \frac{1}{n} \right) x \|_0,$$

we have

$$\sup_{n \to \infty} \| x_n \|_0 \geqq \left( 1 - \frac{1}{n} \right) x \|_0,$$

and also $\sup_{n \to \infty} \| x_n \|_0 \geqq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n \to \infty} \| x_n \|_0 = 0$ if and only if $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geqq 0.$
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \ldots)$ be the elements of $R_0^\perp$ such that

(3.2) \[ [p_{n, \nu}] \uparrow_{\nu = 1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0; \]

(3.3) \[ [p_{n, \nu}]x_{\nu} \geq n[p_{n, \nu}]a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

(3.4) \[ \| [q_m]a \|_0 > \frac{\delta}{2} \text{ and } [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m = 1, 2, \ldots) \]

and

(3.5) \[ n_m[q_m]a \geq [q_m]x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m = 2, 3, \ldots), \]

where $\delta = \| [p_0]a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}][p_0] \uparrow_{\nu = 1}^{\infty} [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ \| [p_{1, \nu}] [p_0]a \|_0 > \frac{\| [p_0]a \|_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1, \nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \ldots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n\alpha - x_{\nu})^+] \uparrow_{n=1}^{\infty} [\alpha]$ and $\| [q_k]a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ \| (n_{k+1}\alpha - x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ \| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1}\alpha - x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1}\alpha - x_{\nu_k})^+[q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad \| [q_{k+1}]a \| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0}\geqq||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0}\geqq||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0}\geqq||[q_{k+1}]a_{0}||_{0}\geqq\frac{\delta}{2}$,

since $[q_{k+1}]\leqq[q_{k}]\leqq[(n_{k}a-x_{\nu_{k-1}})^{+}]$ implies $[q_{k+1}]n_{k}a\geqq[q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu\geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{\downarrow}$ is an $F$-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^{*}$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $||x||_{0} (=\inf \{\xi; \rho^{*}(x/\xi)\leqq \xi\})$ is a quasi-norm on $R_{0}^{\downarrow}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu\geqq 1} \subset R_{0}^{\downarrow}$ be a Cauchy sequence with $0\leqq x_{\nu}\uparrow_{\nu=1,2,\ldots}$. Since $\rho^{*}$ satisfies $(\rho.3)$, there exists $0\leqq x_{0}\in R_{0}^{\downarrow}$ such that $x_{0}=\bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}]=[(x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{\nu=1}^{\infty}[p_{n,v}]=[p_{n}]$, we obtain

$\bigcup_{\nu=1}^{\infty}[p_{n,v}]=x_{\nu}\geqq n[p_{n,v}]x_{0}$ for all $n, \nu\geqq 1$ and $[p_{n}]\downarrow_{n=1}^{\infty}0$. Since $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}]=0$, that is, $\bigcup_{n=1}^{\infty}([x_{\nu}]-[p_{n}])=[x_{0}]$. And

$(1-[p_{n,v}])\geqq(1-[p_{n}]) \quad (n, \nu\geqq 1)$

implies

$n(1-[p_{n,v}])x_{0}\geqq(1-[p_{n}])x_{\nu}\geqq 0$.

Hence we have

$y_{n}=\bigcup_{\nu=1}^{\infty}(1-[p_{n,v}])x_{\nu}\in R_{0}^{\downarrow}$,

because $R_{0}^{\downarrow}$ is universally continuous. As $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$\gamma=\sup_{\nu\geqq 1}||x_{\nu}||_{0}<-\infty$,

which implies

$||y_{n}||_{0}=\sup_{\nu\geqq 1}||(1-[p_{n,v}])x_{\nu}||_{0}\leqq\gamma$

for every $n\geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1}=y_{1}$ and $z_{n}=y_{n}-y_{n-1}$ ($n\geqq 2$). It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu\geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0}=||y_{n}||_{0}\leqq\gamma$. This implies
for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of 
\[
z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}.
\]
This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from
\[
z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - \lfloor p_{\nu} \rfloor) x_{\nu} = \bigcup_{\nu=1}^{\infty} \lfloor x_{\nu} \rfloor x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.
\]
By semi-continuity of \( \| \cdot \|_{0} \), we have
\[
\| z - x_{\nu} \|_{0} \leq \sup_{\mu \geq \nu} \| x_{\mu} - x_{\nu} \|_{0}
\]
and furthermore \( \lim_{\nu \to \infty} \| z - x_{\nu} \|_{0} = 0 \).

Secondly let \( \{ x_{\nu} \}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_{0}^{\perp} \). Then we can find a subsequence \( \{ y_{\nu} \}_{\nu \geq 1} \) of \( \{ x_{\nu} \}_{\nu \geq 1} \) such that
\[
\| y_{\nu+1} - y_{\nu} \|_{0} \leq \frac{1}{2^{\nu}}
\]
for all \( \nu \geq 1 \).

This implies
\[
\| y_{\nu+1} - y_{\nu} \|_{0} \leq \frac{1}{2^{\nu}}
\]
for all \( n > m \geq 1 \).

Putting \( z_{\nu} = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \), we have a Cauchy sequence \( \{ z_{\nu} \}_{\nu \geq 1} \) with \( 0 \leq z_{\nu} \uparrow_{\nu \to \infty} \infty \).

Then by the fact proved just above,
\[
z_{0} = \bigcup_{\nu=1}^{\infty} z_{\nu} = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_{0}^{\perp}
\]
and \( \lim_{\nu \to \infty} \| z_{0} - z_{\nu} \|_{0} = 0 \).

Since \( \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \) is convergent, \( y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and
\[
\| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{\infty} \|_{0} = \| \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \|_{0} \leq \| z_{0} - z_{\infty} \|_{0} \rightarrow 0.
\]

Since \( \{ y_{\nu} \}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{ x_{\nu} \}_{\nu \geq 1} \), it follows that
\[
\lim_{\nu \to \infty} \| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} \|_{0} = 0.
\]
Therefore \( \| \cdot \|_{0} \) is complete in \( R_{0}^{\perp} \), that is, \( R_{0}^{\perp} \) is an F-space with \( \| \cdot \|_{0} \).

Conversely if \( R_{0}^{\perp} \) is an F-space, then for any orthogonal sequence \( \{ x_{\nu} \}_{\nu \geq 1} \in R_{0}^{\perp} \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp} \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy (\( \rho.4' \)).

Q.E.D.

Since \( R_{0} \) contains a normal manifold which is universally complete, if \( R_{0}^{\perp} \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills \( \rho(A') \).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
||x||_1 = \inf_{\xi>0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
\end{equation}

and show that $|| \cdot ||_1$ is also a quasi-norm on $L$ and

\begin{equation}
||x||_0 \leq ||x||_1 \leq 2||x||_0
\end{equation}

for all $x \in L$ hold, where $|| \cdot ||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < +\infty (x \in L)$ and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_0 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} ||x_n||_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} ||\alpha_n x||_1 = 0$ for all $\alpha_n \downarrow_0 0$ and that $\lim_{n \to \infty} ||x_n||_1 = 0$ implies $\lim_{n \to \infty} ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ ||x + y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \]

\[ \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore $||x + y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $|| \cdot ||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\Xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ ||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $|| \cdot ||$ and $|| \cdot ||$ respectively.
This yields (4.2), since we have \[ ||x||_0 \leq \frac{1}{\xi} \] and \[ \rho(\gamma x) > \frac{1}{\eta} \] for every \( \eta \) with ||x||_0 > \frac{1}{\eta}. Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)~(A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( ||\cdot||_1 \) on \( L \) which is equivalent to \( ||\cdot||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

(4.3) \[ ||x||_1 = \inf_{t>0} \left\{ \frac{1}{t} + \rho^*(\xi x) \right\} \quad (x \in R) \]

is a semi-continuous quasi-norm on \( R_0^\perp \) and \( ||\cdot||_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.A' \)), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

(4.4) \[ ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all } x \in R_0^\perp. \]

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except \( (\rho.3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( o\text{-lim}_{\nu \to \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that

\[ |x_\nu - a_\nu| \leq a_\nu \quad (\nu \geq 1) \]

and \( a_\nu \downarrow_{\nu=1}^\infty 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( s\text{-lim}_{\nu \to \infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( o\text{-lim}_{\nu \to \infty} z_\nu = a \).

A quasi-norm \( ||\cdot|| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} ||a_\nu|| = 0 \) for any \( a_\nu \downarrow_{\nu \to 0}^\infty 0 \). In the sequel, we write by \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any \( x \in R \) there exists an orthogonal decomposition \( x = y + z \) such that \( [z]R \) is finite dimensional and \( \rho(y) < +\infty \).

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{[p_n]\}_{n\geq 1} \) such that \( \rho([p_n]x)=+\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( ||[p_n]x||_0>1 \) for all \( n\geq 1 \), which contradicts the continuity of \( ||\cdot||_0 \).

**Sufficiency.** Let \( a_\nu\downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_\nu^\epsilon]=[a_n-\epsilon a_1]^+ \) for any \( \epsilon>0 \) and \( n\geq 1 \).

This implies
\[
\rho^*(\xi a_n)\leq \rho^*(\xi [p_\nu^\epsilon]a_1)+\rho^*(\xi\epsilon(1-[p_\nu^\epsilon])a_1)
\]
for all \( n\geq 1 \) and \( \xi\geq 0 \). In virtue of (5.1) and \( [p_\nu^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \delta \)) such that \( \rho^*([p_\nu^\epsilon]a_1)<+\infty \), and hence \( \inf_{n\geq 1} \rho^*([p_\nu^\epsilon]a_1) =0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain
\[
\inf_{n\geq 1} \rho^*([p_\nu^\epsilon]a_1)\leq \rho^*(\xi a_1).
\]

Since \( \xi \) is arbitrary, \( \lim_{n\rightarrow\infty} \rho^*([p_\nu^\epsilon]a_1)=0 \) follows. Hence we infer that \( \inf_{n\geq 1} ||a_n||_0=0 \) and \( ||\cdot||_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( ||\cdot||_0 \) is continuous, if
\[
(5.2) \quad \rho^*(a_\nu)\rightarrow 0 \text{ implies } \rho^*(\alpha a_\nu)\rightarrow 0 \text{ for every } \alpha\geq 0.
\]

From the definition, it is clear that \( s-\lim_{\nu\rightarrow\infty} x_\nu=0 \) implies \( \lim_{\nu\rightarrow\infty} ||x_\nu||_0=0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu\rightarrow\infty} ||x_\nu||_0=0 \) (or \( \lim_{\nu\rightarrow\infty} ||x_\nu||=0 \)) implies \( s-\lim_{\nu\rightarrow\infty} x_\nu=0 \), if \( ||\cdot||_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu\rightarrow\infty} ||x_\nu||=0 \) by \( \lim_{\nu\rightarrow\infty} \rho(x_\nu)=0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[
(5.3) \quad \rho^*(x)=0 \text{ implies } x=0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_0 \) is complete, \( \rho(a_\nu)\rightarrow 0 \) implies \( s-\lim a_\nu=0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11} \) i.e. \( \rho^*(x)=\sup_{y_1\in A} \rho^*(y_1) \) for any \( 0\leq x_1\in A \). If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x)=\inf_{y_1\in A} \{\sup_{y_j\in A} \rho^*(y_1)\} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x_\nu)\rightarrow 0 \) is equivalent to \( \rho_*(x_\nu)\rightarrow 0 \).

\(^{11} \)
\[
\rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1),
\]
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^\infty |a_\nu| \in \mathcal{R} \) in virtue of \((\rho.3)\).

Now, since
\[
\rho\left( \bigcup_{\nu \geq \nu_v}^\infty |a_\nu| \right) \leq \sum_{\nu \geq \nu_v}^\infty \rho(a_\nu) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^\infty \left( \bigcup_{\nu \geq \nu_v}^\infty |a_\nu| \right) \right) = 0 \) and hence \((5.3)\) implies
\[
\bigcap_{\nu=1}^\infty \left( \bigcup_{\nu \geq \nu_v}^\infty |a_\nu| \right) = 0.
\]
Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \) \((\nu = 1, 2, \cdots)\). Therefore we have \( s\text{-}\lim b_\nu = 0 \). \( \text{Q.E.D.} \)

The latter part of the above proof is quite the same as Lemma 2.1 in \([9]\) (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of \([9]\) and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies \((5.3)\) and \( \|\cdot\|_0 \) is complete and continuous, then \((5.2)\) holds.

**References**


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