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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff’s sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ \[ 0 \leq \rho(x) = \rho(-x) \leq +\infty \] for all $x \in R$;

$(\rho.2)$ \[ \rho(x+y) = \rho(x) + \rho(y) \] for any $x, y \in R$ with $x \perp y$;

$(\rho.3)$ If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_0 \in R$ such that $x_0 = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_0) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

$(\rho.4)$ \[ \lim_{\xi \to 0} \rho(\xi x) < +\infty \] for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) \[ \rho(x) \geq 0 \text{ and } \rho(x) = 0 \text{ if and only if } x = 0; \]

(A.2) \[ \rho(-x) = \rho(x); \]

(A.3) \[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \] for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(A.4) \[ \alpha_n \to 0 \text{ implies } \rho(\alpha_n x) \to 0 \text{ for every } x \in R; \]

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by the formula.

---

1) $x \perp y$ means $|x|\cap|y| = 0$.

2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.

3) For the definition of a modular, see [3].

4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

---
(1.1) 
\[ ||x||_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e., without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \)~(\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)~(A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( || \cdot ||_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( || \cdot ||_0 \) (i.e., \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \left\{ \lim_{\alpha \to 0} \rho(\alpha x) \right\} < +\infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R_0^+ \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4, §83]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( \lfloor p \rfloor \) is a projector: \( \lfloor p \rfloor x = \bigcup_{n=1}^{\infty} (n \lfloor p \rfloor \cap x) \) for all \( x \geq 0 \) and \( 1 - \lfloor p \rfloor \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = \lfloor p \rfloor x + (1 - \lfloor p \rfloor) x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

\[
(2.1) \quad \rho(0) = 0;
\]

\[
(2.2) \quad \rho([p]x) \leq \rho(x) \text{ for all } p, x \in R;
\]

\[
(2.3) \quad \rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x) \text{ for any } [p_{\lambda}]_{\lambda \in \Lambda} \uparrow [p].
\]

In the argument below, we have to use the additional property of \( \rho \):

\[
(\rho.5) \quad \rho(x) \leq \rho(y) \text{ if } |x| \leq |y|, x, y \in R,
\]

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

Proof. We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) be an orthogonal system such that \( \sum_{\lambda \in \Lambda} \rho'(x_{\lambda}) < +\infty \), then

\[
\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \text{ for all } x \in R.
\]

We have

\[
x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in R
\]

and

\[
\rho(x_0) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) \text{ in virtue of } (\rho.3).
\]

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{\lambda \in \Lambda} \rho([x_{\lambda}]y)
\]

\[
= \sum_{\lambda \in \Lambda} \sup_{0 \leq |y| \leq |x_{\lambda}|} \rho([x_{\lambda}]y) = \sum_{\lambda \in \Lambda} \rho'(x_{\lambda})
\]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfill \((\rho.4)\), we have for some \( x_0 \in R \),

\[
\rho'(\frac{1}{n} x_0) = +\infty \text{ for all } n \geq 1.
\]

By \((\rho.2)\) and \((\rho.4)\), \( x_0 \) can not be written as \( x_0 = \sum_{\nu = 1}^{s} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq s \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))

and

\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that \( \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \)

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^\infty y_\nu \in \mathbb{R} \]

and

\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in \$1:\$

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)
\leq \rho(\alpha \lfloor p \rfloor |x| + \alpha(1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor)\beta |y|)
= \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|)
= \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor)y)
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4') \text{ for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty.\]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \text{ and } (\rho.4'') \), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[
\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^\infty i \text{ mes } \{ t : x(t) = \frac{1}{i} \},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{ for all } x \in R.\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[(2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x).\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x+y) = d(x) + d(y) \]
if \( x \perp y \).

Hence, putting
\[
(2.6) \quad \rho^*(x) = \rho'(x) - d(x)
\]
we can see easily that (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.4 \)) and (\( \rho.6 \)) hold true for \( \rho^* \), since
\[ d(x) \leq \rho'(x) \]
and
\[ d(\alpha x) = d(x) \]
for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that (\( \rho.5 \)) is true for \( \rho^* \). First we have to note
\[
(2.7) \quad \inf_{\lambda \in \Lambda} d([p_{\lambda}] x) = 0
\]
for any \([p_{\lambda}] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have
\[ \inf_{\lambda \in A} d([p_{\lambda}] x_0) \geq \alpha > 0 \]
for some \([p_{\lambda}] \downarrow_{\lambda \in A} 0\) and \( x_0 \in R \).

Hence,
\[ \rho'(\frac{1}{\nu} [p_{\lambda}] x_0) \geq d([p_{\lambda}] x_0) \geq \alpha \]
for all \( \nu \geq 1 \) and \( \lambda \in A \). Thus we can find a subsequence \( \{\lambda_m\}_{n \geq 1} \) of \( \{\lambda\}_{\lambda \in A} \) such that
\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]
and
\[ \rho'(\frac{1}{n} ([p_{\lambda_n}] - [p_{\lambda_{n+1}}]) x_0) \geq \frac{\alpha}{2} \]
for all \( n \geq 1 \) in virtue of (\( \rho.2 \)) and (2.3). This implies
\[ \rho'(\frac{1}{n} x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} ([p_{\lambda_m}] - [p_{\lambda_{m+1}}]) x_0) = +\infty , \]
which is inconsistent with (\( \rho.4 \)). Secondly we shall prove
\[
(2.8) \quad d(x) = d(y) , \quad \text{if } [x] = [y] .
\]
We put \([p_n] = [(|x| - n |y|)]^+ \) for \( x, y \in R \) with \([x] = [y] \) and \( n \geq 1 \). Then, \([p_n] \downarrow_{n \geq 1} 0\) and \( \inf_{n=1,2,\ldots} d([p_n] x) = 0 \) by (2.7). Since \( (1 - [p_n]) n |y| \geq (1 - [p_n]) |x| \)
and
\[ d(\alpha x) = d(x) \]
for \( \alpha > 0 \) and \( x \in R \), we obtain
$$d(x) = d([p_n]x) + d((1-[p_n])x)$$
$$\leq d([p_n]x) + d(n(1-[p_n])y)$$
$$\leq d([p_n]x) + d(y).$$

As $n$ is arbitrary, this implies
$$d(x) \leq \inf_{n=1,2,...} d([p_n]x) + d(y),$$
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
$$\rho^*(x) = \rho^*(y) = \rho^*(y)x + \rho^*(([x] - [y])x)$$
$$\geq \rho^*(y) - d(y) + \rho^*(([x] - [y])x)$$
$$\geq \rho^*(y).$$

Thus $\rho^*$ satisfies $(\rho.5)$.

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies $(\rho.3)$ (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

$(\rho.4')$
$$\sup_{x \in X} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty.$$  

Proof. Let $\rho$ satisfy $(\rho.4)$. We need to prove

$(2.9)$
$$\sup_{x \in X} d(x) = \sup_{x \in X} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty,$$
where
$$\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_{\nu}(x) = \rho(x)$ and $n_{\nu}(x) = \rho'(x)$ for $\nu \geq 1$ and $x \in X$. Hence we can find positive numbers $\epsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with
$$\rho(x) \leq \epsilon$$ implies $\rho'(x) \leq \gamma.$

In $N_0$, we have obviously
$$\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.$$

If $\epsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by $(\rho.4')$, and hence there exists always an orthogonal decomposition such that
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$\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z$

where $\frac{\epsilon}{2} < \rho(x_i) \leq \epsilon$ ($i = 1, 2, \cdots, n$), $y_j$ is an atomic element with $\rho(y_j) > \epsilon$ for every $j = 1, 2, \cdots, m$ and $\rho(z) \leq \frac{\epsilon}{2}$. From above, we get $n \leq \frac{4K}{\epsilon}$ and $m \leq \frac{2K}{\epsilon}$. This yields

$$
\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
$$

$$
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
$$

$$
\leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
$$

Hence, we obtain

$$
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \frac{4K + \epsilon}{\epsilon} \gamma + \frac{4K^2}{\epsilon}.
$$

in case of $\epsilon \leq 2K$. If $2K \leq \epsilon$, we have immediately for $x \in N_0^+$

$$
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
$$

Therefore, we obtain

$$
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
$$

where

$$
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0.
$$

Let $\{x_i\}_{i \in A}$ be an orthogonal system with $\sum_{i \in A} \rho^*(x_i) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in A$, we have

$$
\sum_{i=1}^{k} d(x_{i}) = d(\sum_{i=1}^{k} x_i) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_i) \leq \gamma',
$$

which implies $\sum_{i \in A} d(x_i) \leq \gamma'$. It follows that

$$
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
$$

which implies $x_0 = \sum_{i \in A} x_i \in R$ and $\sum_{i \in A} \rho^*(x_i) = \rho^*(x_0)$ by $(\rho.4)$ and (2.7). Therefore $\rho^*$ satisfies $(\rho.3)$.

On the other hand, suppose that $\rho^*$ satisfies $(\rho.3)$ and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_\nu\}_{\nu \geq 1}$ such that

$$
\sum_{\nu=1}^{\mu} d(x_\nu) = d(\sum_{\nu=1}^{\mu} x_\nu) \geq \mu.
$$
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since

$\lim_{t \to 0} \rho^{*}(\xi x) = 0$, there exists $\{\alpha_{\nu}\}_{\nu \geq 1}$ with $0 < \alpha_{\nu}$ ($\nu \geq 1$) and $\sum_{\nu = 1}^{\infty} \rho^{*}(\alpha_{\nu} x_{\nu}) < +\infty$. It follows that $x_{0} = \sum_{\nu = 1}^{\infty} \alpha_{\nu} x_{\nu} \in R$ and $d(x_{0}) = \sum_{\nu = 1}^{\infty} d(\alpha_{\nu} x_{\nu})$ from (\rho.3). For such $x_{0}$, we have for every $\xi \geq 0$,

$\lim_{\epsilon \to 0} \rho^{*}(\xi x) = 0$,

there exists $\{\alpha_{\nu}\}_{\nu \geq 1}$ with $0 < \alpha_{\nu}$ ($\nu \geq 1$) and $\sum_{\nu = 1}^{\infty} \rho^{*}(\alpha_{\nu} x_{\nu}) < +\infty$. It follows that $x_{0} = \sum_{\nu = 1}^{\infty} \alpha_{\nu} x_{\nu} \in R$ and $d(x_{0}) = \sum_{\nu = 1}^{\infty} d(\alpha_{\nu} x_{\nu})$ from (\rho.3).

For such $x_{0_{\iota}}$, we have for every $\xi \geq 0$,

$\rho^{\prime}(\xi x_{0}) = \sum_{\nu = 1}^{\infty} \rho^{\prime}(\xi \alpha_{\nu} x_{\nu}) \geq \sum_{\nu = 1}^{\infty} d(x_{\nu}) = +\infty$,

which is inconsistent with (\rho.4). Therefore we have

$\sup_{x \in R} (\lim_{\epsilon \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty$.

Q.E.D.

§3. Quasi-norms. We denote by $R_{0}$ the set:

$R_{0} = \{x : x \in R, \rho^{*}(nx) = 0 \text{ for all } n \geq 1\}$,

where $\rho^{*}$ is defined by the formula (2.6). Evidently $R_{0}$ is a semi-normal manifold7) of $R$. We shall prove that $R_{0}$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_{0} \ni x_{\lambda} \geq 0$ for all $\lambda \in \Lambda$. Putting $[p_{n,\lambda}] = [(2nx_{\lambda} - nx)^{+}]$, we have $[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]$ and $2n[p_{n,\lambda}] x_{\lambda} \geq [p_{n,\lambda}] nx$, which implies $\rho^{*}(n[p_{n,\lambda}] x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^{*}(n[p_{n,\lambda}] x) = \rho^{*}(nx) = 0$. Hence, we obtain $x \in R_{0}$, that is, $R_{0}$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into

$R = R_{0} \oplus R_{0}^{\perp}$.

In virtue of the definition of $\rho^{*}$, we infer that for any $p \in R_{0}$, $[p] R_{0}$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda} \in A \mid x_{\lambda} \in [p] R_{0}\}$, there exists $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p] R$. Hence we can also verify without difficulty that $R_{0}$ has no universally continuous linear functional except 0, if $R_{0}$ is non-atomic. When $R_{0}$ is discrete, it is isomorphic to $S(\Lambda)^{\perp}$-space. With respect to such a universally complete space $R_{0}$, we can not always construct a linear metric topology on $R_{0}$, even if $R_{0}$ is discrete.

In the following, therefore, we must exclude $R_{0}$ from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_{\lambda} \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_{0}$ and $z \in R_{0}^{\perp}$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^*_0$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.

$$\sup_{i \in \mathcal{A}} \| x_i \|_0 = \| x \|_0$$

for any $0 \leq x_i \uparrow_{i \in \mathcal{A}} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$||x||_0 = \inf \{ \xi; \rho^*(\frac{1}{\xi}x) \leq \xi \}.$$  

Then,

i) $0 \leq \| x \|_0 = \| -x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R^*_0$.

ii) $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{a \to a_0} \| \alpha_n x \|_0 = 0$ and $\lim_{a \to a_0} \| \alpha x_n \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that

$$\lim_{a \to a_0} \| \alpha x \|_0 = \| \alpha_0 x \|_0$$

for all $x \in R^*_0$ and $\alpha_0 \geq 0$. If $x \in R^*_0$ and $[p_{\lambda}] \uparrow_{\lambda \in \mathcal{A}} [p]$, for any positive number $\xi$ with $\| [p]x \|_0 > \xi$ we have

$$\rho^*(\frac{1}{\xi}x) > \xi,$$

which implies $\sup_{\lambda \in \mathcal{A}} \rho^*(\frac{1}{\xi}x_{\lambda}) > \xi$ and hence $\sup_{\lambda \in \mathcal{A}} \| p_{\lambda} \|_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda \in \mathcal{A}} \| [p_{\lambda}]x \|_0 = \| [p]x \|_0,$$

if $[p_{\lambda}] \uparrow_{\lambda \in \mathcal{A}} [p]$.

Let $0 \leq x_{\lambda} \uparrow_{\lambda \in \mathcal{A}} x$. Putting

$$[p_{n,\lambda}] = \left[ (x_{\lambda} - (1 - \frac{1}{n})x)^+ \right]$$

we have

$$[p_{n,\lambda}] \uparrow_{\lambda \in \mathcal{A}} [x]$$

and

$$[p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}](1 - \frac{1}{n})x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda \in \mathcal{A}} \| [p_{n,\lambda}]x_{\lambda} \|_0 \geq \sup_{\lambda \in \mathcal{A}} \| [p_{n,\lambda}](1 - \frac{1}{n})x \|_0 = \| (1 - \frac{1}{n})x \|_0,$$

we have

$$\sup_{\lambda \in \mathcal{A}} \| x_{\lambda} \|_0 \geq \| (1 - \frac{1}{n})x \|_0$$

and also $\sup_{\lambda \in \mathcal{A}} \| x_{\lambda} \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that

$$\lim_{n \to \infty} \| x_n \|_0 = 0$$

if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 


In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\begin{align}
&[p_{n,\nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0; \\
&[p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] a \text{ for all } n, \nu \geq 1.
\end{align}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
&||[q_m] a||_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \\
&n_m [q_m] a \geq [q_m] x_{\nu} m-1, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots),
\end{align}

where $\delta = ||[p_0] a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}][p_0] a||_0 > \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n a - x_{\nu})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k] a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ ||(n_{k+1} a - x_{\nu_k})^+[q_k] a||_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ ||[p_{n_{k+1},\nu_{k+1}}][(n_{k+1} a - x_{\nu_k})^+] [q_k] a||_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}][(n_{k+1} a - x_{\nu_k})^+] [q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad ||[q_{k+1}] a|| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_{k+1}}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| q_{k+1}(x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1}q_{k+1}a - n_{k}q_{k+1}a \|_0 \geq \| q_{k+1}a_0 \|_0 \geq \frac{\delta}{2} , \]

since \( [q_{k+1}] \leq [q_k] \leq [(n_ka - x_{\nu-1})^+] \) implies \( [q_{k+1}]n_{k}a \geq [q_{k+1}]x_{\nu_{k-1}} \) by (3.4). It follows from the above that \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence.

Theorem 3.2. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then \( R_0^\perp \) is an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) satisfies (\( \rho.4' \)).

Proof. If \( \rho \) satisfies (\( \rho.4' \)), \( \rho^* \) is a quasi-modular which fulfills also (\( \rho.5 \)) and (\( \rho.6 \)) in virtue of Theorem 2.3. Since \( \rho^* \) satisfies (\( \rho.3 \)), there exists \( 0 \leq x_0 \in R_0^\perp \) such that \( x_0 = \bigcup_{\nu=1}^{\infty}x_{\nu} \), as is shown in the proof of Theorem 2.3.

Putting \( [p_{n,\nu}] = [(x_{\nu} - nx_{0})^+] \) and \( \bigcup_{\nu=1}^{\infty}[p_{n,\nu}] = [p_n] \), we obtain

(3.6) \[ [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]x_{0} \]

for all \( n, \nu \geq 1 \) and \( [p_n] \uparrow_{n=1}^{\infty}0 \). Since \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we have in virtue of Lemma 2, \( \bigcup_{n=1}^{\infty}[p_{n}] = 0 \), that is, \( \bigcup_{n=1}^{\infty}([x_{\nu}] - [p_{n}]) = [x_{0}] \). And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_{n}]) \]

\[ (n, \nu \geq 1) \]

implies

\[ n(1 - [p_{n}])x_{0} \geq (1 - [p_{n}])x_{\nu} \geq 0 . \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^{\infty}(1 - [p_{n}])x_{\nu} \in R_0^\perp , \]

because \( R_0^\perp \) is universally continuous. As \( \{x_{\nu}\}_{\nu \geq 1} \) is a Cauchy sequence, we obtain from the triangle inequality of \( \| \cdot \|_0 \)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty , \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_{n}])x_{\nu} \|_0 \leq \gamma , \]

for every \( n \geq 1 \) by semi-continuity of \( \| \cdot \|_0 \). We put \( z_1 = y_1 \) and \( z_n = y_n - y_{n-1} \) \( (n \geq 2) \). It follows from the definition of \( y_n \) that \( \{z_{\nu}\}_{\nu \geq 1} \) is an orthogonal sequence with \( \| \sum_{\nu=1}^{n}z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma . \) This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from
\[ z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - [p_{\nu}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} x_{\nu}. \]

By semi-continuity of $|| \cdot ||_{0}$, we have
\[ || z - x_{\nu} ||_{0} \leq \sup_{\nu \geq \rho} || x_{\nu} - x_{\nu} ||_{0} \]
and furthermore \( \lim_{\nu \to \infty} || z - x_{\nu} ||_{0} = 0 \).

Secondly let \( \{ x_{\nu} \}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_{0}^{\perp} \). Then we can find a subsequence \( \{ y_{\nu} \}_{\nu \geq 1} \) of \( \{ x_{\nu} \}_{\nu \geq 1} \) such that
\[ || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}} \]
for all $\nu \geq 1$.

This implies
\[ || \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{n-1}} \]
for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} | y_{\nu+1} - y_{\nu} |$, we have a Cauchy sequence \( \{ z_{n} \}_{n \geq 1} \) with $0 \leq z_{n} \leq \infty$.

Then by the fact proved just above,
\[ z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \bigcup_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_{0}^{\perp} \quad \text{and} \quad \lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0. \]

Since $\sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} |$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and
\[ || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=m}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0. \]

Since $\{ y_{\nu} \}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{ x_{\nu} \}_{\nu \geq 1}$, it follows that
\[ \lim_{\nu \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0. \]

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{\perp}$, that is, $R_{0}^{\perp}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{\perp}$ is an F-space, then for any orthogonal sequence $\{ x_{\nu} \}_{\nu \geq 1} \in R_{0}^{\perp}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. \( \text{Q.E.D.} \)

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0}^{\perp} = 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ defined by (3.1) and $R$ becomes an F-space with $||\cdot||_0$ if and only if $\rho$ fulfils ($\rho$.4').

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
(4.1) \quad ||x||_1 = \inf_{\epsilon > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
\end{equation}

and show that $||\cdot||_1$ is also a quasi-norm on $L$ and

\begin{equation}
(4.2) \quad ||x||_0 \leq ||x||_1 \leq 2||x||_0
\end{equation}

for all $x \in L$

hold, where $||\cdot||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = || -x||_1 < +\infty$ ($x \in L$) and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} ||x_n||_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} ||\alpha_n x||_1 = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n \to \infty} ||x_n||_1 = 0$ implies $\lim_{n \to \infty} ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ 

This yields

$$||x+y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right)$$

$$\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $||x+y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $||\cdot||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$ 

10) For the convex modular $m$, we can define two kinds of norms such as

$$||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \quad \text{for} \ [3 \text{or 4}].$$

For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $||\cdot||$ and $||\cdot||$ respectively.
This yields (4.2), since we have $||x||_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\eta$ with $||x||_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $||\cdot||_1$ on $L$ which is equivalent to $||\cdot||_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$ (4.3) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R) $$

is a semi-continuous quasi-norm on $R_0^+$ and $||\cdot||_1$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$ (4.4) \quad ||x||_0 \leq ||x||_1 \leq 2||x||_0 \quad \text{for all } x \in R_0^+ . $$

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies $(\rho.1)~(\rho.6)$ except $(\rho.3)$ and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called order-convergent to $a$ and denoted by $\lim_{\nu \to \infty} x_\nu = a$, if there exists a sequence of elements $\{a_\nu\}_{\nu \geq 1}$ such that $|x_\nu - a| \leq a_\nu \quad (\nu \geq 1)$ and $a_\nu \downarrow_{\nu=1}^{\infty} 0$. And a sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called star-convergent to $a$ and denoted by $\lim_{\nu \to \infty} x_\nu = a$, if for any subsequence $\{y_\nu\}_{\nu \geq 1}$ of $\{x_\nu\}_{\nu \geq 1}$, there exists a subsequence $\{z_\nu\}_{\nu \geq 1}$ of $\{y_\nu\}_{\nu \geq 1}$ with $\lim_{\nu \to \infty} z_\nu = a$. A quasi-norm $||\cdot||$ on $R$ is termed to be continuous, if $\inf_{\nu \geq 1} ||a_\nu|| = 0$ for any $a_\nu \downarrow_{\nu=1}^{\infty} 0$. In the sequel, we write by $||\cdot||_0$ (or $||\cdot||_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $||\cdot||_0$ (or $||\cdot||_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

$$ (5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty . $$

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \( \{[p_n]\}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n \rightarrow \infty} 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_\nu \downarrow_{\nu \rightarrow \infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1 \).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon]a_1) + \rho^*(\xi(1 - [p_n^\epsilon])a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_n^\epsilon]a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain
\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi\epsilon a_1).
\]
Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \| a_n \|_0 = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if

\[
\rho^*(a_\nu) \rightarrow 0 \quad \text{implies} \quad \rho^*(\alpha a_\nu) \rightarrow 0 \quad \text{for every} \quad \alpha \geq 0.
\]

From the definition, it is clear that s-\( \lim_{\nu \rightarrow \infty} x_\nu = 0 \) implies
\[ \| x_\nu \|_0 = 0 \quad \text{if} \quad \| \cdot \|_0 \]
if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0 \) (or \( \lim_{\nu \rightarrow \infty} \| x_\nu \| = 0 \)) implies s-\( \lim_{\nu \rightarrow \infty} x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu \rightarrow \infty} \| x_\nu \| = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
\rho^*(x) = 0 \quad \text{implies} \quad x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \rightarrow 0 \)
implies s-\( \lim_{\nu \rightarrow \infty} a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{\nu \geq 1} \rho^*(x_\nu) \) for any \( 0 \leq x_\nu \leq x \). If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf \{ \sup_{y \in A} \rho^*(y_\nu) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x_\nu) \rightarrow 0 \) is equivalent to \( \rho_*(x_\nu) \rightarrow 0 \).
\[ \rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_\nu| \in R \) in virtue of (\( \rho,3 \)).

Now, since
\[ \rho\left( \bigcup_{\nu \geq 1}^{\infty} |a_\nu| \right) \leq \sum_{\nu \geq 1}^{\infty} \rho(a_\nu) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_\mu| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_\mu| \right) = 0. \]

Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \quad (\nu = 1, 2, \cdots) \). Therefore we have \( s-\lim_{\nu \to \infty} b_\nu = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_0 \) is complete and continuous, then (5.2) holds.

**References**