ON F-NORMS OF QUASI-MODULAR SPACES

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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff’s sense \[1\]) and $\rho$ be a functional which satisfies the following four conditions:

(\(\rho.1\)) $0 \leqq \rho(x) = \rho(-x) \leqq +\infty$ for all $x \in R$;

(\(\rho.2\)) $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y^1$;

(\(\rho.3\)) If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}^2$, there exists $x_0 \in R$ such that $x_0 = \sum_{\lambda \in \Lambda} x$ and $\rho(x_0) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

(\(\rho.4\)) $\varlimsup_{\xi \rightarrow 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper \[2\], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^3\) $m$ on $R$ for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) $\rho$ \[2\; \text{Theorem 3.1}\].

Recently in \[6\] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geqq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

(A.2) $\rho(-x) = \rho(x)$;

(A.3) $\rho(\alpha x + \beta y) \leqq \rho(x) + \rho(y)$ for every $\alpha, \beta \geqq 0$ with $\alpha + \beta = 1$;

(A.4) $\alpha_n \rightarrow 0$ implies $\rho(\alpha_n x) \rightarrow 0$ for every $x \in R$;

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $|| \cdot ||_0$ defined by the formula;

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1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see \[3\].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow a \in A$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.

4) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano \[3\ and 4\]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geqq 0$ for each $x \in R$. 

\begin{equation}
\|x\|_0 = \inf \left\{ \xi; \rho\left(\frac{1}{\xi} x\right) \leq \xi \right\}
\end{equation}

and \(\|x_n\|_0 \to 0\) is equivalent to \(\rho(\alpha x_n) \to 0\) for all \(\alpha \geq 0\).

In the present paper, we shall deal with a general quasi-modular space \(R\) (i.e. without the assumption that \(R\) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \(R\) and to investigate the condition under which \(R\) is an \(F\)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \(\rho\) on \(R\) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (A.1) \sim (A.4) with those of \(\rho\) [6], we can not apply the formula (1.1) directly to \(\rho\) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \(\rho^*\) which satisfies (A.2) \sim (A.5) on an arbitrary quasi-modular space \(R\) in \(\S\)2 (Theorems 2.1 and 2.2). Since \(R\) may include a normal manifold \(R_0 = \{x: x \in R, \rho^*(\xi x) = 0\) for all \(\xi \geq 0\}\) and we can not define a quasi-norm on \(R_0\) in general, we have to exclude \(R_0\) in order to proceed with the argument further. We shall prove in \(\S\)3 that a quasi-norm \(\|\cdot\|_0\) on \(R_0^+\) defined by \(\rho^*\) according to the formula (1.1) is semi-continuous, and in order that \(R_0^+\) is an \(F\)-space with \(\|\cdot\|_0\) (i.e. \(\|\cdot\|_0\) is complete), it is necessary and sufficient that \(\rho\) satisfies

\[(\rho.4')\quad \sup_{x \in R} \left\{ \lim_{\alpha \to 0} \rho(\alpha x) \right\} < +\infty\]

(Thm. 3.2).

In \(\S\)4, we shall show that we can define another quasi-norm \(\|\cdot\|_1\) on \(R_0^+\) which is equivalent to \(\|\cdot\|_0\) such that \(\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0\) holds for every \(x \in R_0^+\) (Formulas (4.1) and (4.3)). \(\|\cdot\|_1\) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \(\S\)83]. At last in \(\S\)5 we shall add shortly the supplementary results concerning the relations between \(\|\cdot\|_0\)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \(\S\)5 are already known in those cases [8].

Throughout this paper \(R\) denotes a universally continuous semi-ordered linear space and \(\rho\) a quasi-modular defined on \(R\). For any \(p \in R\), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n| p | \cap x)\) for all \(x \geq 0\) and \(1 - [p]\) is a projection operator onto the normal manifold \(N = \{p\}^1\), that is, \(x = [p]x + (1 - [p])x\).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

\begin{align*}
\rho(0) &= 0; \\
\rho([p]x) &\leq \rho(x) \text{ for all } p, x \in R; \\
\rho([p]x) &= \sup_{i \in A} \rho([p_i]x) \text{ for any } [p_i]_{i \in A} \uparrow [p].
\end{align*}

In the argument below, we have to use the additional property of $\rho$:

\begin{align*}
\rho(x) &\leq \rho(y) \text{ if } |x| \leq |y|, x, y \in R,
\end{align*}

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\begin{align*}
\rho'(x) &= \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{align*}

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

\begin{align*}
\sum_{i \in A} \rho(x_i) < +\infty,
\end{align*}

because

\begin{align*}
\rho(x) &\leq \rho'(x) \quad \text{for all } x \in R.
\end{align*}

We have

\begin{align*}
x_0 &= \sum_{i \in A} x_i \in R
\end{align*}

and

\begin{align*}
\rho(x_0) &= \sum_{i \in A} \rho(x_i)
\end{align*}

in virtue of $(\rho.3)$.

For such $x_0$,

\begin{align*}
\rho'(x_0) &= \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) \\
&= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
\end{align*}

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_0 \in R$,

\begin{align*}
\rho'(\frac{1}{n} x_0) &= +\infty \quad \text{for all } n \geq 1.
\end{align*}

By $(\rho.2)$ and $(\rho.4)$, $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{s} \xi_\nu e_\nu$, where $e_\nu$ is an atomic element for each $\nu$ with $1 \leq \nu \leq s$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'\left(\frac{1}{\nu} x_1\right) = +\infty \) \((\nu = 1, 2, \cdots)\) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \),
there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose
\( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'\left(\frac{1}{\nu} x_2\right) = +\infty \] \((\nu = 1, 2, \cdots)\)
and
\[ \rho'\left(\frac{1}{2} x_2'\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way,
we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts \((\rho.4)\). Therefore \( \rho' \) has to satisfy \((\rho.4)\). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the
formula (2.4).

If \( \rho \) satisfies \((\rho.5)\), \( \rho \) does also \((A.3)\) in \( \S 1 \):
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p]=[(|x|−|y|)^+] \), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)
\]
\[
\leq \rho(\alpha [p] |x| + \alpha(1-[p]) |y| + \beta [p] |x| + (1-[p]) \beta |y|)
\]
\[
= \rho([p] |x| + (1-[p]) |y|)
\]
\[
= \rho(\alpha [p] x) + \rho(\alpha(1-[p]) y)
\]
\[
\leq \rho(x) + \rho(y).
\]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition (\( \rho.4 \)). Thus, in the above theorems, we cannot replace (\( \rho.4 \)) by the weaker condition:

(\( \rho.4'' \)) for any \( x \in \mathbb{R} \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.3 \)) and (\( \rho.4'' \)), but does not (\( \rho.4 \)). For this \( \rho_0 \), we obtain

\[
\rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^{\infty} i \text{ mes} \left\{ t : x(t) = \frac{1}{i} \right\},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\( \rho.6 \)) \[ \lim_{\xi \to 0} \rho(\xi x) = 0 \] for all \( x \in \mathbb{R} \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. Let \( \rho \) be a quasi-modular on \( \mathbb{R} \). We can find a functional \( \rho^* \) which satisfies (\( \rho.1 \))~(\( \rho.6 \)) except (\( \rho.3 \)).

Proof. In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies (\( \rho.5 \)). Now we put

(\( \rho.6 \)) \[ d(x) = \lim_{\xi \to 0} \rho'((\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in \mathbb{R} \) and
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\[ d(x+y)=d(x)+d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for \( \rho^* \), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that (\rho.5) is true for \( \rho^* \). First we have to note

\[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0 \]

for any \( [p_{\lambda}] \downarrow_{\lambda \in A} 0 \). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \( [p_{\lambda}] \downarrow_{\lambda \in A} 0 \) and \( x_0 \in R \).

Hence,

\[ \rho'\left(\frac{1}{\nu}[p_{\lambda}]x_0\right) \geq \inf_{\lambda \in A} d([p_{\lambda}]x) \geq \alpha \]

for all \( \nu \geq 1 \) and \( \lambda \in A \). Thus we can find a subsequence \( \{\lambda_n\}_{n \geq 1} \) of \( \{\lambda\}_{\lambda \in A} \) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'\left(\frac{1}{n}[p_{\lambda_n}]x_0\right) - \rho'\left(\frac{1}{n}[p_{\lambda_{n+1}}]x_0\right) \geq \frac{\alpha}{2} \]

for all \( n \geq 1 \) in virtue of (\rho.2) and (2.3). This implies

\[ \rho'\left(\frac{1}{n}x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0\right) = +\infty, \]

which is inconsistent with (\rho.4). Secondly we shall prove

\[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \( [p_n] = [\lfloor x - n \rfloor \lfloor y \rfloor^+] \) for \( x, y \in R \) with \( [x] = [y] \) and \( n \geq 1 \). Then, \( [p_n] \downarrow_{n=1} 0 \) and \( \inf_{n=1,2,...} d([p_n]x) = 0 \) by (2.7). Since \( (1-[p_n])n \lfloor y \rfloor \geq (1-[p_n]) |x| \)

and

\[ d(\alpha x) = d(x) \]

for \( \alpha > 0 \) and \( x \in R \), we obtain
$d(x) = d([p_n]x) + d((1- [p_n])x) \\
\leq d([p_n]x) + d(n(1- [p_n])y) \\
\leq d([p_n]x) + d(y).

As \( n \) is arbitrary, this implies 
\[
d(x) \leq \inf_{n=1,2,...} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then 
\[
\rho^*(x) = \rho^*(y) + \rho^*((x-y)x) \\
= \rho'(y) - d(y) + \rho^*((x-y)x) \\
\geq \rho^*(y).
\]
Hence \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

(\( \rho.4' \))

\[\sup_{x \in R} \lim_{t \to 0} \rho(t x) = K' < +\infty.\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove

\[\sup_{x \in R} d(x) = \sup_{x \in R} \lim_{t \to 0} \rho'(t x) = K' < +\infty,\]

where 
\[\rho'(x) = \sup_{0 \leq y \leq |x|} \rho(y).\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho(\frac{1}{\nu} x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with

\[\rho(x) \leq \epsilon \quad \text{implies} \quad \rho^*(x) \leq \gamma.\]

In \( N_0 \), we have obviously

\[\sup_{x \in N_0} \lim_{t \to 0} \rho'(t x) = \gamma_0 < +\infty.\]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (\( i = 1, 2, \ldots, n \)), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \geq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \).

This yields

\[ \rho \left( \frac{1}{\nu_0} \alpha_0 x_0 \right) \leq \sum_{i=1}^{n} \rho \left( \frac{1}{\nu_0} x_i \right) + \sum_{j=1}^{m} \rho (y_j) + \rho \left( \frac{z}{\nu_0} \right) \]

\[ \leq n \gamma + \sum_{j=1}^{m} \rho (y_j) + \rho \left( \frac{z}{\nu_0} \right) \]

\[ \leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma \]

Hence, we obtain

\[ \lim_{\xi \to 0} \rho^{'}(\xi x_0) \leq \rho^{'}\left( \frac{\alpha_0}{\nu_0} x_0 \right) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right) \]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^* \)

\[ \lim_{\xi \to 0} \rho^{'}(\xi x) \leq \gamma \]

Therefore, we obtain

\[ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^{'}(\xi x) \} \leq \gamma' \]

where

\[ \gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0 \]

Let \( \{x_1\}_{i \in A} \) be an orthogonal system with \( \sum_{x \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[ \sum_{i=1}^{k} d(x_{i}) = d(\sum_{i=1}^{k} x_{i}) = \lim_{\xi \to 0} \rho^{'}(\xi \sum_{i=1}^{k} x_{i}) \leq \gamma' \]

which implies \( \sum_{x \in A} d(x_i) \leq \gamma' \). It follows that

\[ \sum_{x \in A} \rho^{'}(x_i) = \sum_{x \in A} \rho^*(x_i) + \sum_{x \in A} d(x_i) < +\infty \]

which implies \( x_0 = \sum_{x \in A} x_i \in R \) and \( \sum_{x \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[ \sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu \]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since $\lim_{t \to 0} \rho^*(\xi x) = 0$, there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu (\nu \geq 1)$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$,

$$\lim_{\epsilon \to 0} \rho^{*}(\xi x) = 0,$$

there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu (\nu \geq 1)$ and $\sum_{\nu=1}^{\infty} \rho^{*}(\alpha_\nu x_\nu) < +\infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3).

For such $x_0$, we have for every $\xi \geq 0$,

$$\rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty,$$

which is inconsistent with (\rho.4). Therefore we have

$$\sup_{\xi \in R} (\lim_{\epsilon \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.$$  

Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:

$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \},$$

where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in \Lambda$. Putting $[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]$, we have $[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]$ and $2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx$, which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into

$$R = R_0 \oplus R_0^\perp.$$ 8)

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda}(x_\lambda \in [p]R)$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

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7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^0_{\perp}$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.
\[
\sup_{i \in A} \| x_i \|_0 = \| x \|_0
\]
for any $0 \leq x_i \uparrow_{i \in A} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put
\[
(3.1) \quad \| x \|_0 = \inf \left\{ \xi ; \rho^* \left( \frac{1}{\xi} [x] \right) \leq \xi \right\}.
\]
Then,

i) $0 \leq \| x \|_0 = \| -x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R^0_{\perp}$.

ii) $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_n \to 0} \| \alpha_n x \|_0 = 0$ and $\lim_{\alpha \to 0} \| \alpha x \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to 0} \| \alpha x \|_0 = \| \alpha_0 x \|_0$ for all $x \in R^0_{\perp}$ and $\alpha_0 \geq 0$. If $x \in R^0_{\perp}$ and $[p_i] \uparrow_{i \in A} [p]$, for any positive number $\xi$ with $\| [p] x \|_0 > \xi$ we have $\rho^* \left( \frac{1}{\xi} [p] x \right) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^* \left( \frac{1}{\xi} [p] x \right) > \xi$ and hence $\sup_{\lambda \in \Lambda} \| p \|_0 \geq \xi$. Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| p \|_0 = \| [p] x \|_0 , \quad [p] \uparrow_{i \in A} [p].
\]

Let $0 \leq x_i \uparrow_{i \in A} x$. Putting
\[
[p_{n,i}] = \left[ \left( x_i - \left( 1 - \frac{1}{n} \right) x \right) \right]^*
\]
we have
\[
[p_{n,i}] \uparrow_{i \in A} [x] \quad \text{and} \quad [p_{n,i}] x_i \geq [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \quad \text{(n \geq 1)}.
\]
As is shown above, since
\[
\sup_{i \in A} \| [p_{n,i}] x_i \|_0 \geq \sup_{i \in A} \left\| [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \right\|_0 = \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0 ,
\]
we have
\[
\sup_{i \in A} \| x_i \|_0 \geq \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0
\]
and also $\sup_{i \in A} \| x_i \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n \to \infty} \| x_n \|_0 = 0$ if and only if $\lim_{\xi \to 0} \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

On F-Norms of Quasi-Modular Spaces
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \ldots)$ be the elements of $R_0^\perp$ such that

\[(3.2) \quad [p_{n, \nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0 ;
\]

\[(3.3) \quad [p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \text{ for all } n, \nu \geq 1 .
\]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[(3.4) \quad \| [q_m] a \|_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots)
\]

and

\[(3.5) \quad n_m [q_m] a \geq [q_m] x_{\nu_m-1}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots),
\]

where $\delta = \| [p_0] a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[\| [p_{1, \nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2} .\]

We put $[q_1] = [p_{1, \nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n a - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k] a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[\| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2} .\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[\| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2} .\]

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

\[[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] ,
\]

because

\[[q_{k+1}] \subseteq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
\]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$$\|x_{\nu_{k+1}} - x_{\nu_{k-1}}\|_0 \geq \|q_{k+1} (x_{\nu_{k+1}} - x_{\nu_{k-1}})\|_0 \geq$$

$$\|n_{k+1} [q_{k+1}] a - n_{k} [q_{k+1}] a\|_0 \geq \|q_{k+1} a_0\|_0 \geq \frac{\delta}{2},$$

since $[q_{k+1}] \leq [q_{k}] \leq [(n_{k} a - x_{\nu-1})^+]$ implies $[q_{k+1}] n_{k} a \geq [q_{k+1}] x_{\nu_{k-1}}$ by (3.4). It follows from the above that $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^\perp$ is an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ satisfies $(\rho4')$.

**Proof.** If $\rho$ satisfies $(\rho4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho5)$ and $(\rho6)$ in virtue of Theorem 2.3. Since $\|x\|_0 = \inf \{\xi; \rho^* \left( \frac{x}{\xi} \right) \leq \xi\}$ is a quasi-norm on $R_0^\perp$, we need only to verify completeness of $\|\cdot\|_0$. At first let $\{x_{\nu}\}_{\nu \geq 1} \subset R_0^\perp$ be a Cauchy sequence with $0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots}$. Since $\rho^*$ satisfies $(\rho3)$, there exists $0 \leq x_0 \in R_0^\perp$ such that $x_0 = \bigcup_{\nu=1}^\infty x_\nu$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = [(x_{\nu} - nx_0)^+]$ and $\bigcup_{\nu=1}^\infty [p_{n,\nu}] = [p_n]$, we obtain

$$\{p_{n,\nu}\} = \bigcup_{\nu=1}^\infty (1 - [p_{n,\nu}]) x_{\nu} \in R_0^\perp,$$

and $\bigcup_{n=1}^\infty [p_n] = 0$. Since $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^\infty [p_n] = 0$, that is, $\bigcup_{n=1}^\infty ([x_\nu] - [p_n]) = [x_0]$. And

$$(1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1)$$

implies

$$n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_\nu \geq 0.$$ 

Hence we have

$$y_n = \bigcup_{\nu=1}^\infty (1 - [p_n]) x_\nu \in R_0^\perp,$$

because $R_0^\perp$ is universally continuous. As $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $\|\cdot\|_0$

$$\gamma = \sup_{\nu \geq 1} \|x_{\nu}\|_0 < +\infty,$$

which implies

$$\|y_n\|_0 = \sup_{\nu \geq 1} \|1 - [p_n]\| x_\nu \leq \gamma$$

for every $n \geq 1$ by semi-continuity of $\|\cdot\|_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ $(n \geq 2)$. It follows from the definition of $y_n$ that $\{z_{\nu}\}_{\nu \geq 1}$ is an orthogonal sequence with $\|\sum_{\nu=1}^n z_{\nu}\|_0 = \|y_n\|_0 \leq \gamma$. This implies...

\[
\sum_{\nu=1}^{n} \rho^{*}\left(\frac{z_{\nu}}{1+\gamma}\right) = \rho^{*}\left(\frac{y_{n}}{1+\gamma}\right) \leq \gamma
\]

for all \(n \geq 1\) by the formula (3.1). Then \((\rho.3)\) assures the existence of 
\[z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}.\]
This yields 
\[z = \bigcup_{\nu=1}^{\infty} x_{\nu}.\]
Truly, it follows from 
\[z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-\left[p_{n}\right]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \left[x_{0}\right] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.\]
By semi-continuity of ||·||₀, we have
\[||z-x_{\nu}||₀ \leq \sup_{\mu \geq \nu} ||x_{\mu}-x_{\nu}||₀\]
and furthermore \(\lim_{n \to \infty} ||z-x_{\nu}||₀ = 0\).

Secondly, let \(\{x_\nu\}_{\nu \geq 1}\) be an arbitrary Cauchy sequence of \(R_0^\perp\). Then we can find a subsequence \(\{y_\nu\}_{\nu \geq 1}\) of \(\{x_\nu\}_{\nu \geq 1}\) such that
\[||y_{\nu+1}-y_{\nu}||₀ \leq \frac{1}{2^{\nu}}\]
for all \(\nu \geq 1\).
This implies
\[||\sum_{\nu=m}^{n} |y_{\nu+1}-y_{\nu}| ||₀ \leq \sum_{\nu=m}^{n} ||y_{\nu+1}-y_{\nu}||₀ \leq \frac{1}{2^{n-m}}\]
for all \(n > m \geq 1\).

Putting \(z_\nu = \sum_{\nu=1}^{n} |y_{\nu+1}-y_{\nu}|\), we have a Cauchy sequence \(\{z_\nu\}_{\nu \geq 1}\) with \(0 \leq z_\nu \leq \infty\). Then by the fact proved just above,
\[z_0 = \bigcup_{\nu=1}^{\infty} z_\nu = \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \in R_0^\perp \quad \text{and} \quad \lim_{\nu \to \infty} ||z_0-z_\nu||₀ = 0.\]
Since \(\sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}|\) is convergent, \(y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})\) is also convergent and
\[||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_\infty||₀ = ||\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})||₀ \leq ||z_0 - z_\infty||₀ \to 0.\]
Since \(\{y_\nu\}_{\nu \geq 1}\) is a subsequence of the Cauchy sequence \(\{x_\nu\}_{\nu \geq 1}\), it follows that
\[\lim_{\nu \to \infty} ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_\nu||₀ = 0.\]
Therefore \(||·||₀\) is complete in \(R_0^\perp\), that is, \(R_0^\perp\) is an F-space with \(||·||₀\).

Conversely if \(R_0^\perp\) is an F-space, then for any orthogonal sequence \(\{x_\nu\}_{\nu \geq 1} \in R_0^\perp\), we have \(\sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R_0^\perp\) for some real numbers \(\alpha_\nu > 0\) (for all \(\nu \geq 1\)). Hence we can see that \(\sup_{\nu \geq 1} d(\nu) < +\infty\) by the same way applied in Theorem 2.1. It follows that \(\rho\) must satisfy \((\rho.4')\). Q.E.D.

Since \(R_0\) contains a normal manifold which is universally complete, if \(R_0^\perp\), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfils $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[(4.1) \quad \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)} \]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

\[(4.2) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all } x \in L \]

hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_1^\infty 0$ implies $\lim_{n \to +\infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to +\infty} \| x_n \|_1 = 0$ implies $\lim_{n \to +\infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to +\infty} \| \alpha x_n \|_1 = 0$ for all $\alpha_n \downarrow_1^\infty 0$ and that $\lim_{n \to +\infty} \| x_n \|_1 = 0$ implies $\lim_{n \to +\infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ 

This yields

$$\| x + y \|_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$ 

10) For the convex modular $m$, we can define two kinds of norms such as

$$\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \| x \|_1$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\(\sim\)(A.5) in §1, then the formula (4.1) yields a quasi-norm \( \| \cdot \|_1 \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\| \dot{x} \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}
\]

(4.3)

is a semi-continuous quasi-norm on \( R^+_0 \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\]

(4.4)

for all \( x \in R^+_0 \).

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1)\sim(\rho.6) \) except \( (\rho.3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that \( |x_n - a_n| \leq a_n (n \geq 1) \) and \( a_n \downarrow_{n=1}^{\infty} 0 \). And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( \lim_{n \to \infty} z_n = a \). A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} \|a_n\| = 0 \) for any \( \{a_n\}_{n \geq 1} \) with \( \lim_{n \to \infty} a_n = 0 \). In the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \text{ for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_\nu \downarrow 0 \) and put \( [p_\nu^\epsilon] = [(a_\nu - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_\nu^\epsilon]a_1) + \rho^*(\xi \epsilon (1 - [p_\nu^\epsilon])a_1)
\]

for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_\nu^\epsilon] \downarrow 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_\nu^\epsilon]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_\nu^\epsilon]a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \rho^*(\xi a_n) = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if

(5.2) \( \rho^*(a_\nu) \rightarrow 0 \) implies \( \rho^*(\alpha a_\nu) \rightarrow 0 \) for every \( \alpha \geq 0 \).

From the definition, it is clear that s-lim \( x_\nu = 0 \) implies \( \lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} \| x_\nu \|_0 = 0 \) (or \( \lim_{\nu \rightarrow \infty} \| x_\nu \|_{1} = 0 \)) implies s-lim \( x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \))).

If we replace \( \lim_{\nu \rightarrow \infty} \| x_\nu \| = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this reason, we must consider the following condition:

(5.3) \( \rho^*(x) = 0 \) implies \( x = 0 \).

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \rightarrow 0 \) implies s-lim \( a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{y \in A} \rho^*(y) \) for any \( 0 \leq x \in A \). If

11) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf_{y \in A} \{ \sup_{y \in A} \rho^*(y) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_*(x) \rightarrow 0 \).
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of (\( \rho,3 \)).

Now, since
\[
\rho\left(\bigcup_{\nu \geq 1}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu \geq 1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq 1}^{\infty} |a_{\nu}|\right)\right) = 0 \) and hence (5.3) implies
\[
\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq 1}^{\infty} |a_{\nu}|\right) = 0 .
\]

Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \) (\( \nu = 1, 2, \cdots \)). Therefore we have \( s\text{-lim}_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( || \cdot ||_0 \) is complete and continuous, then (5.2) holds.

### References


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(Received September 30, 1960)