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HOKKAIDO UNIVERSITY
§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e., a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

1. \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);
2. \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);
3. If \( \sum_{\lambda \in \Lambda} \rho(x_\lambda) < +\infty \) for a mutually orthogonal system \( \{x_\lambda\}_{\lambda \in \Lambda} \), there exists \( x_0 \in R \) such that \( x_0 = \sum_{\lambda \in \Lambda} x_\lambda \) and \( \rho(x_0) = \sum_{\lambda \in \Lambda} \rho(x_\lambda) \);
4. \( \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular \( m \) on \( R \) for which every universally continuous linear functional is continuous with respect to the norm defined by the modular \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

1. \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);
2. \( \rho(-x) = \rho(x) \);
3. \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);
4. \( \alpha_n \to 0 \) implies \( \rho(\alpha_n x) \to 0 \) for every \( x \in R \);
5. for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by the formula:

\[ \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \] for all \( x \in R \).

---

1. \( x \perp y \) means \( |x| \cap |y| = 0 \).
2. A system of elements \( \{x_\lambda\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_\lambda \perp x_\gamma \) for \( \lambda \neq \gamma \).
3. For the definition of a modular, see [3].
4. A linear functional \( f \) is called universally continuous, if \( \inf_{a_\lambda \in \Lambda} f(a_\lambda) = 0 \) for any \( a_\lambda \in \Lambda \).
5. This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4]. In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
(1.1) \[ \| x \|_0 = \inf \left\{ \xi \mid \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho, 1) \sim (\rho, 4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x \mid x \in R, \; \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0 \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[(\rho, 4') \quad \sup_{x \in R} \{ \lim_{\alpha \to 0} \rho(\alpha x) \} < +\infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0 \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0 \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p] x = \bigcup_{n=1}^{\infty} (n | p | \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N = \{ p \} \), that is, \( x = [p] x + (1 - [p]) x \).

---

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) & \quad \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \quad \text{for any } [p_i]_{i \in A} = [p].
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\( (\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \ x, y \in R \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\]

We have

\[
x_0 = \sum_{i \in A} x_i \in R
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i)
\]

in virtue of \( (\rho.3) \).

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)
\]

\[
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils \( (\rho.3) \).

If \( \rho' \) does not fulfill \( (\rho.4) \), we have for some \( x_0 \in R \),

\[
\rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n \geq 1.
\]

By \( (\rho.2) \) and \( (\rho.4) \), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu}x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu}x_2) = +\infty \] (\( \nu = 1, 2, \ldots \))

and

\[ \rho'(\frac{1}{2}x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho'(\frac{1}{2}y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,...} \) such that

\[ \rho'(\frac{1}{\nu}y_\nu) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,...} \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]

and

\[ \rho'(\frac{1}{\nu}y_\nu) \geq \rho'(\frac{1}{\nu}y_\nu) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in \( \S 1 \):

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(|x| + |y|) \]
\[ \leq \rho(\alpha \lfloor x \rfloor + \alpha(1 - \lfloor x \rfloor) |y| + \beta \lfloor y \rfloor |x| + (1 - \lfloor y \rfloor) \beta |y|) \]
\[ = \rho(\lfloor x \rfloor |y| + (1 - \lfloor x \rfloor) \beta |y|) \]
\[ = \rho(\lfloor x \rfloor |y| + (1 - \lfloor x \rfloor) \beta |y|) \leq \rho(x) + \rho(y) \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[ (\rho.4') \quad \text{for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty. \]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \text{ and } (\rho.4')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \mes \{ t : x(t) = \frac{1}{i} \}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[ (\rho.6) \quad \lim_{t \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6) \) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ (2.5) \quad d(x) = \lim_{t \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
$d(x+y) = d(x) + d(y)$ if $x \perp y$.

Hence, putting

(2.6) \[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R) \]

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for $\rho^*$, since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all $x \in R$ and $\alpha > 0$.

We need to prove that (\rho.5) is true for $\rho^*$. First we have to note

(2.7) \[ \inf_{\lambda \in \Lambda} d(\lfloor p_{\lambda} \rfloor x) = 0 \]

for any $\lfloor p_{\lambda} \rfloor \downarrow_{\lambda \in \Lambda} 0$. In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d(\lfloor p_{\lambda} \rfloor x_0) \geq \alpha > 0 \]

for some $\lfloor p_{\lambda} \rfloor \downarrow_{\lambda \in \Lambda} 0$ and $x_0 \in R$.

Hence,

\[ \rho'\left(\frac{1}{\nu} \lfloor p_{\lambda} \rfloor x_0\right) \geq d(\lfloor p_{\lambda} \rfloor x_0) \geq \alpha \]

for all $\nu \geq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

\[ \lfloor p_{\lambda_n} \rfloor \downarrow \lfloor p_{\lambda_{n+1}} \rfloor \]

and

\[ \rho'\left(\frac{1}{n} \lfloor p_{\lambda_n} \rfloor x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m} \lfloor p_{\lambda_m} \rfloor x_0\right) = +\infty \]

for all $n \geq 1$ in virtue of (\rho.2) and (2.3). This implies

\[ \rho'\left(\frac{1}{n} x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m} \lfloor p_{\lambda_m} \rfloor x_0\right) = +\infty \]

which is inconsistent with (\rho.4). Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } \lfloor x \rfloor = \lfloor y \rfloor. \]

We put $\lfloor p_n \rfloor = \lfloor (|x| - n|y|)^+ \rfloor$ for $x, y \in R$ with $\lfloor x \rfloor = \lfloor y \rfloor$ and $n \geq 1$. Then, $\lfloor p_n \rfloor \downarrow_{n=1}^{\infty} 0$ and $\inf_{n=1, 2, \ldots} d(\lfloor p_n \rfloor x) = 0$ by (2.7). Since $(1 - \lfloor p_n \rfloor)n |y| \geq (1 - \lfloor p_n \rfloor) |x|$ and

\[ d(\alpha x) = d(x) \]

for $\alpha > 0$ and $x \in R$, we obtain
As \( n \) is arbitrary, this implies
\[
d(x) \leq \inf_{n=1,2,\ldots} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*(y) + \rho^*([x]-[y])x)
\]
\[
= \rho^*(y) - d(y) + \rho^*([x]-[y])x)
\]
\[
\geq \rho^*(y).
\]
Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

\[
(\rho.4') \quad \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K < +\infty.
\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[
(2.9) \quad \sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K' < +\infty,
\]
where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_\nu(x) = \rho(x) \) and \( n_\nu(x) = \rho'(\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in \mathbb{R} \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[
\rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma.
\]

In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.
\]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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$$\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z$$

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) for every \( i = 1, 2, \cdots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

$$\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}$$

$$\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}$$

$$\leq \frac{4K}{\epsilon} \gamma + 2K \left( \sup_{\rho(a) \leq \alpha_0} \rho(a) \right) + \gamma.$$ 

Hence, we obtain

$$\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \epsilon}{\epsilon}\right) \gamma + \left(\frac{4K^2}{\epsilon}\right).$$

Hence, in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_0^+ \)

$$\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.$$ 

Therefore, we obtain

$$\sup_{x \in R} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} \leq \gamma'$$

where

$$\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0.$$ 

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

$$\sum_{i=1}^{k} d(x_{i}) = d(\sum_{i=1}^{k} x_{i}) = \lim_{\xi \to 0} \rho'\left(\xi \sum_{i=1}^{k} x_{i}\right) \leq \gamma'$$

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

$$\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,$$

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by \((\rho.4)\) and \((2.7)\). Therefore \( \rho^* \) satisfies \((\rho.3)\).

On the other hand, suppose that \( \rho^* \) satisfies \((\rho.3)\) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

$$\sum_{i=1}^{\infty} d(x_i) = d\left(\sum_{i=1}^{\infty} x_i\right) \geq \mu.$$
for all \( \mu \geq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since 
\[
\lim_{\xi \to 0} \rho^*(\xi x) = 0,
\]
there exists \( \{\alpha_n\}_{n \geq 1} \) with \( 0 < \alpha_n \) (\( n \geq 1 \)) and \( \sum_{n=1}^{\infty} \rho^*(\alpha_n x_n) < +\infty \). It follows that 
\[
x_0 = \sum_{n=1}^{\infty} \alpha_n x_n \in R \quad \text{and} \quad d(x_0) = \sum_{n=1}^{\infty} d(\alpha_n x_n)
\]
from (\( \rho.3 \)).

\[\S 3.\] Quasi-norms. We denote by \( R_0 \) the set:
\[
R_0 = \{x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1\},
\]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_\lambda \) with \( R_0 \ni x_\lambda \geq 0 \) for all \( \lambda \in \Lambda \). Putting 
\[
[p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \quad \text{with} \quad 0 \leq x_\lambda \leq nx,
\]
we have
\[
[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx,
\]
which implies \( \rho^*(n[p_{n,\lambda}]x) = 0 \) and \( \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0 \). Hence, we obtain \( x \in R_0 \), that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp. \quad \text{\( \S 3.\) Q.E.D.}
\]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0 \), \([p]R_0 \) is universally complete, i.e. for any orthogonal system \( \{x_\iota \}_{\iota \in \Lambda}(x_\iota \in [p]R_0) \), there exists \( x_0 = \sum_{\iota \in \Lambda} x_\iota \in [p]R \). Hence we can also verify without difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \) is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda)^{\rho^*} \)-space. With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration. Now we can state the theorems which we aim at.

\(^7\) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, |b| \leq |a|, b \in R \) implies \( b \in S \). Since \( R \) is univerfally continuous, a semi-normal manifold \( S \) is normal if and only if \( \bigcup_{\lambda \in \Lambda} x_\lambda \in R \), \( 0 \leq x_\lambda \in S(\lambda \in \Lambda) \) implies \( \bigcup_{\lambda \in \Lambda} x_\lambda \in S \).

\(^8\) This means that \( x \in R \) is written by \( x = y + z \), \( y \in R_0 \) and \( z \in R_0^\perp \).

\(^9\) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^+_0$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e.

$$\sup_{x_i \in A} ||x_i||_0 = ||x||_0$$

for any $0 \leq x_i \leq x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1)$~$(\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad ||x||_0 = \inf \{\xi; \rho^*(\frac{1}{\xi}x) \leq \xi\}.$$  

Then,

i) $0 \leq ||x||_0 = \| -x \|_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R^+_0$.

ii) $||x + y||_0 \leq ||x||_0 + ||y||_0$ for any $x, y \in R$; follows also from (A.3).

iii) $\lim_{\alpha \to 0} ||\alpha x||_0 = 0$ and $\lim_{\alpha \to \alpha_0} ||\alpha x||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to \alpha_0} ||\alpha x||_0 = ||\alpha_0 x||_0$ for all $x \in R^+_0$ and $\alpha_0 \geq 0$. If $x \in R^+_0$ and $[p]_{i \in A}[p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^*(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{i \in A} \rho^*(\frac{1}{\xi}[p_i]x) > \xi$ and hence $\sup_{i \in A} ||p_i||_0 \geq \xi$. Thus we obtain

$$\sup_{i \in A} ||p_i||_0 = ||[p]x||_0$$

if $[p] \uparrow_{i \in A}[p]$. Let $0 \leq x_i \leq x$. Putting

$$[p_{n,i}] = (\frac{1}{n}x)^*$$

we have

$$[p_{n,i}] \uparrow_{i \in A}[x]$$

As is shown above, since

$$\sup_{i \in A} ||[p_{n,i}]x||_0 \geq \sup_{i \in A} \| [p_{n,i}] \left(1 - \frac{1}{n}\right)x \|_0 = \| (1 - \frac{1}{n})x \|_0,$$

we have

$$\sup_{i \in A} ||x_i||_0 \geq \| (1 - \frac{1}{n})x \|_0$$

and also $\sup_{i \in A} ||x_i||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $||\cdot||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim ||x_i||_0 = 0$ if and only if $\lim \rho(\xi x_i) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm $|| \cdot ||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0$ $(n, \nu = 1, 2, \cdots)$ be the elements of $R_{0}^\perp$ such that

(3.2) $[p_{n,\nu}] \uparrow_{\nu=1}^{\infty} [p_n]$ with $\prod_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0$;

(3.3) $[p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]a$ for all $n, \nu \geq 1$.

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_{0}^\perp$ with respect to $|| \cdot ||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

(3.4) $||[q_m]a||_0 > \frac{\delta}{2}$ and $[q_m]x_{\nu_m} \geq n_m[q_m]a$ $(m = 1, 2, \cdots)$ and

(3.5) $n_m[q_m]a \geq [q_m]x_{\nu_m}$, $n_{m+1} > n_m$ $(m = 2, 3, \cdots)$, where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $|| \cdot ||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$||[p_{1,\nu_1}][p_0]a||_0 > \frac{\delta}{2}$.

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n_a-x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$||(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}$.

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$||[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}$.

in virtue of (3.2) and semi-continuity of $|| \cdot ||_0$. Hence we can put

$[q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+[q_k]]$, because

$[q_{k+1}] \leq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a$

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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\[ ||x_{\nu+1} - x_{\nu-1}||_0 \geq ||[q_{\nu+1}](x_{\nu+1} - x_{\nu-1})||_0 \]
\[ \geq ||n_{\nu+1}[q_{\nu+1}a - n_{\nu}[q_{\nu+1}a])||_0 \geq ||[q_{\nu+1}a_0||_0 \geq \frac{\delta}{2}, \]
since \([q_{\nu+1}] \leq [q_{\nu}] \leq [(n_{\nu}a - x_{\nu-1})^+]\) implies \([q_{\nu+1}]n_{\nu}a \geq [q_{\nu+1}]x_{\nu-1}\) by (3.4). It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R_0^+\) is an F-space with \(||\cdot||_0\) if and only if \(\rho\) satisfies (\(\rho.4'\)).

**Proof.** If \(\rho\) satisfies (\(\rho.4'\)), \(\rho^*\) is a quasi-modular which fulfills also (\(\rho.5\)) and (\(\rho.6\)) in virtue of Theorem 2.3. Since \(\rho^*\) satisfies (\(\rho.3\)), there exists \(0 \leq x_0 \in R_0^+\) such that \(x_0 = \bigcup_{\nu=1}^{\infty}x_\nu\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,\nu}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{\nu=1}^{\infty}[p_{n,\nu}] = [p_n]\), we obtain
\[ (3.6) \]
\[ [p_{n,\nu}]x_\nu \geq n[p_{n,\nu}]x_0 \]
for all \(n, \nu \geq 1\) and \([p_n] \downarrow_{n=1}^{\infty} 0\). Since \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty}[p_n] = 0\), that is, \(\bigcup_{n=1}^{\infty}([x_\nu] - [p_n]) = [x_0]\). And
\[ (1 - [p_{n,\nu}])(1 - [p_n]) \geq (n, \nu \geq 1) \]
implies
\[ n(1 - [p_n])x_0 \geq (1 - [p_n])x_\nu \geq 0. \]
Hence we have
\[ y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n])x_\nu \in R_0^+, \]
because \(R_0^+\) is universally continuous. As \(\{z_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(||\cdot||_0\)
\[ \gamma = \sup_{\nu \geq 1} ||x_\nu||_0 < +\infty, \]
which implies
\[ ||y_n||_0 = \sup_{\nu \geq 1} ||(1 - [p_n])x_\nu||_0 < \gamma \]
for every \(n \geq 1\) by semi-continuity of \(||\cdot||_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geq 1}\) is an orthogonal sequence with \(||\sum_{\nu=1}^{n}z_\nu||_0 = ||y_n||_0 \leq \gamma\). This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z=\sum_{\nu=1}^{\infty}z_{\nu} = \bigcup_{\nu=1}^{\infty}y_{\nu}$. This yields $z= \bigcup_{\nu=1}^{\infty}x_{\nu}$. Truly, it follows from

$$
\sum_{\nu=1}^{n} \rho^{*}\left(\frac{z_{\nu}}{1+\gamma}\right) = \rho^{*}\left(\frac{y_{n}}{1+\gamma}\right) \leq \gamma
$$

for all $n \geq 1$ by the formula (3.1).

Then $(\rho.3)$ assures the existence of $z=\sum_{\nu=1}^{\infty}z_{\nu} = \bigcup_{\nu=1}^{\infty}y_{\nu}$.

This yields $z= \bigcup_{\nu=1}^{\infty}x_{\nu}$.

Truly, it follows from $z= \bigcup_{\nu=1}^{\infty}y_{\nu} = \bigcup_{\nu=1}^{\infty}\bigcup_{\nu=1}^{\infty}(1-[p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty}\bigcup_{\nu=1}^{\infty}(1-[p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty}x_{\nu}$.

By semi-continuity of $||\cdot||_{0}$, we have

$$
||z-x_{\nu}||_{0} \leq \sup_{\mu \geq \nu} ||x_{\mu}-x_{\nu}||_{0}
$$

and furthermore $\lim_{n \to \infty } ||z-x_{\nu}||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{\perp}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$
||y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{\nu}}
$$

for all $\nu \geq 1$.

This implies

$$
||\sum_{\nu=1}^{n} (y_{\nu+1}-y_{\nu})||_{0} \leq \sum_{\nu=1}^{n} ||y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{n-1}}
$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1}-y_{\nu}|$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \uparrow_{n \to \infty} \infty$.

Then by the fact proved just above,

$$
z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \bigcup_{n=1}^{\infty} |y_{\nu+1}-y_{\nu}| \in R_{0}^{\perp} \quad \text{and} \quad \lim_{n \to \infty} ||z_{0}-z_{n}||_{0} = 0.
$$

Since $\sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}|$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})$ is also convergent and

$$
||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_{n}||_{0} = ||\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})||_{0} \leq ||z_{0}-z_{n}||_{0} \to 0.
$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$
\lim_{\nu \to \infty} ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_{\nu}||_{0} = 0.
$$

Therefore $||\cdot||_{0}$ is complete in $R_{0}^{\perp}$, that is, $R_{0}^{\perp}$ is an F-space with $||\cdot||_{0}$.

Conversely if $R_{0}^{\perp}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{\perp}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu}x_{\nu} \in R_{0}^{\perp}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0}^{\perp} = 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills (3.4').

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
\| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}
\end{equation}

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

\begin{equation}
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\end{equation}

hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \rightarrow \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \rightarrow \infty} \| x_n \|_1 = 0$ implies $\lim_{n \rightarrow \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \rightarrow \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ \| x + y \| \leq \frac{\xi \eta}{\xi + \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x+y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \]

\[ \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi}. \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|_1$ respectively.
This yields (4.2), since we have $||x||_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\gamma}$ for every $\gamma$ with $||x||_0 > \frac{1}{\gamma}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1) \sim (A.5) in §1, then the formula (4.1) yields a quasi-norm $|| \cdot ||_1$ on $L$ which is equivalent to $|| \cdot ||_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$||x||_1 = \inf_{\nu \to 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} (x \in R)$$

is a semi-continuous quasi-norm on $R_0^\perp$ and $|| \cdot ||_1$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$||x||_0 \leq ||x||_1 \leq 2 ||x||_0$$

for all $x \in R_0^\perp$.

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$ and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called order-convergent to $a$ and denoted by $o-lim_{\nu \to \infty} x_\nu = a$, if there exists a sequence of elements $\{a_\nu\}_{\nu \geq 1}$ such that $|x_\nu - a_\nu| \leq a_\nu (\nu \geq 1)$ and $a_\nu \downarrow_{\nu=1}^{\infty} 0$. And a sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called star-convergent to $a$ and denoted by $s-lim_{\nu \to \infty} x_\nu = a$, if for any subsequence $\{y_\nu\}_{\nu \geq 1}$ of $\{x_\nu\}_{\nu \geq 1}$, there exists a subsequence $\{z_\nu\}_{\nu \geq 1}$ of $\{y_\nu\}_{\nu \geq 1}$ with $o-lim_{\nu \to \infty} z_\nu = a$. A quasi-norm $|| \cdot ||$ on $R$ is termed to be continuous, if $\inf_{\nu \to 0} ||a_\nu|| = 0$ for any $a_\nu \downarrow_{\nu=1}^{\infty} 0$. In the sequel, we write by $|| \cdot ||_0$ (or $|| \cdot ||_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $|| \cdot ||_0$ (or $|| \cdot ||_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

$$||x||_0 \leq ||x||_1 \leq 2 ||x||_0$$

for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z]R$ is finite dimensional and $\rho(y) < +\infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector $\{[p_n]\}_{n \geq 1}$ such that $\rho([p_n]x) = +\infty$ and $[p_n] \downarrow_{n=1}^{\infty} 0$. Hence by (3.1) it follows that $\| [p_n]x \|_0 > 1$ for all $n \geq 1$, which contradicts the continuity of $\| \cdot \|_0$.

**Sufficiency.** Let $a_\nu \downarrow_{\nu=1}^{\infty} 0$ and put $[p_n^\varepsilon] = [(a_n - \varepsilon a_1)^+]$ for any $\varepsilon > 0$ and $n \geq 1$. It is easily seen that $[p_n^\varepsilon] \downarrow_{n=1}^{\infty} 0$ for any $\varepsilon > 0$ and $a_n = [a_1]a_n = [p_n^\varepsilon]a_n + (1 - [p_n^\varepsilon])a_n \leq [p_n^\varepsilon]a_1 + \varepsilon a_1$.

This implies

$$\rho^*(\xi a_n) \leq \rho^*([\xi [p_n^\varepsilon]a_1]) + \rho^*(\xi(1 - [p_n^\varepsilon])a_1)$$

for all $n \geq 1$ and $\xi \geq 0$. In virtue of (5.1) and $[p_n^\varepsilon] \downarrow_{n=1}^{\infty} 0$, we can find $n_0$ (depending on $\xi$ and $\varepsilon$) such that $\rho^*([\xi [p_n^\varepsilon]a_1]) < +\infty$, and hence $\liminf_{n \geq 1} \rho^*([\xi [p_n^\varepsilon]a_1]) = 0$ by (2.3) in Lemma 1 and (p.2). Thus we obtain

$$\inf_{n \geq 1} \rho^*([\xi a_n]) \leq \rho^*(\xi a_1).$$

Since $\varepsilon$ is arbitrary, $\lim_{n \to \infty} \rho^*([\xi a_n] = 0$ follows. Hence we infer that $\inf_{n \geq 1} \| a_n \|_0 = 0$ and $\| \cdot \|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** $\| \cdot \|_0$ is continuous, if

(5.2) $\rho^*(a_\nu) \to 0$ implies $\rho^*(\alpha a_\nu) \to 0$ for every $\alpha \geq 0$.

From the definition, it is clear that $s-\lim x_\nu = 0$ implies $\lim_{\nu \to \infty} \| x_\nu \| = 0$, if $\| \cdot \|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** $\lim_{\nu \to \infty} \| x_\nu \| = 0$ (or $\lim_{\nu \to \infty} \| x_\nu \| = 0$) implies $s-\lim x_\nu = 0$, if $\| \cdot \|_0$ is complete (i.e. $\rho^*$ satisfies $(p.3)$).

If we replace $\lim_{\nu \to \infty} \| x_\nu \| = 0$ by $\lim_{\nu \to \infty} \rho(x_\nu) = 0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) $\rho^*(x) = 0$ implies $x = 0$.

Truly we obtain

**Theorem 5.3.** If $\rho^*$ satisfies (5.3) and $\| \cdot \|_0$ is complete, $\rho(a_\nu) \to 0$ implies $s-\lim a_\nu = 0$.

**Proof.** We may suppose without loss of generality that $\rho^*$ is semi-continuous, i.e. $\rho^*(x) = \sup_{y \in \mathcal{A}} \rho^*(y)$ for any $0 \leq x \uparrow_{\mathcal{A}}$. If

11) If $\rho^*$ is not semi-continuous, putting $\rho_*(x) = \inf_{y \uparrow_{\mathcal{A}} x} \{\sup_{y \in \mathcal{A}} \rho^*(y)\}$, we obtain a quasi-modular $\rho_*$ which is semi-continuous and $\rho^*(x) \to 0$ is equivalent to $\rho_*(x_\nu) \to 0$. 

\[ \rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^\infty |a_\nu| \in R \) in virtue of \( (\rho, 3) \).

Now, since
\[
\rho\left( \bigcup_{\nu \in \mathbb{N}} |a_\nu| \right) \leq \sum_{\nu \in \mathbb{N}} \rho(a_\nu) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^\infty \left( \bigcup_{\mu \geq \nu} |a_\mu| \right) \right) = 0 \) and hence (5.3) implies
\[
\bigcap_{\nu=1}^\infty \left( \bigcup_{\mu \geq \nu} |a_\mu| \right) = 0.
\]
Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \quad (\nu = 1, 2, \ldots) \). Therefore we have \( s\text{-lim}_{\nu \to \infty} b_\nu = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete and continuous, then (5.2) holds.

**References**


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