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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shōzō KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

1. \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);
2. \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);
3. If \( \sum_{i \in A} \rho(x_i) < +\infty \) for a mutually orthogonal system \( \{x_i\}_{i \in A} \), there exists \( x_0 \in R \) such that \( x_0 = \sum_{i \in A} x_i \) and \( \rho(x_0) = \sum_{i \in A} \rho(x_i) \);
4. \( \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular \( m \) on \( R \) for which every universally continuous linear functional is continuous with respect to the norm defined by the modular in [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

1. \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);
2. \( \rho(-x) = \rho(x) \);
3. \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);
4. \( \alpha_n \to 0 \) implies \( \rho(\alpha_n x) \to 0 \) for every \( x \in R \);
5. for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by the formula:

1) \( x \perp y \) means \( |x| \cap |y| = 0 \).
2) A system of elements \( \{x_i\}_{i \in A} \) is called mutually orthogonal, if \( x_i \perp x_j \) for \( i \neq j \).
3) For the definition of a modular, see [3].
4) A linear functional \( f \) is called universally continuous, if \( \inf_{a \in A} f(a) = 0 \) for any \( a \in A \).
5) This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4].
In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
(1.1) \[ \| x \|_0 = \inf \left\{ \tilde{\xi} ; \rho \left( \frac{1}{\tilde{\xi}} x \right) \leq \xi \right\} \]
and \( \| x_n \|_0 \rightarrow 0 \) is equivalent to \( \rho(\alpha x_n) \rightarrow 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \((\rho.1)\sim(\rho.4)\) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\sim(A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0=\{x:x \in R, \rho^*(\xi x)=0 \text{ for all } \xi \geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^\perp \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^\perp \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^\perp \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^\perp \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \),
\[ [p] \text{ is a projector: } [p]x = \bigcup_{n=1}^{\infty} (n \cdot p \cap x) \text{ for all } x \geq 0 \text{ and } 1-[p] \text{ is a projection operator onto the normal manifold } N=\{p\}^\perp, \text{ that is, } x=[p]x+(1-[p])x. \]

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§ 2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho(\lfloor p \rfloor x) \leq \rho(x) \text{ for all } p, x \in \mathbb{R}; \\
(2.3) & \quad \rho(\lfloor p \rfloor x) = \sup_{i \in A} \rho(\lfloor p_i \rfloor x) \text{ for any } [p_i]_{i \in A}. 
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\( \rho(x) \leq \rho(y) \) if \( |x| \leq |y| \), \( x, y \in \mathbb{R} \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( \mathbb{R} \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x \in \mathbb{R} \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in \mathbb{R}.
\]

We have

\[
x_0 = \sum_{i \in A} x_i \in \mathbb{R}
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of } (\rho.3).
\]

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sum_{i \in A} \rho(\lfloor x_i \rfloor y) = \sum_{i \in A} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in \mathbb{R} \),

\[
\rho' \left( \frac{1}{n} x_0 \right) = +\infty \quad \text{for all } n \geq 1.
\]

By \((\rho.2)\) and \((\rho.4)\), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu} x_2) = +\infty \quad (\nu = 1, 2, \ldots) \]

and

\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho'(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that

\[ \rho'(\frac{1}{\nu} y_\nu) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]

and

\[ \rho'(\frac{1}{\nu} y_0) \geq \rho'(\frac{1}{\nu} y_\nu) \geq \nu, \]

which contradicts \( (\rho.4) \). Therefore \( \rho' \) has to satisfy \( (\rho.4) \). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \( (\rho.5) \), \( \rho \) does also \( (A.3) \) in §1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [(|x| - |y|)^+] \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \]
\[ \leq \rho(\alpha \lceil p \rceil |x| + \alpha(1 - \lceil p \rceil) |y| + \beta \lceil p \rceil |x| + (1 - \lceil p \rceil) \beta |y|) \]
\[ = \rho(\lceil p \rceil |x| + (1 - \lceil p \rceil) |y|) \]
\[ = \rho(\lceil p \rceil x) + \rho((1 - \lceil p \rceil)y) \]
\[ \leq \rho(x) + \rho(y). \]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4'') \quad \text{for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty. \]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \{ t : x(t) = \frac{1}{i} \}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6) \quad \lim_{t \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ (2.5) \quad d(x) = \lim_{t \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
$d(x+y) = d(x)+d(y)$ if $x \perp y$.

Hence, putting

\begin{equation}
\rho^*(x) = \rho'(x) - d(x) \quad (x \in R).
\end{equation}

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for $\rho^*$, since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all $x \in R$ and $\alpha > 0$.

We need to prove that (\rho.5) is true for $\rho^*$. First we have to note

\begin{equation}
\inf_{\lambda \in \Lambda} d([p_\lambda]x) = 0
\end{equation}

for any $[p_\lambda] \downarrow_{\lambda \in A} 0$. In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some $[p_\lambda] \downarrow_{\lambda \in A} 0$ and $x_0 \in R$.

Hence,

\[ \rho'(\frac{1}{\nu} [p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all $\nu \geq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'\left(\frac{1}{n} [p_{\lambda_n}] - [p_{\lambda_{n+1}}]x_0\right) \geq \frac{\alpha}{2} \]

for all $n \geq 1$ in virtue of (\rho.2) and (2.3). This implies

\[ \rho'\left(\frac{1}{n} x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m} [p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0\right) = +\infty, \]

which is inconsistent with (\rho.4). Secondly we shall prove

\begin{equation}
(2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y].
\end{equation}

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n=1}^\infty 0$ and $\inf_{n=1,2,\ldots} d([p_n]x) = 0$ by (2.7). Since $(1-[p_n])n |y| \geq (1-[p_n])|x|$ and

\[ d(\alpha x) = d(x) \]

for $\alpha > 0$ and $x \in R$, we obtain
\[ d(x) = d([p_{n}]x) + d((1 - [p_{n}])x) \leqq d([p_{n}]x) + d(n(1 - [p_{n}])y) \leqq d([p_{n}]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) \leqq \inf_{n=1,2,...} d([p_{n}]x) + d(y), \]
and also \( d(x) \leqq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geqq |y| \), then
\[ \rho^{*}(x) = \rho^{*}([y]x) + \rho^{*}(([x] - [y])x) = \rho^{*}([y]x) - d([y]x) + \rho^{*}(([x] - [y])x) \geqq \rho^{*}(y) - d(y) + \rho^{*}(([x] - [y])x) \geqq \rho^{*}(y). \]

Thus \( \rho^{*} \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^{*} \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^{*} \) is a quasi-modular), if and only if \( \rho \) satisfies

\[(\rho.4') \quad \sup_{x \in K} \{ \lim_{\xi \to 0} \rho(\xi x) \} = K < +\infty.\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[(2.9) \quad \sup_{x \in K} d(x) = \sup_{x \in K} \{ \lim_{\xi \to 0} \rho^{'}(\xi x) \} = K' < +\infty,\]
where
\[ \rho^{'}(x) = \sup_{0 \leqq \rho(y) \leqq |x|} \rho(y). \]

Since \( \rho^{'} \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_{0}(x) = \rho(x) \) and \( n_{\nu}(x) = \rho^{'}(\frac{1}{\nu}x) \) for \( \nu \geqq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_{0} \) and a finite dimensional normal manifold \( N_{0} \) such that \( x \in N_{0}^{\perp} \) with
\[ \rho(x) \leqq \epsilon \text{ implies } \rho^{'}(\frac{1}{\nu_{0}}x) \leqq \gamma. \]

In \( N_{0} \), we have obviously
\[ \sup_{x \in N_{0}} \{ \lim_{\xi \to 0} \rho^{'}(\xi x) \} = \gamma_{0} < +\infty. \]

If \( \epsilon \leqq 2K \), for any \( x_{0} \in N_{0}^{\perp} \), we can find \( \alpha_{0} > 0 \) such that \( \rho(\alpha x_{0}) \leqq 2K \) for all \( 0 \leqq \alpha \leqq \alpha_{0} \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) \((i=1, 2, \ldots, n)\), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j=1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\varepsilon \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\varepsilon \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma',
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_\lambda\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^*(x_\lambda) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_{\lambda}) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho^*(x_{\lambda}) = \sum_{\lambda \in A} \rho^*(x_{\lambda}) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_{\lambda} \in R \) and \( \sum_{\lambda \in A} \rho^*(x_{\lambda}) = \rho^*(x_0) \) by (\( \rho.4 \)) and (\( 2.7 \)). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all \( \mu \geq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since
\[
\lim_{t \to 0} \rho^*(\xi x) = 0,
\]
there exists \( \{\alpha_\nu\}_{\nu \geq 1} \) with \( 0 < \alpha_\nu \) (\( \nu \geq 1 \)) and
\[
\sum_{\nu=1}^\infty \rho^*(\alpha_\nu x_\nu) < +\infty.
\]
It follows that \( x_0 = \sum_{\nu=1}^\infty \alpha_\nu x_\nu \in R \) and \( d(x_0) = \sum_{\nu=1}^\infty d(\alpha_\nu x_\nu) \) from (\( \rho.3 \)). For such \( x_0 \), we have for every \( \xi \geq 0 \),
\[
\rho'(\xi x_0) = \sum_{\nu=1}^\infty \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^\infty d(x_\nu) = +\infty,
\]
which is inconsistent with (\( \rho.4 \)). Therefore we have
\[
\sup_{x \in R} (\lim_{\xi \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.
\]
Q.E.D.

\section{Quasi-norms.} We denote by \( R_0 \) the set:
\[
R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geq 1 \},
\]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_\lambda \) with \( R_0 \ni x_\lambda \geq 0 \) for all \( \lambda \in \Lambda \).

Putting \( [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \), we have \( [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \) and \( 2n[p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}] nx_1 \) which implies \( \rho^*(n[p_{n,\lambda}] x) = 0 \) and \( \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}] x) = \rho^*(nx) = 0 \). Hence, we obtain \( x \in R_0 \), that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp.\quad\text{(8)}
\]
In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0, \ [p]R_0 \) is universally complete, i.e. for any orthogonal system \( \{x_\lambda\}_{\lambda \in \Lambda}(x_\lambda \in [p]R_0) \), there exists \( x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R \). Hence we can also verify without difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \) is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda)^9\)-space. With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, \ |b| \leq |a|, b \in R \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \bigcup_{\lambda \in \Lambda} x_\lambda \in R \) implies \( \bigcup_{\lambda \in \Lambda} x_\lambda \in S(\Lambda) \).

8) This means that \( x \in R \) is written by \( x = y + z, \ y \in R_0 \) and \( z \in R_0^\perp \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e.
$$\sup_{\lambda\in\Lambda} ||x_\lambda||_0 = ||x||_0$$
for any $0 \leq x_\lambda \uparrow_{\lambda\in\Lambda} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad ||x||_0 = \inf\left\{ \xi ; \rho^*\left(\frac{1}{\xi}x\right) \leq \xi \right\}.$$ 

Then,

i) $0 \leq ||x||_0 = ||-x||_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1),(\rho.6), (2.1)$ and the definition of $R_0^\perp$.

ii) $||x+y||_0 \leq ||x||_0 + ||y||_0$ for any $x,y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{n\to 0} ||\alpha_n x||_0 = 0$ and $\lim_{n\to 0} ||\alpha x_n||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_0$ is semi-continuous. From ii) and iii), it follows that $\lim ||\alpha x||_0 = ||\alpha x||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_1] \uparrow_{\lambda\in\Lambda} [p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^*\left(\frac{1}{\xi}[p]x\right) > \xi$, which implies $\sup_{\lambda\in\Lambda} \rho^*\left(\frac{1}{\xi}[p_\lambda]x\right) > \xi$ and hence $\sup_{\lambda\in\Lambda} ||p_\lambda x||_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda\in\Lambda} ||p_\lambda x||_0 = ||[p]x||_0,$$

if $[p_1] \uparrow_{\lambda\in\Lambda} [p]$.

Let $0 \leq x_\lambda \uparrow_{\lambda\in\Lambda} x$. Putting

$$[p_{n,\lambda}] = \left[ (x_\lambda - (1-\frac{1}{n})x)^* \right]$$

we have

$$[p_{n,\lambda}] \uparrow_{\lambda\in\Lambda} [x] \text{ and } [p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}](1-\frac{1}{n})x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda\in\Lambda} ||[p_{n,\lambda}] x_\lambda||_0 \leq \sup_{\lambda\in\Lambda} ||[p_{n,\lambda}](1-\frac{1}{n})x||_0 = \left(1-\frac{1}{n}\right)x_0,$$

we have

$$\sup_{\lambda\in\Lambda} ||x_\lambda||_0 \leq \left(1-\frac{1}{n}\right)x_0$$

and also $\sup_{\lambda\in\Lambda} ||x_\lambda||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $||\cdot||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim ||x_n||_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

On F-Norms of Quasi-Modular Spaces
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^+$ such that

(3.2) \[ [p_{n, \nu}] \uparrow_{\nu=1}^{\infty} [p_n] \quad \text{with} \quad \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0; \]

(3.3) \[ [p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \quad \text{for all} \quad n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^+$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

(3.4) \[ \| [q_m] a \|_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \]

and

(3.5) \[ n_m [q_m] a \geq [q_m] x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots), \]

where $\delta = \| [p_0] a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ \| [p_{1, \nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1, \nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k] a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ \| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ \| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
On F-Norms of Quasi-Modular Spaces

\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \]
\[ \geq \| n_{k+1} [q_{k+1}] a - n_k [q_{k+1}] a \|_0 \geq \| [q_{k+1}] a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_k] \leq (n_k a - x_{\nu-1})^+\) implies \([q_{k+1}] n_k a \geq [q_{k+1}] x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

\textbf{Theorem 3.2.} Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R^1_0\) is an F-space with \(\| \cdot \|_0\) if and only if \(\rho\) satisfies (\(\rho.4'\)).

\textbf{Proof.} If \(\rho\) satisfies (\(\rho.4'\)), \(\rho^*\) is a quasi-modular which fulfills also (\(\rho.5\)) and (\(\rho.6\)) in virtue of Theorem 2.3. Since \(\| x \|_0 = \inf \\left\{ \xi ; \rho^* \left( \frac{x}{\xi} \right) \leq \xi \right\}\)

is a quasi-norm on \(R^1_0\), we need only to verify completeness of \(\| \cdot \|_0\). At first let \(\{x_{\nu}\}_{\nu \geq 1} \subset R^1_0\) be a Cauchy sequence with \(0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots}\). Since \(\rho^*\) satisfies (\(\rho.3\)), there exists \(0 \leq x_0 \in R^1_0\) such that \(x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}\), as is shown in the proof of Theorem 2.3.

Putting \([p_n, \nu] = [(x_{\nu} - nx_0)^+\] and \(\bigcup_{\nu=1}^{\infty} [p_n, \nu] = [p_n]\), we obtain

\[ (3.6) \quad [p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] x_0 \]
\[ \text{for all } n, \nu \geq 1 \]

and \([p_n]_{\nu=1}^{\infty} = 0\). Since \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty} \{[p_n]_{\nu=1}^{\infty} = 0\}, \) that is, \(\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]\). And

\[ (1 - [p_{n, \nu}]) \geq (1 - [p_n]) \]
\[ \text{(n, } \nu \geq 1) \]

implies

\[ n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_0 \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_{\nu} \in R^1_0, \]

because \(R^1_0\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n]) x_{\nu} \|_0 \leq \gamma, \]

for every \(n \geq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2\)). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma\). This implies
\[ \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leqq \gamma \]

for all \( n \geqq 1 \) by the formula (3.1). Then \( (\rho.3) \) assures the existence of \( z = \sum_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\[ z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_{\nu}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} x_{\nu} \]

By semi-continuity of \( ||\cdot||_0 \), we have

\[ \lim_{\nu \to \infty} ||z-x_{\nu}||_0 = 0 \]

and furthermore \( \lim_{\nu \to \infty} ||z-x_{\nu}||_0 = 0 \).

Secondly let \( \{x_{\nu}\}_{\nu \geqq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geqq 1} \) of \( \{x_{\nu}\}_{\nu \geqq 1} \) such that

\[ \forall \nu \geqq 1 \quad ||y_{\nu+1} - y_{\nu}||_0 \leqq \frac{1}{2^{\nu-1}} \]

Putting \( z_n = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}| \), we have a Cauchy sequence \( \{z_n\}_{n \geqq 1} \) with \( 0 \leqq z_n \leqq z_n \leqq \infty \).

Then by the fact proved just above,

\[ z_0 = \bigcup_{n=1}^{\infty} z_n = \bigcup_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_0^\perp \]

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and

\[ ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n||_0 = ||\sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})||_0 \leqq ||z_0 - z_n||_0 \to 0 \]

Since \( \{y_{\nu}\}_{\nu \geqq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geqq 1} \), it follows that

\[ \lim_{\nu \to \infty} ||y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu}||_0 = 0 \]

Therefore \( ||\cdot||_0 \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( ||\cdot||_0 \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geqq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geqq 1 \)). Hence we can see that \( \sup_{x \in R_0^\perp} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \( (\rho.4') \). Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfills ($\rho.A'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz ($\S1$). Here we put for $x \in L$

(4.1) \[ \|x\|_1 = \inf_{\xi > 0} \left( \frac{1}{\xi} + \rho(\xi x) \right) \]

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

(4.2) \[ \|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \]

for all $x \in L$ hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \|x\|_1 = \|\bar{x}\|_1 < +\infty$ ($x \in L$) and that $\|x\|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim \|x_n\|_1 = 0$ implies $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim \|\alpha_n x\|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^{\infty} 0$ and that $\lim \|x_n\|_1 = 0$ implies $\lim \|ax_n\|_1 = 0$ for all $\alpha > 0$. If $\|x\|_1 < \alpha$ and $\|y\|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ \|x+y\|_1 \leq \frac{\xi+\eta}{\xi\eta} + \rho(\frac{\xi\eta}{\xi+\eta} (x+y)) = \frac{1}{\xi} + \frac{1}{\eta} + \rho(\frac{\eta}{\xi+\eta} (\xi x) + \frac{\xi}{\xi+\eta} (\eta y)) \]

\[ \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{m(\xi x) \geq 1} \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|_1$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have $||x||_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $||x||_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)$\sim$(A.5) in §1, then the formula (4.1) yields a quasi-norm $||\cdot||_1$ on $L$ which is equivalent to $||\cdot||_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$||x||_1 = \inf_{\xi \rightarrow \infty} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}$$

is a semi-continuous quasi-norm on $R^+_0$ and $||\cdot||_1$ is complete if and only if $\rho$ satisfies (\rho.4'), where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$||x||_0 \leq ||x||_1 \leq 2||x||_0$$

for all $x \in R^+_0$.

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies (\rho.1)$\sim$(\rho.6) except (\rho.3) and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \neq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $\nu \rightarrow \infty x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a_n| \leq a_n$ ($n \geq 1$) and $a_n \downarrow 0$. And a sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $s-\lim_{n \rightarrow \infty} x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $0-\lim_{n \rightarrow \infty} z_n = a$.

A quasi-norm $||\cdot||$ on $R$ is termed to be continuous, if $\inf_{n \geq 1} ||a_n|| = 0$ for any $a_n \downarrow 0$. In the sequel, we write by $||\cdot||_0$ (or $||\cdot||_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $||\cdot||_0$ (or $||\cdot||_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z]_R$ is finite dimensional and $\rho(y) < + \infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_n 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_\nu \downarrow_{\nu=1}^\infty 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^\infty 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1 \).

This implies \( \rho^*([\xi a_n]) \leq \rho^*([\xi [p_n^\epsilon]a_1]) + \rho^*([\xi (1 - [p_n^\epsilon])a_1]) \) for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^\infty 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*([\xi [p_n^\epsilon]a_1]) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*([\xi a_n]) \leq \rho^*([\xi \epsilon a_1]) \).

Since \( \epsilon \) is arbitrary, \( \lim_{n \to \infty} \rho^*([\xi a_n]) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} [a_n]_{0} = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if

\[
(5.2) \quad \rho^*(a_\nu) \to 0 \implies \rho^*(\alpha a_\nu) \to 0 \quad \text{for every } \alpha \geq 0.
\]

From the definition, it is clear that \( s-\lim_{\nu \to \infty} x_\nu = 0 \) implies \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \) (or \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \)) implies \( s-\lim_{\nu \to \infty} x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (p.3)).

If we replace \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \) by \( \lim_{\nu \to \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x) = 0 \implies x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \to 0 \) implies \( s-\lim_{\nu \to \infty} a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{\alpha \in A} \rho^*(x_\alpha) \) for any \( 0 \leq x \downarrow_{\alpha \in A} \). If

11) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf \{ \sup_{\alpha \in A} \rho^*(y_\alpha) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x_\nu) \to 0 \) is equivalent to \( \rho_*(x_\nu) \to 0 \).
\[ \rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^\infty |a_\nu| \in R \) in virtue of \((\rho.3)\).

Now, since
\[
\rho\left( \bigcup_{\nu \geq \nu}^\infty |a_\nu| \right) \leq \sum_{\nu \geq \nu}^\infty \rho(a_\nu) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho \left( \bigcap_{\nu=1}^\infty \left( \bigcup_{\nu \geq \nu}^\infty |a_\nu| \right) \right) = 0 \) and hence (5.3) implies
\[
\bigcap_{\nu=1}^\infty \left( \bigcup_{\nu \geq \nu}^\infty |a_\nu| \right) = 0.
\]
Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b_\nu'\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu') \leq \frac{1}{2^\nu} \quad (\nu = 1, 2, \cdots) \). Therefore we have \( s-lim b_\nu = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \|\cdot\|_0 \) is complete and continuous, then (5.2) holds.

**References**


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