ON $F$-NORMS OF QUASI-MODULAR SPACES

By

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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;

$(\rho.2)$ $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y^{1)}$;

$(\rho.3)$ If $\sum_{i \in A} \rho(x_i) < +\infty$ for a mutually orthogonal system $\{x_i\}_{i \in A}$, there exists $x_0 \in R$ such that $x_0 = \sum_{i \in A} x_i$ and $\rho(x_0) = \sum_{i \in A} \rho(x_i)$;

$(\rho.4)$ $\lim_{t \to 0} \rho(tx) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

$(A.1)$ $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

$(A.2)$ $\rho(-x) = \rho(x)$;

$(A.3)$ $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

$(A.4)$ $\alpha_n \to 0$ implies $\rho(\alpha_n x) \to 0$ for every $x \in R$;

$(A.5)$ for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $|| \cdot ||_0$ defined by the formula:

1) $x \perp y$ means $|x| \cap |y| = 0$.

2) A system of elements $\{x_i\}_{i \in A}$ is called mutually orthogonal, if $x_i \perp x_j$ for $i \neq j$.

3) For the definition of a modular, see [3].

4) A linear functional $f$ is called universally continuous, if $\inf f(a_i) = 0$ for any $a_i \downarrow 0$. $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) > 0$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.
(1.1) \[ \| x \|_0 = \inf \{ \xi; \rho \left( \frac{1}{\xi} x \right) \leq \xi \} \]
and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho \)1)\( \sim \) (\( \rho \)4) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{x: x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R^+_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R^+_0 \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies
\[ (\rho A') \quad \sup_{x \in R} \{ \lim_{\alpha \to 0} \rho(\alpha x) \} < +\infty \]
(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R^+_0 \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R^+_0 \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S 83 \)]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x) \) for all \( x \geq 0 \) and \( 1- [p] \) is a projection operator onto the normal manifold \( N = \{p\}^1 \), that is, \( x = [p]x + (1- [p])x \).

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6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

(2.1) \[ \rho(0) = 0; \]
(2.2) \[ \rho([p]x) \leq \rho(x) \text{ for all } p, x \in R; \]
(2.3) \[ \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \text{ for any } [p_i] \uparrow_{i \in A} [p]. \]

In the argument below, we have to use the additional property of $\rho$:

(\rho.5) \[ \rho(x) \leq \rho(y) \text{ if } |x| \leq |y|, x, y \in R, \]
which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies (\rho.5).

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which (\rho.5) is valid.

Proof. We put for every $x \in R$,

(2.4) \[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

It is clear that $\rho'$ satisfies the conditions (\rho.1), (\rho.2) and (\rho.5).

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

\[ \sum_{i \in A} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \text{ for all } x \in R. \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \text{ in virtue of (\rho.3)}. \]

For such $x_0$,

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. $\rho'$ fulfils (\rho.3).

If $\rho'$ does not fulfil (\rho.4), we have for some $x_0 \in R$,

\[ \rho'(\frac{1}{n} x_0) = +\infty \text{ for all } n \geq 1. \]

By (\rho.2) and (\rho.4), $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]

where \( \rho'\left(\frac{1}{\nu}x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]

where

\[ \rho'\left(\frac{1}{\nu}x_2\right) = +\infty \quad (\nu = 1, 2, \ldots) \]

and

\[ \rho'\left(\frac{1}{2}x'_2\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho'\left(\frac{1}{2}y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that

\[ \rho'\left(\frac{1}{\nu}y_\nu\right) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]

and

\[ \rho'\left(\frac{1}{\nu}y_0\right) \geq \rho'\left(\frac{1}{\nu}y_\nu\right) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = \lfloor(\|x| - |y|)^+\rfloor\), we obtain
$$\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)$$
\[\leq \rho(\alpha\lfloor p\rfloor |x| + \alpha(1-\lfloor p\rfloor)|y| + \beta\lfloor p\rfloor |x| + (1-\lfloor p\rfloor)\beta |y|)\]
\[= \rho(\lfloor p\rfloor |x| + (1-\lfloor p\rfloor)|y|)\]
\[= \rho(\lfloor p\rfloor x) + \rho((1-\lfloor p\rfloor)y)\]
\[\leq \rho(x) + \rho(y).\]

**Remark 1.** As is shown above, the existence of $\rho'$ as a quasi-modular depends essentially on the condition $(\rho.4)$. Thus, in the above theorems, we cannot replace $(\rho.4)$ by the weaker condition:

$(\rho.4'')$ for any $x \in R$, there exists $\alpha \geq 0$ such that $\rho(\alpha x) < +\infty$.

In fact, the next example shows that there exists a functional $\rho_0$ on a universally continuous semi-ordered linear space satisfying $(\rho.1), (\rho.2), (\rho.3)$ and $(\rho.4'')$, but does not $(\rho.4)$. For this $\rho_0$, we obtain

$$\rho_0'(x) = \sup_{y | y \leq |x|} \rho_0(y) = +\infty$$

for all $x \not= 0$.

**Example.** $L_1[0,1]$ is the set of measurable functions $x(t)$ which are defined in $[0,1]$ with

$$\int_0^1 |x(t)| \, dt < +\infty.$$

Putting

$$\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \operatorname{mes} \{ t : x(t) = \frac{1}{i} \},$$

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: $(A.4)$, namely,

$(\rho.6)$ \quad $\lim_{t \to 0} \rho(\xi x) = 0$ \quad for all $x \in R$.

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let $\rho$ be a quasi-modular on $R$. We can find a functional $\rho^*$ which satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$.

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular $\rho'$ which satisfies $(\rho.5)$. Now we put

$(2.5)$ \quad $d(x) = \lim_{\xi \to 0} \rho'(|x|).$

It is clear that $0 \leq d(x) = d(|x|) < +\infty$ for all $x \in R$ and
\[ d(x+y) = d(x) + d(y) \]
if \( x \perp y \).

Hence, putting
\[ (2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R), \]
we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \( \rho^* \), since
\[ d(x) \leq \rho'(x) \]
and
\[ d(\alpha x) = d(x) \]
for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that \((\rho.5)\) is true for \( \rho^* \). First we have to note
\[ (2.7) \quad \inf_{\lambda \in \Lambda} d(\lfloor p_\lambda \rfloor x) = 0 \]
for any \( \lfloor p_\lambda \rfloor \downarrow_{\lambda \in \Lambda} 0 \). In fact, if we suppose the contrary, we have
\[ \inf_{\lambda \in \Lambda} d(\lfloor p_\lambda \rfloor x_0) \geq \alpha > 0 \]
for some \( \lfloor p_\lambda \rfloor \downarrow_{\lambda \in \Lambda} 0 \) and \( x_0 \in R \).

Hence,
\[ \rho'(\frac{1}{\nu} \lfloor p_\lambda \rfloor x_0) \geq d(\lfloor p_\lambda \rfloor x_0) \geq \alpha \]
for all \( \nu \geq 1 \) and \( \lambda \in \Lambda \). Thus we can find a subsequence \{\lambda_n\}_{n \geq 1} \) of \{\lambda\}_{\lambda \in \Lambda} \) such that
\[ \lfloor p_{\lambda_n} \rfloor \geq \lfloor p_{\lambda_{n+1}} \rfloor \]
and
\[ \rho'(\frac{1}{n} (\lfloor p_{\lambda_n} \rfloor - \lfloor p_{\lambda_{n+1}} \rfloor) x_0) \geq \frac{\alpha}{2} \]
for all \( n \geq 1 \) in virtue of \((\rho.2)\) and \((2.3)\). This implies
\[ \rho'(\frac{1}{n} x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} (\lfloor p_{\lambda_m} \rfloor - \lfloor p_{\lambda_{m+1}} \rfloor) x_0) = +\infty, \]
which is inconsistent with \((\rho.4)\). Secondly we shall prove
\[ (2.8) \quad d(x) = d(y), \quad \text{if} \quad \lfloor x \rfloor = \lfloor y \rfloor. \]

We put \( \lfloor p_n \rfloor = \lfloor (\lfloor x \rfloor - n \lfloor y \rfloor) + \rfloor \) for \( x, y \in R \) with \( \lfloor x \rfloor = \lfloor y \rfloor \) and \( n \geq 1 \). Then, \( \lfloor p_n \rfloor \downarrow_{n \rightarrow \infty} 0 \) and \( \inf_{n=1,2,...} d(\lfloor p_n \rfloor x) = 0 \) by \((2.7)\). Since \((1 - \lfloor p_n \rfloor)n \lfloor y \rfloor \geq (1 - \lfloor p_n \rfloor) \lfloor x \rfloor \)
and
\[ d(\alpha x) = d(x) \]
for \( \alpha > 0 \) and \( x \in R \), we obtain
\[ d(x) = d([p_n]x) + d((1-[p_n])x) \]
\[ \leq d([p_n]x) + d(n(1-[p_n])y) \]
\[ \leq d([p_n]x) + d(y) . \]
As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y) , \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[ \rho^{*}(x) = \rho^{*}([y]x) + \rho^{*}(([x] - [y])x) \]
\[ = \rho'([y]x) - d([y]x) + \rho^{*}(([x] - [y])x) \]
\[ \geq \rho'(y) - d(y) + \rho^{*}(([x] - [y])x) \]
\[ \geq \rho^{*}(y) . \]
Thus \( \rho^{*} \) satisfies (\rho.5).

**Q.E.D.**

**Theorem 2.3.** \( \rho^{*} \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\rho.3) (that is, \( \rho^{*} \) is a quasi-modular), if and only if \( \rho \) satisfies
\[(\rho.4') \quad \sup_{x \in R} [\lim_{\xi \to 0} \rho(\xi x)] = K < +\infty . \]

**Proof.** Let \( \rho \) satisfy (\rho.4). We need to prove
\[ \sup_{x \in R} d(x) = \sup_{x \in R} [\lim_{\xi \to 0} \rho'(\xi x)] = K' < +\infty , \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y) . \]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'\left(\frac{1}{\nu}x\right) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[ \rho(x) \leq \epsilon \quad \text{implies} \quad \rho'\left(\frac{1}{\nu_0}x\right) \leq \gamma . \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} [\lim_{\xi \to 0} \rho'(\xi x)] = \gamma_0 < +\infty . \]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\rho.4'), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) \((i=1, 2, \cdots, n)\), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j=1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho'\left(\frac{1}{\nu_0} x_0\right) \leq \sum_{i=1}^{n} \rho'\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
\]

\[
\leq \sum_{i=1}^{n} \frac{\gamma}{\nu_0} + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
\]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'\left(\frac{1}{\nu_0} x_0\right) \leq \frac{4K + \varepsilon}{\varepsilon} \gamma + \frac{4K^2}{\varepsilon}
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N^+ \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{\lim_{\xi \to 0} \rho'(\xi x)\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_{\lambda}\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^*(x_{\lambda}) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\rho^*(x_0) = \sum_{\lambda \in A} \rho^*(x_{\lambda}) = \lim_{\xi \to 0} \rho'(\xi \sum_{\lambda \in A} x_{\lambda}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_{\lambda}) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho^*(x_{\lambda}) = \sum_{\lambda \in A} \rho^*(x_{\lambda}) + \sum_{\lambda \in A} d(x_{\lambda}) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_{\lambda} \in R \) and \( \sum_{\lambda \in A} \rho^*(x_{\lambda}) = \rho^*(x_0) \) by \((\rho.4)\) and \((2.7)\). Therefore \( \rho^* \) satisfies \((\rho.3)\).

On the other hand, suppose that \( \rho^* \) satisfies \((\rho.3)\) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geq 0$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
$\lim_{t \to 0} \rho^*(\xi x) = 0$, there exists $\{\alpha_n\}_{n \geq 1}$ with $0 < \alpha_n \leq \alpha (\forall \mu \geq 1)$ and $\sum_{n=1}^{\infty} \rho^*(\alpha_n x_n) < +\infty$. It follows that $x_0 = \sum_{n=1}^{\infty} \alpha_n x_n \in R$ and $d(x_0) = \sum_{n=1}^{\infty} d(\alpha_n x_n)$ from $(\rho.3)$. For such $x_0$, we have for every $\xi \geq 0$,

$$\rho'(\xi x) = \sum_{n=1}^{\infty} \rho'(\xi \alpha_n x_n) \geq \sum_{n=1}^{\infty} d(x_n) = +\infty,$$

which is inconsistent with $(\rho.4)$. Therefore we have

$$\sup_{x \in R} \lim_{t \to 0} \rho(\xi x) \leq \sup_{x \in R} d(x) < +\infty.$$

Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:

$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \},$$

where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold

7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{i \in A} x_i$ with $R_0 \ni x_i \geq 0$ for all $i \in A$. Putting

$$[p_{n,1}] = [(2nx_1 - nx)\uparrow_{n \in A}],$$

we have

$$[p_{n,1}] \uparrow_{i \in A} [x] \quad \text{and} \quad 2n[p_{n,1}]x_1 \geq [p_{n,1}]nx,$$

which implies $\rho^*(nx) = 0$ and $\sup_{i \in A} \rho^*(n[p_{n,1}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into

$$R = R_0 \oplus R_0^\perp.$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_i\}_{i \in A} (x_i \in [p]R_0)$, there exists $x_0 = \sum_{i \in A} x_i \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{0 \perp}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{i \in A} x_i \in S(\forall i \in A)$ implies $\bigcup_{i \in A} x_i \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 


Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ which is semi-continuous, i.e.
$$\sup_{i \in I} || x_i ||_0 = || x ||_0$$ for any $0 \leq x, \uparrow_{i \in I} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad || x ||_0 = \inf \{ \xi ; \rho^*(\frac{1}{\xi} x) \leq \xi \} .$$

Then,

i) $0 \leq || x ||_0 = || -x ||_0 < \infty$ and $|| x ||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_0^\perp$.

ii) $|| x + y ||_0 \leq || x ||_0 + || y ||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha \rightarrow 0} || \alpha x ||_0 = 0$ and $\lim || \alpha x_n ||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim || \alpha x ||_0 = || \alpha x ||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p,] \uparrow_{i \in A}[p\!]$, for any positive number $\xi$ with $|| [p] x ||_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p\!] x) > \xi$, which implies $\sup_{i \in A} \rho^*(\frac{1}{\xi} [p\!] x) > \xi$ and hence $\sup || p, x ||_0 \geq \xi$. Thus we obtain

$$\sup_{i \in A} || p, x ||_0 = || [p] x ||_0 ,$$

if $[p,] \uparrow_{i \in A}[p\!]$.

Let $0 \leq x, \uparrow_{i \in A} x$. Putting

$$[p_{n,i}] = \left[ (x_i - \left(1 - \frac{1}{n}\right)x) \right]$$

we have

$$[p_{n,i}] \uparrow_{i \in A}\,[x] \text{ and } [p_{n,i}] x_i \geq [p_{n,i}] \left(1 - \frac{1}{n}\right)x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{i \in A} || [p_{n,i}] x ||_0 \geq \sup_{i \in A} \left|| [p_{n,i}] \left(1 - \frac{1}{n}\right)x \right||_0 = \left|| \left(1 - \frac{1}{n}\right)x \right||_0 ,$$

we have

$$\sup_{i \in A} || x ||_0 \geq \left|| \left(1 - \frac{1}{n}\right)x \right||_0$$

and also $\sup || x_i ||_0 \geq || x ||_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that

$$\lim || x_n ||_0 = 0 \text{ if and only if } \lim \rho(\xi x_n) = 0 \text{ for all } \xi \geq 0.$$
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \ldots)$ be the elements of $R_0^\perp$ such that

\begin{align*}
(3.2) & \quad [p_{n, \nu}] \uparrow_{\nu=1}^\infty [p_n] \quad \text{with} \quad \bigcap_{n=1}^\infty [p_n] a = [p_0] a = 0; \\
(3.3) & \quad [p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \quad \text{for all} \quad n, \nu \geq 1.
\end{align*}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align*}
(3.4) & \quad \| [q_m] a \|_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \ldots) \\
(3.5) & \quad n_m [q_m] a \geq [q_m] x_{\nu_{m-1}} n_{m+1} > n_m \quad (m = 2, 3, \ldots),
\end{align*}

where $\delta = \| [p_0] a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}] [p_0] \uparrow_{\nu=1}^\infty [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$$
\| [p_{1, \nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}.
$$

We put $[q_1] = [p_{1, \nu_1}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \ldots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na - x_{\nu_k})^+] \uparrow_{n=1}^\infty [a]$ and $\| [q_k] a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$$
\| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
$$

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$$
\| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
$$

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

$$
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k],
$$

because

$$
[q_{k+1}] \leq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
$$

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_{k+1}}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0} \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0} \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0} \geqq ||[q_{k+1}]a_{0}||_{0} \geqq \frac{\delta}{2},$

since $[q_{k+1}] \leqq [q_{k}] \leqq [(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{\perp}$ is an F-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^{*}$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $||x||_{0} (= \inf \{\xi; \rho^{*}(\frac{x}{\xi}) \leqq \xi\})$ is a quasi-norm on $R_{0}^{\perp}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^{\perp}$ be a Cauchy sequence with $0 \leqq x_{\nu} \uparrow_{\nu=1,2}, \ldots$. Since $\rho^{*}$ satisfies $(\rho.3)$, there exists $0 \leqq x_{0} \in R_{0}^{\perp}$ such that $x_{0} = \bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}] = [(x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{\nu=1}^{\infty}[p_{n,v}] = [p_{n}]$, we obtain

(3.6) $[p_{n,v}]x_{\nu} \geqq n[p_{n,v}]x_{0}$ for all $n, v \geqq 1$

and $[p_{n}] \downarrow_{n=1}^{\infty} 0$. Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}] = 0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}] - [p_{n}]) = [x_{0}]$. And

$(1 - [p_{n,v}]) \geqq (1 - [p_{n}])$ \hspace{1cm} $(n, v \geqq 1)$

implies

$n(1 - [p_{n}])x_{0} \geqq (1 - [p_{n}])x_{v} \geqq 0$.

Hence we have

$y_{n} = \bigcup_{v=1}^{\infty}(1 - [p_{n}])x_{v} \in R_{0}^{\perp},$

because $R_{0}^{\perp}$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$\gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_{0} < +\infty,$

which implies

$||y_{n}||_{0} = \sup_{\nu \geqq 1} ||(1 - [p_{n}])x_{v}||_{0} \leqq \gamma$

for every $n \geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1} = y_{1}$ and $z_{n} = y_{n} - y_{n-1}$ $(n \geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0} = ||y_{n}||_{0} \leqq \gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} \nu \in 1 + R$. This yields $z = \bigcup_{\nu=1}^{\infty} \nu$. Truly, it follows from

$$z = \bigcup_{\nu=1}^{\infty} \nu = \bigcup_{\nu=1}^{\infty} \nu \nu \mu = \bigcup_{\nu=1}^{\infty} \nu \nu \mu = \bigcup_{\nu=1}^{\infty} \nu \nu \mu.$$

By semi-continuity of $|| \cdot ||_0$, we have

$$|| z - \nu ||_0 \leq \sup_{\nu \geq \nu} || \nu \nu \nu - \nu ||_0$$

and furthermore $\lim_{\nu \to \infty} || z - \nu ||_0 = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_0^\perp$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_0 \leq \frac{1}{2^{\nu-1}}$$

for all $\nu \geq 1$.

This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_0 \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_0 \leq \frac{1}{2^{m-1}}$$

for all $n > m \geq 1$.

Putting $z_n = \sum_{\nu=1}^{n} || y_{\nu+1} - y_{\nu} ||_0$, we have a Cauchy sequence $\{z_n\}_{n \geq 1}$ with $0 \leq z_n \uparrow \infty$.

Then by the fact proved just above,

$$z_0 = \lim_{n \to \infty} z_n = \sum_{\nu=1}^{\infty} || y_{\nu+1} - y_{\nu} ||_0 \in R_0^\perp$$

and $\lim_{n \to \infty} || z_0 - z_n ||_0 = 0$.

Since $\sum_{\nu=1}^{\infty} || y_{\nu+1} - y_{\nu} ||_0$ is convergent, $y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$|| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n ||_0 = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_0 \leq || z_0 - z_n ||_0 \to 0.$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsection of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{\nu \to \infty} || y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_0 = 0.$$

Therefore $\| \cdot \|_0$ is complete in $R_0^\perp$, that is, $R_0^\perp$ is an F-space with $\| \cdot \|_0$.

Conversely if $R_0^\perp$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_0^\perp$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_0$ contains a normal manifold which is universally complete, if $R_0^\perp = 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an F-space with $\|\cdot\|_0$ if and only if $\rho$ fulfills ($\rho.4'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[
10) \quad \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}
\]

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

\[
\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0
\]

for all $x \in L$ hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \|x\|_1 = \|-x\|_1 < +\infty$ ($x \in L$) and that $\|x\|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \|x_n\|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \|\alpha_n x\|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^\infty 0$ and that $\lim_{n \to \infty} \|x_n\|_1 = 0$ implies $\lim_{n \to \infty} \|\alpha x_n\|_1 = 0$ for all $\alpha > 0$. If $\|x\|_1 < \alpha$ and $\|y\|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\|x+y\| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[
\|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \quad [3 \text{ or } 4].
\]

For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm \( \|\cdot\|_1 \) on \( L \) which is equivalent to \( \|\cdot\|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
(4.3) \quad \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^+_0 \) and \( \|\cdot\|_1 \) is complete if and only if \( \rho \) satisfies \((\rho.4')\), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
(4.4) \quad \|x\|_0 \leq \|x\|_1 \leq 2 \|x\|_0 \quad \text{for all } x \in R^+_0.
\]

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that

\[
|a_n - a| \leq a_{n+1} \quad (n \geq 1) \quad \text{and } a_0 = 0.
\]

And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{s \to \infty} x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( \lim_{n \to \infty} z_n = a \). A quasi-norm \( \|\cdot\| \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} \|a_n\| = 0 \) for any \( a_n \neq 0 \). In the sequel, we write by \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( ||[p_n]x||_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( ||\cdot||_0 \).

**Sufficiency.** Let \( a_\nu \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [p_n^\epsilon]a_n = [p_n^\epsilon]a_1 + (1-[p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1 \).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon]a_1) + \rho^*(\xi(1-[p_n^\epsilon])a_1)
\]

for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi a_n) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{\nu \rightarrow \infty} \rho^*(\xi a_\nu) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} ||a_n||_0 = 0 \) and ||\cdot||_0 is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \( ||\cdot||_0 \) is continuous, if

(5.2) \( \rho^*(a_\nu) \rightarrow 0 \) implies \( \rho^*(\alpha a_\nu) \rightarrow 0 \) for every \( \alpha \geq 0 \).

From the definition, it is clear that \( s-\lim_{\nu \rightarrow \infty} x_\nu = 0 \) implies \( \lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0 \) (or \( \lim_{\nu \rightarrow \infty} ||x_\nu|| = 0 \)) implies \( s-\lim_{\nu \rightarrow \infty} x_\nu = 0 \), if \( ||\cdot||_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu \rightarrow \infty} ||x_\nu|| = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \( \rho^*(x) = 0 \) implies \( x = 0 \).

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_0 \) is complete, \( \rho(a_\nu) \rightarrow 0 \) implies \( s-\lim_{\nu \rightarrow \infty} a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{y_j \in A} \rho^*(y_j) \) for any \( 0 \leq x \uparrow_{i \in A}^\nu \). If

11) If \( \rho^* \) is not semi-continuous, putting \( \rho_* = \inf \{ \sup_{y_j \uparrow A} \rho^*(y_j) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_* (x) \rightarrow 0 \).


we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{R}$ in virtue of \((\rho.3)\).

Now, since
$$\rho\left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}$$
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty}\left(\bigcup_{\mu=\nu}^{\infty} |a_{\mu}|\right)\right) = 0$ and hence (5.3) implies
$$\bigcap_{\nu=1}^{\infty}\left(\bigcup_{\mu=\nu}^{\infty} |a_{\mu}|\right) = 0.$$

Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \to 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^{\nu}}$ ($\nu = 1, 2, \cdots$). Therefore we have $s\text{-lim}_{\nu \to \infty} b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete and continuous, then (5.2) holds.

**References**


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