ON F-NORMS OF QUASI-MODULAR SPACES

By
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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\(\rho.1\)) \(0 \leq \rho(x) = \rho(-x) \leq +\infty\) for all \(x \in R\);

(\(\rho.2\)) \(\rho(x+y) = \rho(x) + \rho(y)\) for any \(x, y \in R\) with \(x \perp y\);\(^1\)

(\(\rho.3\)) If \(\sum_{\lambda \in A} \rho(x_{\lambda}) < +\infty\) for a mutually orthogonal system \(\{x_{\lambda}\}_{\lambda \in A}\),\(^2\) there exists \(x_{0} \in R\) such that \(x_{0} = \sum_{\lambda \in A} x_{\lambda}\) and \(\rho(x_{0}) = \sum_{\lambda \in A} \rho(x_{\lambda})\);

(\(\rho.4\)) \(\varlimsup_{\xi \to 0} \rho(\xi x) < +\infty\) for all \(x \in R\).

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^3\) $m$ on $R$ for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) \(\rho(x) \geq 0\) and \(\rho(x) = 0\) if and only if \(x = 0\);

(A.2) \(\rho(-x) = \rho(x)\);

(A.3) \(\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)\) for every \(\alpha, \beta \geq 0\) with \(\alpha + \beta = 1\);

(A.4) \(\alpha_{n} \to 0\) implies \(\rho(\alpha_{n} x) \to 0\) for every \(x \in R\);

(A.5) for any \(x \in L\) there exists \(\alpha > 0\) such that \(\rho(\alpha x) < +\infty\).

They showed that $L$ is a quasi-normed space with a quasi-norm \(\|\cdot\|_{0}\) defined by the formula;

1) \(x \perp y \) means \(|x| \cap |y| = 0\).
2) A system of elements \(\{x_{\lambda}\}_{\lambda \in A}\) is called mutually orthogonal, if \(x_{\lambda} \perp x_{\gamma}\) for \(\lambda \neq \gamma\).
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if \(\inf_{\lambda \in A} f(a_{\lambda}) = 0\) for any $a_{\lambda} \in A$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) = 0$.

This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.\(^6\)
(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_o = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_o \) in general, we have to exclude \( R_o \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S 83 \)]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [3].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n | p |) x \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = [p]x + (1 - [p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho(\lceil p \rceil x) \leq \rho(x) \text{ for all } p, x \in R; \\
(2.3) & \quad \rho(\lceil p \rceil x) = \sup_{i \in A} \rho(\lceil p_i \rceil x) \text{ for any } \lceil p_i \rceil \uparrow_{i \in A} \lceil p \rceil.
\end{align*}

In the argument below, we have to use the additional property of \( \rho \):

\( (\rho.5) \quad \rho(x) \leq \rho(y) \text{ if } |x| \leq |y|, x, y \in R, \)

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

Proof. We put for every \( x \in R \),

\begin{equation}
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{equation}

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[ \sum_{i \in A} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \text{ for all } x \in R. \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \text{ in virtue of } (\rho.3). \]

For such \( x_0 \),

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho(\lceil x_i \rceil y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_i|} \rho(\lceil x_i \rceil y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in R \),

\[ \rho'(\frac{1}{n} x_0) = +\infty \text{ for all } n \geq 1. \]

By \((\rho.2) \) and \((\rho.4) \), \( x_0 \) can not be written as \( x_0 = \sum_{\nu = 1}^{\varepsilon} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \varepsilon \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'\left(\frac{1}{\nu}x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'\left(\frac{1}{\nu}x_2\right) = +\infty \quad (\nu = 1, 2, \ldots) \]
and
\[ \rho'\left(\frac{1}{2}x_2'\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2}y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho\left(\frac{1}{\nu}y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu}y_0\right) \geq \rho\left(\frac{1}{\nu}y_\nu\right) \geq \nu, \]
which contradicts \( (\rho.4) \). Therefore \( \rho' \) has to satisfy \( (\rho.4) \). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \( (\rho.5) \), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [(|x| - |y|)^+] \), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(|x| + |y|) \\
\leq \rho(\alpha\lceil p \rceil |x| + \alpha(1-\lceil p \rceil)|y| + \beta\lceil p \rceil |x| + (1-\lceil p \rceil)\beta|y|) \\
= \rho(\lceil p \rceil |x| + (1-\lceil p \rceil)|y|) \\
= \rho(\lceil p \rceil x) + \rho((1-\lceil p \rceil)y) \\
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4'') \text{ for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty.\]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \text{ and } (\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[\int_0^1 |x(t)| \, dt < +\infty.\]

Putting

\[\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6) \) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[(2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x).\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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$\text{if } x \bot y.$

Hence, putting

(2.6) $\rho^*(x) = \rho'(x) - d(x)$ ($x \in \mathbb{R}$).

we can see easily that $(\rho.1), (\rho.2), (\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all $x \in \mathbb{R}$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

(2.7) $\inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0$ for any $[p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0$. In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some $[p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0$ and $x_0 \in \mathbb{R}$.

Hence,

\[ \rho'\left(\frac{1}{\nu}[p_{\lambda}]x_0\right) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all $\nu \geq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}[p_{\lambda_n}][p_{\lambda_{n+1}}]x_0) \geq \frac{\alpha}{2} \]

for all $n \geq 1$ in virtue of $(\rho.2)$ and (2.3). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}]-[p_{\lambda_{m+1}}]x_0) = +\infty , \]

which is inconsistent with $(\rho.4)$. Secondly we shall prove

(2.8) $d(x) = d(y)$, if $[x] = [y]$.  

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in \mathbb{R}$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n=0} 0$ and $\inf_{n=1, 2, \ldots} d([p_n]x) = 0$ by (2.7). Since $(1 - [p_n])n | y | \geq (1 - [p_n]) | x |$ and

\[ d(\alpha x) = d(x) \]

for $\alpha > 0$ and $x \in \mathbb{R}$, we obtain
\[ d(x) = d([p_n]x) + d((1 - [p_n])x) \]
\[ \leq d([p_n]x) + d(n(1 - [p_n])y) \]
\[ \leq d([p_n]x) + d(y) . \]

As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y) , \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*((x - [y])x) \]
\[ = \rho'([y]x) - d([y]x) + \rho^*([x - [y])x) \]
\[ \leq \rho'(y) - d(y) + \rho^*([x - [y])x) \]
\[ \leq \rho^*(y) . \]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Q.E.D.**

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

\[ \rho^*(x) = \rho^*([y]x) + \rho^*((x - [y])x) \]
\[ \geq \rho'(y) - d(y) + \rho^*([x - [y])x) \]
\[ \geq \rho^*(y) . \]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty , \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y) . \]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_1(x) = \rho(x) \) and \( n_\nu(x) = \rho'(\frac{1}{\nu} x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma, \) a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^+ \) with
\[ \rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0} x) \leq \gamma . \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0^+} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty . \]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^+ \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where $\frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon$ $(i = 1, 2, \cdots, n)$, $y_j$ is an atomic element with $\rho(y_j) > \varepsilon$ for every $j = 1, 2, \cdots, m$ and $\rho(z) \leq \frac{\varepsilon}{2}$. From above, we get $n \leq \frac{4K}{\varepsilon}$ and $m \leq \frac{2K}{\varepsilon}$. This yields

\[
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \\
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \\
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'((\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right)
\]

in case of $\varepsilon \leq 2K$. If $2K \leq \varepsilon$, we have immediately for $x \in N_0^\perp$

\[
\lim_{\xi \to 0} \rho'((\xi x) \leq \gamma
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma
\]

Let $\{x_i\}_{i \in A}$ be an orthogonal system with $\sum_{i \in A} \rho^*(x_i) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in A$, we have

\[
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
\]

which implies $\sum_{i \in A} d(x_i) \leq \gamma'$. It follows that

\[
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies $x_0 = \sum_{i \in A} x_i \in R$ and $\sum_{i \in A} \rho^*(x_i) = \rho^*(x_0)$ by (\rho.4) and (2.7). Therefore $\rho^*$ satisfies (\rho.3).

On the other hand, suppose that $\rho^*$ satisfies (\rho.3) and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_i\}_{i \geq 1}$ such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
$\lim_{t \to 0} \rho^{*}(\xi x) = 0$, there exists $\{\alpha_{\nu}\}_{\nu \geqq 1}$ with $0 < \alpha_{\nu} (\nu \geqq 1)$ and 
$\sum_{\nu=1}^{\infty} \rho^{*}(\alpha_{\nu} x_{\nu}) < +\infty$. It follows that 
$x_{0} = \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R$ and $d(x_{0}) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x_{\nu})$ from $(\rho.3)$. For such 
x$_{0}$. we have for every $\xi \geqq 0$, 
$\rho^{*}(\xi x_{0}) = \sum_{\nu=1}^{\infty} \rho^{*}(\xi \alpha_{\nu} x_{\nu}) \geqq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty$, 
which is inconsistent with $(\rho.4)$. Therefore we have 
$\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leqq \sup_{x \in R} d(x) < +\infty$. Q.E.D.

§3. Quasi-norms. We denote by $R_{0}$ the set:
$R_{0} = \{x : x \in R, \rho^{*}(nx) = 0 \text{ for all } n \geqq 1\}$, 
where $\rho^{*}$ is defined by the formula (2.6). Evidently $R_{0}$ is a semi-normal 
manifold$^{7}$ of $R$. We shall prove that $R_{0}$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_{0} \ni x_{\lambda} \geqq 0$ for all $\lambda \in \Lambda$. Putting 
$[p_{n,\lambda}] = [(2nx_{\lambda} - nx)^{+}]$, we have 
$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]$ and $2n[p_{n,\lambda}]x_{\lambda} \geqq [p_{n,\lambda}]nx$,
which implies $\rho^{*}(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^{*}(n[p_{n,\lambda}]x) = \rho^{*}(nx) = 0$. Hence, we 
obtain $x \in R_{0}$, that is, $R_{0}$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into 
$R = R_{0} \oplus R_{0}^{\perp}$. $^{8}$

In virtue of the definition of $\rho^{*}$, we infer that for any $p \in R_{0}$, $[p]R_{0}$ is 
universally complete, i.e. for any orthogonal system $\{x_{\lambda} \in [p]R_{0} : x_{\lambda} \geqq 0\}$, 
there exists $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R$. Hence we can also verify without difficulty 
that $R_{0}$ has no universally continuous linear functional except 0, if $R_{0}$ 
is non-atomic. When $R_{0}$ is discrete, it is isomorphic to $S(\Lambda)^{9}$-space. 
With respect to such a universally complete space $R_{0}$, we can not always 
construct a linear metric topology on $R_{0}$, even if $R_{0}$ is discrete.

In the following, therefore, we must exclude $R_{0}$ from our consideration. Now we can state the theorems which we aim at.

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7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leqq |a|$, $b \in R$ implies $b \in S$.
Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if 
$0 \leqq x_{\lambda} \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_{0}$ and $z \in R_{0}^{\perp}$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ which is semi-continuous, i.e.

$$\sup_{x \in A} || x_i ||_0 = || x ||_0$$

for any $0 \leq x, y \in A$. 

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$|| x ||_0 = \inf \{ \xi ; \rho^*(\frac{1}{\xi}x) \leq \xi \}.$$ 

Then,

i) $0 \leq || x ||_0 = || -x ||_0 < \infty$ and $|| x ||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1), (\rho.6), (2.1)$ and the definition of $R_0^\perp$.

ii) $|| x + y ||_0 \leq || x ||_0 + || y ||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_n \to 0} || \alpha_n x ||_0 = 0$ and $\lim_{\alpha_n \to 0} || \alpha x_n ||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to \alpha_0} || \alpha x ||_0 = || \alpha_0 x ||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_\lambda]_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $|| [p] x ||_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi} [p_\lambda] x) > \xi$ and hence $\sup_{\lambda \in \Lambda} || [p_\lambda] x ||_0 > \xi$. Thus we obtain

$$\sup_{\lambda \in \Lambda} || [p_\lambda] x ||_0 = || [p] x ||_0,$$

if $[p_\lambda]_{\lambda \in \Lambda} [p]$.

Let $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$. Putting

$$[p_{n,\lambda}] = \left[ x_\lambda - \left(1 - \frac{1}{n} \right) x \right]$$

we have

$$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \text{ and } [p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}] \left(1 - \frac{1}{n} \right) x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda \in \Lambda} || [p_{n,\lambda}] x_\lambda ||_0 \geq \sup_{\lambda \in \Lambda} || [p_{n,\lambda}] \left(1 - \frac{1}{n} \right) x ||_0 = \left(1 - \frac{1}{n} \right) || x ||_0,$$

we have

$$\sup_{\lambda \in \Lambda} || x_\lambda ||_0 \geq \left(1 - \frac{1}{n} \right) || x ||_0$$

and also $\sup_{\lambda \in \Lambda} || x_\lambda ||_0 \leq || x ||_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim || x_n ||_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_\nu \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\[(3.2) \quad [p_{n,\nu}] \uparrow_\nu=1^\infty [p_n]\text{ with } \bigcap_{n=1}^\infty [p_n]a = [p_0]a \neq 0;\]

\[(3.3) \quad [p_{n,\nu}]x_\nu \geq n[p_{n,\nu}]a \text{ for all } n, \nu \geq 1.\]

Then \(\{x_\nu\}_{\nu \geq 1}\) is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_m=1^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[(3.4) \quad ||[q_m]a||_0 > \frac{\delta}{2} \text{ and } [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \cdots)\]

and

\[(3.5) \quad n_m[q_m]a \geq [q_m]x_{\nu_m-1}, \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),\]

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_\nu=1^\infty [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[||[p_{1,\nu_1}]a||_0 > \frac{||[p_0]a||_0}{2} = \frac{\delta}{2}.\]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu})^+] \uparrow_m=1^\infty [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[||(n_{k+1}a-x_{\nu})^+[q_k]a||_0 > \frac{\delta}{2}.\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[||[p_{n_{k+1},\nu_{k+1}}](n_{k+1}a-x_{\nu})^+[q_k]a||_0 > \frac{\delta}{2}.\]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[\quad [q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}](n_{k+1}a-x_{\nu})^+[q_k],\]

because

\[\quad [q_{k+1}] \subseteq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a\]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$$||x_{v_{k+1}}-x_{v_{k-1}}||_0 \geqq ||q_{k+1}(x_{v_{k+1}}-x_{v_{k-1}})||_0 \geqq ||n_{k+1}a-n_{k}a||_0 \geqq ||q_{k+1}a||_0 \geqq \frac{\delta}{2},$$

since $[q_{k+1}] \leqq [q_k] \leqq [(n_{k}a-x_{v-1})^+]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{v_{k-1}}$ by (3.4). It follows from the above that $\{x_{v}\}_{v \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^\perp$ is an F-space with $||\cdot||_0$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $x||_0(=\inf\{\xi; \rho^*(\frac{x}{\xi})\leqq \xi\})$ is a quasi-norm on $R_0^\perp$, we need only to verify completeness of $||\cdot||_0$. At first let $\{x_{v}\}_{v \geqq 1} \subset R_0^\perp$ be a Cauchy sequence with $0 \leqq x_{v} \uparrow_{v=1,2,\ldots}$. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leqq x_0 \in R_0^\perp$ such that $x_0=\bigcup_{v=1}^\infty x_{v}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}]=[(x_{v}-nx_{0})^+]$ and $\bigcup_{v=1}^\infty [p_{n,v}]=[p_{n}]$, we obtain

$$[p_{n,v}]x_{v} \geqq n[p_{n,v}]x_{0} \quad \text{for all } n, v \geqq 1 \quad \text{and } [p_{n}]\downarrow_{n=1}^\infty 0.$$

Since $\{x_{v}\}_{v \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^\infty [p_{n}]=0$, that is, $\bigcup_{n=1}^\infty ([x_{0}]-[p_{n}])=[x_{0}]$. And

$$(1-[p_{n,v}]) \geqq (1-[p_{n}]) \quad (n, v \geqq 1)$$

implies

$$n(1-[p_{n}])x_{0} \geqq (1-[p_{n}])x_{v} \geqq 0.$$

Hence we have

$$y_{n}=\bigcup_{v=1}^\infty (1-[p_{n}])x_{v} \in R_0^\perp,$$

because $R_0^\perp$ is universally continuous. As $\{x_{v}\}_{v \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_0$

$$\gamma=\sup_{v \geqq 1} ||x_{v}||_0 < +\infty,$$

which implies

$$||y_{n}||_0=\sup_{v \geqq 1} ||(1-[p_{n}])x_{v}||_0 \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $||\cdot||_0$. We put $z_{1}=y_{1}$ and $z_{n}=y_{n}-y_{n-1}$ $(n \geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{v}\}_{v \geqq 1}$ is an orthogonal sequence with $||\sum_{v=1}^{n}z_{v}||_0=||y_{n}||_0 \leqq \gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of
\[ z = \sum_{\nu=1}^{\infty} \rho^{*} \left( \frac{z_{\nu}}{1+\gamma} \right) \leq \gamma \]
for all $n \geq 1$ by the formula (3.1).

Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}$.

This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$.

Truly, it follows from $z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-\left[p_{n}\right]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-\left[p_{n}\right]) x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}$.

By semi-continuity of $|| \cdot ||_{0}$, we have
\[ ||z-x_{\nu}||_{0} \leq \sup_{\mu \geq \nu} ||x_{\mu}-x_{\nu}||_{0} \]
and furthermore \( \lim_{n \to \infty} ||z-x_{\nu}||_{0} = 0 \).

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{\perp}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that
\[ ||y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{\nu}} \quad \text{for all } \nu \geq 1. \]

This implies
\[ ||\sum_{\nu=m}^{n} y_{\nu+1}-y_{\nu}||_{0} \leq \sum_{\nu=m}^{n} ||y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{m-1}} \quad \text{for all } n > m \geq 1. \]

Putting $z_{n} = \sum_{\nu=1}^{n} ||y_{\nu+1}-y_{\nu}||_{0}$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \leq \infty$.

Then by the fact proved just above,
\[ z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} ||y_{\nu+1}-y_{\nu}||_{0} \in R_{0}^{\perp} \quad \text{and} \quad \lim_{n \to \infty} ||z_{0}-z_{n}||_{0} = 0. \]

Since $\sum_{\nu=1}^{\infty} ||y_{\nu+1}-y_{\nu}||_{0}$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})$ is also convergent and
\[ ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_{n}||_{0} = ||\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})||_{0} \leq ||z_{0}-z_{n}||_{0} \to 0. \]

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that
\[ \lim_{\nu \to \infty} ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_{\nu}||_{0} = 0. \]

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{\perp}$, that is, $R_{0}^{\perp}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{\perp}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{\perp}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0}^{\perp} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills (3.4').

§ 4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

(4.1) \[ \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10} \]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

(4.2) \[ \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \]

for all $x \in L$, hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim \| \alpha_n x \|_1 = 0$ for all $\alpha_n \downarrow 0$ and that $\lim \| x_n \|_1 = 0$ implies $\lim \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta . \]

This yields

\[ \| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \]

\[ \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta , \]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq -\frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq -\frac{1}{\xi} + \rho(\xi x) \leq -\frac{2}{\xi} . \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \] \[ \text{and} \] \[ \| x \| = \inf_{m(\xi x) \geq 1} \frac{1}{\xi} \]

[3 or 4]. For the general modulares considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|_1$ and $\| \cdot \|_0$ respectively.
This yields (4.2), since we have \( ||x||_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( ||x||_0 \geq \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm \( || \cdot ||_1 \) on \( L \) which is equivalent to \( || \cdot ||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R_0^+ \) and \( || \cdot ||_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.4' \)), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all } x \in R_0^+. 
\]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies (\( \rho.1 \))~(\( \rho.6 \)) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{\nu \uparrow \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a| \leq a_\nu (\nu \geq 1) \) and \( a_\nu \downarrow_{\nu=1}^\infty 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{\nu \rightarrow \infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( \lim_{\nu \rightarrow \infty} z_\nu = a \). A quasi-norm \( || \cdot || \) on \( R \) is termed to be continuous, if \( \inf ||a_\nu|| = 0 \) for any \( a_\nu \downarrow_{\nu=1}^\infty 0 \). In the sequel, we write by \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]_R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
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sequence of projector \([p_n]_{n \geq 1}\) such that \(\rho([p_n]x) = +\infty\) and \([p_n] \downarrow_{n=1}^{\infty} 0\).

Hence by (3.1) it follows that \(|[p_n]x|_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(|\cdot|_0\).

**Sufficiency.** Let \(a_n \downarrow_{n=1}^{\infty} 0\) and put \([p^*_n] = [a_n - \epsilon a_1]^+\) for any \(\epsilon > 0\) and \(n \geq 1\).

This implies

\[\rho^*(\xi a_n) \leq \rho^*(\xi [p^*_n]a_1) + \rho^*(\xi \epsilon [1 - [p^*_n]]a_1)\]

for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p^*_n] \downarrow_{n=1}^{\infty} 0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*([p^*_n]a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*([p^*_n]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain

\[\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1)\]

Since \(\epsilon\) is arbitrary, \(\lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} 1_{a_n} ||_0 = 0\) and \(|\cdot|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** \(|\cdot|_0\) is continuous, if

(5.2) \(\rho^*(a_\nu) \rightarrow 0\) implies \(\rho^*(\alpha a_\nu) \rightarrow 0\) for every \(\alpha \geq 0\).

From the definition, it is clear that s-lim \(x_\nu = 0\) implies \(\lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0\), if \(|\cdot|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0\) (or \(\lim_{\nu \rightarrow \infty} ||x_\nu|| = 0\)) implies s-lim \(x_\nu = 0\) if \(|\cdot|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho.3\)).

If we replace \(\lim_{\nu \rightarrow \infty} ||x_\nu|| = 0\) by \(\lim_{\nu \rightarrow \infty} \rho(x_\nu) = 0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \(\rho^*(x) = 0\) implies \(x = 0\).

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(|\cdot|_0\) is complete, \(\rho(a_\nu) \rightarrow 0\) implies s-lim \(a_\nu = 0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^{11}\) i.e. \(\rho^*(x) = \sup_{\nu \in A} \rho^*(x_\nu)\) for any \(0 \leq x \downarrow_{\nu \in A} x\). If

\(^{11}\) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x) = \inf_{y \downarrow_{\nu \in A} x} \{\sup_{\nu \in A} \rho^*(y_\nu)\}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x_\nu) \rightarrow 0\) is equivalent to \(\rho_*(x_\nu) \rightarrow 0\).
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of \((\rho.3)\).

Now, since

\[
\rho\left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]

holds for each \( \nu \geq 1 \), \( \rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right)\right) = 0 \) and hence \((5.3)\) implies

\[
\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right) = 0.
\]

Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \) \( (\nu = 1, 2, \cdots) \). Therefore we have \( s\text{-lim} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analouges to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies \((5.3)\) and \( \|\cdot\|_{0} \) is complete and continuous, then \((5.2)\) holds.

### References