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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

\( (\rho.1) \) \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);
\( (\rho.2) \) \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);
\( (\rho.3) \) If \( \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \) for a mutually orthogonal system \( \{x_{\lambda}\}_{\lambda \in \Lambda} \), there exists \( x_{0} \in R \) such that \( x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \) and \( \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) \);
\( (\rho.4) \) \( \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^3\) \( m \) on \( R \) for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

\( (A.1) \) \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);
\( (A.2) \) \( \rho(-x) = \rho(x) \);
\( (A.3) \) \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);
\( (A.4) \) \( \alpha_{n} \to 0 \) implies \( \rho(\alpha_{n} x) \to 0 \) for every \( x \in R \);
\( (A.5) \) for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( ||\cdot||_{0} \) defined by the formula;

1) \( x \perp y \) means \( |x| \cap |y| = 0 \).
2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma \).
3) For the definition of a modular, see [3].
4) A linear functional \( f \) is called universally continuous, if \( \inf_{a_{\lambda} \downarrow 0} f(a_{\lambda}) = 0 \) for any \( a_{\lambda} \downarrow 0 \) in \( \Lambda \).
5) \( R \) is called semi-regular, if for any \( x \neq 0, x \in R \), there exists a universally continuous linear functional \( f \) such that \( f(x) \neq 0 \).

This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4]. In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R \).
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(1.1) \[ ||x||_0 = \inf \{ \xi ; \rho\left(\frac{1}{\xi} x \right) \leq \xi \} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an F-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( || \cdot ||_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an F-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \sup_{x \in R} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R_0^+ \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2||x||_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x)\) for all \( x \geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = [p]x + (1 - [p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) & \quad \rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x) \quad \text{for any } \left[ p_{\lambda} \right] \uparrow_{\lambda \in A} [p].
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\[
(\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \quad x, y \in R,
\]

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

*Proof.* We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_{\lambda}\}_{\lambda \in A} \) be an orthogonal system such that \( \sum_{\lambda \in A} \rho'(x_{\lambda}) < +\infty \), then

\[
\sum_{\lambda \in A} \rho(x_{\lambda}) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\]

We have

\[
x_{0} = \sum_{\lambda \in A} x_{\lambda} \in R
\]

and

\[
\rho(x_{0}) = \sum_{\lambda \in A} \rho(x_{\lambda}) \quad \text{in virtue of } (\rho.3).
\]

For such \( x_{0} \),

\[
\rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{\lambda \in A} \rho([x_{\lambda}]y)
\]

\[
= \sum_{\lambda \in A} \sup_{0 \leq |y| \leq |x_{\lambda}|} \rho([x_{\lambda}]y) = \sum_{\lambda \in A} \rho'(x_{\lambda})
\]

holds, i. e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_{0} \in R \),

\[
\rho'(\frac{1}{n} x_{0}) = +\infty \quad \text{for all } n \geq 1.
\]

By \((\rho.2) \) and \((\rho.4) \), \( x_{0} \) can not be written as \( x_{0} = \sum_{\nu=1}^{\xi} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \xi \), namely, we can decompose \( x_{0} \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu} x_2) = +\infty \] (\( \nu = 1, 2, \cdots \))

and

\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2,\ldots} \) such that

\[ \rho(\frac{1}{\nu} y_\nu) \geq \nu \]

and

\[ 0 \leq |y_\nu| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2,\ldots} \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]

and

\[ \rho(\frac{1}{\nu} y_0) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = \left[\left|\left|x\right| - \left|y\right|\right|\right] \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha [p] |x| + \alpha (1 - [p]) |y| + \beta [p] |x| + (1 - [p]) \beta |y|) = \rho([p] |x| + (1 - [p]) |y|) = \rho([p] x) + \rho((1 - [p]) y) \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:
\( \rho.4'' \) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain
\[ \rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with
\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting
\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]
we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,
\( (\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \)

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put
\[ d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ (2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[ (2.7) \quad \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{\nu} [p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n} ([p_{\lambda_n}] - [p_{\lambda_{n+1}}]x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies

\[ \rho'(\frac{1}{n} x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} ([p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0) = +\infty, \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(x - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \(\inf_{n=1, 2, \ldots} d([p_n]x) = 0\) by \((2.7)\). Since \((1 - [p_n])n |y| \geq (1 - [p_n])|x|\) and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
As \( n \) is arbitrary, this implies
\[
d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y]x) \\
= \rho'(y) - d([y]x) + \rho^*([x] - [y]x) \\
\geq \rho'(y) - d([y]x) + \rho^*([x] - [y]x) \\
\geq \rho^*(y).
\]
Thus \( \rho^* \) satisfies (\( \rho.5 \)).

Theorem 2.3. \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

\( \rho.4' \) \hspace{1cm}
\[
\sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho(\xi x) \} = K < +\infty.
\]

Proof. Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[
\sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty,
\]
where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]
Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'(\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in \mathbb{R} \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[
\rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma.
\]
In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.
\]
If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) for every \( i = 1, 2, \ldots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'(\frac{1}{\nu_0} \alpha x_0) \leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'(\frac{z}{\nu_0}) \leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \{ \sup_{0 \leq a \leq a_0} \rho(ax) \} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha x_0}{\nu_0}) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^2}{\epsilon} \right).
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0.
\]

Let \( \{x_\lambda\}_{\lambda \in \Lambda} \) be an orthogonal system with \( \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in \Lambda \), we have

\[
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in \Lambda} d(x_\lambda) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in \Lambda} \rho^*(x_\lambda) = \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) + \sum_{\lambda \in \Lambda} d(x_\lambda) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in R \) and \( \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) = \rho^*(x_0) \) by \( (\rho.4) \) and \( (2.7) \). Therefore \( \rho^* \) satisfies \( (\rho.3) \).

On the other hand, suppose that \( \rho^* \) satisfies \( (\rho.3) \) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_\nu\}_{\nu \geq 1} \) such that

\[
\sum_{\nu = 1}^{\mu} d(x_\nu) = d(\sum_{\nu = 1}^{\mu} x_\nu) \geq \mu
\]

for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since $\lim_{\xi \to 0} \rho^*(\xi x) = 0$, there exists $\{\alpha_v\}_{v \geqq 1}$ with $0 < \alpha_v (v \geqq 1)$ and $\sum_{v=1}^{\infty} \rho^*(\alpha_v x_v) < +\infty$. It follows that $x_0 = \sum_{v=1}^{\infty} \alpha_v x_v \in R$ and $d(x_0) = \sum_{v=1}^{\infty} d(\alpha_v x_v)$ from (\rho.3). For such $x_0$, we have for every $\xi \geqq 0$,

$$\rho'(\xi x) = \sum_{v=1}^{\infty} \rho'(\xi \alpha_v x_v) \geqq \sum_{v=1}^{\infty} d(x_v) = +\infty,$$

which is inconsistent with (\rho.4). Therefore we have $\sup_{x \in R} (\lim_{\xi \to 0} \rho(\xi x)) \leqq \sup_{x \in R} d(x) < +\infty$. Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:

$$R_0 = \{x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geqq 1\},$$

where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^{7)}\) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geqq 0$ for all $\lambda \in \Lambda$. Putting $[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]$, we have $[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]$ and $2n[p_{n,\lambda}]x_\lambda \geqq [p_{n,\lambda}]nx$, which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into

$$R = R_0 \oplus R_0^\perp.\(^8)\)$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda : x_\lambda \in [p]R_0\}$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{\rho}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S, |b| \leqq |a|, b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---

\[^{7}]
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\[^{8}]
This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$. 

\[^{9}]
$S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^{\perp}$ becomes a quasi-normed space with a quasi-norm $||\cdot||_{0}$ which is semi-continuous, i.e.

\[
\sup_{i\in I} ||x_{i}||_{0} = ||x||_{0}
\]

for any $0 \leq x, \uparrow_{i\in I} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^{*}$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put

(3.1) \[ ||x||_{0} = \inf \left\{ \xi ; \rho^{*}\left(\frac{1}{\xi}x\right) \leq \xi \right\} . \]

Then,

i) $0 \leq ||x||_{0} = ||-x||_{0} < \infty$ and $||x||_{0} = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_{0}^{\perp}$.

ii) $||x + y||_{0} \leq ||x||_{0} + ||y||_{0}$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_{n} \uparrow_{\alpha_{0}}^{0}} ||x_{\alpha_{n}}||_{0} = 0$ and $\lim_{\alpha_{n} \uparrow_{\alpha_{0}}^{0}} ||\alpha x_{\alpha_{n}}||_{0} = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_{0}$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha_{n} \uparrow_{\alpha_{0}}^{0}} ||\alpha x||_{0} = ||\alpha_{0} x||_{0}$ for all $x \in R_{0}^{\perp}$ and $\alpha_{0} \geq 0$. If $x \in R_{0}^{\perp}$ and $[p_{\lambda}] \uparrow_{\lambda \in A} [p]$, for any positive number $\xi$ with $||[p_{\lambda}]x||_{0} \geq \xi$ we have $\rho^{*}\left(\frac{1}{\xi} [p_{\lambda}]x\right) > \xi$, which implies $\sup_{\lambda \in A} \rho^{*}\left(\frac{1}{\xi} [p_{\lambda}]x\right) > \xi$ and hence $\sup_{\lambda \in A} ||[p_{\lambda}]x||_{0} \geq \xi$. Thus we obtain $\sup_{\lambda \in A} ||[p_{\lambda}]x||_{0} = ||[p]x||_{0}$, if $[p_{\lambda}] \uparrow_{\lambda \in A} [p]$.

Let $0 \leq x_{1} \uparrow_{\lambda \in A} x$. Putting

$[p_{n,\lambda}]=\left[(x_{\lambda}-(1-\frac{1}{n})x)^{+}\right]$ we have

$[p_{n,\lambda}] \uparrow_{\lambda \in A} [x]$ and $[p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}]\left(1-\frac{1}{n}\right)x$ \hspace{1cm} (n \geq 1).

As is shown above, since

\[
\sup_{\lambda \in A} ||[p_{n,\lambda}]x_{\lambda}||_{0} \geq \sup_{\lambda \in A} \left(1-\frac{1}{n}\right)x_{\lambda} ||_{0} = \left(1-\frac{1}{n}\right)x ||_{0} ,
\]

we have

\[
\sup_{\lambda \in A} ||x_{\lambda}||_{0} \geq \left(1-\frac{1}{n}\right)x ||_{0}
\]

and also $\sup_{\lambda \in A} ||x_{\lambda}||_{0} \geq ||x||_{0}$. As the converse inequality is obvious by iv), $||\cdot||_{0}$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n \to \infty} ||x_{n}||_{0} = 0$ if and only if $\lim_{n \to \infty} \rho(\xi x_{n}) = 0$ for all $\xi \geq 0$. 
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\[ (3.2) \quad [p_{n,\nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0; \]

\[ (3.3) \quad [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[ (3.4) \quad ||[q_m]a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m = 1, 2, \cdots) \]

and

\[ (3.5) \quad n_m[q_m]a \geq [q_m]x_{\nu_m-1}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots), \]

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}][p_0]a||_0 > \frac{||[p_0]a||_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ ||(n_{k+1}a-x_{\nu})^+[q_k]a||_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ ||[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu})^+][q_k]a||_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu})^+][q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and \([q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_{k+1}}\) by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
On F-Norms of Quasi-Modular Spaces

Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^\perp$ is an F-space with $||\cdot||_0$ if and only if $\rho$ satisfies $(\rho.4')$.

Proof. If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $||x||_0 = \inf \{\xi; \rho^*(\frac{x}{\xi}) \leq \xi\}$ is a quasi-norm on $R_0^\perp$, we need only to verify completeness of $||\cdot||_0$. At first, let $\{x_\nu\}_{\nu \geq 1} \subset R_0^\perp$ be a Cauchy sequence with $0 \leq x_\nu \uparrow_{\nu=1,2,\ldots}$. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leq x_0 \in R_0^\perp$ such that $x_0 = \bigcup_{\nu=1}^{\infty} x_\nu$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = [(x_\nu - nx_0)^+]$ and $\bigcup_{\nu=1}^{\infty} [p_{n,\nu}] = [p_n]$, we obtain

$$[p_{n,\nu}] x_\nu \geq n [p_{n,\nu}] x_0$$

for all $n, \nu \geq 1$ and $[p_n] \downarrow_{n=1}^{\infty} 0$. Since $\{x_\nu\}_{\nu \geq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty} [p_n] = 0$, that is, $\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]$. And

$$(1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1)$$

implies

$$n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_\nu \geq 0.$$ 

Hence we have

$$y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_\nu \in R_0^\perp,$$

because $R_0^\perp$ is universally continuous. As $\{x_\nu\}_{\nu \geq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_0$

$$\gamma = \sup_{\nu \geq 1} ||x_\nu||_0 < +\infty,$$

which implies

$$||y_n||_0 = \sup_{\nu \geq 1} ||(1 - [p_n]) x_\nu||_0 \leq \gamma$$

for every $n \geq 1$ by semi-continuity of $||\cdot||_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ ($n \geq 2$). It follows from the definition of $y_n$ that $\{z_\nu\}_{\nu \geq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n} z_\nu||_0 = ||y_n||_0 \leq \gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{n} y_{\nu} = \bigcup_{\nu=1}^{n} x_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from

$$z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - \lfloor p_{\nu} \rfloor) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} \lfloor x_{0} \rfloor x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.$$

By semi-continuity of $|| \cdot ||_{0}$, we have

$$|| z - x_{\nu} ||_{0} \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_{0}$$

and furthermore $\lim_{\mu \to \infty} || z - x_{\nu} ||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{\perp}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$.

This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{n-m}}$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \leq \infty$. Then by the fact proved just above,

$$z_{0} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{\perp} \quad \text{and} \quad \lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0.$$

Since $\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0.$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{\mu \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\mu} ||_{0} = 0.$$

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{\perp}$, that is, $R_{0}^{\perp}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{\perp}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{\perp}$, we have $\sum_{\nu} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$). Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ defined by (3.1) and $R$ becomes an $F$-space with $|| \cdot ||_0$ if and only if $\rho$ fulfills $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[(4.1) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}\]

and show that $|| \cdot ||_1$ is also a quasi-norm on $L$ and

\[(4.2) \quad ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all} \quad x \in L\]

hold, where $|| \cdot ||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < +\infty \quad (x \in L)$ and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_{n} \downarrow_{n=1}^{\infty} 0$ implies

\[\lim_{n \to \infty} \rho(\alpha_{n}x) = 0 \quad \text{for each} \quad x \in L \quad \text{and} \quad \lim_{n \to \infty} ||x_{n}||_1 = 0 \quad \text{implies} \quad \lim_{n \to \infty} \rho(\xi x_{n}) = 0 \quad \text{for all} \quad \xi \geq 0,\]

we obtain that $\lim_{n \to \infty} ||\alpha_{n}x||_1 = 0$ for all $\alpha_{n} \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n \to \infty} ||x_{n}||_1 = 0$ implies $\lim_{n \to \infty} ||\alpha x_{n}||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.\]

This yields

\[||x+y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho \left( \frac{\xi \eta}{\xi + \eta} (x+y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho \left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,\]

in virtue of (A.3). Therefore $||x+y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $|| \cdot ||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi} .\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[||x|| = \inf_{\xi > 0} \frac{1}{\xi} + m(\xi x) \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} ||x||\]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $|| \cdot ||$ and $|| \cdot ||$ respectively.
This yields (4.2), since we have \( ||x||_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( ||x||_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \) (A.5) in §1, then the formula (4.1) yields a quasi-norm \( ||\cdot||_1 \) on \( L \) which is equivalent to \( ||\cdot||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
(4.3) \quad ||x||_1 = \inf_{\xi \to 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^{\perp}_0 \) and \( ||\cdot||_1 \) is complete if and only if \( \rho \) satisfies \((\rho.4')\), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
(4.4) \quad ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all } x \in R^{\perp}_0 .
\]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \leq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( \nu \lim x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a_\nu| \leq a_\nu \) (\( \nu \geq 1 \)) and \( a_\nu \downarrow 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( \sigma \lim x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 0} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( \nu \lim z_\nu = a \).

A quasi-norm \( ||\cdot|| \) on \( R \) is termed to be continuous, if \( \inf \{||a_\nu|| = 0 \) for any \( a_\nu \downarrow a \). In the sequel, we write by \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( ||\cdot||_0 \) (or \( ||\cdot||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z] R \text{ is finite dimensional and } \rho(y) < +\infty .
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector $\{[p_n]\}_{n\geq 1}$ such that $\rho([p_n]x)=+\infty$ and $[p_n]\downarrow_{n=1}^\infty 0$. Hence by (3.1) it follows that $\| [p_n]x \|_0 > 1$ for all $n \geq 1$, which contradicts the continuity of $\| \cdot \|_0$.

**Sufficiency.** Let $a_n \downarrow_{n=1}^\infty 0$ and put $[p_n^{\epsilon}] = [(a_n - \epsilon a_1)^+]$ for any $\epsilon > 0$ and $n \geq 1$. It is easily seen that $[p_n^{\epsilon}] \downarrow_{n=1}^\infty 0$ for any $\epsilon > 0$ and $a_n = [a_1]a_n = [p_n^{\epsilon}]a_n + (1 - [p_n^{\epsilon}])a_n \leq [p_n^{\epsilon}]a_1 + \epsilon a_1$.

This implies

$$\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^{\epsilon}]a_1) + \rho^*(\xi(1 - [p_n^{\epsilon}])a_1)$$

for all $n \geq 1$ and $\xi \geq 0$. In virtue of (5.1) and $[p_n^{\epsilon}] \downarrow_{n=1}^\infty 0$, we can find $n_0$ (depending on $\xi$ and $\epsilon$) such that $\rho^*(\xi [p_n^{\epsilon}]a_1) < +\infty$, and hence $\inf_{n\geq 1} \rho^*(\xi [p_n^{\epsilon}]a_1) = 0$ by (2.3) in Lemma 1 and $(\rho.2)$. Thus we obtain

$$\inf_{n\geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi a_1).$$

Since $\epsilon$ is arbitrary, $\lim_{\nu \to \infty} \rho^*(\xi a_\nu) = 0$ follows. Hence we infer that $\inf_{n\geq 1} \| a_n \|_0 = 0$ and $\| \cdot \|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** $\| \cdot \|_0$ is continuous, if

$$(5.2) \quad \rho^*(a_\nu) \to 0 \text{ implies } \rho^*(\alpha a_\nu) \to 0 \quad \text{for every } \alpha \geq 0.$$ 

From the definition, it is clear that $s\lim x = 0$ implies $\lim_{\nu \to \infty} \| x_\nu \|_0 = 0$, if $\| \cdot \|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** $\lim_{\nu \to \infty} \| x_\nu \|_0 = 0$ (or $\lim_{\nu \to \infty} \| x_\nu \| = 0$) implies $s\lim x_\nu = 0$, if $\| \cdot \|_0$ is complete (i.e. $\rho^*$ satisfies $(\rho.3)$).

If we replace $\lim_{\nu \to \infty} \| x_\nu \| = 0$ by $\lim_{\nu \to \infty} \rho(x_\nu) = 0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

$$(5.3) \quad \rho^*(x) = 0 \text{ implies } x = 0.$$ 

Truly we obtain

**Theorem 5.3.** If $\rho^*$ satisfies (5.3) and $\| \cdot \|_0$ is complete, $\rho(a_\nu) \to 0$ implies $s\lim a_\nu = 0$.

**Proof.** We may suppose without loss of generality that $\rho^*$ is semi-continuous, i.e. $\rho^*(x) = \sup_{y \in A} \rho^*(x_i)$ for any $0 \leq x_i \in A$. If

11) If $\rho^*$ is not semi-continuous, putting $\rho_*(x) = \inf_{y \in A} \{ \sup_{j \in A} \rho^*(y_j) \}$, we obtain a quasi-modular $\rho_*$ which is semi-continuous and $\rho^*(x_\nu) \to 0$ is equivalent to $\rho_*(x_\nu) \to 0$. 

"11)"
\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{R} \) in virtue of \((\rho.3)\).

Now, since
\[ \rho\left( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu=1}^{\infty} |a_{\mu}| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu=1}^{\infty} |a_{\mu}| \right) = 0. \]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu = 1, 2, \ldots) \). Therefore we have \( s\text{-lim}_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete and continuous, then (5.2) holds.

**References**


Mathematical Institute, Hokkaido University

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