ON F-NORMS OF QUASI-MODULAR SPACES

By

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§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff’s sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ \( 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);

$(\rho.2)$ \( \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);

$(\rho.3)$ If \( \sum_{i \in A} \rho(x_i) < +\infty \) for a mutually orthogonal system \( \{x_i\}_{i \in A} \), there exists \( x_0 \in R \) such that \( x_0 = \sum_{i \in A} x_i \) and \( \rho(x_0) = \sum_{i \in A} \rho(x_i) \);

$(\rho.4)$ \( \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^5\) \( m \) on \( R \) for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

(A.1) \( \rho(x) \geq 0 \) and \( \rho(x) = 0 \) if and only if \( x = 0 \);

(A.2) \( \rho(-x) = \rho(x) \);

(A.3) \( \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \);

(A.4) \( \alpha_n \to 0 \) implies \( \rho(\alpha_n x) \to 0 \) for every \( x \in R \);

(A.5) for any \( x \in L \) there exists \( \alpha > 0 \) such that \( \rho(\alpha x) < +\infty \).

They showed that \( L \) is a quasi-normed space with a quasi-norm \( ||\cdot||_0 \) defined by the formula;

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1) \( x \perp y \) means \( |x| \cap |y| = 0 \).

2) A system of elements \( \{x_i\}_{i \in A} \) is called mutually orthogonal, if \( x_i \perp x_j \) for \( i \neq j \).

3) For the definition of a modular, see [3].

4) A linear functional \( f \) is called universally continuous, if \( \inf_{i \in A} f(x_i) = 0 \) for any \( a_i \perp x_i \).

5) \( R \) is called semi-regular, if for any \( x \neq 0, x \in R \), there exists a universally continuous linear functional \( f \) such that \( f(x) = 0 \).

5) This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4]. In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \) for each \( x \in R \).
(1.1) \[ ||x||_0 = \inf \left\{ \xi; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]
and \( ||x_n||_0 \rightarrow 0 \) is equivalent to \( \rho(\alpha x_n) \rightarrow 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \))\( \sim (\rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim (A.5 \)) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x: x \in R, \rho^*(\xi x) = 0 \) for all \( \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( || \cdot ||_0 \) on \( R^+_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R^+_0 \) is an \( F \)-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

(\( \rho.4^* \)) \[ \sup_{x \in R} \rho(\alpha x) < +\infty \]
(Thoerem 3.2).

In §4, we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R^+_0 \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2||x||_0 \) holds for every \( x \in R^+_0 \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \([p] \) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n \cap |p \cap x) \) for all \( x \geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N=\{p\}^1 \), that is, \( x=[p]x+(1-[p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

(2.1) $\rho(0)=0$;
(2.2) $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
(2.3) $\rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x)$ for any $[p_{\lambda}]_{\lambda \in A} \uparrow[p]$.

In the argument below, we have to use the additional property of $\rho$:

(\rho.5) $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,
which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies (\rho.5).

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which (\rho.5) is valid.

Proof. We put for every $x \in R$,

(2.4) $\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y)$.

It is clear that $\rho'$ satisfies the conditions (\rho.1), (\rho.2) and (\rho.5).

Let $\{x_{\lambda}\}_{\lambda \in A}$ be an orthogonal system such that $\sum_{\lambda \in A} \rho'(x_{\lambda}) < +\infty$, then

$\sum_{\lambda \in A} \rho(x_{\lambda}) < +\infty$,

because

$\rho(x) \leq \rho'(x)$ for all $x \in R$.

We have

$x_{0} = \sum_{\lambda \in A} x_{\lambda} \in R$

and

$\rho(x_{0}) = \sum_{\lambda \in A} \rho(x_{\lambda})$ in virtue of (\rho.3).

For such $x_{0}$,

$\rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{\lambda \in A} \rho([x_{\lambda}]y)$

$= \sum_{\lambda \in A} \sup_{0 \leq |y| \leq |x_{\lambda}|} \rho([x_{\lambda}]y) = \sum_{\lambda \in A} \rho'(x_{\lambda})$

holds, i.e. $\rho'$ fulfils (\rho.3).

If $\rho'$ does not fulfil (\rho.4), we have for some $x_{0} \in R$,

$\rho'(\frac{1}{n} x_{0}) = +\infty$ for all $n \geq 1$.

By (\rho.2) and (\rho.4), $x_{0}$ can not be written as $x_{0} = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_{0}$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'\left(\frac{1}{\nu} x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'\left(\frac{1}{\nu} x_2\right) = +\infty \) (\( \nu = 1, 2, \ldots \))
and
\[ \rho'\left(\frac{1}{2} x_2'\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x_1'| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2} \ldots \) such that
\[ \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2} \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = \lfloor(|x| - |y|)^+\rfloor\), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \]
\[ \leq \rho(\alpha \lceil \alpha \rceil |x| + \alpha(1 - \lceil \alpha \rceil) |y| + (1 - \lceil \alpha \rceil) \beta |y|) \]
\[ = \rho(\lceil \alpha \rceil |x| + (1 - \lceil \alpha \rceil) |y|) \]
\[ = \rho(\lceil \alpha \rceil x) + \rho((1 - \lceil \alpha \rceil) y) \]
\[ \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modal depends essentially on the condition (\( \rho.4 \)). Thus, in the above theorems, we cannot replace (\( \rho.4 \)) by the weaker condition:
(\( \rho.4'' \)) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.3 \)) and (\( \rho.4'' \)), but does not (\( \rho.4 \)). For this \( \rho_0 \), we obtain
\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]
for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with
\[ \int_0^1 |x(t)| dt < +\infty. \]
Putting
\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^{\infty} i \text{ mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]
we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (\( A.4 \)), namely,
(\( \rho.6 \)) \[ \lim_{\xi \to 0} \rho(\xi x) = 0 \]
for all \( x \in R \).

A quasi-modal space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modal on \( R \). We can find a functional \( \rho^* \) which satisfies (\( \rho.1 \))~(\( \rho.6 \)) except (\( \rho.3 \)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modal \( \rho' \) which satisfies (\( \rho.5 \)). Now we put
(\( \rho.6 \)) \[ d(x) = \lim_{\xi \to 0} \rho'(\xi x) \]
It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
On F-Norms of Quasi-Modular Spaces

\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[ \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{\nu}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}([p_n]-[p_{n+1}])x_0) \geq \frac{\alpha}{n} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((\rho.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_m]-[p_{m+1}])x_0) = +\infty, \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n \geq 1} 0\) and \(\inf_{n=1, 2, \cdots} d([p_n]x) = 0\) by \((2.7)\). Since \((1-[p_n])n \|y\| \geq (1-[p_n])\|x\|\)

and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
$$d(x) = d([p_n]x) + d((1- [p_n])x)$$
$$\leq d([p_n]x) + d(n(1- [p_n])y)$$
$$\leq d([p_n]x) + d(y).$$

As $n$ is arbitrary, this implies
$$d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y),$$
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
$$\rho^*(x) = \rho^*([y]x) + \rho^*([x]-[y])x)$$
$$= \rho'(y) - d(y) + \rho^*([x]-[y])x)$$
$$\geq \rho^*(y).$$

Thus $\rho^*$ satisfies (\rho.5).

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (\rho.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies (\rho.4)

\[ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K < +\infty. \]

**Proof.** Let $\rho$ satisfy (\rho.4). We need to prove
\[ \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty, \]
where
$$\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_v(x) = \rho'(\frac{1}{\nu}x)$ for $\nu \geq 1$ and $x \in R$. Hence we can find positive numbers $\epsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0$ with
$$\rho(x) \leq \epsilon$$
implies
$$\rho'(\frac{1}{\nu_0}x) \leq \gamma.$$

In $N_0$, we have obviously
$$\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.$$

If $\epsilon \leq 2K$, for any $x_0 \in N_0\dagger$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (\rho.4'), and hence there exists always an orthogonal decomposition such that
\[
\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z
\]
where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) \((i=1, 2, \ldots, n)\), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j=1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields
\[
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
\]
\[
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0}
\]
\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]
Hence, we obtain
\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right)
\]
in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^+ \)
\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]
Therefore, we obtain
\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
\]
where
\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0
\]
Let \( \{x_\lambda\}_{\lambda \in \Lambda} \) be an orthogonal system with \( \sum_{\lambda \in \Lambda} \rho^*(x_\lambda) < + \infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in \Lambda \), we have
\[
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
\]
which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that
\[
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < + \infty,
\]
which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by \((\rho.4)\) and \((2.7)\). Therefore \( \rho^* \) satisfies \((\rho.3)\).

On the other hand, suppose that \( \rho^* \) satisfies \((\rho.3)\) and \( \sup_{x \in R} d(x) = + \infty \). Then we can find an orthogonal sequence \( \{x_\nu\}_{\nu \geq 1} \) such that
\[
\sum_{\nu=1}^{\mu} d(x_\nu) = d(\sum_{\nu=1}^{\mu} x_\nu) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
$$\lim_{t \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_v\}_{v \geq 1}$ with $0 < \alpha_v$ ($v \geq 1$) and
$$\sum_{v=1}^{\infty} \rho^*(\alpha_v x_v) < +\infty.$$  
It follows that $x_0 = \sum_{v=1}^{\infty} \alpha_v x_v \in R$ and $d(x_0) = \sum_{v=1}^{\infty} d(\alpha_v x_v)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$,
$$\rho'(\xi x_0) = \sum_{v=1}^{\infty} \rho'(\xi \alpha_v x_v) \geq \sum_{v=1}^{\infty} d(x_v) = +\infty,$$
which is inconsistent with (\rho.4). Therefore we have
$$\sup_{x \in R} \rho^{*} (\xi x) \leq \sup_{x \in R} d(x) < +\infty.$$  
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^7\) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geq 0$ for all $\lambda \in \Lambda$. Putting
$$(p_{n,\lambda}) = [(2nx_{\lambda} - nx)^+]$$
we have
$$(p_{n,\lambda}) \uparrow_{\lambda \in \Lambda} (p)$$
and
$$2n[p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}]nx_{\lambda},$$
which implies $\rho^*(n[p_{n,\lambda}]x_{\lambda}) = 0$ and $\rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp.$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda}(x_\lambda \in [p]R_0)$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^p$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup x_\lambda \in R$, $0 \leq x_\lambda \in S(\lambda \in \Lambda)$ implies $\cup x_\lambda \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

\(^7\)\(^8\)\(^9\)
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^\perp$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ which is semi-continuous, i.e.
\[
\sup_{1 \leq i \leq d} \| x_i \|_0 = \| x \|_0 \quad \text{for any } 0 \leq x, \sum_{i=1}^{d} x_i.
\]

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put
\[
(3.1) \quad \| x \|_0 = \inf \{ \xi \mid \rho^*(\frac{1}{\xi}x) \leq \xi \}.
\]
Then,
\begin{enumerate}
  \item $0 \leq \| x \|_0 = - \| x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)\sim(\rho.6)$, (2.1) and the definition of $R^\perp$.
  \item $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.
  \item $\lim_{\alpha_n \to 0} \| \alpha_n x \|_0 = 0$ and $\lim_{\| x \|_0 > 0} \| \alpha x \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \to \alpha_0} \| \alpha x \|_0 = \| \alpha_0 x \|_0$ for all $x \in R^\perp$ and $\alpha_0 \geq 0$. If $x \in R^\perp$ and $[p_\lambda] \uparrow_{\lambda \in A} [p]$, for any positive number $\xi$ with $\| [p] x \|_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi} [p_\lambda] x) > \xi$ and hence $\sup_{\lambda \in \Lambda} \| [p_\lambda] x \|_0 \geq \xi$. Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| [p_\lambda] x \|_0 = \| [p] x \|_0, \quad \text{if } [p_\lambda] \uparrow_{\lambda \in A} [p].
\]
Let $0 \leq x_1 \uparrow_{i \in A} x$. Putting
\[
[p_{n,i}] = \left[ x_i - \left(1 - \frac{1}{n} \right)x \right]^+
\]
we have
\[
[p_{n,i}] \uparrow_{i \in A} [x] \quad \text{and} \quad [p_{n,i}] x_i \geq [p_{n,i}] \left(1 - \frac{1}{n} \right)x \quad \text{for } n \geq 1.
\]
As is shown above, since
\[
\sup_{i \in A} \| [p_{n,i}] x_i \|_0 \geq \sup_{i \in A} \| [p_{n,i}] \left(1 - \frac{1}{n} \right)x \|_0 = \left(1 - \frac{1}{n} \right)x \|_0,
\]
we have
\[
\sup_{i \in A} \| x_i \|_0 \geq \left(1 - \frac{1}{n} \right)x \|_0
\]
and also $\sup_{i \in A} \| x_i \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n \to \infty} \| x_n \|_0 = 0 \iff \lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

On F-Norms of Quasi-Modular Spaces

211
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,v}, x_v \geq 0$ and $a \geq 0 (n, v = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\begin{align}
(p_{n,v})\uparrow_{v=1}^\infty (p_n) \\
[p_{n,v}]x_v \geq n[p_{n,v}]a
\end{align}

for all $n, v \geq 1$.

Then $\{x_v\}_{v \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
||[q_m]a||_0 > \frac{\delta}{2} \\
n_m[q_m]x_{\nu_m} \geq n_m[q_m]a
\end{align}

and

\begin{align}
n_m[q_m]a \geq [q_m]x_{n_m-1}, \\
n_{m+1} > n_m (m = 2, 3, \cdots),
\end{align}

where $\delta = ||p_0||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0]\uparrow_{\nu=1}^\infty [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}][p_0]a||_0 > \frac{||p_0||_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu})^+]\uparrow_{n=1}^\infty [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ ||(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ ||p_{n_{k+1},\nu_{k+1}}[n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}[n_{k+1}a-x_{\nu_k})^+[q_k], \]

because

\[ [q_{k+1}] \leq [q_k], ||[q_{k+1}]a|| > \frac{\delta}{2}, [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_{k+1}}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
\[
\| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geqq \| [q_{k+1}](x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geqq \| n_{k+1}[q_{k+1}a - n_{k}[q_{k+1}]a] \|_0 \geqq \| [q_{k+1}]a_0 \|_0 \geqq \frac{\delta}{2},
\]

since \([q_{k+1}] \leqq [q_{k}] \leqq ((n_{k}a - x_{\nu-1})^+)\) implies \([q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geqq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R_{0}^+\) is an F-space with \(\| \cdot \|_0\) if and only if \(\rho\) satisfies \((\rho.4')\).

**Proof.** If \(\rho\) satisfies \((\rho.4')\), \(\rho^*\) is a quasi-modular which fulfills also \((\rho.5)\) and \((\rho.6)\) in virtue of Theorem 2.3. Since \(\| x \|_0 \left(= \inf \{ \xi; \rho^*(\frac{x}{\xi}) \leqq \xi \} \right)\) is a quasi-norm on \(R_{0}^+\), we need only to verify completeness of \(\| \cdot \|_0\). At first let \(\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^+\) be a Cauchy sequence with \(0 \leqq x_{\nu} \uparrow_{\nu=1,2,\ldots}\). Since \(\rho^*\) satisfies \((\rho.3)\), there exists \(0 \leqq x_0 \in R_{0}^+\) such that \(x_0 = \bigcup_{\nu=1}^{\infty}x_{\nu}\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,v}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{\nu=1}^{\infty}[p_{n,v}] = [p_n]\), we obtain
\[
(3.6) \quad [p_{n,v}]x_{\nu} \geqq n[p_{n,v}]x_0 \quad \text{for all } n, \nu \geqq 1
\]
and \([p_n]\downarrow_{n=1}^{\infty}0\). Since \(\{x_{\nu}\}_{\nu \geqq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty}[p_n] = 0\), that is, \(\bigcup_{n=1}^{\infty}([x_0] - [p_n]) = [x_0]\). And
\[
(1-[p_{n,v}]) \geqq (1-[p_n]) \quad \text{for } (n, \nu \geqq 1)
\]
implies
\[
n(1-[p_n])x_0 \geqq (1-[p_n])x_0 \geqq 0.
\]
Hence we have
\[
y_n = \bigcup_{\nu=1}^{\infty}(1-[p_n])x_\nu \in R_{0}^+,
\]
because \(R_{0}^+\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geqq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)
\[
\gamma = \sup_{\nu \geqq 1} \| x_{\nu} \|_0 < +\infty,
\]
which implies
\[
\| y_n \|_0 = \sup_{\nu \geqq 1} \| (1-[p_n])x_\nu \|_0 \leqq \gamma
\]
for every \(n \geqq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geqq 2)\). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geqq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leqq \gamma\). This implies
for all $n \geq 1$ by the formula (3.1). Then ($\rho$3) assures the existence of $z = \sum_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from

$$z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-\left[ p_{n} \right]) x_{\nu} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} [x_{0}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.$$  

By semi-continuity of $|| \cdot ||_{0}$, we have

$$|| z - x_{\nu} ||_{0} \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_{0}$$

and furthermore $\lim_{\nu \to \infty} || z - x_{\nu} ||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu\geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{+}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu\geq 1}$ of $\{x_{\nu}\}_{\nu\geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$.

This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{n-1}}$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|$, we have a Cauchy sequence $\{z_{n}\}_{n\geq 1}$ with $0 \leq z_{n} \uparrow_{n=1}^{\infty}$. Then by the fact proved just above,

$$z_{0} = \sum_{n=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{+},$$

and

$$\lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0.$$  

Since $\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0.$$  

Since $\{y_{\nu}\}_{\nu\geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu\geq 1}$, it follows that

$$\lim_{\nu \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0.$$  

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{+}$, that is, $R_{0}^{+}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{+}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu\geq 1} \in R_{0}^{+}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$). Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy ($\rho.4'$).

Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0}^{+} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ fulfills ($\rho.4'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

(4.1) \[ \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\} \]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

(4.2) \[ \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \]

for all $x \in L$ hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty (x \in L)$ and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow 0 \forall n \in \mathbb{N}$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \|_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi y}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[
\| x \|_1 = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \|_1 = \inf_{m(\xi x) \leq 1} \frac{1}{\xi}.
\]

[3 or 4]. For the general modulaters considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have $\|x\|_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\eta$ with $\|x\|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $\|\cdot\|_0$ on $L$ which is equivalent to $\|\cdot\|_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$\|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad \text{for} \quad x \in R$$

is a semi-continuous quasi-norm on $R^+_0$ and $\|\cdot\|_1$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all} \quad x \in R^+_0.$$

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$ and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \leq x \in R$ (i.e. $R_0 = \{0\})$. A sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called order-convergent to $a$ and denoted by $\nu \to \infty \{x_\nu = a\}$, if there exists a sequence of elements $\{a_\nu\}_{\nu \geq 1}$ such that $|x_\nu - a| \leq a_\nu$ ($\nu \geq 1$) and $a_\nu \to 0$. And a sequence of elements $\{x_\nu\}_{\nu \geq 1}$ is called star-convergent to $a$ and denoted by $s \to \nu \to \infty \{x_\nu = a\}$, if for any subsequence $\{y_\nu\}_{\nu \geq 1}$ of $\{x_\nu\}_{\nu \geq 1}$, there exists a subsequence $\{z_\nu\}_{\nu \geq 1}$ of $\{y_\nu\}_{\nu \geq 1}$ with $s \to \nu \to \infty \{z_\nu = a\}$.

A quasi-norm $\|\cdot\|$ on $R$ is termed to be continuous, if $\inf_{\nu \geq 1} \|a_\nu\| = 0$ for any $a_\nu \to 0$. In the sequel, we write by $||\cdot||_0$ (or $||\cdot||_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $\|\cdot||_0$ (or $\|\cdot||_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

$$\text{(5.1) for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.$$  

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
On F-Norms of Quasi-Modular Spaces

217

sequence of projector \( \{[p_n]\}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^\infty 0 \). Hence by (3.1) it follows that \( \|[p_n]x\|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \|\cdot\|_0 \).

Sufficiency. Let \( a_{\nu} \downarrow_{\nu=1}^\infty 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^\infty 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1] a_n = [p_n^\epsilon] a_n + (1 - [p_n^\epsilon]) a_n \leq [p_n^\epsilon] a_1 + \epsilon a_1 \).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon] a_1) + \rho^*(\xi (1 - [p_n^\epsilon]) a_1)
\]

for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^\infty 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon] a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_n^\epsilon] a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} ||a_n||_0 = 0 \) and \( \|\cdot\|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** \( \|\cdot\|_0 \) is continuous, if

\[
(5.2) \quad \rho^*(a_\nu) \rightarrow 0 \text{ implies } \rho^*(\alpha a_\nu) \rightarrow 0 \text{ for every } \alpha \geq 0.
\]

From the definition, it is clear that \( \text{s-lim}_{\nu} x_\nu = 0 \) implies \( \lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0 \), if \( ||\cdot||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} ||x_\nu||_0 = 0 \) (or \( \lim_{\nu \rightarrow \infty} ||x_\nu|| = 0 \)) implies \( \text{s-lim}_{\nu \rightarrow \infty} x_\nu = 0 \), if \( ||\cdot||_0 \) is complete (i.e. \( \rho^* \) satisfies \( \rho.3 \)).

If we replace \( \lim_{\nu \rightarrow \infty} ||x_\nu|| = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x) = 0 \text{ implies } x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies \( (5.3) \) and \( ||\cdot||_0 \) is complete, \( \rho(a_\nu) \rightarrow 0 \) implies \( \text{s-lim}_{\nu \rightarrow \infty} a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11}\) i.e. \( \rho^*(x) = \sup_{i \in A} \rho^*(x_i) \) for any \( 0 \leq x \leq \sup_{i \in A} x \). If

\(^{11}\) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf_{y \in A} \sup_{i \in A} \rho^*(y_i) \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_*(x) \rightarrow 0 \).
\[ \rho(a_{\nu}) \leq \frac{1}{2^\nu} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of (\( \rho.3 \)).

Now, since
\[ \rho\left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\mu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_{\mu}| \right) \right) = 0 \) and hence (5.3) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\mu \geq \nu}^{\infty} |a_{\mu}| \right) = 0. \]

Thus we see that \( \{a_{\mu}\}_{\mu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^\nu} \quad (\nu=1, 2, \ldots) \). Therefore we have \( s-\lim b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamura) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( || \cdot ||_0 \) is complete and continuous, then (5.2) holds.

**References**


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