<table>
<thead>
<tr>
<th>Title</th>
<th>ON F-NORMS OF QUASI-MODULAR SPACES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Koshi, Shôzô; Shimogaki, Tetsuya</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 15(3-4), 202-218</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1961</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56023">http://hdl.handle.net/2115/56023</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSHIU_15_N3-4_202-218.pdf</td>
</tr>
</tbody>
</table>
ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff’s sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

1. $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;
2. $\rho(x + y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;
3. If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;
4. $\limsup_{\xi \to 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

1. $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;
2. $\rho(-x) = \rho(x)$;
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
4. $\alpha_{n} \to 0$ implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;
5. for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula:

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow a_{0}$.
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.  

1) $\perp$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow a_{0}$.
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.  


\begin{equation}
\|x\|_0 = \inf \{ \xi; \rho\left(\frac{1}{\xi} x\right) \leq \xi \}\)
\end{equation}

and \( \|x_a\|_0 \to 0 \) is equivalent to \( \rho(\alpha x_a) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2) \( \sim \) (A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies
\begin{equation}
(\rho.4') \quad \sup_{x \in R} \{ \lim_{a \to 0} \rho(ax) \} < +\infty
\end{equation}

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S 83 \]). At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n \mid p \mid \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N=\{p\}^1 \), that is, \( x = [p]x + (1 - [p])x. \)

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

(2.1) $\rho(0) = 0$;
(2.2) $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
(2.3) $\rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x)$ for any $[p], [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p]$.

In the argument below, we have to use the additional property of $\rho$:

$(\rho.5)$ $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

(2.4) $\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y)$.

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_{\lambda}\}_{\lambda \in \Lambda}$ be an orthogonal system such that $\sum_{\lambda \in \Lambda} \rho'(x_{\lambda}) < +\infty$, then

$$\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty,$$

because

$$\rho(x) \leq \rho'(x)$$

for all $x \in R$.

We have

$$x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \in R$$

and

$$\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$$

in virtue of $(\rho.3)$.

For such $x_{0}$,

$$\rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{\lambda \in \Lambda} \rho([x_{\lambda}]y)$$

$$= \sum_{\lambda \in \Lambda} \sup_{0 \leq |y| \leq |x_{\lambda}|} \rho([x_{\lambda}]y) = \sum_{\lambda \in \Lambda} \rho'(x_{\lambda})$$

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_{0} \in R$,

$$\rho'(\frac{1}{n} x_{0}) = +\infty$$

for all $n \geq 1$.

By $(\rho.2)$ and $(\rho.4)$, $x_{0}$ can not be written as $x_{0} = \sum_{\nu=1}^{s} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq s$, namely, we can decompose $x_{0}$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'\left(\frac{1}{\nu}x_1\right) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'\left(\frac{1}{\nu}x_2\right) = +\infty \) (\( \nu = 1, 2, \ldots \))
and
\[ \rho'\left(\frac{1}{2}x_2'\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2}y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_{\nu}\}_{\nu=1,2}, \ldots \) such that
\[ \rho\left(\frac{1}{\nu}y_{\nu}\right) \geq \nu \]
and
\[ 0 \leq |y_{\nu}| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_{\nu}\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_{\nu} \in R \]
and
\[ \rho\left(\frac{1}{\nu}y_0\right) \geq \rho\left(\frac{1}{\nu}y_{\nu}\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \\
\leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta |y|) \\
= \rho([p]|x| + (1-[p])|y|) \\
= \rho([p]x) + \rho((1-[p])y) \\
\leq \rho(x) + \rho(y).
\]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| dt < +\infty . \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^{\infty} i \mes \{ t : x(t) = \frac{1}{i} \} , \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\((\rho.6)\)

\[ \lim_{\xi \to 0} \rho(\xi x) = 0 \]

for all \( x \in R \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[(2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
On F-Norms of Quasi-Modular Spaces

\[ d(x + y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

(2.6) \[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for \( \rho^* \), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that (\rho.5) is true for \( \rho^* \). First we have to note

(2.7) \[ \inf_{\lambda \in A} d([p_{\lambda}]x) = 0 \]

for any \([p_{\lambda}] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \([p_{\lambda}] \downarrow_{\lambda \in A} 0\) and \( x_0 \in R \).

Hence,

\[ \rho'\left(\frac{1}{\nu}[p_{\lambda}]x_0\right) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all \( \nu \geq 1 \) and \( \lambda \in A \). Thus we can find a subsequence \( \{\lambda_n\}_{n \geq 1} \) of \( \{\lambda\}_{i \in A} \) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'\left(\frac{1}{n}[p_{\lambda_n}] - [p_{\lambda_{n+1}}]x_0\right) \geq \frac{\alpha}{2} \]

for all \( n \geq 1 \) in virtue of (\rho.2) and (2.3). This implies

\[ \rho'\left(\frac{1}{n}x_0\right) \geq \sum_{m \geq n} \rho'\left(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0\right) = +\infty, \]

which is inconsistent with (\rho.4). Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)\] for \( x, y \in R \) with \([x] = [y] \) and \( n \geq 1 \). Then, \([p_n] \downarrow_{n=1} 0 \) and \( \inf_{n=1,2,\ldots} d([p_n]x) = 0 \) by (2.7). Since \((1 - [p_n])n|y| \geq (1 - [p_n])|x| \)

and

\[ d(\alpha x) = d(x) \]

for \( \alpha > 0 \) and \( x \in R \), we obtain
\[ d(x) = d([p_n]x) + d((1-[p_n])x) \leq d([p_n]x) + d(n(1-[p_n])y) \leq d([p_n]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1,2,...} d([p_n]x) + d(y), \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x
\geq \rho^*(y) - d(y) + \rho^*([x] - [y])x
\geq \rho^*(y).
\]
Thus \( \rho^* \) satisfies (\( \rho.5 \)).

\[ \text{Q.E.D.} \]

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies (\( \rho.4' \))

\[ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K \leq +\infty. \]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K' \leq +\infty, \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho\left(\frac{1}{\nu}x\right) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \varepsilon \), \( \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[ \rho(x) \leq \varepsilon \text{ implies } \rho\left(\frac{1}{\nu_0}x\right) \leq \gamma. \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty. \]

If \( \varepsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
On F-Norms of Quasi-Modular Spaces

\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \ (i = 1, 2, \ldots, n) \), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[ \rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \]

\[ \leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \]

\[ \leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left( \sup_{0 \leq a \leq a_0} \rho(a x) \right) + \gamma \]

Hence, we obtain

\[ \lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right) \]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^\perp \)

\[ \lim_{\xi \to 0} \rho'(\xi x) \leq \gamma \]

Therefore, we obtain

\[ \sup_{x \in R} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} \leq \gamma' \]

where

\[ \gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0 \]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[ \sum_{i=1}^{k} d(x_{i,\nu}) = d(\sum_{i=1}^{k} x_{i,\nu}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{i,\nu}) \leq \gamma' \]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[ \sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty \]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[ \sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu \]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
$$\lim_{t \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_{\nu}\}_{\nu \geq 1}$ with $0 < \alpha_{\nu} (\nu \geq 1)$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu} x_{\nu}) < +\infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x_{\nu})$ from $(\rho.3)$. For such $x_0$, we have for every $\xi \geq 0$,
$$\rho^*(\xi x_0) = \sum_{\nu=1}^{\infty} \rho^*(\xi \alpha_{\nu} x_{\nu}) \geq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty,$$
which is inconsistent with $(\rho.4)$. Therefore we have
$$\sup_{x \in R} (\lim_{\epsilon \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.$$
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geq 0$ for all $\lambda \in \Lambda$. Putting
$$[p_{n,1}] = (2nx_{\lambda} - nx)^+,$$
we have
$$[p_{n,1}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad 2n[p_{n,1}]x_{\lambda} \geq [p_{n,1}]nx,$$
which implies $\rho^*(n[p_{n,1}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,1}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$. Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp .$$
In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}, x_{\lambda} \in [p]R_0$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{p\star}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup_{\lambda \in \Lambda} x_{\lambda} \in S$. Since $R_0$ is the set of all real functions defined on $\Lambda$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---
Theorem 3.1. Let \( R \) be a quasi-modular space. Then \( R_0^\perp \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) which is semi-continuous, i.e.
\[
\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 = \| x \|_0
\]
for any \( 0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x \).

Proof. In virtue of Theorems 2.1 and 2.2, \( \rho^* \) satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\). Now we put
\[
(3.1) \quad \| x \|_0 = \inf \left\{ \xi ; \rho^*(\frac{1}{\xi} x) \leq \xi \right\} .
\]

Then,

i) \( 0 \leq \| x \|_0 = - \| -x \|_0 < \infty \) and \( \| x \|_0 = 0 \) is equivalent to \( x = 0 \); follows from \((\rho.1), (\rho.6), (2.1)\) and the definition of \( R_0^\perp \).

ii) \( \| x + y \|_0 \leq \| x \|_0 + \| y \|_0 \) for any \( x, y \in R \); follows also from \((A.3)\) which is deduced from \((\rho.4)\).

iii) \( \lim_{\alpha_n \rightarrow 0} \| \alpha_n x \|_0 = 0 \) and \( \lim \| \alpha x \|_0 = 0 \); is a direct consequence of \((\rho.5)\). At last we shall prove that \( \| \cdot \|_0 \) is semi-continuous. From ii) and iii), it follows that \( \lim \| \alpha x \|_0 = \| \alpha_0 x \|_0 \) for all \( x \in R_0^\perp \) and \( \alpha_0 \geq 0 \). If \( x \in R_0^\perp \) and \( \lfloor p \rfloor \uparrow_{\lambda \in \Lambda} \lfloor p \rfloor \), for any positive number \( \xi \) with \( \| \lfloor p \rfloor x \|_0 = \xi \) we have \( \rho^*(\frac{1}{\xi} \lfloor p \rfloor x) > \xi \), which implies \( \sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi} \lfloor p \rfloor x) > \xi \) and hence \( \sup_{\lambda \in \Lambda} \| \lfloor p \rfloor x \|_0 \geq \xi \). Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| \lfloor p \rfloor x \|_0 = \| \lfloor p \rfloor x \|_0 , \quad \text{if} \quad \lfloor p \rfloor \uparrow_{\lambda \in \Lambda} \lfloor p \rfloor .
\]

Let \( 0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x \). Putting
\[
\lfloor p_{n,\lambda} \rfloor = \lfloor (x_{\lambda} - (1 - \frac{1}{n}) x)^{+} \rfloor
\]
we have
\[
\lfloor p_{n,\lambda} \rfloor \uparrow_{\lambda \in \Lambda} \lfloor x \rfloor \quad \text{and} \quad \lfloor p_{n,\lambda} \rfloor x_{\lambda} \geq [\lfloor p_{n,\lambda} \rfloor] \left(1 - \frac{1}{n}\right) x
\]
As is shown above, since
\[
\sup_{\lambda \in \Lambda} \| \lfloor p_{n,\lambda} \rfloor x_{\lambda} \|_0 \geq \sup_{\lambda \in \Lambda} \| \lfloor p_{n,\lambda} \rfloor \left(1 - \frac{1}{n}\right) x \|_0 = \| \left(1 - \frac{1}{n}\right) x \|_0,
\]
we have
\[
\sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 \geq \left(1 - \frac{1}{n}\right) x \|_0
\]
and also \( \sup_{\lambda \in \Lambda} \| x_{\lambda} \|_0 \geq \| x \|_0 \). As the converse inequality is obvious by iv), \( \| \cdot \|_0 \) is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that \( \lim \| x_n \|_0 = 0 \) if and only if \( \lim \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \).
In order to prove the completeness of quasi-norm $\| \cdot \|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,v}, x_v \geq 0$ and $a \geq 0 (n, v = 1, 2, \cdots)$ be the elements of $R^\bot_0$ such that

\begin{align}
&[p_{n,v}] \uparrow_{v=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0; \\
&[p_{n,v}] x_v \geq n [p_{n,v}] a \text{ for all } n, v \geq 1.
\end{align}

Then $\{x_v\}_{v \geq 1}$ is not a Cauchy sequence of $R^\bot_0$ with respect to $\| \cdot \|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
&\| [q_m] a \|_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \\
&n_m [q_m] a \geq [q_m] x_{\nu_{m-1}} \quad (m = 2, 3, \cdots)
\end{align}

where $\delta = \| [p_0] a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,v}] [p_0] \uparrow_{v=1}^{\infty} [p_0]$ and $\| \cdot \|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ \| [p_{1,v}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,v}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n a - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k] a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ \| (n_{k+1} a - x_{\nu_k})^+] [q_k] a \|_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ \| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+] [q_k] a \|_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $\| \cdot \|_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+] [q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
On $F$-Norms of Quasi-Modular Spaces

\[ \| x_{k} - x_{k-1} \|_0 \geq \| [q_{k+1}] (x_{k+1} - x_{k-1}) \|_0 \]
\[ \geq \| n_{k+1} [q_{k+1}] a - n_{k} [q_{k+1}] a \|_0 \geq \| [q_{k+1}] a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_{k}] \leq [(n_k a - x_{k-1})^+]\) implies \([q_{k+1}] n_k a \geq [q_{k+1}] x_{k-1}\) by (3.4). It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^+$ is an $F$-space with \(\| \cdot \|_0\) if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $\| x \|_0 = \inf \{ \xi ; \rho^*(\frac{x}{\xi}) \leq \xi \}$ is a quasi-norm on $R_0^+$, we need only to verify completeness of \(\| \cdot \|_0\). At first let \(\{x_{\nu}\}_{\nu \geq 1} \subset R_0^+\) be a Cauchy sequence with $0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots}$. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leq x_0 \in R_0^+$ such that $x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting \([p_{n,\nu}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{\nu=1}^{\infty} [p_{n,\nu}] = [p_n]\), we obtain

\[ (3.6) \quad [p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] x_0 \]

for all $n, \nu \geq 1$ and \([p_n]\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{\nu=1}^{\infty} [p_n] = 0\), that is, \(\bigcup_{\nu=1}^{\infty} ([x_0] - [p_n]) = [x_0]\). And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1) \]

implies

\[ n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_\nu \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_\nu \in R_0^+, \]

because $R_0^+$ is universally continuous. As $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)

\[ \gamma = \sup_{\nu \geq 1} \| x_\nu \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n]) x_\nu \|_0 \leq \gamma \]

for every $n \geq 1$ by semi-continuity of \(\| \cdot \|_0\). We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ ($n \geq 2$). It follows from the definition of $y_n$ that $\{z_{\nu}\}_{\nu \geq 1}$ is an orthogonal sequence with \(\sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma\). This implies
\[ \sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leqq \gamma \]

for all \( n \geqq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of 
\[ z=\bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} (1-[p_{\nu}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} x_{\nu} . \]

By semi-continuity of \( \| \cdot \|_0 \), we have 
\[ \| z-x_{\nu} \|_0 \leqq \sup_{\mu \geqq \nu} \| x_{\mu}-x_{\nu} \|_0 \]
and furthermore \( \lim_{\rightarrow \infty} \| z-x_{\nu} \|_0 = 0 \).

Secondly let \( \{x_{\nu}\}_{\nu \geqq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geqq 1} \) of \( \{x_{\nu}\}_{\nu \geqq 1} \) such that 
\[ \| y_{\nu+1}-y_{\nu} \|_0 \leqq \frac{1}{2^{\nu}} \]
for all \( \nu \geqq 1 \).

This implies 
\[ \| \sum_{\nu=m}^{n} y_{\nu+1}-y_{\nu} \|_0 \leqq \sum_{\nu=m}^{n} \| y_{\nu+1}-y_{\nu} \|_0 \leqq \frac{1}{2^{m-1}} \]
for all \( n>m \geqq 1 \).

Putting \( z_0=\sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \), we have a Cauchy sequence \( \{z_n\}_{n \geqq 1} \) with \( 0 \leqq z_n \uparrow \infty \).

Then by the fact proved just above, 
\[ z_0=\bigcup_{n=1}^{\infty} z_n = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_{\nu}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} \bigcup_{\nu=1}^{\infty} x_{\nu} . \]

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \) is convergent, \( y_1+\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \) is also convergent and
\[ \| y_1+\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})-y_n \|_0 = \| \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \|_0 \leqq \| z_0-z_n \|_0 \to 0 . \]

Therefore \( \| \cdot \|_0 \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( \| \cdot \|_0 \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geqq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geqq 1 \)).

Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \((\rho.4')\).

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.

Q.E.D.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|_{0}$ if and only if $\rho$ fulfills ($\rho.4'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
\| x \|_{1} = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
\end{equation}

and show that $\| \cdot \|_{1}$ is also a quasi-norm on $L$ and

\begin{equation}
\| x \|_{0} \leq \| x \|_{1} \leq 2 \| x \|_{0}
\end{equation}

for all $x \in L$ hold, where $\| \cdot \|_{0}$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_{1} = \| -x \|_{1} < +\infty$ ($x \in L$) and that $\| x \|_{1} = 0$ is equivalent to $x = 0$. Since $\alpha_{n} \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_{n} x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_{n} \|_{1} = 0$ implies $\lim_{n \to \infty} \rho(\xi x_{n}) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha x_{n} \|_{1} = 0$ for all $\alpha_{n} \downarrow_{n=1}^{\infty} 0$ and that $\lim_{n \to \infty} \| x_{n} \|_{1} = 0$ implies $\lim_{n \to \infty} \| \alpha x_{n} \|_{1} = 0$ for all $\alpha > 0$. If $\| x \|_{1} < \alpha$ and $\| y \|_{1} < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ \| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta} (x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y)\right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore $\| x + y \|_{1} \leq \| x \|_{1} + \| y \|_{1}$ holds for any $x, y \in L$ and $\| \cdot \|_{1}$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have $\|x\|_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\eta$ with $\|x\|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $\|\cdot\|_1$ on $L$ which is equivalent to $\|\cdot\|_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$\|x\|_1 = \inf_{t > 0} \left\{ \frac{1}{t} + \rho^*(\xi x) \right\}$$

(4.3)

is a semi-continuous quasi-norm on $R_0^+$ and $\|\cdot\|_1$ is complete if and only if $\rho$ satisfies (ρ.4'), where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$\|x\|_1 \leq \|x\|_0 \leq 2\|x\|_0$$

(4.4)

for all $x \in R_0^+$. 

---

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies (ρ.1)~(ρ.6) except (ρ.3) and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \leq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $\text{o-lim} x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a_n| \leq a_n$ ($n \geq 1$) and $a_n \downarrow 0$. And a sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $\text{s-lim} x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $\text{o-lim} z_n = a$.

A quasi-norm $\|\cdot\|$ on $R$ is termed to be continuous, if $\inf_{n \geq 1} \|a_n\| = 0$ for any $a_n \downarrow 0$. In the sequel, we write by $\|\cdot\|_0$ (or $\|\cdot\|_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $\|\cdot\|_0$ (or $\|\cdot\|_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z]R$ is finite dimensional and $\rho(y) < +\infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty}0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_{\nu} \downarrow_{\nu=1}^{\infty}0 \) and put \( [p^\epsilon_n] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p^\epsilon_n] \downarrow_{n=1}^{\infty}0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p^\epsilon_n]a_n + (1 - [p^\epsilon_n])a_n \leq [p^\epsilon_n]a_1 + \epsilon a_1 \).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*([\xi [p^\epsilon_n]a_1]) + \rho^*(\xi(1 - [p^\epsilon_n])a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p^\epsilon_n] \downarrow_{n=1}^{\infty}0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p^\epsilon_n]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p^\epsilon_n]a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain
\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \rho^*(\xi a_n) = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if
\[
(5.2) \quad \rho^*(a_{\nu}) \rightarrow 0 \quad \text{implies} \quad \rho^*(\alpha a_{\nu}) \rightarrow 0 \quad \text{for every} \quad \alpha \geq 0.
\]

From the definition, it is clear that \( s-\lim_{\nu \rightarrow \infty} x_{\nu} = 0 \) implies \( \lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0 \) (or \( \lim_{\nu \rightarrow \infty} \| x_{\nu} \|_1 = 0 \)) implies \( s-\lim_{\nu \rightarrow \infty} x_{\nu} = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \)).

If we replace \( \lim_{\nu \rightarrow \infty} \| x_{\nu} \| = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_{\nu}) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[
(5.3) \quad \rho^*(x) = 0 \quad \text{implies} \quad x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_{\nu}) \rightarrow 0 \) implies \( s-\lim_{\nu \rightarrow \infty} a_{\nu} = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11} \) i.e. \( \rho^*(x) = \sup_{i \in A} \rho^*(x_i) \) for any \( 0 \leq x_{i \in A} \). If

\(^{11} \) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf \{ \sup_{y_{i \in A}} \rho^*(y_i) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_*(x) \rightarrow 0 \).
\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of \((\rho.3)\).

Now, since
\[ \rho \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]
holds for each \( \nu \geq 1, \rho \left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) \right) = 0 \) and hence \((5.3)\) implies
\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}| \right) = 0. \]

Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu = 1, 2, \cdots) \). Therefore we have \( s-lim_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^{*} \) satisfies \((5.3)\) and \( \| \cdot \|_{0} \) is complete and continuous, then \((5.2)\) holds.

References