ON $F$-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

\begin{align*}
(\rho.1) & \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all } x \in R; \\
(\rho.2) & \quad \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any } x, y \in R \text{ with } x \perp y^{1)}; \\
(\rho.3) & \quad \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \quad \text{for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda}^{2)}, \text{there exists } x_{0} \in R \text{ such that } x_{0} = \sum_{\lambda \in \Lambda} x \text{ and } \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}); \\
(\rho.4) & \quad \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \quad \text{for all } x \in R.
\end{align*}

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $^{3)} m$ on $R$ for which every universally continuous linear functional$^{4)}$ is continuous with respect to the norm defined by the modular$^{5)} m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

\begin{align*}
(A.1) & \quad \rho(x) \geq 0 \quad \text{and } \rho(x) = 0 \quad \text{if and only if } x = 0; \\
(A.2) & \quad \rho(-x) = \rho(x); \\
(A.3) & \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1; \\
(A.4) & \quad \alpha_{n} \to 0 \implies \rho(\alpha_{n} x) \to 0 \quad \text{for every } x \in R; \\
(A.5) & \quad \text{for any } x \in L \text{ there exists } \alpha > 0 \text{ such that } \rho(\alpha x) < +\infty.
\end{align*}

They showed that $L$ is a quasi-normed space with a quasi-norm $|| \cdot ||_{0}$ defined by the formula;

1) $x \perp y \text{ means } |x| \cap |y| = 0.$
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma.$
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0.$
R is called semi-regular, if for any $x \neq 0,$ $x \in R,$ there exists a universally continuous linear functional $f$ such that $f(x) \neq 0.$
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R.$
(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \)~(\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)~(A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \{ \lim_{\alpha \to 0} \rho(\alpha x) \} < +\infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \( \S 83 \)]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R, \ [p] \) is a projector: \( [p] x = \bigcup_{n=1}^{\infty} (n \cap p \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N=[p]^1 \), that is, \( x=[p] x +(1- [p]) x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

1. $\rho(0) = 0$;
2. $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
3. $\rho([p]x) = \sup_{i \in A} \rho([p_i]x)$ for any $[p_i] \uparrow_{i \in A} [p]$.

In the argument below, we have to use the additional property of $\rho$:

\[(\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if} \quad |x| \leq |y|, \quad x, y \in R,\]

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

Proof. We put for every $x \in R$,

\[(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).\]

It is clear that $\rho'$ satisfies the conditions $(\rho.1), (\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

\[\sum_{i \in A} \rho(x_i) < +\infty,\]

because

\[\rho(x) \leq \rho'(x) \quad \text{for all} \quad x \in R.\]

We have

\[x_0 = \sum_{i \in A} x_i \in R\]

and

\[\rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of} \quad (\rho.3).\]

For such $x_0$,

\[\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)\]

\[= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)\]

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_0 \in R$,

\[\rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all} \quad n \geq 1.\]

By $(\rho.2)$ and $(\rho.4)$, $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho\left(\frac{1}{\nu} x_1\right) = +\infty \) \((\nu = 1, 2, \cdots)\) and \( \rho(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho\left(\frac{1}{\nu} x_2\right) = +\infty \quad (\nu = 1, 2, \cdots) \]
and
\[ \rho\left(\frac{1}{2} x_2'\right) > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [\lceil |x| - |y| \rceil^+]\), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(|x| + |y|)
\]
\[
\leq \rho(\alpha \lfloor |x| + \alpha(1-\lfloor |x|) + \beta \lfloor |y| + (1-\lfloor |y|)\beta |y|)
\]
\[
= \rho(\lfloor |x| + (1-\lfloor |y|)\beta |y|)
\]
\[
= \rho(\lfloor x| + \rho((1-\lfloor |y|)y)
\]
\[
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \(\rho'\) as a quasi-modular depends essentially on the condition (\(\rho.4\)). Thus, in the above theorems, we cannot replace (\(\rho.4\)) by the weaker condition:

(\(\rho.4''\)) for any \(x \in R\), there exists \(\alpha \geq 0\) such that \(\rho(\alpha x) < +\infty\).

In fact, the next example shows that there exists a functional \(\rho_0\) on a universally continuous semi-ordered linear space satisfying (\(\rho.1\)), (\(\rho.2\)), (\(\rho.3\)) and (\(\rho.4''\)), but does not (\(\rho.4\)). For this \(\rho_0\), we obtain

\[
\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \(x \neq 0\).

**Example.** \(L_1[0,1]\) is the set of measurable functions \(x(t)\) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| \, dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \text{ mes } \left\{ t : x(t) = \frac{1}{i} \right\},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\(\rho.6\)) \[\lim_{\xi \to 0} \rho(\xi x) = 0\] for all \(x \in R\).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \(F\)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \(\rho\) be a quasi-modular on \(R\). We can find a functional \(\rho^*\) which satisfies (\(\rho.1\)) \(\sim\) (\(\rho.6\)) except (\(\rho.3\)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \(\rho'\) which satisfies (\(\rho.5\)). Now we put

(2.5) \[d(x) = \lim_{\xi \to 0} \rho' (\xi x).\]

It is clear that \(0 \leq d(x) = d(|x|) < +\infty\) for all \(x \in R\) and
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\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

(2.6) \[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for \( \rho^* \), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \( x \in R \) and \( \alpha > 0 \).

We need to prove that (\rho.5) is true for \( \rho^* \). First we have to note

(2.7) \[ \inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0 \]

for any \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \cal A} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \([p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0\) and \( x_0 \in R \).

Hence,

\[ \rho'(\frac{1}{\nu}[p_{\lambda}]_x) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all \( \nu \geq 1 \) and \( \lambda \in \Lambda \). Thus we can find a subsequence \( \{\lambda_n\}_{n \geq 1} \) of \( \{\lambda\}_{\lambda \in \Lambda} \) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}[p_{\lambda_n}]_x) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}]_x) = +\infty, \]

which is inconsistent with (\rho.4). Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)^+]\) for \( x, y \in R \) with \([x] = [y]\) and \( n \geq 1 \). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \( \inf_{n=1,2,\ldots} d([p_n]x) = 0 \) by (2.7). Since \( (1-[p_n])n|y| \geq (1-[p_n])|x| \)
and

\[ d(\alpha x) = d(x) \]

for \( \alpha > 0 \) and \( x \in R \), we obtain
$d(x)=d([p_n]x)+d((1-[p_n])x)$
\leqq d([p_n]x)+d(n(1-[p_n])y)
\leqq d([p_n]x)+d(y)$.

As $n$ is arbitrary, this implies
\[d(x)\leqq \inf_{n=1,2,...} d([p_n]x)+d(y),\]
and also $d(x)\leqq d(y)$. Therefore we conclude that (2.8) holds.

If $|x|\geqq |y|$, then
\[\rho^*(x)=\rho^*([y]x)+\rho^*([x]-[y])x)
=\rho'([y]x)-d([y]x)+\rho^*([x]-[y])x)
\geqq \rho'(y)-d(y)+\rho^*([x]-[y])x)
\geqq \rho^*(y).

Thus $\rho^*$ satisfies $(\rho.5)$.

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies $(\rho.3)$ (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies
\[(\rho.4') \quad \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K < +\infty.
\]

**Proof.** Let $\rho$ satisfy $(\rho.4)$. We need to prove
\[(2.9) \quad \sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty,
\]
where
\[\rho'(x) = \sup_{0 \leqq |y| \leqq |x|} \rho(y).
\]

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho\left(\frac{1}{\nu} x\right)$ for $\nu \geqq 1$ and $x \in \mathbb{R}$. Hence we can find positive numbers $\varepsilon, \gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with
\[\rho(x) \leqq \varepsilon \text{ implies } \rho\left(\frac{1}{\nu_0} x\right) \leqq \gamma.
\]

In $N_0$, we have obviously
\[\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.
\]

If $\varepsilon \leqq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leqq 2K$ for all $0 \leqq \alpha \leqq \alpha_0$ by $(\rho.4')$, and hence there exists always an orthogonal decomposition such that
\[ \alpha_{0}x_{0} = x_{1} + \cdots + x_{n} + y_{1} + \cdots + y_{m} + z \]

where \( \frac{\epsilon}{2} < \rho(x_{i}) \leq \epsilon \) (\( i = 1, 2, \cdots, n \)), \( y_{j} \) is an atomic element with \( \rho(y_{j}) > \epsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'(\frac{1}{\nu_{0}}\alpha_{0}x_{0}) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_{0}}x_{i}) + \sum_{j=1}^{m} \rho'(y_{j}) + \frac{\rho'(z)}{\nu_{0}} \\
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_{j}) + \frac{\rho'(z)}{\nu_{0}} \\
\leq \frac{4K}{\epsilon}\gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_{0}} \rho(\alpha x) \right\} + \gamma 
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_{0}) \leq \rho'(\frac{\alpha_{0}}{\nu_{0}}x_{0}) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^{2}}{\epsilon} \right) 
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_{0}^{\perp} \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma .
\]

Therefore, we obtain

\[
\sup_{x \in \mathcal{R}} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} \leq \gamma' 
\]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^{2}}{\epsilon} + \gamma_{0}. 
\]

Let \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) be an orthogonal system with \( \sum_{\lambda \in \Lambda} \rho^{*}(x_{\lambda}) < +\infty \). Then for arbitrary \( \lambda_{1}, \cdots, \lambda_{k} \in \Lambda \), we have

\[
\sum_{i=1}^{k} d(x_{\lambda_{i}}) = d(\sum_{i=1}^{k} x_{\lambda_{i}}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_{i}}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in \Lambda} d(x_{\lambda}) \leq \gamma' \). It follows that

\[
\sum_{x \in \mathcal{A}} \rho^{*}(x) = \sum_{x \in \mathcal{A}} \rho^{*}(x_{\lambda}) + \sum_{x \in \mathcal{A}} d(x) < +\infty,
\]

which implies \( x_{0} = \sum_{x \in \mathcal{A}} x_{\lambda} \in \mathcal{R} \) and \( \sum_{x \in \mathcal{A}} \rho^{*}(x) = \rho^{*}(x_{0}) \) by \( (\rho.4) \) and \( (2.7) \). Therefore \( \rho^{*} \) satisfies \( (\rho.3) \).

On the other hand, suppose that \( \rho^{*} \) satisfies \( (\rho.3) \) and \( \sup_{x \in \mathcal{R}} d(x) = +\infty \).

Then we can find an orthogonal sequence \( \{x_{i}\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_{i}) = d(\sum_{i=1}^{n} x_{i}) \geq \mu
\]
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
\[ \lim_{\xi \rightarrow 0} \rho^* (\xi x) = 0, \]
there exists $\{\alpha_\nu\}_{\nu \geqq 1}$ with $0 < \alpha_\nu$ ($\nu \geqq 1$) and
\[ \sum_{\nu=1}^{\infty} \rho^* (\alpha_\nu x_\nu) < +\infty. \]
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such
$x_0$, we have for every $\xi \geqq 0$,
\[ \rho' (\xi x_0) = \sum_{\nu=1}^{\infty} \rho' (\xi \alpha_\nu x_\nu) \geqq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty, \]
which is inconsistent with (\rho.4). Therefore we have
\[ \sup_{x \in R} (\lim_{\xi \rightarrow 0} \rho (\xi x)) \leqq \sup_{x \in R} d(x) < +\infty. \]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{x : x \in R, \rho^* (nx) = 0 \text{ for all } n \geqq 1\}, \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\footnote{A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leqq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{x \in R} x \in S$.} of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ with $R_0 \ni x_\lambda \geqq 0$ for all $\lambda \in \Lambda$. Putting
\[ [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \]
we have
\[ [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad 2n[p_{n,\lambda}]x_\lambda \geqq [p_{n,\lambda}]nx, \]
which implies $\rho^* (n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^* (n[p_{n,\lambda}]x) = \rho^* (nx) = 0$. Hence we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
\[ R = R_0 \oplus R_0^\perp. \]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda \in \Lambda \} x_\lambda \in [p]R_0$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leqq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{x \in R} x \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.
$$\sup_{i \in A} \| x_i \|_0 = || x \|_0$$
for any $0 \leq x_i \uparrow_{i \in A} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$|| x \|_0 = \inf \left\{ \xi ; \rho^*(\frac{1}{\xi} x) \leq \xi \right\}. \quad (3.1)$$

Then,

i) $0 \leq || x \|_0 = || - x \|_0 < \infty$ and $|| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1), (\rho.6), (2.1)$ and the definition of $R_0^\perp$.

ii) $|| x + y \|_0 \leq || x \|_0 + || y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_n \uparrow 0} || \alpha_n x \|_0 = 0$ and $\lim_{\| x_n \|_0 \rightarrow \infty} || \alpha x_n \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha \rightarrow \alpha_0} || \alpha x \|_0 = || \alpha_0 x \|_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p \uparrow_{i \in A} [p]$, for any positive number $\xi$ with $|| [p] x \|_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{i \in A} \rho^*(\frac{1}{\xi} [p_i] x) > \xi$ and hence $\sup_{i \in A} || p_i x \|_0 \geq \xi$. Thus we obtain

$$\sup_{i \in A} || [p_i] x \|_0 = || [p] x \|_0,$$

if $[p \uparrow_{i \in A} [p]$. Let $0 \leq x_i \uparrow_{i \in A} x$. Putting

$$[p_{n,i}] = \left[ (x_i - (1 - \frac{1}{n}) x)^+ \right]$$

we have

$$[p_{n,i}] \uparrow_{i \in A} [x] \text{ and } [p_{n,i}] x_i \geq [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{i \in A} || [p_{n,i}] x_i \|_0 \geq \sup_{i \in A} \left\| [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \right\|_0 = \| \left( 1 - \frac{1}{n} \right) x \|_0,$$

we have

$$\sup_{i \in A} || x_i \|_0 \geq \left\| \left( 1 - \frac{1}{n} \right) x \right\|_0$$

and also $\sup_{i \in A} || x_i \|_0 \geq || x \|_0$. As the converse inequality is obvious by iv), $|| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of $(3.1)$, we can see easily that

$$\lim_{n \rightarrow \infty} || x_n \|_0 = 0 \text{ if and only if } \lim_{n \rightarrow \infty} \rho(\xi x_n) = 0 \text{ for all } \xi \geq 0.
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n, \nu}, x_{\nu} \geq 0$ and $a \geq 0$ ($n, \nu = 1, 2, \ldots$) be the elements of $R_0^\perp$ such that

\begin{align}
(3.2) & \quad [p_{n, \nu}] \uparrow_{\nu=1}^{\infty} \cap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0; \\
(3.3) & \quad [p_{n, \nu}]x_{\nu} \geq n[p_{n, \nu}]a \text{ for all } n, \nu \geq 1.
\end{align}

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\begin{align}
(3.4) & \quad ||[q_m]a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \ldots) \\
(3.5) & \quad n_m[q_m]a \geq [q_m]x_{\nu_m} - n_m[q_m]a \quad (m=2, 3, \ldots),
\end{align}

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1, \nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]a$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$$||[p_{1, \nu_1}][p_0]a||_0 > \frac{||[p_0]a||_0}{2} = \frac{\delta}{2}.$$ 

We put $[q_1] = [p_{1, \nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m=1, 2, \ldots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_h})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$$||[p_{n_{k+1}}][n_{k+1}a-x_{\nu_k}]^+[q_k]a||_0 > \frac{\delta}{2}.$$ 

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$$||[p_{n_{k+1}}][n_{k+1}a-x_{\nu_k}]^+[q_k]a||_0 > \frac{\delta}{2}.$$ 

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

$$[q_{k+1}] = [p_{n_{k+1}}, \nu_{k+1}][(n_{k+1}a-x_{\nu_k})^+[q_k]],$$

because

$$[q_{k+1}] \leq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a$$

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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\[ \|x_{\nu_{k+1}} - x_{\nu_{k-1}}\|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1} [q_{k+1}]a - n_k [q_{k+1}]a \|_0 \geq \| [q_{k+1}]a_0 \|_0 \geq \frac{\delta}{2}, \]

since $[q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu-1})^+]$ implies $[q_{k+1}]n_k a \geq [q_{k+1}] x_{\nu_{k-1}}$ by (3.4). It follows from the above that $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^+$ is an $F$-space with $\| \cdot \|_0$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $\| x \|_0 = \inf \{ \xi; \rho^* (\frac{x}{\xi}) \leq \xi \}$ is a quasi-norm on $R_0^+$, we need only to verify completeness of $\| \cdot \|_0$. At first let $\{x_{\nu}\}_{\nu \geq 1} \subset R_0^+$ be a Cauchy sequence with $0 \leq x_{\nu} \uparrow_{\nu=1,2}, \ldots$. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leq x_0 \in R_0^+$ such that $x_0 = \bigcup_{\nu=1}^\infty x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = [x_{\nu} - nx_0]^+$ and $\bigcup_{\nu=1}^\infty [p_{n,\nu}] = [p_n]$, we obtain

\[ [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]x_0 \quad \text{for all } n, \nu \geq 1 \]

and $\bigcap_{n=1}^\infty [p_n] = 0$. Since $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^\infty [p_n] = 0$, that is, $\bigcup_{n=1}^\infty ([x_0] - [p_n]) = [x_0]$. And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1) \]

implies

\[ n(1 - [p_n])x_0 \geq (1 - [p_n])x_{\nu} \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^\infty (1 - [p_n])x_{\nu} \in R_0^+, \]

because $R_0^+$ is universally continuous. As $\{x_{\nu}\}_{\nu \geq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $\| \cdot \|_0$

\[ r = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n])x_{\nu} \|_0 \leq r \]

for every $n \geq 1$ by semi-continuity of $\| \cdot \|_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ ($n \geq 2$). It follows from the definition of $y_n$ that $\{z_{\nu}\}_{\nu \geq 1}$ is an orthogonal sequence with $\| z_{\nu} \|_0 = \| y_n \|_0 \leq r$. This implies
\[
\sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_{n}}{1+\gamma} \right) \leqq \gamma
\]
for all \( n \geqq 1 \) by the formula (3.1). Then \((\rho.3)\) assures the existence of 
\[
z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}.
\]
This yields 
\[
z = \bigcup_{\nu=1}^{\infty} x_{\nu}.
\]
Truly, it follows from \( z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} (1-[p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} \).

By semi-continuity of \( ||\cdot||_{0} \), we have 
\[
||z-x_{\nu}||_{0} \leqq \sup_{\mu\geqq \nu} ||x_{\mu}-x_{\nu}||_{0}
\]
and furthermore \( \lim_{\nu \to \infty} ||z-x_{\nu}||_{0} = 0 \).

Secondly let \( \{x_{\nu}\}_{\nu \geqq 1} \) be an arbitrary Cauchy sequence of \( R_{0}^{\perp} \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geqq 1} \) of \( \{x_{\nu}\}_{\nu \geqq 1} \) such that 
\[
||y_{\nu+1}-y_{\nu}||_{0} \leqq \frac{1}{2^{\nu}} \quad \text{for all } \nu \geqq 1.
\]
This implies 
\[
|| \sum_{\nu=m}^{n} y_{\nu+1}-y_{\nu} ||_{0} \leqq \sum_{\nu=m}^{n} ||y_{\nu+1}-y_{\nu}||_{0} \leqq \frac{1}{2^{n-m}} \quad \text{for all } n > m \geqq 1.
\]
Putting \( z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1}-y_{\nu}| \), we have a Cauchy sequence \( \{z_{n}\}_{n \geqq 1} \) with \( 0 \leqq z_{n} \uparrow_{n=1}^{\infty} \).
Then by the fact proved just above, 
\[
z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \in R_{0}^{\perp} \quad \text{and } \lim_{n \to \infty} ||z_{0}-z_{n}||_{0} = 0.
\]
Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \) is convergent, \( y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \) is also convergent and 
\[
|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=n+1}^{\infty} (y_{\nu+1}-y_{\nu}) ||_{0} \leqq ||z_{0}-z_{n}||_{0} \to 0.
\]
Since \( \{y_{\nu}\}_{\nu \geqq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geqq 1} \), it follows that 
\[
\lim_{\rho \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_{\rho} ||_{0} = 0.
\]
Therefore \( \cdot ||_{0} \) is complete in \( R_{0}^{\perp} \), that is, \( R_{0}^{\perp} \) is an F-space with \( ||\cdot||_{0} \).

Conversely if \( R_{0}^{\perp} \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geqq 1} \in R_{0}^{\perp} \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{\perp} \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geqq 1 \)).
Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy \((\rho.4')\). Q.E.D.

Since \( R_{0} \) contains a normal manifold which is universally complete, if \( R_{0} \neq 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|$ defined by (3.1) and $R$ becomes an F-space with $\| \cdot \|$ if and only if $\rho$ fulfils (\(\rho^4\)).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[
\| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10) \}
\]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\]

for all $x \in L$ hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < +\infty (x \in L)$ and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha_n x \|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^\infty 0$ and that $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[
\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{|\xi|}
\]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have \( \| x \|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( \| x \|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)–(A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( \| \cdot \|_1 \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}
\]

is a semi-continuous quasi-norm on \( R \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho.4') \), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
\]

for all \( x \in R_0^+ \).

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho.1) \sim (\rho.6) \) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{ x_n \}_{n=1}^{\infty} \) is called order-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if there exists a sequence of elements \( \{ a_n \}_{n=1}^{\infty} \) such that

\[
| x_n - a_n | \leq a_n \quad (n \geq 1)
\]

and \( a_n \downarrow 0 \). And a sequence of elements \( \{ x_n \}_{n=1}^{\infty} \) is called star-convergent to \( a \) and denoted by \( \lim_{n \to \infty} x_n = a \), if for any subsequence \( \{ y_n \}_{n=1}^{\infty} \) of \( \{ x_n \}_{n=1}^{\infty} \), there exists a subsequence \( \{ z_n \}_{n=1}^{\infty} \) of \( \{ y_n \}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} z_n = a \). A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{r \geq 1} \| a_r \| = 0 \) for any \( a_r \downarrow 0 \). In the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any \( x \in R \) there exists an orthogonal decomposition \( x = y + z \) such that \( [z] R \) is finite dimensional and \( \rho(y) < +\infty \).

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \{p_n\}_{n \geq 1} such that \(\rho([p_n]x) = +\infty\) and \([p_n] \downarrow_{n=1}^\infty 0\). Hence by (3.1) it follows that \(\| [p_n]x \|_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(\| \cdot \|_0\).

**Sufficiency.** Let \(a_{\nu} \downarrow_{\nu=1}^\infty 0\) and put \(p_{\nu}^* = (a_{\nu} - \varepsilon a_1)^+\) for any \(\varepsilon > 0\) and \(n \geq 1\). It is easily seen that \(p_{\nu}^* \downarrow_{n=1}^\infty 0\) for any \(\varepsilon > 0\) and \(a_n = [a_1]a_n = [p_{\nu}^*]a_n + (1 - [p_{\nu}^*])a_n \leq [p_{\nu}^*]a_1 + \varepsilon a_1\).

This implies

\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_{\nu}^*]a_1) + \rho^*(\xi \varepsilon (1 - [p_{\nu}^*])a_1)
\]

for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p_{\nu}^*] \downarrow_{n=1}^\infty 0\), we can find \(n_0\) (depending on \(\xi\) and \(\varepsilon\)) such that \(\rho^*(\xi [p_{\nu}^*]a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*(\xi [p_{\nu}^*]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \varepsilon a_1).
\]

Since \(\varepsilon\) is arbitrary, \(\lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} \| a_n \|_0 = 0\) and \(\| \cdot \|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \(\| \cdot \|_0\) is continuous, if

(5.2) \(\rho^*(a_{\nu}) \rightarrow 0\) implies \(\rho^*(\alpha a_{\nu}) \rightarrow 0\) for every \(\alpha \geq 0\).

From the definition, it is clear that \(s\)-lim \(x_{\nu} = 0\) implies \(\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0\), if \(\| \cdot \|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0\) (or \(\lim_{\nu \rightarrow \infty} \| x_{\nu} \| = 0\)) implies \(s\)-lim \(x_{\nu} = 0\), if \(\| \cdot \|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho.3\)).

If we replace \(\lim_{\nu \rightarrow \infty} \| x_{\nu} \| = 0\) by \(\lim_{\nu \rightarrow \infty} \rho(x_{\nu}) = 0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \(\rho^*(x) = 0\) implies \(x = 0\).

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(\| \cdot \|_0\) is complete, \(\rho(a_{\nu}) \rightarrow 0\) implies \(s\)-lim \(a_{\nu} = 0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous, i.e. \(\rho^*(x) = \sup_{y \downarrow_{\in A} x} \rho^*(y)\) for any \(0 \leq x \downarrow_{\in A} x\). If

11) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x) = \inf_{y \downarrow_{\in A} x} \sup_{y_1 \downarrow_{\in A} y} \rho^*(y_1)\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x_{\nu}) \rightarrow 0\) is equivalent to \(\rho_*(x_{\nu}) \rightarrow 0\).
\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{R} \) in virtue of (\( \rho.3 \)).

Now, since
\[
\rho\left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}|\right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\mu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}|\right)\right) = 0 \) and hence (5.3) implies
\[
\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}|\right) = 0.
\]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu = 1, 2, \cdots) \). Therefore we have \( s-\lim b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analoguous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( ||\cdot||_{0} \) is complete and continuous, then (5.2) holds.

**References**


Mathematical Institute,
Hokkaido University

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