ON F-NORMS OF QUASI-MODULAR SPACES

By

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§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

\( (\rho.1) \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \) for all \( x \in R \);

\( (\rho.2) \quad \rho(x+y) = \rho(x) + \rho(y) \) for any \( x, y \in R \) with \( x \perp y \);

\( (\rho.3) \quad \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \text{ for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda}, \text{ there exists } x_{0} \in R \text{ such that } x_{0} = \sum_{\lambda \in \Lambda} x \text{ and } \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda});\)

\( (\rho.4) \quad \lim_{\xi \to 0} \rho(\xi x) < +\infty \) for all \( x \in R \).

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^3\) \( m \) on \( R \) for which every universally continuous linear functional\(^4\) is continuous with respect to the norm defined by the modular\(^5\) \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

\( (A.1) \quad \rho(x) \geq 0 \text{ and } \rho(x) = 0 \text{ if and only if } x = 0;\)

\( (A.2) \quad \rho(-x) = \rho(x);\)

\( (A.3) \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \) for every \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1;\)

\( (A.4) \quad \alpha_{n} \to 0 \text{ implies } \rho(\alpha_{n} x) \to 0 \text{ for every } x \in R;\)

\( (A.5) \quad \text{for any } x \in L \text{ there exists } \alpha > 0 \text{ such that } \rho(\alpha x) < +\infty.\)

They showed that \( L \) is a quasi-normed space with a quasi-norm \( \| \cdot \|_{0} \) defined by the formula;

1) \( x \perp y \) means \( |x| \cap |y| = 0.\)
2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma.\)
3) For the definition of a modular, see [3].
4) A linear functional \( f \) is called universally continuous, if \( \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \) for any \( a_{\lambda} \downarrow 0.\)
5) \( R \) is called semi-regular, if for any \( x \neq 0, x \in R, \) there exists a universally continuous linear functional \( f \) such that \( f(x) \neq 0.\)

This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4]. In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R.\)
\[(1.1) \quad \|x\|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]  

and \( \|x_n\|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim \)(A.5) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^\perp \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^\perp \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[(\rho.4^{'}) \quad \sup_{x \in R} \rho(\alpha x) < +\infty \]

(\text{Theorem 3.2}).

In \( \S 4 \), we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^\perp \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^\perp \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \S 83]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a \textit{universally continuous semi-ordered linear space} and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a \textit{projector}: \( [p]x = \bigcup_{n=1}^{\infty} (n \cdot p \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a \textit{projection operator} onto the normal manifold \( N = \{ p \}^\perp \), that is, \( x = [p]x + (1 - [p])x \).

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6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular $\rho$, we have

1. $\rho(0) = 0$;
2. $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
3. $\rho([p]x) = \sup_{i \in A} \rho([p_i]x)$ for any $[p_i] \uparrow_{i \in A} [p]$.

In the argument below, we have to use the additional property of $\rho$:

$(\rho.5)$ $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

**Theorem 2.1.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

**Proof.** We put for every $x \in R$,

$$
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
$$

It is clear that $\rho'$ satisfies the conditions $(\rho.1)$, $(\rho.2)$ and $(\rho.5)$.

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

$$
\sum_{i \in A} \rho(x_i) < +\infty,
$$

because

$$
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
$$

We have

$$
x_0 = \sum_{i \in A} x_i \in R
$$

and

$$
\rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of } (\rho.3).
$$

For such $x_0$,

$$
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)
$$

$$
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
$$

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_0 \in R$,

$$
\rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n \geq 1.
$$

By $(\rho.2)$ and $(\rho.4)$, $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{s} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq s$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))
and
\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho(\frac{1}{\nu} y_0) \geq \rho(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)).

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [(|x| - |y|)^+ \] \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta |y|) \]
\[ = \rho([p]|x| + (1-[p])|y|) \]
\[ = \rho([p]x) + \rho((1-[p])y) \]
\[ \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition (\( \rho.4 \)). Thus, in the above theorems, we cannot replace (\( \rho.4 \)) by the weaker condition:

(\( \rho.4'' \)) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.3 \)) and (\( \rho.4'' \)), but does not (\( \rho.4 \)). For this \( \rho_0 \), we obtain

\[ \rho_0(x) = \sup_{\|y\| \leq |x|} \rho_0(y) = +\infty \]
for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1] \) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\( \rho.6 \)) \[ \lim_{\xi \to 0} \rho(\xi x) = 0 \]
for all \( x \in R \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies (\( \rho.1 \))~(\( \rho.6 \)) except (\( \rho.3 \)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies (\( \rho.5 \)). Now we put

(2.5) \[ d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x + y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ \rho^*(x) = \rho'(x) - d(x) \quad \text{if } x \in R. \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[ \inf_{\lambda \in A} d([p_{\lambda}]x) = 0 \]

for any \([p_{\lambda}] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_{\lambda}]x_0) \geq \alpha > 0 \]

for some \([p_{\lambda}] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty, \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty, \]

(2.8)

\[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \(\inf_{n=1, 2, \ldots} d([p_n]x) = 0\) by \((2.7)\). Since \((1 - [p_n])n |y| \geq (1 - [p_n]) |x|\)

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
$d(x) = d([p_n]x) + d((1-[p_n])x)$
$\leq d([p_n]x) + d(n(1-[p_n])y)$
$\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies
$$d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y),$$
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
$$\rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y]x)$$
$$= \rho'([y]x) - d([y]x) + \rho^*([x] - [y]x)$$
$$\geq \rho'(y) - d(y) + \rho^*([x] - [y]x)$$
$$\geq \rho^*(y).$$
Thus $\rho^*$ satisfies (2.5).

**Q.E.D.**

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (2.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

$$\rho^*(x) = \sup_{0 \leq \gamma \leq |x|} \rho(y).$$

Proof. Let $\rho$ satisfy (2.4). We need to prove

$$\sup_{x \in \mathbf{R}} d(x) = \sup_{x \in \mathbf{R}} \{\lim_{\xi \to 0} \rho'({\xi}x)\} = K < +\infty,$$

where

$$\rho'(x) = \sup_{0 \leq \gamma \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'(\frac{1}{\nu}x)$ for $\nu \geq 1$ and $x \in \mathbf{R}$. Hence we can find positive numbers $\epsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with

$$\rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma.$$

In $N_0$, we have obviously

$$\sup_{x \in N_0} \{\lim_{\xi \to 0} \rho'(\xi x)\} = \gamma_0 < +\infty.$$

If $\epsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (2.4'), and hence there exists always an orthogonal decomposition such that
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\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (i = 1, 2, \ldots, n), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho' \left( \frac{1}{\nu_0} \alpha_0 x_0 \right) \leq \sum_{i=1}^{n} \rho' \left( \frac{1}{\nu_0} x_i \right) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma .
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho' \left( \frac{\alpha_0}{\nu_0} x_0 \right) \leq \left( \frac{4K + \varepsilon}{\varepsilon} \right) \gamma + \left( \frac{4K^2}{\varepsilon} \right)
\]
in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N^+_0 \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma .
\]

Therefore, we obtain

\[
\sup_{x \in R} \lim_{\xi \to 0} \rho'(\xi x) \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma .
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{i_\nu}) = d(\sum_{i=1}^{k} x_{i_\nu}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{i_\nu}) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho^*(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty ,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (2.4) and (2.7). Therefore \( \rho^* \) satisfies (ρ.3).

On the other hand, suppose that \( \rho^* \) satisfies (ρ.3) and \( \sup_{x \in R} d(x) = +\infty \).

Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]

...
for all \( \mu \geq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since 
\[
\lim_{t \to 0} \rho^*(\xi x) = 0,
\]
there exists \( \{\alpha_{\nu}\}_{\nu \geq 1} \) with \( 0 < \alpha_{\nu} \) \( \nu \geq 1 \) and 
\[
\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu} x_{\nu}) < +\infty.
\]
It follows that 
\[
x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R
\]
and 
\[
d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x_{\nu})
\]
from \( (\rho.3) \). For such \( x_0 \), we have for every \( \xi \geq 0 \)
\[
\rho^*(\xi x) = \sum_{\nu=1}^{\infty} \rho^*(\xi \alpha_{\nu} x_{\nu}) \geq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty,
\]
which is inconsistent with \( (\rho.4) \). Therefore we have 
\[
\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.
\]
Q.E.D.

\section{3. Quasi-norms}

We denote by \( R_0 \) the set:
\[
R_0 = \{x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1\},
\]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let 
\[
x = \bigcup_{\lambda \in \Lambda} x_{\lambda}
\]
with \( \forall \lambda \in \Lambda \), \( x_{\lambda} \geq 0 \). Putting
\[
[p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+],
\]
we have 
\[
[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x],
\]
which implies \( \rho^*([p_{n,\lambda}] x) = 0 \) and 
\[
\sup_{\lambda \in \Lambda} \rho^*([p_{n,\lambda}] x) = \rho^*(nx) = 0.
\]
Hence, we obtain 
\[
x \in R_0,
\]
that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp.\(^8\)
\]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0 \), \([p]R_0 \)

is universally complete, i.e. for any orthogonal system \( \{x_{\lambda} \}_{\lambda \in \Lambda} \in [p]R_0 \),

there exists \( x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R \). Hence we can also verify without
difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \)

is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda) \)^9-space.

With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration.

Now we can state the theorems which we aim at.

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7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, |b| \leq |a|, b \in R \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \bigcup_{\lambda \in \Lambda} x_{\lambda} \in R \), \( 0 \leq x_{\lambda} \in S(\lambda \in \Lambda) \) implies \( \bigcup_{\lambda \in \Lambda} x_{\lambda} \in S \).

8) This means that \( x \in R \) is written by \( x = y + z \), \( y \in R_0 \) and \( z \in R_0^\perp \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e. 
\[
\sup_{\lambda \in \Lambda} ||x_\lambda||_0 = ||x||_0 \quad \text{for any } 0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x.
\]

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

\[(3.1) \quad ||x||_0 = \inf \left\{ \xi; \rho^*\left(\frac{1}{\xi} x\right) \leq \xi \right\}.
\]

Then,

i) $0 \leq ||x||_0 = ||-x||_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_0^\perp$.

ii) $||x + y||_0 \leq ||x||_0 + ||y||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim \sup_{a \rightarrow \alpha_0} ||\alpha x||_0 = 0$ and $\lim \sup_{a \rightarrow \alpha_0} ||\alpha x_n||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_0$ is semi-continuous. From ii) and iii), it follows that $\lim \sup_{a \rightarrow \alpha_0} ||\alpha x||_0 = ||\alpha_0 x||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^*(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*\left(\frac{1}{\xi} [p_\lambda] x\right) > \xi$ and hence $\sup_{\lambda \in \Lambda} ||p_\lambda x||_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda \in \Lambda} ||p_\lambda x||_0 = ||[p]x||_0,$$

if $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$.

Let $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$. Putting

$$[p_{n,\lambda}] = \left( x_\lambda - \left( 1 - \frac{1}{n} \right) x \right)^*$$

we have

$$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad [p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}] \left( 1 - \frac{1}{n} \right) x \quad \text{for all } n \geq 1.$$

As is shown above, since

$$\sup_{\lambda \in \Lambda} \left\{ [p_{n,\lambda}] x_\lambda \right\} \leq \sup_{\lambda \in \Lambda} \left\{ [p_{n,\lambda}] \left( 1 - \frac{1}{n} \right) x \right\} = \left\{ 1 - \frac{1}{n} \right\} x,$$

we have

$$\sup_{\lambda \in \Lambda} ||x_\lambda||_0 \geq \left\{ 1 - \frac{1}{n} \right\} x_0$$

and also $\sup_{\lambda \in \Lambda} ||x_\lambda||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $||\cdot||_0$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim ||x_n||_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

\[\text{On F-Norms of Quasi-Modular Spaces 211}\]
In order to prove the completeness of quasi-norm \( ||\cdot||_0 \), the next Lemma is necessary.

**Lemma 2.** Let \( p_{n,\nu}, \ x_{\nu} \geq 0 \) and \( a \geq 0 \) \((n, \nu = 1, 2, \cdots)\) be the elements of \( R_0^\perp \) such that

\[
(p_{n,\nu}) \uparrow_{\nu=1}^{\infty} (p_{n}) \quad \text{with} \quad \bigcap_{n=1}^{\infty} [p_{n}] a = [p_{0}] a = 0;
\]

\[
[p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] a \quad \text{for all} \ n, \nu \geq 1.
\]

Then \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence of \( R_0^\perp \) with respect to \( ||\cdot||_0 \).

**Proof.** We shall show that there exist a sequence of projectors \( [q_m] \downarrow_{m=1}^{\infty} (m \geq 1) \) and sequences of natural numbers \( \nu_m, n_m \) such that

\[
||[q_m] a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m=1, 2, \cdots)
\]

and

\[
n_m [q_m] a \geq [q_m] x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),
\]

where \( \delta = ||[p_0] a||_0 \).

In fact, we put \( \nu_1 = 1 \). Since \( [p_{1,\nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0] \) and \( ||\cdot||_0 \) is semi-continuous, we can find a natural number \( \nu_1 \) such that

\[
||[p_{1,\nu_1}] [p_0] a||_0 > \frac{||[p_0] a||_0}{2} = \frac{\delta}{2}.
\]

We put \( [q_1] = [p_{1,\nu_1}] [p_0] \). Now, let us assume that \( [q_m], \nu_m, n_m \) \((m=1, 2, \cdots, k)\) have been taken such that (3.4) and (3.5) are satisfied.

Since \( [(na-x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a] \) and \( ||[q_k] a||_0 > \frac{\delta}{2} \), there exists \( n_{k+1} \) with

\[
||(n_{k+1}a-x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2}.
\]

For such \( n_{k+1} \), there exists also a natural number \( \nu_{k+1} \) such that

\[
||[p_{n_{k+1}, \nu_{k+1}}] (n_{k+1}a-x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of \( ||\cdot||_0 \). Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1}a-x_{\nu_k})^+ [q_k],
\]

because

\[
[q_{k+1}] \geq [q_k], \quad ||[q_{k+1}] a|| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
\]

by (3.3) and \( [q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_{k+1}} \) by (3.5).

For the sequence thus obtained, we have for every \( k \geq 3 \).
$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0} \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0} \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0} \geqq ||[q_{k+1}]a_{0}||_{0} \geqq \frac{\delta}{2}$, since $[q_{k+1}] \leqq [q_{k}] \leqq [(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4). It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{\perp}$ is an F-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies (\rho.4').

**Proof.** If $\rho$ satisfies (\rho.4'), $\rho^{*}$ is a quasi-modular which fulfills also (\rho.5) and (\rho.6) in virtue of Theorem 2.3. Since $||x||_{0} = \inf \{\xi; \rho^{*}(\frac{x}{\xi}) \leqq \xi\}$ is a quasi-norm on $R_{0}^{\perp}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^{\perp}$ be a Cauchy sequence with $0 \leqq x_{\nu} \uparrow_{\nu=1,2}, \ldots$. Since $\rho^{*}$ satisfies (\rho.3), there exists $0 \leqq x_{0} \in R_{0}^{\perp}$ such that $x_{0} = \bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,\nu}] = ([x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{\nu=1}^{\infty}[p_{n,\nu}] = [p_{n}]$, we obtain

$$[p_{n,\nu}]x_{\nu} \geqq n[p_{n,\nu}]x_{0}$$

for all $n, \nu \geqq 1$ and $[p_{n}] \downarrow_{n=1}^{\infty}0$. Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}] = 0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}]-[p_{n}]) = [x_{0}]$. And

$$1-[p_{n}] \geqq (1-[p_{n}]) \quad (n, \nu \geqq 1)$$

implies

$$n(1-[p_{n}])x_{0} \geqq (1-[p_{n}])x_{\nu} \geqq 0.$$ 

Hence we have

$$y_{n} = \bigcup_{\nu=1}^{\infty}(1-[p_{n}])x_{\nu} \in R_{0}^{\perp},$$

because $R_{0}^{\perp}$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$$\gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_{0} < +\infty,$$

which implies

$$||y_{n}||_{0} = \sup_{\nu \geqq 1} ||(1-[p_{n}])x_{\nu}||_{0} \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1} = y_{1}$ and $z_{n} = y_{n} - y_{n-1}$ $(n \geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0} = ||y_{n}||_{0} \leqq \gamma$. This implies
\[
\sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma
\]
for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of 
\[
z = \bigcup_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}.
\]
This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from
\[
z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - [p_{n}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} [x_{0}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.
\]
By semi-continuity of \( \| \cdot \|_{0} \), we have
\[
\| z - x_{\nu} \|_{0} \leq \sup_{\mu \geq \nu} \| x_{\mu} - x_{\nu} \|_{0}
\]
and furthermore \( \lim_{\nu \to \infty} \| z - x_{\nu} \|_{0} = 0 \).

Secondly let \( \{ x_{\nu} \}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_{0}^{+} \). Then we can find a subsequence \( \{ y_{\nu} \}_{\nu \geq 1} \) of \( \{ x_{\nu} \}_{\nu \geq 1} \) such that
\[
\| y_{\nu+1} - y_{\nu} \|_{0} \leq \frac{1}{2^{\nu}}
\]
for all \( \nu \geq 1 \).

This implies
\[
\| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} \|_{0} \leq \sum_{\nu=m}^{n} \| y_{\nu+1} - y_{\nu} \|_{0} \leq \frac{1}{2^{n-m}} \quad \text{for all } n \geq m \geq 1.
\]
Putting \( z_{n} = \sum_{\nu=1}^{n} | y_{\nu+1} - y_{\nu} | \), we have a Cauchy sequence \( \{ z_{n} \}_{n \geq 1} \) with \( 0 \leq z_{n} \uparrow \infty \).

Then by the fact proved just above,
\[
z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \bigcup_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_{0}^{+} \quad \text{and} \quad \lim_{n \to \infty} \| z_{0} - z_{n} \|_{0} = 0.
\]

Since \( \bigcup_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \) is convergent, \( y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and
\[
\| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} \|_{0} = \| \sum_{\nu=1}^{n} (y_{\nu+1} - y_{\nu}) \|_{0} \leq \| z_{0} - z_{n} \|_{0} \to 0.
\]

Since \( \{ y_{\nu} \}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{ x_{\nu} \}_{\nu \geq 1} \), it follows that
\[
\lim_{\nu \to \infty} \| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} \|_{0} = 0.
\]
Therefore \( \| \cdot \|_{0} \) is complete in \( R_{0}^{+} \), that is, \( R_{0}^{+} \) is an F-space with \( \| \cdot \|_{0} \).

Conversely if \( R_{0}^{+} \) is an F-space, then for any orthogonal sequence \( \{ x_{\nu} \}_{\nu \geq 1} \in R_{0}^{+} \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+} \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy (\( \rho.4' \)). Q.E.D.

Since \( R_{0} \) contains a normal manifold which is universally complete, if \( R_{0} \perp 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ defined by (3.1) and $R$ becomes an $F$-space with $||\cdot||_0$ if and only if $\rho$ fulfils $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz ($\S1$). Here we put for $x\in L$

\begin{equation}
||x||_1 = \inf_{\xi>0}\left\{\frac{1}{\xi} + \rho(\xi x)\right\} \tag{4.1}
\end{equation}

and show that $||\cdot||_1$ is also a quasi-norm on $L$ and

\begin{equation}
||x||_0 \leq ||x||_1 \leq 2||x||_0 \tag{4.2}
\end{equation}

hold, where $||\cdot||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < +\infty$ ($x \in L$) and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^{\infty} 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} ||x_n||_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} ||\alpha x_n||_1 = 0$ for all $\alpha > 0$ and that $||x_n||_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.\]

This yields

\[||x+y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore $||x+y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $||\cdot||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{m(\xi x)} \tag{3 or 4}.
\]

For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $||\cdot||_1$ and $||\cdot||_0$ respectively.
This yields (4.2), since we have \(\|x\|_0 \leq \frac{1}{\xi}\) and \(\rho(\gamma x) > \frac{1}{\eta}\) for every \(\eta\) with \(\|x\|_0 > \frac{1}{\eta}\). Therefore we can obtain from above

**Theorem 4.1.** If \(L\) is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm \(\|\cdot\|_1\) on \(L\) which is equivalent to \(\|\cdot\|_0\) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \(R\) is a quasi-modular space with a quasi-modular \(\rho\), then

\[
(4.3) \quad \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \(R_{\partial}^\perp\) and \(\|\cdot\|_1\) is complete if and only if \(\rho\) satisfies \((\rho.4')\), where \(\rho^*\) and \(R_0\) are the same as in §2 and §3. And further we have

\[
(4.4) \quad \|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all} \quad x \in R_{\partial}^\perp.
\]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular \(\rho^*\) on \(R\) satisfies \((\rho.1)\sim(\rho.6)\) except \((\rho.3)\) and \(\rho^*(\xi x)\) is not identically zero as a function of \(\xi \geq 0\) for each \(0 \leq x \in R\) (i.e. \(R_0 = \{0\}\)). A sequence of elements \(\{x_\nu\}_{\nu \geq 1}\) is called order-convergent to \(a\) and denoted by \(o-lim_{\nu \to \infty} x_\nu = a\), if there exists a sequence of elements \(\{a_\nu\}_{\nu \geq 1}\) such that \(\|x_\nu - a\| \leq a_\nu \quad (\nu \geq 1)\) and \(a_\nu \downarrow_{\nu=1}^\infty 0\). And a sequence of elements \(\{x_\nu\}_{\nu \geq 1}\) is called star-convergent to \(a\) and denoted by \(s-lim_{\nu \to \infty} x_\nu = a\), if for any subsequence \(\{y_\nu\}_{\nu \geq 1}\) of \(\{x_\nu\}_{\nu \geq 1}\), there exists a subsequence \(\{z_\nu\}_{\nu \geq 1}\) of \(\{y_\nu\}_{\nu \geq 1}\) with \(o-lim_{\nu \to \infty} z_\nu = a\). A quasi-norm \(\|\cdot\|\) on \(R\) is termed to be **continuous**, if \(\inf_{\nu \geq 1} \|a_\nu\| = 0\) for any \(a_\nu \downarrow_{\nu=1}^\infty 0\). In the sequel, we write by \(\|\cdot\|_0\) (or \(\|\cdot\|_1\)) the quasi-norm defined on \(R\) by \(\rho^*\) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \(\|\cdot\|_0\) (or \(\|\cdot\|_1\)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any} \ x \in R \text{ there exists an orthogonal decomposition} \ x = y + z \text{ such that} \ [z]R \text{ is finite dimensional and} \ \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \(x \in R\), we can find a
sequence of projector $\{[p_n]\}_{n \geq 1}$ such that $\rho([p_n]x) = +\infty$ and $[p_n] \downarrow_{n=1}^{\infty} 0$. Hence by (3.1) it follows that $\| [p_n]x \|_0 > 1$ for all $n \geq 1$, which contradicts the continuity of $\| \cdot \|_0$.

**Sufficiency.** Let $a_{\nu} \downarrow_{\nu=1}^{\infty} 0$ and put $[p_n^\epsilon] = [(a_n - \epsilon a_1)^+]$ for any $\epsilon > 0$ and $n \geq 1$. It is easily seen that $[p_n^\epsilon] \downarrow_{n=1}^{\infty} 0$ for any $\epsilon > 0$ and $a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^e]a_1 + \epsilon a_1$.

This implies

$$\rho^*(\xi a_n) \leq \rho^*(\xi[p_n^\epsilon]a_1) + \rho^*(\xi(1 - [p_n^\epsilon])a_1)$$

for all $n \geq 1$ and $\xi \geq 0$. In virtue of (5.1) and $[p_n^\epsilon] \downarrow_{n=1}^{\infty} 0$, we can find $n_0$ (depending on $\xi$ and $\epsilon$) such that $\rho^*(\xi[p_n^\epsilon]a_1) < +\infty$, and hence $\inf_{n \geq 1} \rho^*(\xi[p_n^\epsilon]a_1) = 0$ by (2.3) in Lemma 1 and (\rho.2). Thus we obtain

$$\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi\epsilon a_1).$$

Since $\epsilon$ is arbitrary, $\lim_{n \rightarrow \infty} \rho^*(\xi a_n) = 0$ follows. Hence we infer that $\inf_{n \geq 1} 1a_n \|_0 = 0$ and $\| \cdot \|_0$ is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** $\| \cdot \|_0$ is continuous, if

(5.2) $\rho^*(a_\nu) \rightarrow 0$ implies $\rho^*(\alpha a_\nu) \rightarrow 0$ for every $\alpha \geq 0$.

From the definition, it is clear that $\mathrm{s-lim} x_{\nu} = 0$ implies $\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0$, if $\| \cdot \|_0$ is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** $\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0$ (or $\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_1 = 0$) implies $\mathrm{s-lim} x_{\nu} = 0$, if $\| \cdot \|_0$ is complete (i.e. $\rho^*$ satisfies (\rho.3)).

If we replace $\lim_{\nu \rightarrow \infty} \| x_{\nu} \|_0 = 0$ by $\lim_{\nu \rightarrow \infty} \rho(x_{\nu}) = 0$, Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) $\rho^*(x) = 0$ implies $x = 0$.

Truly we obtain

**Theorem 5.3.** If $\rho^*$ satisfies (5.3) and $\| \cdot \|_0$ is complete, $\rho(a_\nu) \rightarrow 0$ implies $\mathrm{s-lim} a_{\nu} = 0$.

**Proof.** We may suppose without loss of generality that $\rho^*$ is semi-continuous,\footnote{If $\rho^*$ is not semi-continuous, putting $\rho_*(x) = \inf_{y_1 \uparrow_{j \in \mathcal{A}} x} \{\sup_{j \in \mathcal{A}} \rho^*(y_j)\}$, we obtain a quasi-modular $\rho_*$ which is semi-continuous and $\rho^*(x) \rightarrow 0$ is equivalent to $\rho_*(x) \rightarrow 0$.} i.e. $\rho^*(x) = \sup_{j \in \mathcal{A}} \rho^*(x_j)$ for any $0 \leq x_{i \in \mathcal{A}}$. If
\[
\rho(a_\nu) \leq \frac{1}{2^\nu} \quad (\nu \geq 1),
\]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^\infty |a_\nu| \in \mathcal{R} \) in virtue of (\( \rho.3 \))

Now, since

\[
\rho\left( \bigcup_{\nu=1}^\infty |a_\nu| \right) \leq \sum_{\nu=1}^\infty \rho(a_\nu) \leq \frac{1}{2^{\nu-1}}
\]

holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^\infty \left( \bigcup_{\nu=1}^\infty |a_\nu| \right) \right) = 0 \) and hence (5.3) implies

\[
\bigcap_{\nu=1}^\infty \left( \bigcup_{\nu=1}^\infty |a_\nu| \right) = 0.
\]

Thus we see that \( \{a_\nu\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b_\nu) \to 0 \), we can find a subsequence \( \{b'_\nu\}_{\nu \geq 1} \) of \( \{b_\nu\}_{\nu \geq 1} \) with \( \rho(b'_\nu) \leq \frac{1}{2^\nu} \) (\( \nu = 1, 2, \ldots \)). Therefore we have \( \text{s-lim}_{\nu \to \infty} b_\nu = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^* \) satisfies (5.3) and \( \|\cdot\|_0 \) is complete and continuous, then (5.2) holds.

**References**


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