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ON F-NORMS OF QUASI-MODULAR SPACES

By Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ \[ 0 \leq \rho(x) = \rho(-x) \leq +\infty \] for all $x \in R$;

$(\rho.2)$ \[ \rho(x+y) = \rho(x) + \rho(y) \] for any $x, y \in R$ with $x \perp y$;\footnote{1)}

$(\rho.3)$ If \[ \sum_{\lambda \in A} \rho(x_{\lambda}) < +\infty \] for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in A}$,\footnote{2)} there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in A} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in A} \rho(x_{\lambda})$;

$(\rho.4)$ \[ \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \] for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\footnote{3)} $m$ on $R$ for which every universally continuous linear functional\footnote{4)} is continuous with respect to the norm defined by the modular\footnote{5)} $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

(A.2) $\rho(-x) = \rho(x)$;

(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(A.4) $\alpha_{n} \to 0$ implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula;

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in A}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in A} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.\footnote{0}
(1.1) \[ \| x \|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies \( (A.2) \sim (A.5) \) on an arbitrary quasi-modular space \( R \) in \$\S\$ 2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0=\{x : x \in R, \rho^*(\xi x)=0 \ \text{for all} \ \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \$\S\$ 3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4) \quad \sup_{x \in R} \lim_{\alpha \to 0} \rho(\alpha x) < +\infty \]

(Theorem 3.2).

In \$\S\$ 4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \$\S\$ 83]. At last in \$\S\$ 5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \$\S\$ 5 are already known in those cases [3].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x) \) for all \( x \geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N=\{p\}^1 \), that is, \( x=[p]x+(1-[p])x \).

---

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular $\rho$, we have

1. $\rho(0) = 0$;
2. $\rho(\lfloor p \rfloor x) \leq \rho(x)$ for all $p, x \in R$;
3. $\rho(\lfloor p \rfloor x) = \sup_{i \in A} \rho(\lfloor p_i \rfloor x)$ for any $\lfloor p \rfloor \uparrow_{i \in A} \lfloor p \rfloor$.

In the argument below, we have to use the additional property of $\rho$:

4. $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies (4).

**Theorem 2.1.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which (4) is valid.

**Proof.** We put for every $x \in R$,

\begin{equation}
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\end{equation}

It is clear that $\rho'$ satisfies the conditions (1), (2) and (4).

Let $\{x_i\}_{i \in A}$ be an orthogonal system such that $\sum_{i \in A} \rho'(x_i) < +\infty$, then

\[ \sum_{i \in A} \rho(x_i) < +\infty, \]

because

\[ \rho(x) \leq \rho'(x) \quad \text{for all } x \in R. \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of (3)}. \]

For such $x_0$,

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho(\lfloor x_i \rfloor y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho(\lfloor x_i \rfloor y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. $\rho'$ fulfils (3).

If $\rho'$ does not fulfil (4), we have for some $x_0 \in R$,

\[ \rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n \geq 1. \]

By (2) and (4), $x_0$ can not be written as $x_0 = \sum_{\nu=1}^{\kappa} \xi_\nu e_\nu$, where $e_\nu$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))

and
\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho'(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho'(\frac{1}{\nu} y_0) \geq \rho'(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts \( \rho(4) \). Therefore \( \rho' \) has to satisfy \( \rho(4) \). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \( \rho(5) \), \( \rho \) does also (A.3) in \( \S 1 \):
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [\{|x| - |y|^{+}\}] \), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \\
\leq \rho(\alpha \lceil p \rceil |x| + \alpha(1 - \lceil p \rceil) |y| + \beta \lceil p \rceil |x| + (1 - \lceil p \rceil) \beta |y|) \\
= \rho(\lceil p \rceil |x| + (1 - \lceil p \rceil) |y|) \\
= \rho(\lceil p \rceil x) + \rho((1 - \lceil p \rceil) y) \\
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

- \((\rho.4'')\) for any \( x \in \mathbb{R} \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[
\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| dt + \sum_{i=1}^{\infty} i \text{ mes} \left\{ t : x(t) = \frac{1}{i} \right\},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[
(\rho.6) \quad \lim_{t \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.
\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[
d(x) = \lim_{t \to 0} \rho'(\xi x).
\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x+y)=d(x)+d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ (2.6) \quad \rho^*(x)=\rho'(x)-d(x) \quad (x \in \mathbb{R}). \]

we can see easily that (\rho.1), (\rho.2), (\rho.4) and (\rho.6) hold true for \( \rho^* \), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \( x \in \mathbb{R} \) and \( \alpha > 0 \).

We need to prove that (\rho.5) is true for \( \rho^* \). First we have to note

\[ (2.7) \quad \inf_{\lambda \in \Lambda} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow \lambda \in A\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow \lambda \in A\) and \( x_0 \in \mathbb{R} \).

Hence,

\[ \rho'(\frac{1}{\nu}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \( \nu \geq 1 \) and \( \lambda \in A \). Thus we can find a subsequence \( \{\lambda_n\}_{n \geq 1} \) of \( \{\lambda\}_{\lambda \in \Lambda} \) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2} \]

for all \( n \geq 1 \) in virtue of (\rho.2) and (2.3). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty, \]

which is inconsistent with (\rho.4). Secondly we shall prove

\[ (2.8) \quad d(x)=d(y), \quad \text{if } [x]=[y]. \]

We put \([p_n]=[(|x|-n|y|)^+]\) for \( x, y \in \mathbb{R} \) with \([x]=[y] \) and \( n \geq 1 \). Then, \([p_n] \downarrow n=0 \) and \( \inf_{n=1,2,...} d([p_n]x) = 0 \) by (2.7). Since \((1-[p_n])|y| \geq (1-[p_n])|x| \) and

\[ d(\alpha x) = d(x) \]

for \( \alpha > 0 \) and \( x \in \mathbb{R} \), we obtain
\[ d(x) = d([p_n]x) + d((1 - [p_n])x) \]
\[ \leq d([p_n]x) + d(n(1 - [p_n])y) \]
\[ \leq d([p_n]x) + d(y) \].

As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y) \],
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x) \]
\[ = \rho'(y)[y]x - d([y]x) + \rho^*([x] - [y])x \]
\[ \geq \rho'(y) - d(y) + \rho^*([x] - [y])x \]
\[ \geq \rho^*(y) \].

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

(\( \rho.4' \))
\[ \sup_{x \in K} \lim_{\xi \to 0} \rho(\xi x) = K + \infty \].

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ \rho^*(x) = \sup_{0 \leq |y| \leq |x|} \rho(y) \].

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho(\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[ \rho(x) \leq \epsilon \] implies \( \rho'(\frac{1}{\nu_0}x) \leq \gamma \).

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} \lim_{\xi \to 0} \rho'(\xi x) = \gamma_0 + \infty \].

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) (\( i = 1, 2, \ldots, n \)), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'(\frac{1}{\nu_0}x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0}x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \\
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \\
\leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0}x_0) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^2}{\epsilon} \right).
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \rightarrow 0} \rho'(\xi x) \} \leq \gamma' \]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0.
\]

Let \( \{x_i\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_i) = d(\sum_{i=1}^{k} x_i) = \lim_{\xi \rightarrow 0} \rho'(\xi \sum_{i=1}^{k} x_i) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_\lambda) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho'(x_\lambda) = \sum_{\lambda \in A} \rho^*(x_\lambda) + \sum_{\lambda \in A} d(x_\lambda) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_\lambda \in R \) and \( \sum_{\lambda \in A} \rho^*(x_\lambda) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu.
\]
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
\[ \lim_{t \to 0} \rho^*(\xi x) = 0, \]
there exists \( \{ \alpha_\nu \}_{\nu \geqq 1} \) with \( 0 < \alpha_\nu (\nu \geqq 1) \) and \( \sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty \).

It follows that \( x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R \) and \( d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu) \) from (\( \rho.3 \)). For such \( x_0 \), we have for every \( \xi \geqq 0 \),
\[ \rho^*(\xi x_0) = \sum_{\nu=1}^{\infty} \rho^*(\xi \alpha_\nu x_\nu) \geqq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty, \]
which is inconsistent with (\( \rho.4 \)). Therefore we have
\[ \sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leqq \sup_{x \in R} d(x) < +\infty. \]

Q.E.D.

§3. Quasi-norms. We denote by \( R_0 \) the set:
\[ R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \ \text{for all} \ n \geqq 1 \}, \]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_0 \) is a semi-normal manifold\(^7\) of \( R \). We shall prove that \( R_0 \) is a normal manifold of \( R \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_\lambda \) with \( R_0 \ni x_\lambda \geqq 0 \) for all \( \lambda \in \Lambda \).

Putting \[ [p_{n,\lambda}] = [(2nx_\lambda - nx)^+] \]
we have \([p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \) and \( 2n[p_{n,\lambda}]x_\lambda \geqq [p_{n,\lambda}] nx \), which implies \( \rho^*(n[p_{n,\lambda}] x) = 0 \) and \( \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}] x) = \rho^*(nx) = 0 \). Hence, we obtain \( x \in R_0 \), that is, \( R_0 \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[ R = R_0 \oplus R_0^{\perp}. \]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_0, [p]R_0 \) is universally complete, i.e. for any orthogonal system \( \{ x_\lambda \}_{\lambda \in \Lambda} (x_\lambda \in [p]R_0) \), there exists \( x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R \). Hence we can also verify without difficulty that \( R_0 \) has no universally continuous linear functional except 0, if \( R_0 \) is non-atomic. When \( R_0 \) is discrete, it is isomorphic to \( S(\Lambda)^9 \)-space. With respect to such a universally complete space \( R_0 \), we can not always construct a linear metric topology on \( R_0 \), even if \( R_0 \) is discrete.

In the following, therefore, we must exclude \( R_0 \) from our consideration. Now we can state the theorems which we aim at.

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7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, \ |b| \leqq |a|, b \in R \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \cup x_\lambda \in R, 0 \leqq x_\lambda \in S(\lambda \in \Lambda) \) implies \( \cup x_\lambda \in S \).

8) This means that \( x \in R \) is written by \( x = y + z, y \in R_0 \) and \( z \in R_0^{\perp} \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_0^\perp$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.
\[
\sup_{x \in A} \| x \|_0 = \| x \|_0
\]
for any $0 \leq x \uparrow_{\lambda \in \Lambda} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1)\sim(\rho.6)$ except $(\rho.3)$. Now we put
\[
(3.1) \quad \| x \|_0 = \inf \left\{ \xi ; \rho^*(\frac{1}{\xi}x) \leq \xi \right\}.
\]

Then,

i) $0 \leq \| x \|_0 = -\| -x \|_0 < \infty$ and $\| x \|_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_0^\perp$.

ii) $\| x + y \|_0 \leq \| x \|_0 + \| y \|_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_0 \uparrow 0} \| \alpha x \|_0 = 0$ and $\lim_{\alpha_0 \uparrow 0} \| \alpha x_n \|_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha_0 \uparrow 0} \| \alpha x \|_0 = \| \alpha_0 x \|_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $\| [p]x \|_0 > \xi$ we have $\rho^*(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi}[p_\lambda]x) > \xi$ and hence $\sup_{\lambda \in \Lambda} \| [p_\lambda]x \|_0 \geq \xi$. Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| [p_\lambda]x \|_0 = \| [p]x \|_0 , \text{ if } [p_\lambda] \uparrow_{\lambda \in \Lambda} [p].
\]

Let $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$. Putting
\[
[p_{n,\lambda}] = \left[ (x_\lambda - (1-\frac{1}{n})x)^+ \right]
\]
we have
\[
[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad [p_{n,\lambda}] x = [p_{n,\lambda}] (1-\frac{1}{n})x \quad (n \geq 1).
\]

As is shown above, since
\[
\sup_{\lambda \in \Lambda} \| [p_{n,\lambda}] x \|_0 \geq \sup_{\lambda \in \Lambda} \| [p_{n,\lambda}] (1-\frac{1}{n})x \|_0 = \| (1-\frac{1}{n})x \|_0
\]
we have
\[
\sup_{\lambda \in \Lambda} \| x_\lambda \|_0 \geq \| (1-\frac{1}{n})x \|_0
\]
and also $\sup_{\lambda \in \Lambda} \| x_\lambda \|_0 \geq \| x \|_0$. As the converse inequality is obvious by iv), $\| \cdot \|_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim \| x_n \|_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

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In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \cdots)$ be the elements of $R_0^\perp$ such that

\[ [p_{n,\nu}] \uparrow_{\nu=1}^\infty [p_n] \text{ with } \bigcap_{n=1}^\infty [p_n] a = [p_0] a \neq 0; \]

\[ [p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^\infty (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[ ||[q_m] a||_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots) \]

and

\[ n_m [q_m] a \geq [q_m] x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots), \]

where $\delta = ||[p_0] a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^\infty [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}][p_0] a||_0 > \frac{\delta}{2}, \]

there exists $n_1$ with

\[ \| (n_1 a - x_{\nu_1})^+ [q_1] a \|_0 > \frac{\delta}{2}. \]

For such $n_1$, there exists also a natural number $\nu_{k+1}$ such that

\[ [p_{n_{k+1}+1}, \nu_{k+1}] [(n_{k+1} a - x_{\nu_k})^+] [q_k] a \|_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}+1}, \nu_{k+1}] [(n_{k+1} a - x_{\nu_k})^+] [q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad ||[q_{k+1}] a|| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}] (x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \]
\[ \geq \| n_{k+1} [q_{k+1}] a - n_k [q_{k+1}] a \|_0 \geq \| [q_{k+1}] a_0 \|_0 \geq \frac{\delta}{2}, \]

since \([q_{k+1}] \leq [q_k] \leq (n_k a - x_{\nu_{k-1}}^+)\) implies \([q_{k+1}] n_k a \geq [q_{k+1}] x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \([x_{\nu}]_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R_0^+\) is an F-space with \(\| \cdot \|_0\) if and only if \(\rho\) satisfies \((\rho.4')\).

**Proof.** If \(\rho\) satisfies \((\rho.4')\), \(\rho^*\) is a quasi-modular which fulfills also \((\rho.5)\) and \((\rho.6)\) in virtue of Theorem 2.3. Since \(||x||_0 = \inf \{\xi; \rho^* (\frac{x}{\xi}) \leq \xi\}\) is a quasi-norm on \(R_0^+\), we only need to verify completeness of \(\| \cdot \|_0\). At first let \([x_{\nu}]_{\nu \geq 1} \subset R_0^+\) be a Cauchy sequence with \(0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots}\). Since \(\rho^*\) satisfies \((\rho.3)\), there exists \(0 \leq x_0 \in R_0^+\) such that \(x_0 = \bigcup_{\nu=1}^\infty x_{\nu}\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,\nu}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{\nu=1}^\infty [p_{n,\nu}] = [p_n]\), we obtain

\[ [p_{n,\nu}] x_{\nu} \geq n [p_{n,\nu}] x_0 \quad \text{for all } n, \nu \geq 1 \]

and \([p_n] \downarrow_{n=1}^\infty 0\). Since \([x_{\nu}]_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^\infty [p_n] = 0\), that is, \(\bigcup_{n=1}^\infty (\{x_0\} - [p_n]) = [x_0]\). And

\[ (1 - [p_{n,\nu}]) \geq (1 - [p_n]) \quad (n, \nu \geq 1) \]

implies

\[ n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_{\nu} \geq 0. \]

Hence we have

\[ y_n = \bigcup_{\nu=1}^\infty (1 - [p_n]) x_{\nu} \in R_0^+, \]

because \(R_0^+\) is universally continuous. As \([x_{\nu}]_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\| \cdot \|_0\)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_n \|_0 = \sup_{\nu \geq 1} \| (1 - [p_n]) x_{\nu} \|_0 \leq \gamma \]

for every \(n \geq 1\) by semi-continuity of \(\| \cdot \|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \([z_{\nu}]_{\nu \geq 1}\) is an orthogonal sequence with \(\| \sum_{\nu=1}^n z_{\nu} \|_0 = \| y_n \|_0 \leq \gamma\). This implies
for all $n \geq 1$ by the formula (3.1). Then (3.3) assures the existence of
\[ z = \sum_{\nu=1}^{\infty}z_{\nu} = \bigcup_{\nu=1}^{\infty}y_{\nu}. \]
This yields $z = \bigcup_{\nu=1}^{\infty}x_{\nu}$. Truly, it follows from
\[ z = \bigcup_{n=1}^{\infty}y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty}(1 - [p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty}[x_{0}]x_{\nu} = \bigcup_{\nu=1}^{\infty}x_{\nu}. \]
By semi-continuity of $|| \cdot ||_{0}$, we have
\[ ||z - x_{\nu}||_{0} \leq \sup_{\mu \geq \nu}||x_{\mu} - x_{\nu}||_{0} \]
and furthermore $\lim_{\nu \to \infty}||z - x_{\nu}||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R^{+}_{0}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that
\[ ||y_{\nu+1} - y_{\nu}||_{0} \leq \frac{1}{2^{\nu}} \]
for all $\nu \geq 1$.

This implies
\[ ||\sum_{\nu=m}^{n}y_{\nu+1} - y_{\nu}||_{0} \leq \sum_{\nu=m}^{n}||y_{\nu+1} - y_{\nu}||_{0} \leq \frac{1}{2^{n-m}} \]
for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n}||y_{\nu+1} - y_{\nu}||_{0}$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \uparrow \infty$.

Then by the fact proved just above,
\[ z_{0} = \bigcup_{n=1}^{\infty}z_{n} = \sum_{n=1}^{\infty}||y_{\nu+1} - y_{\nu}||_{0} \in R^{+}_{0} \]
and $\lim_{n \to \infty}||z_{0} - z_{n}||_{0} = 0$.

Since $\sum_{\nu=1}^{\infty}||y_{\nu+1} - y_{\nu}||_{0}$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty}(y_{\nu+1} - y_{\nu})$ is also convergent and
\[ ||y_{1} + \sum_{\nu=1}^{\infty}(y_{\nu+1} - y_{\nu}) - y_{n}||_{0} = ||\sum_{\nu=1}^{\infty}(y_{\nu+1} - y_{\nu})||_{0} \leq ||z_{0} - z_{n}||_{0} \to 0. \]

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that
\[ \lim_{\nu \to \infty}||y_{1} + \sum_{\nu=1}^{\infty}(y_{\nu+1} - y_{\nu}) - x_{\nu}||_{0} = 0. \]

Therefore $|| \cdot ||_{0}$ is complete in $R^{+}_{0}$, that is, $R^{+}_{0}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R^{+}_{0}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R^{+}_{0}$, we have $\sum_{\nu=1}^{\infty}\alpha_{\nu}x_{\nu} \in R^{+}_{0}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that sup $d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy (3.4). Q.E.D.

Since $R^{+}_{0}$ contains a normal manifold which is universally complete, if $R^{+}_{0} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let \( R \) be a quasi-modular space which includes no universally complete normal manifold. Then \( R \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by (3.1) and \( R \) becomes an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) fulfils (\( \rho.4' \)).

§4. Another Quasi-norm. Let \( L \) be a modular space in the sense of Musielak and Orlicz (§1). Here we put for \( x \in L \)

\[ (4.1) \quad \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\} \]

and show that \( \| \cdot \|_1 \) is also a quasi-norm on \( L \) and

\[ (4.2) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \]

for all \( x \in L \), where \( \| \cdot \|_0 \) is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that \( 0 \leq \| x \|_1 = \| -x \|_1 < + \infty \) \((x \in L)\) and that \( \| x \|_1 = 0 \) is equivalent to \( x = 0 \). Since \( \alpha_n \downarrow_{n=1}^{\infty} 0 \) implies \( \lim_{n \to \infty} \rho(\alpha_n x) = 0 \) for each \( x \in L \) and \( \lim_{n \to \infty} \| x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \), we obtain that \( \lim_{n \to \infty} \| \alpha x_n \|_1 = 0 \) for all \( \alpha > 0 \). If \( \| x \|_1 < \alpha \) and \( \| y \|_1 < \beta \), there exist \( \xi, \eta > 0 \) such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta. \]

This yields

\[ \| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \]

\[ \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta, \]

in virtue of (A.3). Therefore \( \| x + y \|_1 \leq \| x \|_1 + \| y \|_1 \) holds for any \( x, y \in L \) and \( \| \cdot \|_1 \) is a quasi-norm on \( L \). If \( \xi \rho(\xi x) \leq 1 \) for some \( \xi > 0 \) and \( x \in L \), we have \( \rho(\xi x) \leq \frac{1}{\xi} \) and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}. \]

10) For the convex modular \( m \), we can define two kinds of norms such as

\[ \| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \]

[3 or 4]. For the general modulads considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing \( m(\xi x) \) by \( \xi \rho(\xi x) \) in \( \| \cdot \| \) and \( \| \cdot \| \) respectively.
This yields (4.2), since we have $||x||_{0} \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $||x||_{0} > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $||\cdot||_{1}$ on $L$ which is equivalent to $||\cdot||_{0}$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$||x||_{1} = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^{*}(\xi x) \right\} \quad (x \in R)$$

is a semi-continuous quasi-norm on $R_{0}^{+}$ and $||\cdot||_{1}$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^{*}$ and $R_{0}$ are the same as in §2 and §3. And further we have

$$||x||_{0} \leq ||x||_{1} \leq 2||x||_{0} \quad \text{for all } x \in R_{0}^{+}.$$

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular $\rho^{*}$ on $R$ satisfies $(\rho.1)$~$(\rho.6)$ except $(\rho.3)$ and $\rho^{*}(\xi x)$ is not identically zero as a function of $\xi > 0$ for each $0 \neq x \in R$ (i.e. $R_{0} = \{0\}$). A sequence of elements $\{x_{\nu}\}_{\nu \geq 1}$ is called order-convergent to $a$ and denoted by $\lim_{\nu \to \infty} x_{\nu} = a$, if there exists a sequence of elements $\{a_{\nu}\}_{\nu \geq 1}$ such that $|x_{\nu} - a| \leq a_{\nu}$ ($\nu \geq 1$) and $a_{\nu} \downarrow_{\nu = 1}^{\infty} 0$. And a sequence of elements $\{x_{\nu}\}_{\nu \geq 1}$ is called star-convergent to $a$ and denoted by $s-\lim_{\nu \to \infty} x_{\nu} = a$, if for any subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$, there exists a subsequence $\{z_{\nu}\}_{\nu \geq 1}$ of $\{y_{\nu}\}_{\nu \geq 1}$ with $o-\lim_{\nu \to \infty} z_{\nu} = a$. A quasi-norm $||\cdot||$ on $R$ is termed to be continuous, if $\inf_{\nu \geq 1} ||a_{\nu}|| = 0$ for any $a_{\nu} \downarrow_{\nu \to \infty} 0$. In the sequel, we write by $||\cdot||_{0}$ (or $||\cdot||_{1}$) the quasi-norm defined on $R$ by $\rho^{*}$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $||\cdot||_{0}$ (or $||\cdot||_{1}$) is continuous, it is necessary and sufficient that the following condition is satisfied:

(5.1) for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[x]R$ is finite dimensional and $\rho(y) < +\infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \( \{ p_n \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_n \downarrow_{n=1}^{\infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1 \).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon] a_1) + \rho^*(\xi \epsilon (1 - [p_n^\epsilon]) a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon] a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_n^\epsilon] a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain
\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1) \leq \rho^*(\xi a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{\nu \to \infty} \rho^*(\xi a_\nu) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \| a_n \|_0 = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if
\[
(5.2) \quad \rho^*(a_\nu) \to 0 \text{ implies } \rho^*(\alpha a_\nu) \to 0 \text{ for every } \alpha \geq 0.
\]

From the definition, it is clear that \( s-\lim_{\nu \to \infty} x_\nu = 0 \) implies \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in \( [3] \))

**Theorem 5.2.** \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \) (or \( \lim_{\nu \to \infty} \| x_\nu \|_1 = 0 \)) implies \( s-\lim_{\nu \to \infty} x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \))).

If we replace \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \) by \( \lim_{\nu \to \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[
(5.3) \quad \rho^*(x) = 0 \text{ implies } x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \to 0 \) implies \( s-\lim_{\nu \to \infty} a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11}\) i.e. \( \rho^*(x) = \sup_{\nu \in A} \rho^*(x_\nu) \) for any \( 0 \leq x = \sum_{\nu \in A} x_\nu \). If
\(^{11}\) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf \{ \sup_{\nu \in A} \rho^*(y_\nu) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \to 0 \) is equivalent to \( \rho_*(x) \to 0 \).
\[
\rho(a_{\nu}) \leq \frac{1}{2^\nu} \quad (\nu \geq 1),
\]
we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in \mathcal{R} \) in virtue of \((\rho.3)\).

Now, since
\[
\rho \left( \bigcup_{\nu \geq 1}^{\infty} |a_{\nu}| \right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \[ \rho \left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) \right) = 0 \] and hence \((5.3)\) implies
\[
\bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) = 0.
\]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \rightarrow 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^\nu} \) \((\nu = 1, 2, \cdots)\). Therefore we have \( s\text{-}\lim_{\nu \rightarrow \infty} b_{\nu} = 0 \). \quad \text{Q.E.D.}

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^{*} \) satisfies \((5.3)\) and \( || \cdot ||_{0} \) is complete and continuous, then \((5.2)\) holds.

**References**


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