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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;

(\rho.2) $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;

(\rho.3) If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;

(\rho.4) $\limsup_{\xi \to 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;

(A.2) $\rho(-x) = \rho(x)$;

(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(A.4) $\alpha_{n} \to 0$ implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;

(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $||\cdot||_{0}$ defined by the formula;

\begin{enumerate}
\item $x \perp y$ means $|x| \cap |y| = 0$.
\item A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
\item For the definition of a modular, see [3].
\item A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.
\item $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.
\item This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.
\end{enumerate}


\[(1.1) \quad \|x\|_0 = \inf \left\{ \xi ; \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\}^{6}\]

and \(\|x_n\|_0 \to 0\) is equivalent to \(\rho(\alpha x_n) \to 0\) for all \(\alpha \geq 0\).

In the present paper, we shall deal with a general quasi-modular space \(R\) (i.e. without the assumption that \(R\) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \(R\) and to investigate the condition under which \(R\) is an \(F\)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \(\rho\) on \(R\) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \((\rho.1)\sim(\rho.4)\) with those of \(\rho\) [6], we can not apply the formula (1.1) directly to \(\rho\) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \(\rho^*\) which satisfies (A.2)\~(A.5) on an arbitrary quasi-modular space \(R\) in §2 (Theorems 2.1 and 2.2). Since \(R\) may include a normal manifold \(R_o = \{x : x \in R, \rho^*(\xi x) = 0\} \forall \xi \geq 0\} \) and we can not define a quasi-norm on \(R_o\) in general, we have to exclude \(R_o\) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \(\|\cdot\|_o\) on \(R_o^+\) defined by \(\rho^*\) according to the formula (1.1) is semi-continuous, and in order that \(R_o^+\) is an \(F\)-space with \(\|\cdot\|_o\) (i.e. \(\|\cdot\|_o\) is complete), it is necessary and sufficient that \(\rho\) satisfies

\[(\rho.4') \quad \sup_{x \in R} \lim_{\alpha \to 0} \rho(\alpha x) < +\infty \quad .\]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \(\|\cdot\|_1\) on \(R_o^+\) which is equivalent to \(\|\cdot\|_o\) such that \(\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0\) holds for every \(x \in R_o^+\) (Formulas (4.1) and (4.3)). \(\|\cdot\|_1\) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4 ; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \(\|\cdot\|_o\)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \(R\) denotes a universally continuous semi-ordered linear space and \(\rho\) a quasi-modular defined on \(R\). For any \(p \in R\), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x)\) for all \(x \geq 0\) and \(1-[p]\) is a projection operator onto the normal manifold \(N=\{p\}^1\), that is, \(x=[p]x+(1-[p])x\).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
\begin{align*}
(2.1) & \quad \rho(0) = 0; \\
(2.2) & \quad \rho([p]x) \leq \rho(x) \text{ for all } p, x \in R; \\
(2.3) & \quad \rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x) \text{ for any } [p_{\lambda}]_{\lambda \in \Lambda} \uparrow [p].
\end{align*}
\]

In the argument below, we have to use the additional property of \( \rho \):

\[
(\rho.5) \quad \rho(x) \leq \rho(y) \text{ if } |x| \leq |y|, \quad x, y \in R,
\]

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x \in R \),

\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1)\), \((\rho.2)\) and \((\rho.5)\).

Let \( \{x_{i}\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_{i}) < +\infty \), then

\[
\sum_{i \in A} \rho(x_{i}) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x)
\]

for all \( x \in R \).

We have

\[
x_{0} = \sum_{i \in A} x_{i} \in R
\]

and

\[
\rho(x_{0}) = \sum_{i \in A} \rho(x_{i})
\]

in virtue of \((\rho.3)\).

For such \( x_{0} \),

\[
\rho'(x_{0}) = \sup_{0 \leq |y| \leq |x_{0}|} \rho(y) = \sup_{0 \leq |y| \leq |x_{0}|} \sum_{i \in A} \rho([x_{i}]y) = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_{0}|} \rho([x_{i}]y) = \sum_{i \in A} \rho'(x_{i})
\]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_{0} \in R \),

\[
\rho'(\frac{1}{n} x_{0}) = +\infty
\]

for all \( n \geq 1 \).

By \((\rho.2)\) and \((\rho.4)\), \( x_{0} \) cannot be written as \( x_{0} = \sum_{\nu=1}^{s} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq s \), namely, we can decompose \( x_{0} \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \ldots \))
and
\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2. \) In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1. \)

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^\infty y_\nu \in \mathbb{R} \]
and
\[ \rho(\frac{1}{\nu} y_0) \geq \rho(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1. \)

Because, putting \( \lceil p \rceil = \lceil (|x| - |y|)^+ \rceil \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \leq \rho(\alpha [p] |x| + \alpha (1 - [p]) |y| + \beta [p] |x| + (1 - [p]) \beta |y|) \]
\[ = \rho([p] |x| + (1 - [p]) |y|) = \rho([p] x) + \rho((1 - [p]) y) \leq \rho(x) + \rho(y). \]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4'') \text{ for any } x \in R, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty.\]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain
\[ \rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]
for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with
\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting
\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \text{ mes } \left\{ t : x(t) = \frac{1}{i} \right\}, \]
we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((\text{A.4})\), namely,

\[ \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R. \]

\((\rho.6)\)

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x+y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting
\[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]
we can see easily that $(\rho.1)$, $(\rho.2)$, $(\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since
\[ d(x) \leq \rho'(x) \]
and
\[ d(\alpha x) = d(x) \]
for all $x \in R$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note
\[ \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]
for any $[p_\lambda] \downarrow_{\lambda \in A} 0$. In fact, if we suppose the contrary, we have
\[ \inf_{\lambda \in A} d([p_\lambda]x_0) \geq \alpha > 0 \]
for some $[p_\lambda] \downarrow_{\lambda \in A} 0$ and $x_0 \in R$.

Hence,
\[ \rho'(\frac{1}{\nu}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]
for all $\nu \geq 1$ and $\lambda \in A$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in A}$ such that
\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]
and
\[ \rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2} \]
for all $n \geq 1$ in virtue of $(\rho.2)$ and (2.3). This implies
\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty, \]
which is inconsistent with $(\rho.4)$. Secondly we shall prove
\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in R$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n=1}^{\infty} 0$ and $\inf_{n=1,2,...} d([p_n]x) = 0$ by (2.7). Since $(1 - [p_n])n | y | \geq (1 - [p_n]) | x |$
and
\[ d(\alpha x) = d(x) \]
for $\alpha > 0$ and $x \in R$, we obtain
\[ d(x) = d([p_n]x) + d((1-[p_n])x) \leq d([p_n]x) + d(n(1-[p_n])y) \leq d([p_n]x) + d(y). \]

As \( n \) is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y), \]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \(|x| \geq |y|\), then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y]x) \]
\[ = \rho'(y) - d(y) + \rho^*([x] - [y]x) \geq \rho^*(y). \]
Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies
\[ (\rho.4') \sup_{x \in R} \{ \lim_{\xi \to 0} \rho^*(\xi x) \} = K < +\infty. \]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[ (2.9) \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty, \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]
Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'\left(\frac{1}{\nu}x\right) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^+ \) with
\[ \rho(x) \leq \epsilon \text{ implies } \rho'(\frac{1}{\nu_0}x) \leq \gamma. \]

In \( N_0 \), we have obviously
\[ \sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty. \]
If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^+ \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
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\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) for every \( i = 1, 2, \cdots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \cdots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]
\[
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]
\[
\leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^2}{\epsilon} \right)
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_0^\perp \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0.
\]

Let \( \{ x_i \}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \cdots, \lambda_k \in A \), we have

\[
\sum_{\nu=1}^{k} d(x_{\lambda_{\nu}}) = d(\sum_{\nu=1}^{k} x_{\lambda_{\nu}}) = \lim_{\xi \to 0} \rho'(\xi \sum_{\nu=1}^{k} x_{\lambda_{\nu}}) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{ x_i \}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
\[
\lim_{t \to 0} \rho^* (\xi x) = 0,
\]
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu (\nu \geq 1)$ and
\[
\sum_{\nu = 1}^{\infty} \rho^* (\alpha_\nu x_\nu) < +\infty.
\]
It follows that $x_0 = \sum_{\nu = 1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu = 1}^{\infty} d(\alpha_\nu x_\nu)$ from $(\rho.3)$. For such
$x_0$, we have for every $\xi \geq 0$,
\[
\rho^* (\xi x_0) = \sum_{\nu = 1}^{\infty} \rho^* (\xi \alpha_\nu x_\nu) \geq \sum_{\nu = 1}^{\infty} d(\alpha_\nu x_\nu) = +\infty,
\]
which is inconsistent with $(\rho.4)$. Therefore we have
\[
\sup_{x \in R} (\lim_{\epsilon \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.
\]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[
R_0 = \{x : x \in R, \rho^* (nx) = 0 \text{ for all } n \geq 1\},
\]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal
manifold\textsuperscript{7} of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In
fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geq 0$ for all $\lambda \in \Lambda$.
Putting $[p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+]$, we have
\[
[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]
\]
and $2n[p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}]nx$, which implies $\rho^* (n[p_{n,\lambda}]x) = 0$ and
\[
\sup_{\lambda \in \Lambda} \rho^* (n[p_{n,\lambda}]x) = \rho^* (nx) = 0.
\]
Therefore, $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Hence we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp.
\]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$
is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in \Lambda}$ with $x_\lambda \in [p]R_0$,
there exists $x_0 = \sum_{\lambda \in \Lambda} x_\lambda \in [p]R$. Hence we can also verify without difficulty
that $R_0$ has no universally continuous linear functional except 0, if $R_0$
is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^0$-space.
With respect to such a universally complete space $R_0$, we can not always
construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration.
Now we can state the theorems which we aim at.

\textsuperscript{7} A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in \Lambda} x_\lambda \in S(\lambda \in \Lambda)$ implies $\bigcup_{\lambda \in \Lambda} x_\lambda \in S$.

\textsuperscript{8} This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

\textsuperscript{9} $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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\textsuperscript{8}) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$. 

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Theorem 3.1. Let \( R \) be a quasi-modular space. Then \( R_{0}^{\perp} \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_{0} \) which is semi-continuous, i.e.
\[
\sup_{i \in \Lambda} \| x_{i} \|_{0} = \| x \|_{0} \quad \text{for any } 0 \leq x_{i} \uparrow_{i \in \Lambda} x.
\]

Proof. In virtue of Theorems 2.1 and 2.2, \( \rho^* \) satisfies \((\rho.1)\sim(\rho.6)\) except \((\rho.3)\). Now we put
\[
(3.1) \quad \| x \|_{0} = \inf \left\{ \xi ; \rho^* \left( \frac{1}{\xi} x \right) \leq \xi \right\}.
\]

Then,

i) \( 0 \leq \| x \|_{0} = \| -x \|_{0} < \infty \) and \( \| x \|_{0} = 0 \) is equivalent to \( x = 0 \); follows from \((\rho.1), (\rho.6), (2.1)\) and the definition of \( R_{0}^{\perp} \).

ii) \( \| x + y \|_{0} \leq \| x \|_{0} + \| y \|_{0} \) for any \( x, y \in R \); follows also from \((A.3)\) which is deduced from \((\rho.4)\).

iii) \( \lim_{\alpha_{n} \uparrow 0} \| \alpha_{n} x \|_{0} = 0 \) and \( \lim_{\| x_{n} \|_{0} \uparrow 0} \| \alpha x_{n} \|_{0} = 0 \); is a direct consequence of \((\rho.5)\). At last we shall prove that \( \| \cdot \|_{0} \) is semi-continuous. From ii) and iii), it follows that \( \lim_{\alpha \uparrow a_{0}} \| \alpha x \|_{0} = \| \alpha_{0} x \|_{0} \) for all \( x \in R_{0}^{\perp} \) and \( \alpha_{0} \geq 0 \). If \( x \in R_{0}^{\perp} \) and \([p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p]\), for any positive number \( \xi \) with \( \| [p] x \|_{0} > \xi \) we have \( \rho^* \left( \frac{1}{\xi} [p] x \right) > \xi \), which implies \( \sup_{\lambda \in \Lambda} \rho^* \left( \frac{1}{\xi} [p_{\lambda}] x \right) > \xi \) and hence \( \sup_{\lambda \in \Lambda} \| [p_{\lambda}] x \|_{0} \geq \xi \). Thus we obtain
\[
\sup_{\lambda \in \Lambda} \| [p_{\lambda}] x \|_{0} = \| [p] x \|_{0}, \quad \text{if } [p_{\lambda}] \uparrow_{\lambda \in \Lambda} [p].
\]

Let \( 0 \leq x_{i} \uparrow_{i \in \Lambda} x \). Putting
\[
[p_{n,i}] = \left[ (x_{i} - (1 - \frac{1}{n}) x) \right]
\]
we have
\[
[p_{n,i}] \uparrow_{i \in \Lambda} [x] \quad \text{and} \quad [p_{n,i}] x_{i} \geq [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).
\]

As is shown above, since
\[
\sup_{i \in \Lambda} \| [p_{n,i}] x_{i} \|_{0} \geq \sup_{i \in \Lambda} \| [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \|_{0} = \| (1 - \frac{1}{n}) x \|_{0},
\]
we have
\[
\sup_{i \in \Lambda} \| x_{i} \|_{0} \geq \| (1 - \frac{1}{n}) x \|_{0}
\]
and also \( \sup_{i \in \Lambda} \| x_{i} \|_{0} \geq \| x \|_{0} \). As the converse inequality is obvious by iv), \( \| \cdot \|_{0} \) is semi-continuous. Q.E.D.

Remark 2. By the definition of \((3.1)\), we can see easily that \( \lim_{n \to \infty} \| x_{n} \|_{0} = 0 \) if and only if \( \lim_{\xi \to \infty} \rho(\xi x_{n}) = 0 \) for all \( \xi \geq 0 \).
In order to prove the completeness of quasi-norm \( || \cdot ||_0 \), the next Lemma is necessary.

**Lemma 2.** Let \( p_{n, \nu}, x_{\nu} \geq 0 \) and \( a \geq 0 \) \((n, \nu = 1, 2, \cdots)\) be the elements of \( R_0^\perp \) such that

\[
(p_{n, \nu})^{\uparrow_{\nu=1}^{\infty}} [p_n] a = [p_0] a \neq 0; \tag{3.2}
\]

\[
[p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \text{ for all } n, \nu \geq 1. \tag{3.3}
\]

Then \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence of \( R_0^\perp \) with respect to \( || \cdot ||_0 \).

**Proof.** We shall show that there exist a sequence of projectors \( [q_m]^{\downarrow_{m=1}^{\infty}} (m \geq 1) \) and sequences of natural numbers \( \nu_m, n_m \) such that

\[
||[q_m] a||_0 > \frac{\delta}{2} \tag{3.4}
\]

and

\[
n_m [q_m] a \geq n_{m+1} [q_m] a \tag{3.5}
\]

where \( \delta = ||[p_0] a||_0 \).

In fact, we put \( n_1 = 1. \) Since \( [p_{1, \nu_1}] [p_0]^{\uparrow_{\nu=1}^{\infty}}[p_0] \) and \( || \cdot ||_0 \) is semi-continuous, we can find a natural number \( \nu_1 \) such that

\[
||[p_{1, \nu_1}] [p_0] a||_0 > \frac{\delta}{2}. \tag{3.4}
\]

We put \( [q_1] = [p_{1, \nu_1}] [p_0]. \) Now, let us assume that \( [q_m], \nu_m, n_m (m = 1, 2, \cdots, k) \) have been taken such that (3.4) and (3.5) are satisfied.

Since \( ([n a - x_{\nu_k}]^+)^{\uparrow_{n=1}^{\infty}}[a] \) and \( ||[q_k] a||_0 > \frac{\delta}{2} \), there exists \( n_{k+1} \) with

\[
||([n_{k+1} a - x_{\nu_k}]^+) [q_k] a ||_0 > \frac{\delta}{2}. \tag{3.6}
\]

For such \( n_{k+1} \), there exists also a natural number \( \nu_{k+1} \) such that

\[
||[p_{n_{k+1}, \nu_{k+1}}] ([n_{k+1} a - x_{\nu_k}]^+) [q_k] a ||_0 > \frac{\delta}{2}. \tag{3.7}
\]

in virtue of (3.2) and semi-continuity of \( || \cdot ||_0 \). Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] ([n_{k+1} a - x_{\nu_k}]^+) [q_k],
\]

because

\[
[q_{k+1}] \leq [q_k], \quad ||[q_{k+1}] a || > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
\]

by (3.3) and \( [q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k} \) by (3.5).

For the sequence thus obtained, we have for every \( k \geq 3 \)
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\[
\|x_{\nu_{k+1}} - x_{\nu_{k-1}}\|_0 \geq \|[q_{k+1}](x_{\nu_{k+1}} - x_{\nu_{k-1}})\|_0 \\
\geq \|n_{k+1}[q_{k+1}]a - n_{k}[q_{k+1}]a\|_0 \geq \|[q_{k+1}]a_0\|_0 \geq \frac{\delta}{2},
\]

since \([q_{k+1}] \leq [q_k] \leq [(n_k a - x_{\nu-1})^+]\) implies \([q_{k+1}]n_k a \geq [q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R^+_0\) is an F-space with \(\|\cdot\|_0\) if and only if \(\rho\) satisfies (\(\rho.4'\)).

**Proof.** If \(\rho\) satisfies (\(\rho.4'\)), \(\rho^*\) is a quasi-modular which fulfills also (\(\rho.5\)) and (\(\rho.6\)) in virtue of Theorem 2.3. Since \(x\|_0 = \inf \{\xi ; \rho^*(\frac{x}{\xi}) \leq \xi\}\) is a quasi-norm on \(R^+_0\), we need only to verify completeness of \(\|\cdot\|_0\). At first let \(\{x_{\nu}\}_{\nu \geq 1} \subset R^+_0\) be a Cauchy sequence with \(0 \leq x_{\nu} \uparrow_{\nu=1,2,...}\). Since \(\rho^*\) satisfies (\(\rho.3\)), there exists \(0 \leq x_0 \in R^+_0\) such that \(x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,v}] = [(x_{\nu} - nx_0)^+]\) and \(\bigcup_{\nu=1}^{\infty} [p_{n,v}] = [p_n]\), we obtain

\[
[p_{n,v}] x_{\nu} \geq n[p_{n,v}] x_0
\]

for all \(n, \nu \geq 1\) and \([p_n]\downarrow_{n=1}^{\infty} 0\). Since \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty} [p_n] = 0\), that is, \(\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]\). And

\[
(1 - [p_{n,v}]) \geq (1 - [p_n])
\]

\((n, \nu \geq 1)\)

implies

\[
n(1 - [p_n]) x_0 \geq (1 - [p_n]) x_{\nu} \geq 0.
\]

Hence we have

\[
y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_{\nu} \in R^+_0,
\]

because \(R^+_0\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(\|\cdot\|_0\)

\[
\gamma = \sup_{\nu \geq 1} \|x_{\nu}\|_0 < +\infty,
\]

which implies

\[
\|y_n\|_0 = \sup_{\nu \geq 1} \|(1 - [p_n]) x_{\nu}\|_0 \leq \gamma
\]

for every \(n \geq 1\) by semi-continuity of \(\|\cdot\|_0\). We put \(z_1 = y_1\) and \(z_n = y_n - y_{n-1}\) \((n \geq 2)\). It follows from the definition of \(y_n\) that \(\{z_{\nu}\}_{\nu \geq 1}\) is an orthogonal sequence with \(\|\sum_{\nu=1}^{n} z_{\nu}\|_0 = \|y_n\|_0 \leq \gamma\). This implies
\[
\sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma
\]
for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of \( z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\[
z = \bigcup_{\nu=1}^{n} y_{\nu} = \bigcup_{\nu=1}^{n} (1 - [p_n]) x_{\nu} = \bigcup_{\nu=1}^{n} [x_0] x_{\nu} = \bigcup_{\nu=1}^{n} x_{\nu}.
\]

By semi-continuity of \( || \cdot ||_{0} \), we have

\[
|| z - x_{\nu} ||_{0} \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_{0}
\]
and furthermore \( \lim_{\nu \to \infty} || z - x_{\nu} ||_{0} = 0 \).

Secondly let \( \{ x_{\nu} \}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_0^\perp \). Then we can find a subsequence \( \{ y_{\nu} \}_{\nu \geq 1} \) of \( \{ x_{\nu} \}_{\nu \geq 1} \) such that

\[
|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}} \quad \text{for all } \nu \geq 1.
\]

This implies

\[
|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{n-m}} \quad \text{for all } n > m \geq 1.
\]

Putting \( z_n = \sum_{\nu=1}^{n} | y_{\nu+1} - y_{\nu} | \), we have a Cauchy sequence \( \{ z_n \}_{n \geq 1} \) with \( 0 \leq z_n \leq \infty \).

Then by the fact proved just above,

\[
z_0 = \lim_{n \to \infty} z_n = \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \in R_0^\perp \quad \text{and} \quad \lim_{n \to \infty} || z_0 - z_n ||_{0} = 0.
\]

Since \( \sum_{\nu=1}^{\infty} | y_{\nu+1} - y_{\nu} | \) is convergent, \( y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) \) is also convergent and

\[
|| y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_n ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_0 - z_n ||_{0} \to 0.
\]

Since \( \{ y_{\nu} \}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{ x_{\nu} \}_{\nu \geq 1} \), it follows that

\[
\lim_{\nu \to \infty} || y_1 + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0.
\]

Therefore \( || \cdot ||_{0} \) is complete in \( R_0^\perp \), that is, \( R_0^\perp \) is an F-space with \( || \cdot ||_{0} \).

Conversely if \( R_0^\perp \) is an F-space, then for any orthogonal sequence \( \{ x_{\nu} \}_{\nu \geq 1} \in R_0^\perp \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_0^\perp \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)).

Hence we can see that \( \sup_{x \in K} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy (\( \rho.4^* \)).

Q.E.D.

Since \( R_0 \) contains a normal manifold which is universally complete, if \( R_0^\perp = 0 \), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|$ defined by (3.1) and $R$ becomes an $F$-space with $\| \cdot \|$ if and only if $\rho$ fulfills ($\rho.A'$).

§ 4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[ (4.1) \quad \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\} \]

and show that $\| \cdot \|_1$ is also a quasi-norm on $L$ and

\[ (4.2) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \]

hold, where $\| \cdot \|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \| x \|_1 = \| -x \|_1 < + \infty$ ($x \in L$) and that $\| x \|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \to \infty} \| \alpha_n x \|_1 = 0$ for all $\alpha_n \downarrow_{n=1}^\infty 0$ and that $\lim_{n \to \infty} \| x_n \|_1 = 0$ implies $\lim_{n \to \infty} \| \alpha x_n \|_1 = 0$ for all $\alpha > 0$. If $\| x \|_1 < \alpha$ and $\| y \|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left( \frac{\xi \eta}{\xi + \eta} (x+y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore $\| x + y \|_1 \leq \| x \|_1 + \| y \|_1$ holds for any $x, y \in L$ and $\| \cdot \|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular $m$, we can define two kinds of norms such as

\[
\| x \| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \cdot
\]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\| \cdot \|$ and $\| \cdot \|$ respectively.
This yields (4.2), since we have \( \|x\|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \|x\|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\( \sim \) (A.5) in §1, then the formula (4.1) yields a quasi-norm \( \|\cdot\|_1 \) on \( L \) which is equivalent to \( \|\cdot\|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\|x\|_1 = \operatorname{inf}_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}
\quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R_0^\perp \) and \( \|\cdot\|_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.4' \)), where \( \rho^* \) and \( R_0 \) are the same as in §2 and §3. And further we have

\[
\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all } x \in R_0^\perp.
\]

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies (\( \rho.1 \)\( \sim \) (\( \rho.6 \)) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \leq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-\lim_{\nu \to \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a_\nu| \leq a_\nu \quad (\nu \geq 1) \) and \( a_\nu \downarrow 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-\lim_{\nu \to \infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( o-\lim_{\nu \rightarrow \infty} z_\nu = a \).

A quasi-norm \( \|\cdot\| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} \|a_\nu\| = 0 \) for any \( a_\nu \downarrow 0 \). In the sequel, we write by \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that \( \|\cdot\|_0 \) (or \( \|\cdot\|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } \[z\]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_{\nu} \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p^\epsilon_n] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p^\epsilon_n] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1] a_n = [p^\epsilon_n] a_n + (1 - [p^\epsilon_n]) a_n \leq [p^\epsilon_n] a_1 + \epsilon a_1 \).

This implies \( \rho^*(\xi a_n) \leq \rho^*(\xi [p^\epsilon_n] a_1) + \rho^*(\xi \epsilon (1 - [p^\epsilon_n]) a_1) \) for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p^\epsilon_n] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*([p^\epsilon_n] a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*([p^\epsilon_n] a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*([p^\epsilon_n] a_1) \leq \rho^*(\xi a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{\nu \rightarrow \infty} \rho^*([p^\epsilon_n] a_1) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} 1a_n ||_0 = 0 \) and \( || \cdot ||_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( || \cdot ||_0 \) is continuous, if

\[
(5.2) \quad \rho^*(a_{\nu}) \rightarrow 0 \implies \rho^*(\alpha a_{\nu}) \rightarrow 0 \quad \text{for every } \alpha \geq 0.
\]

From the definition, it is clear that \( \text{s-lim } x = 0 \) implies \( \lim_{\nu \rightarrow \infty} || x_{\nu} || = 0 \), if \( || \cdot ||_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \rightarrow \infty} || x_{\nu} || = 0 \) (or \( \lim_{\nu \rightarrow \infty} || x_{\nu} ||_1 = 0 \)) implies \( \text{s-lim } x = 0 \), if \( || \cdot ||_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \))).

If we replace \( \lim_{\nu \rightarrow \infty} || x_{\nu} || = 0 \) by \( \lim_{\nu \rightarrow \infty} \rho(x_{\nu}) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x) = 0 \implies x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( || \cdot ||_0 \) is complete, \( \rho(a_{\nu}) \rightarrow 0 \) implies \( \text{s-lim } a_{\nu} = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous, i.e. \( \rho^*(x) = \sup_{y \in A} \rho^*(x_i) \) for any \( 0 \leq x \uparrow_{i \in A} x \). If

11) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf_{y \uparrow_{i \in A} x} \{ \sup_{j \in A} \rho^*(y_j) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x) \rightarrow 0 \) is equivalent to \( \rho_*(x) \rightarrow 0 \).
$\rho(a_{\nu}) \leq \frac{1}{2^\nu}$ \quad ($\nu \geq 1$),
we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R$ in virtue of $(\rho.3)$.

Now, since
$$\rho\left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu=1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}$$
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right)\right) = 0$ and hence (5.3) implies
$$\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu=1}^{\infty} |a_{\nu}|\right) = 0.$$ 
Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \to 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^\nu}$ \quad ($\nu = 1, 2, \cdots$). Therefore we have $s\lim_{\nu \to \infty} b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $|| \cdot ||_0$ is complete and continuous, then (5.2) holds.

**References**


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