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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;
(\rho.2) $\rho(x + y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;
(\rho.3) If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;
(\rho.4) $\lim_{\xi \rightarrow 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;
(A.2) $\rho(-x) = \rho(x)$;
(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
(A.4) $\alpha_{n} \rightarrow 0$ implies $\rho(\alpha_{n} x) \rightarrow 0$ for every $x \in R$;
(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula;

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.
5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4].

\( R \) is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) = 0$.

In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 
(1.1) \[ ||x||_0 = \inf \left\{ \xi; \rho \left( \frac{1}{\xi} x \right) = \xi \right\} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: \( (\rho.1) \sim (\rho.4) \) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies \( (A.2) \sim (A.5) \) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \rho^*(\xi x) = 0 \ \text{for all} \ \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( || \cdot ||_0 \) on \( R^+_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R^+_0 \) is an \( F \)-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \left\{ \lim_{\alpha \to 0} \rho(\alpha x) \right\} < + \infty \]

(Theorem 3.2).

In \( \S 4 \), we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R^+_0 \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2||x||_0 \) holds for every \( x \in R^+_0 \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4; \S 83]. At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \( [p] \) is a projector: \( [p]x = \bigcup_{n=1}^{\infty} (n \cdot [p] x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N = [p]^{\perp} \), that is, \( x = [p]x + (1 - [p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

(2.1) \( \rho(0) = 0 \);
(2.2) \( \rho([p]x) \leq \rho(x) \) for all \( p, x \in R \);
(2.3) \( \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \) for any \( [p_i] \uparrow_{i \in A} [p] \).

In the argument below, we have to use the additional property of \( \rho \):

(\( \rho.5 \)) \( \rho(x) \leq \rho(y) \) if \( |x| \leq |y|, x, y \in R \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies (\( \rho.5 \)).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which (\( \rho.5 \)) is valid.

Proof. We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions (\( \rho.1 \)), (\( \rho.2 \)) and (\( \rho.5 \)).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\]

We have

\[
x_0 = \sum_{i \in A} x_i \in R
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i)
\]

in virtue of (\( \rho.3 \)).

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)
\]

\[
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils (\( \rho.3 \)).

If \( \rho' \) does not fulfil (\( \rho.4 \)), we have for some \( x_0 \in R \),

\[
\rho' \left( \frac{1}{n} x_0 \right) = +\infty \quad \text{for all } n \geq 1.
\]

By (\( \rho.2 \)) and (\( \rho.4 \)), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \kappa \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \ldots \))
and
\[ \rho'(\frac{1}{2} x'_2) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho'(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho'(\frac{1}{\nu} y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in \mathbb{R} \]
and
\[ \rho'(\frac{1}{\nu} y_0) \geq \rho'(\frac{1}{\nu} y_\nu) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( \lfloor p \rfloor = \lfloor (|x| - |y|)^{+} \rfloor \), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \\
\leq \rho(\alpha [p]|x| + \alpha(1-[p])|y| + \beta [p]|x| + (1-[p])\beta |y|) \\
= \rho([p]|x| + (1-[p])|y|) \\
= \rho([p]x) + \rho((1-[p])y) \\
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3) \) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[
\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| \, dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^\infty i \text{ mes } \left\{ t : x(t) = \frac{1}{i} \right\},
\]
we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[(\rho.6) \quad \lim_{t \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.\]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[(2.5) \quad d(x) = \lim_{t \to 0} \rho'(\xi x).\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and...
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$$d(x+y)=d(x)+d(y)$$ \text{if } x \perp y.

Hence, putting
\begin{equation}
(2.6) \quad \rho^*(x)=\rho'(x)-d(x) \quad (x \in R).
\end{equation}
we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since
\[
d(x) \leq \rho'(x)
\]
and
\[
d(\alpha x) = d(x)
\]
for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note
\begin{equation}
(2.7) \quad \inf_{A} d([p_\lambda]x) = 0
\end{equation}
for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have
\[
\inf_{A} d([p_\lambda]x_0) \geq \alpha > 0
\]
for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,
\[
\rho'(\frac{1}{\nu}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha
\]
for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \({\lambda_n}_{n \geq 1}\) of \({\lambda}_{2 \in A}\) such that
\[
[p_{\lambda_n}] \geq [p_{\lambda_{n+1}}]
\]
and
\[
\rho'(\frac{1}{n}([p_{\lambda_n}]-[p_{\lambda_{n+1}}]x_0) \geq \frac{\alpha}{2}
\]
for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies
\[
\rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}]-[p_{\lambda_{m+1}}]x_0) = +\infty,
\]
which is inconsistent with \((\rho.4)\). Secondly we shall prove
\begin{equation}
(2.8) \quad d(x)=d(y), \quad \text{if } [x]=[y].
\end{equation}
We put \([p_n]=[(x-n|y|)]\) for \(x, y \in R\) with \([x]=[y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n \geq 1} 0\) and \(\inf_{n \geq 1} d([p_n]x) = 0\) by \((2.7)\). Since \((1-[p_n])n \geq (1-[p_n])|x|\)
and
\[
d(\alpha x) = d(x)
\]
for \(\alpha > 0\) and \(x \in R\), we obtain
As \( n \) is arbitrary, this implies
\[
d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*([y]x) + \rho^*(([x] - [y])x)
\geq \rho'(y) - d(y) + \rho^*([x] - [y])x
\geq \rho^*(y).
\]
Thus \( \rho^* \) satisfies (\( \rho.5 \)). Q.E.D.

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

(\( \rho.4' \))
\[
\sup_{x \in \mathbb{R}} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} = K' < +\infty,
\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
\[
\sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} = K' < +\infty,
\]
where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho\left(\frac{1}{\nu} x\right) \) for \( \nu \geq 1 \) and \( x \in \mathbb{R} \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[
\rho(x) \leq \epsilon \quad \text{implies} \quad \rho\left(\frac{1}{\nu_0} x\right) \leq \gamma.
\]

In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \left\{ \lim_{\xi \to 0} \rho'(\xi x) \right\} = \gamma_0 < +\infty.
\]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
$\alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z$

where $\frac{\epsilon}{2} < \rho(x_i) \leq \epsilon$ for $i=1, 2, \cdots, n$, $y_j$ is an atomic element with $\rho(y_j) > \epsilon$ for every $j=1, 2, \cdots, m$ and $\rho(z) \leq \frac{\epsilon}{2}$. From above, we get $n \leq \frac{4K}{\epsilon}$ and $m \leq \frac{2K}{\epsilon}$. This yields

$$
\rho'(\frac{1}{\nu_0} \alpha_0 x_0) \leq \sum_{i=1}^{n} \rho'(\frac{1}{\nu_0} x_i) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0} \\
\leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma 
$$

Hence, we obtain

$$
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'(\frac{\alpha_0}{\nu_0} x_0) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^2}{\epsilon} \right)
$$

in case of $\epsilon \leq 2K$. If $2K \leq \epsilon$, we have immediately for $x \in N_0^+$

$$
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma
$$

Therefore, we obtain

$$
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
$$

where

$$
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0
$$

Let $\{x_i\}_{i \in A}$ be an orthogonal system with $\sum_{i \in A} \rho^*(x_i) < +\infty$. Then for arbitrary $\lambda_1, \cdots, \lambda_k \in A$, we have

$$
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
$$

which implies $\sum_{i \in A} d(x_i) \leq \gamma'$. It follows that

$$
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
$$

which implies $x_0 = \sum_{i \in A} x_i \in R$ and $\sum_{i \in A} \rho^*(x_i) = \rho^*(x_0)$ by $(\rho.4)$ and (2.7). Therefore $\rho^*$ satisfies $(\rho.3)$.

On the other hand, suppose that $\rho^*$ satisfies $(\rho.3)$ and $\sup_{x \in R} d(x) = +\infty$. Then we can find an orthogonal sequence $\{x_i\}_{i \geq 1}$ such that

$$
\sum_{i=1}^{m} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
$$
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since $\lim_{t \to 0} \rho^*(tx) = 0$, there exists $\{\alpha_{\nu}\}_{\nu \geq 1}$ with $0 < \alpha_{\nu}$ ($\nu \geq 1$) and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu}x_{\nu}) < +\infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu}x_{\nu} \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu}x_{\nu})$ from ($\rho.3$). For such $x_{0\iota}$, we have for every $\xi \geq 0$,

$$\rho'(\xi x_{0\iota}) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_{\nu}x_{\nu}) \geq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty,$$

which is inconsistent with ($\rho.4$). Therefore we have $\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty$. Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:

$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geq 1 \},$$

where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\(^7\) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geq 0$ for all $\lambda \in \Lambda$. Putting $[p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+]$, we have $[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} x$ and $2n[p_{n,\lambda}]x_{\lambda} \geq [p_{n,\lambda}]nx$, which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into

$$R = R_0 \oplus R_0^\perp.$$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}, x_{\lambda} \in [p]R_0)$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{p\iota}-$space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold $S$ is said to be semi-normal, if $a \in S, \ b \leq |a|, b \in R$ implies $b \in S$. Since $R$ is univerally continuous, a semi-normal manifold $S$ is normal if and only if $\cup x_{\lambda} \in R, 0 \leq x_{\lambda} \in S(\lambda \in \Lambda)$ implies $\cup x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

---
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^\perp$ becomes a quasi-normed space with a quasi-norm $|| \cdot ||_0$ which is semi-continuous, i.e.

$$\sup_{i \in A} || x_i ||_0 = || x ||_0$$

for any $0 \leq x_i \uparrow_{i \in A} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$|| x ||_0 = \inf \{ \xi ; \rho^*(\frac{1}{\xi} x) \leq \xi \} .$$

Then,

i) $0 \leq || x ||_0 = || -x ||_0 < \infty$ and $|| x ||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1),(\rho.6),(2.1)$ and the definition of $R_{0}^\perp$.

ii) $|| x + y ||_0 \leq || x ||_0 + || y ||_0$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim || \alpha x ||_0 = \lim || \alpha x_n ||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim || \alpha x ||_0 = \lim || \alpha x_n ||_0$ for all $x \in R_{0}^\perp$ and $\alpha_0 \geq 0$. If $x \in R_{0}^\perp$ and $[p_{\lambda}]_\uparrow_{\lambda \in A} [p]$, for any positive number $\xi$ with $|| [p] x ||_0 > \xi$ we have $\rho^*(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{\lambda \in A} \rho^*(\frac{1}{\xi} [p_{\lambda}] x) > \xi$ and hence $\sup || p_{\lambda} x ||_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda \in A} || p_{\lambda} x ||_0 = || [p] x ||_0 ,$$

if $[p_{\lambda}]_\uparrow_{\lambda \in A} [p]$. Let $0 \leq x_{1, \uparrow_{\lambda \in A} x}$. Putting

$$[p_{n,1}] = \left[ (x_{1} - (1 - \frac{1}{n}) x) \right],$$

we have

$$[p_{n,1}]_\uparrow_{\lambda \in A} [x] \quad \text{and} \quad [p_{n,1}] x_{1} = [p_{n,1}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda \in A} || [p_{n,1}] x_{1} ||_0 \geq \sup_{\lambda \in A} || [p_{n,1}] \left( 1 - \frac{1}{n} \right) x ||_0 = || \left( 1 - \frac{1}{n} \right) x ||_0 ,$$

we have

$$\sup_{\lambda \in A} || x_{1} ||_0 \geq \left( 1 - \frac{1}{n} \right) x ||_0$$

and also $\sup || x_{1} ||_0 \geq || x ||_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim || x_n ||_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}, x_{\nu} \geq 0$ and $a \geq 0 (n, \nu = 1, 2, \ldots)$ be the elements of $R_0^\perp$ such that

(3.2) \[ [p_{n,\nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \supseteq 0; \]

(3.3) \[ [p_{n,\nu}]x_{\nu} \geq n[p_{n,\nu}]a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

(3.4) \[ ||[q_m]a||_0 > \frac{\delta}{2} \text{ and } [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \ldots) \]

and

(3.5) \[ n_m[q_m]a \geq [q_m]x_{\nu_m} \quad n_{m+1} > n_m \quad (m=2, 3, \ldots), \]

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,\nu}][p_0] \uparrow_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[ ||[p_{1,\nu_1}][p_0]a||_0 > \frac{\delta}{2}. \]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m=1, 2, \ldots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(na-x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[ ||(n_{k+1}a-x_{\nu_k})^+[q_k]a||_0 > \frac{\delta}{2}. \]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[ ||[p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k]a||_0 > \frac{\delta}{2}. \]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1},\nu_{k+1}}][(n_{k+1}a-x_{\nu_k})^+][q_k], \]

because

\[ [q_{k+1}] \subseteq [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a \]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0}\geqq||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0}\geqq||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0}\geqq||[q_{k+1}]a||_{0}\geqq\frac{\delta}{2},$

since $[q_{k+1}]\leqq[q_{k}]\leqq[(n_{k}a-x_{\nu-1})^{+}]$ implies $[q_{k+1}]n_{k}a\geqq[q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu\geqq 1}$ is not a Cauchy sequence.

Theorem 3.2. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^{\perp}$ is an F-space with $||\cdot||_{0}$ if and only if $\rho$ satisfies $(\rho.4')$.

Proof. If $\rho$ satisfies $(\rho.4')$, $\rho^{*}$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $||x||_{0} (=\inf\{\xi;\rho^{*}(\frac{x}{\xi})\leqq\xi\})$ is a quasi-norm on $R_{0}^{\perp}$, we need only to verify completeness of $||\cdot||_{0}$. At first let $\{x_{\nu}\}_{\nu\geqq 1}\subset R_{0}^{\perp}$ be a Cauchy sequence with $0\leqq x_{\nu}\uparrow_{\nu=1,2,\ldots}$. Since $\rho^{*}$ satisfies $(\rho.3)$, there exists $0\leqq x_{0}\in R_{0}^{\perp}$ as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}]=[(x_{\nu}-nx_{0})^{+}]$ and $\bigcup_{\nu=1}^{\infty}[p_{n,v}]=[p_{n}]$, we obtain

$$[p_{n,v}]x_{\nu}\geqq n[p_{n,v}]x_{0}$$

for all $n, v\geqq 1$ and $[p_{n}]^{\perp}_{n=1}\neq 0$. Since $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}]=0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}]-[p_{n}])=[x_{0}]$. And

$$(1-[p_{n,v}])\geqq(1-[p_{n}])$$

implies

$$n(1-[p_{n,v}])x_{0}\geqq(1-[p_{n}])x_{v}\geqq 0.$$ 

Hence we have

$$y_{n}\uparrow_{\nu=1}^{\infty}(1-[p_{n,v}])x_{\nu}\in R_{0}^{\perp},$$

because $R_{0}^{\perp}$ is universally continuous. As $\{x_{\nu}\}_{\nu\geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_{0}$

$$\gamma=\sup_{\nu\geqq 1}||x_{\nu}||_{0}<+\infty,$$

which implies

$$||y_{n}||_{0}=\sup_{\nu\geqq 1}||(1-[p_{n}]x_{\nu}||_{0}\leqq\gamma$$

for every $n\geqq 1$ by semi-continuity of $||\cdot||_{0}$. We put $z_{1}=y_{1}$ and $z_{n}=y_{n}-y_{n-1}$ ($n\geqq 2$). It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu\geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_{0}=||y_{n}||_{0}\leqq\gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z=\sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from

$$z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} (1 - [p_{\nu}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} [x_{\nu}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.$$

By semi-continuity of $|| \cdot ||_{0}$, we have

$$|| z - x_{\nu} ||_{0} \leq \sup_{\nu \geq \mu} || x_{\mu} - x_{\nu} ||_{0}$$

and furthermore

$$\lim_{\nu \to \infty} || z - x_{\nu} ||_{0} = 0.$$

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{+}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$.

This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{m-1}}$$

for all $n > m \geq 1$.

Putting $z_{n}=\sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \leq \infty$. Then by the fact proved just above,

$$z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{+} \quad \text{and} \quad \lim_{n \to \infty} || z_{0} - z_{n} ||_{0} = 0.$$

Since $\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \to 0.$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{\nu \to \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0.$$

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{+}$, that is, $R_{0}^{+}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{+}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$).

Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0} = 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ defined by (3.1) and $R$ becomes an $F$-space with $||\cdot||_0$ if and only if $\rho$ fulfills $(\rho.4')$.

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\begin{equation}
(4.1) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}
\end{equation}

and show that $||\cdot||_1$ is also a quasi-norm on $L$ and

\begin{equation}
(4.2) \quad ||x||_0 \leq ||x||_1 \leq 2 ||x||_0
\end{equation}

hold, where $||\cdot||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < + \infty$ ($x \in L$) and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_1^\infty 0$ implies $\lim_{n \rightarrow \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim_{n \rightarrow \infty} ||x_n||_1 = 0$ implies $\lim_{n \rightarrow \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim_{n \rightarrow \infty} ||\alpha_n x||_1 = 0$ for all $\alpha_n \downarrow_1^\infty 0$ and that $\lim_{n \rightarrow \infty} ||x_n||_1 = 0$ implies $\lim_{n \rightarrow \infty} ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

\begin{equation}
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\end{equation}

This yields

\begin{equation}
||x + y||_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\end{equation}

in virtue of (A.3). Therefore $||x + y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $||\cdot||_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\begin{equation}
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\end{equation}

10) For the convex modular $m$, we can define two kinds of norms such as

\begin{equation}
||x|| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} [3 \text{ or } 4].
\end{equation}

For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $||\cdot||$ and $||\cdot||$ respectively.
This yields (4.2), since we have \( \| x \|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( \| x \|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1)\(~\) (A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( \| \cdot \|_1 \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in \([6]\) as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
\| \dot{x} \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R_0^\perp \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies \( (\rho,4') \), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all } x \in R_0^\perp.
\]

**§5. A quasi-norm-convergence.** Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \( (\rho,1)\sim(\rho,6) \) except \( (\rho,3) \) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{\nu \to \infty} x_\nu = a \), if there exists a sequence of elements \( \{a_\nu\}_{\nu \geq 1} \) such that \( |x_\nu - a| \leq a_\nu \) (\( \nu \geq 1 \)) and \( a_\nu \downarrow_\nu 0 \). And a sequence of elements \( \{x_\nu\}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( s\lim_{\nu \to \infty} x_\nu = a \), if for any subsequence \( \{y_\nu\}_{\nu \geq 1} \) of \( \{x_\nu\}_{\nu \geq 1} \), there exists a subsequence \( \{z_\nu\}_{\nu \geq 1} \) of \( \{y_\nu\}_{\nu \geq 1} \) with \( \lim_{\nu \to \infty} z_\nu = a \).

A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} \| a_\nu \| = 0 \) for any \( a_\nu \downarrow_\nu 0 \) in the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
\text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \([\{[p_n]\}]_{n\geq 1}\) such that \(\rho([p_n]x)=+\infty\) and \([p_n]\downarrow_{n=1}^{\infty}0\). Hence by (3.1) it follows that \(\| [p_n]x \|_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(\| \cdot \|_0\).

**Sufficiency.** Let \(a_n \downarrow_{n=1}^{\infty}0\) and put \([p_n^\epsilon]=[[a_n-\epsilon a_1]^+]\) for any \(\epsilon>0\) and \(n \geq 1\). It is easily seen that \([p_n^\epsilon]\downarrow_{n=1}^{\infty}0\) for any \(\epsilon>0\) and \(a_n=[a_1]a_n=[p_n^\epsilon]a_n+(1-[p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1+\epsilon a_1\).

This implies \(\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon]a_1)+\rho^*(\xi \epsilon (1-[p_n^\epsilon])a_1)\) for all \(n \geq 1\) and \(\xi \geq 0\). In virtue of (5.1) and \([p_n^\epsilon]\downarrow_{n=1}^{\infty}0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*(\xi [p_n^\epsilon]a_1)<+\infty\), and hence \(\inf_{n \geq 1}\rho^*(\xi [p_n^\epsilon]a_1)\) = 0 by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain \(\inf_{n \geq 1}\rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1)\).

Since \(\epsilon\) is arbitrary, \(\lim_{n \rightarrow \infty}\rho^*(\xi a_n)=0\) follows. Hence we infer that \(\inf_{n \geq 1}\| a_n \|_0=0\) and \(\| \cdot \|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \(\| \cdot \|_0\) is continuous, if

**(5.2)** \(\rho^*(a_n) \rightarrow 0\) implies \(\rho^*(\alpha a_n) \rightarrow 0\) for every \(\alpha \geq 0\).

From the definition, it is clear that \(s-\lim_{v \rightarrow \infty} x_v = 0\) implies \(\lim_{v \rightarrow \infty} \| x_v \|_0 = 0\), if \(\| \cdot \|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{v \rightarrow \infty} \| x_v \|_0 = 0\) (or \(\lim_{v \rightarrow \infty} \| x_v \|_1 = 0\)) implies \(s-\lim_{v \rightarrow \infty} x_v = 0\), if \(\| \cdot \|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho.3\))).

If we replace \(\lim_{v \rightarrow \infty} \| x_v \| = 0\) by \(\lim_{v \rightarrow \infty} \rho(x_v)=0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

**(5.3)** \(\rho^*(x)=0\) implies \(x=0\).

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(\| \cdot \|_0\) is complete, \(\rho(a_n) \rightarrow 0\) implies \(s-\lim a_n=0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^{11}\) i.e. \(\rho^*(x)=\sup_{i \in A} \rho^*(x_i)\) for any \(0 \leq x \uparrow_{i \in A} x\). If

\(^{11}\) If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x)=\inf_{y \uparrow_{j \in A} x} \{\sup \rho^*(y_i)\}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \rightarrow 0\) is equivalent to \(\rho_*(x) \rightarrow 0\).
$\rho(a_{\nu}) \leq \frac{1}{2^\nu}$ \ (\nu \geq 1),
we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^\infty |a_{\nu}| \in R$ in virtue of $(\rho.3)$.

Now, since

$$\rho\left(\bigcup_{\nu \geq \nu^*}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu = \nu^*}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu^*-1}}$$
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu^*}^{\infty} |a_{\nu}|\right)\right) = 0$ and hence (5.3) implies

$$\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu^*}^{\infty} |a_{\nu}|\right) = 0.$$
Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \rightarrow 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^\nu}$ \ (\nu = 1, 2, \ldots). Therefore we have $s\text{-}\lim b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analogous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete and continuous, then (5.2) holds.

**References**


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