§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

(\rho.1) \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all } x \in R;

(\rho.2) \quad \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any } x, y \in R \text{ with } x \perp y^{1)};

(\rho.3) \quad \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \quad \text{for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda}^{2)}; \text{ there exists } x_{0} \in R \text{ such that } x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \text{ and } \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda});

(\rho.4) \quad \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \quad \text{for all } x \in R.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular$^{3)} m$ on $R$ for which every universally continuous linear functional$^{4)}$ is continuous with respect to the norm defined by the modular$^{5)} m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) \quad \rho(x) \geq 0 \quad \text{and } \rho(x) = 0 \quad \text{if and only if } x = 0;

(A.2) \quad \rho(-x) = \rho(x);

(A.3) \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1;

(A.4) \quad \alpha_{n} \to 0 \implies \rho(\alpha_{n} x) \to 0 \quad \text{for every } x \in R;

(A.5) \quad \text{for any } x \in L \text{ there exists } \alpha > 0 \text{ such that } \rho(\alpha x) < +\infty.

They showed that $L$ is a quasi-normed space with a quasi-norm $\|\cdot\|_{0}$ defined by the formula;

---

1) \( x \perp y \) means \( |x| - |y| = 0 \).

2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma \).

3) For the definition of a modular, see [3].

4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow \lambda \in \Lambda$.

5) This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$.
For

results

semi-

pro-

investi-

For

the

of

argument

as

by

similar

from

is

the

continuous

general,

supple-

differ

the

space

universally

shortly

by

quasi-norm

necessary

in

manifold

those

quasi-norm

quasi-modular

that

manifold

normal

and

amplitude

normal

and

a

formula

we

can

formula

we

can

formula

The

of

a

paper,

manifold

and

Nakano

amplitude

the

F-space

non-atomic

Proceed

Since

a

arbitrary

quasi-norm

and

a

semi-continuous,

a

order

a

formula

shall

show,

however,

that

we

can

always

a

quasi-

which

satisfies

(A.2)\sim(A.5) on an arbitrary quasi-modular space \( R \) in §2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0=\{x : x\in R, \rho^*(\xi x)=0 \text{ for all } \xi\geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in §3 that a quasi-norm \( \|\cdot\|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \|\cdot\|_0 \) (i.e. \( \|\cdot\|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ \sup_{x\in R} \rho(ax) < +\infty \]  

(\( \rho,4' \))

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm \( \|\cdot\|_1 \) on \( R_0^+ \) which is equivalent to \( \|\cdot\|_0 \) such that \( \|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \) holds for every \( x\in R_0^+ \) (Formulas (4.1) and (4.3)). \( \|\cdot\|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4 ; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between \( \|\cdot\|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p\in R \), \( [p] \) is a projector: \( [p]x=\bigcup\limits_{n=1}^{\infty} (n[p] \cap x) \) for all \( x\geq 0 \) and \( 1-[p] \) is a projection operator onto the normal manifold \( N=\{p\}^1 \), that is, \( x=[p]x+(1-[p])x \).

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular \( \rho \), we have

(2.1) \( \rho(0) = 0 \);
(2.2) \( \rho([p]x) \leq \rho(x) \) for all \( p, x \in R \);
(2.3) \( \rho([p]x) = \sup_{l \in A} \rho([p_l]x) \) for any \( [p_l] \uparrow_{l \in A} [p] \).

In the argument below, we have to use the additional property of \( \rho \):

(\( \rho.5 \)) \( \rho(x) \leq \rho(y) \) if \( |x| \leq |y| \), \( x, y \in R \),

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies (\( \rho.5 \)).

Theorem 2.1. Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which (\( \rho.5 \)) is valid.

Proof. We put for every \( x \in R \),

(2.4) \[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y) . \]

It is clear that \( \rho' \) satisfies the conditions (\( \rho.1 \)), (\( \rho.2 \)) and (\( \rho.5 \)).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < + \infty \), then

\[ \sum_{i \in A} \rho(x_i) < + \infty , \]

because

\[ \rho(x) \leq \rho'(x) \quad \text{for all } x \in R . \]

We have

\[ x_0 = \sum_{i \in A} x_i \in R \]

and

\[ \rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of (\( \rho.3 \)).} \]

For such \( x_0 \),

\[ \rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y) \]

\[ = \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i) \]

holds, i.e. \( \rho' \) fulfils (\( \rho.3 \)).

If \( \rho' \) does not fulfil (\( \rho.4 \)), we have for some \( x_0 \in R \),

\[ \rho'(\frac{1}{n} x_0) = + \infty \quad \text{for all } n \geq 1 . \]

By (\( \rho.2 \)) and (\( \rho.4 \)), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\varepsilon} \xi \varepsilon \nu \), where \( \varepsilon \nu \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \varepsilon \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]

where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \ldots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]

where

\[ \rho'(\frac{1}{\nu} x_2) = +\infty \quad (\nu = 1, 2, \ldots) \]

and

\[ \rho'(\frac{1}{2} x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_{\nu}\}_{\nu=1, 2}, \ldots \) such that

\[ \rho(\frac{1}{\nu} y_{\nu}) \geq \nu \]

and

\[ 0 \leq |y_{\nu}| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_{\nu}\}_{\nu=1, 2}, \ldots \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_{\nu} \in R \]

and

\[ \rho(\frac{1}{\nu} y_0) \geq \rho(\frac{1}{\nu} y_{\nu}) \geq \nu, \]

which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in \( \S 1 \):

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [\langle x, -y \rangle^+]\), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)
\]
\[
\leq \rho(\lfloor \alpha \rfloor |x| + \alpha(1 - \lfloor \alpha \rfloor) |y| + \beta |x| + (1 - \lfloor \alpha \rfloor) \beta |y|)
\]
\[
= \rho(\lfloor \alpha \rfloor |x| + (1 - \lfloor \alpha \rfloor) |y|)
\]
\[
= \rho(\lfloor \alpha \rfloor x) + \rho((1 - \lfloor \alpha \rfloor) y)
\]
\[
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\[(\rho.4'') \quad \text{for any } x \in \mathbb{R}, \text{ there exists } \alpha \geq 0 \text{ such that } \rho(\alpha x) < +\infty.\]

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1), (\rho.2), (\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[\rho_0'(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty. \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \text{ mes} \left\{ t : x(t) = \frac{1}{i} \right\}, \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[ (\rho.6) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in \mathbb{R}. \]

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( \mathbb{R} \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ (2.5) \quad d(x) = \lim_{\xi \to 0} \rho'(\xi x). \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in \mathbb{R} \) and
On F-Norms of Quasi-Modular Spaces

\[ d(x + y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

(2.6) \[ \rho^*(x) = \rho'(x) - d(x) \quad (x \in R). \]

we can see easily that (\(\rho.1\), (\(\rho.2\)), (\(\rho.4\)) and (\(\rho.6\)) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in R\) and \(\alpha > 0\).

We need to prove that (\(\rho.5\)) is true for \(\rho^*\). First we have to note

(2.7) \[ \inf_{\lambda \in A} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in A} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,

\[ \rho'(\frac{1}{\nu} [p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in A}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n} [p_{\lambda_n}] - [p_{\lambda_{n+1}}]x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of (\(\rho.2\)) and (2.3). This implies

\[ \rho'(\frac{1}{n} x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m} [p_{\lambda_m}] - [p_{\lambda_{m+1}}]x_0) = +\infty, \]

which is inconsistent with (\(\rho.4\)). Secondly we shall prove

(2.8) \[ d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = \lceil |x| - n |y| \rceil\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n=1} 0\) and \(\inf_{n=1,2,...} d([p_n]x) = 0\) by (2.7). Since \((1 - [p_n])n |y| \geq (1 - [p_n])x|\)

and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in R\), we obtain
As $n$ is arbitrary, this implies
\[ d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y), \]
and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then
\[ \rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y])x \]
\[ = \rho'([y]x) - d([y]x) + \rho^*([x] - [y])x \]
\[ \geq \rho'(y) - d(y) + \rho^*([x] - [y])x \]
\[ \geq \rho^*(y). \]

Thus $\rho^*$ satisfies (\rho.5).

**Q.E.D.**

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (\rho.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

(\rho.4') \quad \sup_{x \in \mathcal{R}} \{\lim_{\xi \to 0} \rho^\prime(\xi x)\} = K < +\infty.

**Proof.** Let $\rho$ satisfy (\rho.4). We need to prove
\[ \sup_{x \in \mathcal{R}} d(x) = \sup_{x \in \mathcal{R}} \{\lim_{\xi \to 0} \rho^\prime(\xi x)\} = K' < +\infty, \]
where
\[ \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y). \]

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho'\left(\frac{1}{\nu}x\right)$ for $\nu \geq 1$ and $x \in \mathcal{R}$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with
\[ \rho(x) \leq \varepsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma. \]

In $N_0$, we have obviously
\[ \sup_{x \in N_0} \{\lim_{\xi \to 0} \rho^\prime(\xi x)\} = \gamma_0 < +\infty. \]

If $\varepsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by (\rho.4'), and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\epsilon}{2} < \rho(x_i) \leq \epsilon \) (i = 1, 2, \ldots, n), \( y_j \) is an atomic element with \( \rho(y_j) > \epsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'\left( \frac{1}{\nu_0} \alpha_0 x_0 \right) \leq \sum_{i=1}^{n} \rho'\left( \frac{1}{\nu_0} x_i \right) + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \\
\leq n\gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho'\frac{z}{\nu_0} \\
\leq \frac{4K}{\epsilon} \gamma + \frac{2K}{\epsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma 
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_0) \leq \rho'\left( \frac{\alpha_0}{\nu_0} x_0 \right) \leq \left( \frac{4K + \epsilon}{\epsilon} \right) \gamma + \left( \frac{4K^2}{\epsilon} \right) 
\]

in case of \( \epsilon \leq 2K \). If \( 2K \leq \epsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \to 0} \rho'(\xi x) \leq \gamma. 
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \epsilon}{\epsilon} + \frac{4K^2}{\epsilon} + \gamma_0. 
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_i) = d(\sum_{i=1}^{k} x_i) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_i) \leq \gamma', 
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho'(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty, 
\]

which implies \( x_0 = \sum_{i \in A} \rho^*(x_i) = x \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore, \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu. 
\]
for all $\mu \geq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
\[
\lim_{t \to 0} \rho^*(\xi x) = 0,
\]
there exists $\{\alpha_\nu\}_{\nu \geq 1}$ with $0 < \alpha_\nu (\nu \geq 1)$ and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < +\infty$.
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geq 0$,
\[
\rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geq \sum_{\nu=1}^{\infty} d(x_\nu) = +\infty,
\]
which is inconsistent with (\rho.4). Therefore we have
\[
\sup_{x \in R} (\lim_{t \to 0} \rho(\xi x)) \leq \sup_{x \in R} d(x) < +\infty.
\]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[
R_0 = \{x : x \in R, \rho^*(nx) = 0 \textrm{ for all } n \geq 1\},
\]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold\footnote{A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in A} x_\lambda \in S$ (\lambda \in A) implies $\bigcup_{\lambda \in A} x_\lambda \in S$.
0 \leq x_\lambda \in S(\lambda \in A) implies $\bigcup_{\lambda \in A} x_\lambda \in S$.}$ of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in A} x_\lambda$ with $R_0 \ni x_\lambda \geq 0$ for all $\lambda \in A$. Putting
\[
[p_{n,\lambda}] = [(2nx_\lambda - nx)^+]\],
we have
\[
[p_{n,\lambda}] \uparrow_{\lambda \in A} [x] \quad \text{and} \quad 2n[p_{n,\lambda}]x_\lambda \geq [p_{n,\lambda}]nx,
\]
which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in A} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
\[
R = R_0 \oplus R_0^\perp.\footnote{This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.}$

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_\lambda\}_{\lambda \in A}, x_\lambda \in [p]R_0$, there exists $x_0 = \sum_{\lambda \in A} x_\lambda \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(A)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S$, $|b| \leq |a|$, $b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{\lambda \in A} x_\lambda \in S(\lambda \in A)$ implies $\bigcup_{\lambda \in A} x_\lambda \in S$.
8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.
9) $S(A)$ is the set of all real functions defined on $A$.\]
Theorem 3.1. Let \( R \) be a quasi-modular space. Then \( R_0^\perp \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) which is semi-continuous, i.e.

\[
\sup_{i \in A} \| x_i \|_0 = \| x \|_0
\]

for any \( 0 \leq x_i \uparrow_{i \in A} x \).

Proof. In virtue of Theorems 2.1 and 2.2, \( \rho^* \) satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\). Now we put

\[
(3.1) \quad \| x \|_0 = \inf \left\{ \xi ; \rho^* \left( \frac{1}{\xi} x \right) \leq \xi \right\}.
\]

Then,

i) \( 0 \leq \| x \|_0 = \| -x \|_0 < \infty \) and \( \| x \|_0 = 0 \) is equivalent to \( x = 0 \); follows from \((\rho.1), (\rho.6), (2.1)\) and the definition of \( R_0^\perp \).

ii) \( \| x + y \|_0 \leq \| x \|_0 + \| y \|_0 \) for any \( x, y \in R \); follows also from \((A.3)\) which is deduced from \((\rho.4)\). At last we shall prove that \( \| \cdot \|_0 \) is semi-continuous. From ii) and iii), it follows that \( \lim_{\alpha \to \alpha_0} \| \alpha x \|_0 = \| \alpha_0 x \|_0 \) for all \( x \in R_0^\perp \) and \( \alpha_0 \geq 0 \). If \( x \in R_0^\perp \) and \( [p_j] \uparrow_{i \in A} [p] \), for any positive number \( \xi \) with \( \| [p] x \|_0 > \xi \) we have \( \rho^* \left( \frac{1}{\xi} [p] x \right) > \xi \), which implies \( \sup_{i \in A} \rho^* \left( \frac{1}{\xi} [p_i] x \right) > \xi \) and hence \( \sup_{i \in A} \| [p_i] x \|_0 \geq \| [p] x \|_0 \geq \xi \). Thus we obtain

\[
\sup_{i \in A} \| [p_i] x \|_0 = \| [p] x \|_0 , \quad \text{if} \quad [p_i] \uparrow_{i \in A} [p].
\]

Let \( 0 \leq x_i \uparrow_{i \in A} x \). Putting

\[
[p_{n,i}] = \left[ (x_i - \left( 1 - \frac{1}{n} \right) x )^* \right]
\]

we have

\[
[p_{n,i}] \uparrow_{i \in A} [x] \quad \text{and} \quad [p_{n,i}] x_i \geq [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).
\]

As is shown above, since

\[
\sup_{i \in A} \| [p_{n,i}] x_i \|_0 \geq \sup_{i \in A} \| [p_{n,i}] \left( 1 - \frac{1}{n} \right) x \|_0 = \| \left( 1 - \frac{1}{n} \right) x \|_0 ,
\]

we have

\[
\sup_{i \in A} \| x_i \|_0 \geq \left( 1 - \frac{1}{n} \right) x \|_0
\]

and also \( \sup_{i \in A} \| x_i \|_0 \geq \| x \|_0 \). As the converse inequality is obvious by iv), \( \| \cdot \|_0 \) is semi-continuous. Q.E.D.

Remark 2. By the definition of \((3.1)\), we can see easily that \( \lim \| x_n \|_0 = 0 \) if and only if \( \lim \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \).
In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}$, $x_{\nu} \geq 0$ and $a \geq 0$ ($n, \nu = 1, 2, \cdots$) be the elements of $R_0^+$ such that

\[(p_{n,\nu})_{\nu=1}^{\infty} \subset (p_n)_{n=1}^{\infty} \text{ with } \bigcap_{n=1}^{\infty} [p_n]a = [p_0]a \neq 0; \]

\[(p_{n,\nu})x_{\nu} \geq n[p_{n,\nu}]a \text{ for all } n, \nu \geq 1. \]

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^+$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m]_{m=1}^{\infty}$ ($m \geq 1$) and sequences of natural numbers $\nu_m, n_m$ such that

\[||[q_m]a||_0 > \frac{\delta}{2} \text{ and } [q_m]x_{\nu_m} \geq n_m[q_m]a \quad (m=1, 2, \cdots) \]

and

\[n_m[q_m]a \geq [q_m]x_{\nu_{m-1}}, \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),\]

where $\delta = ||[p_0]a||_0$.

In fact, we put $n_1=1$. Since $[p_{1,\nu}][p_0]_{\nu=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[||[p_{1,\nu_1}][p_0]a||_0 > \frac{\delta}{2}.\]

We put $[q_1] = [p_{1,\nu_1}][p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $((na-x_{\nu})^+)_{n=1}^{\infty} [a]$ and $||[q_k]a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[||([n_{k+1}a-x_{\nu_k}]^+)[q_k]a||_0 > \frac{\delta}{2}.\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[||[p_{n_{k+1}}, \nu_{k+1}]/([n_{k+1}a-x_{\nu_k}]^+)[q_k]a||_0 > \frac{\delta}{2}.\]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[[q_{k+1}] = [p_{n_{k+1}}, \nu_{k+1}]/([n_{k+1}a-x_{\nu_k}]^+)[q_k],\]

because

\[[q_{k+1}] \subset [q_k], \quad ||[q_{k+1}]a|| > \frac{\delta}{2}, \quad [q_{k+1}]x_{\nu_{k+1}} \geq n_{k+1}[q_{k+1}]a\]

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
On F-Norms of Quasi-Modular Spaces

\[ ||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_{0} \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_{0} \]

\[ \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_{0} \geqq ||[q_{k+1}]a_{0}||_{0} \geqq \frac{\delta}{2}, \]

since \([q_{k+1}]\leqq [q_{k}]\leqq [(n_{k}a-x_{\nu-1})^{+}]\) implies \([q_{k+1}]n_{k}a\geqq[q_{k+1}]x_{\nu_{k-1}}\) by (3.4).

It follows from the above that \(\{x_{\nu}\}_{\nu \geqq 1}\) is not a Cauchy sequence.

**Theorem 3.2.** Let \(R\) be a quasi-modular space with quasi-modular \(\rho\). Then \(R_{0}^{+}\) is an F-space with \(||\cdot||_{0}\) if and only if \(\rho\) satisfies \((\rho.4')\).

**Proof.** If \(\rho\) satisfies \((\rho.4')\), \(\rho^{*}\) is a quasi-modular which fulfils also \((\rho.5)\) and \((\rho.6)\) in virtue of Theorem 2.3. Since \(x||_{0}(=\inf\{\xi;\rho^{*}(\frac{x}{\xi})\leqq\xi\})\) is a quasi-norm on \(R_{0}^{+}\), we need only to verify completeness of \(||\cdot||_{0}\). At first let \(\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^{+}\) be a Cauchy sequence with \(0 \leqq x_{\nu} \uparrow_{\nu=1,2}, \ldots\). Since \(\rho^{*}\) satisfies \((\rho.3)\), there exists \(0 \leqq x_{0} \in R_{0}^{+}\) such that \(x_{0} = \bigcup_{\nu=1}^{\infty}x_{v}\), as is shown in the proof of Theorem 2.3.

Putting \([p_{n,v}] = [(x_{v} - nx_{0})^{+}]\) and \(\bigcup_{\nu=1}^{\infty}[p_{n,v}] = [p_{n}]\), we obtain

\[ (1-[p_{n,v}]) \geqq (1-[p_{n}]) \quad (n, \nu \geqq 1) \]

and \([p_{n}]\downarrow_{n=1}^{\infty} 0\). Since \(\{x_{\nu}\}_{\nu \geqq 1}\) is a Cauchy sequence, we have in virtue of Lemma 2, \(\bigcap_{n=1}^{\infty}[p_{n}] = 0\), that is, \(\bigcup_{n=1}^{\infty}([x_{0}] - [p_{n}]) = [x_{0}]\). And

\[ n(1-[p_{n,v}]) \geqq (1-[p_{n}])x_{v} \geqq 0. \]

Hence we have

\[ y_{n} = \bigcup_{v=1}^{\infty} (1-[p_{n}])x_{v} \in R_{0}^{+}, \]

because \(R_{0}^{+}\) is universally continuous. As \(\{x_{\nu}\}_{\nu \geqq 1}\) is a Cauchy sequence, we obtain from the triangle inequality of \(||\cdot||_{0}\)

\[ \gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_{0} < +\infty, \]

which implies

\[ ||y_{n}||_{0} = \sup_{\nu \geqq 1} ||(1-[p_{n}])x_{\nu}||_{0} \leqq \gamma \]

for every \(n \geqq 1\) by semi-continuity of \(||\cdot||_{0}\). We put \(z_{1} = y_{1}\) and \(z_{n} = y_{n} - y_{n-1}\) \((n \geqq 2)\). It follows from the definition of \(y_{n}\) that \(\{z_{\nu}\}_{\nu \geqq 1}\) is an orthogonal sequence with \(||\sum_{\nu=1}^{n} z_{\nu}||_{0} = ||y_{n}||_{0} \leqq \gamma\). This implies
$$\sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_{n}}{1+\gamma} \right) \leq \gamma$$

for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from

$$z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-\lfloor p_{n} \rfloor) x_{\nu} = \bigcup_{n=1}^{\infty} [x_{0}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} .$$

By semi-continuity of $|| \cdot ||_{0}$, we have

$$|| z - x_{\nu} ||_{0} \leq \sup_{\mu \geq \nu} || x_{\mu} - x_{\nu} ||_{0}$$

and furthermore $\lim_{\nu \rightarrow \infty} || z - x_{\nu} ||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^\perp$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$|| y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$. This implies

$$|| \sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu} ||_{0} \leq \sum_{\nu=m}^{n} || y_{\nu+1} - y_{\nu} ||_{0} \leq \frac{1}{2^{m-1}}$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \uparrow_{n=1}^{\infty}$. Then by the fact proved just above,

$$z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^\downarrow \quad \text{and} \quad \lim_{n \rightarrow \infty} || z_{0} - z_{n} ||_{0} = 0 .$$

Since $\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$|| y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n} ||_{0} = || \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) ||_{0} \leq || z_{0} - z_{n} ||_{0} \rightarrow 0 .$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{\nu \rightarrow \infty} || y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu} ||_{0} = 0 .$$

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^\downarrow$, that is, $R_{0}^\downarrow$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^\perp$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^\downarrow$, we have $\sum_{\nu=1}^{\infty} \alpha \cdot x_{\nu} \in R_{0}^\downarrow$ for some real numbers $\alpha > 0$ (for all $\nu \geq 1$). Hence we can see that $\sup_{x \in K} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$. Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0}^\downarrow = 0$, we can conclude directly from Theorems 3.1 and 3.2.
**Corollary.** Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfils ($\rho.A'$).

**§4. Another Quasi-norm.** Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

$$||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}$$

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

$$||x||_0 \leq ||x||_1 \leq 2||x||_0$$

for all $x \in L$ hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_1 = ||-x||_1 < +\infty$ ($x \in L$) and that $||x||_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim ||x_n||_1 = 0$ implies $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim ||\alpha x_n||_1 = 0$ for all $\alpha > 0$. If $||x||_1 < \alpha$ and $||y||_1 < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ 

This yields

$$||x+y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta} (x+y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y)\right)$$

$$\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $||x+y||_1 \leq ||x||_1 + ||y||_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$ 

10) For the convex modular $m$, we can define two kinds of norms such as

$$||x|| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \geq 1} \frac{1}{\xi|\xi|}$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $||\cdot||$ and $||\cdot||$ respectively.
This yields (4.2), since we have $\| x \|_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $\| x \|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $\| \cdot \|_1$ on $L$ which is equivalent to $\| \cdot \|_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

$$
\| \dot{x} \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\}
$$

(4.3)

is a semi-continuous quasi-norm on $R^+_0$ and $\| \cdot \|_1$ is complete if and only if $\rho$ satisfies $(\rho.4')$, where $\rho^*$ and $R_0$ are the same as in §2 and §3. And further we have

$$
\| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0
$$

(4.4) for all $x \in R^+_0$.

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies $(\rho.1)$~(\rho.6) except $\rho.3$ and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \leq x \in R$ (i.e. $R_0 = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $\limsup_{n \to \infty} x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a| \leq a_n$ ($n \geq 1$) and $a_n \downarrow_{n=1}^{\infty} 0$. And a sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $\lim_{n \to \infty} x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $\lim_{n \to \infty} z_n = a$.

A quasi-norm $\| \cdot \|$ on $R$ is termed to be continuous, if $\inf_{n \geq 1} \| a_n \| = 0$ for any $a_n \downarrow_{n=0}^{\infty} 0$. In the sequel, we write by $\| \cdot \|_0$ (or $\| \cdot \|_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $\| \cdot \|_0$ (or $\| \cdot \|_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

$$
\| x \|_R \text{ is finite dimensional and } \rho(y) < +\infty.
$$

(5.1) for any $x \in R$ there exists an orthogonal decomposition $x = y + z$ such that $[z] R$ is finite dimensional and $\rho(y) < +\infty$.

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( \lim_{n \to \infty} \rho([p_n]) = 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_\nu \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1] a_n = [p_n^\epsilon] a_n + (1 - [p_n^\epsilon]) a_n \leq [p_n^\epsilon] a_1 + \epsilon a_1 \).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^\epsilon] a_1) + \rho^*(\xi (1 - [p_n^\epsilon]) a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*(\xi [p_n^\epsilon] a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*(\xi [p_n^\epsilon] a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain
\[
\inf_{n \geq 1} \rho^*(\xi a_n) \leq \rho^*(\xi \epsilon a_1) \leq \rho^*(\xi a_1)
\]
for all \( n \geq 1 \) and \( \xi \geq 0 \).

Since \( \epsilon \) is arbitrary, \( \lim_{n \to \infty} \rho^*(\xi a_n) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \| a_n \|_0 = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S \). Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** \( \| \cdot \|_0 \) is continuous, if
\[
(5.2) \quad \rho^*(a_\nu) \to 0 \quad \text{implies} \quad \rho^*(\alpha a_\nu) \to 0 \quad \text{for every} \quad \alpha \geq 0 .
\]

From the definition, it is clear that \( s-\lim x_\nu = 0 \) implies \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \) (or \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \)) implies \( s-\lim x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies \( \rho.3 \)).

If we replace \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \) by \( \lim \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:
\[
(5.3) \quad \rho^*(x) = 0 \quad \text{implies} \quad x = 0 .
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \to 0 \) implies \( s-\lim a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11}\) i.e. \( \rho^*(x) = \sup_{x_\nu \uparrow_{\nu \in \Lambda}} \rho^*(x_\nu) \) for any \( 0 \leq x \uparrow_{\nu \in \Lambda} \).

\(^{11}\) If \( \rho^* \) is not semi-continuous, putting \( \rho_*(x) = \inf_{y_\nu \downarrow_{\nu \in \Lambda}} \{ \sup_{x_\nu \downarrow_{\nu \in \Lambda}} \rho^*(y_\nu) \} \), we obtain a quasi-modular \( \rho_* \) which is semi-continuous and \( \rho^*(x_\nu) \to 0 \) is equivalent to \( \rho_*(x_\nu) \to 0 \).
$\rho(a_{\nu}) \leq \frac{1}{2^\nu}$ \hspace{1cm} (\nu \geq 1),

we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R$ in virtue of $(\rho.3)$.

Now, since

$$\rho\left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}|\right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}$$

holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}|\right)\right) = 0$ and hence (5.3) implies

$$\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq \nu}^{\infty} |a_{\nu}|\right) = 0.$$

Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \to 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^{\nu}}$ (\nu = 1, 2, \cdots). Therefore we have $s\text{-}\lim_{\nu \to \infty} b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analougous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^*$ satisfies (5.3) and $\|\cdot\|_0$ is complete and continuous, then (5.2) holds.

**References**


Mathematical Institute,
Hokkaido University

(Received September 30, 1960)