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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

$(\rho.1)$ $0 \leq \rho(x) = \rho(-x) \leq +\infty$ for all $x \in R$;
$(\rho.2)$ $\rho(x+y) = \rho(x) + \rho(y)$ for any $x, y \in R$ with $x \perp y$;
$(\rho.3)$ If $\sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty$ for a mutually orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$, there exists $x_{0} \in R$ such that $x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda}$ and $\rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda})$;
$(\rho.4)$ $\varlimsup_{t \to 0} \rho(\xi x) < +\infty$ for all $x \in R$.

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular $m$ on $R$ for which every universally continuous linear functional is continuous with respect to the norm defined by the modular $m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

(A.1) $\rho(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = 0$;
(A.2) $\rho(-x) = \rho(x)$;
(A.3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;
(A.4) $\alpha_{n} \to 0$ implies $\rho(\alpha_{n} x) \to 0$ for every $x \in R$;
(A.5) for any $x \in L$ there exists $\alpha > 0$ such that $\rho(\alpha x) < +\infty$.

They showed that $L$ is a quasi-normed space with a quasi-norm $|| \cdot ||_{0}$ defined by the formula;

---

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow 0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, there exists a universally continuous linear functional $f$ such that $f(x) = 0$.

This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

\[ ||x||_0 = \inf \left\{ \xi : \rho\left(\frac{1}{\xi} x\right) \leq \xi \right\} \]

and \( ||x_n||_0 \to 0 \) is equivalent to \( \rho(ax_n) \to 0 \) for all \( a \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \))\( \sim (\rho.4 \) with those of \( \rho [6] \), we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim (A.5 \) on an arbitrary quasi-modular space \( R \) in \( \S 2 \) (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{x : x \in R, \rho^*(\xi x) = 0 \text{ for all } \xi \geq 0\} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \( \S 3 \) that a quasi-norm \( || \cdot ||_0 \) on \( R^+_0 \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R^+_0 \) is an \( F \)-space with \( || \cdot ||_0 \) (i.e. \( || \cdot ||_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\( (\rho.4') \)

\[ \sup_{x \in R} \{ \lim_{a \to 0} \rho(ax) \} < +\infty \]

(\( \text{Theorem 3.2} \)).

In \( \S 4 \), we shall show that we can define another quasi-norm \( || \cdot ||_1 \) on \( R^+_0 \) which is equivalent to \( || \cdot ||_0 \) such that \( ||x||_0 \leq ||x||_1 \leq 2||x||_0 \) holds for every \( x \in R^+_0 \) (Formulas (4.1) and (4.3)). \( || \cdot ||_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano \([4; \S 83]\). At last in \( \S 5 \) we shall add shortly the supplementary results concerning the relations between \( || \cdot ||_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \( \S 5 \) are already known in those cases \([8]\).

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \), \([p]\) is a projector: \([p]x = \bigcup_{n=1}^{\infty} (n|p| \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N = \{p\}^1 \), that is, \( x = [p]x + (1 - [p])x \).

\(6\) This quasi-norm was first considered by S. Mazur and W. Orlicz \([5]\) and discussed by several authors \([6 \text{ or } 7]\).
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular \( \rho \), we have

\[
(2.1) \quad \rho(0) = 0; \\
(2.2) \quad \rho([p]x) \leq \rho(x) \quad \text{for all } p, x \in R; \\
(2.3) \quad \rho([p]x) = \sup_{i \in A} \rho([p_i]x) \quad \text{for any } \ [p_i] \uparrow_{i \in A} [p].
\]

In the argument below, we have to use the additional property of \( \rho \):

\[
(\rho.5) \quad \rho(x) \leq \rho(y) \quad \text{if } |x| \leq |y|, \ x, y \in R,
\]

which is not valid for an arbitrary \( \rho \) in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular \( \rho \) satisfies \((\rho.5)\).

**Theorem 2.1.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then there exists a quasi-modular \( \rho' \) for which \((\rho.5)\) is valid.

**Proof.** We put for every \( x \in R \),

\[
(2.4) \quad \rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

It is clear that \( \rho' \) satisfies the conditions \((\rho.1), (\rho.2) \) and \((\rho.5)\).

Let \( \{x_i\}_{i \in A} \) be an orthogonal system such that \( \sum_{i \in A} \rho'(x_i) < +\infty \), then

\[
\sum_{i \in A} \rho(x_i) < +\infty,
\]

because

\[
\rho(x) \leq \rho'(x) \quad \text{for all } x \in R.
\]

We have

\[
x_0 = \sum_{i \in A} x_i \in R
\]

and

\[
\rho(x_0) = \sum_{i \in A} \rho(x_i) \quad \text{in virtue of } (\rho.3).
\]

For such \( x_0 \),

\[
\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{i \in A} \rho([x_i]y)
\]

\[
= \sum_{i \in A} \sup_{0 \leq |y| \leq |x_0|} \rho([x_i]y) = \sum_{i \in A} \rho'(x_i)
\]

holds, i.e. \( \rho' \) fulfils \((\rho.3)\).

If \( \rho' \) does not fulfil \((\rho.4)\), we have for some \( x_0 \in R \),

\[
\rho'(\frac{1}{n} x_0) = +\infty \quad \text{for all } n \geq 1.
\]

By \((\rho.2)\) and \((\rho.4)\), \( x_0 \) can not be written as \( x_0 = \sum_{\nu=1}^{\xi} \xi \nu e_{\nu} \), where \( e_{\nu} \) is an atomic element for each \( \nu \) with \( 1 \leq \nu \leq \xi \), namely, we can decompose \( x_0 \) into
an infinite number of orthogonal elements. First we decompose into

\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]

where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) \((\nu = 1, 2, \cdots)\) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into

\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]

where

\[ \rho'(\frac{1}{\nu} x_2) = +\infty \] \((\nu = 1, 2, \cdots)\)

and

\[ \rho'(\frac{1}{2} x'_2) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho\left(\frac{1}{2} y_2\right) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_{\nu}\}_{\nu=1,2}, \ldots \) such that

\[ \rho\left(\frac{1}{\nu} y_{\nu}\right) \geq \nu \]

and

\[ 0 \leq |y_{\nu}| \leq |x| \]

for all \( \nu \geq 1 \).

Since \( \{y_{\nu}\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)

\[ y_0 = \sum_{\nu=1}^{\infty} y_{\nu} \in R \]

and

\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_{\nu}\right) \geq \nu, \]

which contradicts \((\rho.4)\). Therefore \( \rho' \) has to satisfy \((\rho.4)\). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies \((\rho.5)\), \( \rho \) does also \((A.3)\) in §1:

\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]

for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p] = [(|x| - |y|)^+]\), we obtain
\[\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|)\]
\[\leq \rho(\alpha \lfloor p \rfloor |x| + \alpha (1 - \lfloor p \rfloor) |y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) \beta |y|)\]
\[= \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor) |y|)\]
\[= \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor) y)\]
\[\leq \rho(x) + \rho(y).\]

**Remark 1.** As is shown above, the existence of \(\rho'\) as a quasi-modular depends essentially on the condition (\(\rho.4\)). Thus, in the above theorems, we cannot replace (\(\rho.4\)) by the weaker condition:

(\(\rho.4''\)) for any \(x \in \mathbb{R}\), there exists \(\alpha \geq 0\) such that \(\rho(\alpha x) < +\infty\).

In fact, the next example shows that there exists a functional \(\rho_0\) on a universally continuous semi-ordered linear space satisfying (\(\rho.1\)), (\(\rho.2\)), (\(\rho.3\)) and (\(\rho.4''\)), but does not (\(\rho.4\)). For this \(\rho_0\), we obtain

\[\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty\]

for all \(x \neq 0\).

**Example.** \(L_1[0,1]\) is the set of measurable functions \(x(t)\) which are defined in \([0,1]\) with

\[\int_{0}^{1} |x(t)| dt < +\infty.\]

Putting \(\rho_0(x) = \rho_0(x(t)) = \int_{0}^{1} |x(t)| dt + \sum_{i=1}^{\infty} i \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\}\),

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\(\rho.6\)) \[\lim_{\xi \to 0} \rho(\xi x) = 0\] for all \(x \in \mathbb{R}\).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \(\rho\) be a quasi-modular on \(\mathbb{R}\). We can find a functional \(\rho^*\) which satisfies (\(\rho.1\))~(\(\rho.6\)) except (\(\rho.3\)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \(\rho'\) which satisfies (\(\rho.5\)). Now we put

(2.5) \[d(x) = \lim_{\xi \to 0} \rho'(\xi x).\]

It is clear that \(0 \leq d(x) = d(|x|) < +\infty\) for all \(x \in \mathbb{R}\) and
Hence, putting
\[(2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in R),\]
we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since
\[d(x) \leq \rho'(x)\]
and
\[d(\alpha x) = d(x)\]
for all \(x \in R\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note
\[(2.7) \quad \inf_{\lambda \in A} d([p_{\lambda}]x) = 0\]
for any \([p_{\lambda}] \downarrow_{\lambda \in A} 0\). In fact, if we suppose the contrary, we have
\[\inf_{\lambda \in A} d([p_{\lambda}]x_0) \geq \alpha > 0\]
for some \([p_{\lambda}] \downarrow_{\lambda \in A} 0\) and \(x_0 \in R\).

Hence,
\[\rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha\]
for all \(\nu \geq 1\) and \(\lambda \in A\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in A}\)
such that
\[[p_{\lambda_n}] \geq [p_{\lambda_{n+1}}]\]
and
\[\rho'(\frac{1}{n}([p_{\lambda_n}] - [p_{\lambda_{n+1}}])x_0) \geq \frac{\alpha}{2}\]
for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies
\[\rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}([p_{\lambda_m}] - [p_{\lambda_{m+1}}])x_0) = +\infty,\]
which is inconsistent with \((\rho.4)\). Secondly we shall prove
\[(2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y].\]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in R\) with \([x] = [y]\) and \(n \geq 1\). Then,
\([p_n] \downarrow_{n=1}^\infty 0\) and \(\inf_{n=1,2,...} d([p_n]x) = 0\) by \((2.7)\). Since \((1 - [p_n])n |y| \geq (1 - [p_n])|x|\)
and
\[d(\alpha x) = d(x)\]
for \(\alpha > 0\) and \(x \in R\), we obtain
\[
d(x) = d([p_n]x) + d((1-[p_n])x) \\
\leq d([p_n]x) + d(n(1-[p_n])y) \\
\leq d([p_n]x) + d(y).
\]

As \( n \) is arbitrary, this implies
\[
d(x) \leq \inf_{n=1,2,\ldots} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*([y]x) + \rho^*([x] - [y]x) \\
= \rho'(y)(1) - d([y]x) + \rho^*([x] - [y]x) \\
\geq \rho'(y) - d(y) + \rho^*([x] - [y]x) \\
\geq \rho^*(y).
\]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

**Theorem 2.3.** \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

(\( \rho.4' \)) \[
\sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K < +\infty.
\]

**Proof.** Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove

(2.9) \[
\sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty,
\]

where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'(\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, x, x_{0}, \) a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^\perp \) with
\[
\rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq x.
\]

In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.
\]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^\perp \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq x \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]

where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) for every \( i = 1, 2, \ldots, n \), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]

\[
\leq n \gamma + \sum_{j=1}^{m} \rho'(y_j) + \rho' \frac{z}{\nu_0}
\]

\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho(\alpha x) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x_0) \leq \rho'\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]

in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^+ \)

\[
\lim_{\xi \rightarrow 0} \rho'(\xi x) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \{ \lim_{\xi \rightarrow 0} \rho'(\xi x) \} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_\lambda\}_{\lambda \in A} \) be an orthogonal system with \( \sum_{\lambda \in A} \rho^*(x_\lambda) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{\lambda_i}) = d(\sum_{i=1}^{k} x_{\lambda_i}) = \lim_{\xi \rightarrow 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_i}) \leq \gamma',
\]

which implies \( \sum_{\lambda \in A} d(x_\lambda) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in A} \rho'(x_\lambda) = \sum_{\lambda \in A} \rho^*(x_\lambda) + \sum_{\lambda \in A} d(x_\lambda) < +\infty,
\]

which implies \( x_0 = \sum_{\lambda \in A} x_\lambda \in R \) and \( \sum_{\lambda \in A} \rho^*(x_\lambda) = \rho^*(x_0) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^* \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^* \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since 
$$\lim_{t \to 0} \rho^*(\xi x) = 0,$$
there exists $\{\alpha_{\nu}\}_{\nu \geqq 1}$ with $0 < \alpha_{\nu}$ ($\nu \geqq 1$) and 
$$\sum_{\nu=1}^{\infty} \rho^*(\alpha_{\nu} x_{\nu}) < +\infty.$$ 
It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_{\nu} x_{\nu})$ from $(\rho.3)$. For such 
$x_0$, we have for every $\xi \geqq 0$,
$$\rho'(\xi x) \geqq \sum_{\nu=1}^{\infty} d(x_{\nu}) = +\infty,$$
which is inconsistent with $(\rho.4)$. Therefore we have
$$\sup_{x \in R} \rho'(\xi x) \leqq \sup_{x \in R} d(x) < +\infty.$$ 
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
$$R_0 = \{ x : x \in R, \rho^*(nx) = 0 \text{ for all } n \geqq 1 \},$$
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold7) of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geqq 0$ for all $\lambda \in \Lambda$. Putting 
$$[p_{n,1}] = [(2nx_{\lambda} - nx)^+]$$
we have 
$$[p_{n,1}] \uparrow_{\lambda \in \Lambda} [x] \text{ and } 2n[p_{n,1}] x_{\lambda} \geqq [p_{n,1}] nx,$$
which implies $\rho^*(n[p_{n,1}] x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,1}] x) = \rho^*(nx) = 0$. Hence, we obtain 
$x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
$$R = R_0 \oplus R_0^\perp.$$ 
In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p] R_0$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}$ ($x_{\lambda} \in [p] R_0$), there exists $x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p] R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^{\prime\prime}$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

7) A linear manifold $S$ is said to be semi-normal, if $a \in S, |b| \leqq |a|, b \in R$ implies $b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\bigcup_{x \in R} x_{\lambda} \in S(\lambda \in A)$ implies $\bigcup_{x \in R} x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z, y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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Theorem 3.1. Let $R$ be a quasi-modular space. Then $R_{0}^{\perp}$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_{0}$ which is semi-continuous, i.e. $\sup_{i \in A} \| x_{i} \|_{0} = \| x \|_{0}$ for any $0 \leq x_{i} \uparrow_{i \in A} x$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^{*}$ satisfies $(\rho.1)$~$(\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad \| x \|_{0} = \inf \left\{ \xi ; \rho^{*}(\frac{1}{\xi} x) \leq \xi \right\}.$$ 

Then,

i) $0 \leq \| x \|_{0} = \| -x \|_{0} < \infty$ and $\| x \|_{0} = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R_{0}^{\perp}$.

ii) $\| x + y \|_{0} \leq \| x \|_{0} + \| y \|_{0}$ for any $x, y \in R$; follows also from (A.3) which is deduced from $(\rho.4)$.

iii) $\lim_{\alpha_{n \uparrow} > 0} \| \alpha_{n} x \|_{0} = 0$ and $\lim_{\alpha_{n \uparrow} > 0} \| \alpha_{n} x \|_{0} = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $\| \cdot \|_{0}$ is semi-continuous. From ii) and iii), it follows that $\lim_{\alpha_{n \uparrow} > 0} \| \alpha_{n} x \|_{0} = \| \alpha_{n} x \|_{0}$ for all $x \in R_{0}^{\perp}$ and $\alpha_{n} \geq 0$. If $x \in R_{0}^{\perp}$ and $[p_{n}] \uparrow_{n \in A} [p]$, for any positive number $\xi$ with $\| [p] x \|_{0} > \xi$ we have $\rho^{*}(\frac{1}{\xi} [p] x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^{*}(\frac{1}{\xi} [p_{\lambda}] x) > \xi$ and hence $\sup_{\lambda \in \Lambda} \| [p_{\lambda}] x \|_{0} \geq \xi$. Thus we obtain

$$\sup_{\lambda \in \Lambda} \| [p_{\lambda}] x \|_{0} = \| [p] x \|_{0}, \quad \text{if} \quad [p_{n}] \uparrow_{n \in A} [p].$$

Let $0 \leq x_{1} \uparrow_{n \in A} x$. Putting

$$[p_{n,1}] = \left[ (x_{1} - (1 - \frac{1}{n}) x)^{+} \right]$$

we have

$$[p_{n,1}] \uparrow_{n \in A} [x] \quad \text{and} \quad [p_{n,1}] x_{1} \geq [p_{n,1}] \left( 1 - \frac{1}{n} \right) x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{n \in A} \| [p_{n,1}] x_{1} \|_{0} \geq \sup_{n \in A} \| [p_{n,1}] \left( 1 - \frac{1}{n} \right) x \|_{0} = \| (1 - \frac{1}{n}) x \|_{0},$$

we have

$$\sup_{n \in A} \| x_{1} \|_{0} \geq \| (1 - \frac{1}{n}) x \|_{0}$$

and also $\sup_{n \in A} \| x_{1} \|_{0} \geq \| x \|_{0}$. As the converse inequality is obvious by iv), $\| \cdot \|_{0}$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim_{n \in \infty} \| x_{n} \|_{0} = 0$ if and only if $\lim \rho(\xi x_{n}) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm $\|\cdot\|_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,\nu}$, $x_{\nu} \geq 0$ and $a \geq 0$ ($n, \nu = 1, 2, \cdots$) be the elements of $R_0^\perp$ such that

(3.2) $[p_{n,\nu}] \uparrow_{\nu=1}^{\infty}[p_n]$ with $\bigcap_{n=1}^{\infty}[p_n]a = [p_0]a = 0$;

(3.3) $[p_{n,\nu}]x_{\nu} \geq n [p_{n,\nu}]a$ for all $n, \nu \geq 1$.

Then $\{x_{\nu}\}_{\nu \geq 1}$ is not a Cauchy sequence of $R_0^\perp$ with respect to $\|\cdot\|_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m] \downarrow_{m=1}^{\infty}(m \geq 1)$ and sequences of natural numbers $\nu_m$, $n_m$ such that

(3.4) $\| [q_m]a \|_0 > \frac{\delta}{2}$ and $[q_m]x_{\nu_m} \geq n_m [q_m]a$ ($m = 1, 2, \cdots$) and

(3.5) $n_m [q_m]a \geq [q_m]x_{\nu_{m-1}}$, $n_{m+1} > n_m$ ($m = 2, 3, \cdots$),

where $\delta = \| [p_0]a \|_0$.

In fact, we put $n_1 = 1$. Since $[p_{n,\nu}[p_0] \uparrow_{\nu=1}^{\infty}[p_0]$ and $\|\cdot\|_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

$$\| [p_{n,\nu_1}[p_0]a \|_0 > \frac{\| [p_0]a \|_0}{2} = \frac{\delta}{2}. $$

We put $[q_1] = [p_{n,\nu_1}[p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m = 1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $\left( (na - x_{\nu_k})^+ \right) \uparrow_{n=1}^{\infty} [a]$ and $\| [q_k]a \|_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

$$\| (n_{k+1}a - x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}. $$

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

$$\| [p_{n_{k+1}, \nu_{k+1}}(n_{k+1}a - x_{\nu_k})^+[q_k]a \|_0 > \frac{\delta}{2}. $$

in virtue of (3.2) and semi-continuity of $\|\cdot\|_0$. Hence we can put

$$[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}(n_{k+1}a - x_{\nu_k})^+[q_k],$$

because

$$[q_{k+1}] \leq [q_k], \| [q_{k+1}]a \| > \frac{\delta}{2}, \ [q_{k+1}]x_{\nu_k} \geq n_{k+1} [q_{k+1}]a$$

by (3.3) and $[q_{k+1}]n_{k+1}a \geq [q_{k+1}]x_{\nu_k}$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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\[ \| x_{\nu_{k+1}} - x_{\nu_{k-1}} \|_0 \geq \| [q_{k+1}](x_{\nu_{k+1}} - x_{\nu_{k-1}}) \|_0 \geq \| n_{k+1}[q_{k+1}]a - n_{k}[q_{k+1}]a \|_0 \geq \| [q_{k+1}]a_0 \|_0 \geq \frac{\delta}{2}, \]

since \( [q_{k+1}] \leq [q_k] \leq [(n_{k}a - x_{\nu-1})^+] \) implies \( [q_{k+1}]n_{k}a \geq [q_{k+1}]x_{\nu_{k-1}} \) by (3.4). It follows from the above that \( \{ x_{\nu} \}_{\nu \geq 1} \) is not a Cauchy sequence.

**Theorem 3.2.** Let \( R \) be a quasi-modular space with quasi-modular \( \rho \). Then \( R^{\perp}_{0} \) is an F-space with \( \| \cdot \|_0 \) if and only if \( \rho \) satisfies \( \rho^{4}' \).

**Proof.** If \( \rho \) satisfies \( \rho^{4}' \), \( \rho^{\ast} \) is a quasi-modular which fulfills also \( \rho^{5} \) and \( \rho^{6} \) in virtue of Theorem 2.3. Since \( \| x \|_0 = \inf \{ \xi ; \rho^{\ast}(\frac{x}{\xi}) \leq \xi, \xi > 0 \} \) is a quasi-norm on \( R^{\perp}_{0} \), we need only to verify completeness of \( \| \cdot \|_0 \). At first let \( \{ x_{\nu} \}_{\nu \geq 1} \subset R^{\perp}_{0} \) be a Cauchy sequence with \( 0 \leq x_{\nu} \uparrow_{\nu=1,2,\ldots} \). Since \( \rho^{\ast} \) satisfies \( \rho^{3} \), there exists \( 0 \leq x_{0} \in R^{\perp}_{0} \) such as is shown in the proof of Theorem 2.3.

Putting \( [p_{n,v}] = [(x_{\nu} - nx_{0})^+] \) and \( \bigcup_{v=1}^{\infty}[p_{n,v}] = [p_{n}] \), we obtain

\[ (3.6) \quad [p_{n,v}]x_{\nu} \geq n[p_{n,v}]x_{0} \quad \text{for all } n, v \geq 1 \]

and \( [p_{n}] \downarrow_{n=1}^{\infty} 0 \). Since \( \{ x_{\nu} \}_{\nu \geq 1} \) is a Cauchy sequence, we have in virtue of Lemma 2, \( \bigcap_{n=1}^{\infty}[p_{n}] = 0 \), that is, \( \bigcup_{n=1}^{\infty}([x_{0}] - [p_{n}]) = [x_{0}] \). And

\[ (1 - [p_{n,v}]) \geq (1 - [p_{n}]) \quad (n, v \geq 1) \]

implies

\[ n(1 - [p_{n,v}])x_{0} \geq (1 - [p_{n}])x_{v} \geq 0. \]

Hence we have

\[ y_{n} = \bigcup_{v=1}^{\infty} (1 - [p_{n,v}])x_{v} \in R^{\perp}_{0}, \]

because \( R^{\perp}_{0} \) is universally continuous. As \( \{ x_{\nu} \}_{\nu \geq 1} \) is a Cauchy sequence, we obtain from the triangle inequality of \( \| \cdot \|_0 \)

\[ \gamma = \sup_{\nu \geq 1} \| x_{\nu} \|_0 < +\infty, \]

which implies

\[ \| y_{n} \|_0 = \sup_{\nu \geq 1} \| (1 - [p_{n,v}])x_{v} \|_0 \leq \gamma \]

for every \( n \geq 1 \) by semi-continuity of \( \| \cdot \|_0 \). We put \( z_{1} = y_{1} \) and \( z_{n} = y_{n} - y_{n-1} \) \( (n \geq 2) \). It follows from the definition of \( y_{n} \) that \( \{ z_{\nu} \}_{\nu \geq 1} \) is an orthogonal sequence with \( \| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_{n} \|_0 \leq \gamma \). This implies
\[
\sum_{\nu=1}^{n} \rho^{*}\left(\frac{z_{\nu}}{1+\gamma}\right) = \rho^{*}\left(\frac{y_{n}}{1+\gamma}\right) \leq \gamma
\]
for all \(n \geq 1\) by the formula (3.1). Then \((\rho.3)\) assures the existence of \(z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}\). This yields \(z = \bigcup_{\nu=1}^{\infty} x_{\nu}\). Truly, it follows from

\[
z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1 - [p_{n}]) x_{\nu} = \bigcup_{\nu=1}^{\infty} \bigcup_{n=1}^{\infty} [x_{0}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}.
\]

By semi-continuity of \(\|\cdot\|_{0}\), we have

\[
\|z - x_{\nu}\|_{0} \leq \sup_{\mu \geq \nu} \|x_{\mu} - x_{\nu}\|_{0}
\]
and furthermore \(\lim_{\nu \to \infty} \|z - x_{\nu}\|_{0} = 0\).

Secondly let \(\{x_{\nu}\}_{\nu \geq 1}\) be an arbitrary Cauchy sequence of \(R_{0}^{+}\). Then we can find a subsequence \(\{y_{\nu}\}_{\nu \geq 1}\) of \(\{x_{\nu}\}_{\nu \geq 1}\) such that

\[
\|y_{\nu+1} - y_{\nu}\|_{0} \leq \frac{1}{2^{\nu}}
\]
for all \(\nu \geq 1\).

This implies

\[
\|\sum_{\nu=m}^{n} y_{\nu+1} - y_{\nu}\|_{0} \leq \sum_{\nu=m}^{n} \|y_{\nu+1} - y_{\nu}\|_{0} \leq \frac{1}{2^{n-m}}
\]
for all \(n > m \geq 1\).

Putting \(z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1} - y_{\nu}|\), we have a Cauchy sequence \(\{z_{n}\}_{n \geq 1}\) with \(0 \leq z_{n} \uparrow \infty\).

Then by the fact proved just above,

\[
z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}| \in R_{0}^{+}
\]
and \(\lim_{n \to \infty} \|z_{0} - z_{n}\|_{0} = 0\).

Since \(\sum_{\nu=1}^{\infty} |y_{\nu+1} - y_{\nu}|\) is convergent, \(y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})\) is also convergent and

\[
\|y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{n}\|_{0} = ||\sum_{\nu=1}^{\nu} (y_{\nu+1} - y_{\nu})||_{0} \leq \|z_{0} - z_{n}\|_{0} \to 0.
\]

Since \(\{y_{\nu}\}_{\nu \geq 1}\) is a subsequence of the Cauchy sequence \(\{x_{\nu}\}_{\nu \geq 1}\), it follows that

\[
\lim_{\nu \to \infty} \|y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{\nu}\|_{0} = 0.
\]

Therefore \(\|\cdot\|_{0}\) is complete in \(R_{0}^{+}\), that is, \(R_{0}^{+}\) is an F-space with \(\|\cdot\|_{0}\).

Conversely if \(R_{0}^{+}\) is an F-space, then for any orthogonal sequence \(\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+}\), we have \(\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}\) for some real numbers \(\alpha_{\nu} > 0\) (for all \(\nu \geq 1\)).

Hence we can see that \(\sup_{x \in \mathbb{R}} d(x) < +\infty\) by the same way applied in Theorem 2.1. It follows that \(\rho\) must satisfy \((\rho.4')\). Q.E.D.

Since \(R_{0}\) contains a normal manifold which is universally complete, if \(R_{0} \neq 0\), we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $\|\cdot\|_0$ defined by (3.1) and $R$ becomes an $F$-space with $\|\cdot\|_0$ if and only if $\rho$ fulfills (\rho.4').

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[ (4.1) \quad \|x\|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\} \]

and show that $\|\cdot\|_1$ is also a quasi-norm on $L$ and

\[ (4.2) \quad \|x\|_0 \leq \|x\|_1 \leq 2 \|x\|_0 \quad \text{for all } x \in L \]

hold, where $\|\cdot\|_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq \|x\|_1 = \|-x\|_1 < +\infty \quad (x \in L)$ and that $\|x\|_1 = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow_{n=1}^\infty 0$ implies $\lim n \rightarrow \infty \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim n \rightarrow \infty \|x_n\|_1 = 0$ implies $\lim n \rightarrow \infty \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim n \rightarrow \infty \|\alpha x_n\|_1 = 0$ for all $\alpha \downarrow_{n=1}^\infty 0$ and that $\lim n \rightarrow \infty \|x_n\|_1 = 0$ implies $\lim n \rightarrow \infty \|\alpha x_n\|_1 = 0$ for all $\alpha > 0$. If $\|x\|_1 < \alpha$ and $\|y\|_1 < \beta$, there exist $\xi, \eta > 0$ such that

\[ \frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta . \]

This yields

\[ \|x + y\| \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta , \]

in virtue of (A.3). Therefore $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ holds for any $x, y \in L$ and $\|\cdot\|_1$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

\[ \frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq 2 \frac{1}{\xi} . \]

10) For the convex modular $m$, we can define two kinds of norms such as

\[ \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad \text{and} \quad \|x\| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \]

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $\|\cdot\|$ and $\|\cdot\|$ respectively.
This yields (4.2), since we have \( \| x \|_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \eta \) with \( \| x \|_0 > \frac{1}{\eta} \). Therefore we can obtain from above

Theorem 4.1. If \( L \) is a modular space with a modular satisfying \((A.1)\sim(A.5)\) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( \| \cdot \|_1 \) on \( L \) which is equivalent to \( \| \cdot \|_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

Theorem 4.2. If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
(4.3) \quad \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^\perp_0 \) and \( \| \cdot \|_1 \) is complete if and only if \( \rho \) satisfies \((\rho.4')\), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
(4.4) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all } x \in R^\perp_0.
\]

\( \S 5 \). A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies \((\rho.1)\sim(\rho.6)\) except \((\rho.3)\) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( 0 \neq x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{ x_\nu \}_{\nu \geq 1} \) is called order-convergent to \( a \) and denoted by \( \lim_{\nu \to +\infty} x_\nu = a \), if there exists a sequence of elements \( \{ a_\nu \}_{\nu \geq 1} \) such that

\[
|x_\nu - a| \leq a_\nu \quad (\nu \geq 1)
\]

and \( a_\nu \downarrow_{\nu = 1}^{\infty} 0 \). And a sequence of elements \( \{ x_\nu \}_{\nu \geq 1} \) is called star-convergent to \( a \) and denoted by \( \lim_{\nu \to \infty} x_\nu = a \), if for any subsequence \( \{ y_\nu \}_{\nu \geq 1} \) of \( \{ x_\nu \}_{\nu \geq 1} \), there exists a subsequence \( \{ z_\nu \}_{\nu \geq 1} \) of \( \{ y_\nu \}_{\nu \geq 1} \) with \( \lim_{\nu \to \infty} z_\nu = a \).

A quasi-norm \( \| \cdot \| \) on \( R \) is termed to be continuous, if \( \inf_{\nu \geq 1} \| a_\nu \| = 0 \) for any \( a_\nu \downarrow_{\nu = 1}^{\infty} 0 \). In the sequel, we write by \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

Theorem 5.1. In order that \( \| \cdot \|_0 \) (or \( \| \cdot \|_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]_0 \text{ is finite dimensional and } \rho(y) < +\infty.
\]

Proof. Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \( \{ [p_n] \}_{n \geq 1} \) such that \( \rho([p_n]x) = +\infty \) and \( [p_n] \downarrow_{n=1}^{\infty} 0 \). Hence by (3.1) it follows that \( \| [p_n]x \|_0 > 1 \) for all \( n \geq 1 \), which contradicts the continuity of \( \| \cdot \|_0 \).

**Sufficiency.** Let \( a_{\nu} \downarrow_{\nu=1}^{\infty} 0 \) and put \( [p_n^\epsilon] = [(a_n - \epsilon a_1)^+] \) for any \( \epsilon > 0 \) and \( n \geq 1 \). It is easily seen that \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \) for any \( \epsilon > 0 \) and \( a_n = [a_1]a_n = [p_n^\epsilon]a_n + (1 - [p_n^\epsilon])a_n \leq [p_n^\epsilon]a_1 + \epsilon a_1 \).

This implies

\[
\rho^*([p_n^\epsilon]a_1) \leq \rho^*([p_n^\epsilon]a_1) + \rho^*(\xi(1-[p_n^\epsilon])a_1)
\]

for all \( n \geq 1 \) and \( \xi \geq 0 \). In virtue of (5.1) and \( [p_n^\epsilon] \downarrow_{n=1}^{\infty} 0 \), we can find \( n_0 \) (depending on \( \xi \) and \( \epsilon \)) such that \( \rho^*([p_n^\epsilon]a_1) < +\infty \), and hence \( \inf_{n \geq 1} \rho^*([p_n^\epsilon]a_1) = 0 \) by (2.3) in Lemma 1 and (\( \rho.2 \)). Thus we obtain

\[
\inf_{n \geq 1} \rho^*([p_n^\epsilon]a_1) \leq \rho^*([p_n^\epsilon]a_1).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \to \infty} \rho^*([a_n]) = 0 \) follows. Hence we infer that \( \inf_{n \geq 1} \| a_n \|_0 = 0 \) and \( \| \cdot \|_0 \) is continuous in view of Remark 2 in \( \S 3 \). Q.E.D.

**Corollary.** \( \| \cdot \|_0 \) is continuous, if

\[
(5.2) \quad \rho^*(a_\nu) \to 0 \text{ implies } \rho^*(\alpha a_\nu) \to 0 \text{ for every } \alpha \geq 0.
\]

From the definition, it is clear that \( s\lim_{\nu \to \infty} x_\nu = 0 \) implies \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \), if \( \| \cdot \|_0 \) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \) (or \( \lim_{\nu \to \infty} \| x_\nu \| = 0 \)) implies \( s\lim_{\nu \to \infty} x_\nu = 0 \), if \( \| \cdot \|_0 \) is complete (i.e. \( \rho^* \) satisfies (\( \rho.3 \))).

If we replace \( \lim_{\nu \to \infty} \| x_\nu \|_0 = 0 \) by \( \lim_{\nu \to \infty} \rho(x_\nu) = 0 \), Theorem 5.2 may fail to be valid in general. By this reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x) = 0 \text{ implies } x = 0.
\]

Truly we obtain

**Theorem 5.3.** If \( \rho^* \) satisfies (5.3) and \( \| \cdot \|_0 \) is complete, \( \rho(a_\nu) \to 0 \) implies \( s\lim a_\nu = 0 \).

**Proof.** We may suppose without loss of generality that \( \rho^* \) is semi-continuous,\(^{11}\) i.e. \( \rho^*(x) = \sup_{i \in A} \rho^*(x_i) \) for any \( 0 \leq i \leq A \). If

\[
11) \quad \text{If } \rho^* \text{ is not semi-continuous, putting } \rho_\ast(x) = \inf_{y_1 \in A} \{ \sup_{y \in A} \rho^*(y_i) \}, \text{ we obtain a quasi-modular } \rho_\ast \text{ which is semi-continuous and } \rho^*(x_\nu) \to 0 \text{ is equivalent to } \rho_\ast(x_\nu) \to 0.
\]


\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} |a_{\nu}| \in R \) in virtue of \((\rho.3)\).

Now, since
\[
\rho\left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\mu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each \( \nu \geq 1 \), \( \rho\left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) \right) = 0 \) and hence \((5.3)\) implies
\[
\bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} |a_{\mu}| \right) = 0.
\]
Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b_{\nu}^{'},\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}^{'}) \leq \frac{1}{2^{\nu}} \quad (\nu = 1, 2, \cdots) \). Therefore we have \( s\lim_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition \((5.2)\) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analouges to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^{*} \) satisfies \((5.3)\) and \( || \cdot ||_{0} \) is complete and continuous, then \((5.2)\) holds.

**References**


