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ON F-NORMS OF QUASI-MODULAR SPACES

By
Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let \( R \) be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and \( \rho \) be a functional which satisfies the following four conditions:

\begin{align*}
(\rho.1) & \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all} \ x \in R; \\
(\rho.2) & \quad \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any} \ x, y \in R \text{ with } x \perp y \\
(\rho.3) & \quad \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \quad \text{for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda} \\
(\rho.4) & \quad \varlimsup_{\xi \to 0} \rho(\xi x) < +\infty \quad \text{for all} \ x \in R.
\end{align*}

Then, \( \rho \) is called a quasi-modular and \( R \) is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if \( R \) is a non-atomic quasi-modular space which is semi-regular, then we can define a modular \( m \) on \( R \) for which every universally continuous linear functional is continuous with respect to the norm defined by the modular \( m \) [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular \( \rho \) on a linear space \( L \) which satisfies the following conditions:

\begin{align*}
(A.1) & \quad \rho(x) \geq 0 \quad \text{and} \quad \rho(x) = 0 \quad \text{if and only if} \ x = 0; \\
(A.2) & \quad \rho(-x) = \rho(x) \\
(A.3) & \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every} \ \alpha, \beta \geq 0 \ \text{with} \ \alpha + \beta = 1; \\
(A.4) & \quad \alpha_{n} \to 0 \ \text{implies} \ \rho(\alpha_{n} x) \to 0 \quad \text{for every} \ x \in R; \\
(A.5) & \quad \text{for any} \ x \in L \ \text{there exists} \ \alpha > 0 \ \text{such that} \ \rho(\alpha x) < +\infty.
\end{align*}

They showed that \( L \) is a quasi-normed space with a quasi-norm \( \|\cdot\|_{0} \) defined by the formula;

1) \( x \perp y \) means \( |x| \wedge |y| = 0 \).
2) A system of elements \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) is called mutually orthogonal, if \( x_{\lambda} \perp x_{\gamma} \) for \( \lambda \neq \gamma \).
3) For the definition of a modular, see [3].
4) A linear functional \( f \) is called universally continuous, if \( \inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0 \) for any \( a_{\lambda} \downarrow 0 \).
5) This modular \( \rho \) is a generalization of a modular \( m \) in the sense of Nakano [3 and 4].
In the latter, there is assumed that \( m(\xi x) \) is a convex function of \( \xi \geq 0 \) for each \( x \in R. \)
(1.1) \[ \| x \|_0 = \inf \left\{ \xi : \rho \left( \frac{1}{\xi} x \right) \leq \xi \right\} \]

and \( \| x_n \|_0 \to 0 \) is equivalent to \( \rho(\alpha x_n) \to 0 \) for all \( \alpha \geq 0 \).

In the present paper, we shall deal with a general quasi-modular space \( R \) (i.e. without the assumption that \( R \) is non-atomic or semi-regular). The aim of this paper is to construct a quasi-norm on \( R \) and to investigate the condition under which \( R \) is an \( F \)-space with this quasi-norm by making use of the above formula (1.1). Since a quasi-modular \( \rho \) on \( R \) does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (\( \rho.1 \))\( \sim \)(\( \rho.4 \)) with those of \( \rho \) [6], we can not apply the formula (1.1) directly to \( \rho \) to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular \( \rho^* \) which satisfies (A.2)\( \sim \)(A.5) on an arbitrary quasi-modular space \( R \) in \S 2 (Theorems 2.1 and 2.2). Since \( R \) may include a normal manifold \( R_0 = \{ x : x \in R, \ \rho^*(\xi x) = 0 \ \text{for all} \ \xi \geq 0 \} \) and we can not define a quasi-norm on \( R_0 \) in general, we have to exclude \( R_0 \) in order to proceed with the argument further. We shall prove in \S 3 that a quasi-norm \( \| \cdot \|_0 \) on \( R_0^+ \) defined by \( \rho^* \) according to the formula (1.1) is semi-continuous, and in order that \( R_0^+ \) is an \( F \)-space with \( \| \cdot \|_0 \) (i.e. \( \| \cdot \|_0 \) is complete), it is necessary and sufficient that \( \rho \) satisfies

\[ (\rho.4') \quad \sup_{x \in R} \rho(ax) < +\infty \]

(Theorem 3.2).

In \S 4, we shall show that we can define another quasi-norm \( \| \cdot \|_1 \) on \( R_0^+ \) which is equivalent to \( \| \cdot \|_0 \) such that \( \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \) holds for every \( x \in R_0^+ \) (Formulas (4.1) and (4.3)). \( \| \cdot \|_1 \) has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4]; \S 83]. At last in \S 5 we shall add shortly the supplementary results concerning the relations between \( \| \cdot \|_0 \)-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in \S 5 are already known in those cases [8].

Throughout this paper \( R \) denotes a universally continuous semi-ordered linear space and \( \rho \) a quasi-modular defined on \( R \). For any \( p \in R \),

\[ [p] \]

is a projector: \( [p] x = \bigcup_{n=1}^{\infty} (n \cdot |p| \cap x) \) for all \( x \geq 0 \) and \( 1 - [p] \) is a projection operator onto the normal manifold \( N = \{ p \}^1 \), that is, \( x = [p] x + (1 - [p]) x \).

---

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

Lemma 1. For any quasi-modular $\rho$, we have

(2.1) $\rho(0)=0$;
(2.2) $\rho([p]x) \leq \rho(x)$ for all $p, x \in R$;
(2.3) $\rho([p]x) = \sup_{\lambda \in \Lambda} \rho([p_\lambda]x)$ for any $[p_\lambda] \uparrow_{\lambda \in A} [p]$.

In the argument below, we have to use the additional property of $\rho$:

(\rho.5) $\rho(x) \leq \rho(y)$ if $|x| \leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies (\rho.5).

Theorem 2.1. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which (\rho.5) is valid.

Proof. We put for every $x \in R$,

(2.4) $\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y)$.

It is clear that $\rho'$ satisfies the conditions (\rho.1), (\rho.2) and (\rho.5).

Let $\{x_\lambda\}_{\lambda \in A}$ be an orthogonal system such that $\sum_{\lambda \in A} \rho'(x_\lambda) < +\infty$, then

$$\sum_{\lambda \in A} \rho(x_\lambda) < +\infty,$$

because

$$\rho(x) \leq \rho'(x)$$

for all $x \in R$.

We have

$$x_0 = \sum_{\lambda \in A} x_\lambda \in R$$

and

$$\rho(x_0) = \sum_{\lambda \in A} \rho(x_\lambda)$$

in virtue of (\rho.3).

For such $x_0$,

$$\rho'(x_0) = \sup_{0 \leq |y| \leq |x_0|} \rho(y) = \sup_{0 \leq |y| \leq |x_0|} \sum_{\lambda \in A} \rho([x_\lambda]y)$$

$$= \sum_{\lambda \in A} \sup_{0 \leq |y| \leq |x_\lambda|} \rho([x_\lambda]y) = \sum_{\lambda \in A} \rho'(x_\lambda)$$

holds, i.e. $\rho'$ fulfils (\rho.3).

If $\rho'$ does not fulfil (\rho.4), we have for some $x_0 \in R$,

$$\rho''\left(\frac{1}{n} x_0\right) = +\infty$$

for all $n \geq 1$.

By (\rho.2) and (\rho.4), $x_0$ cannot be written as $x_0 = \sum_{\nu=1}^\varepsilon \xi_\nu e_\nu$, where $e_\nu$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \varepsilon$, namely, we can decompose $x_0$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x'_1, \quad x_1 \perp x'_1, \]
where \( \rho'(\frac{1}{\nu} x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x'_1) > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x'_1| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x'_2, \quad x_2 \perp x'_2, \]
where
\[ \rho'(\frac{1}{\nu} x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))
and
\[ \rho'(\frac{1}{2} x'_2) > 2. \]

There exists also \( 0 \leq y_2 \leq |x'_2| \) such that \( \rho'(\frac{1}{2} y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{ y_\nu \}_{\nu=1,2}, \ldots \) such that
\[ \rho'\left(\frac{1}{\nu} y_\nu\right) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{ y_\nu \}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho\left(\frac{1}{\nu} y_0\right) \geq \rho\left(\frac{1}{\nu} y_\nu\right) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \([p]=[(|x|-|y|)^+]\), we obtain
\[
\rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \\
\leq \rho(\alpha \lfloor p \rfloor |x| + \alpha(1 - \lfloor p \rfloor)|y| + \beta \lfloor p \rfloor |x| + (1 - \lfloor p \rfloor)\beta |y|) \\
= \rho(\lfloor p \rfloor |x| + (1 - \lfloor p \rfloor)|y|) \\
= \rho(\lfloor p \rfloor x) + \rho((1 - \lfloor p \rfloor)y) \\
\leq \rho(x) + \rho(y).
\]

**Remark 1.** As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition (\( \rho.4 \)). Thus, in the above theorems, we cannot replace (\( \rho.4 \)) by the weaker condition:

(\( \rho.4'' \)) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying (\( \rho.1 \)), (\( \rho.2 \)), (\( \rho.3 \)) and (\( \rho.4'' \)), but does not (\( \rho.4 \)). For this \( \rho_0 \), we obtain

\[
\rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty
\]

for all \( x \neq 0 \).

**Example.** \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[
\int_0^1 |x(t)| \, dt < +\infty.
\]

Putting

\[
\rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \, \text{mes} \left\{ t : x(t) = \frac{1}{i} \right\},
\]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: (A.4), namely,

(\( \rho.6 \)) \quad \lim_{\xi \to 0} \rho(\xi x) = 0 \quad \text{for all } x \in R.

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an F-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

**Theorem 2.2.** Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies (\( \rho.1 \))\( \sim \)(\( \rho.6 \)) except (\( \rho.3 \)).

**Proof.** In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies (\( \rho.5 \)). Now we put

\[
d(x) = \lim_{\xi \to 0} \rho'(\xi x).
\]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
$d(x+y) = d(x) + d(y)$ if $x \perp y$.

Hence, putting

\[(2.6) \quad \rho^*(x) = \rho'(x) - d(x) (x \in \mathbb{R}).\]

we can see easily that $(\rho.1), (\rho.2), (\rho.4)$ and $(\rho.6)$ hold true for $\rho^*$, since

\[d(x) \leq \rho'(x)\]

and

\[d(\alpha x) = d(x)\]

for all $x \in \mathbb{R}$ and $\alpha > 0$.

We need to prove that $(\rho.5)$ is true for $\rho^*$. First we have to note

\[(2.7) \quad \inf_{\lambda \in \Lambda} d([p_{\lambda}]x) = 0\]

for any $[p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0$. In fact, if we suppose the contrary, we have

\[\inf_{\lambda \in \Lambda} d([p_{\lambda}]x_0) \geq \alpha > 0\]

for some $[p_{\lambda}] \downarrow_{\lambda \in \Lambda} 0$ and $x_0 \in \mathbb{R}$.

Hence,

\[\rho'(\frac{1}{\nu}[p_{\lambda}]x_0) \geq d([p_{\lambda}]x_0) \geq \alpha\]

for all $\nu \geq 1$ and $\lambda \in \Lambda$. Thus we can find a subsequence $\{\lambda_n\}_{n \geq 1}$ of $\{\lambda\}_{\lambda \in \Lambda}$ such that

\[[p_{\lambda_n}] \geq [p_{\lambda_n+1}]\]

and

\[\rho'(\frac{1}{n}[p_{\lambda_n}]x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_n}]x_0) = +\infty,\]

for all $n \geq 1$ in virtue of $(\rho.2)$ and $(2.3)$. This implies

\[\rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}[p_{\lambda_m}] - [p_{\lambda_n}]x_0) = +\infty,\]

which is inconsistent with $(\rho.4)$. Secondly we shall prove

\[(2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y].\]

We put $[p_n] = [(|x| - n|y|)^+]$ for $x, y \in \mathbb{R}$ with $[x] = [y]$ and $n \geq 1$. Then, $[p_n] \downarrow_{n=1}^\infty 0$ and $\inf_{n=1, 2, \ldots} d([p_n]x) = 0$ by $(2.7)$. Since $(1 - [p_n])n |y| \geq (1 - [p_n])|x|$ and

\[d(\alpha x) = d(x)\]

for $\alpha > 0$ and $x \in \mathbb{R}$, we obtain
$d(x) = d([p_n]x) + d((1-[p_n])x)$
$\leq d([p_n]x) + d(n(1-[p_n])y)$
$\leq d([p_n]x) + d(y)$.

As $n$ is arbitrary, this implies

$$d(x) \leq \inf_{n=1,2,\ldots} d([p_n]x) + d(y),$$

and also $d(x) \leq d(y)$. Therefore we conclude that (2.8) holds.

If $|x| \geq |y|$, then

$$\rho^*(x) = \rho^*([y]x) + \rho^*(([x]-[y])x)$$
$$= \rho^*([y]x) - d([y]x) + \rho^*(([x]-[y])x)$$
$$\geq \rho^*(y) - d(y) + \rho^*(([x]-[y])x)$$
$$\geq \rho^*(y).$$

Thus $\rho^*$ satisfies ($\rho.5$).

**Theorem 2.3.** $\rho^*$ (which is constructed from $\rho$ according to the formulas (2.4), (2.5) and (2.6)) satisfies (\rho.3) (that is, $\rho^*$ is a quasi-modular), if and only if $\rho$ satisfies

($\rho.4'$)

$$\sup_{x \in \mathbb{R}} \{\lim_{\xi \to 0} \rho^*(\xi x)\} = K' < + \infty.$$ 

**Proof.** Let $\rho$ satisfy ($\rho.4$). We need to prove

(2.9) $$\sup_{x \in \mathbb{R}} d(x) = \sup_{x \in \mathbb{R}} \{\lim_{\xi \to 0} \rho^*(\xi x)\} = K' < + \infty,$$

where

$$\rho^*(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).$$

Since $\rho'$ is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put $n_0(x) = \rho(x)$ and $n_\nu(x) = \rho\left(\frac{1}{\nu}x\right)$ for $\nu \geq 1$ and $x \in \mathbb{R}$. Hence we can find positive numbers $\varepsilon$, $\gamma$, a natural number $\nu_0$ and a finite dimensional normal manifold $N_0$ such that $x \in N_0^\perp$ with

$$\rho(x) \leq \varepsilon \quad \text{implies} \quad \rho\left(\frac{1}{\nu_0}x\right) \leq \gamma.$$ 

In $N_0$, we have obviously

$$\sup_{x \in N_0^\perp} \{\lim_{\xi \to 0} \rho^*(\xi x)\} = \gamma_0 < + \infty.$$ 

If $\varepsilon \leq 2K$, for any $x_0 \in N_0^\perp$, we can find $\alpha_0 > 0$ such that $\rho(\alpha x_0) \leq 2K$ for all $0 \leq \alpha \leq \alpha_0$ by ($\rho.4'$), and hence there exists always an orthogonal decomposition such that
\[ \alpha_{0}x_{0} = x_{1} + \cdots + x_{n} + y_{1} + \cdots + y_{m} + z \]

where \( \frac{\epsilon}{2} < \rho(x_{i}) \leq \epsilon \quad (i = 1, 2, \ldots, n) \), \( y_{j} \) is an atomic element with \( \rho(y_{j}) > \epsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\epsilon}{2} \). From above, we get \( n \leq \frac{4K}{\epsilon} \) and \( m \leq \frac{2K}{\epsilon} \). This yields

\[
\rho'\left(\frac{1}{\nu_{0}}\alpha_{0}x_{0}\right) \leq \sum_{i=1}^{n} \rho'\left(\frac{1}{\nu_{0}}x_{i}\right) + \sum_{j=1}^{m} \rho'(y_{j}) + \rho'\frac{z}{\nu_{0}} \leq n\gamma + \sum_{j=1}^{m} \rho'(y_{j}) + \rho'\frac{z}{\nu_{0}} \leq \frac{4K}{\epsilon}\gamma + \frac{2K}{\epsilon}\left\{ \sup_{0 \leq a \leq a_{0}} \rho(\alpha x) \right\} + \gamma .
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho'(\xi x_{0}) \leq \rho'\left(\frac{\alpha_{0}}{\nu_{0}}x_{0}\right) \leq \left(\frac{4K + \epsilon}{\epsilon}\right)\gamma + \left(\frac{4K^{2}}{\epsilon}\right) + \gamma_{0} .
\]

Let \( \{x_{\lambda}\}_{\lambda \in \Lambda} \) be an orthogonal system with \( \sum_{\lambda \in \Lambda} \rho^{*}(x_{\lambda}) < +\infty \). Then for arbitrary \( \lambda_{1}, \ldots, \lambda_{k} \in \Lambda \), we have

\[
\sum_{i=1}^{k} d(x_{\lambda_{i}}) = d(\sum_{i=1}^{k} x_{\lambda_{i}}) = \lim_{\xi \to 0} \rho'(\xi \sum_{i=1}^{k} x_{\lambda_{i}}) \leq \gamma' ,
\]

which implies \( \sum_{\lambda \in \Lambda} d(x_{\lambda}) \leq \gamma' \). It follows that

\[
\sum_{\lambda \in \Lambda} \rho'(x_{\lambda}) = \sum_{\lambda \in \Lambda} \rho^{*}(x_{\lambda}) + \sum_{\lambda \in \Lambda} d(x_{\lambda}) < +\infty ,
\]

which implies \( x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \in R \) and \( \sum_{\lambda \in \Lambda} \rho^{*}(x_{\lambda}) = \rho^{*}(x_{0}) \) by (\( \rho.4 \)) and (2.7). Therefore \( \rho^{*} \) satisfies (\( \rho.3 \)).

On the other hand, suppose that \( \rho^{*} \) satisfies (\( \rho.3 \)) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \) such that

\[
\sum_{\nu=1}^{\mu} d(x_{\nu}) = d(\sum_{\nu=1}^{\mu} x_{\nu}) \geq \mu
\]
for all \( \mu \geqq 1 \) in virtue of (2.8) and the orthogonal additivity of \( d \). Since \( \lim_{t \to 0} \rho^*(\xi x) = 0 \), there exists \( \{\alpha_{v}\}_{v \geqq 1} \) with \( 0 < \alpha_{v} (v \geqq 1) \) and \( \sum_{v=1}^{\infty} \rho^*(\alpha_{v} x_{v}) < +\infty \). It follows that \( x_{0} = \sum_{v=1}^{\infty} \alpha_{v} x_{v} \in \mathbb{R} \) and \( d(x_{0}) = \sum_{v=1}^{\infty} d(\alpha_{v} x_{v}) \) from (\( \rho.3 \)). For such \( x_{0} \), we have for every \( \xi \geqq 0 \),
\[
\rho'(\xi x_{0}) = \sum_{v=1}^{\infty} \rho'(\xi \alpha_{v} x_{v}) \geqq \sum_{v=1}^{\infty} d(x_{v}) = +\infty,
\]
which is inconsistent with (\( \rho.4 \)). Therefore we have \( \sup_{x \in \mathbb{R}} (\lim_{t \to 0} \rho(\xi x)) \leqq \sup_{x \in \mathbb{R}} d(x) < +\infty \). Q.E.D.

§3. Quasi-norms. We denote by \( R_{0} \) the set:
\[
R_{0} = \{ x : x \in \mathbb{R}, \rho^*(nx) = 0 \text{ for all } n \geqq 1 \},
\]
where \( \rho^* \) is defined by the formula (2.6). Evidently \( R_{0} \) is a semi-normal manifold\(^7\) of \( \mathbb{R} \). We shall prove that \( R_{0} \) is a normal manifold of \( \mathbb{R} \). In fact, let \( x = \bigcup_{\lambda \in \Lambda} x_{\lambda} \) with \( R_{0} \ni x_{\lambda} \geqq 0 \) for all \( \lambda \in \Lambda \). Putting \( [p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+] \), we have \( [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \) and \( 2n[p_{n,\lambda}]x_{\lambda} \geqq [p_{n,\lambda}]nx_{1} \), which implies \( \rho^*(n[p_{n,\lambda}]x) = 0 \) and \( \sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0 \). Hence, we obtain \( x \in R_{0} \), that is, \( R_{0} \) is a normal manifold of \( R \).

Therefore, \( R \) is orthogonally decomposed into
\[
R = R_{0} \oplus R_{0}^{\perp}.\quad (8)
\]

In virtue of the definition of \( \rho^* \), we infer that for any \( p \in R_{0} \), \( [p]R_{0} \) is universally complete, i.e. for any orthogonal system \( \{x_{\lambda}\}_{\lambda \in \Lambda} (x_{\lambda} \in [p]R_{0}) \), there exists \( x_{0} = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R \). Hence we can also verify without difficulty that \( R_{0} \) has no universally continuous linear functional except \( 0 \), if \( R_{0} \) is non-atomic. When \( R_{0} \) is discrete, it is isomorphic to \( S(\Lambda)^{\ast} \)-space. With respect to such a universally complete space \( R_{0} \), we can not always construct a linear metric topology on \( R_{0} \), even if \( R_{0} \) is discrete.

In the following, therefore, we must exclude \( R_{0} \) from our consideration. Now we can state the theorems which we aim at.

---

7) A linear manifold \( S \) is said to be semi-normal, if \( a \in S, |b| \leqq |a|, b \in \mathbb{R} \) implies \( b \in S \). Since \( R \) is universally continuous, a semi-normal manifold \( S \) is normal if and only if \( \cup \{x_{\lambda}\}_{\lambda \in \Lambda} \in \mathbb{R} \) implies \( \cup \{x_{\lambda}\}_{\lambda \in \Lambda} \in S \).

8) This means that \( x \in \mathbb{R} \) is written by \( x = y + z, y \in R_{0} \) and \( z \in R_{0}^{\perp} \).

9) \( S(\Lambda) \) is the set of all real functions defined on \( \Lambda \).
Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^\perp_0$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ which is semi-continuous, i.e.

$$\sup_{i \in I} ||x_i||_0 = ||x||_0$$

for any $0 \leq x, x_i \in R$.

Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad ||x||_0 = \inf \{ \xi; \rho^*(\frac{1}{\xi}x) \leq \xi \}.$$ 

Then,

i) $0 \leq ||x||_0 = ||-x||_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1), (\rho.6), (2.1)$ and the definition of $R^\perp_0$.

ii) $\sup_{\alpha_0 \geq 0} ||\alpha x||_0 = ||\alpha x||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $||\cdot||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{a \rightarrow \alpha_0} ||\alpha x||_0 = ||\alpha_0 x||_0$ for all $x \in R_0^\perp$ and $\alpha_0 \geq 0$. If $x \in R_0^\perp$ and $[p_i] \uparrow_{i \in A} [p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^*(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{\lambda \in A} \rho^*(\frac{1}{\xi}[p_{\lambda}]x) > \xi$ and hence $\sup_{\lambda \in A} ||[p_{\lambda}]x||_0 > \xi$. Thus we obtain

$\sup_{\lambda \in A} ||[p_{\lambda}]x||_0 > ||[p]x||_0$, if $[p_i] \uparrow_{i \in A} [p]$.

Let $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$. Putting

$[p_{n,\lambda}] = [(x_\lambda - (1 - \frac{1}{n})x)^+]$ 

we have

$[p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x]$ and $[p_{n,\lambda}] x_\lambda \geq [p_{n,\lambda}] (1 - \frac{1}{n})x \quad (n \geq 1)$.

As is shown above, since

$$\sup_{\lambda \in A} ||[p_{n,\lambda}]x_\lambda||_0 \geq \sup_{\lambda \in A} \sup_{\alpha_0 \geq 0} \sup_{i \in A} ||[p_{n,\lambda}] (1 - \frac{1}{n})x||_0 = ||(1 - \frac{1}{n})x||_0,$$

we have

$$\sup_{\lambda \in A} ||x_\lambda||_0 \geq ||(1 - \frac{1}{n})x||_0$$

and also $\sup_{\lambda \in A} ||x_\lambda||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $||\cdot||_0$ is semi-continuous.

Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that

$\lim_{n \rightarrow \infty} ||x_n||_0 = 0$ if and only if $\lim_{\xi \rightarrow 0} \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm \( \| \cdot \|_0 \), the next Lemma is necessary.

**Lemma 2.** Let \( p_{n, \nu}, x_{\nu} \geq 0 \) and \( a \geq 0 \) \((n, \nu = 1, 2, \cdots)\) be the elements of \( R_0^\perp \) such that

(3.2) \[
[p_{n, \nu}] \uparrow_{\nu=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0;
\]
(3.3) \[
[p_{n, \nu}] x_{\nu} \geq n [p_{n, \nu}] a \text{ for all } n, \nu \geq 1.
\]

Then \( \{x_{\nu}\}_{\nu \geq 1} \) is not a Cauchy sequence of \( R_0^\perp \) with respect to \( \| \cdot \|_0 \).

**Proof.** We shall show that there exist a sequence of projectors \([q_m] \downarrow_{m=1}^{\infty} (m \geq 1)\) and sequences of natural numbers \( \nu_m, n_m \) such that

(3.4) \[
\| [q_m] a \|_0 > \frac{\delta}{2} \text{ and } [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m = 1, 2, \cdots)
\]
and

(3.5) \[
n_m [q_m] a \geq [q_m] x_{\nu_m-1}, \quad n_{m+1} > n_m \quad (m = 2, 3, \cdots),
\]
where \( \delta = \| [p_0] a \|_0 \).

In fact, we put \( n_1 = 1 \). Since \([p_{1, \nu}] [p_0] \uparrow_{\nu=1}^{\infty} [p_0] \) and \( \| \cdot \|_0 \) is semi-continuous, we can find a natural number \( \nu_1 \) such that

\[
\| [p_{1, \nu_1}] [p_0] a \|_0 > \frac{\| [p_0] a \|_0}{2} = \frac{\delta}{2}.
\]

We put \([q_1] = [p_{1, \nu_1}] [p_0]\). Now, let us assume that \([q_m], \nu_m, n_m \) \((m = 1, 2, \cdots, k)\) have been taken such that (3.4) and (3.5) are satisfied.

Since \([ (n a - x_{\nu_k})^+ ] \uparrow_{n=1}^{\infty} [a] \) and \( \| [q_k] a \|_0 > \frac{\delta}{2} \), there exists \( n_{k+1} \) with

\[
\| (n_{k+1} a - x_{\nu_k})^+ [q_k] a \|_0 > \frac{\delta}{2}.
\]

For such \( n_{k+1} \), there exists also a natural number \( \nu_{k+1} \) such that

\[
\| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_{k+1}})^+ [q_k] a \|_0 > \frac{\delta}{2}.
\]

in virtue of (3.2) and semi-continuity of \( \| \cdot \|_0 \). Hence we can put

\[
[q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_{k+1}})^+ [q_k],
\]

because

\[
[q_{k+1}] \subseteq [q_k], \quad \| [q_{k+1}] a \| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a
\]
by (3.3) and \([q_{k+1}] n_{k+1} a \geq [q_{k+1}] x_{\nu_k} \) by (3.5).

For the sequence thus obtained, we have for every \( k \geq 3 \)
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$$||x_{\nu_{k+1}}-x_{\nu_{k-1}}||_0 \geqq ||[q_{k+1}](x_{\nu_{k+1}}-x_{\nu_{k-1}})||_0 \geqq ||n_{k+1}[q_{k+1}]a-n_{k}[q_{k+1}]a)||_0 \geqq ||[q_{k+1}]a_0||_0 \geqq \frac{\delta}{2},$$

since $[q_{k+1}] \leqq [q_{k}] \leqq [(n_{k}a-x_{\nu-1})^+]$ implies $[q_{k+1}]n_{k}a \geqq [q_{k+1}]x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

**Theorem 3.2.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_{0}^\perp$ is an $F$-space with $||\cdot||_0$ if and only if $\rho$ satisfies $(\rho.4')$.

**Proof.** If $\rho$ satisfies $(\rho.4')$, $\rho^*$ is a quasi-modular which fulfills also $(\rho.5)$ and $(\rho.6)$ in virtue of Theorem 2.3. Since $x||_{0}(=\inf\{\xi;\rho^{*}(\frac{x}{\xi})\leqq\xi\})$ is a quasi-norm on $R_{0}^\perp$, we only need to verify completeness of $||\cdot||_0$. At first let $\{x_{\nu}\}_{\nu \geqq 1} \subset R_{0}^\perp$ be a Cauchy sequence with $0 \leqq x_{\nu} \uparrow_{\nu=1,2}, \ldots$. Since $\rho^*$ satisfies $(\rho.3)$, there exists $0 \leqq x_{0} \in R_{0}^\perp$ such that $x_{0} = \bigcup_{\nu=1}^{\infty}x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}]=[x_{\nu}-nx_{0})^+]$ and $\bigcup_{n=1}^{\infty}[p_{n,v}]=[p_{n}]$, we obtain

$$\sum_{\nu=1}^{\infty}[p_{n,v}]x_{\nu} \geqq n[p_{n,v}]x_{0} \quad \text{for all } n, \nu \geqq 1$$

and $[p_{n}]\downarrow_{n=1}^{\infty}0$. Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty}[p_{n}]=0$, that is, $\bigcup_{n=1}^{\infty}([x_{0}]-[p_{n}])=[x_{0}]$. And

$$(1-[p_{n,v}]) \geqq (1-[p_{n}]) \quad (n, \nu \geqq 1)$$

implies

$$n(1-[p_{n,v}])x_{0} \geqq (1-[p_{n}])x_{\nu} \geqq 0.$$ 

Hence we have

$$y_{n} = \bigcup_{\nu=1}^{\infty}(1-[p_{n,v}])x_{\nu} \in R_{0}^\perp,$$

because $R_{0}^\perp$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $||\cdot||_0$

$$\gamma = \sup_{\nu \geqq 1} ||x_{\nu}||_0 < +\infty,$$

which implies

$$||y_{n}||_0 = \sup_{\nu \geqq 1} ||(1-[p_{n,v}])x_{\nu}||_0 \leqq \gamma$$

for every $n \geqq 1$ by semi-continuity of $||\cdot||_0$. We put $z_{1}=y_{1}$ and $z_{n}=y_{n}-y_{n-1} \quad (n \geqq 2)$. It follows from the definition of $y_{n}$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $||\sum_{\nu=1}^{n}z_{\nu}||_0 = ||y_{n}||_0 \leqq \gamma$. This implies
for all $n \geq 1$ by the formula (3.1). Then $(\rho.3)$ assures the existence of $z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu}$. This yields $z = \bigcup_{\nu=1}^{\infty} x_{\nu}$. Truly, it follows from $z = \bigcup_{\nu=1}^{\infty} y_{\nu} = \bigcup_{\nu=1}^{\infty} \left(1 - [p_{n}]\right) x_{\nu} = \bigcup_{\nu=1}^{\infty} [x_{0}] x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu}$.

By semi-continuity of $|| \cdot ||_{0}$, we have

$$||z - x_{\nu}||_{0} \leq \sup_{\mu \geq \nu} ||x_{\mu} - x_{\nu}||_{0}$$

and furthermore $\lim_{n \to \infty} ||z - x_{n}||_{0} = 0$.

Secondly let $\{x_{\nu}\}_{\nu \geq 1}$ be an arbitrary Cauchy sequence of $R_{0}^{+}$. Then we can find a subsequence $\{y_{\nu}\}_{\nu \geq 1}$ of $\{x_{\nu}\}_{\nu \geq 1}$ such that

$$||y_{\nu+1} - y_{\nu}||_{0} \leq \frac{1}{2^{\nu}}$$

for all $\nu \geq 1$.

This implies

$$||\sum_{\nu=m}^{n} (y_{\nu+1} - y_{\nu})||_{0} \leq \frac{1}{2^{m-1}}$$

for all $n > m \geq 1$.

Putting $z_{n} = \sum_{\nu=1}^{n} ||y_{\nu+1} - y_{\nu}||_{0}$, we have a Cauchy sequence $\{z_{n}\}_{n \geq 1}$ with $0 \leq z_{n} \leq \infty$.

Then by the fact proved just above,

$$z_{0} = \sum_{\nu=1}^{\infty} ||y_{\nu+1} - y_{\nu}||_{0} \in R_{0}^{+}$$

and

$$\lim_{n \to \infty} ||z_{0} - z_{n}||_{0} = 0.$$

Since $\sum_{\nu=1}^{\infty} ||y_{\nu+1} - y_{\nu}||_{0}$ is convergent, $y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})$ is also convergent and

$$||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - y_{\infty}||_{0} = ||\sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu})||_{0} \leq ||z_{0} - z_{n}||_{0} \to 0.$$

Since $\{y_{\nu}\}_{\nu \geq 1}$ is a subsequence of the Cauchy sequence $\{x_{\nu}\}_{\nu \geq 1}$, it follows that

$$\lim_{n \to \infty} ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1} - y_{\nu}) - x_{n}||_{0} = 0.$$

Therefore $|| \cdot ||_{0}$ is complete in $R_{0}^{+}$, that is, $R_{0}^{+}$ is an F-space with $|| \cdot ||_{0}$.

Conversely if $R_{0}^{+}$ is an F-space, then for any orthogonal sequence $\{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+}$, we have $\sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+}$ for some real numbers $\alpha_{\nu} > 0$ (for all $\nu \geq 1$). Hence we can see that $\sup_{x \in R} d(x) < +\infty$ by the same way applied in Theorem 2.1. It follows that $\rho$ must satisfy $(\rho.4')$.

Q.E.D.

Since $R_{0}$ contains a normal manifold which is universally complete, if $R_{0} \neq 0$, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let \( R \) be a quasi-modular space which includes no universally complete normal manifold. Then \( R \) becomes a quasi-normed space with a quasi-norm \( \| \cdot \|_0 \) defined by (3.1) and \( R \) becomes an \( F \)-space with \( \| \cdot \|_0 \) if and only if \( \rho \) fulfills (\( \rho.4^\prime \)).

§4. Another Quasi-norm. Let \( L \) be a modular space in the sense of Musielak and Orlicz (§1). Here we put for \( x \in L \)

\[
(4.1) \quad \| x \|_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10)}
\]

and show that \( \| \cdot \|_1 \) is also a quasi-norm on \( L \) and

\[
(4.2) \quad \| x \|_0 \leq \| x \|_1 \leq 2 \| x \|_0 \quad \text{for all} \quad x \in L
\]

hold, where \( \| \cdot \|_0 \) is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that \( 0 \leq \| x \|_1 = \| -x \|_1 < +\infty \) \((x \in L)\) and that \( \| x \|_1 = 0 \) is equivalent to \( x = 0 \). Since \( \lim_{n \to \infty} \| x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \rho(\xi x_n) = 0 \) for all \( \xi \geq 0 \), we obtain that \( \lim_{n \to \infty} \| \alpha x_n \|_1 = 0 \) for all \( \alpha > 0 \) and that \( \| \alpha x_n \|_1 = 0 \) implies \( \lim_{n \to \infty} \| x_n \|_1 = 0 \) holds for any \( x, y \in L \) and \( \| \cdot \|_1 \) is a quasi-norm on \( L \). If \( \| x \|_1 < \alpha \) and \( \| y \|_1 < \beta \), there exist \( \xi, \eta > 0 \) such that

\[
\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.
\]

This yields

\[
\| x + y \|_1 \leq \frac{\xi + \eta}{\xi \eta} + \rho \left( \frac{\xi \eta}{\xi + \eta} (x + y) \right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho \left( \frac{\eta}{\xi + \eta} (\xi x) + \frac{\xi}{\xi + \eta} (\eta y) \right)
\]

\[
\leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,
\]

in virtue of (A.3). Therefore \( \| x + y \|_1 \leq \| x \|_1 + \| y \|_1 \) holds for any \( x, y \in L \) and \( \| \cdot \|_1 \) is a quasi-norm on \( L \). If \( \xi \rho(\xi x) \leq 1 \) for some \( \xi > 0 \) and \( x \in L \), we have \( \rho(\xi x) \leq 1/\xi \) and hence

\[
\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.
\]

10) For the convex modular \( m \), we can define two kinds of norms such as

\[
\| x \| = \inf_{\xi > 0} \left\{ \frac{1 + m(\xi x)}{\xi} \right\} \quad \text{and} \quad \| x \| = \inf_{m(\xi x) \leq 1} \left\{ \frac{1}{\xi} \right\}
\]

[3 or 4]. For the general modulares considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing \( m(\xi x) \) by \( \xi \rho(\xi x) \) in \( \| \cdot \| \) and \( \| \cdot \| \) respectively.
This yields (4.2), since we have $\|x\|_0 \leq \frac{1}{\xi}$ and $\rho(\gamma x) > \frac{1}{\eta}$ for every $\gamma$ with $\|x\|_0 > \frac{1}{\eta}$. Therefore we can obtain from above

**Theorem 4.1.** If $L$ is a modular space with a modular satisfying (A.1)~(A.5) in §1, then the formula (4.1) yields a quasi-norm $\|\cdot\|_1$ on $L$ which is equivalent to $\|\cdot\|_0$ defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in §2, we obtain by the same way as in §3

**Theorem 4.2.** If $R$ is a quasi-modular space with a quasi-modular $\rho$, then

\[
\|x\|_1 = \inf_{\xi \geq 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on $R^+_0$ and $\|\cdot\|_1$ is complete if and only if $\rho$ satisfies (\rho.4'), where $\rho^*$ and $R_o$ are the same as in §2 and §3. And further we have

\[
\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \quad \text{for all } x \in R^+_0.
\]

§5. A quasi-norm-convergence. Here we suppose that a quasi-modular $\rho^*$ on $R$ satisfies (\rho.1)~(\rho.6) except (\rho.3) and $\rho^*(\xi x)$ is not identically zero as a function of $\xi \geq 0$ for each $0 \leq x \in R$ (i.e. $R_o = \{0\}$). A sequence of elements $\{x_n\}_{n \geq 1}$ is called order-convergent to $a$ and denoted by $o\lim x_n = a$, if there exists a sequence of elements $\{a_n\}_{n \geq 1}$ such that $|x_n - a| \leq a_n$ ($n \geq 1$) and $a_n \downarrow_{n=1}^\infty 0$. And a sequence of elements $\{x_n\}_{n \geq 1}$ is called star-convergent to $a$ and denoted by $s\lim x_n = a$, if for any subsequence $\{y_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{y_n\}_{n \geq 1}$ with $o\lim z_n = a$. A quasi-norm $\|\cdot\|$ on $R$ is termed to be continuous, if $\inf_{r \in R} \|a_r\| = 0$ for any $a_r \downarrow_{r \in R} 0$. In the sequel, we write by $\|\cdot\|_0$ (or $\|\cdot\|_1$) the quasi-norm defined on $R$ by $\rho^*$ in §3 (resp. in §4).

Now we prove

**Theorem 5.1.** In order that $\|\cdot\|_0$ (or $\|\cdot\|_1$) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \text{ for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some $x \in R$, we can find a
sequence of projector \{[p_n]\}_{n\geqq 1} such that \(\rho([p_n]x)=+\infty\) and \([p_n]\downarrow_{n=1}^{\infty}0\).

Hence by (3.1) it follows that \(\| [p_n]x \|_0 > 1\) for all \(n\geqq 1\), which contradicts the continuity of \(\| \cdot \|_0\).

**Sufficiency.** Let \(a_{\nu}\downarrow_{\nu=1}^{\infty}0\) and put \([p_n^*]=[a_n-\epsilon a_1]^*\) for any \(\epsilon>0\) and \(n\geqq 1\).

This implies
\[
\rho^*(\xi a_n) \leq \rho^*(\xi [p_n^*]a_1) + \rho^*(\xi \epsilon (1-[p_n^*])a_1)
\]
for all \(n\geqq 1\) and \(\xi \geqq 0\). In virtue of (5.1) and \([p_n^*]\downarrow_{n=1}^{\infty}0\), we can find \(n_0\) (depending on \(\xi\) and \(\epsilon\)) such that \(\rho^*(\xi [p_n^*]a_1) < +\infty\), and hence \(\inf_{n\geqq 1} \rho^*(\xi [p_n^*]a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho.2\)). Thus we obtain
\[
\inf_{n\geqq 1} \rho^*(\xi a_n) \leq \rho^*(\epsilon a_1).
\]

Since \(\epsilon\) is arbitrary, \(\lim_{n\rightarrow\infty} \rho^*(\xi a_n) = 0\) follows. Hence we infer that \(\inf_{n\geqq 1} \| a_n \|_0 = 0\) and \(\| \cdot \|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

In view of the proof of the above theorem we get obviously

**Corollary.** \(\| \cdot \|_0\) is continuous, if
\[
(5.2) \quad \rho^*(a) \rightarrow 0 \quad \text{implies} \quad \rho^*(\alpha a) \rightarrow 0 \quad \text{for every} \quad \alpha \geqq 0.
\]

From the definition, it is clear that \(\text{s-lim} x=0\) implies \(\lim_{y\rightarrow\infty} \| x \|_0 = 0\), if \(\| \cdot \|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{y\rightarrow\infty} \| x \|_0 = 0\) (or \(\lim_{y\rightarrow\infty} \| x \|_1 = 0\)) implies \(\text{s-lim} x=0\), if \(\| \cdot \|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho.3\)).

If we replace \(\lim_{y\rightarrow\infty} \| x \| = 0\) by \(\lim_{y\rightarrow\infty} \rho(x)=0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

\[
(5.3) \quad \rho^*(x)=0 \quad \text{implies} \quad x=0.
\]

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(\| \cdot \|_0\) is complete, \(\rho(a) \rightarrow 0\) implies \(\text{s-lim} a=0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous, i.e. \(\rho^*(x) = \sup_{i\in A} \rho^*(x_i)\) for any \(0 \leq x_{i\in A} x\). If

11 If \(\rho^*\) is not semi-continuous, putting \(\rho_*(x)=\inf_{y_{i\in A}} \{ \sup_{j\in A} \rho^*(y_{j}) \}\), we obtain a quasi-modular \(\rho_*\) which is semi-continuous and \(\rho^*(x) \rightarrow 0\) is equivalent to \(\rho_*(x) \rightarrow 0\).
we can prove by the similar way as in the proof of Lemma 2 that there exists $\bigcup_{\nu=1}^{\infty} a_{\nu} \in R$ in virtue of $(\rho.3)$.

Now, since
\[
\rho\left(\bigcup_{\nu \geq 1}^{\infty} a_{\nu}\right) \leq \sum_{\nu \geq 1}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}}
\]
holds for each $\nu \geq 1$, $\rho\left(\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\mu \geq \nu}^{\infty} a_{\mu}\right)\right) = 0$ and hence $(5.3)$ implies
\[
\bigcap_{\nu=1}^{\infty} \left(\bigcup_{\nu \geq 1}^{\infty} a_{\nu}\right) = 0.
\]
Thus we see that $\{a_{\nu}\}_{\nu \geq 1}$ is order-convergent to 0.

For any $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b_{\nu}) \to 0$, we can find a subsequence $\{b'_{\nu}\}_{\nu \geq 1}$ of $\{b_{\nu}\}_{\nu \geq 1}$ with $\rho(b'_{\nu}) \leq \frac{1}{2^{\nu}}$ ($\nu = 1, 2, \ldots$). Therefore we have $s\text{-lim}_{\nu \to \infty} b_{\nu} = 0$. Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition $(5.2)$ with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analougous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If $\rho^{*}$ satisfies (5.3) and $||\cdot||_{0}$ is complete and continuous, then (5.2) holds.

**References**


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