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ON F-NORMS OF QUASI-MODULAR SPACES

By

Shôzô KOSHI and Tetsuya SHIMOGAKI

§1. Introduction. Let $R$ be a universally continuous semi-ordered linear space (i.e. a conditionally complete vector lattice in Birkhoff's sense [1]) and $\rho$ be a functional which satisfies the following four conditions:

\[ (\rho.1) \quad 0 \leq \rho(x) = \rho(-x) \leq +\infty \quad \text{for all } x \in R; \]
\[ (\rho.2) \quad \rho(x+y) = \rho(x) + \rho(y) \quad \text{for any } x, y \in R \text{ with } x \perp y^{1)}; \]
\[ (\rho.3) \quad \text{If } \sum_{\lambda \in \Lambda} \rho(x_{\lambda}) < +\infty \quad \text{for a mutually orthogonal system } \{x_{\lambda}\}_{\lambda \in \Lambda}^{2)}, \]
\[ \text{there exists } x_{0} \in R \text{ such that } x_{0} = \sum_{\lambda \in \Lambda} x \text{ and } \rho(x_{0}) = \sum_{\lambda \in \Lambda} \rho(x_{\lambda}); \]
\[ (\rho.4) \quad \varlimsup_{\xi \rightarrow 0} \rho(\xi x) < +\infty \quad \text{for all } x \in R. \]

Then, $\rho$ is called a quasi-modular and $R$ is called a quasi-modular space.

In the previous paper [2], we have defined a quasi-modular space and proved that if $R$ is a non-atomic quasi-modular space which is semi-regular, then we can define a modular\(^3) m$ on $R$ for which every universally continuous linear functional\(^4) is continuous with respect to the norm defined by the modular\(^5) m$ [2; Theorem 3.1].

Recently in [6] J. Musielak and W. Orlicz considered a modular $\rho$ on a linear space $L$ which satisfies the following conditions:

\[ (A.1) \quad \rho(x) \geq 0 \quad \text{and} \quad \rho(x) = 0 \quad \text{if and only if } x = 0; \]
\[ (A.2) \quad \rho(-x) = \rho(x); \]
\[ (A.3) \quad \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \quad \text{for every } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1; \]
\[ (A.4) \quad \alpha_{n} \rightarrow 0 \text{ implies } \rho(\alpha_{n} x) \rightarrow 0 \quad \text{for every } x \in R; \]
\[ (A.5) \quad \text{for any } x \in L \text{ there exists } \alpha > 0 \text{ such that } \rho(\alpha x) < +\infty. \]

They showed that $L$ is a quasi-normed space with a quasi-norm $\| \cdot \|_{0}$ defined by the formula;

---

1) $x \perp y$ means $|x| \cap |y| = 0$.
2) A system of elements $\{x_{\lambda}\}_{\lambda \in \Lambda}$ is called mutually orthogonal, if $x_{\lambda} \perp x_{\gamma}$ for $\lambda \neq \gamma$.
3) For the definition of a modular, see [3].
4) A linear functional $f$ is called universally continuous, if $\inf_{\lambda \in \Lambda} f(a_{\lambda}) = 0$ for any $a_{\lambda} \downarrow_{\lambda \in \Lambda} 0$.
5) $R$ is called semi-regular, if for any $x \neq 0$, $x \in R$, there exists a universally continuous linear functional $f$ such that $f(x) \neq 0$.

---

This modular $\rho$ is a generalization of a modular $m$ in the sense of Nakano [3 and 4]. In the latter, there is assumed that $m(\xi x)$ is a convex function of $\xi \geq 0$ for each $x \in R$. 

for 3.

We obtain the results of the argument as seen by similar formula (1.1). Since a quasi-modular $\rho$ on $R$ does not satisfy the conditions (A.1), (A.2), (A.4) and (A.5) in general, as is seen by comparing the conditions: (A.1) $\sim$ (A.4) with those of $\rho$ [6], we can not apply the formula (1.1) directly to $\rho$ to obtain a quasi-norm. We shall show, however, that we can construct always a quasi-modular $\rho^*$ which satisfies (A.2) $\sim$ (A.5) on an arbitrary quasi-modular space $R$ in §2 (Theorems 2.1 and 2.2). Since $R$ may include a normal manifold $R_0=\{x: x\in R, \rho^*(\xi x)=0 \text{ for all } \xi\geq 0\}$ and we can not define a quasi-norm on $R_0$ in general, we have to exclude $R_0$ in order to proceed with the argument further. We shall prove in §3 that a quasi-norm $||\cdot||_0$ on $R_0^+$ defined by $\rho^*$ according to the formula (1.1) is semi-continuous, and in order that $R_0^+$ is an $F$-space with $||\cdot||_0$ (i.e. $||\cdot||_0$ is complete), it is necessary and sufficient that $\rho$ satisfies

\[(\rho.4')\quad \sup_{x\in R} \rho(\alpha x) < +\infty\]

(Theorem 3.2).

In §4, we shall show that we can define another quasi-norm $||\cdot||_1$ on $R_0^+$ which is equivalent to $||\cdot||_0$ such that $||x||_0 \leq ||x||_1 \leq 2||x||_0$ holds for every $x\in R_0^+$ (Formulas (4.1) and (4.3)). $||\cdot||_1$ has a form similar to that of the first norm (due to I. Amemiya) of (convex) modular in the sense of Nakano [4 ; §83]. At last in §5 we shall add shortly the supplementary results concerning the relations between $||\cdot||_0$-convergence and order-convergence. The matter does not essentially differ from the case of the (convex) modular on semi-ordered linear spaces and the results stated in §5 are already known in those cases [8].

Throughout this paper $R$ denotes a universally continuous semi-ordered linear space and $\rho$ a quasi-modular defined on $R$. For any $p\in R$, $[p]$ is a projector: $[p]x=\bigcup_{n=1}^{\infty} (n|p|\cap x)$ for all $x\geq 0$ and $1-[p]$ is a projection operator onto the normal manifold $N=\{p\}^1$, that is, $x=[p]x+(1-[p])x$.

---

6) This quasi-norm was first considered by S. Mazur and W. Orlicz [5] and discussed by several authors [6 or 7].
§2. The conversion of a quasi-modular. From the definition of a quasi-modular in §1, the following lemma is immediately deduced.

**Lemma 1.** For any quasi-modular $\rho$, we have

(2.1) $\rho(0)=0$;
(2.2) $\rho([p]x)\leq \rho(x)$ for all $p, x \in R$;
(2.3) $\rho([p]x)=\sup_{\lambda \in \Lambda} \rho([p_{\lambda}]x)$ for any $[p_{\lambda}]_{\lambda \in \Lambda}[p]$.

In the argument below, we have to use the additional property of $\rho$:

(\rho.5) $\rho(x)\leq \rho(y)$ if $|x|\leq |y|$, $x, y \in R$,

which is not valid for an arbitrary $\rho$ in general.

The next theorem, however, shows that we may suppose without loss of generality that a quasi-modular $\rho$ satisfies $(\rho.5)$.

**Theorem 2.1.** Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then there exists a quasi-modular $\rho'$ for which $(\rho.5)$ is valid.

**Proof.** We put for every $x \in R$,

(2.4) $\rho'(x)=\sup_{0\leq |y|\leq |x|} \rho(y)$.

It is clear that $\rho'$ satisfies the conditions $(\rho.1)$, $(\rho.2)$ and $(\rho.5)$.

Let $\{x_{\lambda}\}_{\lambda \in \Lambda}$ be an orthogonal system such that $\sum_{\lambda \in \Lambda} \rho'(x_{\lambda})<+\infty$, then

$$\sum_{\lambda \in \Lambda} \rho(x_{\lambda})<+\infty$$

because

$$\rho(x)\leq \rho'(x)$$

for all $x \in R$.

We have

$$x_{0}=\sum_{\lambda \in \Lambda} x_{\lambda} \in R$$

and

$$\rho(x_{0})=\sum_{\lambda \in \Lambda} \rho(x_{\lambda})$$

in virtue of $(\rho.3)$.

For such $x_{0}$,

$$\rho'(x_{0})=\sup_{0\leq |y|\leq |x_{0}|} \rho(y)=\sup_{0\leq |y|\leq |x_{0}|} \sum_{\lambda \in \Lambda} \rho([x_{\lambda}]y)$$

$$=\sum_{\lambda \in \Lambda} \sup_{0\leq |y|\leq |x_{\lambda}|} \rho([x_{\lambda}]y)=\sum_{\lambda \in \Lambda} \rho'(x_{\lambda})$$

holds, i.e. $\rho'$ fulfils $(\rho.3)$.

If $\rho'$ does not fulfil $(\rho.4)$, we have for some $x_{0} \in R$,

$$\rho'(\frac{1}{n}x_{0})=+\infty$$

for all $n \geq 1$.

By $(\rho.2)$ and $(\rho.4)$, $x_{0}$ can not be written as $x_{0}=\sum_{\nu=1}^{\kappa} \xi_{\nu} e_{\nu}$, where $e_{\nu}$ is an atomic element for each $\nu$ with $1 \leq \nu \leq \kappa$, namely, we can decompose $x_{0}$ into
an infinite number of orthogonal elements. First we decompose into
\[ x_0 = x_1 + x_1', \quad x_1 \perp x_1', \]
where \( \rho'(\frac{1}{\nu}x_1) = +\infty \) (\( \nu = 1, 2, \cdots \)) and \( \rho'(x_1') > 1 \). For the definition of \( \rho' \), there exists \( 0 \leq y_1 \leq |x_1'| \) such that \( \rho(y_1) \geq 1 \). Next we can also decompose \( x_1 \) into
\[ x_1 = x_2 + x_2', \quad x_2 \perp x_2', \]
where
\[ \rho'(\frac{1}{\nu}x_2) = +\infty \) (\( \nu = 1, 2, \cdots \))
and
\[ \rho'(\frac{1}{2}x_2') > 2. \]

There exists also \( 0 \leq y_2 \leq |x_2'| \) such that \( \rho(\frac{1}{2}y_2) \geq 2 \). In the same way, we can find by induction an orthogonal sequence \( \{y_\nu\}_{\nu=1,2}, \ldots \) such that
\[ \rho(\frac{1}{\nu}y_\nu) \geq \nu \]
and
\[ 0 \leq |y_\nu| \leq |x| \]
for all \( \nu \geq 1 \).

Since \( \{y_\nu\}_{\nu=1,2}, \ldots \) is order-bounded, we have in virtue of (2.3)
\[ y_0 = \sum_{\nu=1}^{\infty} y_\nu \in R \]
and
\[ \rho(\frac{1}{\nu}y_0) \geq \rho(\frac{1}{\nu}y_\nu) \geq \nu, \]
which contradicts (\( \rho.4 \)). Therefore \( \rho' \) has to satisfy (\( \rho.4 \)). Q.E.D.

Hence, in the sequel, we denote by \( \rho' \) a quasi-modular defined by the formula (2.4).

If \( \rho \) satisfies (\( \rho.5 \)), \( \rho \) does also (A.3) in §1:
\[ \rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \]
for \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \).

Because, putting \( [p] = [(|x| - |y|)^+] \), we obtain
\[ \rho(\alpha x + \beta y) \leq \rho(\alpha |x| + \beta |y|) \]
\[ \leq \rho(\alpha \lceil p \rceil |x| + \alpha (1 - \lceil p \rceil) |y| + \beta \lceil p \rceil |x| + (1 - \lceil p \rceil) \beta |y|) \]
\[ = \rho(\lceil p \rceil |x| + (1 - \lceil p \rceil) |y|) \]
\[ = \rho(\lceil p \rceil x) + \rho((1 - \lceil p \rceil) y) \]
\[ \leq \rho(x) + \rho(y) \]

Remark 1. As is shown above, the existence of \( \rho' \) as a quasi-modular depends essentially on the condition \((\rho.4)\). Thus, in the above theorems, we cannot replace \((\rho.4)\) by the weaker condition:

\((\rho.4'')\) for any \( x \in R \), there exists \( \alpha \geq 0 \) such that \( \rho(\alpha x) < +\infty \).

In fact, the next example shows that there exists a functional \( \rho_0 \) on a universally continuous semi-ordered linear space satisfying \((\rho.1)\), \((\rho.2)\), \((\rho.3)\) and \((\rho.4'')\), but does not \((\rho.4)\). For this \( \rho_0 \), we obtain

\[ \rho_0(x) = \sup_{|y| \leq |x|} \rho_0(y) = +\infty \]

for all \( x \neq 0 \).

Example. \( L_1[0,1] \) is the set of measurable functions \( x(t) \) which are defined in \([0,1]\) with

\[ \int_0^1 |x(t)| \, dt < +\infty . \]

Putting

\[ \rho_0(x) = \rho_0(x(t)) = \int_0^1 |x(t)| \, dt + \sum_{i=1}^{\infty} i \text{ mes} \left\{ t : x(t) = \frac{1}{i} \right\} , \]

we have an example satisfying the above conditions.

In order to define the quasi-norm, we need one more additional condition: \((A.4)\), namely,

\[ \lim_{\xi \to 0} \rho(\xi x) = 0 \]

for all \( x \in R \).

A quasi-modular space becomes, as is shown below, always a quasi-normed space excluding the trivial part, but not an \( F \)-space in general. This fact is based upon the following Theorems 2.2 and 2.3.

Theorem 2.2. Let \( \rho \) be a quasi-modular on \( R \). We can find a functional \( \rho^* \) which satisfies \((\rho.1) \sim (\rho.6)\) except \((\rho.3)\).

Proof. In virtue of Theorem 2.1, there exists a quasi-modular \( \rho' \) which satisfies \((\rho.5)\). Now we put

\[ d(x) = \lim_{\xi \to 0} \rho'(\xi x) . \]

It is clear that \( 0 \leq d(x) = d(|x|) < +\infty \) for all \( x \in R \) and
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\[ d(x + y) = d(x) + d(y) \quad \text{if } x \perp y. \]

Hence, putting

\[ (2.6) \quad \rho^*(x) = \rho'(x) - d(x) \quad (x \in \mathbb{R}). \]

we can see easily that \((\rho.1), (\rho.2), (\rho.4)\) and \((\rho.6)\) hold true for \(\rho^*\), since

\[ d(x) \leq \rho'(x) \]

and

\[ d(\alpha x) = d(x) \]

for all \(x \in \mathbb{R}\) and \(\alpha > 0\).

We need to prove that \((\rho.5)\) is true for \(\rho^*\). First we have to note

\[ (2.7) \quad \inf_{\lambda \in \Lambda} d([p_\lambda]x) = 0 \]

for any \([p_\lambda] \downarrow_{\lambda \in \Lambda} 0\). In fact, if we suppose the contrary, we have

\[ \inf_{\lambda \in \Lambda} d([p_\lambda]x_0) \geq \alpha > 0 \]

for some \([p_\lambda] \downarrow_{\lambda \in \Lambda} 0\) and \(x_0 \in \mathbb{R}\).

Hence,

\[ \rho'(\frac{1}{\nu}[p_\lambda]x_0) \geq d([p_\lambda]x_0) \geq \alpha \]

for all \(\nu \geq 1\) and \(\lambda \in \Lambda\). Thus we can find a subsequence \(\{\lambda_n\}_{n \geq 1}\) of \(\{\lambda\}_{\lambda \in \Lambda}\) such that

\[ [p_{\lambda_n}] \geq [p_{\lambda_{n+1}}] \]

and

\[ \rho'(\frac{1}{n}(p_{\lambda_n} - p_{\lambda_{n+1}})x_0) \geq \frac{\alpha}{2} \]

for all \(n \geq 1\) in virtue of \((\rho.2)\) and \((2.3)\). This implies

\[ \rho'(\frac{1}{n}x_0) \geq \sum_{m \geq n} \rho'(\frac{1}{m}(p_{\lambda_m} - p_{\lambda_{m+1}})x_0) = +\infty, \]

which is inconsistent with \((\rho.4)\). Secondly we shall prove

\[ (2.8) \quad d(x) = d(y), \quad \text{if } [x] = [y]. \]

We put \([p_n] = [(|x| - n|y|)^+]\) for \(x, y \in \mathbb{R}\) with \([x] = [y]\) and \(n \geq 1\). Then, \([p_n] \downarrow_{n=1}^{\infty} 0\) and \(\inf_{n=1,2,...} d([p_n]x) = 0\) by \((2.7)\). Since \((1-[p_n])n \geq (1-[p_n])|x|\) and

\[ d(\alpha x) = d(x) \]

for \(\alpha > 0\) and \(x \in \mathbb{R}\), we obtain
\[
d(x) = d([p_n]x) + d((1-[p_n])x) \\
\leq d([p_n]x) + d(n(1-[p_n])y) \\
\leq d([p_n]x) + d(y).
\]

As \( n \) is arbitrary, this implies
\[
d(x) \leq \inf_{n=1, 2, \ldots} d([p_n]x) + d(y),
\]
and also \( d(x) \leq d(y) \). Therefore we conclude that (2.8) holds.

If \( |x| \geq |y| \), then
\[
\rho^*(x) = \rho^*(y) + \rho^*(([x] - [y])x) \\
= \rho'(y) - d(y) + \rho^*(([x] - [y])x) \\
\geq \rho^*(y).
\]

Thus \( \rho^* \) satisfies (\( \rho.5 \)).

Theorem 2.3. \( \rho^* \) (which is constructed from \( \rho \) according to the formulas (2.4), (2.5) and (2.6)) satisfies (\( \rho.3 \)) (that is, \( \rho^* \) is a quasi-modular), if and only if \( \rho \) satisfies

(\( \rho.4' \)) \[ \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K < +\infty. \]

Proof. Let \( \rho \) satisfy (\( \rho.4 \)). We need to prove
(2.9) \[ \sup_{x \in R} d(x) = \sup_{x \in R} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = K' < +\infty, \]
where
\[
\rho'(x) = \sup_{0 \leq |y| \leq |x|} \rho(y).
\]

Since \( \rho' \) is also a quasi-modular, Lemma 2 in [2] or [8] can be applicable, if we put \( n_0(x) = \rho(x) \) and \( n_\nu(x) = \rho'(\frac{1}{\nu}x) \) for \( \nu \geq 1 \) and \( x \in R \). Hence we can find positive numbers \( \epsilon, \gamma \), a natural number \( \nu_0 \) and a finite dimensional normal manifold \( N_0 \) such that \( x \in N_0^+ \) with
\[
\rho(x) \leq \epsilon \quad \text{implies} \quad \rho'(\frac{1}{\nu_0}x) \leq \gamma.
\]

In \( N_0 \), we have obviously
\[
\sup_{x \in N_0} \{ \lim_{\xi \to 0} \rho'(\xi x) \} = \gamma_0 < +\infty.
\]

If \( \epsilon \leq 2K \), for any \( x_0 \in N_0^+ \), we can find \( \alpha_0 > 0 \) such that \( \rho(\alpha x_0) \leq 2K \) for all \( 0 \leq \alpha \leq \alpha_0 \) by (\( \rho.4' \)), and hence there exists always an orthogonal decomposition such that
\[ \alpha_0 x_0 = x_1 + \cdots + x_n + y_1 + \cdots + y_m + z \]
where \( \frac{\varepsilon}{2} < \rho(x_i) \leq \varepsilon \) (i = 1, 2, \ldots, n), \( y_j \) is an atomic element with \( \rho(y_j) > \varepsilon \) for every \( j = 1, 2, \ldots, m \) and \( \rho(z) \leq \frac{\varepsilon}{2} \). From above, we get \( n \leq \frac{4K}{\varepsilon} \) and \( m \leq \frac{2K}{\varepsilon} \). This yields

\[
\rho\left(\frac{1}{\nu_0} \alpha_0 x_0\right) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\nu_0} x_i\right) + \sum_{j=1}^{m} \rho\left(y_j\right) + \rho\left(\frac{z}{\nu_0}\right)
\]
\[
\leq n\gamma + \sum_{j=1}^{m} \rho\left(y_j\right) + \rho\left(\frac{z}{\nu_0}\right)
\]
\[
\leq \frac{4K}{\varepsilon} \gamma + \frac{2K}{\varepsilon} \left\{ \sup_{0 \leq a \leq a_0} \rho\left(\alpha x\right) \right\} + \gamma.
\]

Hence, we obtain

\[
\lim_{\xi \to 0} \rho\left(\xi x_0\right) \leq \rho\left(\frac{\alpha_0}{\nu_0} x_0\right) \leq \left(\frac{4K + \varepsilon}{\varepsilon}\right) \gamma + \left(\frac{4K^2}{\varepsilon}\right)
\]
in case of \( \varepsilon \leq 2K \). If \( 2K \leq \varepsilon \), we have immediately for \( x \in N_0^\perp \)

\[
\lim_{\xi \to 0} \rho\left(\xi x\right) \leq \gamma.
\]

Therefore, we obtain

\[
\sup_{x \in R} \left\{ \lim_{\xi \to 0} \rho\left(\xi x\right) \right\} \leq \gamma'
\]

where

\[
\gamma' = \frac{4K + \varepsilon}{\varepsilon} + \frac{4K^2}{\varepsilon} + \gamma_0.
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for arbitrary \( \lambda_1, \ldots, \lambda_k \in A \), we have

\[
\sum_{i=1}^{k} d(x_{i_\nu}) = d(\sum_{i=1}^{k} x_{i_\nu}) = \lim_{\xi \to 0} \rho\left(\xi \sum_{i=1}^{k} x_{i_\nu}\right) \leq \gamma',
\]

which implies \( \sum_{i \in A} d(x_i) \leq \gamma' \). It follows that

\[
\sum_{i \in A} \rho^*(x_i) = \sum_{i \in A} \rho^*(x_i) + \sum_{i \in A} d(x_i) < +\infty,
\]

which implies \( x_0 = \sum_{i \in A} x_i \in R \) and \( \sum_{i \in A} \rho^*(x_i) = \rho^*(x_0) \) by \( (\rho.4) \) and \( (2.7) \). Therefore \( \rho^* \) satisfies \( (\rho.3) \).

On the other hand, suppose that \( \rho^* \) satisfies \( (\rho.3) \) and \( \sup_{x \in R} d(x) = +\infty \). Then we can find an orthogonal sequence \( \{x_i\}_{i \geq 1} \) such that

\[
\sum_{i=1}^{n} d(x_i) = d(\sum_{i=1}^{n} x_i) \geq \mu
\]

Let \( \{x_i\}_{i \in A} \) be an orthogonal system with \( \sum_{i \in A} \rho^*(x_i) < +\infty \). Then for
for all $\mu \geqq 1$ in virtue of (2.8) and the orthogonal additivity of $d$. Since
\[ \lim_{\xi \to 0} \rho^*(\xi x) = 0, \]
there exists $\{\alpha_\nu\}_{\nu \geqq 1}$ with $0 < \alpha_\nu$ ($\nu \geqq 1$) and $\sum_{\nu=1}^{\infty} \rho^*(\alpha_\nu x_\nu) < + \infty$. It follows that $x_0 = \sum_{\nu=1}^{\infty} \alpha_\nu x_\nu \in R$ and $d(x_0) = \sum_{\nu=1}^{\infty} d(\alpha_\nu x_\nu)$ from (\rho.3). For such $x_0$, we have for every $\xi \geqq 0$,
\[ \rho'(\xi x_0) = \sum_{\nu=1}^{\infty} \rho'(\xi \alpha_\nu x_\nu) \geqq \sum_{\nu=1}^{\infty} d(x_\nu) = + \infty, \]
which is inconsistent with (\rho.4). Therefore we have
\[ \sup_{x \in R} \left( \lim_{\xi \to 0} \rho(\xi x) \right) \leqq \sup_{x \in R} d(x) < + \infty. \]
Q.E.D.

§3. Quasi-norms. We denote by $R_0$ the set:
\[ R_0 = \{ x : x \in R, \ \rho^*(nx) = 0 \text{ for all } n \geqq 1 \}, \]
where $\rho^*$ is defined by the formula (2.6). Evidently $R_0$ is a semi-normal manifold$^7$ of $R$. We shall prove that $R_0$ is a normal manifold of $R$. In fact, let $x = \bigcup_{\lambda \in \Lambda} x_{\lambda}$ with $R_0 \ni x_{\lambda} \geqq 0$ for all $\lambda \in \Lambda$. Putting
\[ [p_{n,\lambda}] = [(2nx_{\lambda} - nx)^+] \]
we have
\[ [p_{n,\lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad 2n[p_{n,\lambda}]x_{\lambda} \geqq [p_{n,\lambda}]nx, \]
which implies $\rho^*(n[p_{n,\lambda}]x) = 0$ and $\sup_{\lambda \in \Lambda} \rho^*(n[p_{n,\lambda}]x) = \rho^*(nx) = 0$. Hence, we obtain $x \in R_0$, that is, $R_0$ is a normal manifold of $R$.

Therefore, $R$ is orthogonally decomposed into
\[ R = R_0 \oplus R_0^\perp. \]

In virtue of the definition of $\rho^*$, we infer that for any $p \in R_0$, $[p]R_0$ is universally complete, i.e. for any orthogonal system $\{x_{\lambda}\}_{\lambda \in \Lambda}(x_{\lambda} \in [p]R_0)$, there exists $x_0 = \sum_{\lambda \in \Lambda} x_{\lambda} \in [p]R$. Hence we can also verify without difficulty that $R_0$ has no universally continuous linear functional except 0, if $R_0$ is non-atomic. When $R_0$ is discrete, it is isomorphic to $S(\Lambda)^9$-space. With respect to such a universally complete space $R_0$, we can not always construct a linear metric topology on $R_0$, even if $R_0$ is discrete.

In the following, therefore, we must exclude $R_0$ from our consideration. Now we can state the theorems which we aim at.

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7) A linear manifold $S$ is said to be semi-normal, if $a \in S, \ |b| \leqq |a|, b \in R \implies b \in S$. Since $R$ is universally continuous, a semi-normal manifold $S$ is normal if and only if $\cup_{\lambda \in \Lambda} x_{\lambda} \in S$.

8) This means that $x \in R$ is written by $x = y + z$, $y \in R_0$ and $z \in R_0^\perp$.

9) $S(\Lambda)$ is the set of all real functions defined on $\Lambda$. 

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Theorem 3.1. Let $R$ be a quasi-modular space. Then $R^\perp_0$ becomes a quasi-normed space with a quasi-norm $\| \cdot \|_0$ which is semi-continuous, i.e.,

$$\sup_{x, \lambda \in A} \| x \|_0 = \| x \|_0$$

for any $0 \leq x, \lambda \in A$. $\| x \|_0$ is semi-continuous, i.e.

$$\sup_{\lambda \in \Lambda} \| x_\lambda \|_0 = \| x \|_0$$

for any $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$. Proof. In virtue of Theorems 2.1 and 2.2, $\rho^*$ satisfies $(\rho.1) \sim (\rho.6)$ except $(\rho.3)$. Now we put

$$(3.1) \quad ||x||_0 = \inf \{ \xi ; \rho^*(\frac{1}{\xi}x) \leq \xi \}.$$ 

Then,

i) $0 \leq ||x||_0 = ||-x||_0 < \infty$ and $||x||_0 = 0$ is equivalent to $x = 0$; follows from $(\rho.1)$, $(\rho.6)$, (2.1) and the definition of $R^\perp_0$.

ii) $\lim_{\alpha_0 \to 0} ||\alpha x||_0 = 0$ and $\lim_{||x||_0 \to 0} ||\alpha x||_0 = 0$; is a direct consequence of $(\rho.5)$. At last we shall prove that $|| \cdot ||_0$ is semi-continuous. From ii) and iii), it follows that $\lim_{a \to \alpha_0} ||ax||_0 = ||ax||_0$ for all $x \in R^\perp_0$ and $\alpha_0 \geq 0$. If $x \in R^\perp_0$ and $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$, for any positive number $\xi$ with $||[p]x||_0 > \xi$ we have $\rho^*(\frac{1}{\xi}[p]x) > \xi$, which implies $\sup_{\lambda \in \Lambda} \rho^*(\frac{1}{\xi}[p_\lambda]x) \geq \xi$ and hence $\sup_{\lambda \in \Lambda} ||p_\lambda||_0 \geq \xi$. Thus we obtain

$$\sup_{\lambda \in \Lambda} ||p_\lambda||_0 = ||[p]x||_0$$

if $[p_\lambda] \uparrow_{\lambda \in \Lambda} [p]$.

Let $0 \leq x, \lambda \in A$. Putting

$$[p_{n, \lambda}] = \left[ x_\lambda - \left( 1 - \frac{1}{n} \right)x \right]$$

we have

$$[p_{n, \lambda}] \uparrow_{\lambda \in \Lambda} [x] \quad \text{and} \quad [p_{n, \lambda}] x_\lambda \geq [p_{n, \lambda}] \left( 1 - \frac{1}{n} \right)x \quad (n \geq 1).$$

As is shown above, since

$$\sup_{\lambda \in \Lambda} ||[p_{n, \lambda}] x_\lambda||_0 \geq \sup_{\lambda \in \Lambda} ||[p_{n, \lambda}] \left( 1 - \frac{1}{n} \right)x||_0 = \left( 1 - \frac{1}{n} \right)||x||_0,$$

we have

$$\sup_{\lambda \in \Lambda} ||x||_0 \geq \left( 1 - \frac{1}{n} \right)||x||_0$$

and also $\sup_{\lambda \in \Lambda} ||x_\lambda||_0 \geq ||x||_0$. As the converse inequality is obvious by iv), $|| \cdot ||_0$ is semi-continuous. Q.E.D.

Remark 2. By the definition of (3.1), we can see easily that $\lim ||x_n||_0 = 0$ if and only if $\lim \rho(\xi x_n) = 0$ for all $\xi \geq 0$. 

In order to prove the completeness of quasi-norm $||\cdot||_0$, the next Lemma is necessary.

**Lemma 2.** Let $p_{n,v}, x_v \geq 0$ and $a \geq 0$ $(n, v=1, 2, \cdots)$ be the elements of $R_0^+$ such that

\[(3.2) \quad [p_{n,v}] \uparrow_{v=1}^{\infty} [p_n] \text{ with } \bigcap_{n=1}^{\infty} [p_n] a = [p_0] a \neq 0; \]

\[(3.3) \quad [p_{n,v}] x_v \geq n [p_{n,v}] a \quad \text{for all } n, v \geq 1.\]

Then \{x_v\}_{v \geq 1} is not a Cauchy sequence of $R_0^+$ with respect to $||\cdot||_0$.

**Proof.** We shall show that there exist a sequence of projectors $[q_m]_{m=1}^{\infty} (m \geq 1)$ and sequences of natural numbers $\nu_m, n_m$ such that

\[(3.4) \quad ||[q_m] a||_0 > \frac{\delta}{2} \quad \text{and} \quad [q_m] x_{\nu_m} \geq n_m [q_m] a \quad (m=1, 2, \cdots)\]

and

\[(3.5) \quad n_m [q_m] a \geq n_{m+1} [q_m] a \quad \text{and} \quad n_{m+1} > n_m \quad (m=2, 3, \cdots),\]

where $\delta = ||[p_0] a||_0$.

In fact, we put $n_1 = 1$. Since $[p_{1,v}] [p_0] \uparrow_{v=1}^{\infty} [p_0]$ and $||\cdot||_0$ is semi-continuous, we can find a natural number $\nu_1$ such that

\[||[p_{1,v}] [p_0] a||_0 > \frac{||[p_0] a||_0}{2} = \frac{\delta}{2}.\]

We put $[q_1] = [p_{1,v}] [p_0]$. Now, let us assume that $[q_m], \nu_m, n_m (m=1, 2, \cdots, k)$ have been taken such that (3.4) and (3.5) are satisfied.

Since $[(n a - x_{\nu_k})^+] \uparrow_{n=1}^{\infty} [a]$ and $||[q_k] a||_0 > \frac{\delta}{2}$, there exists $n_{k+1}$ with

\[|| (n_{k+1} a - x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2}.\]

For such $n_{k+1}$, there exists also a natural number $\nu_{k+1}$ such that

\[|| [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k] a||_0 > \frac{\delta}{2}.\]

in virtue of (3.2) and semi-continuity of $||\cdot||_0$. Hence we can put

\[ [q_{k+1}] = [p_{n_{k+1}, \nu_{k+1}}] (n_{k+1} a - x_{\nu_k})^+ [q_k], \]

because

\[ [q_{k+1}] \leq [q_k], \quad ||[q_{k+1}] a|| > \frac{\delta}{2}, \quad [q_{k+1}] x_{\nu_{k+1}} \geq n_{k+1} [q_{k+1}] a \]

by (3.3) and $[q_{k+1}] n_{k+1} a \geq [q_{k+1}] a$ by (3.5).

For the sequence thus obtained, we have for every $k \geq 3$
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$\|x_{\nu_{k+1}}-x_{\nu_{k-1}}\|_0 \geqq \|q_{k+1}(x_{\nu_{k+1}}-x_{\nu_{k-1}})\|_0 \geqq \|n_{k+1} q_{k+1} a - n_{k} q_{k+1} a\|_0 \geqq \|q_{k+1} a_0\|_0 \geqq \frac{\delta}{2},$

since $[q_{k+1}] \leqq [q_k] \leqq [n_{k} a - x_{\nu-1}^+]$ implies $[q_{k+1}] n_{k} a \geqq [q_{k+1}] x_{\nu_{k-1}}$ by (3.4).

It follows from the above that $\{x_{\nu}\}_{\nu \geqq 1}$ is not a Cauchy sequence.

Theorem 3.2. Let $R$ be a quasi-modular space with quasi-modular $\rho$. Then $R_0^+$ is an F-space with $\| \cdot \|_0$ if and only if $\rho$ satisfies (\rho.4').

Proof. If $\rho$ satisfies (\rho.4'), $\rho^*$ is a quasi-modular which fulfills also (\rho.5) and (\rho.6) in virtue of Theorem 2.3. Since $\| x \|_0 \left(= \inf \{ \xi ; \rho^*(x/\xi) \leqq \xi \} \right)$ is a quasi-norm on $R_0^+$, we need only to verify completeness of $\| \cdot \|_0$. At first let $\{x_{\nu}\}_{\nu \geqq 1} \subset R_0^+$ be a Cauchy sequence with $0 \leqq x_{\nu} \uparrow_{\nu=1,2,\ldots}$. Since $\rho^*$ satisfies (\rho.3), there exists $0 \leqq x_0 \in R_0^+$ such that $x_0 = \bigcup_{\nu=1}^{\infty} x_{\nu}$, as is shown in the proof of Theorem 2.3.

Putting $[p_{n,v}] = [x_{\nu} - nx_0^+]$ and $\bigcup_{\nu=1}^{\infty} [p_{n,v}] = [p_n]$, we obtain
\begin{equation}
[p_{n,v}] x_{\nu} \geqq n [p_{n,v}] x_0 \quad \text{for all } n, \nu \geqq 1 \quad \text{and} \quad [p_n]_{\nu=1}^{\infty} = 0.
\end{equation}

Since $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we have in virtue of Lemma 2, $\bigcap_{n=1}^{\infty} [p_n] = 0$, that is, $\bigcup_{n=1}^{\infty} ([x_0] - [p_n]) = [x_0]$. And
\begin{equation}
(1 - [p_{n,v}]) \geqq (1 - [p_n]) \quad (n, \nu \geqq 1)
\end{equation}
implies
\begin{equation}
n(1 - [p_n]) x_0 \geqq (1 - [p_n]) x_{\nu} \geqq 0.
\end{equation}

Hence we have
\begin{equation}
y_n = \bigcup_{\nu=1}^{\infty} (1 - [p_n]) x_{\nu} \in R_0^+,
\end{equation}
because $R_0^+$ is universally continuous. As $\{x_{\nu}\}_{\nu \geqq 1}$ is a Cauchy sequence, we obtain from the triangle inequality of $\| \cdot \|_0$
\begin{equation}
\gamma = \sup_{\nu \geqq 1} \| x_{\nu} \|_0 < +\infty,
\end{equation}
which implies
\begin{equation}
\| y_n \|_0 = \sup_{\nu \geqq 1} \| (1 - [p_n]) x_{\nu} \|_0 \leqq \gamma
\end{equation}
for every $n \geqq 1$ by semi-continuity of $\| \cdot \|_0$. We put $z_1 = y_1$ and $z_n = y_n - y_{n-1}$ \((n \geqq 2)\). It follows from the definition of $y_n$ that $\{z_{\nu}\}_{\nu \geqq 1}$ is an orthogonal sequence with $\| \sum_{\nu=1}^{n} z_{\nu} \|_0 = \| y_n \|_0 \leqq \gamma$. This implies
$$\sum_{\nu=1}^{n} \rho^* \left( \frac{z_{\nu}}{1+\gamma} \right) = \rho^* \left( \frac{y_n}{1+\gamma} \right) \leq \gamma$$
for all \( n \geq 1 \) by the formula (3.1). Then (\( \rho.3 \)) assures the existence of \( z = \sum_{\nu=1}^{\infty} z_{\nu} = \bigcup_{\nu=1}^{\infty} y_{\nu} \). This yields \( z = \bigcup_{\nu=1}^{\infty} x_{\nu} \). Truly, it follows from

\( z = \bigcup_{n=1}^{\infty} y_{n} = \bigcup_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} (1-[p_{n}])x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{0}x_{\nu} = \bigcup_{\nu=1}^{\infty} x_{\nu} \).

By semi-continuity of \( || \cdot ||_{0} \), we have

\( ||z-x_{\nu}||_{0} \leq \sup_{\mu \geq \nu} ||x_{\mu}-x_{\nu}||_{0} \)

and furthermore \( \lim_{\mu \rightarrow \infty} ||z-x_{\nu}||_{0} = 0 \). Secondy let \( \{x_{\nu}\}_{\nu \geq 1} \) be an arbitrary Cauchy sequence of \( R_{0}^{+} \). Then we can find a subsequence \( \{y_{\nu}\}_{\nu \geq 1} \) of \( \{x_{\nu}\}_{\nu \geq 1} \) such that

\( ||y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{\nu-1}} \) for all \( \nu \geq 1 \).

This implies

\( ||\sum_{\nu=m}^{n} y_{\nu+1}-y_{\nu}||_{0} \leq \frac{1}{2^{m-1}} \) for all \( n \geq m \geq 0 \).

Putting \( z_{n} = \sum_{\nu=1}^{n} |y_{\nu+1}-y_{\nu}| \), we have a Cauchy sequence \( \{z_{n}\}_{n \geq 1} \) with \( 0 \leq z_{n} \uparrow_{n=1}^{\infty} \). Then by the fact proved just above,

\( z_{0} = \bigcup_{n=1}^{\infty} z_{n} = \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \in R_{0}^{+} \) and \( \lim_{n \rightarrow \infty} ||z_{0}-z_{n}||_{0} = 0 \).

Since \( \sum_{\nu=1}^{\infty} |y_{\nu+1}-y_{\nu}| \) is convergent, \( y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) \) is also convergent and

\( ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - y_{n}||_{0} = ||\sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu})||_{0} = ||z_{0} - z_{n}||_{0} \rightarrow 0 \).

Since \( \{y_{\nu}\}_{\nu \geq 1} \) is a subsequence of the Cauchy sequence \( \{x_{\nu}\}_{\nu \geq 1} \), it follows that

\( \lim_{\nu \rightarrow \infty} ||y_{1} + \sum_{\nu=1}^{\infty} (y_{\nu+1}-y_{\nu}) - x_{\nu}||_{0} = 0 \).

Therefore \( || \cdot ||_{0} \) is complete in \( R_{0}^{+} \), that is, \( R_{0}^{+} \) is an F-space with \( || \cdot ||_{0} \).

Conversely if \( R_{0}^{+} \) is an F-space, then for any orthogonal sequence \( \{x_{\nu}\}_{\nu \geq 1} \in R_{0}^{+} \), we have \( \sum_{\nu=1}^{\infty} \alpha_{\nu} x_{\nu} \in R_{0}^{+} \) for some real numbers \( \alpha_{\nu} > 0 \) (for all \( \nu \geq 1 \)). Hence we can see that \( \sup_{x \in R} d(x) < +\infty \) by the same way applied in Theorem 2.1. It follows that \( \rho \) must satisfy (\( \rho.4' \)).

Q.E.D.

Since \( R_{0} \) contains a normal manifold which is universally complete, if \( R_{0} \) is complete, we can conclude directly from Theorems 3.1 and 3.2.
Corollary. Let $R$ be a quasi-modular space which includes no universally complete normal manifold. Then $R$ becomes a quasi-normed space with a quasi-norm $||\cdot||_0$ defined by (3.1) and $R$ becomes an F-space with $||\cdot||_0$ if and only if $\rho$ fulfills ($\rho'$).

§4. Another Quasi-norm. Let $L$ be a modular space in the sense of Musielak and Orlicz (§1). Here we put for $x \in L$

\[ ||x||_i = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho(\xi x) \right\}^{10} \]

and show that $||\cdot||_i$ is also a quasi-norm on $L$ and

\[ ||x||_0 \leq ||x||_i \leq 2 ||x||_0 \]

for all $x \in L$ hold, where $||\cdot||_0$ is a quasi-norm defined by the formula (1.1).

From (A.1), (A.2) and (A.5), it follows that $0 \leq ||x||_i = ||-x||_i < +\infty$ $(x \in L)$ and that $||x||_i = 0$ is equivalent to $x = 0$. Since $\alpha_n \downarrow 0$ implies $\lim_{n \to \infty} \rho(\alpha_n x) = 0$ for each $x \in L$ and $\lim ||x_n||_i = 0$ implies $\lim_{n \to \infty} \rho(\xi x_n) = 0$ for all $\xi \geq 0$, we obtain that $\lim ||\alpha_n x||_i = 0$ for all $\alpha_n \downarrow 0$ and that $\lim ||x_n||_i = 0$ implies $\lim ||\alpha x_n||_i = 0$ for all $\alpha > 0$. If $||x||_i < \alpha$ and $||y||_i < \beta$, there exist $\xi, \eta > 0$ such that

$$\frac{1}{\xi} + \rho(\xi x) < \alpha \quad \text{and} \quad \frac{1}{\eta} + \rho(\eta y) < \beta.$$ \n
This yields

$$||x + y|| \leq \frac{\xi + \eta}{\xi \eta} + \rho\left(\frac{\xi \eta}{\xi + \eta}(x + y)\right) = \frac{1}{\xi} + \frac{1}{\eta} + \rho\left(\frac{\eta}{\xi + \eta}(\xi x) + \frac{\xi}{\xi + \eta}(\eta y)\right) \leq \frac{1}{\xi} + \rho(\xi x) + \frac{1}{\eta} + \rho(\eta y) < \alpha + \beta,$$

in virtue of (A.3). Therefore $||x + y||_i \leq ||x||_i + ||y||_i$ holds for any $x, y \in L$ and $||\cdot||_i$ is a quasi-norm on $L$. If $\xi \rho(\xi x) \leq 1$ for some $\xi > 0$ and $x \in L$, we have $\rho(\xi x) \leq \frac{1}{\xi}$ and hence

$$\frac{1}{\xi} \leq \frac{1}{\xi} + \rho(\xi x) \leq \frac{2}{\xi}.$$ \n
10) For the convex modular $m$, we can define two kinds of norms such as

$$||x|| = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + m(\xi x) \right\} \quad \text{and} \quad ||x|| = \inf_{m(\xi x) \leq 1} \frac{1}{\xi} \left| \frac{1}{\xi} \right|$$

[3 or 4]. For the general modulars considered here, the formulas (3.1) and (4.1) are nothing but ones obtained by replacing $m(\xi x)$ by $\xi \rho(\xi x)$ in $||\cdot||$ and $||\cdot||$ respectively.
This yields (4.2), since we have \( ||x||_0 \leq \frac{1}{\xi} \) and \( \rho(\gamma x) > \frac{1}{\eta} \) for every \( \gamma \) with \( ||x||_0 > \frac{1}{\eta} \). Therefore we can obtain from above

**Theorem 4.1.** If \( L \) is a modular space with a modular satisfying (A.1) \( \sim \) (A.5) in \( \S 1 \), then the formula (4.1) yields a quasi-norm \( || \cdot || \) on \( L \) which is equivalent to \( || \cdot ||_0 \) defined by Musielak and Orlicz in [6] as is shown in (4.2).

From the above theorem and the results in \( \S 2 \), we obtain by the same way as in \( \S 3 \)

**Theorem 4.2.** If \( R \) is a quasi-modular space with a quasi-modular \( \rho \), then

\[
(4.3) \quad ||x||_1 = \inf_{\xi > 0} \left\{ \frac{1}{\xi} + \rho^*(\xi x) \right\} \quad (x \in R)
\]

is a semi-continuous quasi-norm on \( R^+_0 \) and \( || \cdot ||_1 \) is complete if and only if \( \rho \) satisfies (\( \rho.A' \)), where \( \rho^* \) and \( R_0 \) are the same as in \( \S 2 \) and \( \S 3 \). And further we have

\[
(4.4) \quad ||x||_0 \leq ||x||_1 \leq 2 ||x||_0 \quad \text{for all } x \in R^+_0.
\]

\( \S 5. \) A quasi-norm-convergence. Here we suppose that a quasi-modular \( \rho^* \) on \( R \) satisfies (\( \rho.1 \) \( \sim \) (\( \rho.6 \)) except (\( \rho.3 \)) and \( \rho^*(\xi x) \) is not identically zero as a function of \( \xi \geq 0 \) for each \( x \in R \) (i.e. \( R_0 = \{0\} \)). A sequence of elements \( \{x_n\}_{n \geq 1} \) is called order-convergent to \( a \) and denoted by \( o-lim x_n = a \), if there exists a sequence of elements \( \{a_n\}_{n \geq 1} \) such that

\[ |x_n - a| \leq a_n \quad (n \geq 1) \quad \text{and} \quad a_\nu \downarrow_{\nu=1}^\infty 0. \]

And a sequence of elements \( \{x_n\}_{n \geq 1} \) is called star-convergent to \( a \) and denoted by \( s-lim x_n = a \), if for any subsequence \( \{y_n\}_{n \geq 1} \) of \( \{x_n\}_{n \geq 1} \), there exists a subsequence \( \{z_n\}_{n \geq 1} \) of \( \{y_n\}_{n \geq 1} \) with \( o-lim z_n = a \).

A quasi-norm \( || \cdot || \) on \( R \) is termed to be continuous, if \( \inf_{n \geq 1} ||a_n|| = 0 \) for any \( a_n \downarrow_{n=0}^\infty 0 \). In the sequel, we write by \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) the quasi-norm defined on \( R \) by \( \rho^* \) in \( \S 3 \) (resp. in \( \S 4 \)).

Now we prove

**Theorem 5.1.** In order that \( || \cdot ||_0 \) (or \( || \cdot ||_1 \)) is continuous, it is necessary and sufficient that the following condition is satisfied:

\[
(5.1) \quad \text{for any } x \in R \text{ there exists an orthogonal decomposition } x = y + z \text{ such that } [z]R \text{ is finite dimensional and } \rho(y) < +\infty.
\]

**Proof.** Necessity. If (5.1) is not true for some \( x \in R \), we can find a
sequence of projector \([p_n]\)_{n \geq 1}^\infty\) such that \(\rho([p_n]x) = +\infty\) and \([p_n] \downarrow_{n=1}^\infty 0\). Hence by (3.1) it follows that \(\| [p_n]x \|_0 > 1\) for all \(n \geq 1\), which contradicts the continuity of \(\| \cdot \|_0\).

**Sufficiency.** Let \(a_n \downarrow_{n=1}^\infty 0\) and put \([p_n'] = [(a_n - \varepsilon a_1)^+]\) for any \(\varepsilon > 0\) and \(n \geq 1\). It is easily seen that \([p_n'] \downarrow_{n=1}^\infty 0\) for any \(\varepsilon > 0\) and

\[
 a_n = [a_1]a_n = [p_n']a_n + (1 - [p_n'])a_n \leq [p_n']a_1 + \varepsilon a_1.
\]

This implies

\[
 \rho^*([a_n]x) \leq \rho^*([p_n']a_1) + \rho^*(\varepsilon(1 - [p_n'])a_1)
\]

for all \(n \geq 1\) and \(\varepsilon \geq 0\). In virtue of (5.1) and \([p_n'] \downarrow_{n=1}^\infty 0\), we can find \(n_0\) (depending on \(\xi\) and \(\varepsilon\)) such that \(\rho^*([p_n']a_1) < +\infty\), and hence \(\inf_{n \geq 1} \rho^*([p_n']a_1) = 0\) by (2.3) in Lemma 1 and (\(\rho, 2\)). Thus we obtain

\[
 \inf_{n \geq 1} \rho^*([p_n']a_1) \leq \rho^*(\varepsilon a_1).
\]

Since \(\varepsilon\) is arbitrary, \(\lim_{n \rightarrow \infty} \rho^*([p_n']a_1) = 0\) follows. Hence we infer that \(\inf_{n \geq 1} \| a_n \|_0 = 0\) and \(\| \cdot \|_0\) is continuous in view of Remark 2 in §3. Q.E.D.

**Corollary.** \(\| \cdot \|_0\) is continuous, if

(5.2) \(\rho^*(a_v) \rightarrow 0\) implies \(\rho^*(\alpha a_v) \rightarrow 0\) for every \(\alpha \geq 0\).

From the definition, it is clear that s-lim \(x_v = 0\) implies \(\lim_{v \rightarrow \infty} \| x_v \|_0 = 0\), if \(\| \cdot \|_0\) is continuous. Conversely we have, by making use of the well-known method (cf. Theorem 33.4 in [3])

**Theorem 5.2.** \(\lim_{v \rightarrow \infty} \| x_v \|_0 = 0\) (or \(\lim_{v \rightarrow \infty} \| x_v \|_1 = 0\)) implies s-lim \(x_v = 0\), if \(\| \cdot \|_0\) is complete (i.e. \(\rho^*\) satisfies (\(\rho, 3\))).

If we replace \(\lim_{v \rightarrow \infty} \| x_v \|_0\) by \(\lim_{v \rightarrow \infty} \rho(x_v) = 0\), Theorem 5.2 may fail to be valid in general. By this, reason, we must consider the following condition:

(5.3) \(\rho^*(x) = 0\) implies \(x = 0\).

Truly we obtain

**Theorem 5.3.** If \(\rho^*\) satisfies (5.3) and \(\| \cdot \|_0\) is complete, \(\rho(a_v) \rightarrow 0\) implies s-lim \(a_v = 0\).

**Proof.** We may suppose without loss of generality that \(\rho^*\) is semi-continuous,\(^{11}\) i.e. \(\rho^*(x) = \sup_{\lambda \in \Lambda} \rho^*(x_{\lambda})\) for any \(0 \leq x_{\lambda} \in A\). If

\(^{11}\) If \(\rho^*\) is not semi-continuous, putting \(\rho^*(x) = \inf_{\lambda \in \Lambda} [\sup_{\lambda \in \Lambda} \rho^*(x_{\lambda})]\), we obtain a quasi-modular \(\rho^*_v\) which is semi-continuous and \(\rho^*(x_v) \rightarrow 0\) is equivalent to \(\rho^*_v(x_v) \rightarrow 0\).
\[ \rho(a_{\nu}) \leq \frac{1}{2^{\nu}} \quad (\nu \geq 1), \]

we can prove by the similar way as in the proof of Lemma 2 that there exists \( \bigcup_{\nu=1}^{\infty} a_{\nu} \in R \) in virtue of (\( \rho.3 \)).

Now, since

\[ \rho \left( \bigcup_{\nu \geq \nu}^{\infty} a_{\nu} \right) \leq \sum_{\nu \geq \nu}^{\infty} \rho(a_{\nu}) \leq \frac{1}{2^{\nu-1}} \]

holds for each \( \nu \geq 1 \), \( \rho \left( \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} a_{\nu} \right) \right) = 0 \) and hence (5.3) implies

\[ \bigcap_{\nu=1}^{\infty} \left( \bigcup_{\nu \geq \nu}^{\infty} a_{\nu} \right) = 0. \]

Thus we see that \( \{a_{\nu}\}_{\nu \geq 1} \) is order-convergent to 0.

For any \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b_{\nu}) \to 0 \), we can find a subsequence \( \{b'_{\nu}\}_{\nu \geq 1} \) of \( \{b_{\nu}\}_{\nu \geq 1} \) with \( \rho(b'_{\nu}) \leq \frac{1}{2^{\nu}} \) (\( \nu = 1, 2, \ldots \)). Therefore we have \( s\text{-lim}_{\nu \to \infty} b_{\nu} = 0 \). Q.E.D.

The latter part of the above proof is quite the same as Lemma 2.1 in [9] (due to S. Yamamuro) concerning the condition (5.2) with respect to (convex) modulars on semi-ordered linear spaces. From Theorem 5.2 and 5.3, we can obtain further the next theorem which is analoguous to the above lemma of [9] and considered as the converse of Corollary of Theorem 5.1 at the same time.

**Theorem 5.4.** If \( \rho^{*} \) satisfies (5.3) and \( \| \cdot \|_{0} \) is complete and continuous, then (5.2) holds.

**References**


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