EXAMPLES OF SINGULAR POINTS

By

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The purpose of the present paper is to construct examples of singular points of first and of second kind.

PART I

In this part we shall construct some Riemann surface to construct singular points, non N-minimal points and to show that there exists a Riemann surface which has the set of singular points of first kind of the power of continuum.

1. Concentrating ring C.R. (α, ε). If a Riemann surface with finite genus has two circular relative boundaries, we call it a *generalized ring*. We call an ordinary ring *a simple ring*. Let $R: 1<|z|<\exp\alpha$ be a simple ring with module $\alpha: 1<|z|<\exp\alpha$. We shall construct from R a generalized ring G.R. such that any harmonic function such that $|U(z)|\leqq 1$ in G.R. must satisfy the condition

$$\left|\max_{|z|=\exp\frac{\alpha}{2}} U(z)-\min_{|z|=\exp\frac{\alpha}{2}} U(z)\right|<\varepsilon$$

on a circle $|z|=\exp\frac{\alpha}{2}$ $\left(\text{in the following we say that } U(z) \textit{ has aberration } <\varepsilon \text{ on } |z|=\exp\frac{\alpha}{2}\right)$. We call such a ring a *concentrating ring* and denote it by C.R. (α, ε).

a). *Determining an integer* $m: m\geqq 4$. Let $U(z)$ be a harmonic function in a simple ring S.R: $\exp\frac{\alpha}{3}<|z|<\exp\frac{2\alpha}{3}$ such that $0\leqq U(z)\leqq 1$ and continuous on $|z|=\exp\frac{\alpha}{3}$ and on $|z|=\exp\frac{2\alpha}{3}$. Then

$$U(z)=\frac{1}{2\pi}\int_{\partial(\text{S.R.})} U(z)\frac{\partial}{\partial n}G(\zeta, z)\,ds,$$

where $G(\zeta, z)$ is the Green's function of the above ring.

Since $\frac{\partial}{\partial n}G(\zeta, z)$ is a continuous functions of z on $|z|=\exp\frac{\alpha}{3}$ and on $|z|=\exp\frac{2\alpha}{3}$ for fixed ζ and

$$|U(z_1)-U(z_2)|\leqq\frac{1}{2\pi}\int_{\partial(\text{S.R.})}|U(\zeta)|\left|\frac{\partial}{\partial n}G(\zeta, z_1)-\frac{\partial}{\partial\omega}G(\zeta, z_2)\right|ds,$$

there exists an integer $m(\alpha)$ such that $|\arg z_1-\arg z_2|\leqq\frac{2\pi}{2^m}$ and $|z_1|=|z_2|=\exp\frac{\alpha}{2}$ imply $|U(z_1)-U(z_2)|<\frac{\varepsilon}{4}$. *Fix m at present.*

b) Determining a constant β. Let D be a rectangle: $|\mathrm{Re}\, z|\leqq\frac{1}{2\beta}$, $0\leqq\mathrm{Im}\, z\leqq 1$ on the z-plane. Let $\omega(z)$ be the harmonic measure of two sides of D: $|\mathrm{Re}\, z|=\frac{1}{2\beta}$, i.e. $w(z)$ is a positive harmonic function in D such that $\omega(z)=1$ on $|\mathrm{Re}\, z|=\frac{1}{2\beta}$ and $w(z)=0$ on $\mathrm{Im}\, z=0$ and $\mathrm{Im}\, z=1$. Now $\omega(z)\to 0$ uniformly on $|\mathrm{Re}\, z|=0$ as $\beta\to 0$. Let β^* be a constant such that $w(z)<\varepsilon'$ on $\mathrm{Re}\, z=0$ for $\beta\leqq\beta^*$. Then any harmonic function $U(z)$ in D such that $U(z)=0$ on $\mathrm{Im}\, z=0$ and $\mathrm{Im}\, z=1$ and $|U(z)|<1$ satisfies $|U(z)|<\varepsilon'$ on $|\mathrm{Re}\, z|=0$, where

$$\varepsilon'=\frac{\varepsilon}{8m}.$$

Put $\gamma=\frac{\alpha}{3m}$ and let H_i, H_i', $\widehat{H}_i$, $\widehat{\widehat{H}}_i$ and H be rings and Γ_i, Γ_i' $(i=1,2,\cdots,n)$ and Γ be circles cited below:

$$H_i:\ \exp(i-1)\gamma<|z|<\exp i\gamma,\quad H_i':\ \exp(\alpha-i\gamma)>|z|>\exp(\alpha-(i-1)\gamma)$$

$$\widehat{\widehat{H}}_i:\ \exp(i-1)\gamma<|z|<\exp(\alpha-(i-1)\gamma),$$

$$\widehat{H}_i:\ \exp\left(i-\frac{1}{2}\right)\gamma<|z|<\exp\left(\alpha-\left(i-\frac{1}{2}\right)\gamma\right),$$

$$H:\ \exp\frac{\alpha}{3}<|z|<\exp\frac{2\alpha}{3}$$

$$\Gamma_i:\ |z|=\exp\left(i-\frac{1}{2}\right)\gamma,\quad \Gamma_i':\ z=\exp\left(\alpha-\left(i-\frac{1}{2}\right)\gamma\right)$$

$$\Gamma:\ |z|=\exp\frac{\alpha}{2}.$$

c) Slits for identifying. Let $s_{i,j}$ and s_{ij}' $(i=1,2,\cdots,m)$, $i=1,2,3,\cdots,2^{ml}$ (where l is an integer such that

$$0<\frac{\frac{2\pi}{2^{ml}}}{\frac{\gamma}{2}}\leqq\beta^*$$

and that $s_{i,j}$ and s'_{ij} are contained in H_i and H'_i respectively, i.e.

$$s_{i,j}:\ \exp\left(i-1+\frac{1}{4}\right)\gamma<|z|<\exp\left(i-1+\frac{3}{4}\right)\gamma: \qquad \arg z=\frac{2\pi}{2^{ml}}(j-1)$$

$$s'_{i,j}:\ \exp\left(\alpha-\left(i-1+\frac{3}{4}\right)\gamma\right)<|z|<\exp\left(\alpha-\left(i-1+\frac{1}{4}\right)\gamma\right):\ \arg z=\frac{2\pi}{2^{ml}}(j-1).$$

Let $H_i(s)$ be H_i with $\sum s_{ij}$. Then $H_i(s)$ and $H'_i(s)$ $(i=1,2,\cdots,m)$ are all conformally equivalent.

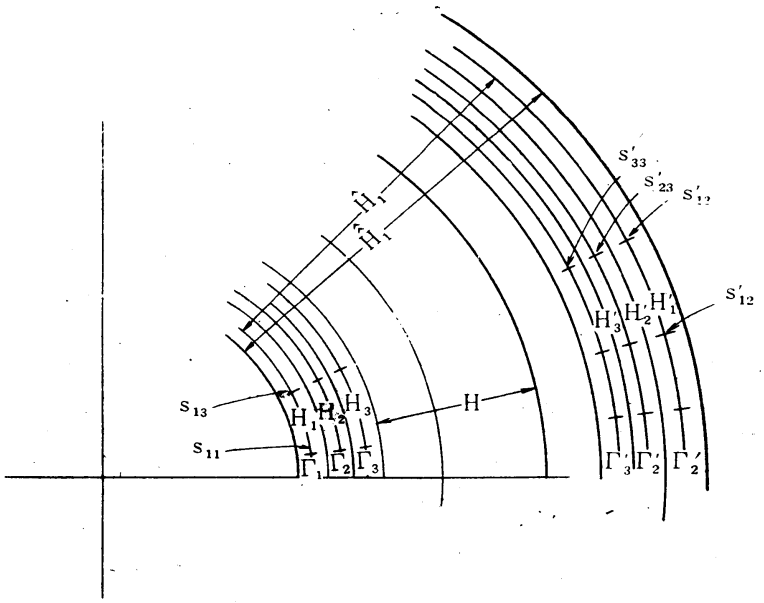

Fig. 1.

d) Construction of a concentrating ring: C.R. (α,ε). At first in $H_1+H'_1$ identify two edges of s_{1j} and s'_{1j} lying symmetrically with respect to $I_{1,1}$ (real axis). Next in $H_2+H'_2$ identify two edges of s_{2j} and s'_{2j} lying symmetrically with respect to $I_{2,1}$ (imaginary axis). Let $E_{i,k}$ be a circular echelon $(i=1,2,\cdots,m)$, $k=1,2,3,\cdots 2^{i-1})$ and $I_{i,k}$ be a diameter such that

$$E_{i,k}:\ \exp(i-1)\gamma<|z|<\exp(\alpha-(i-1)\gamma),\quad \frac{2\pi}{2^{i-1}}(k-1)<\arg z<\frac{2k\pi}{2^{i-1}}.$$

$$I_{i,k}:\ 0\leqq|z|<\exp(\alpha-(i-1)\gamma),\ \arg z=\frac{2\pi}{2^{i-1}}(k-1)+\frac{2\pi}{2^i}\ \text{or}\ \frac{2\pi}{2^{i-1}}(k-1)+\frac{2\pi}{2^i}+\pi.$$

In every $E_{ik}\frown(H_i+H'_i)$ $(i=3,4,\cdots,m)$ identify two edges of $\sum(s_{ij}+\sum s'_{ij})$ lying symmetrically with respect to I_{ik} $(k=1,2,3,\cdots,2^{i-1})$. Then we have identified every s_{ij} and s'_{ij} contained in the simple ring and we have a generalized ring G.R. We shall prove such G.R. is a C.R. (α,ε). Let $U(z)$ be a harmonic function in G.R. such that $|U(z)|\leqq 1$.

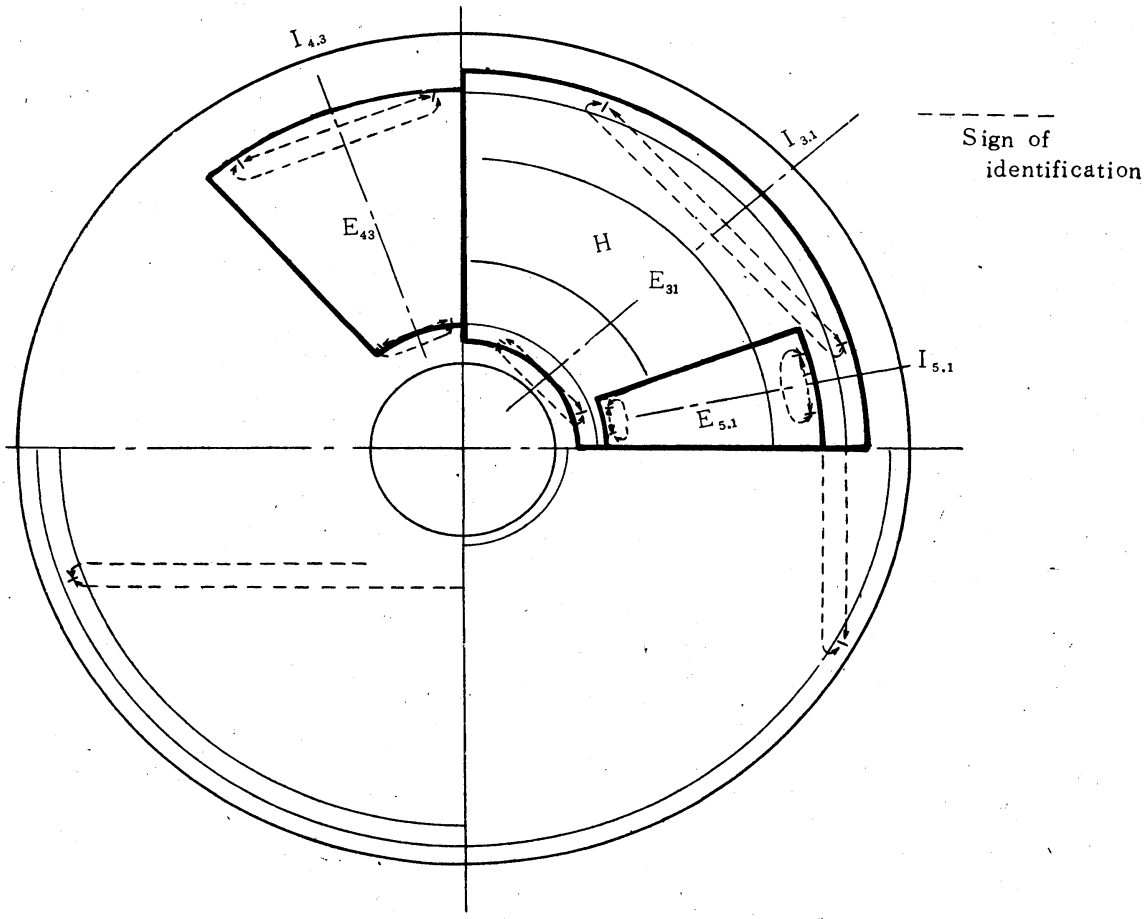

Fig. 2.

Let $T_1(z)$ be the inversion with respect to I_{11} (real axis). Since G.R. has symmetric structure, $U(T_1(z))-U(z)$ is harmonic in G.R. and $U(T_1(z))-U(z)=0$ on $\sum(s_{1j}+s'_{1j})$. Now the echelons

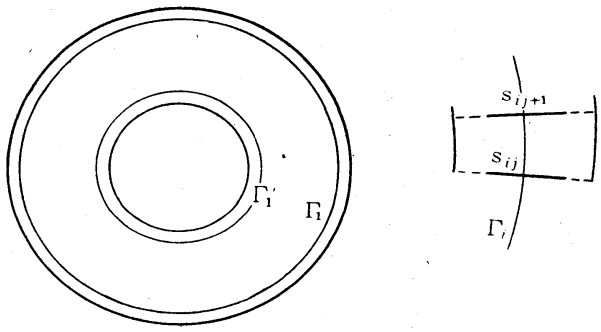

Fig. 3.

$$\exp\left(i-1+\frac{1}{4}\right)\gamma<|z|<\exp\left(i-1+\frac{3}{4}\right)\gamma,\quad \frac{2\pi}{2^m}(j-1)<\arg z<\frac{2\pi}{2^m}j,$$

and

$$\exp\left(\alpha-\left(i-1+\frac{3}{4}\right)\gamma\right)<|z|<\exp\left(\alpha-\left(i-1+\frac{1}{4}\right)\gamma\right),\ \frac{2\pi}{2^m}(j-1)<\arg z<\frac{2\pi}{2^m}j$$

are conformally equivalent to the rectangle:

$$|\operatorname{Re}\zeta|<\frac{1}{2\beta'},\ 0\leqq I_m\zeta\leqq 1,\ \beta'\leqq\beta^*.$$

Consider $U(z)-U(T_1(z))$ in the above rectangle. Then

$$|U(z)-U(T_1(z))|<\varepsilon' \text{ on } \operatorname{Re}\zeta=0.$$

Hence $|U(T_1(z))-U(z)|<\varepsilon'$ on $\Gamma_1+\Gamma_1'$, whence by the maximum principle

$$|U(T_1(z))-U(z)|<\varepsilon' \quad \text{on} \quad \hat{H}_1\supset\hat{\hat{H}}_2. \tag{1}$$

Let $T_2(z)$ be the inversion with respect to I_{21} (imaginary axis). Then similarly as above

$$|U(T_2(z))-U(z)|<\varepsilon' \quad \text{on} \quad \hat{H}_2. \tag{2}$$

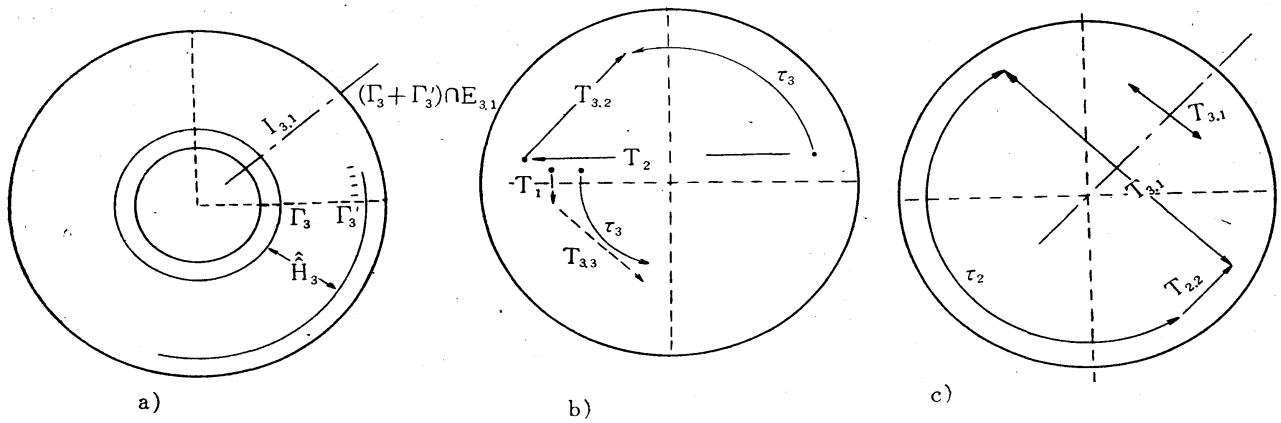

Fig. 4.

Let $\tau_2(z)$ be the rotation about $z=0$ such that $\arg\tau_2(z)-\arg z=\pi$.
Then $\tau_2(z)=T_1T_2(z)$ for $z\in E_{21}$ and $\tau_2(z)=T_1T_2(z)$ for $z\in E_{22}$.
Hence $|U(\tau_2(z))-U(z)|<|U(T_1T_2(z))-U(T_1(z))|$
$+|U(T_1(z)-U(z)|<2\varepsilon'$ on $\Gamma_2+\Gamma_2'$.

$\hat{H}_2$ is invariant with respect to $\tau_2(z)$ and $U(\tau_2(z))-U(z)$ is harmonic in $\hat{H}_2$, whence by the maximum principle

$$|U(\tau_2(z))-U(z)|<2\varepsilon' \quad \text{on} \quad \hat{H}_2$$

Similarly $$|U(\tau_{-2}(z))-U(z)|<2\varepsilon' \text{ on } \hat{H}_2, \tag{3}$$

where $\tau_{-2}(z)$ is the rotation about $z=0$ such that $\arg\tau_{-2}(z)-\arg z=-\pi$.

Let $T_{3,k}(z)$ be the inversion with respect to $I_{3,k}$ $(k=1,2)$. Then $\hat{\hat{H}}_3$ has symmetric structure, $U(T_{3,k}(z))-U(z)$ is harmonic in $\hat{\hat{H}}_3$ and vanishes

only on $(E_{3,k}+E_{3,k'})\frown(\sum(s_{3,j}+s'_{3,j}))$, where $k'=k+2$ and $|U(T_{3,k}(z))-U(z)|\leqq 1$ on $(\sum(s_{3,j}+s'_{3,j}))-((E_{3,k}+E_{3,k'})\frown(\sum(s_{3,j}+s'_{3,j})))$.

Hence $\quad |U(T_{3,k}(z))-U(z)|<\varepsilon' \quad$ on $\quad (\Gamma_3+\Gamma'_3)\frown(E_{3,k}+E_{3,k'}),$ (4)

Let $\tau_3(z)$ be the rotation about $z=0$ such that $\arg\tau_3(z)-\arg z=\frac{\pi}{2}$. Then

$$\tau_3(z)=T_{3,2}T_2(z) \text{ for } z\in E_{3,2}+E_{3,3} \text{ and } \tau_3(z)=T_{3,3}T_1(z) \text{ for } z\in E_{3,2}+E_{3,4}.$$

Now $T_{3,2}T_2(z)$ and $T_2(z)$ for $z\in E_{31,}$(or $E_{3,3}$) are contained in the same $E_{3,4}$ (or $E_{3,2}$), whence by (4)

$$|U(T_{3,2}T_2(z))-U(T_2(z))|<\varepsilon' \quad \text{on} \quad (\Gamma_3+\Gamma'_3)\frown(E_{3,1}+E_{3,3})$$

Similarly $\quad |U(T_{3,1}T_1(z))-U(T_1(z))|<\varepsilon' \quad$ on $\quad (\Gamma_3+\Gamma'_3)\frown(E_{3,2}+E_{3,4})$.

Hence by (4), (1) and (2)

$$|U(\tau_3(z))-U(z)|<|U(T_{3,i}T_i(z))-U(T_i(z))|+|U(T_i(z))-U(z)|<2\varepsilon' \text{ on } \Gamma_3+\Gamma'_3,$$

where $\quad i=1$ for $z\in(E_{3,2}+E_{3,4})$ and $i=2$ for $z\in(E_{3,1}+E_{3,3})$.

$\hat{H}_3$ is invariant with respect to $\tau_3(z)$ and $U(\tau_3(z))-U(z)$ is harmonic in $\hat{H}_3$. Hence by the maximum principle

$$|U(\tau_3(z))-U(z)|<2\varepsilon' \quad \text{on} \quad \hat{H}_3. \tag{5}$$

Similarly $\quad |U(\tau_{-3}(z))-U(z)|<2\varepsilon' \quad$ on $\quad \hat{H}_3$.

Consider $U(T_{3,1}(z))-U(z)$ in $\hat{H}_3$.

At first, $\quad |U(T_{3,1}(z))-U(z)|<\varepsilon'$ for $z\in(\Gamma_3+\Gamma'_3)\frown(E_{3,1}+E_{3,3})$ (6)

Let z and z' be points in $E_{3,2}$ and $E_{3,4}$ such that $T_{3,1}(z)=z'$. Then there exists $\tau_2(z)$ such that $T_{3,4}(\tau_2(z))=z'$ i.e. $T_{3,4}(\tau_2(z))=T_{3,1}(z)$. Now $T_{3,4}(\tau_2(z))$ and $\tau_2(z)$ are contained in $E_{3,4}+E_{3,2}$ and $T_{3,4}(\tau_2(z))=z'$. Hence

$$|T_{3,4}(\tau_2(z))-U(\tau_2(z))|<\varepsilon' \text{ for } z\in(\Gamma_3+\Gamma'_3)\frown(E_{3,1}+E_{3,3}).$$

Whence by (5)

$$|U(T_{3,1}(z)-U(z)|<|U(T_{3,4}(\tau_2(z))-U(\tau_2(z))| \\ +|U(\tau_2(z))-U(z)|<3\varepsilon' \text{ for } z\in(\Gamma_3+\Gamma'_3)\frown(E_{3,2}+E_{3,4})$$

Hence by (6) $|U(T_{3,1}(z)-U(z)|<3\varepsilon'$ on $(\Gamma_3+\Gamma'_3)$ and by the maximum principle

$$|U(T_{31}(z))-U(z)|<3\varepsilon' \quad \text{in} \quad \hat{H}_3. \tag{7}$$

Let $T_{i,j}(z)$ be the inversion with respect to $I_{i,j}$ and $\tau_i(z)(\tau_{-i}(z))$ be the rotation about $z=0$ such that $\arg\tau_i(z)-\arg z=\frac{2\pi}{2^i}\left(\arg\tau_{-i}(z)-z=\frac{-2\pi}{2^i}\right)$. *Then*

$$\begin{aligned} &|U(\tau_i(z))-U(z)|<(2i-2)\varepsilon' \quad on \quad \hat{H}_i, \\ &|U(\tau_{-i}(z))-U(z)|<(2i-2)\varepsilon' \quad on \quad \hat{H}_i, \end{aligned} \tag{8}$$

$$|U(T_{i,j}(z))-U(z)|<(2i-1)\varepsilon' \quad on \quad \hat{H}_i. \tag{9}$$

Proof. Assume $|U(\tau_s(z))-U(z)|<2(s-1)\varepsilon'$,

$$|U(T_{s,j}(z)-U(z)|<(2s-1)\varepsilon' \quad \text{on} \quad \hat{H}_s(s=1,2,\cdots,i).$$

Then we shall prove that the above inequalities hold for $i+1$, Similarly as (6)

$$|U(T_{i+1,j}(z))-U(z)|<\varepsilon' \quad \text{on} \quad (\Gamma_{i+1}+\Gamma'_{i+1})\frown(E_{i+1,j}+E_{i+1,j+2^i}). \tag{10}$$

Let $\tau_{i+1}(z)$ be the rotation about $z=0$ such that $\arg\tau_{i+1}(z)-\arg z=\frac{2\pi}{2^{i+1}}$. Since $I_1+I_{2,1}+(I_{3,1}+I_{3,2})+(I_{4,1}+I_{4,2}+I_{4,3}+I_{4,4})+\cdots+(I_{i+1,1}+I_{i+1,2}+\cdots+I_{i+1,2^{i-l}})$ separates H_{i+1} into 2^{i+1} parts, for any $z\in(\Gamma_{i+1}+\Gamma'_{i+1})$ there exist $T_{s,t}(s\leqq i+1)$ and $T_{i+1,j}(z)$ such that $\tau_{i+1}(z)=T_{i+1,j}T_{s,t}(z)$ and both $T_{i+1,j}T_{s,t}(z)$ and $T_{s,t}(z)$ are contained in the same $E_{i+1,j}$. Hence $|U(\tau_{i+1}(z))-U(z)|<|U(T_{i+1}T_{s,t}(z))-U(T_{s,t}(z))|+|U(T_{s,t}(z))-U(z)|\leqq 2i\varepsilon'$ on $(\Gamma_{i+1}+\Gamma'_{i+1})\frown F_{s,k}$ by (9) and (10). $\hat{H}_{i+1}$ is invariant with respect to $\tau_{i+1}(z)$ and $U(\tau_{i+1}(z))-U(z)$ is harmonic in $\hat{H}_{i+1}$, whence by the maximum principle

$$|U(\tau_{i+1}(z))-U(z)|<2i\varepsilon' \quad \text{on} \quad \hat{H}_{i+1}. \tag{11}$$

Similarly $\quad |U(_-\tau_{i+1}(z))-U(z)|<2i\varepsilon' \quad \text{on} \quad \hat{H}_{i+1}.$

Let z and z' be points such that $T_{i+1,1}(z)=z'$. If z and z' are contained in $E_{i+1,1}$, $|U(T_{i+1,1}(z))-U(z)|<\varepsilon'$ by (10) on $(\Gamma_{i+1}+\Gamma'_{i+1})\frown E_{i+1,1}$. If z and z' are not contained in $E_{i+1,1}$, there exists a $\tau_s(z)(\tau_{-s}(z))$ $(s\leqq i+1)$ such that $\tau_s(z)$ and z are contained in the same $E_{i+1,k}$ and $T_{i+1,k}(\tau_s(z))=z'=T_{i+1,1}(z)$. Hence $|U(T_{i+1,1}(z))-U(z)|=|U(T_{i+1,k}(\tau_s(z))-U(z)|\leqq|U_{i+1,k}(\tau_s(z))-U(\tau_s(z))|+|U(\tau_s(z))-U(z)|<(2i+1)\varepsilon'=(2i+1')\varepsilon'$ on $(\Gamma_{i+1}+\Gamma'_{i+1})\frown E_{i+1,k}$. $\hat{H}_{i+1}$ has symmetric structure with respect to $I_{i+1,1}$ and $U(T_{i+1,1}(z))-U(z)$ is harmonic, whence by the maximum principle

$$|U(T_{i+1,1}(z))-U(z)|<(2i+1)\varepsilon' \quad \text{on} \quad \hat{H}_{i+1}.$$

Similarly

$$|U(T_{i+1,j})z))-U(z)|<(2i+1)\varepsilon' \quad \text{on} \quad H_{i+1,1}.$$

$$(i=1,2,3,\cdots,m) \text{ and } j=1,2,\cdots,2^m$$

Thus (8) and (9) are valid for every i and j. By $H\subset\bigcap_{i=1}^{m}\hat{H}_i$

$$|U(T_{i,j}(z))-U(z)|<2m\varepsilon'=\frac{\varepsilon}{4}:z\in H \quad \text{for } i=1,2,\cdots,m,\ j=1,2,\cdots,2^m. \tag{12}$$

Put $\theta_0=\frac{2\pi}{2^{m+2}}$, $\theta_k=\frac{2\pi}{2^m}k-\frac{2\pi}{2^{m+1}}:k=1,2,\cdots,2^m$ and $r=\exp\frac{\alpha}{2}\left(\Gamma:|z|=\exp\frac{\alpha}{2}\right)$

Then there exists $T_{i,j}(z)$ such that $T_{i,j}(re^{i\theta_k})=re^{i\theta_0}$.

Hence

$$|U(re^{i\theta_k})-U(re^{i\theta_0})|<\frac{\varepsilon}{4}. \tag{13}$$

Next for every point z on Γ, there exists a point $re^{i\theta_k}$ such that

$$|\arg z-\theta_k|<\frac{2\pi}{2^m}.$$

On the other hand, $U(z)$ is harmonic in the ring $H: \exp\frac{\alpha}{3}<|z|<\exp\frac{2\alpha}{3}$. Hence

$$|U(z')-U(z'')|<\frac{\varepsilon}{4} \text{ for } |\arg z'-\arg z''|<\frac{2\pi}{2^m} \text{ and } |z'|=|z''|=\exp\frac{\alpha}{2}.$$

Thus by (13)

$$\max_{z\in\Gamma} U(z)-\min_{z\in\Gamma} U(z)<\varepsilon,$$

and G.R. is a concentrating ring C.R. (α, ε).

Let R be a Riemann surface with compact relative boundary ∂R_0 and let $\{R_n\}$ be an exhaustion of R with compact relative boundary ∂R_n $(n=1,2,\cdots)$. $R-R_n$ is composed of a finite number of non compact domains $G_{n,i}: i=1,2,\cdots i(n)$. We make an ideal boundary component $\mathfrak{p}$ correspond to $\{G_{1,i_1}, G_{2,i_2}, G_{3,i_3},\cdots\}: G_{n,i_n}\supset G_{n+1,i_{n+1}}$, where $G_{1,i_1}\supset G_{2,i_2}\supset,\cdots$ and $\partial G_{n,i}$ is compact and contained in ∂R_n. For any given $G_{n,i}$, if there exists a number j_0 such that $p_j\in G_{n_i}$ for $j\geqq j_0$, we say that a sequence $\{p_j\}$ $(j=1,2,\cdots)$ converges to $\mathfrak{p}$. Let $U(z)$ be a harmonic function in $R-R_1$ such that $U(z)$ has M.D.I. over $R-R_1$ among all harmonic functions with value $U(z)$ on ∂R_1, we say that $U(z)$ is $\overline{\text{h}}$armonic in $R-R_1$.

Lemma 1. *a). Suppose N-Martin's topology is defined in $R-R_0$ and the distance $\delta(p,q)$ is defined as*

$$\delta(p,q)=\sup_{z\in R_1}\left|\frac{N(z,p)}{1+N(z,p)}-\frac{N(z,p)}{1+N(z,q)}\right|.$$

Let $\mathfrak{p}$ be an ideal boundary component such that $\mathfrak{p}=(G_{1,i_1}, G_{2,i_2},\cdots)$. If for any $\overline{\text{h}}$arm onic function[1] *$U(z)$ in $R-R_1$ $|\max_{z\in\partial G_{n',i}} U(z)-\min_{z\in\partial G_{n',i}} U(z)|\to 0$ as $n\to\infty$, there exists only one ideal boundary point of N-Martin's topology on $\mathfrak{p}$.*

b). Suppose that there exists only one boundary point of N-Martin's topology on a boundary component $\mathfrak{p}=\{G_{1,i_1}, G_{2,i_2},\cdots\}$, then for any given number l, there exists a number m such that $\upsilon_l(p)\supset G_{m,i_m}$, where $\upsilon_l(p)=E\left[z\in\overline{R}: \delta(z,p)<\frac{1}{l}\right]$.

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1) Let $U(z)$ be a harmonic function in $R-R_1$. If $U(z)$ has minimal Dirichlet integral among all harmonic functions with value $U(z)$ on ∂R_1, we say that $U(z)$ is $\overline{\text{h}}$armonic in $R-R_1$

Proof of a). Let $U_n(z)$ be a harmonic function in R_n-R_2 such that $U_n(z)=U(z)$ on ∂R_2 and $\frac{\partial}{\partial n}U_n(z)=0$ on ∂R_n. Then by the maximum principle

$$\max_{z\in\partial G_m,\, i_m} U_n(z)\geqq \sup_{z\in G_m,\, i_m} U_n(z)\geqq \inf_{z\in G_m,\, i_m} U_n(z)\geqq \min_{z\in\partial G_m,\, i_m} U_n(z) \quad \text{for } n\geqq m.$$

By Lemma 1 of P[2] $U_n(z)\Rightarrow U(z)$, hence

$$\max_{z\in\partial G_m,\, i_m} U(z)\geqq \sup_{z\in G_m,\, i_m} U(z)\geqq \inf_{z\in G_m,\, i_m} U(z)\geqq \min_{z\in\partial G_m,\, i_m} U(z).$$

Hence by the assumption $U(z)$ converges as $z\to\mathfrak{p}$. Assume that there exist two points p and q of N-Martin's boundary points on $\mathfrak{p}$. Then we can find a point z_0 in R_1 such that $N(z_0, p)\neq N(z_0, q)$. Let $\{p_i\}$ and $\{q_i\}$ be fundamental sequences determing p and q respectively. Then both $\{p_i\}$ and $\{q_i\}\to\mathfrak{p}$. By $N(z_0, p_i)=N(p_i, z_0)$ and $N(z_0, q_i)=N(q_i, z_0)$ we consider $N(p_i, z_0)$ and $N(q_i, z_0)$ instead of $N(z_0, p_i)$ and $N(z_0, q_i)$ respectively. $N(z, z_0)<M<\infty$ in $R-R_2$ and harmonic in $R-R_2$. Hence $N(p_i, z_0)\to N(p, z_0)=N(q, z_0)\leftarrow N(q_i, z)$ by the assumption, whence $N(z_0, p_0)=N(z_0, q)$. This is a contradiction. Hence only one point of N-Martin's topology exists on $\mathfrak{p}$.

Proof of b). Assume that $b)$ is false. We can find a sequence $\{p_i\}$ such that $p_i\notin \upsilon_l(p)$ and $p_i\to\mathfrak{p}$. This means that $G_{i,i_1}, G_{2,i_2},\cdots$ has at least one point of N-Martin's point outside of $\upsilon_l(p)$. This contradicts that $\{G_{i_1,1}, G_{2,i_2}\cdots\}$ has only one point of N-Martin's topology. Hence we have $b)$.

Example 1. Let $1=r_1<r_2<r_3,\cdots,\ \lim_n r_n=2$. Let A_n and B_n be simple rings cited below:

$$A_n:\ r_{2n+1}<|z|<r_{2n+2}:\ \alpha_n=\log\frac{r_{2n+2}}{r_{2n+1}}$$

$$B_n:\ r_{2n+2}<|z|<r_{2n+3}:\ \beta_n=\log\frac{r_{2n+3}}{r_{2n+1}}.$$

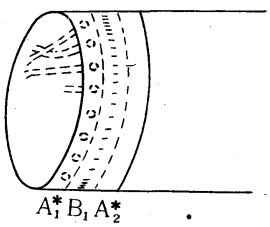

Fig. 5.

We make a concentrating ring C.R. $\left(\alpha_n, \frac{1}{n}\right)=A_n^*$ from A_n.

Let s_{nj} be slits in B_n such that

$$\exp\left(\log r_{2n+2}+\frac{1}{4}\beta_n\right)<|z|<\exp\left(\log r_{2n+2}+\frac{3}{4}\beta_n\right) \text{ and } \arg z=\frac{2\pi}{2^{m_n}}j:$$

$$j=1,2,3,\cdots,\ 2^{m_n}-1,$$

where m_n is a number such that every positive harmonic function $\leqq 1$ in B_n vanishing on $\sum_j s_{n,j}$ must satisfy

$$U(z)<\frac{1}{n} \text{ on } |z|=(r_{2n+2}, r_{2n+3})^{\frac{1}{2}}. \tag{14}$$

2) We abreviate the previous paper "Potentials on Riemann surfaces" by P.

Denote by B_n^* the ring B_n with slits $\sum_j s_{n,j}$ and put

$$R_n(A_n^*)=A_1^*+B_1^*+A_2^*+,\cdots B_{n-1}^*+A_n^*,$$

$$R_n(B_n^*)=R(A_n^*)+B_n^*. \quad \text{Put } R=\bigcup_1^\infty R_n(A_n^*)=\bigcup_1^\infty R_n(B_n^*).$$

$R-R_n(A_n^*)$ is a non compact domain and $\{R-R_n(A_n^*)\}$: $n=1,2,\cdots$ determines an ideal boundary component $\mathfrak{p}$ and R has the following properties:

a). There exists only one point p of N-Martin's boundary point on $\mathfrak{p}$.

b). $\omega(p,z)>0$, i.e. p is a singular point.

c). $w(p,z)=0$, i.e. p is a singular point of first kind.

Proof of a). Let $U(z)$ be a harmonic function in $R-R_2(A_2^*)$ such that $0<U(z)<M$. Then since A_n^* is a concentrating ring,

$$|\max_{z\in\partial A^*_{n+1}} U(z)-\min_{z\in\partial A^*_{n+1}} U(z)|<|\max_{|z|=\sqrt{r_{2n+1}\cdot r_{2n+2}}} U(z)-\min_{|z|=\sqrt{r_{2n+1}\cdot r_{2n+2}}} U(z)|\to 0 \text{ as } n\to\infty.$$

Hence by Lemma 1. *a)* we have *a)*.

Proof of b). By Lemma 1. *b)* there exists a number n such that $\upsilon_l(p)\supset R-R_n(A_n^*)$, where $\upsilon_l(p)=E\left[z\in\overline{R}:\delta(z,p)<\frac{1}{l}\right]$. Let $\omega_n(z)$ be a harmonic function in $R_n(A_n^*)$ such that $\omega_n(z)=0$ on $C=E[z:|z|=1]$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\sum_{i,j} s_{i,j}$ contained in $R_n(A_n^*)$ and $\omega_n(z)=1$ on $\partial(R_n(A_n^*))$. Then by the Dirichlet principle

$$D(w(\upsilon_l(p),z))\geqq D(\omega_n(z)),$$

where $\omega(\upsilon_l(p),z)$ is C.P. of $\upsilon_l(p)$.

Clearly $\omega_n(z)=\frac{\log|z|}{\log r_{2n+1}}$, since $\sum s_{ij}$ are radial slits. Hence

$$\frac{2\pi}{\log 2}=\lim_n D(\omega_n(z))\leqq\lim_{l=\infty} D(\upsilon_l(p),z)=D(\omega(p,z)).$$

Thus *p is a singular point.*

Proof of c). Let $w_n(z)$ be a harmonic function in $R_n(A_n^*)$ such that $w_n(z)=0$ on $C+\sum s_{ij}$ and $w_n(z)=1$ on $\partial(R(A_n^*))$. Then $w(\upsilon_l(p),z)\leqq w_n(z)$, where $w(\upsilon_l(p),z)$ is H.M. (harmonic measure) of $\upsilon_l(p)$ and $\upsilon_l(p)\subset(R-R_n(A_n^*))$. Consider $w_n(z)$ in B_{m-1}^*. Then by (14)

$$w_n(z)\leqq\frac{1}{m-1} \quad \text{on} \quad |z|=(r_{2m},r_{2m+1})^{\frac{1}{2}}$$

Hence by the maximum principle $w_n(z)\leqq\frac{1}{m-1}$ on $R_n(A_{m-1}^*)$. Let $l\to\infty$ and then $n\to\infty$ and then $m\to\infty$. Then

$$w(p, z) \leqq \lim w_n(z) = 0 \quad \text{in} \quad R.$$

Thus p is a singular point of first kind.

Remark. In Example 1, if we do not make slits s_{ij} in B_n, i.e. B_n has no slits and B_n is a simple ring. Then it is easily seen that $\omega(p, z) = w(p, z)$. Hence in this case p is a *singular point of second kind.*

Example 2. Let $1 = r_1 < r_2 < r_3, \cdots, \lim_n r_n = 2$. Let A_n and B_n be simple rings:

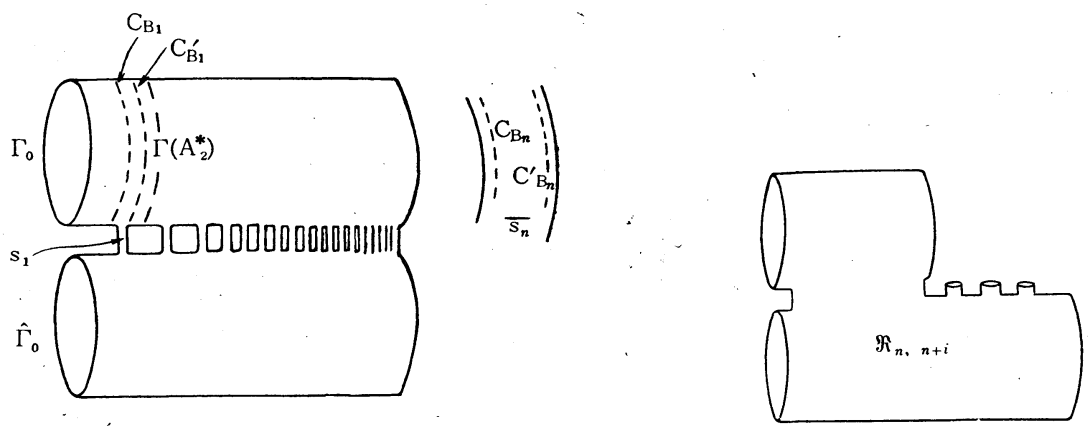

Fig. 6.

$$A_n:\ r_{2n+1} < |z| < r_{2n+2}, \quad \alpha_n = \log \frac{r_{2n+2}}{r_{2n+1}}$$

$$B_n:\ r_{2n+2} \leqq |z| < r_{2n+3}, \quad \beta_n = \log \frac{r_{2n+3}}{r_{2n+2}}$$

Let A_n^* be the same concentrating ring C.R. $\left(\alpha_n, \dfrac{1}{n}\right)$ defined in Example 1. Let $C_{B,n}$ and $C'_{B,n}$ be circles in B_n cited below:

$$C_{Bn}:\ |z| = \exp\left(\log r_{2n+2} + \frac{1}{4}\beta_n\right), \quad C'_{Bn}:\ |z| = \exp\left(\log r_{2n+2} + \frac{3}{4}\beta_n\right).$$

Put $R = A_1^* + B_1 + A_2^* +, \cdots$. Let $\omega(s_n, z)$ be C.P. of s_n in R, i.e. $\omega(s_n, z)$ is a harmonic function in R such that $\omega(s_n, z) = 0$ on $|z| = 1$, $\omega(s_n, z) = 1$ on s_n and has M.D.I. over R, where s_n is a slit in B_n:

$$s_n:\ r_{2n+2} \exp\left(\frac{\beta_n}{2} - t_n\right) < |z| < r_{2n+2} \exp\left(\frac{\beta_n}{2} + t_n\right), \ \arg z = 0.$$

Clearly $\omega(s_n, z) \downarrow 0$ as $t_n \to 0$, i.e. as the length of $s_n \to 0$. Hence there exists a number m_n such that $\omega(s_n, z) < \dfrac{1}{2^{n+2}}$ on $C_{Bn} + C'_{Bn}$ for $t \leqq m_n$. Put $t_n = m_n$. Then $\omega(s_n, z) < \dfrac{1}{2^{n+1}}$ on $C_{Bn} + C'_{Bn}$ and by the maximum principle[3)]

3) See "Potentials on Riemann surfaces".

$$\omega(s_n, z) < \frac{1}{2^{n+1}} \quad \text{in} \quad R - B_n. \tag{15}$$

In every B_n we make a slit s_n mentioned above and denote by B_n^* the B_n with a slit s_n.

Put $R(A_n^*) = A_1^* + B_1^* +, \cdots, B_{n-1}^* + A_n^*$ and $R(B_n^*) = R(A_n^*) + B_n^*$ and $R^* = \bigcup^{\infty} R_n(A_n^*)$.

Let $\hat{B}_n^*$ be a ring with one slit s_n which is symmetric to B_n^* with respect to the imaginary axis. Let $\tilde{A}_n^*$ be the ring with is dentical to A_n^*.

Put $\hat{R}(\hat{A}_n^*) = \hat{A}_1^* + \hat{B}_1^* + \hat{A}_2^*, \cdots \hat{B}_{n-1}^* + \hat{A}_n^*$ and $\hat{R}(\hat{B}_n^*) = \hat{R}(\hat{A}_n^*) + \hat{B}^*$ and $\hat{R}^* = \bigcup \hat{R}(\hat{A}_n^*)$.

Connect R and $\hat{R}$ by identifying two edges of s_n and $\hat{s}_n$ $(n = 1, 2 \cdots)$. Then we have a Riemann surface $\mathfrak{R}$ which is symmetric with respect to $\sum s_n$. Then $\mathfrak{R}$ has the following properties:

a) $\mathfrak{R} \in H_0.2.B$ *and* $H_0.2.D.$[4]

b) $\mathfrak{R}$ *has no irregular points for the Green's function of* $\mathfrak{R}$ *and* $\mathfrak{R}$ *has no N-minimal harmonic function* $N(z, p)$ *such that* $\sup_{z \in R} N(z, p) = \infty$ *and* $p \in B$ *(B is the set of boundary point of* $\mathfrak{R}$*).*

c) $\mathfrak{R}$ *has two singular N-minimal points of second kind, which form the whole set of N-minimal boundary points and* $\mathfrak{R}$ *has non N-minimal points.*

Proof of a). Let $\Gamma(A_n^*)$ be a circle in A_n^* such that

$$|z| = (r_{2n+1}, r_{2n+2})^{\frac{1}{2}}.$$

Let Γ_0 be the compact relative boundary of $\mathfrak{R}$ on $|z| = 1$.
Let $U(z)$ be a bounded harmonic function in $\mathfrak{R} - \sum s_n$ such that $U(z) = 0$ on $\Gamma_0 + \hat{\Gamma}_0$ and $|U(z)| \leqq M$. Put $a_n = \max_{z \in \Gamma(A^*_n)} U(z)$ and $b_n = \min_{z \in \Gamma(A^*_n)} U(z)$. Then since A_n^* is a concentrating ring,

$$0 \leqq a_n - b_n \leqq \frac{M}{n} \tag{16}$$

We show $\lim_n a_n$ *and* $\lim_n b_n$ *exist and* $\lim_n a_n = \lim_n b_n$.

Assume $\overline{\lim_u} a_n = a' > \underline{\lim_n} a_n = a''$. Let $0 < \varepsilon < \frac{a' - a''}{10}$ Since $\overline{\lim} a_n = a'$ and $\underline{\lim} a_n = a''$, there exist numbers n', n'' and n''' such that

$$|a' - a_{n'}| < \varepsilon, \ |a'' - a_{n''}| < \varepsilon, \ |a' - a_{n'''}| < \varepsilon \text{ and } n' < n'' < n''' \text{ and } \frac{M}{2^{n'-1}} < \frac{4M}{n} < \varepsilon. \tag{17}$$

4) We denote by $H_0.2.B.(H_0.2.D.)$ the class of Riemann surfaces with compact relative boundaries ∂R_0 such that there exist 2 linearly independent bounded (Dirichlet bounded) harmonic functions vanishing on ∂R_0.

Let $(\Gamma(A^*_{n'}), \Gamma(A^*_{n'''}))$ be the part of R^* bounded by $\Gamma(A^*_{n'})+\Gamma(A^*_{n'''})+\sum_{n'}^{n'''-1} s_i$. Let $U'(z)$ be a harmonic function in $(\Gamma(A^*_{n'}), \Gamma(A^*_{n'''}))$ such that $U'(z)=b_{n'}$ on $\Gamma(A^*_{n'})$, $U'(z)=0$ on $\sum_{n'}^{n'''-1} s_i$ and $U'(z)=b_{n'''}$ on $\Gamma(A^*_{n'''})$. Let $\omega(s_n, z)$ be C.P. of s_n relative $R=A_1^*+B_1+A_2^*+B_2,\cdots$. Then

$$U(z)\geqq U'(z)-M\sum_{n'}^{n'''-1}\omega(s_i, z). \tag{18}$$

Now by (15)

$$M\sum_{i=n'}^{n'''-1}\omega(s_i, z)<\frac{M}{2^{n'+1}}\left(1+\frac{1}{2}+\frac{1}{2^2}+,\cdots\right)\leqq\frac{M}{2^{n'}} \text{ on } (\Gamma(A^*_{n'}), \Gamma(A^*_{n'''}))-\sum_{n'}^{n'''-1}B_n.$$

Hence by (17)

$$\begin{aligned} U(z)&\geqq\min\left(b_{n'}-\frac{M}{2^{n'}},\ b_{n'''}-\frac{M}{2^{n'}}\right)\\ &\geqq\min\left(a_{n'}-\frac{M}{n}-\frac{M}{2^{n'}},\ a_{n'''}-\frac{M}{n}-\frac{M}{2^{n'}}\right)\geqq a'-\varepsilon-\frac{M}{n} \text{ on } \Gamma(A^*_{n''}).\end{aligned}$$

Whence $(a''+\varepsilon>a_{n''}>)\ b_{n''}=\min_{\Gamma(A^*_{n''})} U(z)>a'-2\varepsilon$. This contradicts (17). Hence $a'=a''$ and $\lim_n a_n$ exists. Similarly $\lim_n b_n$ exists and by (16) $\lim_n a_n=\lim_n b_n$.

Put $\mathfrak{R}_{n,n+i}=R(A_n^*)+\hat{R}(\hat{A}^*_{n+1})$ (see Fig. 6). Then $\mathfrak{R}_{n,n+i}$ is bounded by $\hat{\Gamma}_0+\Gamma_0+\sum_n^{n+i-1}s_i+\Gamma(A_n^*)+\hat{\Gamma}(\hat{A}^*_{n+i})$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $\mathfrak{R}_{n,n+i}$ such that $\omega_{n,n+1}(z)=0$ on $\Gamma_0+\hat{\Gamma}_0$, $\omega_{n,n+i}(z)=1$ on $\Gamma(A_n^*)$ and $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\hat{\Gamma}(\hat{A}^*_{n+i})+\sum^{n+i-1}s_i$. Let $\tilde{\omega}_n(z)$ be a harmonic function in $R^*(A_n)$ such that $\tilde{\omega}_n(z)=1$ on Γ_0, $\tilde{\omega}_n(z)=1$ on $\Gamma(A_n^*)$ and $\frac{\partial}{\partial n}\tilde{\omega}_n(z)=0$ on $\sum^n s_i$. Then $\tilde{\omega}_n(z)=\dfrac{\log|z|}{\frac{1}{2}\log r_{2n+1}r_{2n+2}}$ and

$$D(\omega_{n,n+i}(z))\geqq D(\tilde{\omega}_n(z))\geqq\frac{2\pi}{\log 2}. \tag{19}$$

Let $i\to\infty$ and then $n\to\infty$. Then $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ and $\omega_n(z)\Rightarrow$C.P. determined by a sequence $\{R^*-R(A_n^*)\}$ $(n=1,2,\cdots)$ which we denote by $\omega(B, z)$. Then by (19) $\omega(B, z)>0$. Consider $\omega_{n,n+i}(z)$ in $\hat{R}(\hat{A}^*_{n+i})$. Now $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\hat{\Gamma}(\hat{A}^*_{n+i})$ and <1 on $\sum s_i$ and $\omega_{n,n+i}(z)=0$ on $\hat{\Gamma}_0$. Let $\hat{\omega}(s_m, z)$ be C.P. of s_m in $\hat{R}$. Then $\hat{\omega}_{n+i}(s_m, z)\to\omega(s_m, z)$ as $n+i\to\infty$, where $\hat{\omega}_{n+i}(s_m, z)$ is a harmonic function in $\hat{R}_{n+i}$ (symmetric surface of R_{n+i}) such that $\hat{\omega}_{n+i}(s_m, z)$

$=1$ on s_m, $\hat{\omega}_{n+i}(s_m, z)=0$ on $\hat{\Gamma}_0$ and $\frac{\partial}{\partial n}\hat{\omega}_{n+i}(s_m, z)=0$ on $\hat{\Gamma}(\hat{A}^*_{n+i})$. Hence by the maximum principle

$$\omega_{n,\,n+i}(z) \leqq \sum_{m=1}^{n+i} \hat{\omega}_{n+i}(s_m, z) \text{ in } \hat{R}(\hat{A}^*_{n+i}).$$

Let $i \to \infty$ and then $n \to \infty$. Then

$$\omega(B, z) \leqq \sum_{m=1}^{\infty} \hat{\omega}_{n+1}(s_m, z) \text{ in } \hat{R}^*.$$

By (15) $\omega(B, z) \leqq \sum_{n=1}^{\infty} \hat{\omega}(s_m, z) \leqq \frac{1}{2^2}$ on $\hat{\Gamma}(\hat{A}^*_n)$. Since $\omega(B, z) \leqq 1$ in $\hat{R}^*$, $\lim_{n=\infty} \omega(B, z)$ exists and $\lim_{n=\infty} \omega(B, z) \leqq \frac{1}{4}$ i.e. $\lim_{n=\infty} \max_{z \in \Gamma(A^*_n)} \omega(B,z) = \lim_{n=\infty} \min_{z \in \hat{\Gamma}(\hat{A}^*_n)} \omega(B, z) \leqq \frac{1}{4}$.

Now $\omega(B, z) > 0$ implies $\sup \omega(B, z) = 1$ by P.C.2.[5] Consider $\omega(B, z)$ in $R^*(A^*_n) + \hat{R}^*(\hat{A}^*_n)$. Then $\omega(B, z) \leqq \max_{z \in (\Gamma(A^*_n + \hat{\Gamma}(\hat{A}^*_n))} \omega(B, z)$ in $R^*(A^*_n) + \hat{R}^*(\hat{A}^*_n)$. Hence $\lim_{n=\infty} \max_{z \in \Gamma(A^*_n)} \omega(B, z) = \lim_{n=\infty} \min_{z \in \Gamma(A^*_n)} \omega(B, z) = 1$, since A^*_n is C.R. $\left(\alpha, \frac{1}{n}\right)$.

Let $\omega(\hat{B}, z)$ be C.P. of the ideal boundary determined by the sequence $\{\hat{R}^* - \hat{R}^*(A^*_n)\}$ similarly as $\omega(B, z)$. Then

$$\omega(B, z) \neq \omega(\hat{B}, z).$$

If we denote $\lim_{z \in \Gamma(A^*_n)} \omega(B, z)$ and $\lim_{z \in \hat{\Gamma}(\hat{A}^*_n)} \omega(B, z)$ by *values of* $\omega(B, z)$ *at* B *and* $\hat{B}$ respectively. Then

$$\text{value of } \omega(B, z) \text{ at } B = \text{value of } \omega(\hat{B}, z) \text{ at } \hat{B} = 1$$
$$\text{value of } \omega(B, z) \text{ at } \hat{B} = \text{value of } \omega(\hat{B}, z) \text{ at } B \leqq \frac{1}{4}.$$

Let $U(z)$ be a harmonic function in $\mathfrak{R}$ such that $U(z)=0$ on $\Gamma_0 + \hat{\Gamma}_0$ and $|U(z)| \leqq M$. Then $\lim_{n=\infty} \max_{z \in \Gamma(A^*_n)} U(z) = \lim_{n=\infty} \min_{z \in \Gamma(A^*_n)} U(z)$ and $\lim_{n=\infty} \max_{z \in \hat{\Gamma}(\hat{A}^*_n)} U(z) = \lim_{n=\infty} \min_{z \in \hat{\Gamma}(\hat{A}^*_n)} U(z)$ which we denote by α and $\hat{\alpha}$ respectively. Then it is proved as before that there exist two constant β and $\hat{\beta}$ such that

$$U(z) = \beta\omega(B, z) + \hat{\beta}\omega(\hat{B}, z).$$

Hence $\mathfrak{R} \in H_0.2.B$, because $\omega(B, z)$ and $\omega(\hat{B}, z)$ are linearly independent.

Map the universal covering surface $\mathfrak{R}^\infty$ of $\mathfrak{R}$ onto $|\xi| < 1$. Suppose

5) See the properties of capacitary potentials and harmonic measures of "Potentials on Riemann Surfaces".

$U(z)\in H_0.D$ ($H_0.D.$ means the class of Dirichlet bounded harmonic functions vanishing on $\Gamma_0+\hat{\Gamma}_0$). Then $U(z)$ is Poisson's integrable. Hence $U(z)$ must be a linear form of $\omega(B,z)$ and $\omega(\hat{B},z)$. Hence $\mathfrak{R}\in H_0.2.D$, because $\omega(B,z)$ and $\omega(\hat{B},z)\in H_0D$ and linearly independent.

Proof of b and c). N-Martin's topology can be defined in $\mathfrak{R}$. Let B_N the ideal boundary points $\mathfrak{R}$ with respect to N-Martin's topology. Let $\omega_n(z)$ be a harmonic function in $\hat{R}(\hat{A}_n^*)+R(A_n^*)$ such that $\omega_n(z)=0$ on $\Gamma_0+\hat{\Gamma}_0$ and $\omega_n(z)=1$ on $\Gamma(A_n^*)+\hat{\Gamma}(A_n^*)$. Then $\lim_n \omega_n(z)=\omega(B_N,z)=w(B_N,z)$, where $\omega(B_N,z)$ and $w(B_N,z)$ are C.P. and H.M. of B_N.

Clearly $\omega_n(z)\geqq\omega(B,z)$ and $\geqq\omega(\hat{B},z)$. Hence for any given positive number ε there exists a number n_0 such that $\omega(B,z)>1-\varepsilon$ on $\Gamma(A_n^*)$ and $\omega(B,z)>1-\varepsilon$ on $\hat{\Gamma}(\hat{A}_n^*)$ for $n\geqq n_0$. Hence by the minimum principle $\omega_n(z)>1-\varepsilon$ on $\mathfrak{R}-(R(A_n^*)+\hat{R}(\hat{A}_n^*)$ and $\omega(B_N,z)=w(B_N,z)>1-\varepsilon$ in $\mathfrak{R}-(R(A_n^*)+R(\hat{A}_n^*))$, whence every point of B_N is regular for the Green's function. Hence by Theorem 11. *b*) of the previous paper "Singular Pointsof Riemann Surfaces" there exists no N-minimal function $N(z,p)$ such that $\sup_{z\in R} N(z,p)=\infty$ by $\mathfrak{R}\in H_0.2.D$ and by Theorem 16. *c*) of the same paper B_N has two singular points of second kind, beccuse $\mathfrak{R}\in H_0.2.D$. Let $\{p_i\}$ be a sequence contained in $\sum_{i=1}^{\infty} s_i$ such that $\{N(z,p_i)\}$ converges uniformly to $N(z,p)$. By the symmetry of $\mathfrak{R}$, $N(z,p)=c(\omega(B,z)+\omega(\hat{B},z))$, where c is a constant. $\omega(B,z)+\omega(\hat{B},z)$ is not N-minimal, since $(\omega(B,z)+\omega(\hat{B},z))-\omega(B,z)$ and $\omega(B,z)$ are superharmonic in $\mathfrak{R}$. Hence $N(z,p)$ is non N-minimal, whence $\{p_i\}$ determines a non N-minimal point and $\mathfrak{R}$ has non minimal points.

2. Intense connection. Let J be a simple ring: $1<|z|<\exp\alpha$ and s_j be slits:

$$s_j:\ \exp\frac{\alpha}{4}<|z|<\exp\frac{3\alpha}{4},\ \arg z=\frac{2\pi(j-1)}{2m},\ j=1,2,\cdots,2m.$$

Let $J(s)$ be the ring J with slits $\sum s_j$ and let $\hat{J}(s)$ be an identical examplar as $J(s)$. Connect $\hat{J}(s)$ and $J(s)$ crosswise on $\sum s_j$. Then we have a two-sheeted Riemann surface $R=J(s)+\hat{J}(s)$. Let $U(z)$ be a harmonic function in R and let $P(z)$ be the transformation in R such that $P(z)$ and z have the same projection. Then $U(P(z))-U(z)$ is harmonic in R and vanishes on branch points (endpoints of $\sum s_j$). If $|U(p(z))-U(z)|<M$ in R, then there exists a constant $\lambda<1$ depending on $\mathfrak{M}$ such that

$$|U(P(z))-U(z)|<M\lambda \quad \text{on} \quad |z|=\exp\frac{\alpha}{2},$$

if $0<\mathfrak{M}<\dfrac{\alpha}{\frac{2\pi}{m}}<3\mathfrak{M}$.

In the present paper we fix $\mathfrak{M}$ and suppose that the ratio $\dfrac{\alpha}{\frac{2\pi}{m}}$ of any intense connection used in this paper satisfies

$$\mathfrak{M}<\frac{\alpha}{\frac{2\pi}{m}}<3\mathfrak{M}. \tag{20}$$

In fact, map R onto $0<\operatorname{Re}\zeta<\alpha$ by $\zeta=\log z$. Assume that there exists no constant λ mentioned above, we can construct a two-sheeted Riemann surface R_i with $\left(\frac{\alpha_i}{m}\right)\to\left(\frac{\alpha}{m}\right)$: $\mathfrak{M}<\dfrac{\alpha_i}{\frac{2\pi}{m}}<3\mathfrak{M}$ and we can also find a sequence of harmonic functions $U_i(z)$ on R_i (R_i varies) such that $\max|(U(P(z))-U(z))|=M$ on $\operatorname{Re}\zeta=\frac{\alpha}{2}$ Now $U(z)$ is a normal family in $\{R_i\}$: the set of Riemann surface. Choose a subsequence $U_i(z)$ of $\{U_i(z)\}$ such that $U_i(z)$ converges uniformly to $U(z)$ such that $|U(P(z))-U(z)|=M$ on $|z|=\exp\frac{\alpha}{2}$. Clearly $U(z)$ is a harmonic function in the limit surface $R=\lim_i R_i$ and $U(P(z))-U(z)=0$ on endpoints of $\sum s_j$, whence $U(z)$ is non constant. This contradicts the maximum principle. Hence

$$|U(P(z))-U(z)|<M\lambda \quad \text{on} \quad |z|=\exp\alpha, \text{ if } \mathfrak{M}<\frac{\alpha}{\frac{2\pi}{m}}<3\mathfrak{M}.$$

Intense connection a for two-sheeted Riemann surface. Let J be a simple ring: $1<|z|<\exp\alpha$ and let m be an integer such that $\lambda^m<\varepsilon$.

Let H_i, H_i' and Γ_i, Γ_i' $(i=1,2,\cdots,m)$ and Γ be rings and circles cited below:

$$H_i:\ \exp(i-1)\gamma<|z|<\exp i\gamma, \qquad H_i':\ \exp(\alpha-i\gamma)<|z|\exp(\alpha-i-1)\gamma).$$

$$\Gamma_i:\ |z|=\exp\left(i-\frac{1}{2}\right)\gamma, \qquad \Gamma_i':\ |z|=\exp\left(\alpha-\left(i-\frac{1}{2}\right)\gamma\right),$$

$$\Gamma:\ |z|=\exp\frac{\alpha}{2},$$

Let $s_{i,j}(s'_{i,j})$ $(j=1,2,\cdots,2l)$ be slits in $H_i(H'_i)$ such that where $\gamma=\dfrac{\alpha}{2m+1}$.

$$s_{i,j}:\ \exp\left((i-1)+\frac{1}{4}\right)\gamma<|z|<\exp\left(i-1+\frac{3}{4}\right)\gamma, \qquad \arg z=\frac{2\pi}{l}(j-1).$$

$$s'_{i,j}:\ \exp\left(\alpha-\left(i-1+\frac{3}{4}\right)\gamma\right)<|z|<\exp\left(\alpha-\left(i-1+\frac{1}{4}\right)\gamma\right), \quad \arg z=\frac{2\pi}{l}(j-1),$$

where the ratio $\dfrac{\gamma}{\frac{2\pi}{l}}$ satisfies $\mathfrak{M}<\dfrac{\gamma}{\frac{2\pi}{l}}<3\mathfrak{M}$.

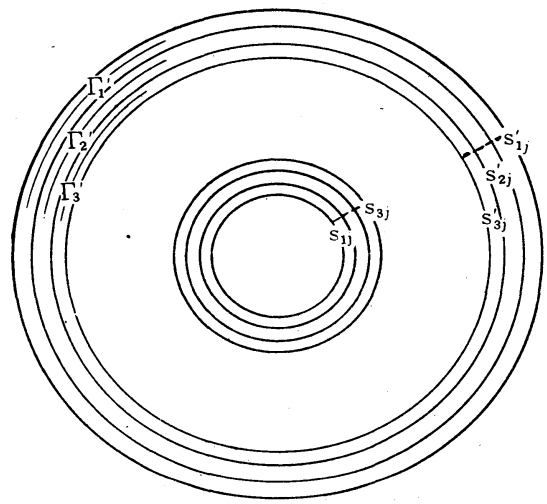

Fig. 7.

Let $J(s)$ be the ring with slits $\sum(s_{ij}+s'_{ij})$ and let $\hat{J}(s)$ be the identical leaf as $J(s)$.

Connect $\hat{J}(s)$ and $J(s)$ crosswise on $\sum_{ij}(s_{ij}+s'_{ij})$. Then we have a two-sheeted Riemann surface $J(s)+\hat{J}(s)$. Let $P(z)$ be the transformation in $R=J(s)+\hat{J}(s)$ such that $\operatorname{proj} P(z)=\operatorname{proj} z$. Then since R is invariant with respect to $P(z)$, $U(P(z))-U(z)$ is harmonic in R and $|U(P(z))-U(z)|<M\lambda$ on $\Gamma_1+\Gamma'_1$, whence by the maximum principle $|U(P(z))-U(z)|<M\lambda$ on the part of R bounded by $\Gamma_1+\Gamma'_1$. Also $|U(P(z))-U(z)|<M\lambda$ in $H_2+H'_2$ implies $|U(P(z)-U(z)|<M\lambda^2$ on the part of R bounded by $\Gamma_2+\Gamma'_2$. In this way we have

$$|U(P(z))-U(z)|<M\lambda^m \quad \text{on} \quad |z|=\exp\frac{\alpha}{2}.$$

Every harmonic function in R such that $0<U(z)<M$ has almost equal values with projectional deviation $<\dfrac{\varepsilon}{M}$ i.e.

$$|U(z_1)-U(z_2)|<\frac{\varepsilon}{M} \quad \text{for} \quad \operatorname{proj} z_1=\operatorname{proj} z_2\in\Gamma.$$

Such a connection will be called *an intense connection in the ring* $1<|z|<\exp\alpha$ *with projectional deviation* $<\dfrac{\varepsilon}{M}$.

Intense connection of 2^{q_n} *number of rings with projectional deviation* ε_n.

Let J be a simple ring $1<|z|<\exp\alpha$. Let H_l, H_l' be rings Γ_l, Γ_l', Γ be circles $(l=1,2,\cdots,q_n)$ cited belows:

$$H_l:\ \exp(l-1)\beta<|z|<\exp l\beta,\quad H_l':\ \exp(\alpha-(l\beta))<|z|<\exp(\alpha-(l-1)\beta),$$

$$\Gamma_l:\ |z|=\exp\left(l-1+\frac{1}{2}\right)\beta,\qquad \Gamma_l':\ |z|=\exp\left(\alpha-\left(l-1+\frac{1}{2}\right)\beta\right).$$

$$\Gamma:\ z=\exp\frac{\alpha}{2},$$

where $\beta=\dfrac{1}{2q_n+1}$.

H_l and H_l' $(l=1,2,\cdots,q_n)$ are conformally equivalent. In every $H_l(H_l')$ we make slits $\sum_{j,k}(s_{lj,k}+s'_{lj,k})$ which are conformally equivalent relative

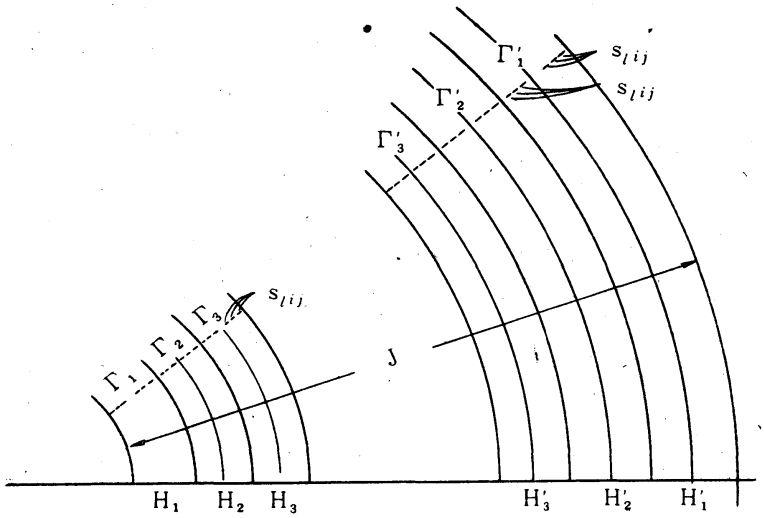

Fig. 9.

H_l (H_l') independent of l for intense connection so that *every harmonic function* $0<U(z)<M$ *in* $H_l+\hat{H}_l(H_l'+\hat{H}_l')$ *must have the projectional deviation* ε' $\left(\varepsilon'=\dfrac{\varepsilon_n}{q_n}\right)$ *on* $\Gamma_l+\Gamma_l'$, where $\hat{H}_l(H_l')$ is the examplar which is identical to $H_l(H_l')$ and H_l and $\hat{H}_l(H_l'$ and $\hat{H}_l')$ are cennected crosswise on $\sum_{j,k}(s_{l,kj}+s'_{l,jk})$ contained in $H_l(H_l')$. (see Fig. 9)

We denote be $I(1)$ the ring J with slits for intense connection. Let $I(2), I(3),\cdots, I(2^{q_n-1})$ and $\hat{I}(1), \hat{I}(2), \hat{I}(3)\cdots, \hat{I}(2^{q_n-1})$ be identical examplars as $I(1)$.

First connection and first order group.

Connect $I(i)$ and $\hat{I}(i)$ on slits crosswise on the slits contained in H_1+H_1' as follows:

$$\begin{array}{ccccc} I(1) & I(2),\cdots\cdots & \left\{\begin{array}{c} I(i) \\ \updownarrow \\ \widehat{I}(i) \end{array}\right\},\cdots & , I(2^{q_n-1}) \\ \updownarrow & \updownarrow & & \updownarrow \\ \widehat{I}(1) & \widehat{I}(2),\cdots\cdots & & , \widehat{I}(2^{q_n-1}). \end{array}$$

first order group.

2-nd connection and 2-nd order group.

$$\begin{array}{ccc} I(1) \quad I(2),\cdots & \left\{\begin{array}{cc} I(2j+1) & I(2j+2) \\ \widehat{I}(2j+1) & \widehat{I}(2j+2) \end{array}\right\},\cdots & \begin{array}{cc} I(2^{q_n-1}-1) & I(2^{q_n-1}) \\ \widehat{I}(2^{q_n-1}-1) & \widehat{I}(2^{q_n-1}) \end{array} \\ \widehat{I}(1) \quad \widehat{I}(2),\cdots & & \end{array}$$

2-nd order group

$I(2j+1)$ and $\widehat{I}(2j+1)$ and $I(2j+2)$ and $\widehat{I}(2j+2)$ are crosswise on the slits contained in H_2+H_2'.

3-rd connection and 3-rd order group.

$$\begin{array}{ccc} I(1) \quad I(2) \quad I(3) \quad I(4),\cdots & \left\{\begin{array}{cccc} I(2j+1) & I(2j+2) & I(2j+3) & I(2j+4) \\ I(2j+1) & I(2j+2) & I(2j+3) & I(2j+4) \end{array}\right\},\cdots \\ \widehat{I}(1) \quad I(2) \quad I(3) \quad I(4),\cdots & \end{array}$$

3-rd order group

$$j=0,1,2,\cdots\frac{2^{q_n-1}-4}{4}.$$

Examplars with an arrow are connected crosswise on the slits contained in H_3 and H_3'.

$\cdots\cdots$

$\cdots\cdots$

i-th connection and i-th order group

$$\left\{\begin{array}{cccc} I(2^{i-1}j+1) & I(2^{i-1}j+2),\cdots I(2^{i-1}j+2^{i-2}) & I(2^{i-1}j+2^{i-2}+1),\cdots I(2^{i-1}j+2^{i-1}) \\ \widehat{I}(2j+1) & \widehat{I}(2j+2),\cdots\cdots \widehat{I}(2j+2^{i-1}) & \widehat{I}(2j+2^{i-2}+1),\cdots\cdots \widehat{I}(2^{i-1}+2^{i-1}) \end{array}\right\}$$

i-th order group

$$j=0,1,2,\cdots\ \frac{2^{q_n-1}-2^{i-1}}{2^{i-1}}.$$

Examplars contained in the i-th groups with an arrow are connected crosswise on the slits contained in H_i+H_i'.

.....

q_n-th connection and q_n-th order group

In this case q_n-th group is only one and

$$I(1)\ I(2),\cdots\cdots,\ I(2^{q_n-2})\ I(2^{q_n-2}+1),\cdots\cdots\ I(2^{q_n-1})$$

$$\hat{I}(1)\ \hat{I}(2),\cdots\cdots,\ \hat{I}(2^{q_n-2})\ \hat{I}(2^{q_n-2}+2),\cdots\cdots\ \hat{I}(2^{q_n-1})$$

Examplars with an arrow are connected crosswise on the slits contained in $H_{q_n}+H_{q_n}'$. Then we have a 2^{q_n}-sheeted covering surface R over $I(1)=J$.

Let $P_1(z)$ be the transformation such that $\operatorname{proj} P_1(z)=\operatorname{proj} z$ and $P(z)$ and z are contained in the examplars with an arrow in the first connection. Then $U(P_1(z))-U(z)$ is harmonic in R and $|U(P(z))-U(z)|<\varepsilon'$ on $\Gamma_1+\Gamma_1'$, whence by the maximum principle $|U(P(z))-U(z)|<\varepsilon'$ on Γ, that is

$$|U(z')-U(z'')|<\varepsilon',\ \text{if}\ \operatorname{proj} z'=\operatorname{proj} z''\in\Gamma\ \text{and}\ z'\in I(i)\ \text{and}\ z''\in I(i). \tag{21}$$

Let $P_2(z)$ be the transformation such that $\operatorname{proj} P(z)=\operatorname{proj} z$ and z are contained in the examplars with an arrow in the second connection. Then as above

$|U(z')-U(z')|<\varepsilon'$ if $\operatorname{proj} z'=\operatorname{proj} z''\in\Gamma$ and z' and z'' are contained in the examplars with an arrow in the second connection. (22)

By (21) and (22)

$$|U(z')-U(z'')|<2\varepsilon':\ \operatorname{proj} z'=\operatorname{proj} z''\in\Gamma,$$

where z' and z'' are contained in the examplars in the same group of second order. (23)

.....

Let $P_i(z)$ be the transformation corresponding to i-th connection as above. Then

$|U(P_i(z))-U(z)|<\varepsilon'$ on Γ, whence similarly as above

$$|U(z')-U(z'')|<i\varepsilon':\ \operatorname{proj} jz'=\operatorname{proj} z''\in\Gamma,$$

where z and z'' are contained in the examplars in the same group of i-th order. Now the q_n-th group contains all examplars. Hence

$$|U(z')-U(z'')|<q_n\varepsilon'=\varepsilon_n:\ \operatorname{proj} z'=\operatorname{proj} z''\in\Gamma, \tag{24}$$

where z' and z'' any points of 2^{q_n} examplars.

Thus $U(z)$ has almost equal value on $|z|=\exp\dfrac{\alpha}{2}$ *with projectional devia*

tion ε_n. We call such a connection 2^{q_n}-*sheets intense connection with projectional deviation* ε_n.

3. Folded concentrating ring of 2^{q_n} number of sheets with aberration $<\varepsilon_n$.

We shall combine the two operations: concentrating and intense connection.

Let G be a simple ring: $1<|z|<\exp\alpha$. Let $L_1, L_2, L_3, L_1', L_2', L_3'$ be rings and $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_1', \Gamma_2', \Gamma_3'$ be circles cited below:

$L_1: 1<|z|<\exp\beta_n$, $L_2: \exp\beta_n<|z|<\exp 2\beta_n$, $L_3: \exp 2\beta_n<|z|<\exp 3\beta_n$.
$L_1': \exp\alpha>|z|>\exp(\alpha-\beta_n)$, $L_2': \exp(\alpha-\beta_n)<|z|<\exp(\alpha-2\beta_n)$,
$L_3': \exp(\alpha-2\beta_n)<|z|<\exp(\alpha-3\beta_n)$.

$$\Gamma_1: \exp\frac{\beta_n}{2}=|z|, \qquad \Gamma_2: |z|=\exp\frac{3\beta_n}{2}, \qquad \Gamma_3: |z|=\exp\frac{5}{2}\beta_n.$$

$$\Gamma_1': |z|=\exp\left(\alpha-\frac{\beta_n}{2}\right), \quad \Gamma_2': z=\exp\left(\alpha-\frac{3\beta_n}{2}\right), \quad \Gamma_3': |z|=\exp\left(\alpha-\frac{5}{2}\beta_n\right),$$

where $7\beta_n<\alpha$.

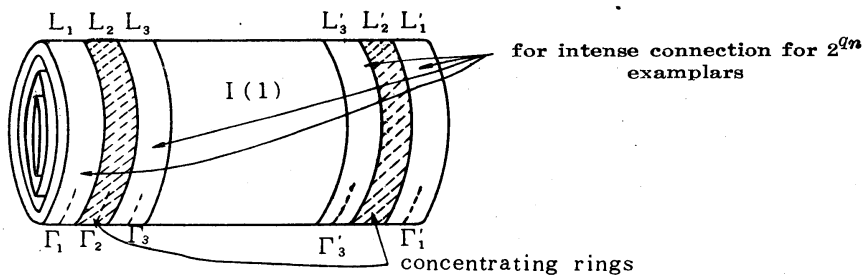

Fig. 9.

In L_i and L_i' $(i=1,3)$ we make slits for intense connection for 2^{q_n}-examplars with *projectinal deviation* $<\frac{\varepsilon_n}{2}$ on Γ_i and Γ_i' $(i=1,3)$. These slits contained in L_i and L_i' must be conformally equivalent. In L_2 and L_2' we make slits and identify their edges so that every harmonic function $|U(z)|<1$ in L_2 (L_2') has *circular aberration* $<\frac{\varepsilon_n}{2}$ on Γ_2 (Γ_2'). Hence we have a generalized ring $I(1)$ with two concentrating ring L_2 and L_2' and has slits for intense connection for 2^{q_n}-examplars with projectional deviation $<\frac{\varepsilon_n}{2}$. Let $I(2), I(3), \cdots, I(2^{q_n-1}), \hat{I}(1), \hat{I}(2), \cdots, I(2^{q_n-1})$ be $2^{q_n}-1$ number of generalized rings with slits which are identical to $I(1)$. Perform the

intense connection about $I(1),\cdots I(2^{q_n-1}), \widehat{I}(1),\cdots \widehat{I}(2^{q_n-1})$, i.e. they will be connected crosswise on the slits for intense connection mentioned as 2) ("Intense connection"). Then we have a covering surface of 2^{q_n} sheets over $I(1)$. We denote it by R. Then R has the following property.

Let $U(z)$ be a harmonic function in R such that $0\leqq U(z)\leqq 1$. Let $P_i(z)$ be the projectional transformation such that $I(2^{i-1}j+k)\leftrightarrow \widehat{I}(2j^{i-1}+2^{i-2}+k)$. Then $U(P_i(z))-U(z)$ is harmonic in R, Then $|U(P_i(z))-U(z)|<\varepsilon'$, $\varepsilon'=\frac{\varepsilon_n}{2^{q_n}}$ on $\Gamma_1+\Gamma_3(\Gamma'_1+\Gamma'_3)$. Hence by the maximum principle $|U(P_i(z))-U(z)|<\varepsilon'$ on $L_2+L'_2$. Hence $U(z)$ has *projectional deviation* $<\frac{\varepsilon_n}{2}$ on $\Gamma_2+\Gamma'_2$. Next $L_2(L'_2)$ is a concentrating ring. $U(z)$ has *circular aberration* $<\frac{\varepsilon_n}{2}$ on $\Gamma_2(\Gamma'_2)$. Whence we see that there exist constants a' and a'' such that

$$|U(z)-a'|<\varepsilon_n \text{ for } z\in R \text{ and } \log|z|=\frac{3}{2}\beta_n, \tag{25}$$

$$|U(z)-a''|<\varepsilon_n \text{ for } z\in R \text{ and } \log|z|=\alpha-\frac{3}{2}\beta_n.$$

We denote such a 2^{q_n}-sheeted covering surface over a generalized ring by $(\alpha, \beta_n, q_n, \varepsilon_n)$ and call *a folded concentrating ring.*

Lemma 2. *Let S be a sector such that $1<|z|<\exp\gamma$, $0<arg\, z<\Theta$ with a finite number of radial slits. Let $U(z)$ be a harmonic function in S with boundary values $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ on $|z|=1$ and $|z|=exp\,\gamma=r$, where $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ are continous functions of z. Then*

$$D(U(z))\geqq\frac{1}{2\pi\gamma}\int_0^\theta|\varphi(e^{i\theta})-\varphi(re^{i\theta})|^2\,d\theta.$$

Proof. We divide S into sufficiently narrow circular rectangles $A_j: 1<|z|<\exp\gamma$, $\theta_j<\arg z\leqq\theta_{j+1}$ $(j=1,2,3,\cdots m)$ such that $\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})-\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})<\frac{1}{n}$ and $\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(re^{i\theta})-\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(re^{i\theta})<\frac{1}{n}$.

Let $\{A'_j\}$ and $\{A''_j\}$ be $\{A''_j\}$ such that $\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})\leqq\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(re^{i\theta})$ and $\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})\geqq\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(re^{i\theta})$ respectively and let $\{A''_j\}$ be rectangles contained neither in $\{A'_j\}$ nor in $\{A''_j\}$. Suppose $A_j\in\{A'_j\}$. Let $\widetilde{U}_j(z)$ be a harmonic function in A_j such that $\widetilde{U}_j(z)=\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})$ on $|z|=1$, $\widetilde{U}_j(z)=\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}(re^{i\theta})$ on $|z|=\exp\gamma$ and $\frac{\partial}{\partial n}\widetilde{U}_j(z)=0$ on two segments: $1<|z|<\exp\gamma$, $\arg z=\theta_j$

and $1<|z|<\exp\gamma$, $\arg z=\theta_{j+1}$ and on radial slits in A_j. Let $\tilde{\tilde{U}}_j(z)$ be a harmonic function in A' such that $\tilde{\tilde{U}}(z)=\varphi(e^{j\theta})$ on $|z|=1$ and $\tilde{\tilde{U}}_j(z)=\varphi(re^{i\theta})$ on $|z|=\exp\gamma$ and $\frac{\partial}{\partial n}\tilde{\tilde{U}}_j(z)=0$ on two segments: $1<|z|<\exp\gamma$, $\arg z=\theta_j$ and on $1<|z|<\exp\gamma$, $\arg z=\theta_{j+1}$ and on radial slits contained in A_j. Then since $U_j(z)-U_j(z)\geqq 0$ and $\frac{\partial}{\partial n}\tilde{U}_j(z)\geqq 0$ on $|z|=1$ and $\tilde{U}_j(z)-\tilde{\tilde{U}}_j(z)\leqq 0$ and $\frac{\partial}{\partial n}U_j(z)\leqq 0$ on $|z|=\exp\gamma$, we have

$$D_{A_j}(\tilde{U}_j(z),\ \tilde{U}_j(z)-\tilde{\tilde{U}}_j(z))\geqq 0,\ \text{whence}\ D_{A_j}(\tilde{\tilde{U}}_j(z))\geqq D_{A_j}(\tilde{U}_j(z)).$$

Clearly

$$D(\tilde{U}_j(z))=\frac{(\theta_{j+1}-\theta_j)}{2\pi\gamma}\left|\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(e^{i\theta})-\min_{\theta_j\leqq\theta\leqq\theta_{j+1}}\varphi(re^{i\theta})\right|^2.$$

On the other hand, $D_{A_j}(U(z)-\tilde{\tilde{U}}_j(z),\ \tilde{\tilde{U}}_j(z))=0$, whence $D_{A_j}(U(z))\geqq D_{A_j}(\tilde{\tilde{U}}_j(z))$.

Hence $$D_{A_j}(U(z))\geqq D_{A_j}(\tilde{\tilde{U}}_j(z))\geqq\frac{1}{2\pi\gamma}\int_{\theta_j}^{\theta_{j+1}}|\varphi(e^{i\theta})-\varphi(re^{i\theta})|^2-\frac{4(\theta_{j+1}-\theta_j)}{2\pi\gamma n^2}.$$

In case $A_j\in\{A_j''\}$, we have similarly the above inequality. In case $A_j\in\{A_j'''\}$, let $\tilde{U}_j(z)=0$.

Then $$D_{A_j}(U(z))\geqq 0\geqq\frac{1}{2\pi\gamma}\int_{\theta_j}^{\theta_{j+1}}|\varphi(e^{i\theta})-\varphi(re^{i\theta})|^2-\frac{4(\theta_{j+1}-\theta_j)}{2\pi\gamma n^2}.$$

Hence $$D(U(z))\geqq\sum_j D(\tilde{U}_j(z))\geqq\frac{1}{2\pi\gamma}\int_0^{\Theta}|\varphi(e^{i\theta})-\varphi(re^{i\theta})|^2-\frac{4\Theta}{2\pi\gamma n}.$$

Let $n\to\infty$. Then we have the lemma.

Example 3. We shall construct a Riemann surface which has singular points of first kind of power of continuum.

3.1. Let A_0 be a rectangle: $0<\mathrm{Re}\,z<\log 2$, $0<\mathrm{Im}\,z<2\pi$.

Let $A_{i_1,i_2,\cdots,i_n}$ ($i_1=1,2.\ i_2=1,2\cdots,i_n=1,2$) be a rectangle such that

$$(\log 2)\left(1+\frac{1}{2}+\cdots,\frac{1}{2^{n-1}}\right)<\mathrm{Re}\,z<(\log 2)\left(1+\frac{1}{2},\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}\right),$$

$$2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_n-1}{2^n}\right)$$
$$<\mathrm{Im}\,z<2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_n-1}{2^n}+\frac{1}{2^n}\right).$$

For every $A_{i_1,i_2,\cdots,i_n}$ identify the two segments

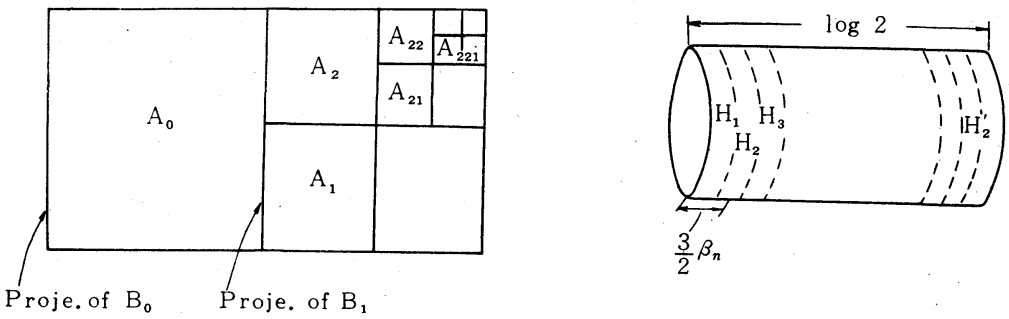

Fig. 10.

$$(\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right)<\mathrm{Re}\, z<\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n}}\right),$$

$$\mathrm{Im}\, z=2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_n-1}{2^n}\right)$$

and

$$(\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right)<\mathrm{Re}\, z<(\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n}}\right),$$

$$\mathrm{Im}\, z=2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_n-1}{2^n}+\frac{1}{2^n}\right).$$

Then $A_{i_1\cdots i_n}$ becomes a ring $A^R_{i_1\cdots i_n}$. $A_{i_1\cdots i_{n+1,1}}$ and $A_{i_1\cdots i_{n+1,2}}$ have boundary $B_{i_1\cdots i_{n-1}}$ lying on $\mathrm{Re}\, z=(\log 2)\left(1+\frac{1}{2^n}\right)$, $2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_{n-1}-1}{2^{n-1}}\right)$ $\leqq \mathrm{Im}\, z \leqq 2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2^2}+\cdots+\frac{i_{n-1}-1}{2^{n-1}}+\frac{1}{2^{n-1}}\right)$. Connect $(A^R_{i_1\cdots i_{n-1,1}}+A^R_{i_1\cdots i_{n-1,2}})$ with $A^R_{i_1,i_2\cdots i_{n-1}}$ on $B_{i_1\cdots i_{n-1}}$ for every $A_{i_1\cdots i_n}: n\leqq m$. Then we have a Riemann surface A^R_m. Put $A^R=\bigcup_{n=0}^{\infty} A^R_m$. Then A^R is a Riemann surface of planer character *and has* $2^{\aleph_0}=\mathfrak{X}$ *number of boundary components.* We call a ring $A^R_{i_1\cdots i_n}$ a ring of n-th grade. Let $A^R_{n\text{-th}}$ $(n\geqq 1)$ be one of n-th grade. We shall construct a generalized ring (folded concentrating ring). Every $A^R_{i_1\cdots i_n}$ is conformally equivalent to $A^R_{n\text{-th}}$. Map $A^R_{i_1\cdots i_n}$ onto ζ-plane as follows:

$$0<\mathrm{Re}<\log 2,\ 0\leqq \mathrm{Im}<2\pi.$$

Let H_1, H_2, H_3, H'_1, H'_2, H'_3 and Γ, Γ' be rings and circle cited belows:

$H_1: 0<\mathrm{Re}\,\zeta<\beta_n$. $H_2: \beta_n<\mathrm{Re}\,\zeta<2\beta_n$. $H_3: 2\beta_n<\mathrm{Re}\,\zeta<3\beta_n$.

$H'_1: \log 2>\mathrm{Re}\,\zeta>(\log 2)-\beta_n$. $H'_2: (\log 2)-\beta_n>\mathrm{Re}\,\zeta>(\log 2)-2\beta_n$.

$H'_3: (\log 2)-2\beta_n>\mathrm{Re}>(\log 2)-3\beta_n$.

$$\Gamma : \mathrm{Re}\,\zeta=\frac{3}{2}\beta_n. \qquad \Gamma' : \mathrm{Re}\,\zeta=(\log 2)-\frac{3}{2}\beta_n,$$

where

$$\beta_n=\frac{\log 2}{2^{2n}\times 6} \tag{26}$$

In $H_2\,(H_2')$ we make slits and identify their edges so that $H_2\,(H_2')$ in $A^R_{n_1\cdots i_n}$ becomes a concentrating ring $\left(\beta_n, \frac{1}{10^n}\right)$, i.e. circular aberration on $\Gamma(\Gamma')\leqq\frac{1}{10^n}$ (see Fig. 10). In H_1, H_3 and H_1', H_3' we make slits for intense connection for 2^{q_n} sheets so that the projectional deviation $<\frac{1}{10^n}$ on $\Gamma(\Gamma')$. When we perform the intense connection, a covering surface over $A^R_{i_1\cdots i_n}$ becomes a 2^{q_n}-folded concentrating ring $\left(\log 2, \beta_n, q_n, \frac{1}{10^n}\right)$, *where* $q_n=2n$. We remark: *the ratio of slits for intense connection is between* $\mathfrak{M}$ *and* $3\mathfrak{M}$.

Let $s_{1,j,k}$ be slits in H_1 for the intense connection (H_3, H_1', H_3'). Then $s_{1,j,k}$ is as follows:

let $h_{1,j}(h_{1j}')$ be a ring

$h_{1,j}$: $(j-1)\gamma\leqq\mathrm{Re}\,\zeta\leqq j\gamma$, $0\leqq\mathrm{Im}\,\zeta\leqq 2\pi$. $\quad h_{1j}'$: $\beta_n-i\gamma\leqq\mathrm{Re}\,\zeta\leqq\beta_n-(i-1)\gamma$, $0\leqq\mathrm{Im}\,\zeta\leqq 2\pi$, where $\gamma=\frac{\beta_n}{2q_n+1}$.

$s_{1,j,k}$ is a slit in $h_j\,(h_j')$ as follows:

$$s_{1,j,k}:\left(j-1+\frac{1}{4}\right)\gamma\leqq\mathrm{Re}\,\zeta\leqq\left(j-1+\frac{3}{4}\right)\gamma,\ \mathrm{Im}\,\zeta=2\pi\frac{k}{l_n}:k=1,\cdots l_n(j=1,2,\cdots,q_n)$$

and the ratio $\dfrac{\gamma}{\frac{2\pi}{l_n}}$ satisfies $\mathfrak{M}<\dfrac{\gamma}{\frac{2\pi}{l_n}}<3\mathfrak{M}$ independently of n. *Especially we observe* $s_{1,1,k}$ in a ring h_1 which is used for the first intense connection, where $h_1: 0\leqq\mathrm{Re}\,\zeta\leqq\gamma$, $0\leqq\mathrm{Im}\,\zeta\leqq 2\pi$. $\quad s_{1,1,k}: \frac{\delta}{4}\leqq\mathrm{Re}\,\zeta\leqq\frac{3}{4}$, $\mathrm{Im}\,\zeta=\frac{2\pi k}{l_n}$. (see Fig. 11)

Put $I(1)=A^R_{i_1\cdots,i_n}$ let $I(2),\cdots,I(2^{q_n-1})$ and $\hat{I}(1),\hat{I}(2),\cdots,\hat{I}(2^{q_n-1})$ be examplars which are identical to $A^R_{i_1\cdots,i_n}$ and perform the 2^{q_n}-intense connection by use of slits contained in H_1, H_3, H_1' and H_3'. Then we have a 2^{q_n} sheeted-folded concentrating ring $\left(\log 2, \beta_n, q_n, \frac{1}{10^n}\right)$. We denote it by $A^{f,C,R}_{i_1\cdots,i_n}$. All $A^{f,C,R}_{i_1\cdots,i_n}$ of n-th grade folded concentrating ring are coformally equivalent to $A^{f,C,R}_{i_1\cdots,i_n}$.

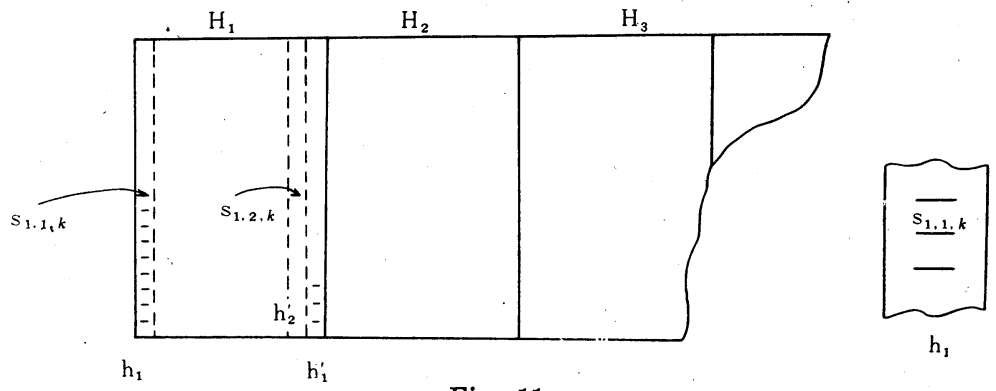

Fig. 11.

Let $A^{G,R}_{i_1\cdots,i_n}$ be a generalized ring with slits for intense connection and put $A^{G,R}_n = A^R_0 + (A_1+A_2) + (A_{11}+A_{12}+A_{21}+A_{22}) + \cdots + \underbrace{(A_{i_1 i_2 \cdots i_n} + A_{i_1, i_n} +, \cdots A)}_{2^n \text{ number}}$ and $A^{G,R} = \bigcup^{\infty} A^{C,R}_n$.

3.2. *Covering surface over* $A^{G,R}$.

Put $CA^{G,G}_n = A^{G,R} - A^{G,R}_{n-1}$. Then $CA^{G,R}_n$ is composed of 2^n components, i.e.

$$CA^{G,R}_n = \sum_{i_1 \cdots i_n} (A_{i_1 \cdots i_n} + A_{i_1 i_2 \cdots i_n, i_{n+1}} +, \cdots).$$

We denote by m CA m number of examplars which are identical to CA. Put $\mathfrak{R}^{G,R} = A_0 + (2^{q_1}-1)CA^{G,R}_1 + (2^{q_2}-2^{q_1})CA^{G,R}_2 +, \cdots (2^{q_n} - {}^{q_{n-1}})CA_n + \cdots$ Then $\mathfrak{R}^{G,R}$ is a covering surface (consisting of infinitely many disjoint components) over $A^{G,R}$. We shall construct a Riemann surface $\mathfrak{R}$ from $\mathfrak{R}^{G,R}$ by intense connection. Every $A^{G,R}_{i_1,\cdots,i_n}$ (of n-th grade $A^{G,R}_n$) is covered by $\mathfrak{R}^{G,R}$ 2^{q_n}-times. We shall perform the intense connection among 2^{q_n} number of sheets over $A^{G,R}_{i_1\cdots,i_n}$ as follows:

0-th step. $I^0(1) = A_0 + \sum_{i_1} A_{i_1} + \sum_{i_1, i_2} A_{i_1, i_2} + \cdots$, where A_0 is a simple ring.

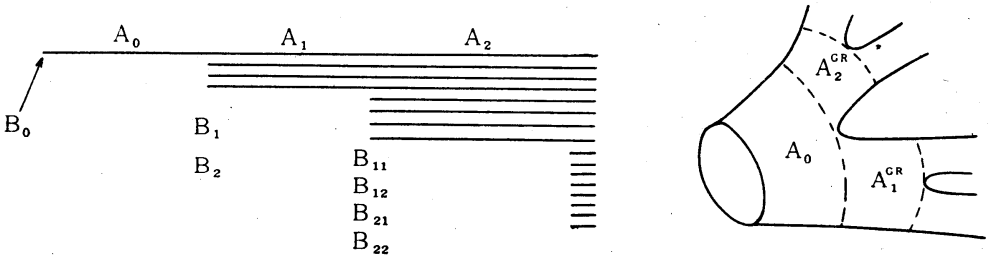

Fig. 12.

First step. Put $I^1(1) = CA_0$, $I^1(2) = I^1(3) = \cdots = I^1(2^{q_1-1}) = \hat{I}^1(1) = \hat{I}^1(2) \cdots = I^1(2^{q_1-1}) = I^1(1)$. $A^{G,R}_1 + A^{G,R}_2$ is covered by $\sum I^1(j) + \hat{I}^1(j)$. We perform the intense connection among the above 2^{q_1}-sheeted covering surface by

the slits contained in A_1+A_2 (the slits are contained in $H_1+H_1'+H_3+H_3'$ of A_i (1-st grade A_i). Then we have a Riemann surface $\mathfrak{R}_1$ which has the following properties.

a). $\mathfrak{R}_1$ has the following relative boundary B_0 which lies on $Re\, z=0$, $0<\mathrm{Im}\, z<2\pi$ and B_1 consisting of $2^{q_1}-1$ number of components lying on $\mathrm{Re}\, z=\log 2$, $0<\mathrm{Im}\, z<\pi$, and B_2 consisting of $2^{q_1}-1$ number of components lying on $\mathrm{Re}\, z=\log 2$, $\pi<\mathrm{Im}\, z<2\pi$.

b). The part of $\mathfrak{R}_1$ lying over $\mathrm{Re}\, z>(\log 2)\left(1+\frac{1}{2}\right)$ is now a generalized ring with slits for intense connection.

c). The part of $\mathfrak{R}_1$ lying over $\log 2<\mathrm{Re}\, z<(\log 2)\left(1+\frac{1}{2}\right)$ has no slits, since the intense connection is performed and the part over $A_1(A_2)$ is a folded concentrating ring $\left(\log 2,\ \beta_1\, q_2,\ \frac{1}{10}\right)$.

Suppose n-1-th step is performed.

n-th step is as follows: Let $I^n(i)$ be examplars cited below:

$$
\begin{array}{ll}
I^n(1) & =I^{n-1}(1) \\
\cdots\cdots & \\
\cdots\cdots & \\
\cdots\cdots & \\
\cdots\cdots & \\
I^n(2^{q_{n-1}-1}) & =I^{n-1}(2^{q_{n-1}-1}) \\
I^n(2^{q_{n-1}-1}+1) & =\widehat{I}^{n-1}(1) \\
\cdots\cdots & \\
\cdots\cdots & \\
I^n(2^{q_{n-1}}) & =\widehat{I}^{n-1}(2^{q_{n-1}-1}) \\
I^n(2^{q_{n-1}}+1) & \\
\cdots & \\
\cdots & \\
I^n(2^{n_{n-1}}) & \\
\widehat{I}^n(1) & \\
\cdots & \\
\cdots & \\
\cdots & \\
\cdots & \\
\cdots & \\
\cdots & \\
\widehat{I}^n(2^{q_{n-1}}) &
\end{array}
$$

newly added at the n-th step and $I^n(j)$ is identical to $CA_{n-1}^{G,R}$.

We perform the intense connection (for 2^{q_n} sheets) by slits contained in $A^{G,R}_{i_1\cdots i_n}$. Let $\mathfrak{R}_n$ be the above Riemann surface obtained after the intense connection and put

$$\mathfrak{R}=\bigcup_{n=1}^{\infty}\mathfrak{R}_n.$$

a). $\mathfrak{R}_n$ has relative boundary $B_0+(B_1+B_2)+\cdots\sum_{i_1,i_2,\cdots,i_n}B_{i_1\cdots i_n}$ and is open. $B_{i_1\cdots i_l}$ is composed of $2^{q_l}-2^{q_{l-1}}(l\leqq n)$ components lying over

$$\operatorname{Re} z=(\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{l-2}}\right),\ 2\pi\left(\frac{i_1-1}{2}+\cdots+\frac{i_{l-1}-1}{2^l}\right)$$
$$<\operatorname{Im} z<2\pi\left(\frac{i_1-1}{2}+\frac{i_2-1}{2}+\frac{i_{l-1}-1}{2^l}+\frac{1}{2^l}\right).$$

b). The part of $\mathfrak{R}_n$ lying over $\operatorname{Re} z>(\log 2)\left(1+\frac{1}{2}+\cdots+\frac{1}{2^n}\right)$ is now a generalized ring with slits (about which the intense connection is not performed) and consists of infinitely many components.

A). We denote by $A^{f,C,R}_{i_1\cdots i_n}$ the part of $\mathfrak{R}_n$ over $A^{G,R}_{i_1\cdots i_n}$. Then $A^{f,C,R}_{i_1\cdots i_n}$ is a folded concentrating ring $\left(\log 2,\ \beta_n,\ q_n,\ \frac{1}{10^n}\right)$.

We map $A^{i_1\cdots i_n}$ onto $0<\operatorname{Re}\zeta<\log 2,\ 0<\operatorname{Im}\zeta<2\pi$ on the ζ-plane. $A^{f,C,R}_{i_1\cdots i_n}$ is a 2^{q_n}-sheeted covering surface consisting of

$$I^n(1),\cdots,I(2^{q_{n-1}}),\underbrace{I^n(2^{q_{n-1}}+1)\cdots I^n(2^{q_n}),\ I^n(1)\cdots,\ \hat{I}^n(q_{n-1})}_{N({}^nI(i))}$$

of which $2^{q_n}-2^{q_{n-1}}$ number of sheets $(N({}^nI(i)))$ are newly added at the n-th step and $B_{i_1\cdots i_n}$ of $\mathfrak{R}$ is the relative boundary of $N(I(i))$ (the set of $I(i)$ and $I(i)$ which are added newly at the n-th step). Every $I^n(i)$ and $\hat{I}^n(i)$ are connected at the first connection, that is, $I^n(i)$ and $\hat{I}^n(i)$ are connected crosswise on $\sum_{j,k}s_{1,j,k}$. We observe $s_{1,1,k}$, where $\sum_k s_{1,1,k}$ are slits in $h_1+\hat{h}_1$ and $h_1(\hat{h}_1):0<\operatorname{Re}\zeta<\gamma,\ 0<\operatorname{Im}\zeta<2\pi$ (see Fig. 13).

$s_{1,1,k}:\frac{\delta}{4}<\operatorname{Re}\zeta<\frac{3}{4}\gamma,\ \operatorname{Im}\zeta=\frac{k}{2l_n}\ (k=1,2,\cdots l_n)$ the ratio $\dfrac{\gamma}{\frac{2\pi}{l_n}}$ is bounded so as

$$\mathfrak{M}<\frac{\gamma}{\frac{2\pi}{l_n}}<3\mathfrak{M}:\ \gamma=\frac{\beta_n}{2^{q_{n+1}}}. \tag{27}$$

$\hat{h}_1$ is the set which is identical to h_1 in ${}^n\hat{I}(i)$.

Now $h_1+\hat{h}_1$ is a generalized ring connected crosswise on $\sum_k s_{1,1,k}$ and has four relative boundaries β_1, β_2, β_3, β_4 (see Fig. 13).

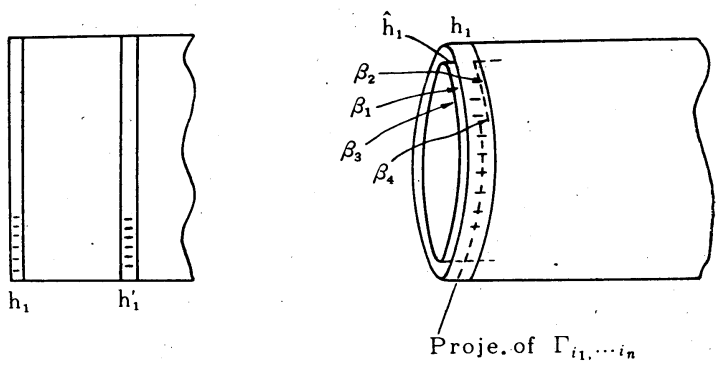

Fig. 13.

Let $w(z)$ be a harmonic function in $h+\hat{h}_1$ such that $0<w(z)<1$ and $=0$ on one of β_1, β_2, β_3 and β_4. Then since $\mathfrak{M}<\dfrac{\gamma}{\dfrac{2\pi}{l_n}}<3\mathfrak{M}$, there exists a constant μ such that

$$0<w(z)<\mu<1 \text{ on } \operatorname{Re}\zeta=\frac{\gamma}{2} \tag{28}$$

Let $\Gamma_{i_1\cdots i_n}$ be the set of $\mathfrak{R}$ lying over $\operatorname{Re}\zeta=\dfrac{\gamma}{2}$. Then $\Gamma_{i_1\cdots i_n}$ consists of 2^{q_n} number of circles which are contained in $A^{f,C,R}_{i_1\cdots i_n}$. Let $w(z)$ be a harmonic function in $\mathfrak{R}$ such that $0<w(z)<1$ and $w(z)=0$ on $B_0+(B_1+B_2)+(B_{11}+B_{12}+B_{21}+B_{12})+,\cdots$. Then since at least one of $I^n(i)$ and $\hat{I}^n(i)$ is added newly at the n-th step, one of β_1, β_2, β_3, β_4 must be contained in $B_{i_1\cdots i_n}$ on which $w(z)=0$. Now $h_1+\hat{h}_1$ in $I^n(i)+\hat{I}^n(i)$ of every $A_{i_1\cdots i_n}$ is conformally equivalent to that of $I^n(1)+\hat{I}^n(1)$. Hence by (28)

$$w(z)<\lambda<1 \quad \text{on} \quad \Gamma_{i_1\cdots i_n}. \tag{29}$$

Let $\mathfrak{R}^\Gamma_n$ be the component of $\mathfrak{R}$ containing B_0 divided by $\sum_{i_1=1,2,\ i_2=1,2,\cdots i_n=1,2} \Gamma_{i_1\cdots i_n}$ from $\mathfrak{R}$. Then $\mathfrak{R}^\Gamma_n$ $(n=1,2,\cdots)$ is an exhaustion of $\mathfrak{R}$, i.e.

$$\bigcup \mathfrak{R}^\Gamma_n=\mathfrak{R}. \tag{30}$$

B. Let ${}^1\Gamma^D_{i_1\cdots i_n}$ and ${}^2\Gamma^D_{i_1\cdots i_n}$ be the sets of $A^{f,C,R}_{i_1\cdots i_n}$ lying over the ring

$$\operatorname{Re}\zeta=\frac{3\beta_n}{2},\ 0<\operatorname{Im}\zeta<2\pi \text{ and } \operatorname{Re}\zeta=(\log 2)-\frac{3}{2}\beta_n,\ 0<\operatorname{Im}\zeta<2\pi.$$

The part $A^{f,C,R}_{i_1\cdots i_n}$ of $\mathfrak{R}$ over $A^{G,R}_{i_1\cdots i_n}$ is a folded concentrating ring. Let $\omega(z)$

be a harmonic function in $A^{f,C,R}_{i_1\cdots i_n}$ such that $0<\omega(z)<1$. Then there exist two constants $a^1_{i_1\cdots i_n}$ and $a^2_{i_1\cdots i_n}$ such that

$$|\omega(z)-a^1_{i_1\cdots i_n}|<\frac{1}{10^n} \quad \text{on } {}^1\Gamma^D_{i_1\cdots i_n}.$$

$$|\omega(z)-a^2_{i_1\cdots i_n}|<\frac{1}{10^n} \quad \text{on } {}^2\Gamma^D_{i_1\cdots i_n}.$$

Suppose $|a^1_{i_1\cdots i_n}-a^2_{i_1\cdots i_n}|>\dfrac{2}{10^n}$. Then the slits used for the intense connection are radial in $A^{G,R}_{i_1\cdots i_n}$ and $A^{f,C,R}_{i_1\cdots i_n}$ is composed of 2^{q_n} sheets. Hence by Lemma 2

$$D(\omega(z))>\frac{2^{q_n}\left(a^1_{i_1\cdots i_n}-a^2_{i_1\cdots i_n}-\dfrac{2}{10^n}\right)^2\times 2\pi}{(\log 2)-3\beta_n}\geqq\frac{2\pi\left(a^1_{i_1\cdots i_n}-a^2_{i_1\cdots i_n}-\dfrac{2}{10^n}\right)^2 2^{q_n}}{\log 2}$$

$$: q_n=2n \qquad (31)$$

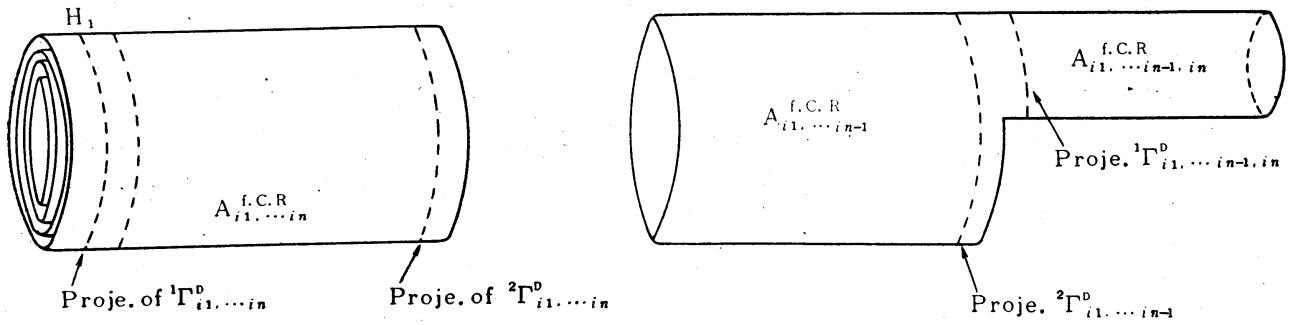

Fig. 14.

C). Let $\omega(z)$ be a harmonic function in $A^{f,C,R}_{i_1\cdots i_{n-1}}+A^{f,C,R}_{i_1\cdots i_{n-1},i_n}$ such that $0<\omega(z)<1$. Then there exist constants $a^2_{i_1\cdots i_{n-1}}$ and $a^1_{i_1\cdots i_{n-1},i_n}$ such that $|\omega(z)-a^2_{i_1\cdots i_{n-1}}|<\dfrac{1}{10^{n-1}}$ on ${}^2\Gamma^D_{i_1\cdots i_{n-1}}$ and $|\omega(z)-a^1_{i_1\cdots i_{n-1},i_n}|<\dfrac{1}{10^n}$ on ${}^1\Gamma^D_{i_1\cdots i_{n-1},i_n}$.

Then as above if $|a^2_{i_1\cdots i_{n-1}}-a^1_{i_1\cdots i_{n-1},i_n}|>\dfrac{2}{10^{n-1}}$,

$$D(\omega(z))\geqq\frac{\left(a^2_{i_1\cdots i_{n-1}}-a^1_{i_1\cdots i_{n-1},i_n}-\dfrac{2}{10^{n-1}}\right)^2\pi}{3\beta_{n-1}}=\frac{\left(a^2_{i_1\cdots i_n,i_{n-1}}-a^1_{i_1\cdots i_{n-1},i_n}-\dfrac{2}{10^{n-1}}\right)^2}{\dfrac{1}{2}\,\dfrac{\log 2}{2^{2n-2}}}$$

$$: \beta_n=\frac{\log 2}{6\times 2^{2n}} \qquad (32)$$

$\mathfrak{R}$ has relative boundary $B_0+(B_1+B_2)+(B_{1,1}+B_{1,2}+B_{2,1}+B_{2,2})\cdots$. $\mathfrak{R}^\Gamma_n$ is

compact and $\mathfrak{R}-\mathfrak{R}_n^{\Gamma}$ is composed of 2^n number of non compact components which we denote by $G_{i_1\cdots i_n}$ $(i_1=1,2,\ i_2=1,2,\cdots)$, where $G_{i_1\cdots i_n}$ is the part of $\mathfrak{R}$ lying over $(A_{i_1\cdots i_n,1}+A_{i_1\cdots i_n,2})+(A_{i_1\cdots i_n,1,1}+A_{i_1\cdots i_n,1,2}+A_{i_1\cdots i_n,2,1}+A_{i_1\cdots i_n,2,2})+\cdots$. $G_{i_1\cdots i_n}\supset G_{i_1\cdots i_n,i_{n+1}}\supset G_{i_1\cdots i_n,i_{n+1},i_{n+2}}\cdots$ determines an ideal boundary component $\mathfrak{p}(i_1,j_2\cdots,i_n,\cdots)$. The set B^* of $\mathfrak{p}$ clearly *the power of continuum.* Then $\mathfrak{R}$ has the following properties.

a). Let $N(z,p)$ be the N-Green's function of $\mathfrak{R}$ and suppose that N-Martin's topology is defined in $\mathfrak{R}$ and

$$\delta(p,q)=\sup_{z\in A_0}\left|\frac{N(z,p)}{1+N(z,p)}-\frac{N(z,q)}{1+N(z,q)}\right|.$$

Then there exists only one ideal boundary point p of N-Martin's topology on a boundary component $\mathfrak{p}(i_1,i_2\cdots)$.

b). $\omega(p,z)>0$, i.e. p is a singular point.

c). $w(B^*,z)=0$.

Hence $B^*\ni p$ and by $w(p,z)\leqq w(B^*,z)\leqq 0$. Every point p is a singular point of first kind. Hence $\mathfrak{R}$ has the set of singular points of first kind of power of continuum.

Proof of a). Let $\mathfrak{R}_n^{D\Gamma}$ be the component of $\mathfrak{R}$ containing B_0 divided by $\sum_{i_1=1,2,\cdots\cdots,i_n=1,2}{}^1\Gamma^D_{i_1\cdots i_n}$ from $\mathfrak{R}$. $\mathfrak{R}-\mathfrak{R}_n^{D\Gamma}$ is composed of 2^n number of non compact components $G^D_{i_1\cdots i_n}$ lying over $A^{f,C,R}_{i_1\cdots i_n}+(A_{i_1\cdots i_n,1}+A_{i_1\cdots i_n,2})+\cdots$. Clearly $\{G_{i_1\cdots i_n}\}$ and $\{G^D_{i_1,\cdots i_n}\}$ equivalent and $G^D_{i_1\cdots i_n}\supset G^D_{i_1\cdots i_n,i_{n+1}}G_{i_1,i_n,i_{n+1},i_{n+2}}\supset\cdots$. determines also $\mathfrak{p}(i_1,i_2,\cdots)$. Let $U(z)$ be a harmonic function in $\mathfrak{R}-A_0$ and $0<U(z)<M$. Then $A^{f,C,R}_{i_1\cdots i_n}$ is a folded concentrating ring. Hence

$$\left|\max_{z\in{}^1\Gamma^D_{i_1\cdots i_n}}U(z)-\min_{z\in{}^1\Gamma^D_{i_1\cdots i_n}}U(z)\right|<\frac{1}{10^n}\to 0 \text{ as } n\to\infty:$$

Hence by Lemma 1, there exists only one point p of N-Martin's topology on $\mathfrak{p}(i_1,i_2,\cdots)$.

Proof of b). Let $w(B^*,z)$ be a harmonic measure of B^*. Then $w(B^*,z)=0$ on $B_0+(B_1+B_2)+\cdots$ and $\leqq 1$. Hence by (29) $w(B^*,z)<\mu<1$ on $\Gamma_{i_1\cdots i_n}$. By the maximum principle $w(B^*,z)\leqq\mu$ on $\mathfrak{R}_n^{\Gamma}$, whence $\sup_{z\in R}w(B^*,z)<\mu<1$. Hence by P.H.3.[5] $w(B^*,z)=0$.

Proof of c). Let p be an ideal boundary point of N-Martin's topology over a boundary component $\mathfrak{p}(i_1,i_2,\cdots)$. Put $\upsilon_l(p)=E\left[z\in\mathfrak{R}:\ \delta(z,p)>\frac{1}{l}\right]$. Then by Lemma 1 there exists a number m such that $\upsilon_l(p)\supset G_{i_1\cdots i_m}$. Let

5) See 4)

$\omega(\upsilon_l(p), z)$ and $\omega(G_{i_1\cdots i_m}, z)$ be C.P.'s of $\upsilon_l(p)$ and $G_{i_1\cdots i_m}$. Then

$$\omega(p, z)=\lim_{l=\infty}\omega(\upsilon_l(p), z)\geqq\lim_{m=\infty}\omega(G_{i_1\cdots i_m}, z).$$

For simplicity put $\omega_m(z)=\omega(G_{i_1\cdots i_m}, z)$. Now $\omega_m(z)$ has M.D.I. among all harmonic functions with value 0 on B_0 and $\omega_m(z)=1$ on $G_{i_1\cdots i_m}$. Hence by the Dirichlet principle

$$D(\omega_m(z))\leqq D(\tilde{\omega}(z))=\frac{2\pi}{\log 2}, \tag{33}$$

where $\tilde{\omega}(z)$ is a harmonic function in A_0 such that $\tilde{\omega}(z)=0$ on $\operatorname{Re} z=0$ and $\tilde{\omega}(z)=1$ on $\operatorname{Re} z=\log 2$.

Since $A^{f,C,R}_{i_1\cdots i_n}$ $(n\geqq 1)$ is a folded concentrating ring, there exist constants $a^1_{i_1\cdots i_n}$ and $a^2_{i_1\cdots i_n}$ such that

$$\begin{aligned} &|\omega_m(z)-a^1_{i_1\cdots i_n}|<\frac{1}{10^n} \text{ for } z\in{}^1\Gamma^D_{i_1\cdots i_n}\ n\geqq 1 \text{ and } m>n,\\ &|\omega_m(z)-a^2_{i_1\cdots i_n}|<\frac{1}{10^n} \text{ for } z\in{}^2\Gamma^D_{i_1\cdots i_n}\ n\geqq 1 \text{ and } m>n, \end{aligned} \tag{34}$$

where $a^1_{i_1\cdots i_n}$ and $a^2_{i_1\cdots i_n}$ depend on $\omega_m(z)$.
Put $A_m(i_1, i_2, \cdots, i_n)=\max|\omega_m(z')-\omega_m(z'')|$, where $z'\in{}^1\Gamma^D_{i_1\cdots i_n}$ and $z''\in{}^2\Gamma^D_{i_1\cdots i_n}$: $n\geqq 1$ (see Fig. 14).

Then by (34) $$|A_m(i_1, i_2, \cdots, i_n)|\leqq|a^1_{i_1\cdots i_n}-a^2_{i_1\cdots i_n}|+\frac{2}{10^n}. \tag{35}$$

Put $A'_m(i_1, i_2, \cdots, i_n)=\max|\omega_m(z')-\omega_m(z'')|$, where $z'\in{}^2\Gamma^D_{i_1\cdots i_{n-1}}$ and $z''\in{}^1\Gamma^D_{i_1,\cdots i_{n-1}, i_n}$: $n\geqq 2$.

Then by (34) $$|A'_m(i_1, i_2, \cdots, i_n)|<|a^2_{i_1\cdots i_{n-1}}-a^1_{i_1\cdots i_n}|+\frac{2}{10^{n-1}}.$$

Now by (34), (33) and (31)

$$|a^1_{i_1\cdots i_n}-a^2_{i_1\cdots i_n}|<\sqrt{\left(\frac{2\pi}{\log 2}\right)\frac{\log 2}{2^{q_n}\times 2\pi}}+\frac{2}{10^n}=\frac{1}{2^n}+\frac{2}{10^n}. \tag{36}$$
$$: q_n=2n.$$

Similarly by (34), (33) and (32)

$$|a^2_{i_1\cdots i_{n-1}}-a^1_{i_1\cdots i_{n-1}, i_n}|<\sqrt{\left(\frac{2\pi}{\log 2}\right)\times\frac{\log 2}{2\pi\times 2^{2(n-1)}}}+\frac{2}{10^{n-1}}=\frac{1}{2^{n-1}}+\frac{2}{10^{n-1}}. \tag{37}$$

We show $\omega(p, z)>0$. Assume $\lim_m \omega_m(z)=0$. Then for any given positive number $\varepsilon>0$ and for any given ${}^D\Gamma^2_{i_1\cdots i_{n_0}}$, there exists a number m $(i_1, i_2, \cdots i_{n_0}, \varepsilon)$ such that

$$\omega_m(z)<\varepsilon \text{ on } {}^D\Gamma^2_{i_1\cdots i_{n_0}} \text{ and } \omega_m(z)=1 \text{ on } {}^D\Gamma^2_{i_1\cdots i_m}.$$

Now

$$\varepsilon > \min_{z\in {}^D\Gamma^2_{i_1\cdots i_{n_0}}} \omega_m(z) \geqq 1-\sum_{n=n_0}^{m} A(i_1,\cdots,i_n)-\sum_{n=n_0}^{m} A'(i_1,\cdots,i_n) > 1-\sum_{n=n_0}^{m}\left(\frac{4}{10^{n-1}}+\frac{1}{2^{n-2}}\right).$$

This is a contradiction for $\sum_{n=n_0}^{m}\left(\frac{4}{10^{n-1}}+\frac{1}{2^{n-2}}\right)<\frac{1}{2}$ for $n_0\geqq 4$. Hence $\omega(p,z)\geqq\lim_m \omega_m(z)>0$.

PART II

We shall construct a covering surface over the z-plane which has finite area and has a singular point of first kind. Theorem 12. *c*) of "Singular points of Riemann surfaces" shows that there exists no singular point, if the number of sheets of the covering surface is bounded and the Theorem 12. *a*) shows that there exists no singular point of second kind, if the area of the covering surface is finite. Hence this example means that the condition that the number of sheets of the covering surface is bounded is necessary for the validity of Theorem 12. *c*.

Example 4. Let A be a disc $0<|z|<1$. Map A onto $-\infty<\mathrm{Re}\,\xi<0$, $0<\mathrm{Im}\,\xi<2\pi$. Let A_n be a ring (A_n is a rectangle in itself, but for the simplicity in this paper we call a ring) such that

$$-n(\log 100)<\mathrm{Re}\,\xi<-(n-1)\log 100,\ 0<\mathrm{Im}\,\xi<2\pi.$$

For $n\geqq 2$, we shall construct a $2^{q_n}(q_n=4n)$-sheeted covering surface over A_n by intense connection. Let H_n, H'_n, and Γ_n, Γ'_n, $\hat{\Gamma}_n$, Γ^*_n be rings and circles cited below:

H_n: $-(n-1)(\log 100)-2\gamma<\mathrm{Re}\,\xi<-(n-1)\log 100,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

H'_n: $-(n\log 100)<\mathrm{Re}\,\xi<-n\log 100+2\gamma,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

$\hat{H}_n$: $2\gamma-n\log 100<\mathrm{Re}\,\xi<-2\gamma-(n-1)\log 100,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

Γ_n: $\mathrm{Re}\,\xi=-(n-1)(\log 100)-\gamma,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

Γ'_n: $\mathrm{Re}\,\xi=-n(\log 100)+\gamma,\ 0\leqq\mathrm{Im}\,\zeta<2\pi.$

$\hat{\Gamma}_n$: $\mathrm{Re}\,\xi=-\left(n-1+\frac{1}{2}\right)\log 100,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

Γ^*_n: $\mathrm{Re}\,\xi=-c_n:-2\gamma+n\log 100>c_n>2\gamma+(n-1)\log 100,\ 0\leqq\mathrm{Im}\,\xi<2\pi.$

where $\gamma=\frac{\log 100}{8}$.

We make slits in H_n and H'_n $(n \geqq 2)$ parallel to the real axis for intense connection for 2^{q_n} sheets so that

$$|U(z')-U(z'')|<\frac{1}{10^n}$$

for z' and $z'' \in A_n(1)+\cdots, A_n(2^{q_n-1})+\hat{A}_n(1)+\cdots+\hat{A}_n(2^{q_n-1})$ and proj $z'=$proj $z'' \in (\Gamma_n+\Gamma'_n)$ for any harmonic function $U(z)$ such that $0<U(z)<1$, where $A_n(1),\cdots, A_n(2^{q_n-1})$ are examplars which are identical to $A_n(1)$. Hence by the maximum principle (maximum principle is used for the projectional transformations $z' \leftrightarrow z''$)

$$|U(z')-U(z'')|<\frac{1}{10^n} \text{ for proj } z'=\text{proj } z'' \in \hat{H}_n.$$

Put $\mathfrak{G}_n=A_1+A_2+,\cdots+A_{n-1}$ and $C\mathfrak{G}_n=A-(A_1+A_2+,\cdots+A_{n-1})$. We denote by $mC\mathfrak{G}_n$ the m number of surfaces $C\mathfrak{G}_n$ with slits for intense connection. As in case of the Example 3 we construct a Riemann surface.

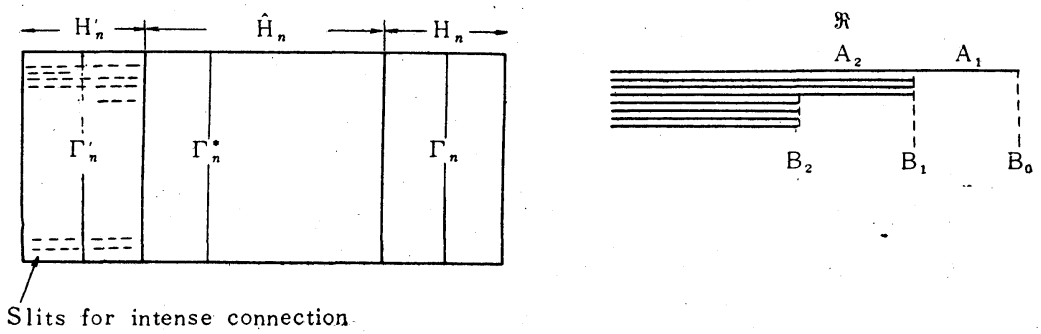

Fig. 15.

First step. Put $I'(1)=A$.

. . . . . .

. . . . . .

n-th step.

$I^n(1)=I^{n-1}(1)$

. . . . . .

. . . . . .

$I^n(2^{q_{n-1}-1}) \quad =I^{n-1}(2^{q_{n-1}-1})$

$I^n(2^{q_{n-1}}+1)=\hat{I}^{n-1}(1)$

$$\left.\begin{array}{l} I^n(2^{q_{n-1}})=\widehat{I}^{n-1}(2^{q_{n-1}-1}) \\ \quad\cdots\cdots\cdots\cdots \\ \quad\cdots\cdots\cdots\cdots \\ I^n(2^{q-1}) \\ \widehat{I}^n(1) \\ \quad\cdots\cdots \\ \quad\cdots\cdots \\ \quad\cdots\cdots \\ \quad\cdots\cdots \\ \widehat{I}^n(2^{q_{n-1}}) \end{array}\right\}$$

newly added at the n-th step. These are identical to $C\mathfrak{G}_n$.

We perform the intense connection by slits in A_n, i.e. by slits contained in H_n+H_n' of A_n. Let $\mathfrak{R}_n$ be the above Riemann surface obtained after the n-th intense connection and put

$$\mathfrak{R}=\bigcup\mathfrak{R}_n.$$

Then $\mathfrak{R}_n$ has relative boundary $B_0+B_1,\cdots+B_{n-1}$ and is open. B_{n-1} is composed of $2^{q_n}-2^{q_{n-1}}$ circles lying over $\mathrm{Re}\,\xi=-(n-1)\log 100$, $0\leqq\mathrm{Re}\,\xi<2\pi$. Now the part of $\mathfrak{R}$ lying over $\mathrm{Re}\,\xi<-n\log 100$ is a generalized ring with slits of which the intense connection is not performed. We denote by $\widetilde{A}_n$ the part of $\mathfrak{R}$ over A_n. Then $\widetilde{A}_n$ is a Riemann surface of 2^{q_n} sheets and $0\leqq U(z)<1$ has projectional deviation $<\dfrac{1}{10^n}$ on the part of $\mathfrak{R}$ lying over $\widehat{H}_n$, i.e. $|U(z')|<-U(z'')<\dfrac{1}{10^n}$ for $\mathrm{proj}\,z'=\mathrm{proj}\,z''\in\widehat{H}_n$ and z' and $z''\in\mathfrak{R}$.

$\mathfrak{R}$ has the following properties:

a). $\mathfrak{R}$ has only one ideal boundary point p of N-Martin's topology on $z=0$.

b). $\mathfrak{R}$ is a covering surface over $|z|<1$ and has a finite area.

c). p is a singular point of first kind.

Proof of a). Let $U(z)$ be a harmonic function in $\sum_{n=1}^{\infty}\widehat{H}_n\subset A:\widehat{H}_n=E\left[z:\ -\exp\left(n-\frac{1}{4}\right)\log 100<|z|<-\exp\left(n-\frac{3}{4}\right)\log 100\right]$ (see Fig. 15) such that $D_{\Sigma\widehat{H}_n}(U(z))<\infty$. Then there exists a sequence of circles $\Gamma_{n_i}^*:\ |z|=r_{n_i}$ in $\sum\widehat{H}_n$ such that

$$\int_{\Gamma_{n_i}^*}\left|\frac{\partial}{\partial n}U(z)\right|ds\to 0 \text{ as } n_i\to\infty.$$

In fact, put $re^{i\theta}=-\log|z|$ and put $L(r)=\int\left|\frac{\partial}{\partial r}U(re^{i\theta})\right|r\,d\theta$. Assume $\underline{\lim}\,L(r)>\delta_0>0$ for $r\in\sum E_n$, where E_n is the circular projection of $(-\log|z|:z\in H_n)$ on the real axis, i.e. $E_n=E\left[r:\left(\frac{1}{4}\right)\log 100>r>\left(n-\frac{3}{4}\right)\log 100\right]$. Then

$$\delta_0\int_{r_0}\frac{1}{r}dr\leqq\int_{r_0}^{r}\frac{L(r)}{r}dr\leqq\iint_{\sum\hat{H}_n}\left\{\left(\frac{\partial}{\partial r}U(z)\right)^2+\left(\frac{\partial}{\partial\theta}U(z)\right)^2r^2\right\}r\,drd\theta=D(U(z)).$$

Let $r\to\infty$. Then $D(U(z))=\infty$. This is a contradiction. Hence there exists a sequence $\{\Gamma_{n_i}\}$ such that

$$\int_{\Gamma_{n_i}}\left|\frac{\partial}{\partial n}U(z)\,ds\right|\to 0\quad\text{as}\quad n_i\to\infty.$$

This sequence $\{\Gamma_{n_i}\}$ depends on the function $U(z)$.

$$\text{Put}\ \varepsilon_{n_i}=\int_{\Gamma^*_{n_i}}\left|\frac{\partial}{\partial n}U(z)\right|ds. \tag{38}$$

Let $\tilde{\Gamma}^*_{n_i}$ be the set of $\mathfrak{R}$ lying over $\Gamma^*_{n_i}$ (see Fig. 15). Then $\tilde{\Gamma}^*_{n_i}$ is composed of $2^{q_{n_i}}$ circles. $\tilde{\Gamma}^*_{n_i}$ divides $\mathfrak{R}$ into two components. Let $\mathfrak{R}_{\tilde{\Gamma}^*_{n_i}}$ be one component of them containing B_0. Then $\{\mathfrak{R}_{\tilde{\Gamma}^*_{n_i}}\}$ is an exhaustion with compact relative boundary $B_0+B_1+\cdots+B_{n_{i-1}}$ and $\tilde{\Gamma}^*_{n_i}$, where B_i lies over $\mathrm{Re}\,\xi=-i\log 100$. Hence

$$\mathfrak{R}=\bigcup\mathfrak{R}_{\tilde{\Gamma}^*_{n_i}}.$$

Now $\hat{H}_{n_i}$ $(\subset A_{n_i})$ and A_{n_i} is intensely connected with $2^{q_{n_i}}-1$ examplars of $\mathfrak{R}$ on the slits contained in H_{n_i} and H'_{n_i} of A_{n_i}, hence $|U(z')-U(z'')|<\frac{1}{10^{n_i}}$ for $\mathrm{proj}\,z'=\mathrm{proj}\,z''\in\hat{H}_{n_i}$ and z' and z'' are contained in $\mathfrak{R}$. Hence by (38)

$$\left|\max_{z\in\tilde{\Gamma}^*_{n_i}}U(z)-\min_{z\in\tilde{\Gamma}^*_{n_i}}U(z)\right|<\varepsilon_{n_i}+\frac{2}{10^{n_i}}. \tag{39}$$

Let $\tilde{\Gamma}_n$ be the set of $\mathfrak{R}$ over Γ_n. Then $\tilde{\Gamma}_n$ is composed of 2^{q_n} circles over Γ_n. $\tilde{\Gamma}_n$ divides $\mathfrak{R}$ into two parts. Let $\mathfrak{R}_{\tilde{\Gamma}_n}$ be one of them containing B_0. Then $\{\mathfrak{R}_{\tilde{\Gamma}_n}\}$ is also an exhaustion of $\mathfrak{R}$, i.e. $\mathfrak{R}=\bigcup\mathfrak{R}_{\tilde{\Gamma}_n}$.
Clearly $\{\mathfrak{R}_{\tilde{\Gamma}^*_{n_i}}\}$ and $\{\mathfrak{R}_{\tilde{\Gamma}_n}\}$ are equivalent and $\{\mathfrak{R}-\mathfrak{R}_{\tilde{\Gamma}_n}\}$ determines an ideal boundary component $\mathfrak{p}$ on $z=0$.

Let $N(z,p)$ be the N-Green's function of $\mathfrak{R}$ and $\delta(p,q)$ be defined by

$$\delta(p,q)=\sup_{z\in A_1}\left|\frac{N(z,p)}{1+N(z,p)}-\frac{N(z,q)}{1+N(z,q)}\right|.$$

Consider $N(z, p) : p \in A_1$ in $A - A_2$. Then $N(z, p)$ is harmonic and $D(N(z, p)) < \infty$. Hence there exists a sequence $\{\Gamma^*_{n_i}\}$ such that $\sum_{n=2}^{\infty} \hat{H}_n$

$$\int_{\Gamma^*_{n_i}} \left| \frac{\partial}{\partial n} N(z, p) \right| ds \to 0 \text{ as } n_i \to \infty \text{ and } \Gamma^*_{n_i} \in \hat{H}_{n_i} \text{ of } A_{n_i}.$$

Hence by (39)

$$| \max_{z \in \tilde{\Gamma}^*_{n_i}} N(z, p) - \min_{z \in \tilde{\Gamma}^*_{n_i}} N(z, p) | \to 0 \text{ as } n_i \to \infty.$$

Hence by Lemma 1 there exists only one point of Martin's topology on the boundary component $\mathfrak{p}$.

Proof of b). Since the area of $A_n = \pi\left(\frac{1}{100^{2(n-1)}} - \frac{1}{100^{2n}}\right) = \frac{9900\pi}{100^{2n}}$, area of $\mathfrak{R} = 9900\pi \sum_{n=1}^{\infty} \left(\frac{2^{q_n}}{100^{2n}}\right) = 9900\pi \sum_{n=1}^{\infty} \left(\frac{2^4}{100^2}\right)^n < \infty$, for $q_n = 4n$.

Proof of c) Let $U(z)$ be a harmonic function in $\mathfrak{R}_{\tilde{\Gamma}_m} (m > n)$ such that $D(U(z)) \leqq \frac{2\pi}{\log 100}$. Let A_n^{n-1} be the ring: $-\exp\left(n - 1 + \frac{1}{2}\right) \log 100 < |z| < -\exp\left(n - 1 - \frac{1}{2}\right) \log 100$.

Assume $|U(r_{n-1}e^{i\theta}) - U(r_n e^{i\theta})| > \frac{3}{10^{\frac{n-1}{2}}}$ *on the set of* θ of *angular measure* $> \frac{2\pi}{2^n}$ *on* A_n^{n-1} as a function of z on A_n^{n-n}, where $r_{n-1}e^{i\theta} \in \hat{\Gamma}_{n-1}$ and $r_n e^{i\theta} \in \hat{\Gamma}_n$ (Fig. 16).

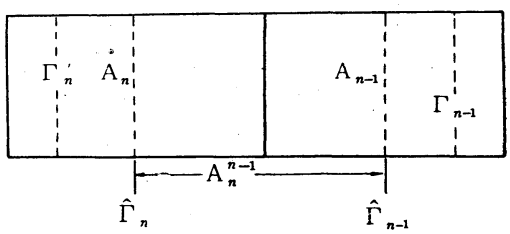

Fig. 16.

Now there exist 2^{q_n} number of examplars identical to A_n which are connected intensely with projectional deviation $< \frac{1}{10^n}$ on $\hat{\Gamma}_n$, whence

$$| U(z') - U(z'') | < \frac{1}{10^n} \text{ for } \operatorname{proj} z' = \operatorname{proj} z'' \in \hat{\Gamma}_n \text{ and } z' \text{ and } z'' \in \mathfrak{R}.$$

Hence

$$| U(r_n e^{i\theta}) - U(r_{n-1}e^{i\theta}) | > \frac{1}{10^{\frac{n-1}{2}}} \text{ for } \theta \text{ of angular measure } > \frac{2\pi}{2^n} \tag{40}$$

on $2^{q_{n-1}}$ examplars lying over A_n^{n-1} where $r_{n-1}e^{i\theta}$ lie and $r_n e^{i\theta}$ on the same examplar.

Let $\tilde{A}_n^{n-1}$ be the part of $\mathfrak{R}$ lying over A_n^{n-1}. Then by Lemma 2 and by (40)

$$\frac{2\pi}{\log 100} \geqq D(U(z)) \geqq \frac{2^{q_{n-1}} \times 2\pi}{2\pi \times (\log 100) \times 10^{n-1} \times 2^n} > \frac{2^{4(n-1)}}{10^{n-1}(\log 100)} > \frac{2\pi}{\log 100}.$$

This is a contradiction for $n \geqq 3$. Hence

$$|U(r_{n-1}e^{i\theta}) - U(r_n e^{i\theta})| < \frac{3}{10^{\frac{n-1}{2}}} : n \geqq 3, \tag{41}$$

except at most an exceptional set $\Theta_{n,n-1}(E)$ *of angular measure* $< \frac{2\pi}{2^n}$ *on the ring* A_n^{n-1}.

Let $\tilde{\hat{\Gamma}}_n$ be the set of $\mathfrak{R}$ lying over $\hat{\Gamma}_n$. Then $\tilde{\hat{\Gamma}}_n$ divides $\mathfrak{R}$ into two parts. Let $\mathfrak{R}_{\tilde{\hat{\Gamma}}_n}$ be the one of them containing B_0. Then $\{\mathfrak{R}_{\tilde{\hat{\Gamma}}_n}\}$ is an exhaustion and $\{\mathfrak{R}_{\tilde{\hat{\Gamma}}_n}\}$ is equivalent to $\{\mathfrak{R}_{\tilde{\Gamma}^*_{n_i}}\}$. $\{\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_n}\}$ determines the ideal boundary component $\mathfrak{p}$ and by a), by Lemma 1 for any given $\upsilon_l(p)$ there exists a number m_l such that

$$(\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_m}) \subset \upsilon_l(p) = E\left[z \in \mathfrak{R} : \delta(z, p) < \frac{1}{l}\right] \text{ for } m \geqq m_l.$$

Hence

$$\text{C.P. of } p \quad \omega(p, z) = \lim_l \omega(\upsilon_l(p), z) \geqq \lim_m \omega(\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_m}, z),$$

where $\omega(\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_m}, z)$ is C.P. of $\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_m}$.
For simplicity put $\omega_m(z) = \omega(\mathfrak{R} - \mathfrak{R}_{\tilde{\hat{\Gamma}}_m}, z)$. Assume $\lim_m \omega_m(z) = 0$. Then for any given positive number $\varepsilon < \frac{1}{10}$, there exists a number m_0 such that $\omega_m(z) < \varepsilon$ on $\hat{\Gamma}_5 = E\left[|z| = -\exp\left(4 + \frac{1}{2}\right) \log 100\right]$ for $m \geqq m_0$. Now $\omega_m(z) = 1$ on $\Gamma_m (\subset \tilde{\hat{\mathfrak{R}}} - A_1)$ and has M.D.I. over $\mathfrak{R}$. Hence by the Dirichlet principle

$$D(\omega_m(z)) < D(\tilde{\omega}(z)) = \frac{2\pi}{\log 100},$$

where $\tilde{\omega}(z)$ is a harmonic function in A_1 such that $\tilde{\omega}(z) = 0$ on $|z| = 1$ and $\tilde{\omega}(z) = 1$ on $|z| = \frac{1}{100}$. Hence by (41)

$$|U(r_n e^{i\theta}) - U(r_{n-1}e^{i\theta})| < \frac{3}{10^{\frac{n-1}{2}}} \text{ except } \Theta_{n,n-1}(E), \tag{42}$$

where $\Theta_{n,n-1}(E)$ depends on $\omega_m(z)$.

Let Φ be the complementary set of $\sum_{n=5}^{m} \Theta_{n,n-1}(E)$. Then angular measure of $\Phi > 2\pi \left(1 - \sum_{n=4}^{\infty} \frac{1}{2^n}\right) > \frac{15\pi}{8}$. Suppose $\theta \in \Phi$. Then by (42)

$$|\omega(r_m\theta^{i\theta})-\omega(r_4e^{i\theta})|<\sum_{n=5}^{m}\frac{3}{10^{\frac{n-1}{2}}}<\frac{3}{10}.$$

On the other hand, $\omega_m(z)=1$ on $\widehat{\Gamma}_m$, i.e. $\omega(r_me^{i\theta})=1$, whence

$$\frac{1}{10}>\varepsilon>\max_{z\in\hat{\Gamma}_5}\omega_m(z)>1-\frac{3}{10} \text{ on } \widehat{\Gamma}_5.$$

This is a contradiction. Hence $0<\lim_m \omega_m(z)<\omega(p, z)$.

Next by Theorem 12. *a*) (of the previous paper "On Singular points" $\mathfrak{R}$ has no singular point of second kind. Hence p is a singular point of first kind. Thus we have *c*).

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