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Author(s)
Kuramochi, Zenjiro

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EXAMPLES OF SINGULAR POINTS

By

Zenjiro KURAMOCHI

The purpose of the present paper is to construct examples of singular points of first and of second kind.

PART I

In this part we shall construct some Riemann surface to construct singular points, non N-minimal points and to show that there exists a Riemann surface which has the set of singular points of first kind of the power of continuum.

1. Concentrating ring C.R. \( (\alpha, \varepsilon) \). If a Riemann surface with finite genus has two circular relative boundaries, we call it a generalized ring. We call an ordinary ring a simple ring. Let \( R : 1 < |z| < \exp \alpha \) be a simple ring with module \( \alpha : 1 < |z| < \exp \alpha \). We shall construct from \( R \) a generalized ring G.R. such that any harmonic function such that \( |U(z)| \leq 1 \) in G.R. must satisfy the condition

\[
| \max_{|z| = \exp \frac{\alpha}{2}} U(z) - \min_{|z| = \exp \frac{\alpha}{2}} U(z) | < \varepsilon
\]

on a circle \( |z| = \exp \frac{\alpha}{2} \) (in the following we say that \( U(z) \) has aberration \( < \varepsilon \) on \( |z| = \exp \frac{\alpha}{2} \)). We call such a ring a concentrating ring and denote it by C.R. \((\alpha, \varepsilon)\).

a). Determining an integer \( m : m \geq 4 \). Let \( U(z) \) be a harmonic function in a simple ring S.R: \( \exp \frac{\alpha}{3} < |z| < \exp \frac{2\alpha}{3} \) such that \( 0 \leq U(z) \leq 1 \) and continuous on \( |z| = \exp \frac{\alpha}{3} \) and on \( |z| = \exp \frac{2\alpha}{3} \). Then

\[
U(z) = \frac{1}{2\pi} \int_{\partial(S.R.)} U(z) \frac{\partial}{\partial n} G(\zeta, z) ds,
\]

where \( G(\zeta, z) \) is the Green’s function of the above ring.
Since $\frac{\partial}{\partial n}G(\zeta, z)$ is a continuous functions of $z$ on $|z| = \exp \frac{\alpha}{3}$ and on $|z| = \exp \frac{2\alpha}{3}$ for fixed $\zeta$ and

$$|U(z_1) - U(z_2)| \leq \frac{1}{2\pi} \int_{\partial(S.B.)} |U(\zeta)||\frac{\partial}{\partial n}G(\zeta, z_1) - \frac{\partial}{\partial \omega}G(\zeta, z_2)|\,ds,$$

there exists an integer $m(\alpha)$ such that $|\arg z_1 - \arg z_2| \leq \frac{2\pi}{2^m}$ and $|z_1| = |z_2| = \exp \frac{\alpha}{2}$ imply $|U(z_1) - U(z_2)| < \frac{\epsilon}{4}$. Fix $m$ at present.

b) Determining a constant $\beta$. Let $D$ be a rectangle: $|\text{Re} z| \leq \frac{1}{2\beta}$, $0 \leq \text{Im} z \leq 1$ on the $z$-plane. Let $w(z)$ be the harmonic measure of two sides of $D$: $|\text{Re} z| = \frac{1}{2\beta}$, i.e. $w(z)$ is a positive harmonic function in $D$ such that $w(z) = 1$ on $|\text{Re} z| = \frac{1}{2\beta}$ and $w(z) = 0$ on $\text{Im} z = 0$ and $\text{Im} z = 1$. Now $w(z) \rightarrow 0$ uniformly on $|\text{Re} z| = 0$ as $\beta \rightarrow 0$. Let $\beta^*$ be a constant such that $w(z) < \epsilon'$ on $\text{Re} z = 0$ for $\beta \leq \beta^*$. Then any harmonic function $U(z)$ in $D$ such that $U(z) = 0$ on $\text{Im} z = 0$ and $\text{Im} z = 1$ and $|U(z)| < 1$ satisfies $|U(z)| < \epsilon'$ on $|\text{Re} z| = 0$, where

$$\epsilon' = \frac{\epsilon}{8m}.$$

Put $\gamma = \frac{\alpha}{3m}$ and let $H_i, H'_i, \hat{H}_i, \hat{H}'_i$ and $H$ be rings and $\Gamma_i, \Gamma'_i (i = 1, 2, \cdots, n)$ and $\Gamma$ be circles cited below:

$H_i$: $\exp ((i - 1)\gamma) < |z| < \exp (i\gamma), \quad H'_i$: $\exp (\alpha - i\gamma) > |z| > \exp (\alpha - (i - 1)\gamma)$

$\hat{H}_i$: $\exp ((i - 1)\gamma) < |z| < \exp (\alpha - (i - 1)\gamma)$,

$\hat{H}'_i$: $\exp (\alpha - (i - 1)\gamma) < |z| < \exp (\alpha - (i - \frac{1}{2})\gamma)$,

$H$: $\exp \frac{\alpha}{3} < |z| < \exp \frac{2\alpha}{3}$

$\Gamma_i$: $|z| = \exp (i - \frac{1}{2})\gamma, \quad \Gamma'_i$: $z = \exp (\alpha - (i - \frac{1}{2})\gamma)$

$\Gamma$: $|z| = \exp \frac{\alpha}{2}$.

c) Slits for identifying. Let $s_{i,j}$ and $s'_{i,j} (i = 1, 2, \cdots, m)$, $i = 1, 2, 3, \cdots, 2^{mi}$ (where $l$ is an integer such that

$$\frac{2\pi}{2^{ml}} \leq \beta^*$$

$$0 < \frac{\gamma}{2} \leq \frac{\gamma}{2^l}.$$
and that $s_{i,j}$ and $s'_{i,j}$ are contained in $H_i$ and $H'_i$ respectively, i.e.

$s_{i,j}: \exp\left(\frac{i-1+\frac{1}{4}}{4}\right)\gamma < |z| < \exp\left(\frac{i-1+\frac{3}{4}}{4}\right)\gamma: \quad \arg z = \frac{2\pi}{2^m}(j-1)$

$s'_{i,j}: \exp\left(\alpha - \left(\frac{i-1+\frac{3}{4}}{4}\right)\gamma\right) < |z| < \exp\left(\alpha - \left(\frac{i-1+\frac{1}{4}}{4}\right)\gamma\right): \quad \arg z = \frac{2\pi}{2^m}(j-1)$

Let $H_i(s)$ be $H_i$ with $\sum s_{i,j}$. Then $H_i(s)$ and $H'_i(s)$ ($i=1,2,\cdots, m$) are all conformally equivalent.

\[ \text{Fig. 1.} \]

\textbf{d) Construction of a concentrating ring: C.R.} $(\alpha, \epsilon)$. At first in $H_1 + H'_1$ identify two edges of $s_{1,j}$ and $s'_{1,j}$ lying symmetrically with respect to $I_{1,1}$ (real axis). Next in $H_2 + H'_2$ identify two edges of $s_{2,j}$ and $s'_{2,j}$ lying symmetrically with respect to $I_{2,1}$ (imaginary axis). Let $E_{i,k}$ be a circular echelon ($i=1,2,\cdots, m), k=1,2,3,\cdots 2^{i-1}$) and $I_{i,k}$ be a diameter such that

$E_{i,k}: \exp((i-1)\gamma) < |z| < \exp(\alpha-(i-1)\gamma), \quad \frac{2\pi}{2^{i-1}}(k-1) < \arg z < \frac{2k\pi}{2^{i-1}}.$

$I_{i,k}: 0 \leq |z| < \exp(\alpha-(i-1)\gamma), \quad \arg z = \frac{2\pi}{2^{i-1}}(k-1) + \frac{2\pi}{2^{i}} \quad \text{or} \quad \frac{2\pi}{2^{i-1}}(k-1) + \frac{2\pi}{2^{i}} + \pi.$

In every $E_{i,k} \cap (H_i+H'_i)$ ($i=3,4,\cdots, m$) identify two edges of $\sum(s_{i,j}+\sum s'_{i,j})$ lying symmetrically with respect to $I_{i,k}$ ($k=1,2,3,\cdots, 2^{i-1}$). Then we have identified every $s_{i,j}$ and $s'_{i,j}$ contained in the simple ring and we have a generalized ring G.R. We shall prove such G.R. is a C.R. $(\alpha, \epsilon)$. Let $U(z)$ be a harmonic function in G.R. such that $|U(z)| \leq 1$. 

Let $T_1(z)$ be the inversion with respect to $I_{11}$ (real axis). Since G.R. has symmetric structure, $U(T_1(z)) - U(z)$ is harmonic in G.R. and $U(T_1(z)) - U(z) = 0$ on $\sum (s_{1j} + s_{1j}')$. Now the echelons

$\exp\left(i - 1 + \frac{1}{4}\right) \gamma < |z| < \exp\left(i - 1 + \frac{3}{4}\right) \gamma$, \quad $\frac{2\pi}{2^m} (j-1) < \arg z < \frac{2\pi}{2^m} j$,
exp \((\alpha-(i-1+\frac{3}{4})r)<|z|<\exp(\alpha-(i-1+\frac{1}{4})r),\ \frac{2\pi}{2^n}(j-1)<\arg z<\frac{2\pi}{2^n}j\)

are conformally equivalent to the rectangle:

\[ |\Re \zeta|<\frac{1}{2\beta'},\ 0 \leq I_n \zeta \leq 1, \ \beta' \leq \beta^*.\]

Consider \(U(z)-U(T_1(z))\) in the above rectangle. Then

\[ |U(z)-U(T_1(z))|<\epsilon' \text{ on } \Re \zeta=0.\]

Hence \(|U(T_1(z))-U(z)|<\epsilon'\) on \(\Gamma_1+\Gamma_1'\), whence by the maximum principle

\[ |U(T_1(z))-U(z)|<\epsilon' \text{ on } \hat{H_1} \supset \hat{H_2}. \tag{1}\]

Let \(T_2(z)\) be the inversion with respect to \(I_{21}\) (imaginary axis). Then similarly as above

\[ |U(T_2(z))-U(z)|<\epsilon' \text{ on } \hat{H_2}. \tag{2}\]

Let \(\tau_2(z)\) be the rotation about \(z=0\) such that \(\arg \tau_2(z)-\arg z=\pi\).

Then \(\tau_2(z)=T_1T_2(z)\) for \(z \in E_{21}\) and \(\tau_2(z)=T_1T_2(z)\) for \(z \in E_{22}\).

Hence \(|U(\tau_2(z))-U(z)|<\epsilon'\) on \(\Gamma_2+\Gamma_2'\).

\(\hat{H_2}\) is invariant with respect to \(\tau_2(z)\) and \(U(\tau_2(z))-U(z)\) is harmonic in \(\hat{H_2}\), whence by the maximum principle

\[ |U(\tau_2(z))-U(z)|<2\epsilon' \text{ on } \hat{H_2}.\]

Similarly \(|U(\tau_2'(z))-U(z)|<2\epsilon' \text{ on } \hat{H_2}\).

(3)

where \(\tau_2'(z)\) is the rotation about \(z=0\) such that \(\arg \tau_2'(z)-\arg z=-\pi\).

Let \(T_{3,k}(z)\) be the inversion with respect to \(I_{3,k} (k=1,2)\). Then \(\hat{H_3}\) has symmetric structure, \(U(T_{3,k}(z))-U(z)\) is harmonic in \(\hat{H_3}\) and vanishes
only on $(E_{s,k}+E_{s,k'}) \cap (\sum (s_{3,j}+s_{3,j}'))$, where $k' = k + 2$ and $|U(T_{s,k}(z)) - U(z)| \leq 1$ on $(\sum (s_{3,j}+s_{3,j}') - ((E_{s,k}+E_{s,k'}) \cap (\sum (s_{3,j}+s_{3,j}')))$. Hence

\[ |U(T_{s,k}(z)) - U(z)| < \varepsilon' \text{ on } (\Gamma_{3} + \Gamma_{3}') \cap (E_{3,k}+E_{3,k'}). \]  

(4)

Let $\tau_{3}(z)$ be the rotation about $z=0$ such that $\arg \tau_{3}(z) = \arg z = \frac{\pi}{2}$.

Then

$\tau_{3}(z) = T_{3,2}T_{2}(z)$ for $z \in E_{3,2} + E_{3,3}$ and $\tau_{3}(z) = T_{3,3}T_{1}(z)$ for $z \in E_{3,1} + E_{3,3}$.

Now $T_{3,2}T_{2}(z)$ and $T_{2}(z)$ for $z \in E_{3,2} + E_{3,4}$ and $T_{3,3}T_{1}(z)$ for $z \in E_{3,1} + E_{3,3}$.

Consider $U(T_{3,1}(z)) - U(z)$ in $\hat{H}_{3}$.

At first, $|U(T_{3,1}(z)) - U(z)| < \varepsilon'$ for $z \in (\Gamma_{3} + \Gamma_{3}') \cap (E_{3,1} + E_{3,3})$.

(6)

Let $z$ and $z'$ be points in $E_{3,2}$ and $E_{3,4}$ such that $T_{3,1}(z) = z'$. Then there exists $\tau_{2}(z)$ such that $T_{3,4}(\tau_{2}(z)) = z'$. Hence

$|U(T_{3,1}(z)) - U(z)| < \varepsilon'$ for $z \in (\Gamma_{3} + \Gamma_{3}') \cap (E_{3,1} + E_{3,3})$.

Whence by (5)

$|U(T_{3,1}(z)) - U(z)| < \varepsilon'$ for $z \in (\Gamma_{3} + \Gamma_{3}') \cap (E_{3,1} + E_{3,3})$.

Hence by (6) $|U(T_{3,1}(z)) - U(z)| < 3\varepsilon'$ on $(\Gamma_{3} + \Gamma_{3}')$ and by the maximum principle

$|U(T_{3,1}(z)) - U(z)| < 3\varepsilon'$ in $\hat{H}_{3}$.

(7)

Let $T_{i,j}(z)$ be the inversion with respect to $I_{i,j}$ and $\tau_{i}(z)(\tau_{-i}(z))$ be the rotation about $z=0$ such that $\arg \tau_{i}(z) - \arg z = \frac{2\pi}{2^{i}}(\arg \tau_{-i}(z) - \arg z = \frac{-2\pi}{2^{i}})$. Then

$|U(\tau_{i}(z)) - U(z)| < (2i-2)\varepsilon'$ on $\hat{H}_{i}$.

$|U(\tau_{-i}(z)) - U(z)| < (2i-2)\varepsilon'$ on $\hat{H}_{i}$.

$|U(T_{i,j}(z)) - U(z)| < (2i-1)\varepsilon'$ on $\hat{H}_{i}$.

(8)
Proof. Assume $|U(\tau_{i}(z)) - U(z)| < 2(s-1)\epsilon'$,  
$|U(T_{s,i}(z) - U(z))| < (2s-1)\epsilon'$ on $\hat{H}_{i}(s=1,2,\cdots,i)$.

Then we shall prove that the above inequalities hold for $i+1$,  
Similarly as (6) 
$|U(T_{i+1,j}(z)) - U(z)| < \epsilon'$ on $(\Gamma_{i+1} + \Gamma'_{i+1}) \cap (E_{i+1,j} + E_{i+1,j+2^{i}})$. (10) 

Let $\tau_{i+1}(z)$ be the rotation about $z=0$ such that $\arg \tau_{i+1}(z) - \arg z = \frac{2\pi}{2^{i+1}}$. Since $I_{1} + I_{2} + (I_{3,1} + I_{3,2}) + (I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4}) + \cdots + (I_{i+1,1} + I_{i+1,2} + \cdots + I_{i+1,2^{i-1}})$ separates $H_{i+1}$ into $2^{i+1}$ parts, for any $z \in (\Gamma_{i+1} + \Gamma'_{i+1}) \cap (E_{i+1,j} + E_{i+1,j+2^{i}})$ by (9) and (10), $\hat{H}_{i+1}$ is invariant with respect to $\tau_{i+1}(z)$ and $U(\tau_{i+1}(z)) - U(z)$ is harmonic in $\hat{H}_{i+1}$, whence by the maximum principle 
$|U(\tau_{i+1}(z)) - U(z)| < 2i\epsilon'$ on $\hat{H}_{i+1}$. (11) 

Similarly 
$|U(\tau_{i+1}(z)) - U(z)| < 2i\epsilon'$ on $\hat{H}_{i+1}$. 

Let $z$ and $z'$ be points such that $T_{i+1,1}(z) = z'$. If $z$ and $z'$ are contained in $E_{i+1,1}$, $|U(T_{i+1,1}(z)) - U(z)| < \epsilon'$ by (10) on $(\Gamma_{i+1} + \Gamma'_{i+1}) \cap E_{i+1,1}$. If $z$ and $z'$ are not contained in $E_{i+1,1}$, there exists a $\tau_{s}(z)$ such that $\tau_{s}(z)$ and $z$ are contained in the same $E_{i+1,k}$ and $T_{i+1,k}(\tau_{s}(z)) = z' = T_{i+1,1}(z)$. Hence $|U(T_{i+1,1}(z)) - U(z)| = |U(T_{i+1,k}(\tau_{s}(z)) - U(z)| \leq |U(\tau_{s}(z)) - U(z)| < (2i+1)\epsilon' = (2i+1)\epsilon'$ on $(\Gamma_{i+1} + \Gamma'_{i+1}) \cap E_{i+1,k}$ by (9) and (10). $\hat{H}_{i+1}$ has symmetric structure with respect to $I_{i+1,1}$ and $U(T_{i+1,1}(z)) - U(z)$ is harmonic, whence by the maximum principle 
$|U(T_{i+1,1}(z)) - U(z)| < (2i+1)\epsilon'$ on $\hat{H}_{i+1}$. 

Similarly 
$|U(T_{i+1,j}(z)) - U(z)| < (2i+1)\epsilon'$ on $\hat{H}_{i+1}$. 

Thus (8) and (9) are valid for every $i$ and $j$. By $H \subset \cap_{i=1}^{m} \hat{H}_{i} 

|U(T_{i,j}(z)) - U(z)| < 2m\epsilon' = \frac{\epsilon}{4} : z \in H$ for $i=1,2,\cdots,m, j=1,2,\cdots,2^{m}$. (12) 

Put $\theta_{0} = \frac{2\pi}{2^{m+2}}$, $\theta_{k} = \frac{2\pi}{2^{m}}k - \frac{2\pi}{2^{m+1}} : k=1,2,\cdots,2^{m}$ and $r = \exp \frac{\alpha}{2} \left( \Gamma : |z| = \exp \frac{\alpha}{2} \right)$. 

Then there exists $T_{i,j}(z)$ such that $T_{i,j}(re^{i\theta_{k}}) = re^{i\theta_{0}}$. 

Hence 
$|U(re^{i\theta_{k}}) - U(re^{i\theta_{0}})| < \frac{\epsilon}{4}$. (13)
Next for every point \( z \) on \( \Gamma \), there exists a point \( re^{i\theta_k} \) such that
\[
|\arg z - \theta_k| < \frac{2\pi}{2^m}.
\]
On the other hand, \( U(z) \) is harmonic in the ring \( H : \exp \frac{\alpha}{3} < |z| < \exp \frac{2\alpha}{3} \).
Hence
\[
|U(z') - U(z'')| < \frac{\epsilon}{4} \quad \text{for} \quad |\arg z' - \arg z''| < \frac{2\pi}{2^m} \quad \text{and} \quad |z'| = |z''| = \exp \frac{\alpha}{2}.
\]
Thus by (13)
\[
\max_{z \in \Gamma} U(z) - \min_{z \in \Gamma} U(z) < \epsilon,
\]
and G.R. is a concentrating ring C.R. \((\alpha, \epsilon)\).

Let \( R \) be a Riemann surface with compact relative boundary \( \partial R_0 \) and let \( \{R_n\} \) be an exhaustion of \( R \) with compact relative boundary \( \partial R_n \) \((n=1,2,\cdots)\). \( R-R_n \) is composed of a finite number of non compact domains \( G_{n,i} : i=1,2,\cdots i(n) \). We make an ideal boundary component \( \mathfrak{p} \) correspond to \( \{G_{1,i_1}, G_{2,i_2}, G_{3,i_3}, \cdots \} : G_{n,i} \supset G_{n+1,i} \supset \cdots \) and \( \partial G_{n,i} \) is compact and contained in \( \partial R_n \). For any given \( G_{n,i} \), if there exists a number \( j_0 \) such that \( p_j \notin \mathfrak{p} \) for \( j \geq j_0 \), we say that a sequence \( \{p_j\} \) \((j=1,2,\cdots)\) converges to \( \mathfrak{p} \). Let \( U(z) \) be a harmonic function in \( R-R_1 \) such that \( U(z) \) has M.D.I. over \( R-R_1 \) among all harmonic functions with value \( U(z) \) on \( \partial R_1 \), we say that \( U(z) \) is harmonic in \( R-R_1 \).

**Lemma 1.**

\( a) \). Suppose N-Martin’s topology is defined in \( R-R_0 \) and the distance \( \delta(p,q) \) is defined as
\[
\delta(p,q) = \sup_{z \in R_1} \left| \frac{N(z,p)}{1+N(z,p)} - \frac{N(z,q)}{1+N(z,q)} \right|.
\]
Let \( \mathfrak{p} \) be an ideal boundary component such that \( \mathfrak{p} = (G_{1,i_1}, G_{2,i_2}, \cdots) \). If for any harmonic function \(^1\) \( U(z) \) in \( R-R_1 \) \[ \max_{z \in \partial G_{n,i}} U(z) - \min_{z \in \partial G_{n,i}} U(z) \rightarrow 0 \text{ as } n \rightarrow \infty, \]
there exists only one ideal boundary point of N-Martin’s topology on \( \mathfrak{p} \).

\( b) \). Suppose that there exists only one boundary point of N-Martin’s topology on a boundary component \( \mathfrak{p} = (G_{1,i_1}, G_{2,i_2}, \cdots) \), then for any given number \( l \), there exists a number \( m \) such that \( \nu_l(p) \supset G_{m,i_m} \), where \( \nu_l(p) = E \left[ z \in \overline{R} : \delta(z,p) < \frac{1}{l} \right] \).

\(^1\) Let \( U(z) \) be a harmonic function in \( R-R_1 \). If \( U(z) \) has minimal Dirichlet integral among all harmonic functions with value \( U(z) \) on \( \partial R_1 \), we say that \( U(z) \) is harmonic in \( R-R_1 \).
Examples of Singular Points

Proof of $a)$. Let $U_n(z)$ be a harmonic function in $R_n-R_2$ such that $U_n(z)=U(z)$ on $\partial R_2$ and $\frac{\partial}{\partial n} U_n(z)=0$ on $\partial R_n$. Then by the maximum principle

$$\max_{z \in \partial G_{m_1}} U_n(z) \geq \sup_{z \in G_{m_1}} U_n(z) \geq \inf_{z \in G_{m_1}} U_n(z) \geq \min_{z \in \partial G_{m_1}} U_n(z)$$

for $n \geq m$.

By Lemma 1 of $P^{2})$ $U_n(z) \Rightarrow U(z)$, hence

$$\max_{z \in \partial G_{m_1}} U(z) \geq \sup_{z \in G_{m_1}} U(z) \geq \inf_{z \in G_{m_1}} U(z) \geq \min_{z \in \partial G_{m_1}} U(z).$$

Hence by the assumption $U(z)$ converges as $z \rightarrow \nu$. Assume that there exist two points $p$ and $q$ of N-Martin’s boundary points on $\nu$. Then we can find a point $z_{0}$ in $R_{1}$ such that $N(z_{0}, p)=N(z_{0}, q)$. Let $\{p_{i}\}$ and $\{q_{i}\}$ be fundamental sequences determining $p$ and $q$ respectively. Then both $\{p_{i}\}$ and $\{q_{i}\}$ converge to $\nu$. By $N(z_{0}, p_{i})=N(p_{i}, z_{0})$ and $N(z_{0}, q_{i})=N(q_{i}, z_{0})$ we consider $N(p_{i}, z_{0})$ and $N(q_{i}, z_{0})$ instead of $N(z_{0}, p_{i})$ and $N(z_{0}, q_{i})$ respectively. $N(z_{0}, z_{0})<M<\infty$ in $R-R_{2}$ and harmonic in $R-R_{2}$. Hence $N(p_{i}, z_{0}) \rightarrow N(p, z_{0})=N(q, z_{0})<N(q_{i}, z)$ by the assumption, whence $N(z_{0}, p_{0})=N(z_{0}, q)$. This is a contradiction. Hence only one point of N-Martin’s topology exists on $\nu$.

Proof of $b)$. Assume that $b)$ is false. We can find a sequence $\{p_{i}\}$ such that $p_{i} \notin v_{l}(p)$ and $p_{i} \rightarrow \nu$. This means that $G_{i_{1}, i_{1}}, G_{2, i_{2}}, \cdots$ has at least one point of N-Martin’s point outside of $v_{l}(p)$. This contradicts that $\{G_{i_{1}, i_{1}}, G_{2, i_{2}}, \cdots\}$ has only one point of N-Martin’s topology. Hence we have $b)$.

Example 1. Let $1=r_{1}<r_{2}<r_{3}, \cdots$, $\lim_{n} r_{n}=2$. Let $A_{n}$ and $B_{n}$ be simple rings cited below:

$$A_{n}: r_{2n+1} \leq |z| \leq r_{2n+2}: \alpha_{n}=\log \frac{r_{2n+2}}{r_{2n+1}}$$

$$B_{n}: r_{2n+2} \leq |z| \leq r_{2n+3}: \beta_{n}=\log \frac{r_{2n+3}}{r_{2n+2}}.$$  

We make a concentrating ring $C.R. \left( \alpha_{n}, \frac{1}{n} \right)=A_{n}^{*}$ from $A_{n}$.

Let $s_{n, j}$ be slits in $B_{n}$ such that

$$\exp \left( \log r_{2n+2} + \frac{1}{4} \beta_{n} \right) \leq |z| \leq \exp \left( \log r_{2n+2} + \frac{3}{4} \beta_{n} \right) \quad \text{and} \quad \arg z=\frac{2\pi}{2^{m_{n}}} j$$

$$j=1,2,3, \cdots, 2^{m_{n}}-1,$$

where $m_{n}$ is a number such that every positive harmonic function $\leq 1$ in $B_{n}$ vanishing on $\sum_{j} s_{n, j}$ must satisfy

$$U(z) \leq \frac{1}{n} \quad \text{on} \quad |z|=(r_{2n+2}, r_{2n+3})^{4}.$$  

(14)

2) We abbreviate the previous paper "Potentials on Riemann surfaces" by $P$. 

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Denote by $B_n^*$ the ring $B_n$ with slits $\sum s_{n,j}$ and put
\[
R_n(A^*)=A_1^*+B_1^*+A_2^*+\cdots B_{n-1}^*+A_n^*,
\]
\[
R_n(B^*)=R(A^*)+B_n^*.
\]
Put \( R = \bigcup_{1}^{\infty}R_n(A^*) = \bigcup_{1}^{\infty}R_n(B^*) \).

\( R-R_n(A^*) \) is a non-compact domain and \( \{ R-R_n(A^*) \} : n=1,2,\cdots \) determines an ideal boundary component \( \mathfrak{p} \) and \( R \) has the following properties:

\( a) \). There exists only one point \( p \) of N-Martin's boundary point on \( \mathfrak{p} \).
\( b) \). \( \omega(p,z)>0 \), i.e. \( p \) is a singular point.
\( c) \). \( \omega(p,z)=0 \), i.e. \( p \) is a singular point of first kind.

Proof of \( a) \). Let \( U(z) \) be a harmonic function in \( R-R_n(A^*) \) such that \( 0<\max_{z}U(z)<\infty \) as \( n \to \infty \). Then since \( A_n^* \) is a concentrating ring,

\[
| \text{max}_{z \in \partial A_{n+1}^*} U(z) - \text{min}_{z \in \partial A_{n+1}^*} U(z) | \to 0 \text{ as } n \to \infty.
\]

Hence by Lemma 1. a) we have \( a) \).

Proof of \( b) \). By Lemma 1. b) there exists a number \( n \) such that \( v_l(p) \supset R-R_n(A_n^*) \), where \( v_l(p) = E[z \in \bar{R} : \delta(z, p) < \frac{1}{l}] \). Let \( \omega_n(z) \) be a harmonic function in \( R_n(A_n^*) \) such that \( \omega_n(z)=0 \) on \( C = E[z : |z|=1] \) and \( \frac{\partial}{\partial n} \omega_n(z)=0 \) on \( \sum s_{ij} \) contained in \( R_n(A_n^*) \) and \( \omega_n(z)=1 \) on \( \partial(R_n(A_n^*)) \). Then by the Dirichlet principle

\[
D(\omega_l(p),z) \geq D(\omega_n(z)),
\]

where \( \omega_l(p),z \) is C.P. of \( v_l(p) \).

Clearly \( \omega_n(z) = \frac{\log |z|}{\log r_{2n+1}} \), since \( \sum s_{ij} \) are radial slits. Hence

\[
\frac{2\pi}{\log 2} = \lim_{n \to \infty} D(\omega_n(z)) \leq \lim_{l \to \infty} D(v_l(p),z) = D(\omega(p),z).
\]

Thus \( p \) is a singular point.

Proof of \( c) \). Let \( w_n(z) \) be a harmonic function in \( R_n(A_n^*) \) such that \( w_n(z)=0 \) on \( C+\sum s_{ij} \) and \( w_n(z)=1 \) on \( \partial(R(A_n^*)) \). Then \( w(v_l(p),z) \leq w_n(z) \), where \( w(v_l(p),z) \) is H.M. (harmonic measure) of \( v_l(p) \) and \( v_l(p) \subset (R-R_n(A_n^*)) \). Consider \( w_n(z) \) in \( B_{m-1}^* \). Then by (14)

\[
w_n(z) \leq \frac{1}{m-1} \text{ on } |z|=(r_{2m}, r_{2m+1})^\frac{1}{2}
\]

Hence by the maximum principle \( w_n(z) \leq \frac{1}{m-1} \) on \( R_n(A_{m-1}^*) \). Let \( l \to \infty \) and then \( n \to \infty \) and then \( m \to \infty \). Then
Examples of Singular Points

$w(p, z) \leq \lim w_n(z) = 0$ in $R$.

Thus $p$ is a singular point of first kind.

**Remark.** In Example 1, if we do not make slits $s_{ij}$ in $B_n$, i.e. $B_n$ has no slits and $B_n$ is a simple ring. Then it is easily seen that $\omega(p, z) = w(p, z)$. Hence in this case $p$ is a singular point of second kind.

**Example 2.** Let $1 = r_1 < r_2 < r_3, \cdots, \lim r_n = 2$. Let $A_n$ and $B_n$ be simple rings:

\[ A_n: r_{2n+1} \leq |z| < r_{2n+2}, \quad \alpha_n = \log \frac{r_{2n+2}}{r_{2n+1}} \]

\[ B_n: r_{2n+2} \leq |z| < r_{2n+3}, \quad \beta_n = \log \frac{r_{2n+3}}{r_{2n+2}} \]

Let $A_n^*$ be the same concentrating ring C.R. $(\alpha_n, \frac{1}{n})$ defined in Example 1. Let $C_{B_n}$ and $C_{B_n}'$ be circles in $B_n$ cited below:

\[ C_{B_n}: |z| = \exp \left( \log r_{2n+2} + \frac{1}{4} \beta_n \right), \quad C_{B_n}': |z| = \exp \left( \log r_{2n+2} + \frac{3}{4} \beta_n \right) \]

Put $R = A_1^* + B_1 + A_2^* + \cdots$. Let $\omega(s_n, z)$ be C.P. of $s_n$ in $R$, i.e. $\omega(s_n, z)$ is a harmonic function in $R$ such that $\omega(s_n, z) = 0$ on $|z| = 1$, $\omega(s_n, z) = 1$ on $s_n$ and has M.D.I. over $R$, where $s_n$ is a slit in $B_n$:

\[ s_n: r_{2n+2} \exp \left( \frac{\beta_n}{2} - t_n \right) < |z| < r_{2n+2} \exp \left( \frac{\beta_n}{2} + t_n \right), \quad \arg z = 0. \]

Clearly $\omega(s_n, z) \downarrow 0$ as $t_n \to 0$, i.e. as the length of $s_n \to 0$. Hence there exists a number $m_n$ such that $\omega(s_n, z) < \frac{1}{2^{n+1}}$ on $C_{B_n} + C_{B_n}'$ for $t \leq m_n$. Put $t_n = m_n$.

Then $\omega(s_n, z) < \frac{1}{2^{n+1}}$ on $C_{B_n} + C_{B_n}'$ and by the maximum principle

3) See "Potentials on Riemann surfaces".
\[ \omega(s_n, z) < \frac{1}{2^{n+1}} \quad \text{in} \quad R - B_n. \] (15)

In every \( B_n \) we make a slit \( s_n \) mentioned above and denote by \( B^*_n \) the \( B_n \) with a slit \( s_n \).

Put \( R(A^*_n) = A^*_1 + B^*_1 + \cdots, B^*_{n-1} + A^*_n \) and \( R(B^*_n) = R(A^*_n) + B^*_n \) and \( R^* = \bigcup R_n(A^*_n) \).

Let \( \hat{B}_n^* \) be a ring with one slit \( s_n \) which is symmetric to \( B^*_n \) with respect to the imaginary axis. Let \( \hat{A}_n^* \) be the ring with is identical to \( A^*_n \).

Put \( \hat{R}(\hat{A}_n^*) = \hat{A}_1^* + \hat{B}_1^* + \cdots, \hat{B}_{n-1}^* + \hat{A}_n^* \) and \( \hat{R}(\hat{B}_n^*) = \hat{R}(\hat{A}_n^*) + \hat{B}^* \) and \( \hat{R}^* = \bigcup \hat{R}(\hat{A}_n^*) \).

Connect \( R \) and \( \hat{R} \) by identifying two edges of \( s_n \) and \( \hat{s}_n \) \((n=1,2,\ldots)\). Then we have a Riemann surface \( \mathbb{R} \) which is symmetric with respect to \( \sum s_n \). Then \( \mathbb{R} \) has the following properties:

a) \( \mathbb{R} \in H_{0.2.B} \) and \( H_{0.2.D} \).

b) \( \mathbb{R} \) has no irregular points for the Green's function of \( \mathbb{R} \) and \( \mathbb{R} \) has no N-minimal harmonic function \( N(z, p) \) such that \( \sup_{z \in \mathbb{R}} N(z, p) = \infty \) and \( p \in B \) \((B \) is the set of boundary point of \( \mathbb{R} \)).

c) \( \mathbb{R} \) has two singular N-minimal points of second kind, which form the whole set of N-minimal boundary points and \( \mathbb{R} \) has non N-minimal points.

Proof of a). Let \( \Gamma(A^*_n) \) be a circle in \( A^*_n \) such that

\[ |z| = (r_{2n+1}, r_{2n+2}) \frac{1}{2}. \]

Let \( \Gamma_0 \) be the compact relative boundary of \( \mathbb{R} \) on \( |z| = 1 \).

Let \( U(z) \) be a bounded harmonic function in \( \mathbb{R} - \sum s_n \) such that \( U(z) = 0 \) on \( \Gamma_0 + \hat{\Gamma}_0 \) and \( |U(z)| \leq M \). Put \( a_n = \max_{z \in \Gamma(A^*_n)} U(z) \) and \( b_n = \min_{z \in \Gamma(A^*_n)} U(z) \). Then since \( A^*_n \) is a concentrating ring,

\[ 0 \leq a_n - b_n \leq \frac{M}{n} \] (16)

We show \( \lim_{n} a_n \) and \( \lim_{n} b_n \) exist and \( \lim_{n} a_n = \lim_{n} b_n \).

Assume \( \lim_{n} a_n = a' > \lim_{n} a_n = a'' \). Let \( 0 < \varepsilon < \frac{a' - a''}{10} \). Since \( \lim_{n} a_n = a' \) and \( \lim_{n} a_n = a'' \), there exist numbers \( n', n'' \) and \( n''' \) such that

\[ |a' - a_{n'}| < \varepsilon, \quad |a'' - a_{n''}| < \varepsilon, \quad |a' - a_{n''}| < \varepsilon \text{ and } n' < n'' < n''' \quad \text{and} \quad \frac{M}{2^{n-1}} < \frac{4M}{n} < \varepsilon. \] (17)

4) We denote by \( H_{0.2.B}(H_{0.2.D}) \) the class of Riemann surfaces with compact relative boundaries \( \partial R_0 \) such that there exist 2 linearly independent bounded (Dirichlet bounded) harmonic functions vanishing on \( \partial R_0 \).
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Let \((\Gamma(A^*_n), \Gamma(A^*_{n''}))\) be the part of \(R^*\) bounded by \(\Gamma(A^*_n) + \Gamma(A^*_{n''})\) + \(\sum_{n'}^{n''-1} s_i\). Let \(U'(z)\) be a harmonic function in \((\Gamma(A^*_n), \Gamma(A^*_{n''}))\) such that
\[U'(z) = b_{n'} \text{ on } \Gamma(A^*_n), \quad U'(z) = 0 \text{ on } \sum_{n'}^{n''-1} s_i, \quad \text{and } U'(z) = b_{n''} \text{ on } \Gamma(A^*_{n''}).\]
Let \(\omega(s_n, z)\) be C.P. of \(s_n\) relative \(R = A^*_1 + B_1 + A^*_2 + B_2, \cdots\). Then
\[U(z) \geq U'(z) - M \sum_{n'}^{n''-1} \omega(s_i, z). \quad (18)\]

Now by (15)
\[M \sum_{i=n'}^{n''-1} \omega(s_i, z) < \frac{M}{2^{n'+1}}(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) \leq \frac{M}{2^{n'}}\]
on \((\Gamma(A^*_n), \Gamma(A^*_{n''})) - \sum_{n'}^{n''-1} B_n\). Hence by (17)
\[U(z) \geq \min \left( b_{n'} - \frac{M}{2^{n'}}, \ b_{n''} - \frac{M}{2^{n''}} \right) \geq \min \left( a_{n'} - \frac{M}{n}, \ a_{n''} - \frac{M}{2^{n''}} \right) \geq a' - \varepsilon - \frac{M}{n} \text{ on } \Gamma(A^*_n). \]

Whence \((a'' + \varepsilon > a_{n''})\) \(b_{n''} = \min U(z) > a' - 2\varepsilon\). This contradicts (17). Hence \(a' = a''\) and \(\lim_n a_n\) exists. Similarly \(\lim_n b_n\) exists and by (16) \(\lim_n a_n = \lim b_n\).

Put \(R_{n,n+i} = R(A^*_n) + \hat{R}(\hat{A}^*_{n+i})\) (see Fig. 6). Then \(R_{n,n+i}\) is bounded by \(\hat{\Gamma}_0 + \Gamma_0 + \sum_{n}^{n+i-1} s_i + \Gamma(A^*_n) + \hat{\Gamma}^*(\hat{A}^*_{n+i})\). Let \(\omega_{n,n+i}(z)\) be a harmonic function in \(R_{n,n+i}\) such that \(\omega_{n,n+i}(z) = 0\) on \(\Gamma_0 + \hat{\Gamma}_0\), \(\omega_{n,n+i}(z) = 1\) on \(\Gamma(A^*_n)\) and \(\frac{\partial}{\partial n}\omega_{n,n+i}(z) = 0\) on \(\sum_{n}^{n+i-1} s_i\). Let \(\tilde{\omega}_n(z)\) be a harmonic function in \(R^*(A_n)\) such that \(\tilde{\omega}_n(z) = 1\) on \(\Gamma_0\), \(\tilde{\omega}_n(z) = 1\) on \(\Gamma(A^*_n)\) and \(\frac{\partial}{\partial n}\tilde{\omega}_n(z) = 0\) on \(\sum s_i\). Then \(\tilde{\omega}_n(z) = \frac{\log |z|}{\frac{1}{2} \log r_{2n+1}r_{2n+2}}\) and
\[D(\omega_{n,n+i}(z)) \geq D(\tilde{\omega}_n(z)) \geq \frac{2\pi}{\log 2}. \quad (19)\]

Let \(i \to \infty\) and then \(n \to \infty\). Then \(\omega_{n,n+i}(z) \to \omega_n(z)\) and \(\omega_n(z)\) \(\to\) C.P. determined by a sequence \(\{R^* - R(A^*_n)\} (n = 1, 2, \cdots)\) which we denote by \(\omega(B, z)\). Then by (19) \(\omega(B, z) > 0\). Consider \(\omega_{n,n+i}(z)\) in \(\hat{R}(\hat{A}^*_{n+i})\). Now \(\frac{\partial}{\partial n}\omega_{n,n+i}(z) = 0\) on \(\hat{\Gamma}^*(\hat{A}^*_{n+i})\) and <1 on \(\sum s_i\) and \(\omega_{n,n+i}(z) = 0\) on \(\hat{\Gamma}_0\). Let \(\hat{\omega}(s_m, z)\) be C.P. of \(s_m\) in \(\hat{R}\). Then \(\hat{\omega}_{n+i}(s_m, z) \to \omega(s_m, z)\) as \(n + i \to \infty\), where \(\hat{\omega}_{n+i}(s_m, z)\) is a harmonic function in \(\hat{R}_{n+i}\) (symmetric surface of \(R_{n+i}\)) such that \(\hat{\omega}_{n+i}(s_m, z)\)
$=1$ on $s_m$, $\hat{\omega}_{n+i}(s_m, z)=0$ on $\hat{\Gamma}_0$ and $\frac{\partial}{\partial n}\hat{\omega}_{n+i}(s_m, z)=0$ on $\hat{\Gamma}(\hat{A}_{n+i}^*)$. Hence by the maximum principle

$$\omega_{n,n+i}(z) \leq \sum_{m=1}^{n+i} \hat{\omega}_{n+i}(s_m, z)$$

in $\hat{R}(\hat{A}_{n+i}^*)$. Let $i \to \infty$ and then $n \to \infty$. Then

$$\omega(B, z) \leq \sum_{n=1}^{\infty} \hat{\omega}(s_m, z) \leq \frac{1}{2^2}$$

on $\hat{\Gamma}(\hat{A}_{n}^*)$. Since $\omega(B, z) \leq 1$ in $\hat{R}^*$, we have

$$\lim_{n=\infty} \omega(B, z) \leq \frac{1}{4} z \in \hat{\Gamma}(\hat{A}_{n}^*)$$

and

$$\lim_{n=\infty} \omega(B, z) \geq \frac{1}{4} z \in \hat{\Gamma}(\hat{A}_{n}^*)$$

by (15) $\omega(B, z) \leq \sum_{n=1}^{\infty} \hat{\omega}(s_m, z)$.

Now $\omega(B, z) > 0$ implies $\sup \omega(B, z) = 1$ by P.C.2.5) Consider $\omega(B, z)$ in $R^*(A_n^*) + \hat{R}^*(\hat{A}_n^*)$. Then $\omega(B, z) \leq \max \omega(B, z)$ in $R^*(A_n^*) + \hat{R}^*(\hat{A}_n^*)$. Hence

$$\lim_{n=\infty} z \in \hat{\Gamma}(\hat{A}_{n}^*) \omega(B, z) = \lim_{n=\infty} z \in \hat{\Gamma}(\hat{A}_{n}^*) \min \omega(B, z) = 1$$

since $A_n^*$ is C.R. $\left(\alpha, \frac{1}{\eta}\right)$. Let $\omega(\hat{B}, z)$ be C.P. of the ideal boundary determined by the sequence $\{\hat{R}^* - \hat{R}^*(A_n^*)\}$ similarly as $\omega(B, z)$. Then

$$\omega(B, z) \neq \omega(\hat{B}, z).$$

If we denote $\lim \omega(B, z)$ and $\lim \omega(\hat{B}, z)$ by values of $\omega(B, z)$ at $B$ and $\hat{B}$ respectively. Then

value of $\omega(B, z)$ at $B$ = value of $\omega(\hat{B}, z)$ at $\hat{B}$ = 1

value of $\omega(B, z)$ at $\hat{B}$ = value of $\omega(\hat{B}, z)$ at $B$ $\leq \frac{1}{4}$. Let $U(z)$ be a harmonic function in $R$ such that $U(z) = 0$ on $\Gamma_0 + \hat{\Gamma}_0$ and $|U(z)| \leq M$. Then

$$\lim_{n=\infty} \max_{z \in \hat{\Gamma}(\hat{A}_{n}^*)} U(z) = \lim_{n=\infty} \min_{z \in \hat{\Gamma}(\hat{A}_{n}^*)} U(z)$$

and

$$\lim_{n=\infty} \max_{z \in \hat{\Gamma}(\hat{A}_{n}^*)} U(z) = \lim_{n=\infty} \min_{z \in \hat{\Gamma}(\hat{A}_{n}^*)} U(z)$$

which we denote by $\alpha$ and $\hat{\alpha}$ respectively. Then it is proved as before that there exist two constant $\beta$ and $\hat{\beta}$ such that

$$U(z) = \beta \omega(B, z) + \hat{\beta} \omega(\hat{B}, z).$$

Hence $R \in H_0.2.B$, because $\omega(B, z)$ and $\omega(\hat{B}, z)$ are linearly independent.

Map the universal covering surface $R^\infty$ of $R$ onto $|\xi| < 1$. Suppose

5) See the properties of capacitary potentials and harmonic measures of "Potentials on Riemann Surfaces".
References


\[ |U(P(z)) - U(z)| < M \lambda \text{ on } |z| = \exp \frac{\alpha}{2}, \]

if \(0 < \mathfrak{M} < \frac{\alpha}{2\pi} < 3\mathfrak{M}\).

In the present paper we fix \(\mathfrak{M}\) and suppose that the ratio \(\frac{\alpha}{2\pi} m\) of any intense connection used in this paper satisfies
\[\mathfrak{M} < \frac{\alpha}{2\pi} < 3\mathfrak{M}. \tag{20}\]

In fact, map \(R\) onto \(0 < \text{Re} \zeta < \alpha\) by \(\zeta = \log z\). Assume that there exists no constant \(\lambda\) mentioned above, we can construct a two-sheeted Riemann surface \(R_i\) with \(\left( \frac{\alpha_i}{m} \right) \rightarrow \left( \frac{\alpha}{m} \right)\): \(\mathfrak{M} < \frac{\alpha_i}{2\pi} < 3\mathfrak{M}\) and we can also find a sequence of harmonic functions \(U_i(z)\) on \(R_i\) (\(R_i\) varies) such that \(\max |U(P(z)) - U(z)| = M\) on \(\text{Re} \zeta = \frac{\alpha}{2}\). Now \(U(z)\) is a normal family in \(\{R_i\}\): the set of Riemann surface. Choose a subsequence \(U_i(z)\) of \(\{U_i(z)\}\) such that \(U_i(z)\) converges uniformly to \(U(z)\) such that \(|U(P(z)) - U(z)| = M\) on \(|z| = \exp \frac{\alpha}{2} \).

Clearly \(U(z)\) is a harmonic function in the limit surface \(R = \lim_{i} R_i\) and \(U(P(z)) - U(z) = 0\) on endpoints of \(\sum s_j\), whence \(U(z)\) is non constant. This contradicts the maximum principle. Hence
\[ |U(P(z)) - U(z)| < M \lambda \text{ on } |z| = \exp \alpha, \text{ if } \mathfrak{M} < \frac{\alpha}{2\pi} < 3\mathfrak{M}. \]

**Intense connection a for two-sheeted Riemann surface.** Let \(J\) be a simple ring: \(1 < |z| < \exp \alpha\) and let \(m\) be an integer such that \(\lambda^m < \epsilon\).

Let \(H_i, H_i'\) and \(\Gamma_i, \Gamma_i'\ (i=1,2,\cdots, m)\) and \(\Gamma\) be rings and circles cited below:
\[H_i: \exp (i-1)\gamma < |z| < \exp i\gamma, \quad H_i': \exp (\alpha-i\gamma) < |z| \exp (\alpha-i-1)\gamma).\]
\[\Gamma_i: |z| = \exp \left(i - \frac{1}{2}\right)\gamma, \quad \Gamma_i': |z| = \exp \left(\alpha - \left(i - \frac{1}{2}\right)\gamma\right),\]
\[\Gamma: |z| = \exp \frac{\alpha}{2},\]
Let $s_{i,j}(s'_{ij})$ ($j=1, 2, \cdots, 2l$) be slits in $H_i(H'_i)$ such that

where $\gamma = \frac{\alpha}{2m+1}$.

$s_{i,j}$: $\exp \left((i-1)+\frac{1}{4}\right)\gamma < |z| < \exp \left(i-1+\frac{3}{4}\right)\gamma$, $\arg z = \frac{2\pi}{l}(j-1)$.

$s'_{ij}$: $\exp \left(\alpha - (i-1+\frac{3}{4})\gamma\right) < |z| < \exp \left(\alpha -(i-1+\frac{1}{4})\gamma\right)$, $\arg z = \frac{2\pi}{l}(j-1)$,

where the ratio $\frac{\gamma}{2\pi} \leq \frac{2\pi}{l}$.

Let $J(s)$ be the ring with slits $\sum s_{ij} + s'_{ij}$ and let $\widehat{J}(s)$ be the identical leaf as $J(s)$.

Connect $\widehat{J}(s)$ and $J(s)$ crosswise on $\sum s_{ij} + s'_{ij}$. Then we have a two-sheeted Riemann surface $J(s)+\widehat{J}(s)$.

Let $P(z)$ be the transformation in $R=J(s)+\widehat{J}(s)$ such that $\text{proj} P(z) = \text{proj} z$. Then since $R$ is invariant with respect to $P(z)$, $U(P(z))-U(z)$ is harmonic in $R$ and $|U(P(z))-U(z)| < M\lambda$ on $\Gamma_1+\Gamma'_1$, whence by the maximum principle $|U(P(z))-U(z)| < M\lambda$ on the part of $R$ bounded by $\Gamma_1+\Gamma'_1$. Also $|U(P(z))-U(z)| < M\lambda$ in $H_2+H'_2$ implies $|U(P(z))-U(z)| < M\lambda^2$ on the part of $R$ bounded by $\Gamma_2+\Gamma'_2$. In this way we have

$$|U(P(z))-U(z)| < M\lambda^n \quad \text{on} \quad |z| = \exp\frac{\alpha}{2}.
$$

Every harmonic function in $R$ such that $0 < U(z) < M$ has almost equal values with projectional deviation $< \frac{\varepsilon}{M}$ i.e.

$$|U(z_1)-U(z_2)| < \frac{\varepsilon}{M} \quad \text{for} \quad \text{proj} z_1 = \text{proj} z_2 \in \Gamma.$$

Such a connection will be called an intense connection in the ring $1 < |z| < \exp \alpha$ with projectional deviation $< \frac{\varepsilon}{M}$.

Intense connection of $2^n$ number of rings with projectional deviation $\varepsilon_n$. 

Let $J$ be a simple ring $1 < |z| < \exp \alpha$. Let $H$, $H'$ be rings $\Gamma$, $\Gamma'$, $\Gamma$ be circles $(l=1, 2, \cdots, q_n)$ cited belows:

$H_l: \exp (l-1)\beta < |z| < \exp l\beta$, $H'_l: \exp (\alpha-(l\beta)) < |z| < \exp (\alpha-(l-1)\beta)$,

$\Gamma: |z| = \exp \left(l-1 + \frac{1}{2}\right)\beta$, $\Gamma': |z| = \exp \left(\alpha-(l-1 + \frac{1}{2})\beta\right)$.

where $\beta = \frac{1}{2q_n + 1}$.

$H_l$ and $H'_l$ $(l=1, 2, \cdots, q_n)$ are conformally equivalent. In every $H_l(H'_l)$ we make slits $\sum_{j,k}(s_{lj,k} + s'_{l,j,k})$ which are conformally equivalent relative $\Gamma$ in $H_l+\hat{H}_l(H'_l+\hat{H}'_l)$.

$H_l(H'_l)$ independent of $l$ for intense connection so that every harmonic function $0 < U(z) < M$ in $H_l+\hat{H}_l(H'_l+\hat{H}'_l)$ must have the projectional deviation $\epsilon'$ ($\epsilon' = \frac{\epsilon_n}{q_n}$) on $\Gamma + \Gamma'$, where $\hat{H}_l(H'_l)$ is the examplar which is identical to $H_l(H'_l)$ and $\hat{H}_l$ and $\hat{H}'_l$ and $\hat{H}'_l$ are connected crosswise on $\sum_{j,k}(s_{l,kj} + s'_{l,j,k})$ contained in $H_l(H'_l)$. (see Fig. 9)

We denote be $I(1)$ the ring $J$ with slits for intense connection. Let $I(2), I(3), \cdots, I(2^{q_n - 1})$ and $\hat{I}(1), \hat{I}(2), \hat{I}(3) \cdots, \hat{I}(2^{q_n - 1})$ be identical examplars as $I(1)$.

First connection and first order group.

Connect $I(i)$ and $\hat{I}(i)$ on slits crosswise on the slits contained in $H_l + H'_l$ as follows:
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\[ I(1) \quad I(2) \quad \ldots \quad I(i) \quad \ldots \quad I(2^{n-1}) \]
\[ \hat{I}(1) \quad \hat{I}(2) \quad \ldots \quad \hat{I}(i) \quad \ldots \quad \hat{I}(2^{n-1}). \]

First order group.

2-nd connection and 2-nd order group.

\[ I(1) \quad I(2) \quad \ldots \quad I(2j+1) \quad I(2j+2) \quad \ldots \quad I(2^{n-1}-1) \quad I(2^{n-1}) \]
\[ \hat{I}(1) \quad \hat{I}(2) \quad \ldots \quad \hat{I}(2j+1) \quad \hat{I}(2j+2) \quad \ldots \quad \hat{I}(2^{n-1}-1) \quad \hat{I}(2^{n-1}) \]

2-nd order group

\[ I(2j+1) \quad \text{and} \quad \hat{I}(2j+1) \quad \text{and} \quad I(2j+2) \quad \text{and} \quad \hat{I}(2j+2) \]
are crosswise on the slits contained in \( H_{s} + H'_{s} \).

3-rd connection and 3-rd order group.

\[ I(1) \quad I(2) \quad I(3) \quad I(4) \quad \ldots \quad I(2j+1) \quad I(2j+2) \quad I(2j+3) \quad I(2j+4) \quad \ldots \]
\[ \hat{I}(1) \quad \hat{I}(2) \quad \hat{I}(3) \quad \hat{I}(4) \quad \ldots \quad \hat{I}(2j+1) \quad \hat{I}(2j+2) \quad \hat{I}(2j+3) \quad \hat{I}(2j+4) \quad \ldots \]

3-rd order group

\[ j=0,1,2, \ldots \frac{2^{qn-1}-4}{4} \]

Examplars with an arrow are connected crosswise on the slits contained in \( H_{s} \) and \( H'_{s} \).

\[ \ldots \ldots \]
\[ \ldots \ldots \]

I-th connection and I-th order group

\[ \left\{ I(2^{i-1}j+1) \quad I(2^{i-1}j+2) \quad \ldots \quad I(2^{i-1}j+2^{i-2}) \quad I(2^{i-1}j+2^{i-2}+1) \quad \ldots \quad I(2^{i-1}j+2^{i-1}) \right\} \]
\[ \left\{ \hat{I}(2j+1) \quad \hat{I}(2j+2) \quad \ldots \quad \hat{I}(2j+2^{i-1}) \quad \hat{I}(2j+2^{i-2}+1) \quad \ldots \quad \hat{I}(2^{i-1}+2^{i-1}) \right\} \]

I-th order group

\[ j=0,1,2, \ldots \frac{2^{qn-1}-2^{i-1}}{2^{i-1}}. \]
Examplars contained in the $i$-th groups with an arrow are connected crosswise on the slits contained in $H_i + H'_i$.

$q_n$-th connection and $q_n$-th order group
In this case $q_n$-th group is only one and

\[
\begin{align*}
I(1) & \quad I(2) \quad \cdots \quad I(2^{q_n-2}) \quad I(2^{q_n-2}+1) \quad \cdots \quad I(2^{q_n-1}) \\
\hat{I}(1) & \quad \hat{I}(2) \quad \cdots \quad \hat{I}(2^{q_n-2}) \quad \hat{I}(2^{q_n-2}+2) \quad \cdots \quad \hat{I}(2^{q_n-1})
\end{align*}
\]

Examplars with an arrow are connected crosswise on the slits contained in $H_{qn} + H'_{qn}$. Then we have a $2^{q_n}$-sheeted covering surface $R$ over $I(1)=J$.

Let $P_1(z)$ be the transformation such that $\text{proj} \ P_1(z) = \text{proj} \ z$ and $P(z)$ and $z$ are contained in the examplars with an arrow in the first connection. Then $U(P_1(z))-U(z)$ is harmonic in $R$ and $|U(P(z))-U(z)|<\varepsilon'$ on $I_1 + I'_1$, whence by the maximum principle $|U(P(z))-U(z)|<\varepsilon'$ on $I'$, that is

\[|U(z') - U(z'')|<\varepsilon', \text{ if } \text{proj } z' = \text{proj } z'' \in I' \text{ and } z' \in I(i) \text{ and } z'' \in I(i). \] (21)

Let $P_2(z)$ be the transformation such that $\text{proj} \ P_2(z) = \text{proj} \ z$ and $z$ are contained in the examplars with an arrow in the second connection. Then as above

\[|U(z') - U(z'')|<\varepsilon' \text{ if } \text{proj } z' = \text{proj } z'' \in I' \text{ and } z' \text{ and } z'' \text{ are contained in the examplars with an arrow in the second connection.} \] (22)

By (21) and (22)

\[|U(z') - U(z'')|<2\varepsilon': \text{proj } z' = \text{proj } z'' \in I', \]

where $z'$ and $z''$ are contained in the examplars in the same group of second order. (23)

\[\ldots \ldots \]

Let $P_i(z)$ be the transformation corresponding to $i$-th connection as above. Then

\[|U(P_i(z)) - U(z)|<\varepsilon' \text{ on } I', \text{ whence similarly as above}
\]

\[|U(z') - U(z'')|<i\varepsilon': \text{proj } jz' = \text{proj } z'' \in I', \]

where $z$ and $z''$ are contained in the examplars in the same group of $i$-th order. Now the $q_n$-th group contains all examplars. Hence

\[|U(z') - U(z'')|<q_n\varepsilon' = \varepsilon_n: \text{proj } z' = \text{proj } z'' \in I', \] (24)

where $z'$ and $z''$ any points of $2^{q_n}$ examplars. Thus $U(z)$ has almost equal value on $|z|=\exp \frac{\alpha}{2}$ with projectional devia
Examples of Singular Points

We call such a connection 2\(^n\)-sheets intense connection with projectional deviation \(\epsilon_n\).

3. Folded concentrating ring of 2\(^n\) number of sheets with aberration \(<\epsilon_n\).

We shall combine the two operations: concentrating and intense connection.

Let \(G\) be a simple ring: \(1<|z|<\exp\alpha\). Let \(L_1, L_2, L_3, L_1', L_2', L_3'\) be rings and \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_1', \Gamma_2', \Gamma_3'\) be circles cited below:

\[
\begin{align*}
L_1 &: 1<|z|<\exp\beta_n, \\
L_2 &: \exp\beta_n<|z|<\exp2\beta_n, \\
L_3 &: \exp2\beta_n<|z|<\exp3\beta_n. \\
\end{align*}
\]

\[
\begin{align*}
L_1' &: \exp\alpha>|z|>\exp(\alpha-\beta_n), \\
L_2' &: \exp(\alpha-\beta_n)<|z|<\exp(\alpha-2\beta_n), \\
L_3' &: \exp(\alpha-2\beta_n)<|z|<\exp(\alpha-3\beta_n).
\end{align*}
\]

\[
\begin{align*}
\Gamma_1' &: |z|\exp(\alpha-\frac{\beta_n}{2}), \\
\Gamma_2' &: z=\exp(\alpha-\frac{3\beta_n}{2}), \\
\Gamma_3' &: |z|=\exp(\alpha-\frac{5\beta_n}{2}).
\end{align*}
\]

where \(7\beta_n<\alpha\).

In \(L_i\) and \(L_i'\) \((i=1,3)\) we make slits for intense connection for 2\(^n\)-examplars with projectonal deviation \(<\frac{\epsilon_n}{2}\) on \(\Gamma_i\) and \(\Gamma_i'\) \((i=1,3)\). These slits contained in \(L_i\) and \(L_i'\) must be conformally equivalent. In \(L_2\) and \(L_2'\) we make slits and identify their edges so that every harmonic function \(|U(z)|<1\) in \(L_2\) \((L_2')\) has circular aberration \(<\frac{\epsilon_n}{2}\) on \(\Gamma_2\) \((\Gamma_2')\). Hence we have a generalized ring \(I(1)\) with two concentrating ring \(L_2\) and \(L_2'\) and has slits for intense connection for 2\(^n\)-examplars with projectional deviation \(<\frac{\epsilon_n}{2}\). Let \(I(2), I(3), \ldots, I(2^{q_n-1}), \tilde{I}(1), \tilde{I}(2), \ldots, I(2^{q_n-1})\) be \(2^{q_n}-1\) number of generalized rings with slits which are identical to \(I(1)\). Perform the
intense connection about $I(1), \cdots I(2^{n-1}), \hat{I}(1), \cdots \hat{I}(2^{n-1})$, i.e. they will be connected crosswise on the slits for intense connection mentioned as 2) (“Intense connection”). Then we have a covering surface of $2^n$ sheets over $I(1)$. We denote it by $R$. Then $R$ has the following property.

Let $U(z)$ be a harmonic function in $R$ such that $0 \leq U(z) \leq 1$. Let $P_i(z)$ be the projectional transformation such that $I(2^{i-1}j+k) \mapsto \hat{I}(2^{i-1}j+2^{i-2}+k)$. Then $U(P_i(z)) - U(z)$ is harmonic in $R$. Then $|U(P_i(z)) - U(z)| < \varepsilon'$, $\varepsilon' = \varepsilon_n / 2^{2n}$ on $\Gamma_1 + \Gamma_2(\Gamma_1^* + \Gamma_2^*)$. Hence by the maximum principle $|U(P_i(z)) - U(z)| < \varepsilon'$ on $L_2 + L_4$. Hence $U(z)$ has projectional deviation $< \varepsilon_n / 2$ on $\Gamma_2(\Gamma_2^*)$. Next $L_2(L_4)$ is a concentrating ring. $U(z)$ has circular aberration $< \varepsilon_n / 2$ on $\Gamma_2(\Gamma_2^*)$. Whence we see that there exist constants $\alpha'$ and $\alpha''$ such that

$$|U(z) - \alpha'| < \varepsilon_n \text{ for } z \in R \text{ and } \log |z| = \frac{3}{2} \beta_n,$$

$$|U(z) - \alpha''| < \varepsilon_n \text{ for } z \in R \text{ and } \log |z| = \alpha - \frac{3}{2} \beta_n.$$

We denote such a $2^n$-sheeted covering surface over a generalized ring by $(\alpha, \beta_n, q_n, \epsilon_n)$ and call a folded concentrating ring.

**Lemma 2.** Let $S$ be a sector such that $1 < |z| < \exp \gamma$, $0 < \arg z < \Theta$ with a finite number of radial slits. Let $U(z)$ be a harmonic function in $S$ with boundary values $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ on $|z| = 1$ and $|z| = \exp \gamma = r$, where $\varphi(e^{i\theta})$ and $\varphi(re^{i\theta})$ are continuous functions of $z$. Then

$$D(U(z)) \geq \frac{1}{2\pi} \int_{0}^{\pi} |\varphi(e^{i\theta}) - \varphi(re^{i\theta})|^2 d\theta.$$ 

**Proof.** We divide $S$ into sufficiently narrow circular rectangles $A_j: 1 < |z| < \exp \gamma$, $\theta_j < \arg z \leq \theta_{j+1}$ ($j=1,2,3, \cdots m$) such that $\max_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(e^{i\theta}) = \min_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(e^{i\theta}) < \frac{1}{n}$ and $\max_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(re^{i\theta}) = \min_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(re^{i\theta}) < \frac{1}{n}.$

Let $\{A_j\}$ and $\{A'_j\}$ be $\{A'_j\}$ such that $\max_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(e^{i\theta}) \leq \min_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(re^{i\theta})$ and $\min_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(e^{i\theta}) \geq \max_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(re^{i\theta})$ respectively and let $\{A'_j\}$ be rectangles contained neither in $\{A'_j\}$ nor in $\{A'_j\}$. Suppose $A_j \notin \{A'_j\}$. Let $\tilde{U}_j(z)$ be a harmonic function in $A_j$ such that $\tilde{U}_j(z) = \max_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(e^{i\theta})$ on $|z| = 1$, $\tilde{U}_j(z) = \min_{\theta_j \leq \theta \leq \theta_{j+1}} \varphi(re^{i\theta})$ on $|z| = \exp \gamma$ and $\frac{\partial}{\partial n} \tilde{U}_j(z) = 0$ on two segments: $1 < |z| < \exp \gamma, \arg z = \theta_j$. 


and $1 < |z| < \exp \gamma$, arg $z = \theta_{j+1}$ and on radial slits in $A_j$. Let $\tilde{U}_j(z)$ be a harmonic function in $A'$ such that $\tilde{U}(z) = \varphi(e^{i\theta})$ on $|z| = 1$ and $\tilde{U}_j(z) = \varphi(re^{i\theta})$ on $|z| = \exp \gamma$ and $\frac{\partial}{\partial n} \tilde{U}_j(z) = 0$ on two segments: $1 < |z| < \exp \gamma$, arg $z = \theta_j$ and on $1 < |z| < \exp \gamma$, arg $z = \theta_{j+1}$ and on radial slits contained in $A_j$. Then since $U_j(z) - \tilde{U}_j(z) \geqq 0$ and $\frac{\partial}{\partial n} \tilde{U}_j(z) \geqq 0$ on $|z| = 1$ and $\tilde{U}_j(z) - \overline{\tilde{U}_j(z)} \leqq 0$ and $\frac{\partial}{\partial n} U_j(z) \leqq 0$ on $|z| = \exp \gamma$, we have

$$D_{A_j}(\tilde{U}_j(z), \tilde{U}_j(z) - \overline{\tilde{U}_j(z)}) \geqq 0,$$

whence

$$D_{A_j}(U_j(z)) \geqq D_{A_j}(\overline{\tilde{U}_j(z)}) \approx 0,$$

Clearly

$$D(\tilde{U}_j(z)) = \frac{(\theta_{j+1} - \theta_j)}{2\pi \gamma} \left[ \max_{\theta_j \leqq \theta \leqq \theta_{j+1}} \varphi(e^{i\theta}) - \min_{\theta_j \leqq \theta \leqq \theta_{j+1}} \varphi(e^{i\theta}) \right]^2.$$

On the other hand, $D_{A_j}(U(z) - \tilde{U}_j(z), \tilde{U}_j(z)) = 0$, whence $D_{A_j}(U(z)) \geqq D_{A_j}(\tilde{U}_j(z))$.

Hence

$$D_{A_j}(U(z)) \geqq D_{A_j}(\tilde{U}_j(z)) \geqq \frac{1}{2\pi \gamma} \int_{\theta_j}^{\theta_{j+1}} |\varphi(e^{i\theta}) - \varphi(re^{i\theta})|^2 - \frac{4(\theta_{j+1} - \theta_j)}{2\pi \gamma n^2}.$$

In case $A_j \in \{A_j''\}$, we have similarly the above inequality. In case $A_j \in \{A_j'''\}$, let $\tilde{U}_j(z) = 0$.

Then

$$D_{A_j}(U(z)) \geqq \frac{1}{2\pi \gamma} \int_{\theta_j}^{\theta_{j+1}} |\varphi(e^{i\theta}) - \varphi(re^{i\theta})|^2 - \frac{4(\theta_{j+1} - \theta_j)}{2\pi \gamma n^2}.$$

Hence

$$D(U(z)) \geqq \sum_{j} D(\tilde{U}_j(z)) \geqq \frac{1}{2\pi \gamma} \int_{0}^{\Theta} |\varphi(e^{i\theta}) - \varphi(re^{i\theta})|^2 - \frac{4\Theta}{2\pi \gamma n}.$$
$(\log 2) (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}) < \text{Re } z < (\log 2) (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n}})$

$\text{Im } z = 2\pi \left( \frac{i_1 - 1}{2} + \frac{i_2 - 1}{2^2} + \cdots + \frac{i_n - 1}{2^n} \right)$

and

$(\log 2) (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}) < \text{Re } z < (\log 2) (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n}})$

$\text{Im } z = 2\pi \left( \frac{i_1 - 1}{2} + \frac{i_2 - 1}{2^2} + \cdots + \frac{i_{n-1} - 1}{2^{n-1}} + \frac{1}{2^{n-1}} \right)$.

Then $A_{i_1 \cdots i_n}$ becomes a ring $A_{i_1 \cdots i_n}^R$. $A_{i_1 \cdots i_{n+1}}$ and $A_{i_1 \cdots i_{n+1}}^R$ have boundary $B_{i_1 \cdots i_{n+1}}$ lying on $\text{Re } z = (\log 2) (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n}})$, $2\pi \left( \frac{i_1 - 1}{2} + \frac{i_2 - 1}{2^2} + \cdots + \frac{i_{n-1} - 1}{2^{n-1}} + \frac{1}{2^{n-1}} \right)$.

$0 \leq \text{Re } z < \log 2$, $0 \leq \text{Im } z < 2\pi$.

Let $H_1$, $H_2$, $H_3$, $H_1'$, $H_2'$, $H_3'$ and $\Gamma$, $\Gamma'$ be rings and circle cited belows:

$H_1: 0 < \text{Re } \zeta < \beta_n$. $H_2: \beta_n < \text{Re } \zeta < 2\beta_n$. $H_3: 2\beta_n < \text{Re } \zeta < 3\beta_n$.

$H_1': \text{log 2} > \text{Re } \zeta > (\text{log 2}) - \beta_n$. $H_2': (\text{log 2}) - \beta_n > \text{Re } \zeta > (\text{log 2}) - 2\beta_n$.

$H_3': (\text{log 2}) - 2\beta_n > \text{Re } > (\text{log 2}) - 3\beta_n$.
Examples of Singular Points

\[ \Gamma : \text{Re} \zeta = \frac{3}{2} \beta_n \]
\[ \Gamma' : \text{Re} \zeta = (\log 2) - \frac{3}{2} \beta_n, \]

where
\[ \beta_n = \frac{\log 2}{2^{2n} \times 6} \]

In \( H_2 \) we make slits and identify their edges so that \( H_2 \) becomes a concentrating ring \( \left( \beta_n, \frac{1}{10^n} \right) \), i.e. circular aberration on \( \Gamma(\Gamma') \leq \frac{1}{10^n} \) (see Fig. 10).

When we perform the intense connection, a covering surface over \( A_{i_{1}, i_{2}, \ldots, i_{n}} \) becomes a \( 2^{q_{n}} \)-folded concentrating ring \( \left( \log 2, \beta_n, q_{n}, \frac{1}{10^n} \right) \), where \( q_{n} = 2n \).

We remark: the ratio of slits for intense connection is between \( \mathfrak{M} \) and \( 3\mathfrak{M} \).

Let \( s_{1,i,j,k} \) be slits in \( H_i \) for the intense connection \( (H_3, H_1', H_3') \). Then \( s_{1,i,j,k} \) is as follows:

let \( h_{1,j}(h_{1j}') \) be a ring
\[ h_{1,j} : (j-1)\gamma \leq \text{Re} \zeta \leq j\gamma, \quad 0 \leq \text{Im} \zeta \leq 2\pi. \]
\[ h_{1j}' : \beta_n - i\gamma \leq \text{Re} \zeta \leq \beta_n - (i-1)\gamma, \quad 0 \leq \text{Im} \zeta \leq 2\pi, \]

where \( \gamma = \frac{\beta_n}{2q_{n}+1} \). \( s_{1,i,j,k} \) is a slit in \( h_{1}(h_{1j}') \) as follows:

\[ s_{1,i,j,k} : \left( j-1 + \frac{1}{4} \right)\gamma \leq \text{Re} \zeta \leq \left( j-1 + \frac{3}{4} \right)\gamma, \quad \text{Im} \zeta = \frac{2\pi k}{l_n} : k = 1, \ldots, l_n (j = 1, 2, \ldots, q_{n}) \]

and the ratio \( \frac{\gamma}{2\pi} \) satisfies \( \mathfrak{M} < \frac{\gamma}{2\pi < 3\mathfrak{M} \text{ independently of } n. \)

Especially we observe \( s_{1,i,j,k} \) in a ring \( h_{1} \) which is used for the first intense connection, where \( h_{1} : 0 \leq \text{Re} \zeta \leq \gamma, \quad 0 \leq \text{Im} \zeta \leq 2\pi. \)
\[ s_{1,1,k} : \frac{\delta}{4} \leq \text{Re} \zeta \leq \frac{3}{4}, \quad \text{Im} \zeta = \frac{2\pi k}{l_n}. \]

(See Fig. 11)

Put \( I(1) = A_{i_{1}, i_{2}, \ldots, i_{n}} \), let \( I(2), \ldots, I(2^{q_{n}-1}) \) and \( \hat{I}(1), \hat{I}(2), \ldots, \hat{I}(2^{q_{n}-1}) \) be exemplars which are identical to \( A_{i_{1}, i_{2}, \ldots, i_{n}} \) and perform the \( 2^{q_{n}} \)-intense connection by use of slits contained in \( H_3, H_1', H_3' \) and \( H_3' \). Then we have a \( 2^{q_{n}} \) sheeted-folded concentrating ring \( \left( \log 2, \beta_n, q_{n}, \frac{1}{10^n} \right) \). We denote it by \( A_{i_{1}, i_{2}, \ldots, i_{n}}^{f,C,R} \).

All \( A_{i_{1}, i_{2}, \ldots, i_{n}}^{f,C,R} \) of \( n \)-th grade folded concentrating ring are coformally equivalent to \( A_{i_{1}, i_{2}, \ldots, i_{n}}^{f,C,R} \).
Let $A_{i_{1},\cdots,i_{n}}^{G,R}$ be a generalized ring with slits for intense connection and put $A_{n}^{G,R}=A_{0}^{R}+(A_{1}+A_{2})+(A_{11}+A_{21}+A_{22})+\cdots+(A_{i_{1}i_{2}\cdots i_{n}}+A_{i_{1}\infty\cdots i_{n}})$.

3.2. Covering surface over $A^{G,R}$.

Put $CA_{n}^{G,R}=A^{G,R}-A_{n-1}^{G,R}$. Then $CA_{n}^{G,R}$ is composed of $2^{n}$ components, i.e.

$$CA_{n}^{G,R}=\sum_{i_{1}\cdots i_{n}}(A_{i_{1}\cdots i_{n}}+A_{i_{1}i_{2}\cdots i_{n}},\cdots).$$

We denote by $m$ CA $m$ number of examplars which are identical to $CA$. Put $R_{n}^{G,R}=A_{0}+(2^{q_{1}}-1)CA_{1}^{G,R}+(2^{q_{2}}-2^{q_{1}})CA_{2}^{G,R}+\cdots+(2^{q_{n}}-2^{q_{n-1}})CA_{n}^{G,R}+\cdots$. Then $R_{n}^{G,R}$ is a covering surface (consisting of infinitely many disjoint components) over $A^{G,R}$. We shall construct a Riemann surface $R$ from $R_{n}^{G,R}$ by intense connection. Every $A_{i_{1}i_{2}\cdots i_{n}}^{G,R}$ (of $n$-th grade $A_{n}^{G,R}$) is covered by $R_{n}^{G,R} 2^{q_{n}}$-times. We shall perform the intense connection among $2^{q_{n}}$ number of sheets over $A_{i_{1}i_{2}\cdots i_{n}}^{G,R}$ as follows:

0-th step. $I^{0}(1)=A_{0}+\sum_{i_{1}}A_{i_{1}}+\sum_{i_{1},i_{2}}A_{i_{1}i_{2}}+\cdots$, where $A_{0}$ is a simple ring.

First step. Put $I^{1}(1)=CA_{0}$, $I^{1}(2)=I^{1}(3)=\cdots=I^{1}(2^{q_{1}-1})=\hat{I}^{1}(1)=\hat{I}^{1}(2)$ $\cdots=I^{1}(2^{q_{1}-1})=I^{1}(1)$. $A_{1}^{G,R}+A_{2}^{G,R}$ is covered by $\sum I^{1}(j)+\hat{I}^{1}(j)$. We perform the intense connection among the above $2^{q_{1}}$-sheeted covering surface by
the slits contained in $A_1 + A_2$ (the slits are contained in $H_1 + H'_1 + H_3 + H'_3$ of $A_i$ (1-st grade $A_i$). Then we have a Riemann surface $R_1$ which has the following properties.

a). $R_1$ has the following relative boundary $B_0$ which lies on $\Re z = 0$, $0 < \Im z < 2\pi$ and $B_1$ consisting of $2^{q_1} - 1$ number of components lying on $\Re z = \log 2$, $0 < \Im z < \pi$, and $B_2$ consisting of $2^{q_1} - 1$ number of components lying on $\Re z = \log 2$, $\pi < \Im z < 2\pi$.

b). The part of $R_1$ lying over $\Re z > (\log 2)(1 + \frac{1}{2})$ is now a generalized ring with slits for intense connection.

c). The part of $R_1$ lying over $\log 2 < \Re z < (\log 2)(1 + \frac{1}{2})$ has no slits, since the intense connection is performed and the part over $A_1(A_2)$ is a folded concentrating ring $\left( \log 2, \beta_1 q_2, \frac{1}{10} \right)$.

Suppose $n-1$-th step is performed.

$n$-th step is as follows: Let $I^n(i)$ be examplars cited below:

\[
\begin{align*}
I^n(1) &= I^{n-1}(1) \\
\vdots & \vdots \\
I^n(2^{q_{n-1}-1}) &= I^{n-1}(2^{q_{n-1}-1}) \\
I^n(2^{q_{n-1}-1} + 1) &= \tilde{I}^{n-1}(1) \\
\vdots & \vdots \\
I^n(2^{q_{n-1}}) &= \tilde{I}^{n-1}(2^{q_{n-1}-1}) \\
I^n(2^{q_{n-1}} + 1) &\left\{ \begin{array}{l}
\vdots \\
I^n(2^{q_{n-1}}) \\
\tilde{I}^n(1) \\
\vdots \\
\vdots \\
\vdots \\
\tilde{I}^n(2^{q_{n-1}}) \\
\end{array} \right. \\
\right. \\
\end{align*}
\]

newly added at the $n$-th step and $I^n(j)$ is identical to $CA_{n-1}^{G,R}$. 

We perform the intense connection (for $2^{n}$ sheets) by slits contained in $A_{i_{1},...,i_{n}}^{g,R}$. Let $\mathcal{M}$ be the above Riemann surface obtained after the intense
connection and put

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_{n}.$$  

a). $\mathcal{M}_{n}$ has relative boundary

$$B_{0}+(B_{1}+B_{2})+\cdots+\sum_{i_{1},i_{2},...,i_{n}}B_{i_{1}i_{n}}.$$  

and is open. $B_{i_{1}...i_{l}}$ is composed of

$$2^{q_{l}}-2^{q_{l-1}}(l \leq n).$$

b). The part of $\mathcal{M}_{n}$ lying over

$$\text{Re} \ z>(\log 2)(1+\frac{1}{2}+\cdots+\frac{1}{2^{n}})$$

is now a generalized ring with slits (about which the intense connection is not performed) and consists of infinitely many components.

A). We denote by $A_{i_{1},...,i_{n}}^{f,C,R}$ the part of $\mathcal{M}_{n}$ over $A_{i_{1},...,i_{n}}^{C,R}$. Then $A_{i_{1}}^{f,C}$ is a folded concentrating ring

$$\left(\log 2, \beta_{n}, q_{n}, \frac{1}{10^{n}}\right).$$

We map $A_{i_{1},...,i_{n}}^{f,C,R}$ onto $0<\text{Re} \ z<\log 2, 0<\text{Im} \ z<2\pi$ on the $\zeta$-plane. $A_{i_{1}}^{f,C,R}$ is a $2^{q_{n}}$-sheeted covering surface consisting of

$$I^{n}(1), \ldots, I^{n}(2^{n-1}), I^{n}(2^{n-1}+1)\cdots I^{n}(2^{n-1}), I^{n}(1), \ldots, \hat{I}^{n}(q_{n})$$

of which $2^{n}-2^{n-1}$ number of sheets ($N^{n}(I(i))$) are newly added at the $n$-th step and $B_{i_{1},...,i_{n}}$ of $\mathcal{M}$ is the relative boundary of $N(I(i))$ (the set of $I(i)$ and $I(i)$ which are added newly at the $n$-th step). Every $I^{n}(i)$ and $\hat{I}^{n}(i)$ are connected at the first connection, that is, $I^{n}(i)$ and $\hat{I}^{n}(i)$ are connected crosswise on $\sum_{j_{k}}s_{1,j_{k}}$. We observe

$$s_{1,1,k}, \text{where} \sum_{k}s_{1,1,k}= \text{slits in} \ h_{1}+\hat{h}_{1}.$$  

and $h_{i}(\hat{h}_{i})$: $0<\text{Re} \ z<\gamma, 0<\text{Im} \ z<2\pi$ (see Fig. 13).

$$s_{1,1,k} \geq \frac{\gamma}{4}<\text{Re} \ z<\frac{3}{4} \gamma, \text{Im} \ z=\frac{k}{2l_{n}}(k=1,2, \ldots l_{n})$$

the ratio $\frac{\gamma}{2\pi}$ is bounded so as

$$\frac{\gamma}{2\pi} < 3\mathcal{M} : \gamma = \frac{\beta_{n}}{2^{q_{n+1}}},$$

(27)

$\hat{h}_{1}$ is the set which is identical to $h_{1}$ in $^{n}I(i)$. 

Now $h_1 + \hat{h}_1$ is a generalized ring connected crosswise on $\sum_k s_{1,1,k}$ and has four relative boundaries $\beta_1, \beta_2, \beta_3, \beta_4$ (see Fig. 13).

Let $w(z)$ be a harmonic function in $h + \hat{h}_1$ such that $0 < w(z) < 1$ and $= 0$ on one of $\beta_1, \beta_2, \beta_3$ and $\beta_4$. Then since $\mathfrak{N} < \frac{r}{2\pi} < 3\mathfrak{N}$, there exists a constant $\mu$ such that

$$0 < w(z) < \mu < 1 \text{ on } \Re \zeta = \frac{r}{2}$$

(28)

Let $\Gamma_{i_1 \ldots i_n}$ be the set of $\Re$ lying over $\Re \zeta = \frac{r}{2}$. Then $\Gamma_{i_1 \ldots i_n}$ consists of $2^q$ number of circles which are contained in $A_{i_1 \ldots i_n}^{f,C,R}$. Let $w(z)$ be a harmonic function in $\Re$ such that $0 < w(z) < 1$ and $w(z) = 0$ on $B_0 + (B_1 + B_2) + (B_{11} + B_{12} + B_{21} + B_{12}) + \cdots$. Then since at least one of $I^n(i)$ and $\hat{I}^n(i)$ is added newly at the $n$-th step, one of $\beta_1, \beta_2, \beta_3, \beta_4$ must be contained in $B_{i_1 \ldots i_n}$ on which $w(z) = 0$. Now $h_1 + \hat{h}_1$ in $I^n(i) + \hat{I}^n(i)$ of every $A_{i_1 \ldots i_n}$ is conformally equivalent to that of $I^n(1) + \hat{I}^n(1)$. Hence by (28)

$$w(z) < \lambda < 1 \text{ on } \Gamma_{i_1 \ldots i_n}.$$  

(29)

Let $\Re_n^\Gamma$ be the component of $\Re$ containing $B_0$ divided by $\sum_{i_1=1,2, i_2=1,2, \ldots i_n=1,2} \Gamma_{i_1 \ldots i_n}$ from $\Re$. Then $\Re_n^\Gamma (n=1,2, \cdots)$ is an exhaustion of $\Re$, i.e.

$$\bigcup \Re_n^\Gamma = \Re.$$  

(30)

B. Let $^1\Gamma_{i_1 \ldots i_n}$ and $^2\Gamma_{i_1 \ldots i_n}$ be the sets of $A_{i_1 \ldots i_n}^{f,C,R}$ lying over the ring

$$\Re \zeta = \frac{3\beta_n}{2}, \quad 0 < \Im \zeta < 2\pi \text{ and } \Re \zeta = (\log 2) - \frac{3}{2} \beta_n, \quad 0 < \Im \zeta < 2\pi.$$  

The part $A_{i_1 \ldots i_n}^{f,C,R}$ of $\Re$ over $A_{i_1 \ldots i_n}^{g,R}$ is a folded concentrating ring. Let $w(z)$
be a harmonic function in $A_{i_{1}}^{f,C}$ such that $0 < \omega(z) < 1$. Then there exist two constants $a_{i_{1}...i_{n}}^{1}$ and $a_{i_{1}...i_{n}}^{2}$ such that

$$|\omega(z) - a_{i_{1}...i_{n}}^{1}| < \frac{1}{10^{n}} \text{ on } 1\Gamma_{i_{1}...i_{n}}^{D},$$

$$|\omega(z) - a_{i_{1}...i_{n}}^{2}| < \frac{1}{10^{n}} \text{ on } 2\Gamma_{i_{1}...i_{n}}^{D}.$$

Suppose $|a_{i_{1}...i_{n}}^{1} - a_{i_{1}...i_{n}}^{2}| > \frac{2}{10^{n}}$. Then the slits used for the intense connection are radial in $A_{i_{1}}^{g,R}$ and $A_{i_{1}}^{f,C}$, and it is composed of $2^{n}$ sheets. Hence by Lemma 2

$$D(\omega(z)) > \frac{2^{n} \left( a_{i_{1}...i_{n}}^{1} - a_{i_{1}...i_{n}}^{2} - \frac{2}{10^{n}} \right)^{2} \times 2\pi}{\log 2} = \frac{2\pi \left( a_{i_{1}...i_{n}}^{1} - a_{i_{1}...i_{n}}^{2} - \frac{2}{10^{n}} \right)^{2} 2^{n}}{\log 2}.$$

Suppose $\omega(z)$ be a harmonic function in $A_{i_{1}}^{f,C,R} + A_{i_{1}}^{f,C,R}$ such that $0 < \omega(z) < 1$. Then there exist constants $a_{i_{1}...i_{n-1}}^{2}$ and $a_{i_{1}...i_{n}}^{1}$ such that

$$|\omega(z) - a_{i_{1}...i_{n-1}}^{2}| < \frac{1}{10^{n-1}} \text{ on } 2\Gamma_{i_{1}...i_{n-1}}^{D} \text{ and } |\omega(z) - a_{i_{1}...i_{n-1}}^{1}| < \frac{1}{10^{n}} \text{ on } 1\Gamma_{i_{1}...i_{n-1}}^{D}.$$

Then as above if $|a_{i_{1}...i_{n-1}}^{2} - a_{i_{1}...i_{n-1}}^{1}| > \frac{2}{10^{n-1}},$

$$D(\omega(z)) \geq \frac{\left( a_{i_{1}...i_{n-1}}^{2} - a_{i_{1}...i_{n-1}}^{1} - \frac{2}{10^{n-1}} \right)^{2} \pi}{3\beta_{n-1}} = \frac{1}{2} \frac{\log 2}{2^{2n-2}} \cdot \beta_{n} = \frac{\log 2}{6 \times 2^{2n}} \quad (32)$$

$\Re$ has relative boundary $B_{0} + (B_{1} + B_{2}) + (B_{1,1} + B_{1,2} + B_{2,1} + B_{2,2}) \cdots$. $\Re_{n}^{f}$ is
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compact and \( \mathcal{R} - \mathcal{R}^* \) is composed of \( 2^n \) number of non compact components which we denote by \( G_{i_1 \ldots i_n} \) (\( i_1 = 1, 2, \ldots \)), where \( G_{i_1 \ldots i_n} \) is the part of \( \mathcal{R} \) lying over \( (A_{i_1 \ldots i_n, 1} + A_{i_1 \ldots i_n, 2}) + (A_{i_1 \ldots i_n, 1} + A_{i_1 \ldots i_n, 2}) + \cdots \). \( G_{i_1 \ldots i_n} \supset G_{i_1 \ldots i_n, i_{n+1}} \supset G_{i_1 \ldots i_n, i_{n+1}, i_{n+2}} \cdots \) determines an ideal boundary component \( v(i_1, j_2, \ldots, i_n, \ldots) \). The set \( B^* \) of \( p \) clearly the power of continuum. Then \( \mathcal{R} \) has the following properties.

a). Let \( N(z, p) \) be the N-Green's function of \( \mathcal{R} \) and suppose that N-Martin's topology is defined in \( \mathcal{R} \) and

\[
\delta(p, q) = \sup_{z \in \mathcal{R}} \left| \frac{N(z, p)}{1+N(z, p)} - \frac{N(z, q)}{1+N(z, q)} \right|.
\]

Then there exists only one ideal boundary point \( p \) of N-Martin's topology on a boundary component \( v(i_1, i_2, \ldots) \).

b). \( \omega(p, z) > 0 \), i.e. \( p \) is a singular point.

c). \( w(B^*, z) = 0 \).

Hence \( B^* \ni p \) and by \( w(p, z) \leq w(B^*, z) \leq 0 \). Every point \( p \) is a singular point of first kind. Hence \( \mathcal{R} \) has the set of singular points of first kind of power of continuum.

Proof of a). Let \( \mathcal{R}^D_{\Gamma} \) be the component of \( \mathcal{R} \) containing \( B_0 \) divided by \( \sum_{i_1=1,2, \ldots, i_n=1,2} \). \( \mathcal{R} - \mathcal{R}^D_{\Gamma} \) is composed of \( 2^n \) number of non compact components \( G_{i_1 \ldots i_n} \) lying over \( A_{i_1 \ldots i_n, 1} + A_{i_1 \ldots i_n, 2} \). Clearly \( \{G_{i_1 \ldots i_n}\} \) and \( \{G_{i_1 \ldots i_n}^D\} \) equivalent and \( G_{i_1 \ldots i_n}^D \supset G_{i_1 \ldots i_n, i_{n+1}} \supset G_{i_1 \ldots i_n, i_{n+1}, i_{n+2}} \cdots \) determines also \( v(i_1, i_2, \ldots) \). Let \( U(z) \) be a harmonic function in \( \mathcal{R} - A_0 \) and \( 0 < U(z) < M \). Then \( A_{i_1 \ldots i_n}^D \) is a folded concentrating ring. Hence

\[
\max_{z \in \mathcal{R}} |U(z)| = \min_{z \in \mathcal{R}} |U(z)| < \frac{1}{10^n} \rightarrow 0 \text{ as } n \rightarrow \infty:
\]

Hence by Lemma 1, there exists only one point \( p \) of N-Martin's topology on \( v(i_1, i_2, \ldots) \).

Proof of b). Let \( w(B^*, z) \) be a harmonic measure of \( B^* \). Then \( w(B^*, z) = 0 \) on \( B_0 + (B_1 + B_2) + \cdots \) and \( \leq 1 \). Hence by (29) \( w(B^*, z) < \mu < 1 \) on \( \mathcal{R}^D_{\Gamma} \). By the maximum principle \( w(B^*, z) \leq \mu \) on \( \mathcal{R}^D_{\Gamma} \), whence sup \( w(B^*, z) < \mu < 1 \). Hence by P.H.3.5) \( w(B^*, z) = 0 \).

Proof of c). Let \( p \) be an ideal boundary point of N-Martin's topology over a boundary component \( v(i_1, i_2, \ldots) \). Put \( v_i(p) = E \left[ z \in \mathcal{R} : \delta(z, p) > \frac{1}{l} \right] \).

Then by Lemma 1 there exists a number \( m \) such that \( v_i(p) \supset G_{i_1 \ldots i_n} \). Let

5) See 4)
\( \omega(v_l(p), z) \) and \( \omega(G_{i_1 \cdots i_m}, z) \) be C.P.'s of \( v_l(p) \) and \( G_{i_1 \cdots i_m} \). Then

\[
\omega(p, z) = \lim_{l=\infty} \omega(v_l(p), z) \geq \lim_{m=\infty} \omega(G_{i_1 \cdots i_m}, z).
\]

For simplicity put \( \omega_m(z) = \omega(G_{i_1 \cdots i_m}, z) \). Now \( \omega_m(z) \) has M.D.I. among all harmonic functions with value 0 on \( B_0 \) and \( \omega_m(z) = 1 \) on \( G_{i_1 \cdots i_m} \). Hence by the Dirichlet principle

\[
D(\omega_m(z)) \leq D(\tilde{\omega}(z)) = \frac{2\pi}{\log 2},
\]

where \( \tilde{\omega}(z) \) is a harmonic function in \( A_0 \) such that \( \tilde{\omega}(z) = 0 \) on \( \text{Re } z = 0 \) and \( \tilde{\omega}(z) = 1 \) on \( \text{Re } z = \log 2 \).

Since \( A_1^1 \) (\( n \geq 1 \)) is a folded concentrating ring, there exist constants \( a_{i_1 \cdots i_n}^1 \) and \( a_{i_1 \cdots i_n}^2 \) such that

\[
|\omega_m(z) - a_{i_1 \cdots i_n}^1| < \frac{1}{10^n} \quad \text{for } z \in \Gamma_{i_1 \cdots i_n}^p, \quad n \geq 1 \quad \text{and} \quad m > n, \tag{34}
\]

\[
|\omega_m(z) - a_{i_1 \cdots i_n}^2| < \frac{1}{10^n} \quad \text{for } z \in \Gamma_{i_1 \cdots i_n}^p, \quad n \geq 1 \quad \text{and} \quad m > n,
\]

where \( a_{i_1 \cdots i_n}^1 \) and \( a_{i_1 \cdots i_n}^2 \) depend on \( \omega_m(z) \).

Put \( A_m(i_1, i_2, \cdots, i_n) = \max |\omega_m(z) - \omega_m(z')| \), where \( z' \in \Gamma_{i_1 \cdots i_n}^p \) and \( z'' \in \Gamma_{i_1 \cdots i_n}^p \): \( n \geq 1 \) (see Fig. 14).

Then by (34)

\[
|A_m(i_1, i_2, \cdots, i_n)| \leq |a_{i_1 \cdots i_n}^1 - a_{i_1 \cdots i_n}^2| + \frac{2}{10^n}. \tag{35}
\]

Put \( A'_m(i_1, i_2, \cdots, i_n) = \max |\omega_m(z) - \omega_m(z')| \), where \( z' \in \Gamma_{i_1 \cdots i_n}^p \) and \( z'' \in \Gamma_{i_1 \cdots i_{n-1}, i_n}^p \): \( n \geq 2 \).

Then by (34)

\[
|A'_m(i_1, i_2, \cdots, i_n)| < |a_{i_1 \cdots i_{n-1}}^1 - a_{i_1 \cdots i_n}^2| + \frac{2}{10^{n-1}}. \tag{36}
\]

Now by (34), (33) and (31)

\[
|a_{i_1 \cdots i_n}^1 - a_{i_1 \cdots i_n}^2| < \sqrt{\left( \frac{2\pi}{\log 2} \right) \log 2 + \frac{2}{10^n}} = \frac{1}{2^n} + \frac{2}{10^n}. \tag{37}
\]

Similarly by (34), (33) and (32)

\[
|a_{i_1 \cdots i_{n-1}}^1 - a_{i_1 \cdots i_{n-1}}^2| < \sqrt{\left( \frac{2\pi}{\log 2} \right) \log 2 + \frac{2}{10^{n-1}}} = \frac{1}{2^{n-1}} + \frac{2}{10^{n-1}}. \tag{38}
\]

We show \( \omega(p, z) > 0 \). Assume \( \lim_m \omega_m(z) = 0 \). Then for any given positive number \( \epsilon > 0 \) and for any given \( \gamma \Gamma_{i_1 \cdots i_n}^p \), there exists a number \( m \) \((i_1, i_2, \cdots, i_n, \epsilon)\) such that
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\[ \omega_m(z) < \varepsilon \text{ on } \mathcal{D}_{t_1 \cdots t_n}^2 \text{ and } \omega_m(z) = 1 \text{ on } \mathcal{D}_{i_1 \cdots i_m}^{z}. \]

Now

\[ \varepsilon > \min_{z \in \mathcal{D}_{t_1 \cdots t_n}^2} \omega_m(z) \geq 1 - \sum_{n = n_0}^{m} \left( \frac{4}{10^{n-1}} + \frac{1}{2^{n-2}} \right) \frac{\gamma}{n_0} \text{ for } n_0 \geq 4. \]

This is a contradiction for \( \sum_{n = n_0}^{m} \left( \frac{4}{10^{n-1}} + \frac{1}{2^{n-2}} \right) < \frac{1}{2} \) for \( n_0 \geq 4 \). Hence \( \omega(p, z) \geq \lim_{m} \omega_m(z) > 0 \).

PART II

We shall construct a covering surface over the \( z \)-plane which has finite area and has a singular point of first kind. Theorem 12.\( c \) of "Singular points of Riemann surfaces" shows that there exists no singular point, if the number of sheets of the covering surface is bounded and the Theorem 12.\( a \) shows that there exists no singular point of second kind, if the area of the covering surface is finite. Hence this example means that the condition that the number of sheets of the covering surface is bounded is necessary for the validity of Theorem 12.\( c \).

Example 4. Let \( A \) be a disc \( 0 < |z| < 1 \). Map \( A \) onto \( -\infty < \text{Re} \xi < 0, 0 < \text{Im} \xi < 2\pi \). Let \( A_n \) be a ring (\( A_n \) is a rectangle in itself, but for the simplicity in this paper we call a ring) such that

\[ -n(\log 100) < \text{Re} \xi < -(n-1) \log 100, 0 < \text{Im} \xi < 2\pi. \]

For \( n \geq 2 \), we shall construct a \( 2^{n}(q_n = 4n) \)-sheeted covering surface over \( A_n \) by intense connection. Let \( H_n, H'_n, \Gamma_n, \Gamma'_n, \hat{\Gamma}_n, \Gamma^*_n \) be rings and circles cited below:

\begin{align*}
H_n & : -(n-1)(\log 100)-2\gamma < \text{Re} \xi < -(n-1) \log 100, 0 \leq \text{Im} \xi < 2\pi. \\
H'_n & : -(n \log 100) < \text{Re} \xi < -n \log 100 + 2\gamma, 0 \leq \text{Im} \xi < 2\pi. \\
\hat{H}_n & : 2\gamma - n \log 100 < \text{Re} \xi < -n \log 100 + 2\gamma, 0 \leq \text{Im} \xi < 2\pi. \\
\Gamma_n & : \text{Re} \xi = -(n-1)(\log 100)-\gamma, 0 \leq \text{Im} \xi < 2\pi. \\
\Gamma'_n & : \text{Re} \xi = -(n \log 100)+\gamma, 0 \leq \text{Im} \xi < 2\pi. \\
\hat{\Gamma}_n & : \text{Re} \xi = -(n-1+\frac{1}{2}) \log 100, 0 \leq \text{Im} \xi < 2\pi. \\
\Gamma^*_n & : \text{Re} \xi = -c_n : -2\gamma + n \log 100 > c_n > 2\gamma + (n-1) \log 100, 0 \leq \text{Im} \xi < 2\pi.
\end{align*}

where \( \gamma = \frac{\log 100}{8} \).
We make slits in $H_n$ and $H'_n (n \geq 2)$ parallel to the real axis for intense connection for $2^n$ sheets so that

$$|U(z') - U(z'')| < \frac{1}{10^n}$$

for $z'$ and $z'' \in A_n(1) + \cdots, A_n(2^{q_{n-1}}) + \hat{A}_n(1) + \cdots + \hat{A}_n(2^{q_{n-1}})$ and $\text{proj } z' = \text{proj } z'' \in (\Gamma_n + \Gamma'_n)$ for any harmonic function $U(z)$ such that $0 < U(z) < 1$, where $A_n(1), \cdots, A_n(2^{q_{n-1}})$ are examplars which are identical to $A_n(1)$. Hence by the maximum principle (maximum principle is used for the projectional transformations $z' \leftrightarrow z''$)

$$|U(z') - U(z'')| < \frac{1}{10^n} \quad \text{for } \text{proj } z' = \text{proj } z'' \in \hat{H}_n.$$ 

Put $\mathfrak{G}_n = A_1 + A_2 + \cdots + A_{n-1}$ and $C\mathfrak{G}_n = A - (A_1 + A_2 + \cdots + A_{n-1})$. We denote by $mC\mathfrak{G}_n$ the $m$ number of surfaces $C\mathfrak{G}_n$ with slits for intense connection. As in case of the Example 3 we construct a Riemann surface.


\[ \cdots \cdots \cdots \]

$n$-th step.

$I^n(1) = I^{n-1}(1)$

\[ \cdots \cdots \cdots \]

$I^n(2^{q_{n-1}-1}) = I^{n-1}(2^{q_{n-1}-1})$

$I^n(2^{q_{n-1}} + 1) = \hat{I}^{n-1}(1)$
$I^{n}(2^{q_{n-1}}) = \hat{I}^{n-1}(2^{q_{n-1}})$ 

\[ \begin{array}{c}
I^{n}(2^{q-1}) \\
I^{n}(1) \\
\vdots \\
\vdots \\
\vdots \\
\hat{I}^{n}(2^{q_{n-1}}) \\
\end{array} \]

newly added at the $n$-th step. These are identical to $C\mathfrak{G}_{n}$.

We perform the intense connection by slits in $A_{n}$, i.e. by slits contained in $H_{n} + H'_{n}$ of $A_{n}$. Let $\mathfrak{K}_{n}$ be the above Riemann surface obtained after the $n$-th intense connection and put

$$\mathfrak{K} = \bigcup \mathfrak{K}_{n}. $$

Then $\mathfrak{K}_{n}$ has relative boundary $B_{0} + B_{1} + \cdots + B_{n-1}$ and is open. $B_{n-1}$ is composed of $2^{q_{n}} - 2^{q_{n-1}}$ circles lying over $\Re\xi = -(n-1)\log 100$, $0 \leq \Re\xi < 2\pi$. Now the part of $\mathfrak{K}$ lying over $\Re\xi < -n\log 100$ is a generalized ring with slits of which the intense connection is not performed. We denote by $\tilde{A}_{n}$ the part of $\mathfrak{K}$ over $A_{n}$. Then $\tilde{A}_{n}$ is a Riemann surface of $2^{q_{n}}$ sheets and $0 \leq U(z) < 1$ has projectional deviation $< \frac{1}{10^{n}}$ on the part of $\mathfrak{K}$ lying over $\hat{H}_{n}$, i.e. $|U(z')| < -U(z'') < \frac{1}{10^{n}}$ for proj $z' = \text{proj } z'' \in \hat{H}_{n}$ and $z'$ and $z'' \in \mathfrak{K}$.

$\mathfrak{K}$ has the following properties:

a). $\mathfrak{K}$ has only one ideal boundary point $p$ of N-Martin's topology on $z=0$.

b). $\mathfrak{K}$ is a covering surface over $|z| < 1$ and has a finite area.

c). $p$ is a singular point of first kind.

Proof of a). Let $U(z)$ be a harmonic function in $\sum_{n=1}^{\infty} \hat{H}_{n} \subset A: \hat{H}_{n} = E[z: -\exp(n-\frac{1}{4})\log 100 < |z| < -\exp(3n-\frac{3}{4})\log 100]$ (see Fig. 15) such that $D_{\Sigma \hat{H}_{n}}(U(z)) < \infty$. Then there exists a sequence of circles $\Gamma_{n_{i}}^{*}: |z| = r_{n_{i}}$ in $\sum \hat{H}_{n}$ such that

$$ \int_{\Gamma_{n_{i}}^{*}} \left| \frac{\partial}{\partial n} U(z) \right| ds \to 0 \quad \text{as } n_{i} \to \infty. $$
In fact, put \( r e^{i	heta} = -\log |z| \) and put \( L(r) = \int \frac{\partial}{\partial r} U(r e^{i\theta}) \, r \, d\theta \). Assume \( \lim L(r) > \delta_0 > 0 \) for \( r \in \Sigma E_n \), where \( E_n \) is the circular projection of \( (-\log |z| : z \in H_n) \) on the real axis, i.e. \( E_n = E \left[ r : \left( \frac{1}{4} \right) \log 100 > r > \left( n - \frac{3}{4} \right) \log 100 \right] \). Then

\[
\delta_0 \int_{r_0}^{r} \frac{1}{r} \, dr \leq \int_{r_0}^{r} \frac{L(r)}{r} \, dr \leq \int_{r_0}^{r} \left\{ \left( \frac{\partial}{\partial r} U(z) \right)^2 + \left( \frac{\partial}{\partial \theta} U(z) \right)^2 r^2 \right\} \, dr \, d\theta = D(U(z)).
\]

Let \( r \to \infty \). Then \( D(U(z)) = \infty \). This is a contradiction. Hence there exists a sequence \( \{ \Gamma_{n_i} \} \) such that

\[
\int_{\Gamma_{n_i}} \left| \frac{\partial}{\partial n} U(z) \right| ds \to 0 \quad \text{as} \quad n_i \to \infty.
\]

This sequence \( \{ \Gamma_{n_i} \} \) depends on the function \( U(z) \).

Put \( \varepsilon_{n_i} = \int_{\Gamma_{n_i}} \left| \frac{\partial}{\partial n} U(z) \right| ds \).

Let \( \tilde{\Gamma}_{n_i}^* \) be the set of \( \Re \) lying over \( \Gamma_{n_i}^* \) (see Fig. 15). Then \( \tilde{\Gamma}_{n_i}^* \) is composed of \( 2^{q_{n_i}} \) circles. \( \tilde{\Gamma}_{n_i}^* \) divides \( \Re \) into two components. Let \( \Re_{\tau_{n_i}} \) be one component of them containing \( B_0 \). Then \( \{ \Re_{\tau_{n_i}} \} \) is an exhaustion with compact relative boundary \( B_0 + B_1 + \cdots + B_{n_i-1} \) and \( \tilde{\Gamma}_{n_i}^* \), where \( B_i \) lies over \( \Re \xi = -i \log 100 \). Hence

\[ \Re = \bigcup \Re_{\tau_{n_i}}. \]

Now \( \hat{H}_{n_i} (\subset A_{n_i}) \) and \( A_{n_i} \) is intensely connected with \( 2^{q_{n_i}} - 1 \) examplers of \( \Re \) on the slits contained in \( H_{n_i} \) and \( H'_{n_i} \) of \( A_{n_i} \), hence \( |U(z') - U(z'')| < \frac{1}{10^{n_i}} \) for \( \text{proj} z' = \text{proj} z'' \in \hat{H}_{n_i} \) and \( z' \) and \( z'' \) are contained in \( \Re \). Hence by (38)

\[
| \max_{z \in \hat{H}_{n_i}} U(z) - \min_{z \in \hat{H}_{n_i}} U(z) | < \varepsilon_{n_i} + \frac{2}{10^{n_i}}.
\]

Let \( \tilde{\Gamma}_n \) be the set of \( \Re \) over \( \Gamma_n \). Then \( \tilde{\Gamma}_n \) is composed of \( 2^q \) circles over \( \Gamma_n \). \( \tilde{\Gamma}_n \) divides \( \Re \) into two parts. Let \( \Re_{\tau_n} \) be one of them containing \( B_0 \). Then \( \{ \Re_{\tau_n} \} \) is also an exhaustion of \( \Re \), i.e. \( \Re = \bigcup \Re_{\tau_n} \). Clearly \( \{ \Re_{\tau_n} \} \) and \( \{ \Re_{\tau_n} \} \) are equivalent and \( \{ \Re - \Re_{\tau_n} \} \) determines an ideal boundary component \( \partial \) on \( z = 0 \).

Let \( N(z, p) \) be the \( N \)-Green's function of \( \Re \) and \( \delta(p, q) \) be defined by

\[
\delta(p, q) = \sup_{z \in A_1} \left| \frac{N(z, p)}{1 + N(z, p)} - \frac{N(z, q)}{1 + N(z, q)} \right|.
\]
Consider $N(z, p): p \in A_1$ in $A - A_2$. Then $N(z, p)$ is harmonic and $D(N(z, p)) < \infty$. Hence there exists a sequence $\{\Gamma^*_n\}$ such that
\[ \int_{\Gamma^*_n} \frac{\partial}{\partial n} N(z, p) \, ds \to 0 \text{ as } n_i \to \infty \] and $\Gamma^*_n \in \mathcal{H}_{n_i}$ of $A_{n_i}$.

Hence by (39)
\[ |\max_{z \in \Gamma_n^*} N(z, p) - \min_{z \in \Gamma_n^*} N(z, p)| \to 0 \text{ as } n_i \to \infty. \]

Hence by Lemma 1 there exists only one point of Martin's topology on the boundary component $p$.

**Proof of b).** Since the area of $A_n = \pi \left( \frac{1}{100^{2(n-1)}} - \frac{1}{100^{2n}} \right) = \frac{9900\pi}{100^{2n}}$, area of $\mathcal{R} = 9900\pi \sum_{n=1}^{\infty} \left( \frac{2^{2n}}{100^{2n}} \right) = 9900\pi \sum_{n=1}^{\infty} \left( \frac{2^{4}}{100^{2}} \right)^n < \infty$, for $q_n = 4n$.

**Proof of c)** Let $U(z)$ be a harmonic function in $\mathcal{R}_{\tau}(m > n)$ such that $D(U(z)) \leq \frac{2\pi}{\log 100}$. Let $A_{n-1}$ be the ring: $-\exp(n-1\frac{1}{2}) \log 100 < |z| < -\exp(n-1-\frac{1}{2}) \log 100$. Assume $|U(r_{n-1}e^{i\theta}) - U(r_{n}e^{i\theta})| > \frac{3}{10^{n-1}}$ on the set of $\theta$ of angular measure $\frac{2\pi}{2^n}$ on $A_{n-1}$ as a function of $z$ on $A_{n-1}$, where $r_{n-1}e^{i\theta} \in \hat{\Gamma}_{n-1}$ and $r_{n}e^{i\theta} \in \hat{\Gamma}_{n}$ (Fig. 16).

Now there exist $2^{2n}$ number of examplars identical to $A_n$ which are connected intensely with projectional deviation $< \frac{1}{10^n}$ on $\hat{\Gamma}_n$, whence
\[ |U(z') - U(z'')| < \frac{1}{10^n} \text{ for proj } z' = \text{proj } z'' \in \hat{\Gamma}_n \text{ and } z' \text{ and } z'' \in \mathcal{R}. \]

Hence
\[ |U(r_{n}e^{i\theta}) - U(r_{n-1}e^{i\theta})| > \frac{1}{10^{n-1}} \text{ for } \theta \text{ of angular measure } \frac{2\pi}{2^n} \] on $2^{2n-1}$ examplars lying over $A_{n-1}$ where $r_{n-1}e^{i\theta}$ lie and $r_{n}e^{i\theta}$ on the same examplar.

Let $\tilde{A}_{n-1}$ be the part of $\mathcal{R}$ lying over $A_{n-1}$. Then by Lemma 2 and by (40)
This is a contradiction for $n \geq 3$. Hence

\[
|U(r_{n-1}e^{i\theta}) - U(r_{n}e^{i\theta})| < \frac{3}{n-1} ; \quad n \geq 3,
\]

except at most an exceptional set $\Theta_{n,n-1}(E)$ of angular measure $\frac{2\pi}{2^{n}}$ on the ring $A_{n}^{-1}$.

Let $\tilde{\Gamma}_{n}$ be the set of $\Re$ lying over $\tilde{\Gamma}_{n}$. Then $\tilde{\Gamma}_{n}$ divides $\Re$ into two parts. Let $\tilde{\Re}_{1}, \tilde{\Re}_{m}$ be the one of them containing $B_{0}$. Then $\{\Re_{\tilde{\Gamma}_{n}}\}$ is an exhaustion and $\{\Re_{\tilde{\Re}_{m}}\}$ is equivalent to $\{\Re_{\tilde{\Gamma}_{n}}\}$. $\{\Re - \Re_{\tilde{\Gamma}_{n}}\}$ determines the ideal boundary component $p$ and by $a)$, by Lemma 1 for any given $\nu_{l}(p)$ there exists a number $m_{l}$ such that

\[
(\Re - \Re_{\tilde{\Re}_{m}}) \subset \nu_{l}(p) = E\left[ z \in \Re : \delta(z, p) < \frac{1}{l} \right] \text{ for } m \geq m_{l}.
\]

Hence

\[
\text{C.P. of } p \quad \omega(p, z) = \lim_{l} \omega_{l}(p, z) \geq \lim_{m} \omega(\Re - \Re_{\tilde{\Re}_{m}}, z),
\]

where $\omega(\Re - \Re_{\tilde{\Re}_{m}}, z)$ is C.P. of $\Re - \Re_{\tilde{\Re}_{m}}$.

For simplicity put $\omega_{m}(z) = \omega(\Re - \Re_{\tilde{\Re}_{m}}, z)$. Assume $\lim_{m} \omega_{m}(z) = 0$. Then for any given positive number $\varepsilon < \frac{1}{10}$, there exists a number $m_{0}$ such that $\omega_{m}(z) < \varepsilon$ on $\tilde{\Gamma}_{n} = E\left[ |z| = -\exp \left( 4 + \frac{1}{2} \right) \log 100 \right]$ for $m \geq m_{0}$. Now $\omega_{m}(z) = 1$ on $\Gamma_{m}(\subset \Re - A_{1})$ and has M.D.I. over $\Re$. Hence by the Dirichlet principle

\[
D(\omega_{m}(z)) < D(\tilde{\omega}(z)) = \frac{2\pi}{\log 100},
\]

where $\tilde{\omega}(z)$ is a harmonic function in $A_{1}$ such that $\tilde{\omega}(z) = 0$ on $|z| = 1$ and $\tilde{\omega}(z) = 1$ on $|z| = \frac{1}{100}$. Hence by (41)

\[
|U(r_{n}e^{i\theta}) - U(r_{n-1}e^{i\theta})| < \frac{3}{n-1} \text{ except } \Theta_{n,n-1}(E),
\]

except $\Theta_{n,n-1}(E)$ depends on $\omega_{m}(z)$.

Let $\Phi$ be the complementary set of $\sum_{n=5}^{m} \Theta_{n,n-1}(E)$. Then angular measure of $\Phi > 2\pi \left( 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n}} \right) = 15\pi \frac{8}{8}$. Suppose $\theta \in \Phi$. Then by (42)
$|\omega(r_{m}e^{i\theta})-\omega(r_{4}e^{i\theta})|<\sum_{n=f}^{m}\frac{3}{10^{\frac{n-1}{2}}}<\frac{3}{10}$.

On the other hand, $\omega_{m}(z)=1$ on $\hat{\Gamma}_{m}$, i.e. $\omega(r_{m}e^{i\theta})=1$, whence

$$\frac{1}{10}>\varepsilon>\max_{z\in\hat{\Gamma}_{5}}\omega_{m}(z)>1-\frac{3}{10}$$

on $\hat{\Gamma}_{5}$. This is a contradiction. Hence $0<\lim_{m}\omega_{m}(z)<\omega(p, z)$.

Next by Theorem 12. a) (of the previous paper “On Singular points” \(\Re\) has no singular point of second kind. Hence \(p\) is a singular point of first kind. Thus we have c).

Department of Mathematics
Hokkaido University

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