SINGULAR POINTS OF RIEMANN SURFACES

By

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Let D_r be a ring domain such that $r<|z|<1$. Let $U_r(z)$ be a harmonic function in D_r such that $U_r(z)=0$ on $|z|=1$ and $U_r(z)=1$ on $|z|=r$. Put $M_r=\max\left(\left|\frac{\partial U_r(z)}{\partial x}\right|, \left|\frac{\partial U_r(z)}{\partial y}\right|\right)$: $z=x+iy$. I wrote that $M_r\to 0$ as $r\to 0$. This is false. Hence our proof of Theorem 8 in the previous paper[1] depending on the above fact is incomplete. The purpose of the present paper is to correct the above theorem, to simplify the other theorems and further to discuss new results. For the sake of convenience, I shall begin from the first.

Let R be a Riemann surface with positive boundary. Let $\{R_n\}$ be its exhaustion with compact relative boundaries $\partial R_n (n=0, 1, 2, \cdots)$. We proved the following.

Theorem.[2] *Let* $R\in O_{HB}(O_{HD})-O_G$. *Then* $R-R_0\in O_{AB}(O_{AD})$.

Above theorem means that the boundary of a Riemann surface R which is so complicated as $R\in O_{HB}-O_G(O_{HD}-O_G)$ cannot be represented in any way as a covering surface over a bounded domain (a covering surface with finite area). Hence we propose the following questions:

1) *What part of the boundary does generate the above singular fact?*
2) *What is the method to characterize the singularity of the boundary?*

To solve the questions, we must define the boundary of R. There are various methods to define the ideal boundary of R. But we understand that the following Martin's topologies are most available.

PART I

K-Martin's topology.[3] Let $G(z, p_i)$ be the Green's function of with pole at p_i. Put $K(z, p_i)=\frac{G(z, p_i)}{G(p_0, p_i)}$, where p_0 is a fixed point. Suppose

1) Z. Kuramochi: On the ideal boundaries of Riemann surfaces. Osaka Math. Jour, 1958.
2) Z. Kuramochi: On the behaviour of analytic functions. Osaka Math. Jour, 1955.
3) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. 39, 1941.

$\{p_i\}$ is a divergent sequence of points. We call $\{p_i\}$ a fundamental sequence determining an ideal boundary point, if $\{K(z, p_i)\}$ converges uniformly in every compact domain of R. If $\{K(z, p_i)\}$ and $\{K(z, p_i')\}$ determine the same limit function, we say that $\{p_i\}$ and $\{p_i'\}$ define the same ideal boundary point. We denote by B the set of all the ideal boundary points. We define the distance between two points p and q of $R+B$ (denoted by $\overline{R}$) by

$$\delta(p, q)=\sup_{z\in R_1}\left|\frac{K(z, p)}{1+K(z, p)}-\frac{K(z, q)}{1+K(z, q)}\right|.$$

Let $U(z)$ be a positive superharmonic function in R. If $V(z)=aU(z)$: $0\leqq a\leqq 1$ for every positive superharmonic function $V(z)$ such that $0\leqq V(z)\leqq U(z)$, $U(z)$ is called a K-minimal function. Let $K(z, p)$ be the above function. p is called a K-minimal point or K-non minimal point according as $K(z, p)$ is K-minimal or not. R. S. Martin proved the following:

1)* *Let $K_{v_n(p)}(z, p)$ be the lower envelope of positive superharmonic functons in R larger than $K(z, p)$ in $v_n(p)=E\left[z\in\overline{R}:\ \delta(z, p)<\frac{1}{n}\right]$. Then $\lim_{n=\infty} K_{v_n(p)}(z, p)=K(z, p)$ or 0 according as $K(z, p)$ is K-minimal or not.*

2)* *Every point of R is K-minimal. Let B_1 and B_0 the sets of K-minimal boundary points and non K-minimal points respectively. Then $B=B_1+B_0$ and B_0 is an F_σ set of harmonic measure zero.*

3)* *Let F be a closed set in $\overline{R}$ and let $U(z)$ be a positive superharmonic function and let $U_{F_n}(z)$ be the lower envelope of superharmonic functions larger than $U(z)$ in $F_n=E\left[z\in\overline{R}:\ \delta(z, F)\leqq\frac{1}{n}\right]$. Then $U_F(z)=\lim_{n=\infty}U_{F_n}(z)$ is represented by a positive mass on F.*

4)* *Every positive superharmonic function is represented by a positive mass distribution on $R+B_1$ which is uniquely determined and called a canonical mass distribution.*

5)* *If $U(z)=\int_A K(z,p)d\mu(p)$ is K-minimal, $U(z)$ is a multiple of $K(z,p)$: $p\in A$.*

1. Harmonic measure of sets in $\overline{\mathrm{R}}$. Let F be a closed set in $\overline{R}$ and put $F_n=E\left[z\in R:\ \delta(z, F)\leqq\frac{1}{n}\right]$. Let $w(F_n, z, R_m)$ be a harmonic function in R_m-F_n such that $w(F_n, z, R_m)=0$ on ∂R_m-F_n and $w(F_n, z, R_m)=1$ on $(R_m\frown F_n)$. Put $w(F, z, R)=\lim_n\lim_m w(F_n, z, R_m)$ and call it H.M. (harmonic measure) of F. Let G be an open set in B. We define $w(\mathrm{G}, z, R)$ as

$\lim_n \lim_n w_{m,n}(z)$, where $w_{m,n}(z)$ is a hamonic function in $R_m-(CG)_n$ such that $w_{m,n}(z)=1$ on $\partial R_m-(CG)_n$ and $w_{m,n}(z)=0$ on $\partial((CG)_n)$, where $CG=B-G$ and $(CG)_n=E\left[z\in\bar{R}:\ \delta(z, CG)\leqq\frac{1}{n}\right]$. Clearly we have $w(G, z, R)=1-w(CG, z, R)$. For an F_σ set: $F_\sigma=\bigcup^{\infty}F_i$ we define $w(F_\sigma, z, R)$ by $\lim_{n=\infty} w(\sum^{n}F_i, z, R)$ and for G_δ in B, i.e. $G_\delta=\bigcap^{\infty}G_i$ we define $w(G_\delta, z, R)$ by $\lim_{n=\infty} w(\bigcap_i^n G_i, z, R)$.

Let R^∞ be the universal covering surface of R and map R^∞ conformally onto $|\xi|<1$. If $\sup_{z\in R} K(z, p)<\infty$: $p\in R+B_1$, we call p *a singular point*. Clearly $K(z, p)=aG(z, p)$: $a>0$ for $p\in R$, where $G(z, p)$ is the Green's function of R. Hence every point of R is not singular. We denote by B_s the set of singular points. Then $B_s\subset B_1$.

Put $F=p$ and let $w(p, z, R)$ be the H.M. of p, i.e. $w(p, z, R)=\lim_m \lim_n w(\upsilon_n(p), z, R_m)$. Then

Theorem 1. *a) $w(p, z, R)>0$ if and only if $p\in B_s$ and in this case $w(p, z, R)=aK(z, p)$: $a>0$.*

b) Consider $K(z, p)$: $p\in B_s$ in $|\xi|<1$. Then there exists a set E of positive measure such that $K(z, p)$ has angular limits $=M=\sup_{z\in R} K(z, p)$ on E and $=0$ on CE almost everywhere. Hence $w(p, z, R)=w(E, \xi)$, where $w(E, \xi)$ is H.M. of E with respect to $|\xi|<1$. We call E the image of $p\in B_s$.

c) Let $K(z, p_i)$: $p_i\in B_s$ and let E_i be the image of p_i. Then mes $E_i>0$, mes $(E_i\frown E_j)=0$ for $i\neq j$ and $\sum$ mes $E_i\leqq 2\pi$, whence B_s is at most enumerable. Let $w(\sum^{k} p_i, z, R)=\lim_n \lim_m w(\sum^{k} \upsilon_n(p_i), z, R_m)$. Then $\sum^{k} w(p_i, z, R)=w(\sum^{k} p_i, z, R)$ for $p_i\in B_s$.

d) Let E_i be the image of $p_i\in B_s$. Then $\sum^{\infty}$ mes $E_i=2\pi$ if and only if H. M. of $B-\sum^{\infty} p_i=0$.

e) Let E be the image of a point p of B_s. Then every bounded harmonic (Poisson's integrable) function $U(z)$ has angular limits $=$ const a.e. (almost everywhere) on E.

Proof of a). Assume $w(p, z, R)>0$. Then by 2)* $p\notin B_0$. Hence $p\in B_1+R$ and $K(z, p)$ is K-minimal. Suppose $\sup_{z\in R} K(z, p)=\infty$. Then $w(p, z, R)=\lim_n w(\upsilon_n(p), R_m)$: $\upsilon_n(p)=E\left[z\in R:\ \delta(z, p)<\frac{1}{n}\right]$. Then by 3)* $w(p, z, R)$ is represented by a mass on $\bigcap_n \bar{\upsilon}_n(p)$[4] $=p$, i.e. $w(p, z, R)=aK(z, p)$. Now

4) See 3), where $\bar{\upsilon}_{n(p)}$ means the closure of $\upsilon_{n(p)}$.

$w(p, z, R) \leqq 1$ and $\sup_{z \in R} K(z, p) = \infty$, whence $a = 0$ and $w(p, z, R) = 0$. Next suppose $\sup_{z \in R} K(z, p) \leqq M < \infty$. Then $w(\upsilon_n(p), z, R) \geqq \frac{K_{\upsilon_n(p)}(z, p)}{M}$ by $p \in (R + B_1)$ and $w(p, z, R) \geqq \frac{K(z, p)}{M} > 0$. Hence $w(p, z, R) > 0$ if and only if $p \in B_s$.

Proof of b). Suppose $\sup_{z \in R} K(z, p) = M$. Let E and E' be sets on $|\xi| = 1$ such that $K(z, p)$ has angular limits $\geqq M - \varepsilon$ on E and $K(z, p)$ has angular limits between $M - 2\varepsilon$ and ε on E' for a positive number ε: $0 < \varepsilon < \frac{M}{3}$. Since $K(z, p)$ is representable by Poisson's integral, E is of positive measure. Now E' is a set of meausre zero, because, if it were not so, we construct a harmonic function $U(\xi)$ such that $U(\xi)$ has the same angular limits as $K(z, p)$ on E and $= 0$ on CE. Then $U(\xi)$ is a function in R and is not a multple of $K(z, p)$ and $K(z, p) > U(z) = U(\xi) > 0$, which implies that $K(z, p)$ is not K-minimal. This is a contradiction. Hence by letting $\varepsilon \to 0$, $K(z, p)$ has angular limits $= M = \sup_{z \in R} K(z, p)$ on E and 0 on CE almost everywhere. Hence $w(E, \xi) = \frac{K(z, p)}{M} = w(p, z, R)$.

Proof of c). Suppose mes $(E_i \frown E_j) > 0$. Let $U(\xi)$ be a harmonic function in $|\xi| < 1$ such that $U(\xi)$ has angular limits $= \min(M_i, M_j)$ on $E_i \frown E_j$ and $= 0$ on $C(E_i \frown E_j)$. Then for at least one of $K(z, p_i)$ and $K(z, p_j)$, $0 < U(\xi) < K(z, p_k)$: $k = i$ or j and $U(\xi)$ is not a multiple of $K(z, p_k)$. Hence $K(z, p_k)$ is not K-minimal. This is a contradiction. Hence mes $(E_i \frown E_j) = 0$. On the other hand, by a) mes $E_i > 0$ by $w(p_i, z, R) > 0$ and $\sum$ mes $E_i \leqq 2\pi$. Hence B_s is at most enumerable.

Clearly $\sum^k w(p_i, z, R) \geqq w(\sum^k p_i, z, R) \geqq w(p_i, z, R)$ $i = 1, 2, \cdots k$.

By $\sum^k w(p_i, z, R) \geqq w(\sum^k p_i, z, R)$ we see $w(\sum^k p_i, z, R) = 0$ a.e. on $C(\sum^k E_i)$ and by $w(\sum^k p_i, z, R) \geqq w(p_i, z, R)$ we have $w(\sum^k p_i, z, R) = 1$ a. e. on $\sum^k E_i$. Hence $w(\sum^k p_i, z, R) = \sum^k w(E_i, z) = \sum^k w(p_i, z, R)$.

Proof of d). Let $p_i \in B_s$ and let E_i be the image of p_i. Then $B - \sum^\infty p_i$ is a G_δ set in B. Hence by definition $w(B - \sum^\infty p_i, z, R) = 0$ is equivalent to $w(\sum^\infty p_i, z, R) = 1$. Suppose $\sum^\infty$ mes $E_i < 2\pi$. By $w(\sum^\infty p_i, z, R) = \lim_{n=\infty} w(\sum^n p, z, R)$, for any given positive number $\varepsilon < \frac{\delta}{2}$ $(\delta = 2\pi - \sum^\infty$ mes $E_i)$, there exsts a number n_0 such that

$$w(\sum^\infty p_i, z_0, R) \leqq w(\sum^n p_i, z_0, R) + \varepsilon, \text{ for } n \geqq n_0,$$

where $\xi=0$ is an image of z_0.

Now $w(\sum^n p_i, z, R)=\sum^n w(E_i, \xi)$ implies $w(\sum^\infty p_i, z_0, R)-\varepsilon\leqq w(\sum^n p_i, z_0, R)=1-\frac{\delta}{2\pi}$, whence $w(\sum^\infty p_i, z, R)<1$. Conversely suppose $\sum^\infty \text{mes}\, E_i=2\pi$. Then $w(\sum^\infty p_i, z, R)\geqq w(\sum^n p_i, z, R)=w(\sum^n E_i, z)$ for every n. Hence $w(\sum^\infty p_i, z, R)=1$. Thus we have d).

Proof of e). Assume that $U(z)$ has angular limits being not a constant a. e. on E. Then we can find two sets E_1 and E_2 of positive measure in E such that $U(z)\geqq L$ on E_1 and $<L-\delta$ on E_2 for constants L and δ: $\delta>0$. Let $W(\xi)(=W(z))$ be a harmonic function in $|\xi|<1$ such that $W(\xi)=1$ a. e. on $E_1\subset E$ and $=0$ a. e. on CE_1. Then since $U(z)$ is a function in R, $W(\xi)$ is a function in R. Clearly $0<W(z)<\frac{K(z, p)}{M}$ and $W(z)$ is not a multple of $K(z, p)$. This implies that $K(z, p)$ is not K-minimal. Hence every Poisson's integrable function has angular limits=const a. e. on E.

2. Class H. N. B. We denote by H. N. B. the class of Riemann surfaces on which N number of linearly independent bounded harmonic function exist, where $N\leqq\infty$ and the cardinal number of N is $\aleph_0$.

Theorem 2. *A Riemann surface $R\in$ H. N. B.($N\leqq\infty$) if and only if R has N number of singular points and a set of boundary points of harmonic measure zero.*

Proof. Suppose that R has N number of points $p_1, p_2, \cdots p_N$ and a set of boundary points of harmonic measure zero. Let $U(z)$ be a bounded harmonic function (Poisson's integrable) in R. Then $U(z)$ has angular limits=const a. e. on E_i of the image of p_i by e) of Theorem 1 and by d) of Theorem 1. Whence $U(z)$ is a linear form of $\{K(z, p_i)\}$ and $\{K(z, p_i)\}$ is linearly independent. Hence $R\in$ H. N. B. Conversely suppose $R\in$ H. N. B. Then the harmonic measure of $B-\sum p_i$ is zero, because if $\sum \text{mes}\, E_i<2\pi$, we can construct infinitely (cardinal number$=\aleph$) many linearly independent harmonic functions. Let N': $N\neq N'$ be the number of singular points, then by e) $R\in$ H. N'. B.$\neq$H. N. B. Hence $N'=N$. Hence we have Theorem 2.

3. Harmonic functions and analytic functions in a neighbourhood of a singular point.

Let G be a non compact domain[5] in R and let $U(z)$ be a positive

5) In the present paper we suppose that ∂G of a domain G consists of an enumerably infinite number of analytic curves clustering nowhere in R.

harmonic function in G with $U(z)=0$ on ∂G. Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R-R_n)\frown G))$ such that $U_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$ and $U_{n,n+i}(z)=U(z)$ in $G\frown(R_{n+i}-R_n)$. Then $U_{n,n+i}(z)\uparrow U_n(z)$ as $i\to\infty$ and $U_n(z)\uparrow$ as $n\to\infty$. Put ${}_{ex,G}U(z)=\lim_n \lim_i U_{n,n+i}(z)$. Let $V(z)$ be a positive harmonic function in R. Let ${}_{inexG\frown R_n}V(z)$ be the lower envelope of superharmonic functions in G larger than $V(z)$ in $G\frown(R-R_n)$. Then

$V_n(z)\downarrow$ as $n\to\infty$. Put ${}_{inex,G}V(z)=\lim_n {}_{inexG\frown R_n}V(z)$. Then

if ${}_{ex,G}U(z)<\infty$, then $U(z)={}_{inex,G}({}_{ex,G}U(z))$.[6]

Let $w_{n,n+i}(z)$ be a continuous superharmonic function in $R_n+(\upsilon_m(p)\frown R_{n+i})$ such that $w_{n,n+i}(z)$ is harmonic in $R_n+(\upsilon_m(p)\frown R_{n+i})$, $w_{n,n+i}(z)=w(p,z,R)$ on $(R-R_n)\frown C\upsilon_m(p)$ and $w_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown\upsilon_m(p)$, where $\upsilon_m(p)=E\left[z\in\overline{R}:\ \delta(z,p)<\frac{1}{m}\right]$. Then $w_{n,n+i}(z)\uparrow w_n(z)$ as $i\to\infty$. Now $w_n(z)$ is the lower envelope of superharmonic functions in R larger than $w(p,z,R)$ on $(R-R_n)\frown C\upsilon_m(p)$.[7] We denote it by $w_{C\upsilon_m(p)\frown(R-R_n)}(p,z,R)$. Then $w_{C\upsilon_m(p)\frown(R-R_n)}(p,z,R)\downarrow w_{B\frown C\upsilon_m(p)}(p,z,R)$ as $n\to\infty$.

Let G be a non compact domain and let $p\in B_s$. If $w(p,z,R)-w_{CG}(p,z,R)>0$, we say that G *contains* p *K-approximately.*

Theorem 3. *a) Let $p\in B_s$, i. e. $w(p,z,R)$ is K-minimal and >0. Then $w_{B\frown C\upsilon_m(p)}(p,z,R)=0$.*

b) Let $w_{CG}(p,z,R)$ be the lower envelope of positive superharmonic functions in R larger than $w(p,z,R)$ in CG and let $w(p,z,G)$ be H. M. *of p relative G, i. e. $w(p,z,G)=\lim_m \lim_n w(\upsilon_m(p),z,G\frown R_n)$, where $w(\upsilon_m(p),z,G\frown R_n)$ is a harmonic function in $(G\frown R_n)-\upsilon_m(p)$ such that $w(\upsilon_m(p),z,G\frown R_n)=0$ on $(\partial G\frown R_n)-\upsilon_m(p)+\partial R_n-\upsilon_m(p)$ and $w(\upsilon_m(p),z,GR)=1$ on $\upsilon_m(p)\frown G$. Then*

$$w(p,z,G)={}_{inex,G}w(p,z,R)=w(p,z,R)-{}_{CG}w(p,z,R).$$

c) If $w(p,z,G)>0$, $w(p,z,CG)=0$ in other words at most one of G and CG contains p K-approximately.

d) Let $\upsilon_m(p)$ be a neighbourhood of $p\in B_s$. Then $\upsilon_n(p)$ contains p K-approximately.

Proof of a). By 3)* $w_n(z)$ is represented by a positive mass distribution μ_n on $C\upsilon_m(p)\frown(\overline{R-R_n})$, whence $w_{B\frown C\upsilon_m(p)}(p,z,R)$ is represented by an weak limit μ of $\{\mu_n\}$ on $B\frown C\upsilon_m(p)$ such that $w_{B\frown C\upsilon_m(p)}(p,z,R)=\int_{B\frown C\upsilon_m(p)} K(z,p)d\mu(p)$.

6) Z. Kuramochi: Relations between harmonic Dimensions. Proc. Japan Acad. 1954.

7) $C\upsilon_{m(p)}$ means the complementary set of $\upsilon_{m(p)}$.

Now $p\in B_s$ and by the K-minimality of $w(p, z, R)$, $w_{B\frown C\upsilon_m(p)}(p, z, R)=a\cdot w(p,z,R)=a'K(z,p)$: a and $a'\geqq 0$. Assume $a'>0$. Then we can find a point $q\in C\upsilon_m(p)$ such that the restriction μ_l of μ on $\bar{\upsilon}_l(q)=\left(E\left[z\in\bar{R}:\ \delta(z, q)\leqq\frac{1}{l}\right]\right)>0$ for every l. Also by the minimality of $w(p, z, R)$ $\int K(z, p)d\mu_l(p)$ is also minimal, whence $\dfrac{\int K(z, p)d\mu_l(p)}{\int d\mu_l}=K(z, p)$. Now $\left\{\dfrac{\mu_l}{\int d\mu_l}\right\}$ has an weak limit μ^* at q as $l\to\infty$. Hence $K(z, q)=K(z, p)$. This is a contradiction by $\delta(p, q)\geqq\dfrac{1}{m}>0$. Hence $a'=0$ and $w_{B\frown C\upsilon_m(p)}(p, z, R)=0$.

Proof of b). Compare ${}_{inex,G}w(p, z, R)$ and $w(p, z, R)+w_{B\frown C\upsilon_m(p)}(p, z, R)$ in $(R_n\frown G)-\upsilon_m(p)$.

Then ${}_{inex,G\frown R_n}w(p, z, R)=0=w(p, z, G\frown R_n)$ on ∂G, ${}_{inex\frown R_n}w(p, z, R)=w(p, z, R)=w_{C\upsilon_m(p)\frown(R-R_n)}(p, z, R)$ on $(\partial R_n\frown G)-\upsilon_m(p)$ and ${}_{inex,G\frown R_n}w(p, z, R)\leqq 1=w(\upsilon_m(p), z, G\frown R_n)$ on $\partial\upsilon_m(p)\frown G$. Hence by the maximum principle

$$ {}_{inexG\frown R_n}w(p, z, R)\leqq w(\upsilon_m(p), z, G\frown R_n)+w_{C\upsilon_m(p)\frown(R-R_n)}(p, z, R). $$

Let $n\to\infty$. Then

$$ {}_{inex,G}w(p, z, R)\leqq w(\upsilon_m(p), z,G)+w_{C\upsilon_m(p)\frown B}(p, z, R)=w(\upsilon_m(p), z, G) \text{ by } (a). $$

Let $m\to\infty$. Then

$$ {}_{inex,G}w(p, z, R)\leqq w(p, z, G). $$

Conversely, clearly $w(p, z, R)\geqq w(p, z, G)$. Hence by $w(p. z. G)=0$ on ∂G

$$ {}_{inex,G}w(p, z, R)\geqq {}_{inex,G}w(p, z, G)=w(p, z, G). $$

Thus

$$ w(p, z, G)={}_{inex,G}w(p, z, R). $$

Let $w(\upsilon_m(p), z, R_n)$ be H. M. of $\upsilon_m(p)$ relative R_n i. e. $w(\upsilon_m(p), z, R_n)$ is a superharmonic function in R_n such that $w(\upsilon_m(p), z, R_n)$ is harmonic in $R_n-\upsilon_m(p)$ such that $w(\upsilon_m(p), z, R_n)=1$ on $\upsilon_m(p)$ and $w(\upsilon_m(p), z, R_n)=0$ on $\partial R_n-\upsilon_m(p)$.

Let $w(\upsilon_m(p), z, R)$ be H. M. of $\upsilon_m(p)$ i. e. $w(\upsilon_m(p), z, R)$ is the least positive superharmonic function in R such that $w(\upsilon_m(p),z,R)=1$ on $\upsilon_m(p)$.

Let $V_{m+i,n}(\upsilon_m(p), z)$ be a harmonic function in $(R_n\frown G)-\upsilon_{m+i}(p)$ such that $V_{m+i,n}(\upsilon_m(p), z)=w(\upsilon_m(p), z, R_n)$ on $\partial G\frown R_n$ and $=0$ on $\partial\upsilon_{m+i}(p)+(\partial R_n-\upsilon_{m+i}(p))$. Then

$$ V_{m+i,n}(\upsilon_m(p), z)=\frac{1}{2\pi}\int_{(\partial G\frown R_n)-\upsilon_{m+i}(p)} w(\upsilon_m(p), \xi, R_n)\frac{\partial}{\partial n}G_{m+i,n}(\xi, z)\,ds, $$

wher $G_{m+i,n}(\xi, z)$ is the Green's function of $(R_n\frown G)-\upsilon_{m+i}(p)$.

Now $V_{m+i,n}(\upsilon_m(p), z) \uparrow V_{m+i}(\upsilon_m(p), z)$ as $n \to \infty$ by $w(\upsilon_m(p), z, R_n) \uparrow w(\upsilon_m(p), z, R)$ and $\frac{\partial}{\partial n} G_{m+i,n}(\xi, z) \uparrow \frac{\partial}{\partial n} G_{m+i}(\xi, z)$ on ∂G as $n \to \infty$.
Hence by Lebesgue's theorem

$$V_{m+i}(\upsilon_m(p), z) = \frac{1}{2\pi} \int_{\partial G - \upsilon_{m+i}(p)} w(\upsilon_m(p), \xi, R) \frac{\partial}{\partial n} G_{m+i}(\xi, z) ds,$$

where $G_{m+i}(\xi, z)$ is the Green's function of $G - \upsilon_{m+i}(p)$.

Similarly by $V_{m+i}(\upsilon_m(p), z) \uparrow V(\upsilon_m(p), z)$ as $i \to \infty$,

$$V(\upsilon_m(p), z) = \frac{1}{2\pi} \int_{\partial G} w(\upsilon_m(p), \xi, R) \frac{\partial}{\partial n} G(\xi, z) ds, \tag{1}$$

where $G(\xi, z)$ is the Green's function of G.
Hence by (1) $V(\upsilon_m(p), z)$ is the least positive harmonic function in G with value $w(\upsilon_m(p), R)$ on ∂G i. e. $V(\upsilon_m(p), z) = w_{CG}(\upsilon_m(p), z, R)$ in G.

Let $m \to \infty$. Then as above

$$\lim_m V(\upsilon_m(p), z) = \frac{1}{2\pi} \int_{\partial G} \lim_{m=\infty} w(\upsilon_m(p), z, R) \frac{\partial}{\partial n} G(\xi, z) ds = w_{CG}(p, z, R).$$

Let $V_{m,n}(p, z)$ be a harmonic function in $(G \frown R_n) - \upsilon_m(p)$ such that $V_{m,n}(p, z) = w(p, z, R)$ on $(\partial G \frown R_n) - \upsilon_m(p)$ and $=0$ on $\partial \upsilon_m(p) + \partial R_n - \upsilon_m(p)$. Then

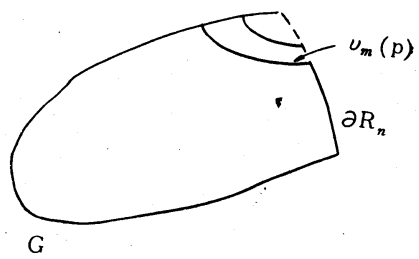

Fig. 1

$$V_{m,n}(p, z) = \frac{1}{2\pi} \int_{(\partial G \frown R_n) - \upsilon_m(\upsilon)} w(p, \xi, R) \frac{1}{2\pi} G_{m,n}(\xi, z) ds,$$

$$\lim_n V_{m,n}(p, z) = V_m(p, z) = \frac{1}{2\pi} \int_{\partial G - \upsilon_m(p)} w(p, \xi, R) \frac{\partial}{\partial n} G_m(\xi, z) ds \text{ and}$$

$$\lim_m V_m(p, z) = \frac{1}{2\pi} \int_{\partial G} w(p, \xi, R) \frac{\partial}{\partial n} G(\xi, z) ds = w_{CG}(p, z, R). \tag{2}$$

Clearly $\quad V_{m,n}(p, z) \leqq V_{m,n}(\upsilon_m(p), z) \leqq V_{m+i,n}(\upsilon_m(p), z)$.
Let $n \to \infty$. Then by $V_{m+i}(\upsilon_m(p), z) \uparrow V(\upsilon_m(p), z) = w_{CG}(\upsilon_m(p), z, R)$

$$V_m(p, z) \leqq V_m(\upsilon_m(p), z) \leqq V_{m+i}(\upsilon_m(p), z) \leqq w_{CG}(\upsilon_m(p), z, R). \tag{3}$$

Let $m \to \infty$. Then by (2)

$$w_{CG}(p, z, R) \leqq \lim_m V_m(\upsilon_m(p), z) \leqq w_{CG}(p, z, R). \tag{4}$$

Hence $\lim_m V_m(\upsilon_m(p), z) = w_{CG}(p, z, R)$.

Now $V_{m,n}(\upsilon_m(p), z) = 0$ on $(G \frown \partial R_n) - \upsilon_m(p) + \partial \upsilon_m(p)$ and $= w(\upsilon_m(p), z, R)$ on $(\partial G \frown R_n) - \upsilon_m(p)$. Hence $w(\upsilon_m(p), z, R_n) - V_{m,n}(\upsilon_m(p), z) = 0 = w(\upsilon_m(p), z, R_n)$ on $(\partial G \frown R_n) + (\partial R_n \frown G) - \upsilon_m(p)$ and $w(\upsilon_m(p), z, R_n) - V_{m,n}(\upsilon_m(p), z) = 1 = w(\upsilon_m$

$(p), z, G\frown R_n)$ on $\partial\upsilon_m(p)\frown G\frown R_n$.

Hence by the maximum principle

$$w(\upsilon_m(p), z, R_n)-V_{m,n}(\upsilon_m(p), z)=w(\upsilon_m(p), z, G\frown R_n).$$

Let $n\to\infty$ and then $m\to\infty$. Then by (3) and (4)

$$w(p, z, R)-w_{CG}(p, z, R)=w(p, z, G).$$

Proof of c). Assume ${}_{inex,G}w(p, z, R)>0$. Then ${}_{exG}({}_{inex,G}w(p, z, R))=w(p, z, R)$. Let $\hat{U}_n(z)$ and $\check{U}_n(z)$ be harmonic functions in R_n such that $\hat{U}_n(z)=w(p,z,R)$, $\check{U}_n(z)=0$ on $\partial R_n\frown G$ and $\hat{U}_n(z)=0$, $\check{U}_n(z)=w(p, z, R)$ on ∂R_n-G. Then $\hat{U}_n(z)+\check{U}_n(z)=w(p,z,R)$. Clearly $\hat{U}_n(z)\leqq w(p, z, R)$ and $\hat{U}_n(z)=w(p, z, R)\geqq{}_{inex,G}w(p, z, R)$ on $\partial R_n\frown G$. Hence $w(p, z, R)\geqq\overline{\lim_n}\,\hat{U}_n(z)\geqq\underline{\lim_n}\,\hat{U}_n(z)\geqq{}_{ex,G}({}_{inex,G}w(p, z, R))=w(p, z, R)$ and

$$\lim_n \hat{U}_n(z)=w(p, z, R) \text{ and } \lim \check{U}_n(z)=0.$$

Similarly as above $\lim_n \check{U}_n(z)={}_{inex,CG}w(p, z, R)$. Hence ${}_{inex,CG}w(p, z, R)=0$.

Proof of d). Assume $w_{C\upsilon_n(p)}(p, z, R)=w(p, z, R)$. Then by (3)* $w(p,z,R)$ is represented by a mass distribution μ on $C\upsilon_n(p)$ and by (4)* $w(p, z, R)=aK(z, q)$: $q\in C\upsilon_n(p)$ and $a>0$. Now $w(p, z, R)=a'K(z, p)$ and by $K(p_0, p)=K(p_0, q)=1$, $a=a'$ and $K(z, p)=K(z, q)$. This contradicts $p\neq q$. Hence $w(p, z, R)-w_{C\upsilon_m(p)}(p, z, R)>0$ i.e. $\upsilon_n(p)$ contains p K-approximately.

Theorem 4. *a) Let G be a domain containing a point $p\in B_s$ K-approximately. Then $w(p, z, G)>0$. Map the universal covering surface G^∞ onto $|\xi|<1$. Then $w(p, z, G)$ has angular limits$=1$ on a set E of positive measure and has angalar limts$=0$ on CE almost everywhere. Let $U(z)$ be a Poisson's integrable harmonic function in G. Then $U(z)$ has angular limits$=$const a. e. on E.*

b) Let G be a domain in a). Then there exists no non constant analytic function of bounded type in G.[8)]

c) Let $\upsilon_n(p)$ be a neighbourhood of $p\in B_s$: $\upsilon_n(p)=E\left[z\in\overline{R}: \delta(z, p)<\frac{1}{n}\right]$. Then $\upsilon_n(p)$ contains p K-approximately and by b) there exists no non constant analytic function of bounded type.

Proof of a). By ${}_{ex,G}w(p, z, G)\leqq w(p, z, R)$, ${}_{ex,G}w(p, z, G)=a\ w(p, z, R)$:

8) Z. Kuramochi: Dirichlet problems on Riemann surfaces. 1. Proc. Japan Acad. 1954. The characteristic $T(z)$ of an analytic function $A(z)$ on a Riemann surface R can be defined. If $T(z)<\infty$, $A(z)$ has angular limits a.e. on $|\xi|=1$, where $|\xi|<1$ is the image of the universal corering surface R^∞ of R.

$a>0$ by the K-minimality of $w(p, z, R)$. We show that $w(p, z, G)$ (>0 by the assumption and by Theorem 3. b)) is K-minimal in G. Suppose there exists a positive harmonic function $U(z)$ such that $U(z) \leqq w(p, z, G)$. Then ${}_{ex,G}U(z) \leqq {}_{ex,G}w(p, z, G) = a\ w(p, z, R)$, whence by the minimality of $w(p, z, R)$, ${}_{ex,G}U(z) = b\ w(p, z, R)$: $b>0$. Hence

$$U(z) = {}_{inex,G}({}_{ex,G}U(z)) = {}_{inex,G}({}_{ex,G}b\ w(p, z, R)) = \frac{b}{a}\, w(p, z, G).$$

Hence $w(p, z, G)$ *is K-minimal in* G. Similarly as b) of Theorem 3, $w(p, z, G)$ has angular limits$=1$ on a set E of positive measure on $|\xi|=1$ and has angular limits$=0$ a. e. on CE. Next as e) of Theorem 1, it is proved that every Poisson's integrable harmonic function $U(z)$ has angular limits $=$const a. e. on E.

Proof of b). Let $A(z)$ be an analytic function of bounded type. Then $ReA(z)$ and $ImA(z)$ have angular limits$=$const a. e. on E. By mes $E>0$ and by Riesz's theorem $A(z)$ must be a constant. Hence we have b).

Proof of c). By Theorem 3. d) $\upsilon_n(p)$ contains p K-approximately. Hence we have c) by b).

PART II

The present part is an application of the previous paper "Potentials on Riemann surfaces".

N-Martin topology. Let $N(z, p)$ be a harmonic function in $R-R_0$ with one logarithmic singularity at $p \in R-R_0$ such that $N(z, p)=0$ on ∂R_0 and $N(z, p)$ has the minimal $\overset{*}{\text{D}}$irichlet integral over $R-R_0$. We use $N(z, p)$ instead of $K(z, p)$ of K-Martin's topology. Then we have N-Martin's topology. The distance between two points p and q of $\overline{R}-R_0$ is given by

$$\delta(p, q) = \sup_{z \in R_1 - R_0} \left| \frac{N(z, p)}{1+N(z, p)} - \frac{N(z, q)}{1+N(z, q)} \right|.$$

We suppose that N-Martin's topology is defined on $\overline{R}-R_0$. We use the same notation as in the previous paper and refer the theorem in "Potentials on Riemann surfaces" with P.

4. Theorem 5. (Separation theorem S. 1). *Let G, G_1 and G_2 be non compact domains such that $G \supset G_1$, $G \supset G_2$, $G_1 \frown G_2 = 0$ and let B' be a closed subset of B. If C. P. $\omega(G_1 \frown B', z, G)>0$ and $\omega(G_2 \frown B', z, G-G_1)$*[9]*$>0$, then* $\omega(G_1 \frown B', z, \tilde{G}) \neq \omega(G_2 \frown B, z, \tilde{G})$ *for any domain* $\tilde{G} \supset G$,

9) See the definition of $\omega(G_2 \frown B^1, z, G)$ of "Potentials".

where $\omega(B'\frown G_1, z, G)=\lim_n \omega(B'_n\frown G_1, z, G)$ *and* $B'_n=E\left[z\in\overline{R}:\ \delta(z, B')\leqq\frac{1}{n}\right]$.

Proof. For simplicity put $\omega^1(z)=\omega(G_1\frown B', z, G)$ and $\omega^2(z)=\omega(G_2\frown B', z, G-G_1)$. Then $\omega^i(z)$ $(i=1, 2)$ has properties from P.C. 1 to P.C. 7.[10]

Put $\Omega=E[z\in G:\ \delta<\omega^2(z)<1-\varepsilon]$, and let C_δ and $C_{1-\varepsilon}$ be regular niveau curves of $\omega^2(z)$. Since $\Omega'\frown G_1=0:\ \Omega'=E[z\in G:\ \omega^2(z)>\delta]$, $\omega^1(z)$ has M.D.I. over $\Omega'\supset\Omega$ by P.C. 1. Then by Lemma 1. *b*) of *P* (we abbreviate Potentials on Riemann surface by *P*) $\omega^1_n(z)\Rightarrow\omega^1(z)$ ($\Rightarrow$means convergence in mean and convergence), where $\omega^1_n(z)$ is a harmonic function in $\Omega'\frown R_n$ such that $\omega^1_n(z)=\omega^1(z)$ on $\partial\Omega'\frown R_n$ and $\frac{\partial}{\partial n}\omega^1_n(z)=0$ on $\partial R_n\frown\Omega'$.

Let $\omega^2_n(z)$ be a harmonic function in $R_n\frown\Omega$ such that $\omega^2_n(z)=\omega^2(z)$ on $\partial\Omega\frown R_n$ and $\frac{\partial}{\partial n}\omega^2_n(z)=0$ on $\partial R_n\frown\Omega$. Now $\omega^2(z)=\omega(\Omega'_{1-\varepsilon}, z, G-G_1)(1-\varepsilon)$ in $C\Omega'_{1-\varepsilon}$: $\Omega'_{1-\varepsilon}=E[z\in G:\ \omega^2(z)>1-\varepsilon]$ by P.C. 4., whence by $\Omega\frown\Omega'_{1-\varepsilon}=0$ $\omega^2(z)$ has M.D.I. over Ω. Hence by Lemma 1. *b*) of *P* $\omega^2_n(z)\Rightarrow\omega^2(z)$ as $n\to\infty$.

Now
$$\int_{C\delta\frown R_n}\frac{\partial}{\partial n}\omega^1_n(z)ds=\int_{\partial R_n\frown\Omega'}\frac{\partial}{\partial n}\omega^1_n(z)ds=0=\int_{(\Omega'-\Omega)\frown\partial R_n}\frac{\partial}{\partial n}\omega^1_n(z)ds$$
$$=\int_{C_{1-\varepsilon}\frown R_n}\frac{\partial}{\partial n}\omega^1_n(z)ds,$$

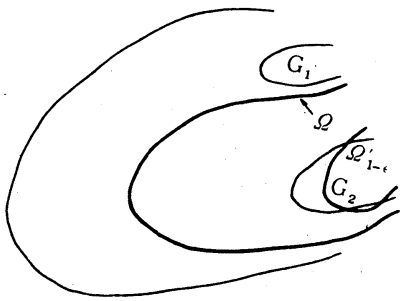

Fig. 2

whence
$$\int_{C\delta\frown R_n}\omega^2_n(z)\frac{\partial}{\partial n}\omega^1_n(z)ds$$
$$=\int_{C_{1-\varepsilon}\frown R_n}\omega^2_n(z)\frac{\partial}{\partial n}\omega^1_n z)ds=0.$$

Hence by the Green's formula

$$\int_{C\delta\frown R_n}\omega^1_n(z)\frac{\partial}{\partial n}\omega^2_n(z)ds=\int_{C_{1-\varepsilon}\frown R_n}\omega^1_n(z)\frac{\partial}{\partial n}\omega^2_n(z)ds. \qquad (5)$$

Since $C_{1-\varepsilon}$ and C_δ are regular, $0<\omega^1_n(z)\,(<1)\to\omega^1(z)$ and $\frac{\partial}{\partial n}\omega^2_n(z)\to\frac{\partial}{\partial n}\omega^2(z)$ on $C_\delta+C_{1-\varepsilon}$ imply by Theoyem 3. *a*) of *P*

$$\int_{C_{1-\varepsilon}}\omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds=\int_{C\delta}\omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds. \qquad (6)$$

Since $C_\delta\subset R$ and $\omega^1(z)<1$ in R, $\int_{C\delta}\omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds<\int_{C\delta}\frac{\partial}{\partial n}\omega^2(z)ds=D(\omega^2(z))$.

Hence there exists a constant $\varepsilon_0>0$ depending only on $\omega^1(z)$ and C_δ such that

10) See 9).

$$\int_{C_\delta} \omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds<(1-\varepsilon_0)D(\omega^2(z)). \tag{7}$$

Now ε is arbitrary so long as $C_{1-\varepsilon}$ is regular. We choose ε so that $0<\varepsilon<\frac{\varepsilon_0}{2}$. Now $\omega^2(z)=1-\varepsilon$ on $C_{1-\varepsilon}$, whence by (6) and (7) we have

$$\int_{C_{1-\varepsilon}} \omega^2(z)\frac{\partial}{\partial n}\omega^2(z)ds=(1-\varepsilon)\int_{C_{1-\varepsilon}}\frac{\partial}{\partial n}\omega^2(z)ds=(1-\varepsilon)D(\omega^2(z))>(1-\varepsilon_0)D(\omega^2(z))$$
$$>\int_{C_\delta}\omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds=\int_{1-\varepsilon}\omega^1(z)\frac{\partial}{\partial n}\omega^2(z)ds.$$

This implies $\omega^1(z)\neq\omega^2(z)$.

Put $\tilde{\omega}^1(z)=\omega(G_1\frown B', z, \tilde{G})$ and $\tilde{\omega}^2(z)=\omega(G_2\frown B', z, \tilde{G})$. Then by $\tilde{G}\supseteq G$ and $\tilde{G}\supset(G-G_1)$ $\tilde{\omega}^i(z)\geqq\omega^i(z)$ and $D(\tilde{\omega}^i(z))\leqq D(\omega^i(z))$ $(i=1, 2)$[11]. Consider the same regular niveau curve of $\omega^2(z)$. Then by $\tilde{\omega}^1(z)<1$ in R, there exists a constant ε_0 such that $\int_{C_\delta}\tilde{\omega}^1(z)\frac{\partial}{\partial n}\omega^2(z)ds<(1-\varepsilon_0)D(\omega^2(z))$.

Similarly as above

$$\int_{C_\delta}\tilde{\omega}^1(z)\frac{\partial}{\partial n}\omega^2(z)ds=\int_{C_{1-\varepsilon}}\tilde{\omega}^1(z)\frac{\partial}{\partial n}\omega^1(z)ds.$$

Choose $0<\varepsilon<\frac{\varepsilon_0}{2}$ such that $C_{1-\varepsilon}$ is regular. Then

$$\int_{C_{1-\varepsilon}}\tilde{\omega}^1(z)\frac{\partial}{\partial n}\omega^2(z)ds=\int_{C_\delta}\tilde{\omega}^1(z)\frac{\partial}{\partial n}\omega^2(z)ds<(1-\varepsilon_0)\,D(\omega^2(z))<(1-\varepsilon)\,D(\omega^2(z))$$
$$=\int_{C_{1-\varepsilon}}\omega^2(z)\frac{\partial}{\partial n}\omega^2(z)ds\leqq\int_{C_{1-\varepsilon}}\tilde{\omega}^2(z)\frac{\partial}{\partial n}\omega^2(z)ds.$$

Thus $\tilde{\omega}^1(\neq)\tilde{\omega}^2(z)$.

Theorem 6. (Separation theorem. S. 2). *Let G, G_1 and G_2 be non compact domains such that $G\supset G_1\supset G_2$ and let B' be a closed subset of B. If $\omega(G_1\frown B', z, G)>\omega(G_2\frown B', z, G)>0$, we can find domains D_1 and D_2 in G*

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11) Put $B'_n=E\left[z\in\bar{R}:\ \delta(z, B')\leqq\frac{1}{n}\right]$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $(G\frown R_{n+i})-B'_n$ such that $\omega_{n,n+i}(z)=0$ on $\partial G\frown R_{n+i}$, $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown(G-B'_n)$ and $\omega_{n.n+i}(z)=1$ on B'_n. Let $\tilde{\omega}_{n,n+i}(z)$ be a harmonic function in $(\tilde{G}\frown R_{n+i})-B'_n$ such that $\tilde{\omega}_{n,n+i}(z)=0$ on $\partial\tilde{G}\frown R_{n+i}$, $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}\frown(\tilde{G}-B'_n)$ and $\tilde{\omega}_{n,n+i}(z)=1$ an B'_n. Then $D(\tilde{\omega}_{n,n+i}(z))\leqq D(\omega_{n,n+i}(z))$. Let $i\to\infty$ ond then $n\to\infty$. Then $\tilde{\omega}_{n,n+i}(z)\Rightarrow\tilde{\omega}^i(z)$ and $\omega_{n,+i}(z)\Rightarrow\omega^i(z)$. Hence $D(\tilde{\omega}^i(z))\leqq D(\omega^i(z))$ and $D(\tilde{\omega}(z))\leqq D(\omega(z))$.

such that $D_1 \supset D_2$ *and that* $0<\omega(D_2 \frown \tilde{G}_1 \frown B', z, G) \neq \omega(D_2 \frown C\tilde{D}_1 \frown B', z, G) > 0$ *for any domain* $\tilde{G} \supset G$.

Proof. For simplicity put $\omega^1(z)=\omega(G_1 \frown B', z, G)$ and $\omega^2(z)=\omega(G_2 \frown B', z, G)$. Let $D_1=E\left[z \in G:\ V(z)>\frac{M}{3}\right]$ and $D_2=E\left[z \in G: V(z)>\frac{2M}{3}\right]$, where $V(z)=\omega^1(z)-\omega^2(z)$ and $M=\sup_{z \in G} V(z)$.

At first we remak

$$D_1 \subset D^2_{\frac{M}{3}}=E\left[z \in G:\ \omega^2(z)<1-\frac{M}{3}\right],$$

because if $\omega^2(z) \geqq 1-\frac{M}{3}$, $V(z) \leqq \frac{M}{3}$ by $\omega^1(z)<1$.

Hence by P.C.2.

$\omega(G_2 \frown B' \frown D^2_{\frac{M}{3}}, z, G)=0$ and by $D_1 \subset D^2_{\frac{M}{3}}$ we have $\omega(B' \frown D_1 \frown G_2, z, G)=0$.

Whence by $\omega(D_1 \frown G_2 \frown B', z, G)+\omega(CD_1 \frown G_2 \frown B', z, G) \geqq \omega(G_2 \frown B', z, G) \geqq \omega(G_2 \frown CD_1 \frown B', z, G)$, we have

$$\omega^2(z) \doteq \omega(G_2 \frown B', z, G)=\omega(G_2 \frown CD_1 \frown B', z, G). \tag{8}$$

Put $\Omega_{n, n+1, n+i+j}=(D_1 \frown R_{n+i+j})-(G_1 \frown (D_1-D_2) \frown B'_{n+i})-(B'_n \frown D_2)$, where $B'_n=E\left[z \in \overline{R}:\ \delta(z, B') \leqq \frac{1}{n}\right]$.

Then by $G_2 \frown CD_1 \frown D_1=0$ and by $\Omega_{n, n+i} \frown (CD_1 \frown G_2)=0$ $\omega^2(z)(=\omega(G_2 \frown CD_1 \frown B', z, G)$ has M. D. I. over $\Omega_{n, n+i}(=\lim_j \Omega_{n, n+j, n+i+j})$ with value $\omega^2(z)$ on $\alpha_{n, n+i}(=(\partial D_1+\partial(B'_{n+i} \frown (D_1-D_2) \frown G_1)+\partial(B'_n \frown D_2 \frown G_1))$ by P.C.2. Hence

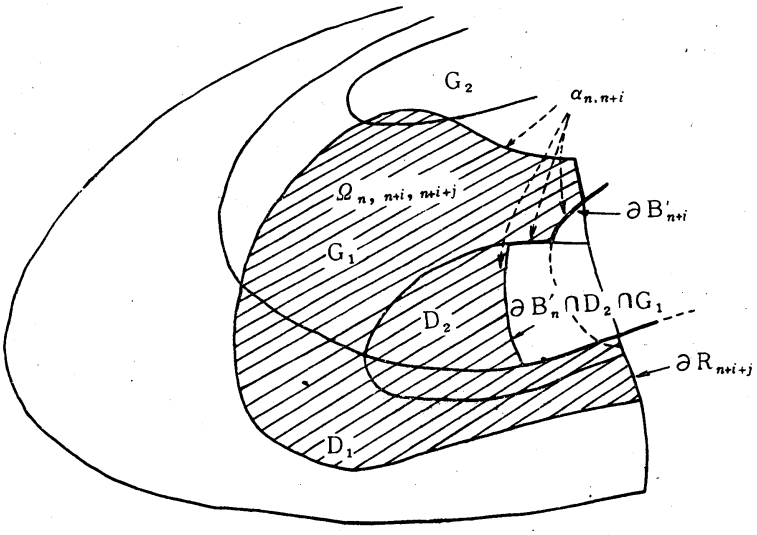

Fig. 3.

$$\omega^2_{n,n+i,n+i+j}(z) \Rightarrow \omega^2(z) \text{ as } j \to \infty,$$

where $\omega^2_{n,n+i,n+i+j}(z)$ is a harmonic function in $\Omega_{n,n+i,n+i+j}$, such that $\omega^2_{n,n+i,n+i+j}(z)=\omega^2(z)$ on $\alpha_{n,n+i} \frown R_{n+i+j}$ and $\frac{\partial}{\partial n}\omega^2_{n,n+i,n+i+j}(z)=0$ on $\partial R_{n+i+j} \frown \Omega_{n,n+i}$.

Similarly $\Omega_{n,n+i} \frown [G_1 \frown \{B'_{n+i} \frown (D_1-D_2)+(B'_n \frown D_2)\}]=0$ implies that $\omega^1(z)$ has M. D. I. over $\Omega_{n,n+i}$ with value $\omega^1(z)$ on $\alpha_{n,n+i}$. Hence

$$\omega^1_{n,n+i,n+i+j}(z) \Rightarrow \omega^1(z) \text{ as } j \to \infty,$$

where $\omega^2_{n,n+i,n+i+j}(z)$ is a harmonic function in $\Omega_{n,n+i,n+i+j}$ such that $\omega^1_{n,n+i,n+i+j}(z)=\omega^1(z)$ on $\alpha_{n,n+i} \frown R_{n+i+j}$ and $\frac{\partial}{\partial n}\omega^1_{n,n+i,n+i+j}(z)=0$ on $\partial R_{n+i+j} \frown \Omega_{n,n+i}$. Thus

$$\omega^1_{n,n+i,n+i+j}(z)-\omega^2_{n,n+i,n+i+j}(z)=V_{n,n+i,n+i+j}(z) \Rightarrow V(z)=\omega_1(z)-\omega_2(z) \text{ as } j \to \infty.$$

Let $\tilde{\omega}_{n,n+i,n+i+j}(z)$ be a harmonic function in $\Omega_{n,n+i,n+i+j}$ such that $\tilde{\omega}_{n,n+i,n+i+j}(z)=0$ on $\partial D_1 \frown \partial\Omega_{n,n+i,n+i+j}$, $\tilde{\omega}_{n,n+i,n+i+j}(z)=1$ on $\partial(D_2 \frown G_1 \frown B'_n)$ and $\frac{\partial}{\partial n}\tilde{\omega}_{n,n+i,n+i+j}(z)=0$ on $\partial R_{n+i+j} \frown \Omega_{n,n+i+j}+\partial B'_{n+i} \frown (D_1-D_3) \frown G_1$.

Then
$$D(\tilde{\omega}_{n,n+i,n+i+j}(z)) \leqq \frac{9}{M^2} D(V(z)) < \infty,$$

because $V'(z)=\frac{3}{M}\left(V(z)-\frac{3}{M}\right)=0$ on ∂D_1, $V'(z) \geqq 1$ on $\overline{D}_2(\supset \partial(D_2 \frown G_1 \frown B'_n))$ and $\tilde{\omega}_{n,n+i,n+i+j}(z)$ has M. D. I. over $\Omega_{n,n+i,n+i+j}$ among all functions with value $\tilde{\omega}_{n,n+i,n+i+j}(z)=\min(1, V'(z))$ on $\alpha_{n,n+i}$.

Hence $\tilde{\omega}_{n,n+i,n+i+j}(z) \Rightarrow \tilde{\omega}_{n,n+i}(z)$ as $j \to \infty$, $\tilde{\omega}_{n,n+i}(z) \Rightarrow \tilde{\omega}_n(z)$ as $i \to \infty$ and

$$\tilde{\omega}_n(z) \Rightarrow \omega(D_2 \frown G_1 \frown B', zD_1) \text{ as } n \to \infty. \tag{9}$$

Let $\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)$ be a harmonic function in $\Omega_{n,n+i,n+i+j}$ such that $\frac{\partial}{\partial n}\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)=0$ on $\Omega_{n,n+i} \frown \partial R_{n+i+j}$ and $\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)=\min\left(V(z), \frac{2M}{3}\right)$ on $\partial\Omega_{n,n+i,n+i+j}-\partial R_{n+i+j}$.

Put $T(z)=V(z)$ in D_1-D_2 and $=\frac{2M}{3}$ in $\Omega_{n,n+i,n+i+j} \frown D_2$. Then by the Dirichlet principle

$$D(\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)) \leqq D(T(z)) \leqq D(V(z)).$$

Hence it can be proved by Lemma 1, *b*) of *P* that

$$\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z) \Rightarrow \tilde{\omega}_{n,n+i}(z) \text{ as } j \to \infty.$$

Compare $\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z)$ and $V_{n,n+i,n+i+j}(z)$ in $\Omega_{n,n+i,n+i+j}$. Then

$$\tilde{\tilde{\omega}}_{n,n+i,n+i+i}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z) \geqq V_{n,n+i,n+i+j}(z) \text{ on } \partial D_1,$$

by $\tilde{\omega}_{n,n+i,n+i+j}(z)=0$ and $\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)=V_{n,n+i,n+i+j}(z)=\dfrac{M}{3}$ on ∂D_1.

$$\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z)\geqq V_{n,n+i,n+i+j}(z) \text{ on } \partial(B'_{n+i}\frown(D_1-D_2)\frown G_1),$$

by $\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)=V(z)=V_{n,n+i,n+i+j}(z)$, because $V(z)\leqq\dfrac{2M}{3}$ on $\partial(B'_{n+i}\frown(D_1-D_2))\frown G_1)$. Hence

$$\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z)\geqq 1\geqq V(z)=V_{n,n+i,n+i+j}(z) \text{ on } \partial(D_2\frown D_1\frown B'_n),$$

by $\tilde{\omega}_{n,n+i,n+i+j}(z)=1$ and $V(z)\leqq 1$ on $\partial(D_2\frown G_1\frown B'_n)$.

$$\frac{\partial}{\partial n}\tilde{\omega}_{n,n+i,n+i+j}(z)=\frac{\partial}{\partial n}\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)=\frac{\partial}{\partial n}V_{n,n+i,n+i+j}(z)=0$$
$$\text{on } \partial R_{n+i+j}\frown\Omega_{n,n+i}.$$

Hence by the maximum principle

$$\tilde{\tilde{\omega}}_{n,n+i,n+i+j}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z)\geqq V_{n,n+i,n+i+j}(z).$$

Let $j\to\infty$. Then

$$\tilde{\tilde{\omega}}_{n,n+i}(z)+\tilde{\omega}_{n,n+i}(z)\geqq\omega^1(z)-\omega^2(z)=V(z).$$

Choose a subsequence $\{i'\}$ such that $\tilde{\tilde{\omega}}_{n,n+i}(z)\to\tilde{\tilde{\omega}}_n(z)$. Further choose a subsequence $\{n'\}$ such that $\tilde{\tilde{\omega}}_{n'}(z)\to\tilde{\tilde{\omega}}(z)$.

Assume $\omega(D_2\frown G_1\frown B', z, D)=0$. Then by (9) $\tilde{\tilde{\omega}}(z)+\omega(D_2\frown G_1\frown B', z, D_1)\geqq V(z)$ and

$$\frac{2M}{3}\geqq\tilde{\tilde{\omega}}(z)\geqq V(z):\ \sup_{z\in G}V(z)=M.$$

This is a contradiction.

Hence $$\omega(D_2\frown G_1\frown B', z, D_1)>0. \tag{10}$$

Now by (8) $$\omega^2(z)=\omega(G_2\frown CD_1\frown B', z, G)>0. \tag{11}$$

Since $(G_2\frown CD_1)\frown D_1=0$, S. 1. is applicable to (10) and (11) and we have

$$\omega(D_2\frown G_1\frown B', z, \tilde{G})\neq\omega(G_2\frown CD_1\frown B', z, \tilde{G}).$$

Thus D_1 and D_2 are required domains.

5. Singular points. Let $p\in R-R_0$. Then $N(z,p)$ is N-minimal and its C. P. $\omega(p,z)=0$ and $N(z,p)$ likes $-\log|z-p|$ in a neighbourhood of p. Hence if $\omega(p,z)>0$ for a point $p\in B$, it is imagined that $N(z,p)$ and other function take queer behaviour in a neighbourhood of p. If $\omega(p,z)>0$, we call p *a singular point* and denote the set of singular points by B_S (see Theorem 7). We call a singular point p *a singular point of first or second kind according as* H. M. (*harmonic measure*) *of* p: $w(p,z)=0$ *or* >0. It is clear $\omega(p,z)\geqq w(p,z)$. It can be thought that such a large set Δ as C. P. $\omega(p,z)$ of $p>0$) is condensed so intensely by genus in a neighbourhood of p that Δ may become one point p in N-Martin's

topology. In other words, p is larger, the more complicated the structure of p is, and the more queer the behaviors of functions are in a neighbourhood of p. As an answer of problem 2, we use the notion singular point. In the present paper the discussions of singular points play the central part.

Let G be a domain[12] in $R-R_0$ and let $\omega(p, z)$ be C. P. of a singular point. Let ${}_{CG}\omega(p, z)$ be a superharmonic function in $\overline{R}-R_0$ such that ${}_{CG}\omega(p, z)=\omega(p, z)$ in CG and ${}_{CG}\omega(p, z)=0$ on ∂R_0 and harmonic in G with M.D.I. over G. If ${}_{CG}\omega(p, z)<\omega(p, z)$, we say that G *contains* p *N-approximately.*

Theorem 7. *a) Suppose that a domain G contains a singular point p N-approximately. Put $V_M^C=E[z\in R:\ {}_{CG}\omega(p, z)>M]$ (V_M^C may be empty). Then there exists a constant $M_0<1$ such that*

$$\omega(p\frown V_M^C, z)=0 \text{ for } M\geqq M_0,$$

where $\omega(p\frown V_M^C, z)$ is C.P. of $p\frown V_M^C$: $\omega(p\frown V_M^C, z)=\lim_{m=\infty}\omega(\upsilon_m(p)\frown V_M^C, z)$.

b) Let $\upsilon_n(p)$ be a neighbourhood of p: $p\in B_S$ such that $\upsilon_n(p)=E\left[z\in\overline{R}:\ \delta(z, p)\leqq\frac{1}{n}\right]$. Then $\upsilon_n(p)$ contains p N-approximately.

By d) of Theorem 9 of P, there exists a number n such that $(\upsilon_n(p)\frown R)\subset V_M(p)$ for any given $M<1$, where $V_M(p)=E[z\in R:\ \omega(p, z)>M]$. But the inverse is not always true. Now as an almost inversed theorem, we shall prove the following:

c) For any given $\upsilon_n(p)$: $p\in B_S$

$$\omega(C\upsilon_n(p)\frown V_M(p), z)\ \downarrow\ 0 \text{ as } M\to 1.$$

This means that $\{\upsilon_n(p)\}$: $n=1, 2,\cdots$ is almost equivalent to $\{V_{M_i}(p)\}$: $M_1<M_2\cdots\lim_i M_i=1$.

Proof of a). If ${}_{CG}\omega(p, z)=0$, our assertion is clear. Suppose ${}_{CG}\omega(p, z)>0$. Then ${}_{CG}\omega(p, z)>M$ on $V_M^C\frown\upsilon_n(p)$ and ${}_{V_M^C\frown\upsilon_n(p)}({}_{CG}\omega(p, z))$[13] and $\omega(\upsilon_n(p)\frown V_M^C, z)$ have M.D.I. over $R-R_0-(\upsilon_n(p)\frown V_M^C)$. Hence by the maximum principle

$${}_{CG}\omega(p, z)\geqq {}_{V_M^C\frown\upsilon_n(p)}({}_{CG}\omega(p, z))\geqq M\omega(\upsilon_n(p)\frown V_M^C, z).$$

Let $n\to\infty$. Then ${}_{CG}\omega(p, z)\geqq M\omega(p\frown V_M^C, z)$.

Now $\omega(p\frown V_M^C, z)$ has mass at only p by $(p\frown V_M^C)\subset p$ by Theorem 5. *b)* of P. i. e.

12) See 5).

13) See "Potentials on Riemann surfaces" for the definition of ${}_{V_M^C\frown\upsilon_n(p)}({}_{CG}\omega(p, z))$.

$$\omega(p\frown V_M^C, z)=K\ \omega(p, z):\ K>0.$$

But by P.C. 2. $\sup_{z\in R}\omega(p\frown V_M^C, z)=1=\sup_{z\in R}\omega(p, z)$. Whence $K=1$ and $\omega(p\frown V_M^C, z)=\omega(p, z)$. Assume $\omega(p\frown V_M^C, z)>0$ for every M such that $M<1$. Then by letting $M\to 1$

$${}_{CG}\omega(p, z)\geqq\lim_{M\to 1} M\omega(p\frown V_M^C, z)=\omega(p, z).$$

This contradicts the assumption. Hence we have *a*).

Proof of b). Assume ${}_{Cv_n(p)}\omega(p, z)=\omega(p, z)$. Then ${}_{Cv_n(p)}\omega(p, z)$ is N-minimal and by Theorem 5. *b*) of P ${}_{Cv_n(p)}\omega(p, z)$ is represented by a mass distribution μ over $R-R_0-v_n(p)$ and by Theorem 9. *a*) of P ${}_{Gv_n(p)}\omega(p, z)$ $=a'N(z, q)$: $a'>0$ and $q\notin v_n(p)$. On the other hand, by $\omega(p, z)=a\ N(z, p)$: $a>0$ and by $\int_{\partial R_0}\frac{\partial}{\partial n}N(z, p)ds=\int_{\partial R_0}\frac{\partial}{\partial n}N(z, p)ds$ and by $\sup_{z\in R}\omega(p, z)=1$ $=\sup_{z\in R}\omega(q, z)$, we have $N(z, p)=N(z, q)$. This is a contradiction. Thus $\omega(p, z)>{}_{Cv_n(p)}\omega(p, z)$.

Proof of c). At first, *we remark* $\omega(p\frown CG)=0$, if G contains p N-approximately. In fact, assume $\omega(p\frown CG, z)>0$, then by $p\supset(p\frown CG)$, $\omega(p\frown CG, z)$ has mass only at p, i.e. $\omega(p\frown CG, z)=K\omega(p, z)$. But by P.C. 2. $\sup_{z\in R}\omega(p\frown CG, z)=1=\sup_{z\in R}\omega(p, z)$,

hence
$$\omega(p\frown CG, z)=\omega(p, z). \tag{a}$$

By *d*) of Theorem 9 of P, there exists a neighbourhood $v_m(p)$ such that $\omega(p, z)>1-\varepsilon$ in $v_m(p)$ for any given positive number ε. Hence by the maximum principle

$$(1-\varepsilon)\ \omega(v_m(p)\frown CG, z)\leqq{}_{v_m(p)\frown CG}\omega(p, z)\leqq{}_{CG}\omega(p, z),$$

because $\omega(p, z)\geqq 1-\varepsilon$ on $v_m(p)\supset(v_m(p)\frown CG)$ implies ${}_{v_m(p)\frown CG}\omega(p, z)=\omega(p, z)$ $\geqq 1-\varepsilon$ on $v_m(p)\frown CG$.

Let $v_m(p)\to p$ and then $\varepsilon\to 0$. Then $\omega(p\frown CG, z)\leqq{}_{CG}\omega(p, z)$. Now by (*a*) ${}_{CG}\omega(p, z)=\omega(p, z)$. This contradicts $\omega(p, z)>{}_{CG}\omega(p, z)$.

Hence
$$\omega(p\frown CG, z)=0. \tag{12}$$

By (12) and by $\omega(p\frown CG, z)+\omega(p\frown G, z)\geqq\omega(p, z)$ we have $\omega(p\frown G, z)=\omega(p, z)$ and by $p\in V_{1-\varepsilon}(p)=E[z\in R:\ \omega(p, z)>1-\varepsilon]$, we have

$$\omega(p, z)=\omega(p\frown V_{1-\varepsilon}(p)\frown G, z)>0,\ \textit{whence}\ V_{1-\varepsilon}(p)\frown G\ \textit{is non void}. \tag{13}$$

By Theorem 7. *a*), there exists $V_{M_0}^C$ such that $\omega(V_{M_0}^C\frown p, z)=0$,

where
$$V_{M_0}^C=E[z\in R:\ {}_{CG}\omega(p, z)>M_0\ \text{and}\ M_0<1]. \tag{14}$$

At present fix M_0. Then by (13) and by (14) and by

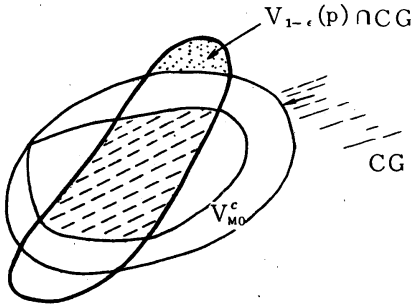

Fig. 4.

$$\omega(p\frown V_{1-\varepsilon}(p)\frown G\frown CV^c_{M_0}, z)$$
$$+\omega(p\frown V_{1-\varepsilon}(p)\frown V^c_{M_0}\frown G, z)$$
$$\geqq\omega(p\frown V_{1-\varepsilon}(p)\frown G, z)=\omega(p, z)$$

we have

$$\omega(p, z)=\omega(p\frown V_{1-\varepsilon}(p)\frown CV^c_{M_0}\frown G, z)$$

and $CV^c_{M_0}\frown V_{1-\varepsilon}(p)\frown G\frown \upsilon_n(p)$ *is non void for every* $V_{1-\varepsilon}(p)$. (15)

Put $\widetilde{V}=CV^c_{M_0}\frown V_{1-\varepsilon}(p)\frown\upsilon_n(p)$ and $G=\upsilon_n(p)$ and we shall prove *c*). Since $\upsilon_n(p)$ contains p N-approximately, (12), (13), (14) and (15) are valid. Let $\widetilde{\omega}(z)=\omega(\widetilde{V}, z, G)$ be C. P. of $\widetilde{V}$ relative G. Since $\widetilde{\omega}(z)$ has M. D. I. $G-\widetilde{V}$ among all functions with value$=1$ on $\widetilde{V}$ and$=0$ on ∂G: $\upsilon_n(p)=G$.

$$D(\widetilde{\omega}(z))\leqq D\left(\min\left(1, \frac{\omega'(p, z)}{1-M_0-\varepsilon}\right)\right)\leqq\frac{1}{(1-M_0-\varepsilon)^2}D(\omega'(p, z)), \tag{16}$$

where $\omega'(p,z)=\omega(p,z)-_{CG}\omega(p,z)$ and $\omega'(p,z)\geqq 1-M_0-\varepsilon$ on $(G\frown V_{1-\varepsilon}(p))-V^c_{M_0}$ and$=0$ on ∂G.

Put $\widetilde{\omega}(z)=1$ in CG and $\widetilde{\widetilde{\omega}}(z)=1-\widetilde{\omega}(z)$ in G. Then $\widetilde{\widetilde{\omega}}(z)=0$ in $\widetilde{V}$ and $\widetilde{\widetilde{\omega}}(z)=1$ in CG and

$$D(\widetilde{\widetilde{\omega}}(z))=D(\widetilde{\omega}(z))<\infty.$$

Consider $\omega^*(z)=\min(\omega(V_{1-\varepsilon}(p), z), \widetilde{\widetilde{\omega}}(z))$. Then $\omega^*(z)=0$ on $\partial R_0+\widetilde{V}$ and $\omega^*(z)=1$ on $CG\frown V_{1-\varepsilon}(p)$ and

$$D(\omega^*(z))\leqq D(\omega(V_{1-\varepsilon}(p), z))+D(\widetilde{\widetilde{\omega}}(z)),$$

because $\left|\dfrac{\partial\omega^*(z)}{\partial x}\right|\leqq\max\left(\left|\dfrac{\partial\omega(V_{1-\varepsilon}(p), z)}{\partial x}\right|, \left|\dfrac{\partial\widetilde{\widetilde{\omega}}(z)}{\partial x}\right|\right)$

and $\left|\dfrac{\partial\omega^*(z)}{\partial y}\right|<\max\left(\left|\dfrac{\partial\omega(V_{1-\varepsilon}(p), z)}{\partial y}\right|, \left|\dfrac{\partial\widetilde{\widetilde{\omega}}(z)}{\partial y}\right|\right)$.

Let $\widehat{\omega}(z)=(V_{1-\varepsilon}(p)\frown CG, z, R-R_0-\widetilde{V})$. Then $\widehat{\omega}(z)$ has M. D. I. over $R-R_0-\widetilde{V}-(V_{1-\varepsilon}(p)\frown CG)$ among all function with value$=1$ on $V_{1-\varepsilon}(p)\frown CG$ and$=0$ on $\partial R_0+\partial\widetilde{V}$, whence

$$D(\widehat{\omega}(z))\leqq D(\omega^*(z))<\infty.$$

Assume $\lim\limits_{\varepsilon\to 0}\omega(V_{1-\varepsilon}(p)\frown CG, z)>0$, where $D(\omega(V_{1-\varepsilon}(p)\frown CG, z))\leqq D(\omega(V_{1-\varepsilon}(p), z)$ $=\dfrac{D(\omega(p, z))}{(1-\varepsilon)^2}$. Then by the Dirichlet principle

$$0<D(\omega(V_{1-\varepsilon}(p)\frown CG, z))\leqq D(\omega(V_{1-\varepsilon}(p)\frown CG, z, R-R_0-\widetilde{V}))\leqq D(\omega^*(z)).$$

Hence

$$\omega(V_{1-\varepsilon}(p)\frown CG, z, R-R_0-\widetilde{V})\Rightarrow\widehat{\widehat{\omega}}(z)>0 \text{ as } \varepsilon\to 0, \tag{16}$$

where $\widehat{\widehat{\omega}}(z)$ is C. P. defined by sequence $\{V_{1-\varepsilon_i}(p)\frown CG\}$: $i=1, 2\cdots$ and $\lim_{i=\infty}\varepsilon_i=0$. (see 4 of P).

Let C_δ and $C_{1-\varepsilon}$ be regular niveau curves of $\widehat{\widehat{\omega}}(z)$ and put $\Omega_\delta^{1-\varepsilon}=E[z\in R-R_0-\widetilde{V}:\ \delta<\widehat{\widehat{\omega}}(z)<1-\varepsilon]$ and $\Omega_\delta=E[z\in R-R_0-\widetilde{V}:\ \widehat{\widehat{\omega}}(z)>\delta]$. Then $\omega(p, z)(=\omega(\widetilde{V}\frown p, z))$ by (15)) has M.D.I. over $R-R_0-\widetilde{V}$ by $(\widetilde{V}\frown p)\frown(R-R_0-\widetilde{V})=0$ by P.C.1.

Hence $$\omega_n^1(z)\Rightarrow\omega(p, z) \text{ as } n\to\infty,$$

where $\omega_n^1(z)$ is a harmonic function in $\Omega_\delta\frown(R_n-R_0)$ such that $\omega_n^1(z)=\omega(p, z)$ on $\partial\Omega_\delta\frown R_n$ and $\frac{\partial}{\partial n}\omega_n^1(z)=0$ on $\partial R_n\frown\Omega_\delta$.

Since by (16) $\widehat{\widehat{\omega}}(z)$ is C.P., $\omega_n^2(z)\Rightarrow\widehat{\widehat{\omega}}(z)$ as $n\to\infty$,
where $\omega_n^2(z)$ is a harmonic function in $\Omega_\delta^{1-\varepsilon}\frown(R_n-R_0)$ such that $\omega_n^2(z)=\widehat{\widehat{\omega}}(z)$ on $\partial\Omega_\delta^{1-\varepsilon}\frown R_n$ and $\frac{\partial}{\partial n}\omega_n^2(z)=0$ on $\partial R_n\frown\Omega_\delta^{1-\varepsilon}$.

Hence by the Green's formula and by letting $n\to\infty$, we have by the regularity of C_δ and $C_{1-\varepsilon}$

$$\int_{C\delta}\omega(p, z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)=\int_{C_{1-\varepsilon}}\omega(p, z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)ds. \tag{17}$$

$\omega(p, z)<1$ on C_δ, whence there exists a constant ε_0 such that $\int_{C\delta}\omega(p, z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)ds<(1-\varepsilon_0)D(\widehat{\widehat{\omega}}(z))$. Above (17) holds so long as $C_{1-\varepsilon}$ is regular. Choose $\varepsilon<\frac{\varepsilon_0}{2}$: Then

$$\begin{aligned}\int_{C\delta}\omega(p, z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)ds&=\int_{C_{1-\varepsilon}}\omega(p, z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)ds\leqq(1-\varepsilon_0)D(\widehat{\widehat{\omega}}(z))<(1-\varepsilon)D(\widehat{\widehat{\omega}}(z))\\&=\int_{C_{1-\varepsilon}}\widehat{\widehat{\omega}}(z)\frac{\partial}{\partial n}\widehat{\widehat{\omega}}(z)ds.\end{aligned} \tag{18}$$

On the other hand, $\omega(p, z)>1-\varepsilon'$ on $V_{1-\varepsilon'}(p)$. Hence by the maximum principle

$$\begin{aligned}\omega(p, z)&>(1-\varepsilon')\omega(V_{1-\varepsilon'}(p), z)\geqq(1-\varepsilon')\,\omega(V_{1-\varepsilon'}(p)\frown CG, z)\\&\geqq(1-\varepsilon')\,\omega(V_{1-\varepsilon'}(p)\frown CG, z, R-R_0-\widetilde{V}).\end{aligned}$$

Let $\varepsilon'\to 0$. Then $\omega(p, z)\geqq\widehat{\widehat{\omega}}(z)$. This contradicts (18).

Hence $$\lim_{\varepsilon \to 0} \omega(V_{1-\varepsilon}(p) \frown C\upsilon_n(p), z) = 0.$$

Let F be a closed set of positive capacity in the z-plane and let $\upsilon(F)$ be a neighbourhood of F. Then $\omega(F, z) > 0$ implies $\omega(F, z, \upsilon(F)) > 0$ by $\sup_{z \in \partial\upsilon(F)} \omega(F, z) < 1$. We show that G has the same property, if G contains $p \in B_S$ N-approximately.

Theorem 8. *a) Suppose that G contains $p \in B_S$ N-approximately. Then there exists a domain $\widetilde{V}$ in G such that $\omega(\widetilde{V} \frown p, z, G) > 0$ and $D(\omega(\widetilde{V}, z, G)) < \dfrac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} < \infty$, where M_0 and ε are constants such that $1-\varepsilon > M_0 > 0$ and $\omega'(p, z) = \omega(p, z) - {}_{CG}\omega(p, z) > 1-M_0-\varepsilon > 0$ in $\widetilde{V}$.*

a') Conversely if there exists a domain $\widetilde{V}$ in G such that $0 < D(\omega(p \frown \widetilde{V}, z, G)) < \infty$, then G contains p N-approximately.

b) Let $\widetilde{V}$ be the domain a). Put $G_\delta = E[z : \omega(\widetilde{V} \frown p, z, G) < \delta : 0 < \delta < 1]$. Then

$$\omega(\widetilde{V} \frown p, z, G) = \omega(p \frown CG_\delta, z, G) = \delta\omega(CG_\delta, z, G) \text{ in } G_\delta.$$

c) If G contains a singular point of second kind, there exists a domain $\widetilde{V}$ in G such that

$$\omega(\widetilde{V} \frown p, z, G) \geqq w(p \frown \widetilde{V}, z, G) > 0 \text{ and } D(\omega(\widetilde{V}, z, G)) \leqq \frac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} < \infty.$$

d) Let $\upsilon(p)$ be a neighbourhood of a point $p \in B_S$. Then $\upsilon(p)$ contains p N-approximately by Theorem 7, b). Hence a) and c) are valid for $\upsilon(p)$.

e) If G contains $p \in B_S$ N-approximately, CG does not contain p N-approximately.

f) If $(i = 1, 2, \cdots i_0)$ contains $p \in B_S$ N-approximately, then $\bigcap^{i_0} G_i$ contains p N-approximately.

Proof of a). By a) of Theorem 7, there exists a domain $\widetilde{V} = CV^C_{M_0} \frown V_{1-\varepsilon}(p)$ such that $V^C_{M_0} = E[z \in R : {}_{CG}\omega(p, z) > M_0] : 1 > M_0 > 0$ and $\omega(p \frown V^C_{M_0}) = 0$. Then

$\omega'(p, z) = \omega(p, z) - {}_{CG}\omega(p, z) > 1-M_0-\varepsilon > 0$ in $\widetilde{V} \frown G \frown \upsilon_n(p)$ and $= 0$ on ∂G for every n by (15).

Since $\omega(\widetilde{V}, z, G)$ has M.D.I. over $G - \widetilde{V}$ among all functions with value 0 on ∂G and 1 on $\partial\widetilde{V}$,

$$\frac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} \geqq D(\omega(\widetilde{V}, z, G)) \geqq D(\omega(\widetilde{V} \frown \upsilon_n(p), z, G)) > 0,$$

because $\omega'(p, z) > 1-M_0-\varepsilon$ on $\widetilde{V}$ and $= 0$ on ∂G.

On the other hand, by the Dirichlet principle and by (15)

$$D(\omega(\widetilde{V}, z, G)) \geqq D(\omega(\widetilde{V} \frown \upsilon_n(p), z, G)) \geqq D(\omega(\widetilde{V} \frown \upsilon_n(p), z)) \geqq D(\omega(\widetilde{V} \frown p, z)) = D(\omega(p, z)) > 0.$$

Let $n \to \infty$. Then $\omega(\widetilde{V} \frown p, z, G) > 0$. Thus $\widetilde{V}$ is the required domain.

Proof of a') Let C_{M_i}: $M_1 < M_2 < 1$ be a regular niveau curves of $\omega(p \frown \widetilde{V}, z, G)$ such that

$$\int_{C_{M_i}} \frac{\partial}{\partial n} \omega(p \frown \widetilde{V}, z, G) ds = D(\omega(p \frown \widetilde{V}, z, G)).$$

Now ${}_{CG}\omega(p, z)$ is harmonic in G (${}_{CG}\omega(p, z)$ is harmonic in G and has M.D.I. over G). Hence

$${}_{CG}\omega_n(p, z) \Rightarrow {}_{CG}\omega(p, z) \text{ as } n \to \infty,$$

where ${}_{CG}\omega_n(p, z)$ is a harmonic function in $\Omega_{M_1}^{M_2} \frown R_n$ such that ${}_{CG}\omega_n(p, z) = {}_{CG}\omega(p, z)$ on $\partial\Omega_{M_1}^{M_2} \frown R_n$ and $\frac{\partial}{\partial n} {}_{CG}\omega_n(p, z) = 0$ on $\partial R_n \frown \Omega_{M_1}^{M_2}$: $\Omega_{M_1}^{M_2} = E[z \in G : M_1 < \omega(p \frown \widetilde{V}, z, G) < M_2]$.

Also $\omega(p \frown \widetilde{V}, z, G)$ has M.D.I. over $\Omega_{M_1}^{M_2}$, whence $\omega_n(p \frown \widetilde{V}, z, G) \Rightarrow \omega(p \frown \widetilde{V}, z, G)$ as $n \to \infty$, where $\omega_n(p \frown \widetilde{V}, z, G)$ is a harmonic function in $\Omega_{M_1}^{M_2} \frown R_n$ such that $\omega_n(p \frown \widetilde{V}, z, G) = \omega(p \frown \widetilde{V}, z, G)$ on $\partial\Omega_{M_1}^{M_2} \frown R_n$ and $\frac{\partial}{\partial n} \omega_n(p \frown \widetilde{V}, z, G) = 0$ on $\Omega_{M_1}^{M_2} \frown \partial R_n$. Then

$$\int_{C_{M_1} \frown R_n} {}_{CG}\omega_n(p, z) \frac{\partial}{\partial n} \omega_n(p \frown \widetilde{V}, z, G) ds = \int_{C_{M_2} \frown R_n} {}_{CG}\omega_n(p, z) \frac{\partial}{\partial n} \omega_n(p \frown \widetilde{V}, z, G) ds.$$

By letting $n \to \infty$ and by Theorem 6 of P and since ${}_{CG}\omega(p, z) < 1$ on $C_{M_1} \frown R$, there exists a constant δ_0 such that

$$D(\omega(p \frown \widetilde{V}, z, G)) - \delta_0 \geqq \int_{C_{M_1}} {}_{CG}\omega(p, z) \frac{\partial}{\partial n} \omega(p \frown \widetilde{V}, z, G) ds = \int_{C_{M_2}} {}_{CG}\omega(p, z) \frac{\partial}{\partial n} \omega(p \frown V, z, G) ds.$$

Hence

$$\lim_{M_2 \uparrow 1} \int_{C_{M_2}} {}_{CG}\omega(p, z) \frac{\partial}{\partial n} \omega(p \frown \widetilde{V}, z, G) ds \leqq D(\omega(p \frown \widetilde{V}, z, G)) - \delta_0. \tag{18}$$

On the other hand, $\omega(p \frown \widetilde{V}, z, G) \to 1$ on C_{M_2} as $M_2 \uparrow 1$ and by $\omega(p, z) \geqq \omega(p \frown \widetilde{V}, z, G)$ we have

$$\lim_{M_2 \to 1} \int_{C_{M_2}} \omega(p, z) \frac{\partial}{\partial n} \omega(p \frown \widetilde{V}, z, G) ds \geqq \lim_{M_2 \to 1} \int_{C_{M_2}} M_2 \frac{\partial}{\partial n} \omega(p \frown \widetilde{V}, z, G) ds$$
$$= D(\omega(p \frown \widetilde{V}, z, G)). \tag{19}$$

(18) means by (19) that ${}_{CG}\omega(p, z) < \omega(p, z)$. Thus G contains p N-approximately.

Proof of b). $D(\omega(\widetilde{V} \frown p \frown G_\delta, z, G)) \leqq D(\omega(\widetilde{V}, z, G)) < \infty$ and by P.C.3. $\omega(\widetilde{V} \frown p \frown G_\delta, z, G) = 0$. Now

$$\omega(p \frown \widetilde{V} \frown G_\delta, z, G) + \omega(p \frown \widetilde{V} \frown CG_\delta, z, G) \geqq \omega(p \frown \widetilde{V}, z, G) \geqq \omega(p \frown \widetilde{V} \frown CG_\delta, z, G).$$

Hence
$$\omega(p \frown \widetilde{V}, z, G) = \omega(p \frown \widetilde{V} \frown CG_\delta, z, G).$$

$\omega(p \frown \widetilde{V}, z, G) = \delta$ on ∂G_δ and has M.D.I. over $G - (\upsilon_n(p) \frown \widetilde{V} \frown CG_\delta)$ for every n.

Hence
$$\omega(p \frown \widetilde{V}, z, G) = \omega(CG_\delta, z, G).$$

Proof of c). We use the same notation in as c) of Theorem 7. Put $V^C_{M_0} = E[z \in R : {}_{CC}\omega(p, z) \geqq M_0]$ and suppose that $\omega(p \frown V^C_{M_0}, z) = 0$. Let $w_{CG}(p, z)$ be a function such that $w_{CG}(p, z) = w(p, z)$ in CG and $w_{CG}(p, z)$ is the positively least harmonic function in G. Then $w_{CG}(p, z) \leqq {}_{CG}\omega(p, z)$ by $w(p, z) \leqq \omega(p, z)$. Put $V^w_{1-\varepsilon}(p) = E[z \in R : w(p, z) > 1 - \varepsilon]$ and $V_{1-\varepsilon}(p) = E[z \in R : \omega(p, z) > 1 - \varepsilon]$. Then

$$V^w_{1-\varepsilon}(p) \subset V_{1-\varepsilon}(p).$$

By P.H.5,

$$w(p \frown G \frown V^w_{1-\varepsilon}(p), z) + w(p \frown G \frown CV^w_{1-\varepsilon}(p), z)$$
$$+ w(p \frown CG, z) \geqq w(p, z) \geqq w(p \frown G \frown V^w_{1-\varepsilon}(p), z).$$

Now $w(p \frown CG, z) \leqq \omega(p \frown CG, z) = 0$ by (12) and $w(p \frown G \frown CV^w_{1-\varepsilon}(p), z) = 0$ by P.H.3. Hence

$$w(p, z) = w(p \frown G \frown V^w_{1-\varepsilon}(p), z). \tag{20}$$

Next $w(p \frown V^C_{M_0} \frown G, z) \leqq w(p \frown V^C_{M_0}, z) \leqq \omega(p \frown V^C_{M_0}, z) = 0$, whence

$$0 < w(p, z) = w(p \frown G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0}), z). \tag{20'}$$

Thus $\quad \upsilon_n(p) \frown G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0})$ *is non void for every* $\upsilon_n(p)$. (21)

$w(p, z) - w_{CG}(p, z) > 1 - M_0 - \varepsilon > 0$ in $G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0})$ (*non void*) by $w_{CG}(p, z) \leqq {}_{CG}\omega(p, z) < M_0$ in $V^C_{M_0}$. Hence

$$w(p, z) - w_{CG}(p, z) > 0. \tag{22}$$

Put $s_m(p) = \upsilon_m(p) \frown G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0})$. Then $s_m(p) \frown G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0}) \frown (G - s_m(p)) = 0$. Hence by P.H.1. and by (20′)

$$w(p, z) (= w(p \frown G \frown (V^w_{1-\varepsilon}(p) - V^C_{M_0}), z) = \lim_n w^1_n(z),$$

where $w_n^1(z)$ is a harmonic function in $(G-s_m(p))\frown(R_n-R_0)$ such that $w_n^1(z)=w(p,z)$ on $(\partial G+\partial s_m(p))\frown R_n$ and $w_n^1(z)=0$ on $\partial R_n\frown(G-s_m(p))$. Since $w_{CG}(p,z)$ is the positively least harmonic function in G with ${}_{CG}w(p,z)=w(p,z)$ on ∂G, by P.H.1. $w_{CG}(p,z)=\lim_n w_n^2(z)$, where $w_n^2(z)$ is a harmonic function in $(G-s_m(p))\frown R_n$ such that $w_n^2(z)=w_{CG}(p,z)$ on $(\partial G+\partial s_m(p))\frown R_n$ and $w_n^2(z)=0$ on $(G-s_m(p))\frown\partial R_n$. Hence $1>w_n^1(z)-w_n^2(z)=w(p,z)-{}_{CG}w(p,z)>1-M_0-\varepsilon$ on $s_m(p)$ and $w_m^1(z)-w_m^2(z)=0$ on $(G-s_m(p))\frown\partial R_n$.

Let $w_n^*(z)$ be a harmonic function in $(G-s_m(p))\frown R_n$ such that $w_n^*(z)=0$ on $\partial G+\partial R_n-s_m(p)$ and $w_n^*(z)=1$ on $\partial s_m(p)\frown R_n$. Then by the maximum principle

$$w_n^*(z)\geqq w_n^1(z)-w_n^2(z).$$

Let $n\to\infty$. Then $w(s_m(p),z,G)\geqq w(p,z)-w_{CG}(p,z)$. Let $m\to\infty$. Then by (22)

$$w(p\frown G\frown(V_{1-\varepsilon}^w(p)-V_{M_0}^C),z,G)\geqq w(p,z)-w_{CG}(p,z)>0. \tag{23}$$

By $\omega(p,z)\geqq w(p,z)$, $\omega'(p,z)=\omega(p,z)-{}_{CG}\omega(p,z)>1-M_0-\varepsilon$ in $V_{1-\varepsilon}^w(p)-V_{M_0}^C$ and $\omega'(p,z)=0$ on ∂G. Hence by the Dirichlet principle

$$D(\omega(p\frown G\frown(V_{1-\varepsilon}^w(p)-V_{M_0}^C),z,G)\leqq D(\omega(V_{1-\varepsilon}(p)-V_{M_0}^C,z,G))\leqq\frac{D(\omega'(p,z))}{(1-M_0-\varepsilon)^2}. \tag{24}$$

Hence by (23) and (24) $G\frown(V_{1-\varepsilon}^w(p)-V_{M_0}^C)$ is the required domain.

Proof of d). $\upsilon_m(p)$ contains p N-approximately by *b)* of Theorm 7, hence we have *d)*.

Proof of e). Suppose G and CG contain p N-approximately. Then there exist V and V' in G and CG respectively such that $\omega(p\frown V,z,G)>0$ and $0<\omega(p\frown V',z,CG)\leqq\omega(p\frown CG,z)$. On the other hand, by (12) $\omega(p\frown CG,z)=0$. This is a contradiction. Hence we have *e)*.

Proof of f). Since G_i contains p N-approximately, by *a)* there exists a domain $\widetilde{V}_i$ in G_i such that $\widetilde{V}_i=CV_{M_{0,i}}^C\frown V_{1-\varepsilon_i}(p)$ and $1-M_{0,i}-\varepsilon_i>0$ and

$$\infty>D(\omega(\widetilde{V}_i,z,G_i))\geqq D(\omega(p\frown\widetilde{V}_i,z,G_i))>0:\ i=1,2,\cdots,i_0.$$

By the Dirichlet principle

$$D(\omega(\widetilde{V}_1\frown V_{M_{0,i}}^C,z,\sum_1^{i_0}G_i))\leqq D(\omega)\widetilde{V}_1,z,G_1))<\infty,\ i=2,\cdots,i_0,$$

because $(\widetilde{V}_1\frown V_{M_{0,i}}^C)\subset\widetilde{V}_1$ and $\sum^{i_0}G_i\supset G_1$. Similarly

$$D(\omega(\widetilde{V}_1\frown CV_{1-\varepsilon_i}(p),z,\sum_1^{i_0}G_i))\leqq D(\omega(V_1,z,G_1))<\infty,\ i=2,\cdots,i_0.$$

Hence by the maximum principle

$$\omega(p\frown\widetilde{V}_1\frown V_{M_{0,i}}^C,z,\sum_1^{i_0}G_i)\leqq\omega(p\frown V_{M_{0,i}}^C,z)=0 \quad \text{by Theorem 7, } a) \text{ and}$$

$$\omega(p\frown\widetilde{V}_1\frown CV_{1-\varepsilon_i}(p), z, \sum_1^{i_0}G_i)\leqq\omega(p\frown CV_{1-\varepsilon_i}(p), z)=0 \text{ by P.C.3.}$$

Hence by $C\widetilde{V}_i\subset(V^C_{M_0,i}+CV_{1-\varepsilon_i}(p))$ and by the Dirichlet principle

$$D(\omega(p\frown\widetilde{V}_1\frown C\widetilde{V}_i, z, \sum_{i=1}^{i_0}G_i))\leqq D(\omega(\widetilde{V}_1, z, G_1))<\infty \text{ and } \omega(p\frown\widetilde{V}_1\frown C\widetilde{V}_i, z, \sum_1^{i_0}G_i)$$

$$\leqq\omega(p\frown V^C_{M_0,i}, z)+\omega(p\frown CV_{1-\varepsilon_i}(p), z)=0. \quad i=2\cdots, i_0.$$

Hence by P.C.5. $\omega(p\frown\widetilde{V}_1\frown(\sum_2^{i_0}C\widetilde{V}_i), z, \sum_1^{i_0}G_i)=0.$

On the other hand,. by $\widetilde{V}_1\frown\{(\bigcap_2^{i_0}\widetilde{V}_i)+\widetilde{V}_1\frown(\sum_2^{i_0}C\widetilde{V}_i)\}=\widetilde{V}_i$

$$\omega(p\frown\widetilde{V}_1\frown(\bigcap_2^{i_0}\widetilde{V}_i), z, \sum_1^{i_0}G_i)+\omega(p\frown\widetilde{V}_1\frown(\sum_2^{i_0}C\widetilde{V}_i), z, \sum_1^{i_0}G_i)$$

$$\geqq\omega(p\frown\widetilde{V}_1, z, \sum_1^{i_0}G_i)\geqq\omega(p\frown\widetilde{V}_1, z, G_1)>0. \quad \text{Hence}$$

$$\omega(p\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \sum_1^{i_0}G_i)>0 \text{ and } D(\omega(p\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \sum_1^{i_0}G_i))<\infty. \tag{25}$$

Put $\omega(z)=\min_i(\omega(\widetilde{V}_i, z, G_i))$. Then $D(\omega(z))\leqq\sum_1^{i_0}D(\omega(\widetilde{V}_i, z, G_i))<\infty$,

because $\left|\frac{\partial\omega(z)}{\partial x}\right|\leqq\max_i\left(\left|\frac{\partial\omega(\widetilde{V}_i, z, G_i)}{\partial x}\right|\right)$ and $\left|\frac{\partial\omega(z)}{\partial y}\right|\leqq\max_i\left(\left|\frac{\partial\omega(\widetilde{V}_i, z, G_i)}{\partial y}\right|\right)$.

Now $\omega(z)$ is a continuons function in $(\bigcap_1^{i_0}G_i)-(\bigcap_1^{i_0}\widetilde{V}_i)$ such that $\omega(z)=0$ on $\partial(\bigcap_1^{i_0}G_i)$ and $\omega(z)=1$ on $(\bigcap_1^{i_0}\widetilde{V}_i)$. Hence

$$0<D(\omega(\bigcap_1^{i_0}\widetilde{V}_i, z, \bigcap_1^{i_0}G_i))\leqq D(\omega(z))<\infty.$$

Whence by the Dirichlet principle and by $(\bigcap_1^{i_0}G_i)\subset(\sum_1^{i_0}G_i)$ and by (25)

$$\infty>D(\omega(\bigcap_1^{i_0}\widetilde{V}_i, z, \bigcap_1^{i_0}G_i)\geqq D(\omega(\upsilon_n(p)\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \bigcap_1^{i_0}G_i)$$

$$\geqq D(\omega(\upsilon_n(p)\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \sum_1^{i_0}G_i)>0.$$

Let $n\to\infty$. Then

$$\infty>D(\omega(\bigcap_1^{i_0}\widetilde{V}_i, z, \bigcap_1^{i_0}G_i))\geqq D(\omega(p\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \bigcap_1^{i_0}G_i))\geqq D(\omega(p\frown(\bigcap_1^{i_0}\widetilde{V}_i), z, \sum_1^{i_0}G_i)>0$$

$$\text{and} \quad \infty>D(\omega(p\frown(\bigcap_1^{i_0}\widetilde{V}_i), z_i), z, \bigcap_1^{i_0}G_i)>0.$$

Hence by a') $\bigcap_1^{i_0}G_i$ contains p N-approximately.

A sufficient condition for a point $B\in p_S$ to be a point of second kind.

Theorem 9. *Suppose that a domain G contains $p\in B_S$ N-approximately and let $\overline{G}$ be the closure of G. If $\overline{G}\frown(B-p)$ is an F_σ set of harmonic*

measure zero, p is a singular point of second kind.

Proof. By (16) we can find a domain $\tilde{V}$ in G such that

$$\infty > D(\omega(\tilde{V} \frown G, z, G)) \geqq D(\omega(p \frown \tilde{V}, z, G)) > 0.$$

Now $(\overline{G}-p) \frown B$ is an F_σ set of harmonic measure zero, i.e. $F_\sigma = \sum_i F^i$, : $w(F^i, z)=0$. Let $w(F^i_m, z)$ be H.M. of $F^i_m = E\left[z \in \overline{R} : \delta(z, F^i) \leqq \frac{1}{m}\right]$. Then for a given point z_0 in $G-(V \frown \upsilon_n(p))$, there exists a number $m(n)$ such that $w(F^i_{m(n)}, z_0) \leqq \frac{1}{2^n}$. Put $w^*(F^i, z) = \sum_{n=0}^{\infty} w(F^i_{m(n)}, z)$ and $w^*(F_\sigma, z) = \sum_{i=1}^{\infty} \frac{1}{2^i} w^*(F^i, z)$. Then $w^*(F_\sigma, z) \to \infty$ as $z \to F_\sigma$. Put $\Omega = E[z \in R : \eta w^*(F_\sigma, z) \leqq 2]$: $\eta > 0$. Then Ω does not tend to $(B-p) \frown G$. Hence for any given positive number η there exists a number l_η such that $\Omega \frown (R - R_{l_\eta} - \upsilon_n(p)) = 0$ for $l > l_\eta$. Let $w_{l_\eta, n}(z)$ be a harmonic function in $G \frown (R_{l_\eta} - \upsilon_n(p))$ such that $w_{l_\eta, n}(z) = 0$ on $(\partial G \frown R_{l_\eta}) + (\partial R_{l_\eta} \frown G) - \upsilon_n(p)$ and $w_{l_\eta, n}(z) = 1$ on $G \frown \partial \upsilon_n(p)$. Then by the maximum principle

$$\omega(\tilde{V} \frown \upsilon_n(p), z, G) - \eta w^*(F_\sigma, z) \leqq w_{l_\eta, n}(z) \leqq w(\upsilon_n(p), z, G),$$

where $w(\upsilon_n(p), z, G)$ is H.M. of $\upsilon_n(p)$ relative G.
Let $n \to \infty$ and then $\eta \to 0$. Then

$$0 < \omega(p \frown \tilde{V}, z, G) \leqq w(p, z, G) \leqq w(p, z).$$

Thus p is a singular point of second kind.

6. Image of a singular point of second kind on the unit circle.

Theorem 10. *a) Let $V_M(p) = E[z \in R : \omega(p, z) > M]$. Then $V_M(p) \downarrow$ as $M \uparrow 1$ and*

$$\lim_n w(\upsilon_n(p), z) = w(p, z) = \lim_{M\ 1} w(V_M(p), z).$$

b) Map the universal covering surface $(R-R_0)^\infty$ of $(R-R_0)$ onto $|\xi|<1$ by $\xi = f(z)$. Let $\omega(p, z)$ be C.P. of a singular point p of second kind and let E be the set where $\omega(p, z)$ has angular limits $=1$ almost everywhere. We call E the image of p. Then mes $E>0$ and $w(p, z) = w(E, \xi)$, where $w(E, \xi)$ is the harmonic measure of E in $|\xi|<1$. Let p_1 and p_2 be singular points of second kind: $p_1 \neq p_2$ and let E_i be its image. Then mes $(E_1 \frown E_2) = 0$. Hence the set of singular points of second kind is at most enumerable.

Proof of a). By Theorem 9. *d)* of P, for any given $V_M(p)(M<1)$, there exists a number n_0 such that $V_M(p) \supset (\upsilon_n(p) \frown R)$ for $n > n_0$, whence

$\lim_{M\uparrow 1} w(V_M(p), z) \geqq \lim_n w(\upsilon_n(p), z) = w(p, z)$. On the other hand, by Theorem 7. *c*) $\omega(C\upsilon_n(p) \frown V_M(p), z) \downarrow 0$ as $M \uparrow 1$.
This implies $w(C\upsilon_n(p) \frown V_M(p), z) \downarrow 0$ as $M \uparrow 1$. Hence by

$$w(V_M(p), z) \leqq w(V_M(p) \frown C\upsilon_n(p), z) + w(V_M(p) \frown \upsilon_n(p), z)$$

we have $\lim_{M=1} w(V_M(p), z) \leqq \lim_{M\ 1} w(V_M(p) \frown \upsilon_n(p), z) \leqq w(\upsilon_n(p), z)$ for every n.
Whence $\lim_{M\uparrow 1} w(V_M(p), z) \leqq \lim_n w(\upsilon_n(p), z) = w(p, z)$.
Thus we have *a*).

Proof of b). $f(\partial R_0)$ consists of an enumerably infinite number of arcs on $|\xi|=1$. Let $G(z, p)$ be the Green's function of $(R-R_0)$ and let E_g be the set on $|\xi|=1$ where $G(z, p)$ has angular limits$=0$ on $|\xi|=1$. Then mes $E_g=2\pi$. Put $\inf_{z\in\partial R_n} G(z, p)=\delta_n$. Then since ∂R_n is compact, $\delta_n>0$. Let E_0 be the set on $|\xi|=1$ such that $E_0 \frown f(\partial R_0)=0$ and $\omega(p, z)$ has angular limits<1.

Assume mes $E_0>0$. In this case, by Egoroff's theorem, for any given positive number ε we can find a closed set $E' \subset (E_0 \frown E_g)$ (by mes $E_g=2\pi$) such that mes $(E'-E_0)<\frac{\varepsilon}{2}$ and $\omega(p, z)$ converges uniformly in an angular domain $A(\theta)=E\left[\xi : \left|\arg \frac{\xi-e^{i\theta}}{e^{i\theta}}\right| < \frac{2\pi}{3}\right]$ for every point $e^{i\theta}$ in E' as $\xi=f(z)$ tends to E' in $A(\theta)$. And further we can find a closed set $E'' \subset E'$ and a constant $M'<1$ and a number m such that

$$\text{mes}\,(E_0-E'')<\varepsilon : \ \omega(p, z)<M' \text{ in } A(\theta) \frown \mathfrak{R}_m \text{ for } e^{i\theta} \in E'', \tag{26}$$

where $\mathfrak{R}_m$ is a ring domain: $1-\frac{1}{m}<|\xi|<1$.

Let D be a domain containing an end part of $A(\theta)$: $e^{i\theta} \in E''$ and bounded by $\sum_{e^{i\theta}\in E''} \partial A(\theta)+E''$ and a circle $\mathfrak{C}_{\frac{1}{2}}$: $|\xi|=\frac{1}{2}$. Then D may consist of at most three components. Hence there exists at least one component D' such that mes $(\partial D' \frown E'')>0$. Let D^* be one of them. For simplicity denote it by D newly. Let $w^D(\xi)$ be a harmonic function in D such that $w^D(\xi)=1$ on $\partial D-E''$ and $w^D(\xi)=0$ on E''. Then by the rectifiableness of ∂D $w^D(\xi)<1$ and $w^D(\xi)$ has angular limits$=0$ a.e. on E''.

Put $V_M(p)=E[z\in R : \omega(p, z)>M]$. Then by *a*) $w(p, z)=\lim_{M=1} w(V_M(p), z)$. Let $w_{M,n}(z)$ be a harmonic function in $(R_n-R_0-V_M(p))$ such that $w_{M,n}(z)=0$ on $\partial R_n-V_M(p)+\partial R_0$ and $w_{M,n}(p)=1$ on $\partial V_M(p)$.

Then $$\lim_{M=1} \lim_{n=\infty} w_{M,n}(z)=w(p, z)$$

Consider the image of $(R_n - R_0 - V_M(p))$ in D.

Since $\omega(p, z) < 1$ in $R - R_0$, $f(\partial V_M(p)) \to \Gamma_1 : |\xi| = 1$ as $M \to 1$. Hence there exists a number M_2 such that $f(V_M(p)) \subset \mathfrak{R}_m$ for $M > M_2$. Hence by (26)

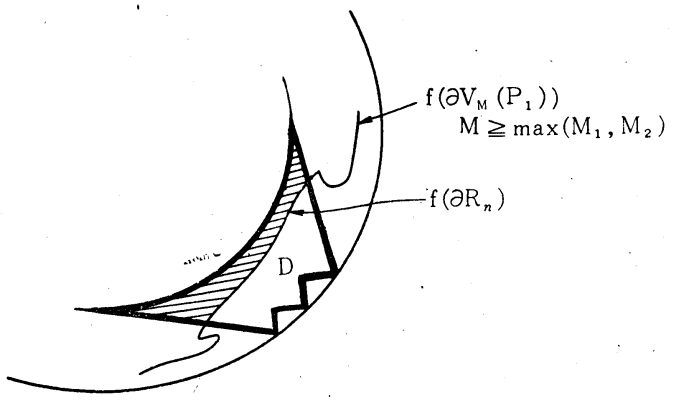

Fig. 5.

$$f(V_M(p)) \frown D = 0 \text{ for } M > M_3 = \max(M_1, M_2). \tag{27}$$

Since $\inf_{z \in \partial R_n} G(z, p) = \delta_n > 0, f(\partial R_n)$ does not tend to E'' in D, (28)

because, if it were so, there existed a sequence $\xi_k \in f(\partial R_n)$ such that $\xi_k \to E''$ as $k \to \infty$ inside of D. But by $E'' \subset E_g$ $G(\xi_k, p) \to 0$. This is a contradiction. Hence $f(\partial R_n)$ separates E'' from $\Gamma_m : |\xi| = 1 - \dfrac{1}{m}$ in D and $\overline{f(R_n - R_0)} \frown E'' = 0$.

Compare $w_{M,n}(z)$ and $w^D(\xi)$ in $D \frown f(R_n - R_0 - V_M(p)) : M > M_3$.
Then $w^D(\xi) = 1$ and $w_{M,n}(z) < 1$ on $\partial D \frown f(R_n - R_0 - V_M(p))$,

$$w^D(\xi) > 0 \text{ and } w_{M,n}(z) = 0 \text{ on } D \frown \partial(f(R_n - R_0 - V_M(p)) (\subset f(\partial R_n)).$$

Hence by the maximum principle

$$w^D(\xi) \geqq w_{M,n}(z) \text{ in } D \frown f(R_n - R_0 - V_M(p)).$$

Let $n \to \infty$ and then $M \uparrow 1$. Then $w^D(\xi) \geqq w(p, z)$.
Now D is an arbitrary component and let $\varepsilon \to 0$. Then $w(p, z)$ has angular limits$=0$ a.e. on E_0 ($w(p, z) = 0$ on $\partial f(\partial R_0)$).

Let E_1 be the set of $|\xi| = 1$ where $\omega(p, z)$ has angular limits$=1$. Similarly as above, we can find a closed set $E' \subset E_1$ and a domain D containing an endpart of $A(\theta) = E\left[\xi : \left|\arg \dfrac{\xi - e^{i\theta}}{e^{i\theta}}\right| < \dfrac{2\pi}{3}\right]$ for $e^{i\theta} \in E'$ and bounded by $\sum_{e^{i\theta} \in E'} \partial A(\theta) + E'$ such that

$$\text{mes}(E_1 - E') < \varepsilon \text{ and } \omega(p, z) \to 1 \text{ as } \xi = f(z) \to E' \text{ in } D. \tag{29}$$

Let D (newly denoted) be one of the components of D such that $\text{mes}\,(\overline{D}\frown E')>0$. Let $w^D(\xi)$ be a harmonic function in $\overline{D}$ such that $w^D(\xi)=0$ on $\partial D-E'$ and $w^D(\xi)=1$ on E'. Then $w^D(\xi)>0$ and $w^D(\xi)$ has angular limits$=1$ a.e. on E'. By (29) $f(\partial V_M(p))$: $M<1)$ does not tend to E' in D. Hence

$$\text{dist}\ (f(\partial V_M(p))\frown D),\ E')=\delta_M>0 \text{ and by (29)}$$

$$\overline{D\frown f(V_M(p))}\supset E'. \tag{30}$$

Since $f(\partial R_n)\to\Gamma_1$: $|\xi|=1$ as $n\to\infty$, there exists a number δ_M such that $\text{dist}\,(f(\partial R_n),\xi=0)>1-\delta_M$. Then by (30) $f(R_n-R_0V_M(p))\supset(D-f(V_M(p))$ for $n>n_M$. Hence as above we have $w^D(\xi)\leqq w_{M,n}(z)$.
Let $n\to\infty$ and then $M\to 1$, $w^D(\xi)\leqq w(p,z)$ and by letting $\varepsilon\to 0$, we see that $w(p,z)$ has angular limits$=1$ a.e. on E. Thus

$$w(p,z)=w(E,\xi) \text{ and mes } E>0 \text{ by } w(p,z)>0.$$

Assume $\text{mes}\,(E_1\frown E_2)>0$. Then as above it is proved that

$$w(E_1\frown E_2,\xi)=w(p_1\frown p_2,z)\ (\leqq\omega(p_1\frown p_2,z)).$$

Now $\omega(p_1\frown p_2,z)$ has mass only at p_1 (i.e. $\omega(p_1\frown p_2,z)=K\ \omega(p,z)$) by $(p_1\frown p_2)\subset p_1$. By P.C.2. $\sup\limits_{z\in R}\omega(p_1\frown p_2,z)=1=\sup\limits_{z\in R}\omega(p_1,z)$. Hence $\omega(p_1\frown p_2,z)=\omega(p_1,z)$. Similarly $\omega(p_1\frown p_2,z)=\omega(p_2,z)$. This implies $N(z,p_1)=N(z,p_2)$. This contradicts $p_1\neq p_2$. Hence $\text{mes}\,(E_1\frown E_2)=0$. Let p_i be a singular point of second kind and let E_i be its image. Then $\text{mes}\,(E_i\frown E_j)=0$ for $i\neq j$. Hence by $\sum\text{mes}\,E_i<2\pi$ the set of singular points of second kind is at most enumerable.

Remark. Toplopical structure of B_S. Let B_n the set of singular points p_i such that $\text{Cap}(p_i)\geqq\dfrac{1}{n}$ and p be a limit point of a sequence $\{p_i\}\in B_n$. Then by $\upsilon_m(p)\ni p_i$ $\text{Cap}(p_i)\leqq\text{Cap}(\upsilon_m(p))$ for every m. Hence $\text{Cap}(p)=\lim\limits_{m=\infty}\text{Cap}(\upsilon_m(p))\geqq\dfrac{1}{n}$ and B_n is closed. Whence *by* $B_S=\bigcup B_n$, B_S *is an* F_σ *set.*

By Theorem 10 the set of singular points of second kind is enumerable, but the set of singular points of first kind is not necessarily enumerable. In reality there exists a Riemann surface which has the set of singular points of first kind of the power of continuum (see the following paper "Examples of singular points").

7. Harmonic functions in a neighbourhood of a singular point of second kind. Harmonic domain and their harmonic measures. Map the universal convering surface G^∞ of a domain G in $R-R_0$ onto $|\xi|<1$

by $\xi=f(z)$ conformally. If a harmonic function $H(z)$ in G has angular limits a.e. on $|\xi|=1$, we say that $H(z)$ *is of F-type. Clearly if* $H(z)\in HB$ *or* $\in HD$, $H(z)$ *is of F-type.*

Consider a system of harmonic functions $H_i(z)$ of F-type $(i=1,2,\cdots l)$. If a domain D^H is defined as $D^H=\bigcap_{i=1}^{l} E[z\in G:\ H_i(z)\geqq a_i]$, we call D^H *a harmonic domain in* G. Let B_i be the set on $|\xi|=1$ where $H_i(z)$ has angular limits$\geqq a_i$. Put $B(D^H)=\bigcap^{l} B_i$. We call $B(D^H)$ *the image of the ideal boundary of* D^H. Let $D_k^H=\bigcap^{l} E\left[z\in G:\ H_i(z)\geqq a_i-\dfrac{1}{k}\right]$. Then $D^H=\bigcap_k D_k^H$.

Put $$w(p\frown[D^H],z,G)=\lim_n \lim_k w(\upsilon_n(p)\frown D_k^H,z,G),$$
where $w(\upsilon_n(p)\frown D_k^H,z,G)$ is H.M. of $\upsilon_n(p)\frown D_k^H$ relative G, i.e. $w(\upsilon_n(p)\frown D_k^H, z,G)$ is the least positive superharmonic function in G larger than 1 on $\upsilon_n(p)\frown D_k^H$.
In the following we distiguish it from $w(p\frown D^H,z,G)=\lim_n w(\upsilon_n(p)\frown D^H,z,G)$ by [ ]. Then it is clear
$$w(p\frown D_k^H,z,G)\geqq w(p\frown[D^H],z,G)\geqq w(p\frown D^H,z,G).$$

Theorem 11. *a) Let* D_k^H *be a harmonic domain in* $G\subset(R-R_0)$. *Put* $\tilde{V}_M^W=E[z\in G:\ w(p\frown[D^H],z,G)>M]$. *Then* $\tilde{V}_M^W\subset V_M^W(p)=E[z\in R:\ w(p,z)>M]\subset V_M(p)=E[z\in R:\ \omega(p,z)>M]$ *by* $w(p\frown[D^H],z,G)\leqq w(p,z)\leqq\omega(p,z)$. *Then for any domain* G' *and* G'' *such that* $G'\subset G$ *and* $G''\subset G$.
$$\lim_{M=1}\lim_{k=\infty} w(\tilde{V}_M^W\frown G''\frown D_k^H,z,G')=\lim_{M=1} w(\tilde{V}_M\frown[D^H]\frown G'',z,G').$$

b) Let G *be a domain in* $R-R_0$ *and* D^H *be a harmonic domain in* G. *Let* E *be the set of positive measure where* $w(p\frown[D^H],z,G)$ *has angular limits*$=1$ *on* $|\xi|=1$ *and let* $B(D^H)$ *be the image of the ideal boundary of* D^H. *Then*
$$\mathrm{mes}(E-B(D))=0,$$
and
$w(p\frown[D^H],z,G)=\lim_{M=1} w(\tilde{V}_M^W,z,G)=\lim_{M=1} w(\tilde{V}_M^W\frown[D^H],z,G)=(\lim_{M=1}\lim_k w(\tilde{V}_M^W\frown D_k^H,z,G))=w(E,\xi)$, *where* $w(E,\xi)$ *is the harmonic measure of* E *with respect to* $|\xi|<1$.

c) Let D^H *be a harmonic domain in* $G\subset R-R_0$. *Let* $U(z)$ *be a harmonic function of F-type in* G *such that* $U(z)$ *has angular limit* $U(e^{i\theta})$ *at* $e^{i\theta}$ *which is not a constant a.e. on* E, *where mes* $E>0$ *and* $w[p\frown[D^H],z,G)$ *has angular limits*$=1$ *a.e. on* E. *Then we can find two constants*

L and δ such that

$$w[p\frown[D^H]\frown G^U_{L+\delta}, z, G^U_L\frown G>0 \text{ and } w(p\frown[D^H]\frown G^L_{L-\delta}, z, G^L_L\frown G)>0,$$

where $G^U_a=E[z\in G:\ U(z)>a]$ *and* $G^L_a=Ez\in G:\ U(z)<a]$.
Whence $w(p\frown D^H_k\frown G^U_{L+\delta}, z, G^U_L\frown G)>0$ *and* $w(p\frown D^H_k\frown G^L_{L-\delta}, z, G\frown G^L_L)>0$ for $k<\infty$.

Proof of a). $w(\widetilde{V}^W_M\frown C\upsilon_n(p)\frown[D^H]\frown G'', z, G')\leqq w(V^W_M(p)\frown C\upsilon_n(p), z)$ $\leqq\omega(V_M(p)\frown C\upsilon_n(p), z)\to 0$ as $M\to 1$ by c) of Theorem 5.
By $w(\widetilde{V}^W_M)\frown C\upsilon_n(p)\frown[D^H]\frown G'', z, G')+w(\widetilde{V}^W_M\frown\upsilon_n(p)\frown[D^H]\frown G'', z, G')$ $\geqq w(\widetilde{V}^W_M\frown[D^H]\frown G'', z, G')$, we have by letting $M\to 1$
$\lim\limits_{M\to 1} w(\widetilde{V}^W_M\frown[D^H]\frown G'', z, G')=\lim\limits_{M\to 1} w(\widetilde{V}^W_M\frown[D^H]\frown\upsilon_n(p)\frown G'', z, G')\leqq w(\upsilon_n(p)$ $\frown[D^H]\frown G'', z, G')$. Let $n\to\infty$. Then

$$\lim_{M\to 1} w(\widetilde{V}^W_M\frown[D^H]\frown G'', z, G')\leqq w(p\frown[D^H]\frown G'', z, G').$$

Proof of b). Let $G(z, p)$ be the Green's function of G and let E_g be the set where $G(z, p)$ has angular limits$=0$. Then mes $E_g=2\pi$. Put Γ_1: $|\xi|=1$. If mes $E_C>0$: $E_C=\Gamma_1-(E\frown E_g\frown B(D^H))$, for any given positive number ε, we can find a closed set E' of positive measur in $(E_C\frown E_g)$ and a constant $\delta>0$ such that mes $(E_C-E')<\dfrac{\varepsilon}{2}$ and at least one of $w(p\frown[D^H]$, $z, G)$ and $H_i(z)(i=1, 2,\cdots, l)$ has angular limits$<1-\delta$ or $<a_i-\delta$ as $f(z)$ $=\xi\to E'$ along Stolz's path. Further we can find two closed sets $E^{(1)}$ and $E^{(2)}$ in E' such that $E^{(1)}\frown E^{(2)}=0$, mes $(E'-(E^{(1)}+E^{(2)}))<\varepsilon$ and domains $D^{(1)}$ and $D^{(2)}$ such that $D^{(i)}(i=1, 2)$ containing an endpart of $A(\theta)=E\left[\xi:\ \left|arg\,\dfrac{\xi-e^{i\theta}}{e^{i\theta}}\right|<\dfrac{2\pi}{3}\right]$ for $e^{i\theta}\in E^{(i)}$ and bounded by $\sum\limits_{e^{i\theta}\in E^{(i)}}\partial A(\theta)+E^{(i)}$ and a ring $\mathfrak{R}_m$: $1-\dfrac{1}{m}<|\xi|<1$ such that $w(p\frown[D^H], G)<1-\dfrac{\delta}{2}$ in $D^{(1)}$ $\frown\mathfrak{R}_m$ and $H_{i_0}(z)<a_{i_0}-\dfrac{\delta}{2}$ in $D^{(2)}\frown\mathfrak{R}_m$ for a number i_0 respectively. Hence $f(\widetilde{V}^W_M)\frown D^{(1)}\frown\mathfrak{R}_m=0$ and $f(D^H_k)\frown D^{(2)}\frown\mathfrak{R}_m=0$ for $M>1-\dfrac{\delta}{2}$ and $\dfrac{1}{k}<\dfrac{\delta}{2}$,

$$\text{where } D^H_k=\bigcap_{i=1}^{l}E\left[z\in G:\ H_i(z)\geqq a_i-\frac{1}{k}\right]. \tag{a}$$

Now $D^{(i)}$ is composed of a finite number of components. Let D' be one of them such that mes$(\overline{D'}\frown E^{(1)})>0$. Now $w(p\frown[D^H], z, G)=\lim\limits_{M\to 1} M\ w(\widetilde{V}^W_H,$ $z, G)=\lim\limits_{M\to 1} w(\widetilde{V}^W_H, z, G)$ (by P.H.4.) and $=\lim\limits_{n}\lim\limits_{k}(\upsilon_n(p)\frown D^H_k, z, G)$. Put G_j

$=E\left[z\in G:\ \delta(\partial G_j, z)>\frac{1}{j}\right]$. Let $w_{M.j.n}(z)$ be a harmonic function in $(R_n \frown G_j)-\widetilde{V}_M^W$ such that $w_{M.n.j}(z)=0$ on $\partial G_j+\partial R_n-\widetilde{V}_M^W$ and $w_{M.j.n}(z)=1$ on $\partial\widetilde{V}_M^W$. Consider the image of $(G_i\frown R_n)-\widetilde{V}_M^W$. Then by $E^{(1)}\subset E_g$, image of $(\partial G_j+\partial R_n)$ separates $\Gamma_{1-\frac{1}{m}}:\ |\xi|=1-\frac{1}{m}$ from $E^{(1)}$ in D', (b)

because $$G(z, p)>\delta_{j.n}>0 \text{ on } \partial G_j+\partial R_n.$$

Hence as usual (see the proof of Theorem 8) by a) and b) and by the maximum principle

$$w_{M.j.n}(z)\leqq w^D(\xi),$$

where $w^D(\xi)$ is a harmonic function in D' such that $w^D(\xi)=1$ on $\partial D'-E^{(1)}$ and $w^D(\xi)=0$ on $\partial D'\frown E^{(2)}$. Let $i\to\infty$ and then $n\to\infty$ and then $M\to 1$. Then

$$w(p\frown[D^H], z, G)=\lim_M w(\widetilde{V}_M^W, z, G)\leqq w^D(\xi). \tag{c}$$

Since $w^D(\xi)$ has angular limits$=0$ a.e. on $E^{(1)}$ and D' is arbitrary, $w(p\frown[D^H], z, G)$ has angular limits$=0$ a.e. on $E^{(1)}$ by letting $\varepsilon\to 0$.

Let $w_{k.j.n}(z)$ be a harmonic function in $(R_n\frown G_j)-D_k^H$ such that $w_{k.j.n}(z)=0$ on $\partial R_n+\partial G_j-D_k^H$ and $w_{k.j.n}(z)=1$ on ∂D_k^H. Then the image of $(G_j\frown R_n)-\partial D_k^H$ separates $|\xi|=1-\frac{1}{m}$ from $E^{(2)}$ and as above

$$w_{k.i.n}(z)\leqq w^D(\xi) \text{ and } w(p\frown[D^H], z, G)\leqq w^D(\xi), \tag{d}$$

where $w^D(\xi)$ is a harmonic function in $D^{(2)}$ such that $w^D(\xi)=1$ on $\partial D^{(2)}-E^{(2)}$ and $w^D(\xi)=0$ on $\partial D^{(2)}\frown E^{(2)}$. Let $\varepsilon\to 0$. Then by (c) and (d) $w(p\frown[D^H], z, G)$ has angular limits$=0$ a.e. on E_C. Hence mes $(E-B(D^H))=0$, because if mes $(E-B(D^H))>0$, we can find a subset of $E-B(D^H)$ of positive measure on which $w(p\frown[D^H], z, G)$ has angular limits$=0$. This contradicts the definition of E on which $w(p\frown[D^H], z, G)$ has angular limits$=1$ almost everywhere.

Now by mes E_C+mes $E=2\pi$ $w(p\frown[D^H], z, G)$ has angular limits$=1$ on E. Hence

$$w(p\frown[D^H], z, G)=w(E, \xi)=\lim_{M\to 1} w(\widetilde{V}_M^W, z, G).$$

$\lim_{M=1} w(\widetilde{V}_M^W, z, G)\geqq\lim_{M=1} w(\widetilde{V}_M^W\frown[D^H], z, G)$ is clear. We show $\lim_{M=1} w(\widetilde{V}_M^W, z, G)=\lim_{M=1} w(\widetilde{V}_M^W\frown[D^H], z, G)=\lim_M\lim_k w(\widetilde{V}_M^W\frown[D_k^H], z, G)$. By mes $(E-B(D^H))=0$, mes $E=$mes $(E\frown B(D^H))$. If $w(E, \xi)=0$, mes $(E\frown B(D^H))=0$.

We have $0=\lim_{M=1} w(\widetilde{V}_M^W, z, G)\geqq\lim_{M=1} w(\widetilde{V}_M^W\frown[D^H], z, G)$ and we have our

assertion. Hence we assume mes $(E\frown B(D^H))>0$. In this case, for any given positive number ε, we can find a closed set $E'\subset(E\frown B(D^H))$ such that mes $((E\frown B(D^H)-E')<\varepsilon$ and a domain D $\Big(D$ is containing and endpart of $A(\theta)$: $e^{i\theta}\in E'$ and bounded by $\sum_{e^{i\theta}\in E'}\partial A(\theta)+E'$ and a circle: $|\xi|=\frac{1}{2}\Big)$ and bounded a number m_0 such that

$$w(p\frown[D^H], z, G)\geqq 1-\frac{1}{2k},\quad H_i(z)\geqq a_i-\frac{1}{2k}\,(i=1,2,\cdots,l) \quad \text{in} \quad D\frown\mathfrak{R}_{m_0}:$$

$\mathfrak{R}_{m_0}=E\Big[[\xi:\ 1-\frac{1}{m_0}<|\xi|<1\Big]$. Hence

$$\text{dist}\ (f(\widetilde{V}^W_{1-\frac{1}{k}}+\partial D^H_k)\frown D, E')=\delta_k>\frac{1}{m_0}>0 \text{ and } f(\widetilde{V}^W_{1-\frac{1}{k}}\frown D^H_k)\supset(D\frown\mathfrak{R}_{m_0}). \quad (a)$$

Since the image of $\partial G_j: G_j=E\Big[z\in G:\ \delta(\partial G, z)>\frac{1}{j}\Big]$ and ∂R_n tend to $|\xi|=1$ as $n\to\infty$ and $j\to\infty$, there exist numbers n_0 and j_0 such that

$$\text{dist}\ (f(\partial R_n)+f(\partial G_j), \xi=0)>1-\delta_k \text{ for } n\geqq n_0 \text{ and } j\geqq j_0. \quad (b)$$

Let D' be a component of $D\frown\mathfrak{R}_m$ such that mes $(\partial D'\frown E')>0$. Then by (b) $f(\partial R_n+\partial G_j)$ does not fall in D' for $n\geqq n_0$ and $j\geqq j_0$.

Let $w^D(\xi)$ be a harmonic function in D' such that $w^D(\xi)=0$ on $\partial D'-E'$ and $w^D(\xi)=1$ on E'. Then by (a) and by the maximum principle

$$w^D(\xi)\leqq w_{n.j.k}(z), \text{ for } n\geqq n_0 \text{ and } j\geqq j_0,$$

where $w_{n.j.k}(z)$ is a harmonic function in $(R_n\frown G_j)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown D_k)$ such that $w_{n.j.k}(z)=0$ on $\partial G_j+\partial R_n-(\widetilde{V}^W_{1-\frac{1}{k}}\frown D^H_k)$ and $w_{n.j.k}(z)=1$ on $\partial(\widetilde{V}^W_{1-\frac{1}{k}}\frown D^H_k)$. Let $n\to\infty$ and $j=\infty$ and then $k\to\infty$. Then $w^D(\xi)\leqq\lim_{M=1}\lim_{k=\infty} w(\widetilde{V}^W_M\frown D^H_k, z, G)$ $\leqq\lim_{M=1} w(\widetilde{V}^W_M\frown D^H_k, z, G)$. But D' is arbitrary and by letting $\varepsilon\to 0$, $\lim_{M=1} w(\widetilde{V}^W_H\frown[D^H], z, G)$ has angular limits$=1$ a.e. on $(E\frown B(D^H))$. On the other hand, $w(E,\xi)=\lim w(\widetilde{V}^W_H, z, G)$ $(\geqq\lim w(\widetilde{V}^W_H\frown[D^H], z, G))$ has angular limits$=0$ a.e. on CE. Thus

$$\lim_{M=1} w(\widetilde{V}^W_M, z, G)=\lim_{M=1} w(\widetilde{V}^W_M\frown[D^H], z, G).$$

Proof of c). Since $U(e^{i\theta})\neq$const a.e. on E, we can find two constants L and $\delta>0$ and two sets E_1 and E_2 of positive measure in E such that the angular limits $U(e^{i\theta})\geqq L+2\delta$ on E_1 and $U(e^{i\theta})<L-2\delta$ on E_2 respectively. For any given positive number ε we can find a closed subset E'

of E_1 and a domain D such that $\mathrm{mes}\,(E_1-E')<\varepsilon$ and D is containing an endpart of $A(\theta)$: $e^{i\theta}\in E'$ and bounded by $E'+\sum_{e^{i\theta}\in E'}\partial A(\theta)$ with the property as follows:

$$U(\xi)\to\geqq L+2\delta \text{ as } \xi\to E' \text{ in } D. \tag{a}$$

$$w(p\frown[D^H], z, G)\to 1 \text{ and } H_i(z)\to\geqq a_i (i=1,2,\cdots,l) \text{ as } \xi\to E' \text{ in } D. \tag{b}$$

Put $G_L^U=E[z\in G:\ U(z)>L]$. Since $U(z)\to\geqq L+2\delta$ as $\xi\to E'$, $f(\partial G_L^U)$ does not tend to E' in D, whence $\mathrm{dist}\,(f(\partial G_L^U)\frown D, E')>\delta_U>0$. Let $\mathfrak{R}_m$ be a ring domain: $1>|\xi|>1-\dfrac{1}{m}$. Then

$$U(z)\geqq L \quad \text{in } D\frown\mathfrak{R}_m \text{ for } \frac{1}{m}<\delta_U. \tag{c}$$

$D\frown\mathfrak{R}_m$ is composed of at most $n(m)$ number of components. Let D' be a component such that $\mathrm{mes}\,(E'\frown\overline{D'})>0$ and let G_L' be the component of G_L^U such that there exists a point ξ in D' with $f(z)=\xi$: $z\in G_L'$. We denote it also by G_L.

By $\mathrm{mes}\,(E-B(D^H))=0$, $w(p\frown[D^H], z, G)\to 1$ and $H_i(z)\to\geqq a_i$ as $\xi\to E'$ in D', $\mathrm{dist}\,(f(\partial\widetilde{V}^W_{1-\frac{1}{k}}+\partial D_k^H)\frown D', E')=\delta_k>0$ for $k>0$ and since $U(z)\to\geqq L+2\delta$ as $\xi\to E'$,

$$\mathrm{dist}\,(f(\partial G_{L+\delta}^U)\frown D', E')=\delta_L'>0. \tag{d}$$

Let $G_j=E\left[z\in G:\ \delta(z,\partial G)>\dfrac{1}{j}\right]$. Then since $f(\partial R_n+\partial G_j)\to|\xi|=1$ as $n\to\infty$ and $j\to\infty$, there exist numbers n_0 and j_0 such that

$$\mathrm{dist}\,(f(\partial R_n+\partial G_j)\frown D', \xi=0)>1-\delta:\ \delta=\min\,(\delta_U, \delta_L', \delta_k) \text{ for } n\geqq n_0 \text{ and } j\geqq j_0. \tag{e}$$

Let $w_{j.n.k}(z)$ be a harmonic function in $(R_n\frown G_j\frown G_L)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^L\frown D_k^H)$ such that $w_{j.n.k}(z)=0$ on $\partial(\partial G_k\frown R_n\frown G_L^U)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^U\frown D_k^H)$ and $w_{j.n.k}(z)=1$ on $\partial(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^U\frown D_k^H)$. Then by (e) for $n\geqq n_0$ and $j\geqq j_0$ the image of $(\partial R_n+\partial G_j)$ does not fall in D' and by (c) ∂G_L^U does not fall in D' and also by (d) $\partial(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^U\frown D_k^H)$ separates E' from $|\xi|=1-\dfrac{1}{m}$.

Let $w^D(\xi)$ be a harmonic function in D' such that $w^D(\xi)=0$ on $\partial D'-E'$ and $w^D(\xi)=1$ on $E'\frown\partial D'$. Consider $w^D(\xi)$ and $w_{j.n.k}(z)$ in $D'\frown f(R_n\frown G_j\frown G_L^U-(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^U\frown D_k^H))$. Then $0=w^D(\xi)\leqq w_{j.n.k}(z)$ on $\partial D'-E'$ and $w^D(\xi)<1=w_{j.n.k}(z)$ on $D\frown f(\widetilde{V}^W_{1-\frac{1}{k}}\frown G_{L+\delta}^U\frown D_k^H)$. Hence by the maximum principle

$$w^D(\xi)\leqq w_{j.n.k}(z).$$

Let $j\to\infty$ and $n\to\infty$ then

$$w^D(\xi)\leqq w(\widetilde{V}^W_{1-\frac{1}{k}}\frown G^U_{L+\delta}\frown D^H_k,\ z,\ G^U_L\frown G)=\lim_n\lim_j w_{j.n.k}(z).$$

Hence by letting $k\to\infty$ and letting $\varepsilon\to 0$, $\lim_{k=\infty} w(\widetilde{V}^W_{1-\frac{1}{k}}\frown D^H_k\frown G^U_{L+\delta},\ z,\ G^U_L\frown G)$ has angular limits$=1$ a.e. on E_1, since $w^D(\xi)$ has angular limits$=1$ a.e. on E_1. Thus by Theorem 11. *a*)

$$0<\lim_{M=1}\lim_{k=\infty} w(\widetilde{V}^W_M\frown D^H_k\frown G^U_{L+\delta},\ z,\ G^U_L\frown G)\leqq w(p\frown[D^H]\frown G^U_{L+\delta},\ z,\ G^U_L\frown G).$$

Similarly $\qquad w(p\frown[D^H]\frown G^L_{L-\delta},\ z,\ G^L_L\frown G)>0$, whence

$$0<w(p\frown D^H_k\frown G^U_{L+\delta},\ z,\ G^U_L\frown G) \text{ and } 0<w(p\frown D^H_k\frown G^L_{L-\delta},\ z,\ G^L_L\frown G).$$

Theorem 12. *a*) *Let p be a singular point of second kind and suppose a domain G contains p N-approximately. Then there exists by Theorem 8. c) a harmonic domain $D^H_k=E\left[z\in G:\ \omega(p,z)>1-\varepsilon-\frac{1}{k}\right]\frown E\left[z\in G:\ -{}_{CG}\omega(p,z)>-M_0-\frac{1}{k}\right]$ such that $0<w(p\frown[D^H],z,G)\leqq\omega(p\frown D^H_k,G)$ and $D(\omega(D^H_k,z,G))<\dfrac{D(\omega(p,z)-{}_{CG}\omega(p,z))}{\left(1-M_0-\frac{2}{k}\right)^2}<\infty$ for $1-M_0-\frac{2}{k}>0$. Then there exists no function $U(z)\in HD$ (HD means the class of harmonic function with bounded Dirichlet integral) with constants L and δ such that $w(p\frown D^H_k,G^U_{L+\delta},z,G^U_L\frown G)>0$ and $w(p\frown D^H_k\frown G^L_{L-\delta},z,G\frown G^L_L)$, where $G^U_a=E[z\in G:\ U(z)>a]$ and $G^L_a=E[z\in G:\ U(z)<a]$.*

b) *Let G be the domain in a) and let D^H_k be also the harmonic domain in a). Map the universal covering surface G^∞ of G onto $|\xi|<1$. Let E be the set where $w(p\frown[D^H],z,G)$ has angular limits$=1$ almost everywhere. Then* mes $E>0$ *by $0<w(p\frown[D^H],z,G)=w(E,\xi)$ by Theorem* 11. *b). Let $U(z)\in HD$. Then $U(z)$ has angular limits$=$const a.e. on E.*

c) *As a special case of b), map the universal covering surface $(R-R_0)^\infty$ of $(R-R_0)$ onto $|\xi|<1$. Let E be the image of a singular point of second kind. Then $U(z)\in HD$ has angular limits$=$const a.e. on E.*

Proof of a). Let $\omega'_{m,n,n+i}(z)$ be a harmonic function in $(G\frown G^U_L\frown R_{n+i})-((R_{n+i}-R_n)\frown G^U_{L+\delta}\frown D^H_k\frown\upsilon_m(p))$ such that $\omega'_{m,n,n+i}(z)=0$ on $\partial G^U_L\frown G\frown R_{n+i}$, $\omega'_{m,n,n+i}(z)=1$ on $\partial((R_{n+i}-R_n)\frown G^U_{L+\delta}\frown D^H_K\frown\upsilon_m(p))=\alpha$ and $\frac{\partial}{\partial n}\omega'_{m,n.n+i}(z)=0$ on $(\partial G\frown R_{n+i})+(\partial R_{n+i}-(G^U_{L+\delta}\frown D^H_K\frown\upsilon_m(p))$. Then

$$\frac{U(z)-L}{\delta}\geqq 1\geqq\omega'_{m,n,n+i}(z) \text{ and } \frac{\partial}{\partial n}\omega'_{m,n,n+i}(z)\geqq 0 \text{ by } \omega'_{m,n,n+i}(z)=1$$

$$=\max_{z\in G}\omega'_{m,n,n+i}(z) \text{ on } \alpha.$$

$$\frac{U(z)-L}{\delta}\leqq 0=\omega'_{m,n,n+i}(z) \text{ and } \frac{\partial}{\partial n}\omega'_{m,n,n+i}(z)\leqq 0 \text{ by } \omega'_{m,n,n+i}(z)$$
$$=\min_{z\in G}\omega'_{m,n,n+i}(z) \text{ on } \partial G_L^U\frown G\frown R_{n+i}.$$

Hence $D\left(\frac{U(z)-L}{\delta}-\omega'_{m,n,n+i}(z),\ \omega'_{m,n,n+i}(z)\right)\geqq 0$, whence

$$D(\omega'_{m,n,n+i}(z))\leqq\frac{1}{\delta^2}D(U(z))<\infty. \tag{31}$$

$D(\omega(D_k^H, z, G))<M<\infty$ implies $D(\omega_{m,n,n+i}(z))\leqq M<\infty$ for every n and m and i, where $\omega_{m,n,n+i}(z)$ is a harmonic function in $(G\frown G_{n+j})-(\upsilon_m(p)\frown D_k^H\frown(R_{n+i}-R_n))$ such that $\omega_{m,n,n+i}(z)=0$ on $\partial G\frown R_{n+i}$, $\frac{\partial}{\partial n}\omega_{m,n,n+i}(z)=0$ on $\partial R_n-(\upsilon_m(p)\frown D_k^H)$ and $\omega_{m,n,n+i}(z)=1$ on $(\upsilon_m(p)\frown D_k^H)\frown(R_{n+i}-R_n)$. Put $\tilde{\omega}_{m,n,n+i}(z)=\min(\omega'_{m,n,n+i}(z),\ \omega_{m,n,n+i}(z))$. Then

$$D(\tilde{\omega}_{m,n,n+i}(z))\leqq D(\omega'_{m,n,n+i}(z))+D(\omega_{m,n,n+i}(z))\leqq M+\frac{1}{\delta^2}D(U(z))<\infty. \tag{32}$$

Let $\omega''_{m,n,n+i}(z)$ be a harmonic function in $(R_{n+i}\frown G_L^U)-(D_k^H\frown\upsilon_n(p)\frown G_{L+\delta}^U\frown(R_{n+i}-R_n))$ such that $\omega''_{m,n,n+i}(z)=0$ on $\partial G+(\partial G_L^U\frown R_{n+i})$, $\frac{\partial}{\partial n}\omega''_{m,n,n+i}(z)=0$ on $\partial R_{n+i}-(D_k^H\frown\upsilon_m(p)\frown(G_{L+\delta}^U\frown(R_{n+i}-R_n))$ and $\omega''_{m,n,n+i}(z)=1$ on $D_k^H\frown\upsilon_m(p)\frown(R_{n+i}-R_n)\frown G_{L+\delta}^U$. Since $\omega''_{m,n,n+i}(z)$ has M.D.I. among all functions with value 1 on $G_{L+\delta}^U\frown D_K^H\frown\upsilon_m(p)\frown(R_{n+i}-R_n)$ and $=0$ on $\partial G+(\partial G_L^U\frown G)$,

$$D(\omega''_{m,n,n+i}(z))\leqq D(\tilde{\omega}_{m,n,n+i}(z))\leqq M+\frac{1}{\delta^2}D(U(z))<\infty$$

and

$$\lim_{m=\infty}\lim_{n=\infty}\lim_{i=\infty}\omega''_{m,n,n+i}(z)=\omega(p\frown D_k^H\frown G_{L+\delta}^U, z, G_L^U\frown G) \text{ exists.} \tag{32'}$$

As above we have also the existence of $\omega(p\frown D_K^H\frown G_{L-\delta}^L, z, G_L^L\frown G)$. Suppose $w(p\frown D_k^H\frown G_{L+\delta}^U, z, G_L^U\frown G)>0$ and $w(p\frown D_k^H\frown G_{L-\delta}^L, z, G_L^L\frown G)>0$. Clearly

$$\omega(p\frown D_k^H\frown G_{L+\delta}^U, z, G_L^U\frown G)\geqq w(p\frown D_k^H\frown G_{L+\delta}^U, z, G_L^U\frown G)>0.$$
$$\omega(p\frown D_k^H\frown G_{L-\delta}^L, z, G_L^L\frown G)\geqq w(p\frown D_k^H\frown G_{L-\delta}^L, z, G_L^L\frown G)>0. \tag{33}$$

Now $G_L^U\frown G_L^L=0$ and $\omega(p\frown D_k^H\frown G_{L+\delta}^U, z, G_L^U\frown G)$ and $\omega(p\frown D_k^H\frown G_{L-\delta}^L, z, G_L^L,\frown G)$ can be consiered as C.P.s. defined by sequences $\{(\upsilon_n(p)\frown D_k^H\frown G_{L+\delta}^U)\}$ and $\{(\upsilon_n(p)\frown D_k^H\frown G_{L-\delta}^L)\}$ $(n=1, 2,\cdots)$ respectively. Hence these have properties from C.P.1. to C.P.6. Hence the Separation Theorem S. 1 is applicable by putting $\tilde{G}=R-R_0$ and $B'=p$, whence

$$\omega(p\frown D_k^H\frown G_{L+\delta}^U, z)\neq\omega(p\frown D_k^H\frown G_{L-\delta}^L, z). \tag{34}$$

On the ther hand, $\omega(p\frown D_k^H\frown G_{L+\delta}^U, z)$ and $\omega(p\frown D_k^H\frown G_{L-\delta}^L, z)$ have their masses only on $\bigcap_{n=1}^{\infty}\bar{\upsilon}_n(p)=p$ by $\upsilon_n(p)\supset(\upsilon_n(p)\frown D_k^H\frown G_{L+\delta,L-\delta}^{U,L})$. But by

$\sup \omega(p \frown D_k^H \frown G_{L+\delta}^U, z) = 1 = \sup \omega(p \frown D_k^H \frown G_{L-\delta}^L, z)$, these are equal to $\omega(p, z)$. This contradicts (34). Hence we have the theorem.

Proof of b). Assume angular limits $U(e^{i\theta}) \neq$const a.e. on E, by Theorem 11. *c)* we can find constants L and δ and domains $G_{L+\delta}^U, G_L^U, G_{L-\delta}^L$ and G_L^L such that

$w(p \frown D_k^H \frown G_{L+\delta}^U, z, G \frown G_L^U) > w(p \frown [D^H] \frown G_{L+\delta}^U, z, G \frown G_L^U) > 0$ and $w(p \frown D_k^H \frown G_{L-\delta}^L, z, G \frown G_L^L) > w(p \frown [D^H) \frown G_{L-\delta}^L, z, G \frown G_L^L) > 0$. By $D(\omega(D_k^H, z, G)) < \infty$ by *a)* and by (32′) and (33), $0 < D(\omega(p \frown D_k^H \frown G_{L+\delta}^U, z, G_L^U \frown G)) < \infty$ and $0 < D(\omega(p \frown D_k^H \frown G_{L-\delta}^L, z, G_L^L \frown G)) < \infty$. Hence we have the theorem, because such consts L and δ do not exist by Theorem 12. *b)*.

Proof of c). Put $G = D_k = R - R_0$. Then by *b)* we have at once *c)*.

8. Analytic functions in a neighbourhood of a singular point.

Theorem 12. *a) Let p be a singular point of second kind and suppose that a domain G contains p N-approximately. Then there exists no non constant analytic function $T(z)$ in G with $D(T(z)) < \infty$.*

b) Let p be a singular point of second kind and let G be a domain containing p N-approximately. Then there exists no non constant analytic function $T(z)$ in G such that the spherical area $A(T(z))$ of $T(z)$ is finite.

c) Suppose a domain G contains a singular point p (of first or second kind) N-approximately. Then there exists no non constant analtytic function $T(z)$ in G such that $n(w) < M < \infty$, where $n(w)$ is the number of zero points of $T(z) - w$ in G.

Remark. Theorem 12. *c)* is Theorem 8 of the previous paper mentioned at the top of the present paper. Since I thought that $M_r \to 0$ as $r \to 0$, the condition $n(w) < M < \infty$ was left unnoticed, where $M_r = \max\left(\left|\frac{\partial U_r(w)}{\partial u}\right|, \left|\frac{\partial U_r(w)}{\partial v}\right|\right)$: $w = \upsilon + iv$ and $U_r(w)$ is a harmonic function in $r < |w| < 1$ such that $U_r(w) = 1$ on $|w| = r$ and $U_r(w) = 0$ on $|w| = 1$. But the condition that $n(w) < M < \infty$ is necessary. In reality, there exists a Riemann surface R such that the area of R is finite and R has a singular point of first kind. (See the example 4 of the following paper "Examples of singular points").

Proof of a). Map the universal covering surface G^∞ of G onto $|\xi| < 1$. Then by Theorem 12. *b)* there exists a set E of positive measure on $|\xi| = 1$ such that $ReT(z)$ and $ImT(z)$ have angular limits $= a$ and b respectively a.e. on E by $D(T(z)) < \infty$. Hence by Riesz's theorem $T(z) \equiv a + ib$. This is a contradiction. Hence we have *a)*.

Proof of b). Put $D_k^M = V^W_{1-\varepsilon-\frac{1}{k}} \frown CV^C_{M-\frac{1}{k}}$, where $V^W_{1-\varepsilon-\frac{1}{k}} = E\left[z \in R: w(p, z) > 1-\varepsilon-\frac{1}{k}\right]$ and $V^C_{M-\frac{1}{k}} = E\left[z \in R: -{}_{CG}\omega(p, z) \leqq -M+\frac{1}{k}\right]$. Then $D^H = (V^W_{1-\varepsilon} \frown CV^C_M) \subset D_k^H$ and D^H and D_k^H are harmonic domains. Then by Theorem 8. *c*)

$$D(\omega(D_k^H, z, G)) \leqq \frac{D(\omega(p, z) - {}_{CG}\omega(p, z))}{\left(1-M-\frac{2}{k}\right)^2} \leqq M^*: \ 1-M-\frac{2}{k} > 0. \tag{35}$$

and $w(p \frown D_k^H, z, G) \geqq \lim_{m=\infty} \lim_{k=\infty} w(\upsilon_m(p) \frown D_k^H, z, G) = w(p \frown [D^H], z, G) \geqq w(p \frown D^H, z, G) \geqq w(p \frown D^H, z, G) > 0$.

Map the universal covering surface G^∞ of G onto $|\xi| < 1$ by $\xi = f(z)$. Then $A(T(z)) < \infty$ implies that $T(z)$ has angular limits a.e. on $|\xi| = 1$. Whence $ReT(z)$ and $I_m T(z)$ are of F-type.

Let Δ_{ij}^W be a rectangle: $E[w: i \leqq Re\, w \leqq i+1, \ j \leqq Im\, w \leqq j+1]$. Then $\Delta_{ij} = E[z \in G: \ T(z) \in \Delta_{ij}^W] = E[z \in G: \ Re(-T(z)) \geqq -i-1] \frown E[z \in G: \ ReT(z) \geqq i] \frown E[z \in G: \ Im(-T(z)) \geqq -j-1] \frown E[z \in G: \ Im\, T(z) \geqq j]$ is also a harmonic domain in G.

By $w(p \frown D^H, z, G) \leqq \sum_{ij} w(p \frown D^H \frown \Delta_{ij}, z, G)$, there exist numbers i_0 and j_0 and at least one component Δ of $\Delta_{i_0 j_0}$ ($\Delta_{i_0 j_0}$ may consist of an enumerably infinite number of components) such that

$$w(p \frown \Delta \frown D^H, z, G) > 0. \tag{36}$$

Without loss of generality, we can suppose that Δ is a component of

$$E[z \in G: \ |Re\, w| \leqq 1 \text{ and } |Im\, w| \leqq 1].$$

Let Δ_k be a component of $E\left[z \in G: \ |Re\, w| \leqq 1+\frac{1}{k} \text{ and } |Im\, w| \leqq 1+\frac{1}{k}\right]$ containing Δ. Then $\Delta = \bigcap_{k=1}^{\infty} \Delta_k$ and $\Delta_k \frown D_k^H$ and $\Delta \frown D^H$ are harmonic domains. Hence $\lim_{m=\infty} \lim_{k=\infty} w(\upsilon_m(p) \frown (\Delta_k \frown D_k^H), z, G) \geqq w(p \frown [\Delta \frown D^H], z, G) \geqq w(p \frown \Delta \frown D^H, z, G) > 0$ and by Theorem 11. *b*)

$$0 < w(p \frown [\Delta \frown D^H], z, G) = w(E, \xi), \text{ whence mes } E > 0,$$

where E is the set on which $w(p \frown [\Delta \frown D^H], z, G) = 1$ almost everywhere. Also by $\text{mes}(E - B(D^H \frown \Delta)) = 0$ by Theorem 11. *b*). Hence

$$w(p \frown [\Delta \frown D^H], z, G) \to 1 \text{ as } \xi \to E \text{ along Stolz's path and} \tag{a}$$

$$T(z)(\xi = f(z)) \to \Delta_k \text{ as } \xi \to E \text{ along Stolz's path.} \tag{b}$$

Hence for any given positive number ε, we can find a closed set E' in E such that $\text{mes}(E - E') < \frac{\varepsilon}{2}$ and the above functions converge uniformly

in angular domains $A(\theta): e^{i\theta}\in E'$. Then we can find a closed set $E''\subset E'$ such that $\mathrm{mes}(E'-E'')<\frac{\varepsilon}{2}$ and a domain D $\Big($containing an endpart of $A(\theta): e^{i\theta}\in E''$ and bounded by $\sum_{e^{i\theta}\in E''}\partial A(\theta)+E''$ and $|\xi|=1-\frac{1}{m}\Big)$ and a number k_0 such that

$$w(p\frown[\Delta\frown D^H], z, G)>1-\frac{1}{k_0} \text{ and} \tag{a'}$$

$$T(z)\in\Delta_{k_0}^W=E\Big[w:\ |Re\, w|<1+\frac{1}{k_0},\ |Im\, w|<1+\frac{1}{k_0}\Big] \text{ in } D\frown R_m \tag{b'}$$

for $f(z)\in D\frown\mathfrak{R}_m$: $\mathfrak{R}_m=E\Big[\xi:\ 1>|\xi|>1-\frac{1}{m}\Big]$.

Let $\Delta_0^W=E[|Re\, w|\leqq 3,\ |Im\, w|\leqq 3]$ and Δ_0 be the component of $T^{-1}(\Delta_0^W)$ containing $\Delta_k(\supset\Delta)$. Put $G_j=E\Big[z\in G:\ \delta(z,\partial G)>\frac{1}{j}\Big]$. Then $f(\partial G_j+(\partial R_n\frown G))$ tends to $|\xi|=1$ as $n\to\infty$ and $j\to\infty$, i.e.

$$\mathrm{dist}(\partial f(\partial G_j+\partial R_n),\ \xi=0)>1-\frac{1}{m} \text{ for } n\geqq n_0 \text{ and } j\geqq j_0 \tag{c}$$

Now by (b') $\qquad f(\partial\Delta_0\frown G)$ does not fall in D. $\qquad (d)$

By (a')

$f(\widetilde{V}^W_{1-\frac{1}{k_0}}\frown D_k^H)$ covers $D\frown\mathfrak{R}_m$ and $f(\partial(V^W_{1-\frac{1}{k_0}}\frown\Delta^H))$ separates E'' in $D\frown\mathfrak{R}_m$,

where $\widetilde{V}^W_{1-\frac{1}{k_0}}=E\Big[z\in G:\ w(p\frown[\Delta\frown D^H], z, G)>1-\frac{1}{k_0}\Big]$. $\qquad (e)$

Let $w_{j,n,k}(z)$ be a harmonic function in $(\Delta_0\frown G\frown R_n\frown G_j)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown\Delta_k\frown D_k^H)$ such that $w_{j,n,k}(z)=0$ $\partial(\Delta_0\frown R_n\frown G_j)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown\Delta_k\frown D_k^H)$ and $w_{j,n,k}(z)=1$ on $\partial(\widetilde{V}^W_{1-\frac{1}{k}}\frown\Delta_k\frown D_k)$. Let $w^D(\xi)$ be a hormonic function in $D\frown\mathfrak{R}_m$ such that $w^D(\xi)=0$ on $\partial(D\frown\mathfrak{R}_m)-E''$ and $w^D(\xi)=1$ on E''. Then by the rectifiableness of ∂D and by mes $E''>0$, $w^D(\xi)>0$.

Then by (c) and (d) $\partial(\Delta_0\frown G\frown R_n\frown G_j)-(\widetilde{V}^W_{1-\frac{1}{k}}\frown\Delta_k\frown D_k^H)$ does not fall in $D\frown\mathfrak{R}_m$ for $n\geqq n_0, k\geqq k_0$ and $j\geqq j_0$ and by (e) $f(\widetilde{V}^W_{1-\frac{1}{k_0}}\frown\Delta_{k_0}\frown D_{k_0}^H)$ covers $E''\frown\partial D$. Hence by the maximum principle

$$w_{j,n,k}(z)\geqq w^D(\xi) \text{ for } n\geqq n_0,\ k\geqq k_0 \text{ and } j\geqq j_0.$$

Let $j\to\infty$ and $n\to\infty$ and then $k\to\infty$. Then by Theorem 12. $a)$

$$w(p\frown\Delta_0\frown\Delta_k\frown D_k^H, z, \Delta_0\frown G)\geqq\lim_{k=\infty} w(p\frown\Delta_0\frown\Delta_k\frown D_k^H, z, \Delta_0\frown G)$$

$$\geqq\lim_{k=\infty} w(\widetilde{V}^W_{1-\frac{1}{k}}\frown\Delta_k\frown D_k^H, z, \Delta_0\frown G)\geqq w^D(\xi). \text{ Hence}$$

$$w(p\frown\Delta_{k_0}\frown D_{k_0}^H, z, \Delta_0\frown G)>0 \text{ for } k_0<\infty. \tag{37}$$

Let $\alpha(w)$ be a continuous function in the w-plane such that $\alpha(w)=1$ in $|w|\leqq 2+\frac{1}{k_0}$, harmonic in $2+\frac{1}{k_0}<|w|<3$ and $\alpha(w)=0$ in $|w|\geqq 3$. Put $\alpha(z)=\alpha(T(z))$. Then $\alpha(z)$ is a continuous function in G and $\alpha(z)=1$ in Δ_{k_0} and $\alpha(z)=0$ on $\partial\Delta_0\frown G$, because $|T(z)|<\sqrt{2}+\frac{1}{k_0}$ for $z\in\Delta_{k_0}$ and $|T(z)|\geqq 3$ for $z\in\partial\Delta_0\frown G$. Clearly

$$\underset{z\in\Delta_0}{D}(\alpha(z))<M^2KA(T(z))<M^{**}<\infty, \tag{38}$$

where $M=\max\left(\left|\frac{\partial\alpha(w)}{\partial u}\right|^2, \left|\frac{\partial\alpha(w)}{\partial v}\right|^2\right)$ and $K=\max\left(\frac{\text{area element}}{\text{spherical area element}}\right)$ in Δ_0 and $w=u+iv$.

Put $\omega'(z)=\min(w(\upsilon_n(p)\frown D^H_{k_0}, z, G), \alpha(z))$. Then $\omega'(z)=0$ on $\partial\Delta_0+\partial G$ and $\omega'(z)=1$ on $\Delta_{k_0}\frown D^H_{k_0}\frown\upsilon_n(p)$, where $\omega(\upsilon_n(p)\frown D^H_{k_0}, z, G)$ is C.P. of $\upsilon_n(p)\frown D^H_{k_0}$ relative G.

By (35) and (38) and as in case of the proof of Theorem 7. *c*)

$$D(\omega'(z))\leqq D(\omega(\upsilon_n(p)\frown D^H_{k_0}, z, G))+D(\omega'(z))\leqq M^*+M^{**} \tag{39}$$

Let $\omega(\upsilon_n(p)\frown D^H_{k_0}\frown\Delta_{k_0}, z, G\frown\Delta_0)$ be C.P. of $(\upsilon_n(p)\frown D^H_{k_0}\frown\Delta_{k_0})$ relative $\Delta_0\frown G$. Then it has M.D.I. among all functions having value 1 on $(\upsilon_n(p)\frown D^H_{k_0}\frown\Delta_{k_0})$ and 0 on $\partial(\partial G\frown\Delta_0)$. Hence by (39)

$$D(\omega(\upsilon_n(p)\frown D^H_{k_0}\frown\Delta_{k_0}, z, G\frown\Delta_0))\leqq D(\omega'(z))<\infty.$$

Now by (37) $\qquad \omega(p\frown D^H_{k_0}\frown\Delta_{k_0}, z, G\frown\Delta_0)>0.$

Hence by Theorem 8, *a*) $G\frown\Delta_0$ contains p N-approximately. On the other hand, clearly $D_{\Delta_0}(T(z))<KA(T(z))<\infty$. Hence by Theorem 12. *a*) $T(z)$ must reduce to be a constant. This is a contradiction. Hence we have *b*).

Proof of c). By Theorem 8. *a*) there exists a harmonic domain $D^H=E[z\in R:\ \omega(p,z)>1-\varepsilon]\frown E[z\in R:\ {}_{CG}\omega(p,z)<M]$ such that $\omega(D^H\frown p, z, G)>0$ and $D(\omega(D^H, z, G))<\frac{D(\omega'(p,z))}{(1-M-\varepsilon)^2}<\infty$.

Let $\mathfrak{S}_n(\Delta_i^{W,n})$ be a triangulation of the w-plane and $\mathfrak{S}_{n+1}(\Delta_i^{W,n+1})$ be a subdivision of $\mathfrak{S}_n$ and becomes as fine as we please as $n\to\infty$. Put $\Delta_i^n=E[z\in G:\ T(z)\in\Delta_i^{W_n}]$. Then $\Delta_{i,j}^n$ may consist of at most enumerably infinite number of components $\Delta_{i,j}^n$. By $\omega(D^H\frown p,z,G)\leqq\sum\omega(D^H\frown\Delta_{ij}^n\frown p,z,G)$ there exists at least one Δ_{i_0,j_0}^n such that $\omega(D^H\frown\Delta_{i_0,j_0}^n\frown p, z, G)>0$. Similarly there exists at least one $(\Delta_{i_0,j_0}^n\supset)\Delta_{i_1,j_1}^{n+1}$ such that $\omega(\Delta_{i_1,j_1}^{n+1}\frown p, D^H, z, G)>0$. In this fasion we can construct a sequence $\Delta_{i_0,j_0}^n\subset\Delta_{i_1,j_1}^{n+1}\subset\cdots$ such that $\omega(p\frown D^H\frown\Delta_{i,j}^n, z, G)>0$ for every n and $i(n)$ and $j(n)$. We denote them by $\Delta_1\supset\Delta_2\supset\cdots$ and suppose diameter of $T(\Delta^n)<\frac{1}{n}$ and put $q=\bigcap_n^\infty\overline{T(\Delta^n)}$.

Let L be a compact curve on ∂G such that $\mathrm{dist}(T(L), T(\Delta^n))>\delta>0$ for $n\geqq n_0$. Such a curve L can be chosen by the fact that $T(\partial G)$ is not a point. Let C_δ and C_n be circles with radii δ and $\frac{2}{n}$ $\left(\delta>\frac{2}{n_0}\right)$ with centres at q such that C_n encloses $T(\Delta_n)$ for $n>n_0$ and $T(L)$ lies outside of C_δ.

Let $\alpha_n(w)$ be a continuous function in the w-plane such that $\alpha_n(w)=0$ outside of C_δ, $\alpha_n(w)$ is harmonic in $C_\delta-C_n$ and $\alpha_n(w)=1$ in C_n. Then $D(\alpha_n(w))\downarrow 0$ as $n\to\infty$. Put $\alpha_n(z)=\alpha_n(T(z))$. Then $\alpha_n(z)$ is a function in G such that $\alpha_n(z)=1$ in Δ_n and $=0$ on L. And by $n(w)\leqq M$, $D(\alpha_n(z))\leqq MD(\alpha_n(w))<\infty$.

Let $U_{n,m}(z)$ be a harmonic function in $(R_m\frown G)-(D^H\frown\upsilon_n(p)\frown\Delta_n)$ such that $U_{n,m}(z)=0$ on $\partial G\frown R^n$, $\frac{\partial}{\partial n}U_{n,m}(z)=0$ on $\partial R_m-(D^H\frown\upsilon_n(p))$ and $U_{n,m}(z)=1$ on $(D^H\frown\upsilon_n(p)\frown\Delta_n)$. Let $U^*_{n,m}(z)$ be a harmonic function in $(R_m\frown G)-(D^H\frown\upsilon_n(p)\frown\Delta_n)$ such that $U^*_{n,m}(z)=0$ on L, $\frac{\partial}{\partial n}U^*_{n,m}(z)=0$ on $((\partial G\frown R_m)-L)+(\partial R_m-(D^H\frown\upsilon_n(p)\frown\Delta_n))$ and $U^*_{n,m}(z)=1$ on $(\Delta_n\frown D^H\frown\upsilon_n(p))$. Then by the maximum principle

$$U_{n,m}(z)\leqq U^*_{n,m}(z).$$

Clearly $D(U^*_{n,m}(z))=\int\limits_{\partial(D^H\frown\upsilon_n(p))}U^*_{n,m}(z)\frac{\partial}{\partial n}U^*_{n,m}(z)ds=\int\limits_L\frac{\partial}{\partial n}U^*_{n,m}(z)ds.$

On the other hand, $\alpha_n(z)=0$ on L and $\alpha_n(z)=1$ on $\Delta_n(\supset(D^H\frown\Delta_n\frown\upsilon_n(p))$. Since $U^*_{n,m}(z)$ has M.D.I. among all functions with value$=0$ on L and$=1$ on $(D^H\frown\upsilon_n(p)\frown\Delta_n)$,

$$D(U^*_{n,m}(z))\leqq D(\alpha_n(z))\leqq MD(\alpha_n(w)).$$

$U_{n,m}(z)\Rightarrow U_n(z)=\omega(\upsilon_n(p)\frown\Delta_n\frown D^H, z, G)$ as $m\to\infty$ and $U^*_{n,m}(z)\Rightarrow U^*_n(z)$ as $m\to\infty$ by $D(U_{n,m}(z))\leqq D(\omega(D^H, z, G))<\infty$ and $D(U^*_{n,m}(z))\leqq D(\alpha_n(z))$ respectively for every m.

Now $\quad U^*_n(z)\geqq\omega(\upsilon_n(p)\frown D^H\frown\Delta_n, z, G)\geqq\omega(p\frown D^H\frown\Delta_n\frown z, G)>0$ and

$$\int\limits_L\frac{\partial}{\partial n}U^*_n(z)ds=D(U^*_n(z))\leqq MD(\alpha_n(w)),$$

because $\omega(p\frown D^H\frown\Delta_n, z, G)>0$ by the assumption.

Since $MD(\alpha_n(w))\downarrow 0$ as $n\to\infty$, $\frac{\partial}{\partial n}U^*_n(z)\downarrow 0$ on L uniformly by the compactness of L as $n\to\infty$. Hence there exists a number n_0 and a point z_0 in a neighbourhood of L such that

$$0<\omega(p\frown D^H\frown\Delta_{n_0}, z_0, G)<U^*_{n_0}(z_0)<\omega(p\frown D^H, z_0, G).$$

In Separation Theorem $S.$ 2 put $B'=p$, $G_1=D^H$, $G_2=D^H\frown\Delta_{n_0}$ and $\widetilde{G}=R$

$-R_0$. Then there exist domains D_1 and D_2 such that

$$0 \neq \omega(p \frown D^H \frown D_2, z) \neq \omega(p \frown CD_1 \frown D^H \frown \Delta_{n_0}, z) \neq 0 \tag{40}$$

But these have their masses only at p, whence as usual they are equal to $\omega(p, z)$. This contradicts (40). Hence we have c).

Corollary 1. *Let $\upsilon_n(p)$ be a neighbourhood of a singular point of p, then there exists no analytic function in $\upsilon_n(p)$ such that $n(w) \leqq M$.*

Corollary 2. *Let R be a Riemann surface of finite genus. Then there exists no singular point.*

Corollarly 1) is evident, since $\upsilon_n(p)$ contains p N-approximately. And the Riemann surface of fininte genus g can be represented as a covering surface of at most $2g$ number of sheets over the w-plane. Hence by Theorem 12. c) we have Corollary 2.

9. Relation between HD unteilbare Menge and N-minimal points.

Let R be a Riemann surface with positive boundary. Map the universal covering surface $(R-R_0)^\infty$ onto $|\xi|<1$, where R_0 is a compact set of R. Constantinescu and Cornea[14] introduced the $\underline{HD}$ class and the maximal $\underline{HD}$ indivisible sets to extend Theorem 1 in some way. We shall consider the relations between theirs and ours.

Let $U(z)$ be a positive harmonic function. If $U(z) \geqq V(z) > 0$ implies $V(z) = KU(z)$, $U(z)$ is called a minimal function.

If there exist decreasing positive harmonic functions $U_n(z)$: $U_n(z) \in HD$[15] such that $U(z) = \lim_n U_n(z)$, $U(z)$ is called a *function in $\underline{HD}$ class.*

Let E be a set on $|\xi|=1$ such that every $U(z) \in HD$ has angular limits$=$const a.e. on E. Then E is called *a HD indivisible set.* If there exists no set E^* such that $E^* \supset E$ and mes $(E^*-E)>0$ and E^* is HD indivisible, E is called a maximal indivisible set. They proved the following

Theorem 13. *If E is a maximal indivisible set, then H.M. (harmonic measure of E) $w(E, \xi)$ is minimal in the class $\underline{HD}$. If $U(z)$ is minimal in $\underline{HD}$ class, $U(z) = K\,w(E, \xi)$, where E is a maximal indivisible set.*

We show that *a maximal indivisible set is the image of a singular point of second kind.*

Theorem 14. *a) Let $\omega(p, z)$ be C.P. of a singular point p of second kind and let E be the image of p, i.e. the set on which $\omega(p, z)$ has anguler limits$=1$ almost everywhere. Then E is maximal indivisible and*

14) C. Constantinescu. A. Cornea: Über den idealen Rand und einige seiner Anwendungen bei der Klassifikation der Riemannschen Flächen. Nagoya Math. J., 1958.

15) HD means the class of Dirichlet bounded harmonic functions.

$w(p, z)$ (H.M. *of* p) *is minimal in* $\underline{HD}$ *class.*

b) If E *is maximal indivisible, there exists a singular point* p *of second kind such that* $w(p, z)=w(E, \xi)$ *and* E *is the set where* $\omega(p, z)$ *has angular limits*$=1$ *almost everywhere.*

Proof of a). At first we show that $w(p, z)$ is in $\underline{HD}$ class. Let p be a singular point of second kind. Let $V_{1-\frac{1}{m}}=\left[z\in R:\ \omega(p, z)>1-\dfrac{1}{m}\right]$. Let $T_{m, n, n+i, n+i+j}(z)$ be a harmonic function in $R_{n+i+j}-((R_{n+i+j}-R_{n+i})\frown(V_{1-\frac{2}{m}}-V_{1-\frac{1}{m}}))-((R_{n+i}-R_n)\frown V_{1-\frac{1}{m}})=\Omega_{m, n, n+i, n+i+j}$ such that $T_{m, n, n+i, n+i+j}(z)=0$ on $\partial R_0+(\partial R_{n+i+j}-V_{1-\frac{2}{m}})+(\partial V_{1-\frac{2}{m}}\frown(R_{n+i+j}-R_{n+i}))$, $T_{m, n, n+i, n+i+j}(z)=1$ on $((R_{n+i}-R_n)\frown V_{1-\frac{1}{m}})$ and $\dfrac{\partial}{\partial n}\ T_{m, n, n+i, n+i+j}(z)=0$ on $\partial R_{n+i}\frown(V_{1-\frac{2}{m}}-V_{1-\frac{1}{m}})$.

Put $S(z)=\dfrac{\omega(p, z)-\left(1-\dfrac{2}{m}\right)}{\dfrac{1}{m}}$. Then $S(z)=0$ on $\partial V_{1-\frac{2}{m}}$ and $=1$ on $\partial V_{1-\frac{1}{m}}$.

Let $S'(z)$ be a function in $\Omega_{m, n, n+i, n+i+j}$ such that $S'(z)=\min(1, S(z))$ in $V_{1-\frac{2}{m}}$ and $S'(z)=0$ in $R-R_0-V_{1-\frac{1}{m}}$. Then by the Dirichlet principle

$$D(S'(z))<D(S(z))\leqq\frac{1}{\left(\dfrac{1}{m}\right)^2}D(\omega(p, z))=M_m<\infty. \tag{41}$$

Let $S^2(z)$ be a harmonic function in $\Omega_{m, n, n+i, n+i+j}$ such that $S^2(z)=S'(z)$ on $\partial R_0+(\partial R_{n+i+j}-V_{1-\frac{2}{m}})+\partial(V_{1-\frac{2}{m}}\frown(R_{n+i+j}-R_{n+i}))+((R_{n+i+j}-R_n)\frown V_{1-\frac{1}{m}}))$. Then by the Dirichlet principle $D(S^2(z))\leqq D(S(z))$. Now

$$D(S^2(z)-T_{m, n, n+i, n+i+j}(z), T_{m, n, n+i, n+i+j}(z))=0, \quad \text{whence}$$
$$D(S^2(z)-T_{m, n, n+i, n+i+j}(z))=D(S^2(z))-D(T_{m, n, n+i, n+i+j}(z)), \text{ hence}$$
$$D(T_{m, n, n+i, n+i+j}(z))\leqq M_m<\infty. \tag{42}$$

Clearly $T_{m, n, n+i, n+i+j}(z)\uparrow T_{m,n,n+i}(z)$ as $j\to\infty$ and by (41) and by Lemma 1. *b)* of P $T_{m, n+i}(z)\Rightarrow T_{m, n}(z)$ as $i\to\infty$ and $T_{m, n}(z)\downarrow T_m(z)$ as $n\to\infty$.

Let E' be the set on which $\omega(p, z)$ has angular limits$<1-\dfrac{2}{m}$ almost everywhere and let E_g be the set where the Green's function $G(z, p)$ has angular limits$=0$. Then we can find for any given positive number ε a closed set $E''\subset(E'\frown E_g)$ such that $\text{mes}\,(E'-E'')<\varepsilon$ and a domain D containing an endpart $A(\theta)$: $e^{i\theta}\in E''$ and bounded by $\sum\limits_{e^{i\theta}\in E'}\partial A(\theta)+E''$ and a ring $\mathfrak{R}_l$: $1>|\xi|>1-\dfrac{1}{l}$ with the following property

$$\omega(p, z) < 1 - \frac{2}{m} - \delta \text{ in } D \frown \mathfrak{R}_l \text{ for a positive constant } \delta.$$

Hence the image of $\partial V_{1-\frac{2}{m}}$ does not fall in $D \frown \mathfrak{R}_l$. (*a*)

Since $E'' \subset E_g$ and $G(z, p) > \delta_{n+i+j} > 0$ on ∂R_{n+i+j}, ∂R_{n+i+j} does not tend to $|\xi| = 1$ and by

$$\partial R_{n+i+j} \to |\xi| = 1 \text{ as } j \to \infty, \ \partial R_{n+i+j} \text{ separates } E'' \text{ from } |\xi| = 1 - \frac{1}{l}. \quad (b)$$

Hence by the maximum principle

$$T_{m, n, n+i, n+i+j}(z) \leqq w^D(\xi)$$

where $w^D(\xi)$ is a harmonic function in $D \frown \mathfrak{R}_l$ such that $w^D(\xi) = 0$ on E'' and $= 1$ on $\partial(D \frown \mathfrak{R}_l) - E''$. Now $w^D(\xi) = 0$ a.e. on E'. Hence by letting $\varepsilon \to 0$ $T_m(z)$ has angular limits $= 0$ a.e. on E'. Similarly $T_m(z)$ has angular limits $= 1$ a.e. on the set where $\omega(p, z)$ has angnlar limits $> 1 - \frac{1}{m}$. Hence $T_m(z) \downarrow w(E, \xi)$ as $m \to \infty$. On the other hand, by (42) $T_m(z) \in H.D.$ Hence $w(E, \xi)$ $(= w(p, z)$ by Theorem 10. *a*)) is $\underline{HD}$ class.

By Theorem 12. *c*) every $U(z) \in HD$ has angular limits $=$ const a.e. on E. Hence E is an indivisible set. Assume E is not maximal, then there exists a set $E^* \supset E$ such that $\text{mes}(E^* - E) > 0$ and E^* is indivisible, whence $T_m(z) \geqq w(p, z) = w(E, \xi)$ implies $T_m(z) \geqq w(E^*, z)$. Whence $\lim_m T_m(z) = w(E, \xi)$ has angular limits $= 1$ a.e. on E^*. This contradicts the fact that $w(E, \xi)$ has angular limits $= 0$ a.e. on CE. Hence E is maximal.

Proof of b). Let E be a maximal indivisible set. Then $w(E, \xi)$ can be considered as a function $w(E, z)$ in $R - R_0$. Put $V_{M_i} = [z \in R : w(E, z) > M_i]$. Then $\{V_{M_i}\} : M_1 < M_2 < M_3 \cdots \lim_i M_i = 1$ is a sequence of decreasing domains. Let $\omega(V_{M_i}, z)$ be *C.P.* of V_{M_i}. Then $\omega(V_{M_i}, z)$ and $\lim \omega(V_{M_i}, z)$ $(= \omega(\{V\}, z))$ is *C.P.* defined by a sequence of decreasing domains) are superharmonic in $\overline{R} - R_0$ by Theorem 5. *a*) of P.

Let $w(V_{M_i}, z)$ be H.M. of V_{M_i}. Then by P.H. 4 M_i $w(V_{M_i}, z) = w(E, z)$ whence $\lim_i w(V_{M_i}, z) = w(\{V\}, z) = w(E, z)$. Clearly $\omega(V_{M_i}, z) \geqq w(V_{M_i}, z)$. Hence

$$1 \geqq \omega(\{V\}, z) \geqq w(\{V\}, z). \quad (43)$$

On the other hand, ${}_{V_{M_i}}\omega(\{V\}, z) = \omega(\{V\}, z)$ for every M_i by P.C.4.

Let $U(z)$ be a positive superharmonic function in $\overline{R} - R_0$ such that $\omega(\{V\}, z) - U(z) = T(z)$ is also superharmonic in $\overline{R} - R_0$. Then

$${}_{V_{M_i}}U(z) \leqq U(z) \text{ and } {}_{V_{M_i}}T(z) \leqq T(z).$$

Now $\omega(\{V\}, z) = U(z) + T(z) \geqq {}_{V_{M_i}}U(z) + {}_{V_{M_i}}T(z) = {}_{V_{M_i}}\omega(\{V\}, z) = \omega(\{V\}, z)$,

whence ${}_{V_{M_i}}U(z)=U(z)$ and ${}_{V_{M_i}}T(z)=T(z)$.

Let $U(z)(T(z))$ be positive superhearmonic in $\overline{R}-R_0$ and harmonic in $R-R_0$. Then by Theorem 5. *a*) and *b*) of P $U(z)=\int N(z,p)d\mu(p)$ and $D(\min(M, U(z))\leqq MK<\infty$. But $\sup_{z\in R} U(z)\leqq 1$ and $\sup_{z\in R} T(z)\leqq 1$, whence $U(z)$ and $T(z)$ are contained in HD.

Since E is indivisible, $U(z)$ and $T(z)$ have angular limits$=c_1$ and c_2 a.e. on E. By (43) and $c_1+c_2=1$ a.e. on E we have $c_1+c_2=1$. $U(z)$ has angular limits$\geqq 0$ on CE and c_1 on E almost everywhere. Hence $U(z)\geqq c_1 w(E,z)$. This implies $U(z)\geqq c_1 M_i$ on V_{M_i}. Hence

$$U(z)={}_{V_{M_i}}U(z)\geqq c_1\omega(V_{M_i}, z) \text{ and } U(z)\geqq c_1\omega(\{V\}, z).$$

Similarly $T(z)\geqq c_2\omega(\{V\}, z)$. Hence

$$\omega(\{V\}, z)=U(z)+T(z)\geqq c_1\omega(\{V\}, z)+c_2\omega(\{V\}, z)=\omega(\{V\}, z).$$

Thus $U(z)=c_1\omega(\{V\}, z)$ and $\omega(\{V\}, z)$ is N-minimal, hence $\omega(\{V\}, z)$ $=K\,N(z,p)$.

Let $\omega(p,z)$ be C.P. of p. Then $\omega(p,z)=K'N(z,p)$. By $\sup\omega(p,z)=1$ $=\sup\omega(\{V\}, z)$, we have $K=K'$ and $\omega(p,z)=\omega(\{V\}, z)$. By $\sup_{z\in R} N(z,p)<\infty$, $p\in B_S$. Let $V_{M_i}(p)=E[z\in R: \omega(p,z)>M_i]$. Then by $\omega(\{V\}, z)\geqq w(\{V\}, z)$ $=w(E,\xi)$, $\omega(\{V\},z)>M_i$ on $V_{M_i},=E[z\in R: w(E,\xi)>M_i]$. whence $V_{M_i}(p)\supset V_{M_i}$ Hence $w(\{V_{M_i}(p)\}, z)$ (H.M. determined by a decreasing sequence $V_{M_i}(p)$: $\lim_{i=\infty} M_i=1$) is larger than $w(\{V\}, z)=w(E,\xi)>0$. Now by Theorem 10, *a*) $w(p,z)=\lim_{M_i=1} w(V_{M_i}(p), z)\geqq w(\{V\}), z)=w(E,\xi)>0$. Whence p is a singular point of second kind. Hence a singular point of second kind corresponds to a maximal indivisible set.

Let $\widetilde{E}$ be the image of p ($\widetilde{E}$ is the set on which $\omega(p,z)=1$ almost everywhere). Then by $\omega(p,z)\geqq w(p,z)\geqq w(E,\xi)$ $\widetilde{E}\supset E$. Now $\widetilde{E}$ is maximal by *a*). By the assumption E is maximal and we have $\widetilde{E}=E$. Next by Theorem 10. *b*) mes $(E\frown E_i)=0$, where E_i is the image of a singular point p_i of second kind such that $p\neq p_i$. Hence a uniquely determined singular point of second kind corresponds to a maximal indivisible set. Thus we have *b*).

10. Class H.N.D. Let H.N.D. be the class of Riemann surfaces on which N number of linearly independent harmonic functions $\in HD$ exist, where $N\leqq\infty$ and the cardinal number of N is $\aleph_0$.

Theorem 14. *The set of HD functions in R is isomorphic to the set of HD functions in $R-R_0$ vanishing on ∂R_0. Hence without loss of generality, we can consider the set of HD functions vanishing on ∂R_0*

instead of HD function in R.

Proof. Let $U(z)$ be a Dirichlet bounded harmonic function in R with positive boundary. Let $U_n(z)$ be a harmonic function in R_n-R_0 such that $U_n(z)=U(z)$ on ∂R_0 and $U_n(z)=0$ on ∂R_n. Then $U_n(z)$ $=\frac{1}{2\pi}\int_{\partial R_0} U(\xi)\frac{\partial}{\partial n}G_n(\xi,z)ds$, where $G_n(\xi,z)$ is the Green's function of $(R_n-R_0.)$ Now $G_n(\xi,z)\to G(\xi,z)$ as $n\to\infty$ and $\frac{\partial}{\partial n}G(\xi,z)$ is continuous on ∂R_0 by the compactness of ∂R_0. Hence $\lim_n U_n(z)$ exists and $\lim_n U_n(z)$ $=\frac{1}{2\pi}\int_{\partial R_0} U(\xi)\frac{\partial}{\partial n}G(\xi,z)ds$, where $G(\xi,z)$ is the Green's function of $R-R_0$. Put $\lim_n U_n(z)=U'(z)$. Since $U_n(z)-U'(z)=0$ on ∂R_0, $U_n(z)-U'(z)$ is harmonic in a neighbourhood of ∂R_0. Now $U_n(z)\to U'(z)$ implies $\frac{\partial}{\partial n}U_n(z)\to\frac{\partial}{\partial n}U'(z)$ uniformly on ∂R_0 and $\frac{\partial}{\partial n}U'(z)$ is continuous on ∂R_0. Hence

$$D(U_n(z))=\int_{\partial R_0} U_n(z)\frac{\partial}{\partial n}U_n(z)ds\to\int_{\partial R_0} U'(z)\frac{\partial}{\partial n}U'(z)ds=D(U'(z))<\infty,$$

whence $U_n(z)\Rightarrow U'(z)$. Put $U^*(z)=U(z)-U'(z)$. Then $U^*(z)$ is uniquely determined by $U(z)$ and $D(U^*(z))=D(U(z)+D(U'(z))-2D(U(z),U'(z))<\infty$.

Put $B_n=R-R_n$ and let $w(B_n,z,R_n-R_0)$ be a harmonic function in R_n-R_0 such that $w(B_n,z,R_n-R_0)=1$ on ∂R_n and $w(B_n,z,R_n-R)=0$ on ∂R_0. Then by the maximum principle

$|U_n(z)|\leqq M(1-w(B_n,z,R_n-R_0))$, where $M=\max|U(z)|$ on ∂R_0. Let $n\to\infty$.

Then $w(B_n,z,R_n-R_0)\to w(B,z,R-R_0)$ (H.M. of B relative $R-R_0$). (44)

Put $G_\delta=E[z\in R:\ w(B,z,R-R_0)<\delta]$. Then by P.H.2. $w(G_\delta\frown B,z,R-R_0)$ $=\lim_n w(G_\delta\frown B_n,z,R-R_0)=0$ for $\delta<1$, where $w(G_\delta\frown B_n,z,R-R_0)$ is H.M. of $G_\delta\frown B_n$ relative $(R-R_0)$. By (44) we have

$$|U'(z)|<M(1-w(B,z,R-R_0)). \tag{45}$$

Let $w(G_\delta\frown B_n,z,R)$ be H.M. of $G_\delta\frown B_n$ relative R, i.e. $w(G_\delta\frown B_n,z,R)$ is the least positive superharmonic function in R larger than 1 in $G_\delta\frown B_n$. *We show* $w(G_\delta\frown B,z,R)=\lim_n w(G_\delta\frown B_n,z,R)=0$.

In fact, $\frac{1-w(B,z,R-R_0)}{1-\delta}\geqq 1$ in G_δ and $\geqq 1$ on ∂R_0. Hence by the maximum principle $\frac{1-w(B,z,R-R_0)}{1-\delta}\geqq w(G_\delta\frown B,z,R)$.

By P.H.2. $\sup w(B,z,R-R_0)=1$, whence $\inf w(G_\delta\frown B,z,R)<1$ and

$w(G_\delta \frown B, z, R) \not\equiv 1$. Assume $w(B \frown G_\delta, z, R) > 0$. Then by $w(B \frown G_\delta, z, R) \not\equiv 1$ $\max w(B \frown G_\delta, z, R) = M_\delta < 1$ on ∂R_0, But by P.H.2. $\sup w(B \frown G_\delta, z, R) = 1$. On the other hand, by the maximum principle

$$w(B \frown G_\delta, z, R - R_0) = \lim_n w(G_\delta \frown B_n, z, R - R_0) \geqq \lim_{n=\infty} w(G_\delta \frown B_n, z, R) - M_\delta > 0.$$

Hence we have $w(B \frown G_\delta, z, R - R_0) > 0$. This is a contradiction. Thus $w(B \frown G_\delta, z, R) = 0$.

Let $U_n''(z)$ be a harmonic function in R_n such that $U_n''(z) = U'(z)$ on ∂R_n. Then by (45)

$$U_n''(z) \leqq M(1-\delta) + Mw(B_n \frown G_\delta, z, R),$$

because $|U'(z)| < 1 - \delta$ in $C\,G_\delta$ and $|U'(z)| \leqq M$ in $R \frown G_\delta$.
Let $n \to \infty$ and then $\delta \to 1$. Then $\lim_n U_n''(z) = 0$, by $w(B \frown G_\delta, z, R) = 0$.

Let $U_n'''(z)$ be a harmonic function in R_n such that $U_n'''(z) = U^*(z) = U(z) - U'(z)$ on ∂R_n. Then $U_n'''(z) = U(z) - U_n''(z)$. Hence by letting $n \to \infty$

$$U_n'''(z) \to U(z).$$

Thus $U(z)$ is uniquely determined by $U^*(z)$ and the sets $\{U(z)\}$ and $\{U^*(z)\}$ are isomorphic hence we have the theorem.

Green's function and generalized Green's function.[16]

Let q be a point of B and let $G(z, p)$ be the Green's function. We call q *a regular* or *irregular point for the Green's function* according as $\lim_{z \to q} G(z, p) = 0$ or $\overline{\lim_{z \to q}}\, G(z, p) > 0$, where $z \to q$ means that z converges with respect to the δ-metric. Let $B_{g,i}$ be the set of irregular points for the Green's function. Then by Theorem 11. *b*) of P $B_{g,i}$ is an F_σ set of capacity zero. Map the universal covering surface $(R - R_0)^\infty$ conformally onto $|\xi| < 1$. If a harmonic function $G(z)$ which is positive and $D(\min(M, G(z))) \leqq MK$: $K < \infty$ and $G(z)$ has angular limits $= 0$ a.e. on $|\xi| = 1$, we call $G(z)$ a *generalized Green's function.*

We proved in the previous paper the following

Theorem.[17] *Let $W(z)$ be positive superharmonic in $\overline{R} - R_0$ and harmonic in $R - R_0$. Then $W(z) = P(z) + G(z)$, where $G(z)$ is a generalized Green's function and $P(z)(\leqq W(z))$ is a harmonic function representable by Poisson's integral.*

Now we shall prove

Theorem 15. *Let $p \in B_{g,r} \frown (B_1 - B_S)$ and let $G(z)$ be a generalized*

16) Z. Kuramochi: On harmonic functions representable by Poisson's integral. Osaka Math. 1959.

17) See 16).

Green's function such that $G(z)\leqq N(z, p)$. *Then* $N(z, p)=P(z)$, *i.e.* $G(z)=0$ *and* $N(z, p)$ *is representable by Poisson's integral, where* $B_{g,r}$ *is the set of regular points for the Green's function.*

Proof. Assume $G(z)>0$. Put $R_\delta=E[z\in R-R_0:\ G(z)>\delta]$ and $R_{\frac{\delta}{2}}=E\left[z\in R-R_0:\ G(z, p)>\frac{\delta}{2}\right]$. Map the universal covering surface $(R-R_0)^\infty$ of $(R-R_0)$ onto $|\xi|<1$. Let $w(B\frown R_{\frac{\delta}{2}}, z)$ be H.M. of $(B\frown R_{\frac{\delta}{2}})$ (H.M. of the boundary determined by $R_{\frac{\delta}{2}}$). Then $w(B\frown R_{\frac{\delta}{2}}, z)$ has angular limits $=0$ a.e. on the set E where $G(z)$ has angular limits $<\frac{\delta}{2}$. In fact for any given positive number ε we can find a closed set $E'\subset E$ such that mes $(E-E')<\varepsilon$ and a domain D containing an endpart of $A(\theta):\ e^{i\theta}\in E'$ and bounded by $\sum_{e^{i\theta}\in E'}\partial A(\theta)+E'$ and a ring $\mathfrak{R}_m:\ 1>|\xi|>1-\frac{1}{m}$ and a number k such that

$$G(z)>\frac{\delta}{2}-\frac{1}{k}\quad\text{in } D\frown\mathfrak{R}_m.$$

Hence usually it can be proved that $w(B\frown R_{\frac{\delta}{2}}, z)=0$ on E'. But $G(z)$ has angular limits $=0$ a.e. on $|\xi|=1$, whence mes $E=2\pi$. Hence $w(B\frown R_{\frac{\delta}{2}}, z)=0$ and

$$0=w(B\frown R_{\frac{\delta}{2}}, z)\geqq w(B\frown R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}})=0. \tag{46}$$

Let $\omega_{n,n+i}(z)$ be a harmonic function in $(R_{\frac{\delta}{2}}\frown R_{n+i})-((R_{n+i}-R_n)\frown R_\delta)$ such that $\omega_{n,n+i}(z)=0$ on $\partial R_{\frac{\delta}{2}}$, $\omega_{n,n+i}(z)=1$ on $((R_{n+i}-R_n)\frown R_\delta)$ and $\frac{\partial}{\partial n}\omega_{m,n+i}(z)=0$ on $\partial R_{n+i}-R_\delta$. Then $D(\omega_{n,n+i}(z))\leqq\frac{4D_{R_{\frac{\delta}{2}}}(G(z))}{\left(\frac{1}{\delta}\right)^2}<\infty$. Hence $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ as $i\to\infty$ and $\omega_n(z)\Rightarrow\omega(B\frown R_\delta, z, R_{\frac{\delta}{2}})$ as $n\to\infty$. On the other hand, $w(B\frown R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}})=\lim_n w_n(z)$, where $w_n(z)$ is a harmonic function in $R_n\frown R_{\frac{\delta}{2}}$ such that $w_n(z)=0$ on $\partial R_{\frac{\delta}{2}}\frown R_n$ and $w_n(z)=1$ on $\partial R_m\frown R_{\frac{\delta}{2}}$. By the maximum principle $w_n(z)\geqq\omega_{n,n+i}(z)$ and $w_n(z)\geqq\omega_n(z)$. Whence by letting $n\to\infty$, $0=w(B\frown R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}})\geqq\omega(B\frown R_\delta, z, R_{\frac{\delta}{2}})$. Whence

$$D(\omega_n(z))\downarrow 0\ \text{ as } n\to\infty. \tag{47}$$

Put $R^*=(R_{\frac{\delta}{2}}\frown R_{n_0})+R_\delta$, where n_0 is a certain number. Let F be a closed compact arc on $\partial R_{\frac{\delta}{2}}\frown R_{n_0}$. Let $\omega'_{n,n+i}(z)$ be a harmonic function in $R^*\frown R_n$ such that $\omega_n^*(z)=0$ on F, $\omega'_n(z)=1$ on $R_\delta\frown\partial R_n$ and $\frac{\partial}{\partial n}\omega'_n(z)=0$ on

$(\partial R^* \frown R_n) - F$. Then $D(\omega_n'(z)) \leqq D(\omega_n(z))$. Hence $\omega_n'(z) \downarrow 0$ as $n \to \infty$ and

$$\lim_n D(\omega_n'(z)) = 0.$$

Hence there exists a sequence of harmonic functions $\omega_n^*(z)$ in $R^* \frown R_n$ such that $\omega_n^*(z) = 0$ on F, $\frac{\partial}{\partial n}\omega_n^*(z) = 0$ on $(\partial R^* \frown R_n) - F$ and $\omega_n^*(z) = M_n$ on $\partial R_n \frown R^*$ and $\int_F \frac{\partial}{\partial n}\omega_n^*(z) ds = 2\pi$ and $M_n \uparrow \infty$ as $n \to \infty$. For any given number n' there exists a number n'' such that $\omega_n^*(z) < \frac{M_n}{2}$ in $R^* \frown (R \frown R_{n'})$ for $n > n''$, i.e. every niveau curve of $\omega_n^*(z)$ with height $\frac{M_n}{2}$ is contained in $R^* \frown (R - R_{n'})$, because $\omega_n'(z) \to 0$ as $n \to \infty$.

Lemma a. *Let G_i be a domain in R^* and let $U_i(z)$ be a function in $R^* - G_i$ which is harmonic in $R^* - G_i$ and on ∂G_i such that the Dirichlet integral of $U_i(z)$ is finite $(i = 1, 2, \cdots i_0)$. Then there exists a sequence of compact curves $\{\gamma_j\}$ such that γ_j separates B from F, $\{\gamma_j\}$ clusters at B and that $\int_{\gamma_j \frown (R^* - G_i)} \left| \frac{\delta}{\partial n} U_i(z) \right| ds \to 0$ as $j \to \infty$ for every i.*

Proof. Let $U(z)$ be one of $\{U_i(z)\}$ and put

$$L(r) = \int_{C_r} \frac{\partial}{\partial n} |U(z)| \, r \, d\theta = \int_{C_r} \frac{\partial}{\partial n} U(z) ds, \quad C_r = E[z \in R^* : \omega_n^*(z) = \log r]$$

where $r e^{i\theta} = e^{\omega_n^*(z) + i\tilde{\omega}_n^*(z)}$ and $\tilde{\omega}_n^*(z)$ is the conjugate function of $\omega_n^*(z)$.

Suppose that there exist two positive constants η and δ and infinitely many numbers n' with the following properties:

There exists a closed set H_n in the interval $(e^{M_n}, e^{\frac{M_n}{2}})$ such that $\frac{\text{mes } H_n}{(e^{M_n}, -e^{\frac{M_n}{2}})} \geqq \eta$ and $L(r) \geqq \delta > 0$ for any $r \in H_n$.

Since $\int_{C_r} d\theta \leqq 2\pi$, we have by Schwarz's inequality

$$\underset{R-G}{D}(U(z)) = \iint \left\{ \left(\frac{\partial U(z)}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial U(z)}{\partial \theta} \right)^2 \right\} r dr \, d\theta \geqq \frac{1}{2\pi} \int_1^{e^{M_n}} \frac{L^2(r)}{r} dr$$

$$\geqq \frac{1}{2\pi} \int_{e^{M_n/2}}^{e^{M_n}} \frac{L^2(r)}{r} dr \geqq \frac{1}{2\pi} \delta \int_{e^{M_n/2 + \eta(M_n) - M_n/2)}}^{e^{M_n}} \frac{1}{r} dr = \frac{M_n}{4\pi}(1 - \eta)\delta^2.$$

Let $n \to \infty$. Then $D_{R-G}(U(z)) \uparrow \infty$. This is a contradiction. Hence there exists a sequence of exceptional sets H_n in the interval $(e^{M_n}, e^{\frac{M_n}{2}})$ such that $\lim_n \frac{\text{mes } H_n}{(e^{M_n} - e^{\frac{M_n}{2}})} \leqq \frac{1}{4i_0}$ and that $r \notin H_n$ implies $L(r) < \delta_n$, where $\lim_{n=\infty} \delta_n = 0$.

Returning to case of $U_i(z)$. Let $\{H_{i,n}\}$ be a sequence of exceptional sets corresponding to $U_i(z)$ and $\delta_{i,n}$ be the corresponding quantities of $H_{i,n}$. Then we see that $\lim_n \frac{\text{men}(\sum_1^{i_0} H_{i,n})}{(e^{M_n}-e^{\frac{M_n}{2}})} \leqq \frac{1}{4}$ and $\max_i \delta_{i,n} \to 0$ as $n \to \infty$. On the other hand, the niveau cuvrs with height$\geqq \frac{M_n}{2}$ are contained in $R^*-R_{n'}$, since $\omega_n^*(z) < \frac{M_n}{2}$ in $(R_{n'} \frown R_\delta)-R_0$.It follows that $\int_{C_r \frown (R-G_1)} \left|\frac{\partial U_i}{\partial n}(z)\right| ds \leqq \max_i \delta_{i,n} (\to 0$ as $n \to \infty)$ and every niveau curve C_r with $r \in (e^{M_n}, e^{\frac{M_n}{2}}) - \sum_i H_{i,n}$ clusters at B as $n \to \infty$. Consider niveau curves C_r mentioned above as γ_j. Then we have the lemma.

Lemma b. *Let* $V_M(p) = E[z \in R : N(z, p) > M] : p \in B_1 - B_S$ *and* $\upsilon_m(p) = E\left[z \in \overline{R} : \delta(z, p) < \frac{1}{m}\right]$. *Then*

$$\lim_{M=\infty} {}_{V_M(p) \frown C\upsilon_m(p)} N(z, p) = 0.$$

Put $N'(z, p) = \lim_{M=\infty} {}_{V_M(p) \frown C\upsilon_m(p)} N(z, p)$. Since $\sup_{z \in R} N(z, p) = \infty$,

$$\lim_{M=\infty} \text{Cap}(V_M(p) \frown \upsilon_m(p)) \leqq \lim_{M=\infty} \text{Cap}\,(V_M(p)) = 0.$$

Hence by Theorem 6. *b*) of P, $N(z, p) - N'(z, p)$ is superharmonic in $\overline{R} - R_0$. Hence $N'(z, p) = aN(z, p)$: $a \geqq 0$ by the N-minimality of $N(z, p)$. By Theorem 9. *a*) of P $N'(z, p) = a' N(z, q)$: $a' \geqq 0$ and $q \in C\upsilon_m(p) \frown B_1$. But by $\int_{\partial R_0} \frac{\partial}{\partial n} N(z, p) ds = \int_{\partial R_0} \frac{\partial}{\partial n} N(z, q) ds$ we have $N(z, p) = N(z, q)$. This is a contradiction. Hence $\lim_{M=\infty} {}_{V_M \frown C\upsilon_m(p)} N(z, p) = 0$.

Let $U_{m,M}(z)$ be a harmonic function in $R_\delta - (R_M \frown \upsilon_m(p))$ such that $U_{m,M}(z) = 0$ on ∂R_δ and $U_{m,M}(z) = M - \delta$ on $\partial(R_M \frown \upsilon_m(p))$ and has M.D.I., where $R_M = E[z \in R_\delta : G(z) > M]$. Then by the Dirichlet principle

$$D(U_{m,M}(z)) \leqq {}_{CR_M} D(G(z)) \leqq MK.$$

Let $U'_{m,M}(z)$ be a harmonic function in $R_\delta - (R_M \frown C\upsilon_m(p))$ such that $U'_{m,M}(z) = 0$ on ∂R_δ and $U'_{m,M}(z) = G(z) - \delta$ on $(R_M \frown C\upsilon_m(p)$ and has M.D.I. over $R_\delta - (R_M \frown C\upsilon_m(p))$. Consider $(U_{m,M}(z) + U'_{m,M}(z))$ and $G(z) - \delta$ in $(R_\delta - R_M) \frown R_n$. Then by the maximum principle

$U_{m,M}(z) + U'_{m,M}(z) + M\, w(B_n \frown R_\delta, z, R_\delta) \geqq G(z) - \delta \geqq U_{m,M}(z) - M\, w(B_n \frown R_\delta, z, R_\delta)$, where $B_n = R - R_n$ and $w(B_n \frown R_\delta, z, R_\delta)$ is H.M. of $B_n \frown R_\delta$ relative to R_δ, i.e. $w(B_n \frown R_\delta, z, R_\delta)$ is a harmonic function in $R_\delta - (B_n \frown R_\delta)$ such that $w(B_n$

$\frown R_\delta, z, R_\delta)=0$ $\partial R_\delta-(B_n\frown R_\delta)$ and $=1$ on $\partial(B_n\frown R_\delta)$. Let $n\to\infty$.
Then $w(B_n\frown R_\delta, z, R_\delta)\to w(B\frown R_\delta, z, R_\delta)\leqq w(B\frown R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}})=0$ by (46). Hence

$$U_{m,M}(z)+U'_{m,M}(z)\geqq G(z)-\delta\geqq U_{m,M}(z).$$

Now by $G(z)\leqq N(z, p)$, $E[z: G(z)>M]=R_M\subset V_M(p)$. $U'_{m,M}(z)$ and $N(z, p)$ have M.D.I. over $R_\delta-(R_M\frown C\upsilon_m(p))$. $U'_{m,M}(z)\leqq G(z)\leqq N(z, p)$ on ∂R_δ $+\partial(R_M\frown C\upsilon_m(p))$, whence by the maximum principle

$$U'_{m,M}(z)\leqq N(z, p).$$

Hence by Lemma b $\quad \lim_{M=\infty} U'_{m,M}(z)\leqq \lim_{M=\infty} {}_{V_M\frown C\upsilon_m(p)}N(z,p)=0.$

Hence
$$\lim_{M=\infty} U_{m,M}(z)=G(z)-\delta. \tag{48}$$

Let $G(z, q)$ be the Green's function of R_δ and let $\upsilon(q)$ be a neighbourhood of q. Then $\underset{R_\delta-\upsilon(q)}{D}(G(z, q))<\infty$. By $D(U_{m,M}(z))\leqq MK$, hence by Lemma a) there exists a sequence $\{\gamma_j\}$ such that

$$\int_{\gamma_j}\left|\frac{\partial}{\partial n}U_{m,M}(z)\right| ds\to 0 \text{ and } \int_{\gamma_j}\left|\frac{\partial}{\partial n}G(z, q)\right| ds\to 0$$

and γ_j clusters at B as $j\to\infty$.
Let R_j be the compact part of R_δ divided by γ_j. Then $R_\delta=\bigcup_j^\infty R_j$.
Put $\Omega_{M_0}^{M_1}=E[z\in R_\delta: M_0<U_{m,M}(z)<M_1]$ and $C_{M_i}=E[z\in R: U_{m,M}(z)=M_i]$.
Then

$$\int_{C_{M_0}\frown R_j}\frac{\partial}{\partial n}U_{m,M}(z)\, ds=\int_{C_{M_1}\frown R_j}\frac{\partial}{\partial n}U_{m,M}(z)\, ds+\int_{\gamma_j\frown\Omega_{M_0}^{M_1}}\frac{\partial}{\partial n}U_{m,M}(z)ds. \tag{49}$$

Since $\frac{\partial}{\partial n}U_{m,M}(z)\geqq 0$ on C_{M_0} and C_{M_1}, we have by (49)

$$\int_{C_{M_0}}\frac{\partial}{\partial n}U_{m,M}(z)\, ds=\lim_{j=\infty}\int_{C_{M_0}\frown R_j}\frac{\partial}{\partial n}U_{m,M}(z)\, ds=\lim_{j=\infty}\int_{C_{M_1}\frown R_j}\frac{\partial}{\partial n}U_{m,M}(z)\, ds$$
$$=\int_{C_{M_1}}\frac{\partial}{\partial n}U_{m,M}(ds). \tag{50}$$

Since $\underset{R_j-(R_M\frown\upsilon_m(p))}{D}(U_{m,M}(z))=\int_{\partial(R_M\frown\upsilon_m(p))\frown R_j}U_{m,M}(z)\frac{\partial}{\partial n}U_{m,M}(z)\, ds+\int_{\gamma_j-(R_M\frown\upsilon_m(p)}U_{m,M}(z)\frac{\partial}{\partial n}U_{m,M}(z)ds$ and $U_{m,M}(z)=M-\delta$ on $\partial(R_M\frown\upsilon_m(p))$, we have by letting $j\to\infty$, $D(U_{m,M}(z))$ $=\int_{\partial(R_M\frown\upsilon_m(p))}U_{m,M}(z)\frac{\partial}{\partial n}U_{m,M}(z)ds$ $(\leqq MK)$. Since $U_{m,M}(z)=M-\delta$ on $\partial(R_M\frown\upsilon_m(p))$, we have by (50)

$$\int_{C_L}\frac{\partial}{\partial n}U_{m,M}(z)ds\leqq 2K \text{ for } \frac{M}{2}>\delta, \text{ for every } L. \tag{51}$$

By the Green's formula

$$\int_{\partial\upsilon(q)+(\gamma_j-(R_M\frown\upsilon_m(p))+(\partial(R_M\frown\upsilon_m(p))\frown R_j)} U_{m,M}(z)\frac{\partial}{\partial n}G(z,q)ds = \int_{\partial\upsilon(q)+(\gamma_j-(R_M\frown\upsilon_m(p))+(\partial(R_M\frown\upsilon_m(p))\frown R_j)} G(z,q)\frac{\partial}{\partial n}U_{m,M}(z)ds.$$

Now $U_{m,M}(z)=M-\delta$ on $\partial(R_M\frown\upsilon_m(p))$, whence

$$\int_{\partial(R_M\frown\upsilon_m(p))\frown R_j} U_{m,M}(z)\frac{\partial}{\partial n}G(z,q)\,ds=(M-\delta)\int_{\partial(R_M\frown\upsilon_m(p))\frown R_j}\frac{\partial}{\partial n}G(z,q)\,ds=(M-\delta)\int_{\gamma_j\frown R_M\frown\upsilon_m(p)}\frac{\partial}{\partial n}G(z,q)\,ds$$

$$\leqq(M-\delta)\int_{\gamma_j}\left|\frac{\partial}{\partial n}G(z,q)\,ds\right|\to 0 \text{ as } j\to\infty.$$

By $U_{m,M}(z)\leqq M-\delta$, $\displaystyle\int_{\gamma_j-(R_M\frown\upsilon_m(p)} U_{m,M}(z)\frac{\partial}{\partial n}G(z,q)ds\leqq M\int_{\gamma_j}\left|\frac{\partial}{\partial n}G(z,q)\,ds\right|\to 0$ as $j\to\infty$.

Similarly by $G(z,q)<L_0$ in $C\upsilon(q)$, $\displaystyle\int_{\gamma_j-(R_M\frown\upsilon_m(p))} G(z,q)\frac{\partial}{\partial n}U_{m,M}(z)ds\to 0$ as $j\to\infty$.

Hence by letting $j\to\infty$, we have

$$U_{m,M}(q)=\frac{1}{2\pi}\int_{\partial(R_M\frown\upsilon_m(p))} G(z,q)\frac{\partial}{\partial n}U_{m,M}(z)ds. \tag{52}$$

p is regular by the assumption. Hence for any given positive number ε there exists a neighbourhood $\upsilon_m(p)$ such that $G(z,q)<\varepsilon$ in $\upsilon_m(p)$. Hence by (51) and (52)

$$U_{m,M}(q)\leqq\frac{1}{\pi}K\varepsilon.$$

Let $M\to\infty$. Then $G(q)=\lim_M U_{m,M}(q)+\delta$ by (48.) But ε and δ are arbitrary. Hence $G(z)=0$. Thus $N(z,p)=P(z)$ and $N(z,p)$ is Poisson's integrable.

Let $U_i(z)$ $((i=1,2,\cdots N)$ and the cardinal number of $N\leqq\aleph)$ be a harmonic function in $R-R_0$ vanishing on ∂R_0. If any finite number of $\{U_i(z)\}$ are linearly independent, we say that $\{U_i(z)\}$ is linearly independent.

Theorem 16. *a)* $\{N(z,p_i)\}$: $p_i\in B_1$ *is linearly independent.*

b) If there exists a point in $B_{g,r}\frown(B_1-B_S)$, *then there exist infinitely many linealy independent H.D. functions vanishing on* ∂R_0, *where* $B_{g,r}$ *is the set of regular points for the Green's function.*

c) $R\in H.N.D.(N=\infty$ *and* $\aleph_0)$ *if and only if both* $B_{g,r}\frown(B_1-B_S)$ *and* $B_{S.1}$ *(set of singular points of first kind) are at most enumerable and the number of points of* $B_{S.2}$ *(set of singular points of second kind)* $=\infty$ *and* $\aleph_0$.

d) $R\in H.N.D.(N<\infty)$ *if and only if* $B_{g,r}\frown(B_1-B_S)=0=B_{S.1}$ *and the number of points of* $B_{S.2}$ *is* N.

Proof of a). Assume $\{N(z,p_i)\}$ is not linearly independent. Then

there exists a linear form $\sum_{j=1}^{l} c_i N(z, p_i)=0$. Suppose $c_1, c_2, \cdots, c_m>0$ and $c_{m+1}, \cdots, c_l<0$.

Then
$$\sum_{1}^{m} c_i N(z, p_i)=\sum_{m+1}^{l} c_i N(z, p_i).$$

Put $U(z)=\sum^{m} c_i N(z, p_i)$ and $F=\sum_{1}^{m} p_i$, $F'=\sum_{m+1}^{l} p_i$. Then F and F' are closed and dist $(F, F')>0$. By Theorem 12. *a*) and *b*) of P
$$U(z)={}_F U(z)>{}_{F'}U(z). \tag{53}$$
On the other hand, $U(z)$ is also $\sum_{m+1}^{l} c_i N(z, p_i)$, whence as above $U(z)={}_{F'}U(z)>{}_F U(z)$. This is a contradiction. Hence $\{N(z, p_i)\}$ is linearly independent.

Proof of b). Map the universal covering surface $(R-R_0)^\infty$ of $R-R_0$ onto $|\xi|<1$. Put $V_M(p)=E[z\in R: N(z, p)>M]$. Then H.M. of $(V_M(p)\frown B)$: $w(B\frown V_M(p), z)=w(E, \xi)$, where E is the set on which $N(z, p)$ has angular limits$\geqq M$. Now by Theorem 15 $N(z, p)$ is Poisson's integrable in $|\xi|<1$, and $\sup N(z, p)=\infty$ by $p\in(B_1-B_S)$, whence mes $E>0$. On the other hand, by the Dirichlet principle $0<D(\omega(B\frown V_M(p), z))\leqq\dfrac{D(\min(M, N(z, p)))}{M^2}$ $\leqq\dfrac{2\pi M}{M^2}$, because $\dfrac{N(z, p)}{M}\geqq 1$ on $V_M(p)$ and $0<w(B\frown V_M(p), z)\leqq\omega(B\frown V_M(p), z)$. Put $D_{M_i}=D(\omega(B\frown V_{M_i}(p), z))$. Let M_{i+1} be a number such that
$$D_{M_i}>\frac{D(\min(M_{i+1}, N(z, p)))}{M_{i+1}^2}<\frac{2\pi}{M_{i+1}}.$$
Then $D(\omega(B\frown V_{M_{i+1}}(p), z))<D_{M_i}$. Hence we can find a sequence M_1, $M_2\cdots$, $\lim_{i=\infty} M_i=\infty$ and $D(\omega(B\frown V_{M_i}(p), z)<D(\omega(B\frown V_{M_{i+1}}(p), z)$. Now $E_1\supset E_2\supset E_3\cdots$, whence such $\{\omega(B\frown V_{M_i}(p), z)\}$ is linearly independent, where E_i is the set on which $N(z, p)$ has angular limits$=M_i$. Hence we have *b*).

Proof of c). By Theorem 14 it is sufficient to consider H_0D functions vanishing on ∂R_0 instead of HD function in R. *Necessity*. Suppose $R\in H.N.D.(N=\infty$ and $N=\aleph_0)$. Suppose the cardinal number of points of $B_{g,r}\frown(B_1-B_S)$ is $\aleph$. Let $p\in B_{g,r}\frown(B_1-B_S)$. Then $N(z, p)$ is Poisson's integrable. Let $U_n(z)$ be a harmonic function in R_n-R_0 such that $U_n(z)=\min(M, N(z, p))$ on $\partial R_0+\partial R_n$. Then $U_n(z)\to U(z, M, p)$ as $n\to\infty$, where
$$U(z, M, p)=\frac{1}{2\pi}\int\varphi^M(e^{i\theta})\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}d(\theta),$$
where $\varphi^M(e^{i\theta})=\min(M, \varphi(e^{i\theta}))$ and $\varphi(e^{i\theta})$ is angular limits of $N(z, p)$ at $e^{i\theta}$. On the other hand, by the Dirichlet principle $D(U(z, M, p))$ $\leqq D(\min(N, N(z, p))=2\pi M$. Hence $U(z, M, p)\in H_0D$.

Let α and α_M be the cardinal numbers of linearly independent functions $\{N(z, p_i)\}$ and $\{U(z, M, p_i)\}: p_i \in B_{g,r} \frown (B_1 - B_S)$ respectively. Assume $\alpha = \aleph$. Since $\alpha = \lim_{M=\infty} \alpha_M$, there exists a constant M_0 such that $\alpha_{M_0} \geqq \aleph$. Hence there exist an $\aleph$ number of linearly independent H_0D functions in $R - R_0$ vanishing on ∂R_0, whence $R \in H.N.D.(N = \aleph)$. This is a contradiction. Hence $B_{g,r} \frown (B_1 - B_S)$ is *at most enumerable.*

Next $N(z, p) = a\ \omega(p, z)$: $a > 0$ for $p \in B_{S.1} (\subset B_1)$ (whence $N(z, p) \in HD$). Hence by *a*) $\{\omega(p, z)\}$ is linearly independent, whence $B_{S.1}$ is also *at most enumerable. We show $B_{S.2}$ consists of infinitely many points* (clearly $\leqq \aleph_0$ by Theorem 10. *b*). Assume that $B_{S.2}$ consists of $n_0 < \infty$ number of points $p_1, p_2, \cdots, p_{n_0}$. If $B_{S.1} \neq 0$, let $q \in B_{S.1}$. Now $B_{S.1} + (B_{g,r} \frown (B_1 - B_S))$ is at most enumerable, $B_{g,i}$ (set of irregular points) is an F_σ set of capacity zero by Theorem 11. *d*) of P, B_0 is also an F_σ set of capacity zero. Whence H.M. of $B_{S.1} + B_{g,i} + B_0 + B_{g,r} \frown (B_1 - B_S)$ is zero. Let $\upsilon_{\delta_0}(q) = E[z \in R: \delta(z, q) < \delta_0]: 2\delta_0 = 2 \min_i \delta(p_i, q)$. Then $B \frown (\upsilon_{\delta_0}(q) - q)$ is an F_σ set of harmonic measure zero. Hence by Theorem 9 $q \in B_{S.2}$. This is a contradiction. *Hence if* $B_{S.2} = p_1 + \cdots + p_{n_0}$, $B_{S.1} = 0$.
Assume $B_{S.2} = p_1 + p + \cdots + p_{n_0}$. Then $B_0 + B_{g,i} + B_{S.1} + (B_{g,r} \frown (B_1 - B_S))$ is an F_σ set of capacity zero (clearly of harmonic measure zero). Hence it can be proved similarly as in Theorem 1. *d*)

$$\sum_{i=1}^{n_0} \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0),$$

where E_i is the image of $p_i \in B_{S.2}$ on $|\xi| = 1$.
By Theorem 12. *c*) every $U(z) \in H.D.$ has angular limits $=$ const a.e. on E_i. Whence $R \in H.N.D.$ $(N = n_0)$. This is a contradiction. Hence $B_{S.2}$ consists of infinitely many points and we have also $\sum_{i=1}^{\infty} \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0)$.

Sufficiency is evident. Thus we have *c*).

Proof of d). Suppose $R \in H.N.D.$ $N < \infty$. Then by *a*) $B_{g,r} \frown (B_1 - B_S) = 0$ and $B_{S.2}$ consists of at most $n_0 \leqq N$ number of points. Whence as in *c*) $B_{S.1} = 0$ and

$$\sum_{i}^{n_0} \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0).$$

where E_i is the image of $p_i \in B_{S.2}$.
Whence $R \in H.n_0.D.$ Hence $n_0 = N$, i.e. $B_{S.2}$ consists of N number of points. Thus we have *d*).

Remark. If $R \in H.N.D.(N < \infty)$, $w(p, z)$ of $p \in B_{S.2}$ is an *H.D.* function. In fact, $\omega(p_i, z) = \sum_j \alpha_{ij}\, w(p_j, z)$. Now $\{\omega(p_i, z)\}$ is linearly independent,

whence the determinant $|\alpha_{ij}|\neq 0$ and $w(p_j, z)=\sum_i \beta_{ji}\omega(p_i, z)$, where $|\beta_{ji}|$ is the inversed matrix of $|\alpha_{ji}|$. Hence $w(p_j, z)\in H.D.$

PART III. On Subdomains.

Let G be a domain[18] in $R-R_0$. Let $w(B\frown G, z, G)$ be $H.M.$ of $(B\frown G)$ relative to G. i.e. $w(B\frown G, z, G)=\lim_n w_n(z)$, where $w_n(z)$ is a superharmonic function in G which is harmonic in $G\frown R_n$, $w_n(z)=0$ on $\partial G\frown R_n$ and $w_n(z)=1$ on $(R-R_n)\frown G$. Map the universal covering surface G^∞ of G onto $|\xi|<1$. *Then* $w(B\frown G, z, G)$ *has angular limits*$=1$ *or* 0 *a.e. on* $|\xi|=1$.

Proof. The image of ∂G is composed of enumerably infinite number of arcs on $|\xi|=1$. Clearly $w(B\frown G, z, G)=0$ on the image of ∂G. For simplicity put $w(z)=w(B\frown G, z, G)$. If $w(z)=0$, our assertion is trivial. Suppose $w(z)>0$. Then by P.H.2. $\sup w(z)=1$. Let E_g be the set where the Green's function $G(z, p)$ of G has angular limits$=0$. Then mes $E_g=2\pi$. Let E be the set where $w(z)$ has angular limits between 0 and 1. Assume mes $E>0$. Then for any given positive number ε, we can find a closed set $E'\subset(E\frown E_g)$, a number $\delta>0$, a domain D and a ring $\mathfrak{R}_m$: $1>|\xi|>1-\frac{1}{m}$ such that mes $(E-E')<\varepsilon$, D is containing an end part of $A(\theta)$ of $e^{i\theta}\in E'$ and bounded by $\sum_{e^{i\theta}\in E'}\partial A(\theta)+E'$ and a circle $|\xi|=1-\frac{1}{m}$ with the following property: $1-\delta>w(z)>\delta$ in $D\frown\mathfrak{R}_m$, i.e. $D\frown\mathfrak{R}_m$ is contained in the image of $\Omega_\delta^{1-\delta}=E[z\in G: \delta<w(z)<1-\delta]$. Since ∂G is *composed of analytic curves* and $w(z)=0$ on ∂G, the image of ∂G does not fall in $D\frown\mathfrak{R}_m$, image of $(\partial R_n\frown G)\to|\xi|=1$ as $n\to\infty$ and $\partial R_n\frown G$ separates E' from $|\xi|=1-\frac{1}{m}$ by $G(z, p)>0$ on $\partial R_n\frown G$. Let $w^D(\xi)$ be a harmonic function in D such that $w^D(\xi)=1$ on E' and$=0$ on $\partial D-E'$. Then as usual

$$w(B\frown\Omega_\delta^{1-\delta}, z, G)\geqq w^D(\xi)>0.$$

On the other hand, $w(\Omega_\delta^{1-\delta}\frown B, z, G)\leqq w(B\frown G, z, G)$ and $\sup w(G\frown\Omega_\delta^{1-\delta}, z, G)\leqq 1-\delta$. Hence by P.H.2. $w(\Omega_\delta^{1-\delta}\frown B, z, G)=0$. This is a contradiction. Hence mes $E=0$. Thus $w(B\frown G, z, G)$ has angular limits$=0$ or 1 a.e. on $|\xi|=1$.

Let $\{z_i\}$ be a sequence such that $z_i\to B$ and $\overline{\lim_i}\, w(z_i)<1$, we say that $\{z_i\}$ converges to the *annexed relative boundary*. Let $E_{K,A}$ be the set where $w(z)$ has angular limits$=0$ and is not the image of ∂G. We call

18) See. 5).

$E_{K,A}$ *K-annexed relative boundary.* We call E_B where $w(z)$ has angular limits$=1$ the image of the ideal boundary. Then by the above fact $w(B\frown G, z, G)=w(E_B, \xi)$, where $w(E_B, \xi)$ is *H.M.* of E_B with respect to $|\xi|<1$. We denote by $H_0.B.$ $(H_0.D)$ the class of bounded (Dirichlet bounded) harmonic functions in a domain G vanishing on ∂G. Also we denote by $O_{H0,B}$ $(O_{H_0,D})$ and $H_0.N.B.$ $(H_0.N.D)$ the class of domain in which there exist no non constant $H_0.B.$ $(H_0.D.)$ function and there exist an N number of linearly independent $H_0.B.$ $(H_0.D.)$ functions respectively.

11. Class O_{H_0B} and H_0.N.B. Since G is a Riemann surface, we can define K-Martin's topology in G and we have at once Theorem 2. But it is more interesting to characterize the class $H_0.N.B.$ by K-Martin's topology of $R\supset G$.

Theorem 17. *a) Every $H_0.B.$ function has angular limits$=0$ a.e. on $E_{K,A}$.*

b) $G\in O_{H_0B}$ if and only if mes $E_B=0$.

c) $G\in H_0.N.B.$ if and only if G contains K-approximately N number of singular points p_i of R and $w((B-\sum_i p_i)\frown G, z, G)=0$.

d) If $R\in H.N.B.$, $G\in H_0.N'.B.$ and $N'\leqq N$.

Proof of a). Let $U(z)\in H_0.B.$ and let $U_n(z)$ be a harmonic function in $G\frown R_n$ such that $U_n(z)=|U(z)|$ on $(\partial R_n\frown G)+(\partial G\frown R_n)$. Then $|U(z)|\leqq|U_n(z)|\leqq M\ w_n(z)$, where $M=\sup_{z\in G} U(z)$ and $w_n(z)$ is a harmonic function in $G\frown R_n$ such that $w_n(z)=0$ on $\partial G\frown R_n$ and $w_n(z)=1$ on $G\frown\partial R_n$. Let $n\to\infty$. Then

$$|U(z)|\leqq M\, w(B\frown G, z, G). \tag{54}$$

Hence we have *a)*.

Proof of b). If mes $E_B=0$, $w(B\frown G, z, G)=0$ and every harmonic function $U(z)$ has angular limits$=0$ by (54) and $G\in O_{H_0B}$. Suppose mes $E_B>0$. Then $w(B\frown G, z, G)>0$ and $G\notin 0_{H_0B}$. Thus we have *b)*.

Proof of c). Suppose G contains $p_i\in B_s$ K-approximately. Then $w(p\frown G, z, G)>0$ and K-minimal in G by Theorem 4. *a)*. Let E_i be the set on which $w(p_i\frown G, z, G)$ has angular limits$=1$ almost everywhere. Then by Theorem 1 $w(p_i\frown G, z, G)=w(E_i, \xi)$ and every $U(z)\in H_0.B.$ has angular limits$=a_i$ a.e. on E_i and mes $E_i\frown E_j=0$ for $p_i\neq p_j$. Hence the only one uniquely determined bounded K-minimal function in G corresponds to every E_i. If $w(B\frown G, z, G)>w(\sum_i p_i\frown G, z, G)$, we can construnct an $\aleph$ number of linearly independent $H_0.B.$ functions. Hence if $G\in H.N.B.$ $(N=\aleph_0)$, mes $(E_B-\sum E_i)=0$. Conversely suppose mes $(E_B-\sum E_i)=0$. Then every function $\in H_0.B$ is a linear form of $\{w(G\frown p_i, z, G)\}$, Hence we have *b)*.

Proof of d). This is clear by c).

12. Class O_{H_0D} and H_0.N.D. Let $N'(z, p)$ be a harmonic function with one logarithmic singularity at p, $N'(z, p)=0$ on ∂G and $N'(z, p)$ has minimal D*irichlet integral. Then $N'(z, p)$ is uniquely determined. We call $N'(z, p)$ N-Green's function of G. Let $G(z, p)$ be the Green's function of G. If $N'(z, p)>G(z, p)$ for at least one point p, we say that G *has the ideal boundary of positive capacity.*

Theorem 18. *a) If $N'(z_0, p)>G(z_0, p)$ for two points p and z_0 then $N'(z, q)>G(z, q)$ for any points q* and z in G.

b) Let $E_{N,A}$ be the set where $N'(z, p)$ has angular limits$=0$ (which is not the image of ∂G), we call $E_{N,A}$ N-annexed relative boundary. Then every $U(z)\in H_0.D.$ function in G has angular limits$=0$ a.e. on $E_{N,A}$ and $E_{N,A}\supset E_{K,A}$.

c) The following conditions are equivalent.

1) G has the ideal boundary of positive capacity, i.e. $N'(z, p)>G(z, p)$[19].

2) $G\notin 0_{H_0D}$.

3) mes $(E_B-E_{N,A})>0$.

Proof of a). $N'(z, p)\geqq G(z, p)$. If for a point z_0 $N'(z_0, p),=G(z_0, p)$, $N'(z, p)\equiv G(z, p)$. This is a contradiction. Hence $N'(z, p)>G(z, p)$ for any points p and z. Next by the Green's formula $N'(p, q)\equiv N'(q, p)$ and $G(p, q)=G(q, p)$, whence $N'(p, q)-G(p, q)>0$ and $N'(z, q)>G(z, q)$. Hence we have a).

Proof of b). If mes $E_{N,A}=0$, our assertion is trivial. Suppose mes $E_{N,A}>0$. Let $U(z)\in H_0.D.$ Then $U(z)$ has angular limits a.e. on $|\xi|=1$. Suppose $U(z)$ has angular limits$\neq 0$ a.e. on $E_{N,A}$. Then we can find a closed set E' in $E_{N,A}$ and a number $a>0$ such that $U(z)$ has angular limits $>a$ or $<-a$ a.e. on E'. Without loss of generality, we can suppose that $U(z)$ has angular limits$>a>0$ a.e. on E'. Put $\Omega_a=E[z\in G: U(z)>a]$ and $\Omega_0=E[z\in G: U(z)>0]$. Let $\omega(G\frown\Omega_a, z, \Omega_0)$ be $C.P.$ of $(G\frown\Omega_a)$ relative Ω_0. Then by the Dirichlet priciple

$$D(\omega(G\frown\Omega_a, z, G))\leqq D(\omega(G\frown\Omega_a, z, \Omega_0))\leqq\frac{D(U(z))}{a^2}\leqq M<\infty.$$

Let ${}_N\Omega_\varepsilon=E[z\in G: N'(z, p)<\varepsilon]$. Then by $({}_N\Omega_\varepsilon\frown\Omega_a)\subset\Omega_a$

$$D(\omega({}_N\Omega_\varepsilon\frown\Omega_a, z, G))\leqq D(\omega(G\frown\Omega_a, z, G))\leqq M.$$

19) This condition is a far simpler for $G\notin 0_{H_0D}$ than the condition given by A. Mori: On the existence of harmonic functions on a Riemann surface. Journ of the Faculty of Science University Tokyo. Vol. VI, 1951.

Now as usual it can be proved that H.M. $w({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z, G)$ of $({}_N\Omega_\varepsilon \frown \Omega_a \frown B) \geqq w(E', \xi) > 0$, because $N'(z, p)$ has angular limits $= 0 < \varepsilon$ on E' and $U(z) > a$ a.e. on E'. Hence

$$0 < w({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z, G) \leqq \omega({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z, G).$$

Let F_n be a closed set in G such that $F_n \uparrow F$. If $D(\omega(F, z, G)) < \infty$. Then $D(\omega(F_n, z, G)) \leqq D(\omega(F, z, G))$. Hence $D(\lim_n \omega(F_n, z, G)) \leqq \lim_n D(\omega(F_n, z, G)) \leqq D(\omega(F, z, G))$. Now $\lim_n \omega(F_n, z, G)) = 1$ on F except at most a set of capacity zero and has M.D.I. On the other hand, $\omega(F, z, G)$ has also M.D.I. Hence by Lemma 1. *b*) of P

$$\omega(F, z, G) = \lim_n \omega(F_n, z, G).$$

Put $G_n^* = E\left[z \in G : \delta(z, \partial G) \geqq \frac{1}{n}\right] \frown R_n$. Then ${}_N\Omega_\varepsilon \frown \Omega_a = \lim_n (G_n^* \frown {}_N\Omega_\varepsilon \frown \Omega_a)$. Hence for the same ε used for ${}_N\Omega_\varepsilon$ and a point $z_0 \notin ({}_N\Omega_\varepsilon \frown \Omega_a)$, there exists a number n_0 such that

$$\omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*, z_0, G) > \omega({}_N\Omega_\varepsilon \frown \Omega_a, z_0, G) - \varepsilon, \tag{55}$$

Then also by the Dirichlet principle

$$D(\omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*, z, G)) \leqq D(\omega({}_N\Omega_\varepsilon \frown \Omega_a, z, G)) \leqq M.$$

Let $N_n'(z, z_0)$ be a harmonic function in $G \frown R_n$ with a logarithmic singularity at $z_0 \notin ({}_N\Omega_\varepsilon \frown \Omega_a)$ such that $N_n'(z, z_0) = 0$ on $\partial G \frown R_n$ and $\frac{\partial}{\partial n} N'_n(z, z_0) = 0$ on $\partial R_n \frown G : n \geqq n_0$. Then $N_n'(z, z_0) \Rightarrow N'(z, z_0)$. Let $\omega_n(z)$ be a harmonic function in $(G \frown R_n) - ({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)$ such that $\omega_n(z) = 0$ on $\partial G \frown R_n$, $\omega_n(z) = 1$ on $({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)$ and $\frac{\partial}{\partial n}\omega_n(z) = 0$ on $\partial R_n \frown G$. Then $\omega_n(z) \Rightarrow \omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*, z, G)$ and by the compactness of $\partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)$

$$\int_{\partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)} \frac{\partial}{\partial n}\omega_n(z) ds = D(\omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*, z, G)) \leqq M.$$

By the Green's formula

$$\int N_n'(z, z_0) \frac{\partial}{\partial n}\omega_n(z) ds = \int \omega_n(z) \frac{\partial}{\partial n} N_n'(z, z_0) ds.$$

By $\int_{\partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)} \omega_n(z) \frac{\partial}{\partial n} N_n'(z, z_0) ds = 0$ we have

$$\omega_n(z_0) = \frac{1}{2\pi} \int_{\partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)} N_n'(z, z_0) \frac{\partial}{\partial n}\omega_n(z) ds \leqq \frac{1}{2\pi} \max_{z \in \partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)} N_n'(z, z_0) \int_{\partial({}_N\Omega_\varepsilon \frown \Omega_a \frown G_{n_0}^*)} \frac{\partial}{\partial n}\omega(z) ds$$

$$\leqq \frac{1}{2\pi} \max_{z \in {}_N\Omega_\varepsilon} N_n'(z, z_0) M. \quad \text{Let } n \to \infty. \quad \text{Then}$$

$$\omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G^*_{n_0}, z_0, G) \leqq \frac{\varepsilon}{2\pi} M \text{ by } N'(z, z_0) < \varepsilon \text{ in } {}_N\Omega_\varepsilon. \tag{56}$$

Take $\varepsilon < \min\left(\frac{1}{M}, \frac{w({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z_0, G)}{4}\right)$. Then by (55) and (56)

$$0 < w({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z_0, G) \leqq \omega({}_N\Omega_\varepsilon \frown \Omega_a \frown G^*_{n_0}, z_0, G) + \varepsilon \leqq \varepsilon + \frac{\varepsilon}{2\pi} M$$
$$\leqq \frac{w({}_N\Omega_\varepsilon \frown \Omega_a \frown B, z_0, G)}{2}.$$

This is a contradiction. Hence mes $E' = 0$ and every $U(z) \in H_0.D.$ has angular limits $= 0$ a.e. on $E_{N,A}$. There exists a number M such that $N(z, p) < M$ in $G \frown (R - R_{n_0})$ for a number n_0, whence $N(z, p) \leqq M\, w(G \frown B, z, G)$ in $(R - R_{n_0})$. Hence $E_{K,A} \subset E_{N,A}$. Thus we have *b*).

Proof of c). Clearly by $\underset{\upsilon(p)}{D}(N(z, p) - G(z, p)) < \infty$, $\underset{G-\upsilon(p)}{D}(N(z, p)) < \infty$ and $\underset{R-\upsilon(p)}{D}(G(z, p)) < \infty$ we have $D(N(z, p) - G(z, p)) < \infty$, where $\upsilon(p)$ is a neighbourhood of p. Hence $N(z, p) - G(z, p) \in H_0.D.$ and 1) implies 2).
Assume mes $(E_B - E_{N,A}) = 0$. Then by *b*) $U(z) \in H_0.D.$ has angular limits $= 0$ a.e. on $|\xi| = 1$, whence $G \in 0_{H_0,D}$. Hence (2) implies (3).
Assume $N'(z, p) = G(z, p)$. Then $N'(z, p)$ has angular limits $= 0$ a.e. on $|\xi| = 1$, whence mes $(E_B - E_{N,A}) = 0$. Hence (3) implies (1).

Suppose that G has the ideal boundary of positive capacity, i.e. $N'(z, p) - G(z, p) > 0$. In G we can introduce N-Martin's topology by use of N-Green's function of G, but $\int_{\partial G} \frac{\partial}{\partial n} N(z, p) ds$ is not necessarily $= 2\pi$ but $\leqq 2\pi$. Then we have Theorem 16. Now as in case of H.N.B. it is more useful to characterize the class H_0.N.D. by the N-Martin's topology of $R - R_0$. Even when $\sup N(z, p) = \infty : p \in R - R_0 + B_1$, we say that G *contains* p *N-approximately, if* $N(z, p) > {}_{CG}N(z, p)$ *and if* ${}_p({}_{CG}N(z, p)) = 0$. We denote by B_S and $B_{g,r}$ the sets of singular points, of regular points for the Green's function of $\overline{R} - R_0$.

Superharmonicity in a domain G. Let $U(z)$ be a function of C_1-class such that $U(z) = 0$ on ∂G, $U(z) > 0$ in G and $D(\min(M, U(z))) < \infty$ for $M < \infty$. Let D be a domain with compact ∂D. Let ${}_{\bar{D}}U^M(z)$ be a function in G such that ${}_{\bar{D}}U^M(z) = \min(M, U(z))$ on D, ${}_{\bar{D}}U^M(z) = 0$ on ∂G and ${}_{\bar{D}}U^M(z)$ has M.D.I. over $G - D$. Put ${}_{\bar{D}}U(z) = \lim_{M=\infty} {}_{\bar{D}}U^M(z)$. If ${}_{\bar{D}}U(z) \leqq U(z)$ for any domain D with compact ∂D, we say that $U(z)$ *is* $\overline{\text{superharmonic}}$ *in* $\overline{G}$. Then it is easily seen that all theorems about $\overline{\text{superharmonic}}$ functions in $\overline{R} - R_0$ are valid for $\overline{\text{superharmonic}}$ functions in $\overline{G}$. Hence for a non compact

domain D ${}_{\tilde{D}}U(z)$ is also defined.

Theorem 19. *a) Let* $\upsilon_n(p)=E\left[z\in R:\delta(z,p)<\frac{1}{n}\right]:p\in R-R_0+B_1-B_S$ *with respect to N-Martin's topology of* $R-R_0$. *Then* $\upsilon_n(p)$ *contains* p *N-approximately.*

b) Let G *be a domain in* $R-R_0$ *such that* G *contains* $p\in(R-R_0+B_1-B_S)$ *N-approximately. Then* $N(z,p)-{}_{CG}N(z,p)$ *is superharmonic in* $\overline{G}$ *and* ${}_{\tilde{p}}\tilde{N}(z,p)=\tilde{N}(z,p)$, *where* $\tilde{N}(z,p)=N(z,p)-{}_{CG}N(z,p)$.

c) If G *contains* $p\in(B_1-B_S)\frown B_{g,r}$ *N-approximately. Then* $\tilde{N}(z,p)$ *is representable by Poisson's integral and* $\sup\tilde{N}(z,p)=\infty$. *Hence there exist infinitely many linearly independent* $H_0.D.$ *functions in* G.

Proof of a). p is the kernel of the canonical mass distribution of $N(z,p)$ and $\operatorname{dist}(p,C\upsilon_n(p))\geqq\frac{1}{n}$. Hence by Theorem 13), *c)* of P $N(z,p)>{}_{C\upsilon_n(p)}N(z,p)$. Since $\operatorname{Cap}(p)=0$, by Theorem 6) *a)* of P ${}_{C\upsilon_n(p)}N(z,p)-{}_p({}_{C\upsilon_n(p)}N(z,p))=U(z)$ is superharmonic in $\overline{R}-R_0$. Let F be the kernel of the canonical mass distribution of $U(z)$. Assume ${}_p({}_{C\upsilon_n(p)}N(z,p))>0$. Then $F+p$ is the kernel of the canonical mass distribution of ${}_{C\upsilon_n(p)}N(z,p)$ $(=U(z)+{}_p({}_{C\upsilon_n(p)}N(z,p))=U(z)+a\ N(z,p):a>0)$. On the other hand, F' (the kernel of the canonical mase distribution of ${}_{C\upsilon_n(p)}N(z,p)$) is contained in $C\upsilon_n(p)$ by Theorem 13 *d)* of P. But by the same theorem $F'=F+p$. This is a contradiction. Hence ${}_p({}_{C\upsilon_n(p)}N(z,p))=0$. Thus $\upsilon_n(p)$ contains p N-approximately.

Proof of b). Put $\tilde{N}(z,p)=N(z,p)-{}_{CG}N(z,p)$. We show that $\tilde{N}(z,p)$ is superharmonic in $\overline{G}$. Put $V_M=E[z\in G:\tilde{N}(z,p)>M]$. ${}_{\upsilon_n(p)\frown CV_M}N(z,p)$ and ${}_{\upsilon_n(p)\frown CV_M}({}_{CG}N(z,p)+M\,\omega(V_M,z))$ are superharmonic in $\overline{R}-R_0$ and harmonic in $R-R_0-(\upsilon_n(p)\frown CV_M)$ and ${}_{\upsilon_n(p)\frown CV_M}N(z,p)=N(z,p)\leqq{}_{CG}N(z,p)+M={}_{\upsilon_n(p)\frown CV_M}({}_{CG}N(z,p)+M\,\omega(V_M,z))$ on $\partial(\upsilon_n(p)\frown CV_M)+\partial R_0$, where $\upsilon_n(p)=E\left[z\in R:\delta(z,p)<\frac{1}{n}\right]$. Whence by the maximum principle

$${}_{\upsilon_n(p)\frown CV_M}N(z,p)\leqq{}_{\upsilon_n(p)\frown CV_M}({}_{CG}N(z,p)+M\,\omega(V_M,z))\leqq{}_{\upsilon_n(p)}({}_{CG}N(z,p)+M\,\omega(\upsilon_n(p),z)).$$

Let $n\to\infty$. Then by the assumption that ${}_p({}_{CG}N(z,p))=0$, we have by $\operatorname{Cap}(p)=0$

$${}_{\upsilon_n(p)\frown CV_M}({}_{CG}N(z,p)+M\,\omega(\upsilon_n(p),z))\downarrow 0 \text{ as } n\to\infty \text{ and } \lim_n {}_{\upsilon_n(p)\frown CV_M}N(z,p)=0.$$

$${}_{\upsilon_n(p)\frown V_M}N(z,p)+{}_{\upsilon_n(p)\frown CV_M}N(z,p)\geqq{}_{\upsilon_n(p)}N(z,p)\geqq{}_{\upsilon_n(p)\frown V_M}N(z,p), \text{ whence}$$

$$N(z,p)\geqq{}_{\upsilon_n(p)\frown V_M}N(z,p)\geqq\lim_n {}_{\upsilon_n(p)\frown V_M}N(z,p)=N(z,p)(=\lim_n {}_{\upsilon_n(p)}N(z,p))$$

Hence $$N(z, p) \geqq {}_{V_M}N(z, p) \geqq {}_{v_n(p)\frown V_M}N(z, p) \geqq N(z, p). \tag{56}$$
By Theorem 4. *c*) of P ${}_{V_M\frown R_n}N(z, p) \uparrow {}_{V_M}N(z, p) = N(z, p)$ as $n \to \infty$.
${}_{V_M\frown R_n}N(z, p)$ and ${}_{CG}({}_{V_M\frown R_n}N(z, p))$ are superharmonic in $\overline{R} - R_0$ by Theorem 4. *a*) of P. Now ${}_{V_M\frown R_n}(N(z, P) = N(z, p)$ on $V_M \frown R_n$ and ${}_{CG}({}_{V_M\frown R_n}N(z, p)) \leqq {}_{CG}N(z, p)$, whence
${}_{V_M\frown R_n}N(z, p) - {}_{CG}({}_{V_M\frown R_n}N(z, p)) \geqq N(z, p) - {}_{CG}N(z, p) = \widetilde{N}(z, p) \geqq M$ on $V_M \frown R_n$.
On the other hand ${}_{V_M\frown R_n}N(z, p) - {}_{CG}({}_{V_M\frown R_n}(z, p)) = 0 = \widetilde{N}(z, p)$ on ∂G. Hence
$$E[z \in G:\ 0 < {}_{V_M\frown R_n}N(z, p) - {}_{CG}({}_{V_M\frown R_n}N(z, p)) < M] = \Omega_n^M \supset \Omega^M$$
$$= E[z \in G:\ 0 < \widetilde{N}(z, p) < M].$$
Since $V_M \frown R_n$ is compact ${}_{V_M\frown R_n}N(z, p) < \max\limits_{z \in V_M\frown R_n} N(z, p) < M^* < \infty$ in $R - R_0 - (V_M \frown R_n)$, whence
$D_{R-R_0-(V_M\frown R_n)}({}_{V_M\frown R_n}N(z, p)) \leqq 2\pi M^*$ and $D_{R-R_0-(V_M\frown R_n)}({}_{CG}({}_{V_M\frown R_n}(z, p)) \leqq 2\pi M^*$ by Lemma 2. *c*) of P. Hence
$$D_{G-(V_M\frown R_n)}({}_{V_M\frown R_n}N(z, p) - {}_{CG}({}_{V_M\frown R_n}N(z, p))) < \infty.$$
Let $A_m(z)$ be a harmonic function in $R_m - R_0 - (V_M \frown R_n)$ such that $A_m(z) = {}_{V_M\frown R_n}N(z, p)$ on $\partial(V_M \frown R_n) + \partial R_0$ and $\dfrac{\partial}{\partial n} A_m(z) = 0$ on ∂R_m. Then $A_m(z) \Rightarrow {}_{V_M\frown R_n}N(z, p)$ as $m \to \infty$. Let $B_m(z)$ be a harmonic function in $G \frown R_m$ such that $B_m(z) = {}_{CG}({}_{V_M\frown R_n}N(z, p))$ on $\partial G \frown R_m$ and $\dfrac{\partial}{\partial n} B_m(z) = 0$ on $\partial R_m \frown G$. Then $B_m(z) \Rightarrow {}_{CG}({}_{V_M\frown R_n}N(z, p))$ as $m \to \infty$. Put
$$\Omega_{n,m}^M = E[z \in G:\ 0 < A_m(z) - B_m(z) < M] \frown R_m.$$
Consider the Dirichlet integral of $(A_m(z) - B_m(z))$ on $\Omega_{n\,m}^M$.
Then $D_{\Omega_{n,m}^M}(A_m(z) - B_m(z)) = M \int\limits_{\partial \Omega_{n,m}^M - \partial G} \dfrac{\partial}{\partial n}(A_m(z) - B_m(z))ds = M \int\limits_{\partial \Omega_{n,m}^M - \partial G} \dfrac{\partial}{\partial n} A_m(z)ds$
$= \int\limits_{\partial R_0} \dfrac{\partial}{\partial n} A_m(z)ds$, because $\int\limits_{\partial \Omega_{n,m}^M - \partial G} \dfrac{\partial}{\partial n} B_m(z)ds = \int\limits_{\partial R_n \frown \Omega_{n,m}^M} B(z)ds = 0.$

$A_m(z) \Rightarrow {}_{V_M\frown R_n}N(z, p) \leqq N(z, p)$ as $m \to \infty$. Hence for any given positive number ε, there exists a number m_0 such that $\int\limits_{\partial R_0} \dfrac{\partial}{\partial n} A_m(z, p)ds$
$\leqq \int\limits_{\partial R_0} \dfrac{\partial}{\partial n} {}_{V_n\frown R_n}N(z, p)ds + \varepsilon \leqq \int\limits_{\partial R_0} \dfrac{\partial}{\partial n} N(z, p)ds + \varepsilon = 2\pi + \varepsilon$, for $m \geqq m_0$. Hence
$$D_{\Omega_{n,m}^M}(A_m(z) - B_m(z)) \leqq (2\pi + \varepsilon)M \text{ for } m \geqq m_0.$$
By $A_m(z) \Rightarrow {}_{V_M\frown R_n}N(z, p)$ and $B_m(z) \Rightarrow {}_{CG}({}_{V_M\frown R_n}N(z, p))$ as $m \to \infty$,
$$\Omega_{m,n}^M \to \Omega_n^M = E[z \in G: 0 < {}_{V_M\frown R_n}N(z, p) - {}_{CG}({}_{V_M\frown R_n}N(z, p))] \text{ as } m \to \infty.$$

Let Ω be a domain completely contained in Ω_n^M. Then for any given number l there exists a number m' such that $(\Omega\frown R_l)\subset\Omega_{n,m}^M$ for $m\geqq m'$. Then by Fatou's lemma

$$D_{\Omega\frown R_l}({}_{V_M\frown R_n}N(z,p)-{}_{CG}({}_{V_M\frown R_n}N(z,p)))\leqq\varliminf_{m=\infty}D_{\Omega_{n,m}^M}(A_m(z)-B_m(z))\leqq 2\pi M.$$

Let $l\to\infty$ and then $\Omega\uparrow\Omega_n^M$. Then

$$D_{\Omega_n^M}({}_{V_M\frown R_n}N(z,p)-{}_{CG}({}_{V_M\frown R_n}N(z,p))\leqq 2\pi M.$$

Next ${}_{V_M\frown R_n}N(z,p)\uparrow{}_{V_M}N(z,p)$ as $n\to\infty$ and by Theorem 4. h) of P ${}_{CG}({}_{V_M\frown R_n}N(z,p))\uparrow{}_{CG}({}_{V_M}N(z,p))={}_{CG}N(z,p)$ as $n\to\infty$. Hence as above

$$D(\min(M,N(z,p))=D_{\Omega_M}(\widetilde{N}(z,p)-{}_{CG}N(z,p))\leqq 2\pi M. \tag{57}$$

Let D be a domain in G with compact ∂D. Then there exists a number n_0 such that $\partial D\subset R_{n_0}$. Let $J_n^M(z)$ be a harmonic function in $(G\frown R_n)-D(n\geqq n_0)$ such that $J_n^M(z)=\min(M,N(z,p))$ on $D+(\partial G\frown R_n)$ and $\dfrac{\partial}{\partial n}J_n^M(z)=0$ on $\partial R_n\frown(G-D)$, where $M>\sup\limits_{z\in\partial D}N(z,p)$. Then $D_{R_n-R_0-D}(J_n^M(z))\leqq 2\pi M$ by Lemma 2. c) of P and $J_n^M(z)\Rightarrow{}_{CG+D}N^M(z,p)$ in $G-D$ as $n\to\infty$, where ${}_{CG+D}N^M(z,p)$ is a function in $R-R_0-(CG+D)$ such that ${}_{CG+D}N^M(z,p)$ has M.D.I. over $(R-R_0-(CG+D))$ and ${}_{CG+D}N^M(z,p)=\min(M,N(z,p))$ on $CG+\partial D$.

Let $H_n^M(z)$ be a harmonic function in $(G\frown R_n)-D$ such that $H_n^M(z)=\min(M,{}_{CG}N(z,p))$ on $(\partial G\frown R_n)+\partial D$ and $\dfrac{\partial}{\partial n}H_n^M(z)=0$ on $\partial R_n\frown(G-D)$. Then $D_{(G\frown R_n)-D}(H_n(z))\leqq 2\pi M$ and $H_n^M(z)\Rightarrow{}_{CG+D}^{M}({}_{CG}N(z,p))$ in $G-D$ as $n\to\infty$.

Then $J_n^M(z)-H_n^M(z)$ is a harmonic function in $(G\frown R_n)-D$ such that $J_n^M(z)-H_n^M(z)=0$ on $\partial G\frown R_n$ (because $N(z,p)={}_{CG}N(z,p)$ on ∂G), $J_n^M(z)-H_n^M(z)=\min(M,N(z,p))=N(z,p)-{}_{CG}N(z,p)$ by $M>\sup\limits_{z\in\partial D}N(z,p)>N(z,p)-{}_{CG}N(z,p)$ on ∂D and $\dfrac{\partial}{\partial n}(J_n^M(z)-H_n^M(z))=0$ on $\partial R_n\frown(G-D)$ and $D(J_n^M(z)-H_n^M(z))\leqq 8\pi M$. Hence $(J_n^M(z)-H_n^M(z))$ has M.D.I. over $(G\frown R_n)-D$ among all functions with value 0 on $\partial G\frown R_n$ and $(N(z,p)-{}_{CG}N(z,p))$ on ∂D. Now $(J_n^M(z)-H_n^M(z))$ converges in mean as $n\to\infty$.

Put ${}_{\tilde{D}}\widetilde{N}(z,p)=\lim\limits_n(J_n^M(z)-H_n^M(z))$. Then ${}_{\tilde{D}}\widetilde{N}(z,p)$ has M.D.I. over $G-D$ with value $N(z,p)-{}_{CG}N(z,p)(=\widetilde{N}(z,p))$ on $\partial G+\partial D$ and

$${}_{\tilde{D}}\widetilde{N}(z,p)=\lim_n(J_n(z)-H_n(z))(={}_{CG+D}N^M(z,p)-{}_{CG+D}^{M}N(z,p)) \text{ for every } M.)$$

Let $M\to\infty$. Then ${}_{\tilde{D}}\widetilde{N}(z,p)={}_{CG+D}N(z,p)-{}_{CG+D}({}_{CG}N(z,p))$.
On the other hand, by Theorem 4. b) of P ${}_{CG}N(z,p)={}_{CG+D}({}_{CG}N(z,p))$. Hence

$$_{\tilde{D}}\tilde{N}(z, p) = {}_{CG+D}N(z, p) - {}_{CG+D}({}_{CG}N(z, p)) \leqq N(z, p) - {}_{CG}N(z, p) = \tilde{N}(z, p). \quad (58)$$

Put $D = \upsilon_m(p) \frown R_n \frown G$. Then $_{\tilde{D}}\tilde{N}(z, p) = {}_{CG+D}N(z, p) - {}_{CG}N(z, p)$. Let $n \to \infty$. Then by $(\upsilon_m(p) + CG) \ni p$ we have $_{\upsilon_m(p)+CG}N(z, p) = N(z, p)$. Whence by letting $m \to \infty$

$$_{\tilde{p}}\tilde{N}(z, p) = N(z, p) - {}_{CG}N(z, p) = \tilde{N}(z, p). \quad (59)$$

Assume $\tilde{N}(z, p) \leqq M < \infty$. Then $_{\tilde{p}}\tilde{N}(z, p) \leqq M\omega(p, z) = 0$. This is a contradiction.

Hence $$\sup \tilde{N}(z, p) = \infty. \quad (60)$$

By (57), (58), (59) and (60) we have *b*).

Proof of c). Put $\tilde{N}(z, p) = N(z, p) - {}_{CG}N(z, p)$. Suppose $p \in B_{g,r} \frown (B_1 - B_S)$. Then there exists no non constant generalized Green's function in $R - R_0$ smaller than $N(z, p)$ by Theorem 15. Assume that there exists a non constant generalized Green's function $\tilde{G}(z)$ in G smaller than $\tilde{N}(z, p)$. Put $V_M = E[z \in G : \tilde{G}(z) > M]$. Let $G_M^n(z)$ be a harmonic function in $R_n - R_0 - V_M$ such that $G_M^n(z) = \min(M, G(z))$ on $V_M + \partial R_n$, $G_M^n(z) = 0$ on $\partial R_0 + (\partial R_n - G)$ and has M.D.I. over $R_n - R_0 - V_M$. Then by the Dirichlet principle $D(G_M^n(z)) \leqq D(\min(M, \tilde{G}(z)) < \infty$. Put $G(z) = \lim_{M=\infty} \lim_n G_M^n(z)$. Then $G(z)$ is a generalized Green's function in $R - R_0$ smaller than $N(z, p)$. This is a contradiction. Hence there exists no generalized Green's function $G(z)$ smaller than $\tilde{N}(z, p)$. Hence $\tilde{N}(z, p)$ is Poisson's integrable and by (60) $\sup \tilde{N}(z, p) = \infty$. Thus by Theorem 16. *b*) we have *c*).

Theorem 20. *a*) *If G contains $p \in B_{S,2}$ N-approximately, G contains p K-approximately by Theorem 12. b) and there exists a harmonic domain V such that $D(\omega(V, z, G)) < \infty$ and $\omega(p \frown V, z, G) > 0$. Map the universal covering surface G^∞ of G onto $|\xi| < 1$. Let E be the set on which $\omega(p \frown V, z, G)$ has angular limits $= 1$ almost everywhere. Then* mes $E > 0$, mes $(E \frown E_{N,A}) = 0$ *and E does not depend on the domain V, in other words, let V_1 and V_2 be domains such that $\infty > D(\omega(V_i, z, G))$ and $\omega(p \frown V_i, z, G) \geqq w(p \frown V_i, z, G) > 0$ $(i = 1, 2)$ and let E_i be the set where $\omega(p \frown V_i, z, G)$ has angular limits $= 1$. Then $E_1 = E_2$. We call E the image of p relative to G. Every $U(z) \in H_0.D.$ has angular limits $=$ const a.e. on E.*

b) *Suppose G contains $p_i \in B_{S,2}$ N-appaoximately. Then $\omega(p_i \frown V, z, G)$ has angular limits $=$ const < 1 a.e. on E_j, where E_j is the image of p_j: $p_i \neq p_j$ and $p_i + p_j \in B_{S,2}$ and G contains p_j N-approximately.*

c) *If G contains p K-approximately and not contains p N-appro-*

ximately, then the set E_W *where* $w(p\frown G, z, G)$ *has angular limits*$=1$ *is contained in* $E_{N,A}$.

Proof of a). By Theorem 12. *b)* mes $E>0$ and $U(z)\in H_0.D.$ has angular limits$=a$ a.e. on E. Next $U(z)\in H_0.D.$ must have angular limits$=0$ a.e. on $E_{N,A}$, whence mes $(E\frown E_{N,A})=0$, because $\omega(p\frown V, z, G)\in H_0.D.$

Assume mes $(E_1\frown E_2)=0$. Then $\omega(p\frown V_1, z, G)=1$ a.e. on E_1 and$=a<1$ a.e. on E_2. Put $\Omega_2=E[z\in G:\ \omega(p\frown V_1, z, G)>1-2\delta]$ and $\Omega_1=E[z\in G:\ \omega(p\frown V_1, z, G)>1-\delta]$, where $\delta=\dfrac{1-a}{3}$. Then

$$D(\omega(\Omega_1, z, \Omega_2))\leqq\frac{1}{\delta^2}D(\omega(p\frown V_1, z, G))<M<\infty.$$

Put $V_{1-\varepsilon}(p)=E[z:\omega(p, z)>1-\varepsilon]$ and $V_{1-\varepsilon}=E[z\in G: w(p\frown V_1, z, G)>1-\varepsilon]$. Then $V_{1-\varepsilon}(p)\supset V_{1-\varepsilon}$. By Theorem 7. *c)* we have

$$w(Cv_n(p)\frown V_{1-\varepsilon}, z, G)\leqq\omega(Cv_n(p)\frown V_{1-\varepsilon}(p), z)\to 0 \text{ as } \varepsilon\to 0. \tag{61}$$

By P.H.3. $w(CV_{1-\varepsilon}\frown p, z, G)=0$ and by $w(p\frown\Omega_1, z, \Omega_2)\leqq w(V_{1-\varepsilon}\frown p\frown\Omega_1, z, \Omega_2)+w(CV_{1-\varepsilon}\frown p\Omega_1, z, \Omega_2)$ we have

$$w(p\frown\Omega_1, z, \Omega_2)=w(V_{1-\varepsilon}\frown p\frown\Omega_1, z, \Omega_2) \text{ for any } \varepsilon>0. \tag{62}$$

$\lim\limits_{\varepsilon=0} w(V_{1+\varepsilon}\frown v_n(p)\frown\Omega_1, z, \Omega_2)+\lim\limits_{\varepsilon=0} w(V_{1-\varepsilon}\frown Cv_n(p)\frown\Omega_1, z, \Omega_2)\geqq\lim\limits_{\varepsilon=0} w(V_{1-\varepsilon}\frown\Omega_1, z, \Omega_2)\geqq\lim\limits_{\varepsilon=0} w(V_{1-\varepsilon}\frown v_{n'}(p)\frown\Omega_1, z, \Omega_2)$. Let $n\to\infty$. Then we have by (62) and (61)

$$w(p\frown\Omega_1, z, \Omega_2)=\lim_{\varepsilon=0} w(V_{1-\varepsilon}\frown\Omega_1, z, \Omega_2).$$

Since mes $E>0$, we can find a closed set E' of positive measure in E such that both $w(p\frown V_1, z, G)$ and $\omega(p\frown V_1, z, G)$ converge uniformly along Stolz's path at every point of E'. Hence we can find a closed set E'' and a domain D containing an endart of $A(\theta)$: $e^{i\theta}\in E''$ and is bounded by $E''+\sum\partial A(\theta)$ and a circle $|\xi|=1-\dfrac{1}{m}$ such that the image of $\partial\Omega_2$ does not fall in D and the image of $\partial(V_{1-\varepsilon})$ separates E''. Hence as usual, we have $w(p\frown\Omega_1, z, \Omega_2))=\lim\limits_{\varepsilon\to 0} w(V_{1-\varepsilon}\frown\Omega_1, z, \Omega_2)>0$. Hence $D(\omega(p\frown\Omega_1, z, \Omega_2))>0$ and $D(\omega(p\frown\Omega_1, z, \Omega_2))\leqq D(\omega(\Omega_1, z, \Omega_2))<M<\infty$.
Similarly we have $\infty>D(\omega(p\frown C\Omega_2, z, G))>0$.
Putting $\widetilde{G}=R-R_0$ and $B'=p$, we have $\omega(p\frown\Omega_1, z)\neq\omega(p\frown C\Omega_2, z)$ by Separation Theorem *S.* 1. But these have their masses at only p and by P.C.3. $\sup\omega(p\frown C\Omega_2, z)=1=\sup\omega(p\frown\Omega_1, z, \Omega_2)$ whence these are equal to $\omega(p, z)$. This is a contradiction. Hence mes $(E_1\frown E_2)>0$. On the other hand, $U(z)\in H_0.D.$ has angular limits a_1 a.e. on E_1 and a_2 on E_2, but by mes $(E_1\frown E_2)$

>0, $a_1=a_2$ and $E_1=E_2$. Thus E does not depend on V.

Proof of b). Map the universal covering surface $(R-R_0)^\infty$ onto $|\xi|<1$. Let E_i be the set where $\omega(p_i, z)$ has angular limits$=1$. Suppose $\omega(p_i, z)$ has angular limits$=a$ a.e. on E_j $(i\neq j)$. Assume $a=1$. Put $V^j_{1-\varepsilon}=[z\in R: w(p_j, z)>1-\varepsilon]$.
By Theorem 10. *b)* $w(p_i, z)\geqq w(E_i, \xi)$. Since $a=1$, $\omega(p_i, z)\geqq w(E_j, \xi)$, whence $\omega(p_i, z)>1-\varepsilon$ on $V^j_{1-\varepsilon}$. Hence by $w(p_j, z)\geqq(1-\varepsilon)$ on $V^j_{1-\varepsilon}$ $\omega(p_i, z)$ $\geqq(1-\varepsilon)\ {}_{V^j_{1-\varepsilon}}\omega(p_j, z)\geqq(1-\varepsilon)\,\omega(V^j_{1-\varepsilon}\frown\upsilon_n(p_j), z)\geqq(1-\varepsilon)\,\omega(p_j, z)>0$. Now $\omega(p_j\frown V^j_{1-\varepsilon}, z)=\lim_n \omega(V^j_{1-\varepsilon}\frown\upsilon_n(p_j), z)$ has its mass only at p_j and by P.C.3. $\sup \omega(p_j\frown V^j_{1-\varepsilon}, z)=1$.Hence $\omega(p_j\frown V^j_{1-\varepsilon}, z)=\omega(p_j, z)$. Let $\varepsilon\to 0$. Then

$$\omega(p_i, z)\geqq\omega(p_j, z). \tag{63}$$

But by Theorem 12. *c)* of P $\omega(p_i, z)<1$ in $R-R_0-p_i$ except at most a set of capacity zero. (63) means that $\omega(p_i, z)=1$ on a set p_j of positive capacity. This is a contradiction. Hence $a<1$.

$\omega(p_i\frown V_i, z, G)\leqq\omega(p_i, z)$ and $\omega(p_j\frown V_j, z, G)\leqq\omega(p_j, z)$. Hence $\omega(p_i, z, G)$ has angular limits<1 on the image of p_j relative to G. Hence $\mathrm{mes}(E_i\frown E_j)=0$, because if $\mathrm{mes}(E_i\frown E_j)>0$, $\omega(p_i\frown V_i, z, G)$ has angular limits$=1$ a.e. on E_i+E_j.

Proof of c). Assume $E_W\not\subset E_{NA}$. Then there exists a function $U(z)\in H_0.D.$ such that $U(z)$ has angular limits$\neq 0$ a.e. on E_W. Then we can suppose without loss of generality that $U(z)$ has angular limits$>a>0$ on a set $E'(\subset E_W)$ of positive measure. Put $\Omega_0=E[z\in G: U(z)>0]$ and $\Omega_a=E\left[z\in G: U(z)>\dfrac{a}{2}\right]$. Then

$$D(\omega(p\frown\Omega, z, G))\leqq D(\omega(\Omega_a, z, \Omega_0))\leqq\frac{D(U(z))}{\left(\dfrac{a}{2}\right)^2}<\infty.$$

As usual we can fiind a closed set E'' $(\subset E')$ of positive measure such that both $U(z)$ and $w(p, z, G)$ converge uniformly along Stolz's path terminating at E'' and it can be proved that

$$w(p\frown\Omega_a, z, G)=\lim_{\varepsilon\,.\,0} w(V^W_{1-\varepsilon}\frown\Omega_a, z\in G: w(p, z, G)>0,$$

where $V^W_{1-\varepsilon}=E[z\in G: w(p, z, G)>[1-\varepsilon]$. Hence

$$\infty>D(\omega(p\frown\Omega_a, z, G))>0.$$

This means that G contains p N-approximately. This contradicts the assumption. Hence we have *c)*.

It is necessary to make many preparations to study the class $H.N.D.$ $(N=\infty$ and $=\aleph_0)$. Hence we consider only the class $H_0.N.D.$ $(N<\infty)$.

Theorem 21. *$G\in H_0.N.D.(N<\infty)$ if and only if G contains N number of points $\in B_{S,2}$ of $R-R_0$ N-approximately and does not contain any point of $B_{g,r}\frown(B_1-B_S)+B_{S,1}$ N-approximately. Map the universal covering surface G^∞ onto $|\xi|<1$. Then the set Γ: $|\xi|=1$ is divided into three kinds of sets, 1° image of ∂G, 2° image of $p(\in B_{S,2})$ contained N-approximately in G and 3° E_{NA} and*

$$\text{mes}\,(E_B-E_{NA})=\sum^N \text{mes}\,E_i,$$

where E_i is the image of p_i relative to G and E_B is the image of the ideal boundary $(B\frown G)$.

Proof. Map the universal covering surface $(R-R_0)^\infty$ onto $|\zeta|<1$. Let $q\in B-B_{S,2}$ and let $\upsilon_n(q)=E\left[z\in R:\ \delta(z,q)<\dfrac{1}{n}\right]$. Let $\omega(\upsilon_n(p)\frown B,z)$ be $C.P.$ of $(\upsilon_n(q)\frown B)$ determined by $\upsilon_n(q)$. We prove the following

Lemma. *For any given point $q\in B-B_{S,2}$ and for any given positive number $\varepsilon>0$, we can find a number $n(q)$ such that the measure of the set $E_n<\varepsilon$, where E_n is the set on which $\omega(\upsilon_n(q)\frown B,z)$ has angular limits$=1$ almost every everywhere.*

Assume $\omega(\upsilon_n(q)\frown B,z)$ has angular limits$=1$ on a set E of positive measure $\delta>0$ for every n. Then by letting $n\to\infty$

$$\omega(q,z)\geqq w(E,\zeta)>0, \tag{64}$$

where $w(E,\zeta)$ is H.M. of E with respect to $|\zeta|<1$.
On the other hand, by Theorem 10. *b*) for $q\in B_{S,1}$, $w(q,z)=\lim\limits_{\varepsilon=0} w(V_{1-\varepsilon}(q),z)$, where $V_{1-\varepsilon}(q)=E[z\in R:\ \omega(q,z)>1-\varepsilon]$, i.e. $w(q,z)$ is H.M. of the set on which $\omega(q,z)$ has angular limits$=1$. But by $q\in B_{S,1}$, $w(q,z)=0$, i.e. $\omega(q,z)$ has angular limits<1 a.e. on $|\xi|=1$. For $q\in B-B_S$, we have $\omega(q,z)=0$. This contradicts (64). Hence the measure of the set E_n (where $\omega(\upsilon_n(q)\frown B,z)$ has angular limits$=1$) $\downarrow 0$ as $n\to\infty$.

We show

$$\text{mes}\,(E_B-E_{NA}-\sum E_i)>0 \ \textit{implies}\ G\in H_0.N.D.\ (N=\infty). \tag{65}$$

Put $E^*=E_B-E_{NA}-\sum E_i$. We cover $B-\sum\limits_i p_i(p_i\in B_{S,2})$ by a system of neighbourhoods $\upsilon_n(q_j)$ of $q_j\in B-B_{S,2}$ such that $\omega(\upsilon_n(q_j)\frown B,z)$ has angular limits$=1$ on the set whose measure $<\dfrac{1}{2}$ ($\upsilon_n(q_j)$ depends on q_j).

By
$$\sum_j w(B\frown\upsilon_n(q_j),z,G)+\sum_i w(p_i,z,G)\geqq w(B,z,G)$$
there exists at least one $\upsilon_n(q_j)$ such that $\text{mes}\,(E_j^W\frown E^*)>0$, where E_j^W is the set on which $w(\upsilon_n(q_j)\frown B,z,G)$ has angular limits$=1$. Denote the

above $\upsilon_n(q_j)$ by $\mathfrak{V}(q_1)$. Next cover $\mathfrak{V}(q_1)$ by $\upsilon_{n'}(q'_j)$: $q'_j \in B - B_{S,2}$ such that the measure of the set (where $\omega(\upsilon_{n'}(q'_j) \frown B, z)$ has angular limits $=1$) $< \frac{1}{2^2}$. Then there exists at least one $\upsilon_{n'}(q'_j)$ such that mes $(E_j^W \frown E^*) > 0$, where E_j^W is the set where $w(\upsilon_{n'}(q'_j) \frown \mathfrak{V}(q_1) \frown B, z, G)$ has angular limits $=1$. Put $\mathfrak{V}_2(q) = \mathfrak{V}_1(q) \frown \upsilon_{n'}(q'_j)$. In this way we can find a sequence

$$\mathfrak{V}_1(q_1) \supset \mathfrak{V}_2(q_2) \supset \cdots .$$

This sequence has the following properties:

1°. *The measure of the set where* $\omega(\mathfrak{V}_n(q_n) \frown B, z)$ *has angular limits* $=1$ $< \frac{1}{2^n}$.

2° *Let* E_n^W *be the set where* $w(\mathfrak{V}_n(q_n) \frown B, z, G)$ *has angular limits* $=1$. *Then* $E_1^W \supset E_2^W \supset \cdots$, mes $(E_n^W \frown E^*) = \alpha_n > 0$ *and by* $w(\mathfrak{V}_n(q_n) \frown B, z, G) < \omega(\mathfrak{V}_n(q_n) \frown B, z)$ and $\alpha_n < \frac{1}{2^n}$ we have $\lim_n \alpha_n = 0$.

From the above sequence we extract a subsequence as follows:

Put $\mathfrak{V}'_1(q_1) = \mathfrak{V}_1(q_1)$. Let n_1 be the least integer such that $\frac{1}{2^{n'}} < \alpha_1$. Put $\mathfrak{V}'_2(q_2) = \mathfrak{V}_{n'}(q_{n'})$. Suppose $\mathfrak{V}'_n(q_n)$ is defined. Let n'' be the least integer such that $\frac{1}{2^{n''}} < \alpha_n$ and put $\mathfrak{V}'_{n+1}(q_{n+1}) = \mathfrak{V}_{n''}(q_{n''})$. Then we have a sequence

$$\mathfrak{V}'_1(q_1) \supset \mathfrak{V}'_2(q_2) \supset \cdots .$$

The above sequence has the following properties:

1°. *The measure of the set where* $\omega(\mathfrak{V}'_n(q_n) \frown B, z)$ *has angular limits* $=1$ $< \alpha_{n-1}$.

2°. *The measure of the set* $(E_n'^W \frown E^*)$ *(where* $w(\mathfrak{V}'_n(q_n) \frown B, z, G)$ *has angular limits* $=1$ on $E_n'^W) = \alpha_n$, $\alpha_n > 0$ and $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$: $\lim_n \alpha_n = 0$.

Let l be a given integer, we shall construct l number of linearly independent $H_0.D.$ functions in G. $N(z, p) - G(z, p) > 0$ on E^*. We can find a closed set $\widehat{E}$ in $E^* \frown E_1^W$ such that mes $((E^* \frown E_1^W) - \widehat{E}) < \varepsilon^0$ and $N(z, p) - G(z, p) > \delta > 0$ on $\widehat{E}$, where E_1^W is the set on which $w(\mathfrak{V}'_1(q_1) \frown B, z, G)$ has angular limits $=1$ and $\varepsilon_0 = \min_{n=1,2,\cdots l-1} (\alpha_n - \alpha_{n+1})$.

Put $\Omega = E\left[z \in G : N(z, p) - G(z, p) > \frac{\delta}{2}\right]$. Then

$$D(\omega(\Omega, z, G)) \leqq \frac{D(N(z, p) - G(z, p))}{\left(\frac{\delta}{2}\right)^2} < M < \infty .$$

Since $w(B \frown \mathfrak{V}'_n(q_n), z, G) = 1$ a.e. on $E_n'^W \frown \widehat{E}$ $(n = 1, 2, \cdots l)$, it can be proved as

usual $M>D(\omega(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G))$, $\omega(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G)\geqq w(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G)$ and $w(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G)=1$ a.e. on $E_n'^W\frown\widehat{E}$, where $E_n'^W$ is the set where $w(B\frown\mathfrak{V}'_n(q_n), z, G)$ has angular limits$=1$.

Let E_n^ω be the subset of E^* where $\omega(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G)=1$ and let $\widehat{\alpha}_n$ be its measure. Then by $\omega(\Omega\frown B\frown\mathfrak{V}'_n(q_n), z, G)\leqq\omega(B\frown\mathfrak{V}'_n(q_{n'}), z)$ $\alpha_{n-1}\geqq\widehat{\alpha}_n$. Next by $\omega(\Omega\frown B\frown\mathfrak{V}'_n(q), z, G)\geqq w(\Omega\frown B\frown\mathfrak{V}'_n(q_{n'}), z, G)$ and $(E_n^\omega\frown\widehat{E})\supset(E_n'^W\frown\widehat{E})$ we have $\widehat{\alpha}_n\geqq\alpha_n-\varepsilon_0$, Hence $\alpha_{n-1}\geqq\widehat{\alpha}_n\geqq\alpha_n-\varepsilon_0$. Whence by the property of ε_0

$$\widehat{\alpha}_1>\widehat{\alpha}_2>\widehat{\alpha}_3>\cdots>\widehat{\alpha}_l. \tag{66}$$

Clearly by $(\Omega\frown B\frown\mathfrak{V}'_n(q_n))\supset(\Omega\frown B\frown\mathfrak{V}'_{n+1}(q_{n+1}))$ and by (66) $\{\omega(\Omega\frown B\frown\mathfrak{V}'_n(q), z, G)\}$ is linearly independent and $\in H_0.D.$ Now l is any integer. Hence $G\in H.N.D.$ $(N=\infty)$.

We show if G contains $p_k\in B_{S,2}$ $(k=1, 2,\cdots n_0)$, then we can construct n_0 number of linearly indepent $H_0.D.$ functions.

Since $C.P.$ of p_k: $\omega(p_k\frown V, z, G)$ dose not depend on V by Theorem 20. a), we denote it by $\omega(p_k, z, G)$ for simplicity. Let E_k be the image of $p_k\in B_{S,2}$ relative to G. Then by Theorem 20. b) $\omega(p_k, z, G)=a_j$ on E_j and $a_j<1$: $k\neq j$. Put $a^k=\max\limits_{\substack{j\neq k\\ j=1,\cdots n_0}} a_j$ and put $\Omega=E[z\in G: \omega(p_k, z, G)>a^k+\varepsilon]$. Let $V=E[z\in G: \omega(p_k, z, G)>1-\varepsilon]$: $1-2\varepsilon>a^k$. Let $\omega^k_{n,n+i,n+i+j}(z)$ be a harmonic function in $(G\frown R_{n+i+j})-((\Omega-V)\frown(R_{n+i+j}-R_{n+i}))-(V\frown(R_{n+i}-R_n))$ such that $\omega_{n,n+i,n+i+j}(z)=0$ on $\partial R_{n+i+j}-\Omega+\partial G$, $\dfrac{\partial}{\partial n}\omega_{n,n+i,n+i+j}(z)=0$ on $\partial R_{n+i}\frown(\Omega-V)$ and $\omega_{n,n+i,n+i+j}(z)=1$ on $((R_{n+i}-R_n)\frown V)$. Then $\omega_{n,n+i,n+i+j}(z)\uparrow\omega_{n,n+i}(z)$, $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ and $\omega_n(z)\downarrow\omega^k(z)$. It can be easily proved that $\omega^k(z)=1$ a.e. on E_k and$=0$ a.e. on $\sum\limits_{j\neq k}E_j$ and $D(\omega^k(z))\leqq\dfrac{D(\omega(p_k, z, G))}{(1-\varepsilon+a)^2}$. Whence $\{\omega^k(z)\}$ is linearly independent and $\in H_0.D.$

Proof of the theorem. Suppose $G\in H_0.N.D.(N<\infty)$. Then by Theorem 19. c) G does not contain any point of $B_{g,r}\frown(B_1-B_S)$ N-approximately and by (65) mes $(E_B-E_{N,A})=$mes $\sum E_i$. Let N' be the number of points $\in B_{S,2}$ contained in G N-approximalely. Then we can construct N' number of linearly independent $H_0.D.$ functions. On the other hand, every $H_0.D.$ function has angular limits$=a_i$ on E_i, whence $G\in H_0.N'.D.$ and $N=N'$. Conversely if G contains $p_i\in B_{S,2}$ $(i=1, 2,\cdots, N)$ N-approximately and mes $(E_B-E_{N,A})=$mes $\sum E_i$, we have $G\in H.N.D.$ Thus we have the theorem.

By Theorems 16 and 20 we have easily the following

Corollary. *If $R-R_0\in H.N.D.$ $(N\leqq\infty$ and $N=\aleph_0)$, $G\in H_0.N'.D.$ and $N'\leqq N$.*

In fact, $w(B-B_{S,2}, z)=0$ implies $\text{mes}\,(E_B-E_{N,A})=\text{mes}\sum_i E_i$, where E_i is the image of $p_i \in B_{S,2}$ contained in G N-approximately. Hence we have by Theorem 12. *b*), $G \in H_0.N.D.$ and $N' \leqq N$.

13. Covering property of Riemann surfaces.

Consider an analytic function $w=f(z)$: $z \in R$. Let w_0 be a point of the w-plane. Then the part of R over $|w-w_0|<r$ consists of at most an enumerably infinite number of components. Such one component is called a *connected piece* on $|w-w_0|<r$. Then

Theorem 22. *a) Let $R \in H.N.B.(N \leqq \infty, N=\aleph_0)$. Then every connected piece G over $|w-w_0|<r$ covers $|w-w_0|<r$ except at most a set of capacity zero.*

b) Let $R \in H.N.D.(N \leqq \infty, N=\aleph_0)$. Let G be a connected piece over $|w-w_0|<r$. If the area of G is finite, G covers $|w-w_0|<r$ except at most a set of capacity zero.

Proof of a). Suppose that G does not cover a set F (clearly closed) of positive capacity. Then there exists a subset F_{r_0} of F of positive capacity such that F_{r_0} is contained in the circle $|w-w_0|<r_0<r$. Let $T(w)$ be a bounded harmonic function in $E[w: |w-w|<r]-F_{r_0}$ such that $T(w)=0$ on $|w-w_0|=r$ and $T(w)=1$ on F_{r_0}. Then $T(z)=w(T(z))>0$ in G and $=0$ on ∂G. Hence $w(B \frown G, z, G) \geqq T(z)>0$.

$$w(B \frown G, z, G) \leqq w(B-B_s, z, G)+w(B_s \frown G, z, G) \leqq \sum w(B \frown p_i, z, G),$$

where $p_i \in B_s$ and $w(B-B_s, z, G)=0$ by $R \in H.N.B$.
Hence there exists at least one point $p \in B_s$ such that $w(p \frown G, z, G)>0$. This means that G contains p K-approximately. Hence by Theorem 4. *b*) there exists no non constant bounded analytic function. On the other hand, $f(z)$ is bounded in G. This is a contradiction. Hence we have *a*).

Proof of b). Suppose that G does not cover a set of positive capacity in $|w-w_0|<r$. Then there exists a number r_0 such that F_{r_0} of F in $|w-w_0|<r_0<r$ is of positive capacity. Put $\Omega=E[z \in G: |f(z)-w_0|<r_0]$. Then $w(\Omega \frown B, z, G) \geqq T(z)$, where $T(z)$ is the function used in *a*). Since $R \in H.N.D.$, $w(B-B_{S,2}, z)=0$, hence there exists at least one point $p \in B_{S,2}$ such that $w(\Omega \frown p, z, G)>0$. Let $S(w)$ be a continuous function in $|w-w_0|<r$ such that $S(w)$ is harmonic in $r_0<|w-w_0|<r$, $S(w)=0$ on $|w-w_0|=r$ and $S(w)=1$ on $|w-w_0| \leqq r_0$. Put $S(z)=S(f(z))$. Then

$$D(S(z)) \leqq M^2 A,$$

where A is the area of G and $M=\max\left\{\left|\frac{\partial S(w)}{\partial u}\right|, \left|\frac{\partial S(w)}{\partial v}\right|\right\}$: $w=u+iv$.

Hence $0<w(\Omega\frown p, z, G)\leqq\omega(\Omega\frown p, z, G)$ and $D(\omega(\Omega\frown p, z, G))\leqq D(S(z))<\infty$. This means G contains $p\in B_{S,2}$ N-approximately. Hence by Theorem 12. a) there exists no non constant analytic function with finite area in G. But $f(z)$ has finite area. This is a contradiction. Hence we have b).

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(Received July 20, 1961)