SINGULAR POINTS OF RIEMANN SURFACES

By

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Let $D_r$ be a ring domain such that $r<|z|<1$. Let $U_r(z)$ be a harmonic function in $D_r$ such that $U_r(z)=0$ on $|z|=1$ and $U_r(z)=1$ on $|z|=r$. Put $M_r = \max \left( \left| \frac{\partial U_r(z)}{\partial x} \right|, \left| \frac{\partial U_r(z)}{\partial y} \right| \right)$: $z=x+iy$. I wrote that $M_r \rightarrow 0$ as $r \rightarrow 0$. This is false. Hence our proof of Theorem 8 in the previous paper depending on the above fact is incomplete. The purpose of the present paper is to correct the above theorem, to simplify the other theorems and further to discuss new results. For the sake of convenience, I shall begin from the first.

Let $R$ be a Riemann surface with positive boundary. Let $\{R_n\}$ be its exhaustion with compact relative boundaries $\partial R_n (n=0,1,2,\cdots)$. We proved the following.

Theorem. Let $R \in O_{HB}(O_{HD})-O_G$. Then $R-R_0 \in O_{AB}(O_{AD})$.

Above theorem means that the boundary of a Riemann surface $R$ which is so complicated as $R \in O_{HB}-O_G(O_{HD}-O_G)$ cannot be represented in any way as a covering surface over a bounded domain (a covering surface with finite area). Hence we propose the following questions:

1) What part of the boundary does generate the above singular fact?
2) What is the method to characterize the singularity of the boundary?

To solve the questions, we must define the boundary of $R$. There are various methods to define the ideal boundary of $R$. But we understand that the following Martin's topologies are most available.

PART I

K-Martin's topology. Let $G(z, p_i)$ be the Green's function of with pole at $p_i$. Put $K(z, p_i) = \frac{G(z, p_i)}{G(p_0, p_i)}$, where $p_0$ is a fixed point. Suppose

\{p_i\} is a divergent sequence of points. We call \{p_i\} a fundamental sequence determining an ideal boundary point, if \{K(z,p_i)\} converges uniformly in every compact domain of \(R\). If \{K(z,p_i)\} and \{K(z,p'_i)\} determine the same limit function, we say that \{p_i\} and \{p'_i\} define the same ideal boundary point. We denote by \(B\) the set of all the ideal boundary points. We define the distance between two points \(p\) and \(q\) of \(R+B\) (denoted by \(\overline{R}\)) by

\[
\delta(p, q) = \sup_{z \in \overline{R}} \left| \frac{K(z,p)}{1+K(z,p)} - \frac{K(z,q)}{1+K(z,q)} \right|
\]

Let \(U(z)\) be a positive superharmonic function in \(R\). If \(V(z)=aU(z)\): \(0\leq a\leq 1\) for every positive superharmonic function \(V(z)\) such that \(0\leq V(z) \leq U(z)\), \(U(z)\) is called a K-minimal function. Let \(K(z,p)\) be the above function. \(p\) is called a K-minimal point or K-non minimal point according as \(K(z,p)\) is K-minimal or not. R.S. Martin proved the following:

1) Let \(K_{vn}(z,p)\) be the lower envelope of positive superharmonic functions in \(R\) larger than \(K(z,p)\) in \(\nu(p) = E \left[ z \in \overline{R} : \delta(z,p) < \frac{1}{n} \right] \). Then \(\lim_{n=\infty} K_{vn}(z,p) = K(z,p)\) or 0 according as \(K(z,p)\) is K-minimal or not.

2) Every point of \(R\) is K-minimal. Let \(B_1\) and \(B_0\) the sets of K-minimal boundary points and non K-minimal points respectively. Then \(B=B_1+B_0\) and \(B_0\) is an \(F_\sigma\) set of harmonic measure zero.

3) Let \(F\) be a closed set in \(\overline{R}\) and let \(U(z)\) be a positive superharmonic function and let \(U_{F_n}(z)\) be the lower envelope of superharmonic functions larger than \(U(z)\) in \(F_n = E \left[ z \in \overline{R} : \delta(z,F) \leq \frac{1}{n} \right] \). Then \(U_F(z) = \lim_{n=\infty} U_{F_n}(z)\) is represented by a positive mass on \(F\).

4) Every positive superharmonic function is represented by a positive mass distribution on \(R+B_1\) which is uniquely determined and called a canonical mass distribution.

5) If \(U(z) = \int_{A} K(z,p) d\mu(p)\) is K-minimal, \(U(z)\) is a multiple of \(K(z,p): p \in A\).

1. Harmonic measure of sets in \(\overline{R}\). Let \(F\) be a closed set in \(\overline{R}\) and put \(F_n = E \left[ z \in R : \delta(z,F) \leq \frac{1}{n} \right] \). Let \(w(F_n, z, R_m)\) be a harmonic function in \(R_m - F_n\) such that \(w(F_n, z, R_m) = 0\) on \(\partial R_m - F_n\) and \(w(F_n, z, R_m) = 1\) on \((R_m \cap F_n)\). Put \(w(F, z, R) = \lim_{n=\infty} \lim_{m=\infty} w(F_n, z, R_m)\) and call it H.M. (harmonic measure) of \(F\). Let \(G\) be an open set in \(B\). We define \(w(G, z, R)\) as
\[
\lim_{n} \lim_{n} w_{m,n}(z), \quad \text{where} \quad w_{m,n}(z) \quad \text{is a harmonic function in} \quad R_{m}-(CG)_{n} \quad \text{such that} \quad w_{m,n}(z) = 1 \quad \text{on} \quad \partial R_{m}-(CG)_{n} \quad \text{and} \quad w_{m,n}(z) = 0 \quad \text{on} \quad \partial((CG)_{n})\text{.}
\]

Clearly we have \(w(G, z, R) = 1 - w(CG, z, R)\). For an \(F_{\sigma}\) set: \(F_{\sigma} = \bigcup_{i} F_{i}\) we define \(w(F_{\sigma}, z, R)\) by \(\lim_{n} w(\sum F_{i}, z, R)\) and for \(G_{\delta}\) in \(B, i, \varepsilon, G_{\delta} = \bigcap_{i} G_{i}\) we define \(w(G_{\delta}, z, R)\) by \(\lim_{n} w(\bigcap_{i} G_{i}, z, R)\).

Let \(R^\infty\) be the universal covering surface of \(R\) and map \(R^\infty\) conformally onto \(|\xi| < 1\). If \(\sup_{z \in R} K(z, p) < \infty\): \(p \in R + B_{1}\), we call \(p\) a singular point. Clearly \(K(z, p) = aG(z, p): a > 0\) for \(p \in R\), where \(G(z, p)\) is the Green's function of \(R\). Hence every point of \(R\) is not singular.

We denote by \(B_{s}\) the set of singular points. Then \(B_{s} \subset B_{1}\).

Put \(F = p\) and let \(w(p, z, R)\) be the H.M. of \(p\), i.e. \(w(p, z, R) = \lim_{n} w(u_{n}(p), z, R)\). Then

\[
\text{Theorem 1.} \quad a) \quad w(p, z, R) > 0 \quad \text{if and only if} \quad p \in B_{s}, \quad \text{and in this case} \quad w(p, z, R) = aK(p, z): a > 0.
\]

\[
b) \quad \text{Consider} \quad K(z, p): p \in B_{s} \quad \text{in} \quad |\xi| < 1. \quad \text{Then there exists a set} \quad E \quad \text{of positive measure such that} \quad K(z, p) \quad \text{has angular limits} = M = \sup_{z \in R} K(z, p) \quad \text{on} \quad E \quad \text{and} = 0 \quad \text{on} \quad CE \quad \text{almost everywhere}. \quad \text{Hence} \quad w(p, z, R) = w(E, \xi), \quad \text{where} \quad w(E, \xi) \quad \text{is H.M. of} \quad E \quad \text{with respect to} \quad |\xi| < 1. \quad \text{We call} \quad E \quad \text{the image of} \quad p. \quad \text{Then} \quad mE_{i} = 0, \quad \text{mes}(E_{i} \cap E_{j}) = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad \sum \text{mes} E_{i} \leq 2\pi, \quad \text{whence} \quad B_{s} \quad \text{is at most enumerable}. \quad \text{Let} \quad w(\sum p_{i}, z, R) = \lim_{n} w(\sum_{i} p_{i}, z, R). \quad \text{Then}
\]

\[
\sum_{i} w(p_{i}, z, R) = w(\sum_{i} p_{i}, z, R) \quad \text{for} \quad p_{i} \in B_{s}.
\]

\[
d) \quad \text{Let} \quad E_{i} \quad \text{be the image of} \quad p_{i} \in B_{s}. \quad \text{Then} \quad \sum \text{mes} E_{i} = 2\pi \quad \text{if and only if} \quad H.M. \quad \text{of} \quad B - \sum p_{i} = 0.
\]

\[
e) \quad \text{Let} \quad E \quad \text{be the image of a point} \quad p \quad \text{of} \quad B_{s}. \quad \text{Then every bounded harmonic (Poisson's integrable) function} \quad U(z) \quad \text{has angular limits} = \text{const} \ a.e. \quad \text{almost everywhere} \quad \text{on} \quad E.
\]

\[
\text{Proof of} \quad a). \quad \text{Assume} \quad w(p, z, R) > 0. \quad \text{Then by} \quad 2) * \quad p \in B_{o}. \quad \text{Hence} \quad p \in B_{1} + R \quad \text{and} \quad K(z, p) \quad \text{is K-minimal}. \quad \text{Suppose} \quad \sup_{z \in R} K(z, p) = \infty. \quad \text{Then} \quad w(p, z, R) = \lim_{n} w(u_{n}(p), R_{m}): u_{n}(p) = E[z \in R: \delta(z, p) < \frac{1}{n}]. \quad \text{Then by} \quad 3) * \quad w(p, z, R) \quad \text{is represented by a mass on} \quad \bigcap_{n} \overline{u}_{n}(p) = p, \quad \text{i.e.} \quad w(p, z, R) = aK(z, p). \quad \text{Now}
\]

4) See 3), where \(\overline{u}_{n}(p)\) means the closure of \(u_{n}(p)\).
$w(p, z, R) \leq 1$ and $\sup_{z \in R} K(z, p) = \infty$, whence $a = 0$ and $w(p, z, R) = 0$. Next suppose $\sup_{z \in R} K(z, p) \leq M < \infty$. Then $w(v_n(p), z, R) \geq \frac{K_{v_n(p)}(z, p)}{M}$ by $p \in (R + B_1)$ and $w(p, z, R) \geq \frac{K(z, p)}{M} > 0$. Hence $w(p, z, R) > 0$ if and only if $p \in B_s$.

**Proof of b.** Suppose $\sup_{z \in R} K(z, p) = M$. Let $E$ and $E'$ be sets on $|\xi| = 1$ such that $K(z, p)$ has angular limits $\geq M - \varepsilon$ on $E$ and $K(z, p)$ has angular limits between $M - 2\varepsilon$ and $\varepsilon$ on $E'$ for a positive number $\varepsilon$: $0 < \varepsilon < \frac{M}{3}$. Since $K(z, p)$ is representable by Poisson's integral, $E$ is of positive measure. Now $E'$ is a set of measure zero, because, if it were not so, we construct a harmonic function $U(\xi)$ such that $U(\xi)$ has the same angular limits as $K(z, p)$ on $E$ and $=0$ on $CE$: Then $U(\xi)$ is a function in $R$ and is not a multiple of $K(z, p)$ and $K(z, p) > U(\xi) = U(\xi) > 0$, which implies that $K(z, p)$ is not $K$-minimal. This is a contradiction. Hence by letting $\varepsilon \to 0$, $K(z, p)$ has angular limits $\sup_{z \in R} = M = \sup_{z \in R} K(z, p)$ on $E$ and $0$ on $CE$ almost everywhere. Hence $w(E, \xi) = \frac{K(z, p)}{M} = w(p, z, R)$.

**Proof of c.** Suppose $\sum \text{mes}(E_i \cap E_j) > 0$. Let $U(\xi)$ be a harmonic function in $|\xi| < 1$ such that $U(\xi)$ has angular limits $= \min(M_i, M_j)$ on $E_i \cap E_j$ and $=0$ on $C(E_i \cap E_j)$. Then for at least one of $K(z, p_i)$ and $K(z, p_j)$, $0 < U(\xi) < K(z, p_k): k = i$ or $j$ and $U(\xi)$ is not a multiple of $K(z, p_k)$. Hence $K(z, p_k)$ is not $K$-minimal. This is a contradiction. Hence $\sum \text{mes} E_i = 0$. On the other hand, by a) $\text{mes} E_i > 0$ by $w(p_i, z, R) > 0$ and $\sum \text{mes} E_i \leq 2\pi$. Hence $B_s$ is at most enumerable.

Clearly $\sum w(p_i, z, R) \geq w(\sum p_i, z, R) \geq w(p_i, z, R) i = 1, 2, \cdots k$.

By $\sum w(p_i, z, R) \geq w(\sum p_i, z, R)$ we see $w(\sum p_i, z, R) = 0$ a.e. on $C(\sum E_i)$ and by $w(\sum p_i, z, R) \geq w(p_i, z, R)$ we have $w(\sum p_i, z, R) = 1$ a.e. on $\sum E_i$. Hence $w(\sum p_i, z, R) = \sum w(E_i, \xi) + \sum w(p_i, z, R)$.

**Proof of d.** Let $p_i \in B_s$ and let $E_i$ be the image of $p_i$. Then $B - \sum p_i$ is a $G_s$ set in $B$. Hence by definition $w(B - \sum p_i, z, R) = 0$ is equivalent to $w(\sum p_i, z, R) = 1$. Suppose $\sum \text{mes} E_i < 2\pi$. By $w(\sum p_i, z, R) = \lim_{n \to \infty} w(\sum p_i, z, R)$, for any given positive number $\varepsilon < \frac{\delta}{2}$ ($\delta = 2\pi - \sum \text{mes} E_i$), there exst a number $n_0$ such that

$w(\sum p_i, z, R) \leq w(\sum p_i, z, R) + \varepsilon$, for $n \geq n_0$,
where $\xi=0$ is an image of $z_0$.

Now $w(\sum p_i, z, R) = \sum w(E_i, \xi)$ implies $w(\sum p_i, z_0, R) - \epsilon \leq w(\sum p_i, z_0, R) = 1 - \frac{\delta}{2\pi}$, whence $w(\sum p_i, z, R) < 1$. Conversely suppose $\sum \mes E_i = 2\pi$.

Then $w(\sum p_i, z, R) \geq w(\sum p_i, z, R) = w(\sum E_i, z)$ for every $n$. Hence $w(\sum p_i, z, R) = 1$. Thus we have $d)$.

**Proof of e).** Assume that $U(z)$ has angular limits being not a constant a.e. on $E$. Then we can find two sets $E_1$ and $E_2$ of positive measure in $E$ such that $U(z) \geq L$ on $E_1$ and $< L - \delta$ on $E_2$ for constants $L$ and $\delta > 0$. Let $W(\xi) := W(z)$ be a harmonic function in $|\xi| < 1$ such that $W(\xi) = 1$ a.e. on $E_1 \subset E$ and $= 0$ a.e. on $CE_i$. Then since $U(z)$ is a function in $R$, $W(\xi)$ is a function in $R$. Clearly $0 < W(z) < \frac{K(z, p)}{M}$ and $W(z)$ is not a multiple of $K(z, p)$. This implies that $K(z, p)$ is not $K$-minimal. Hence every Poisson's integrable function has angular limits = const a.e. on $E$.

2. Class H.N.B. We denote by H.N.B. the class of Riemann surfaces on which $N$ number of linearly independent bounded harmonic function exist, where $N \leq \infty$ and the cardinal number of $N$ is $\mathfrak{Z}_o$.

**Theorem 2.** A Riemann surface $R \in H.N.B. (N \leq \infty)$ if and only if $R$ has $N$ number of singular points and a set of boundary points of harmonic measure zero.

**Proof.** Suppose that $R$ has $N$ number of points $p_1, p_2, \ldots, p_N$ and a set of boundary points of harmonic measure zero. Let $U(z)$ be a bounded harmonic function (Poisson's integrable) in $R$. Then $U(z)$ has angular limits = const a.e. on $E_i$ of the image of $p_i$ by $e)$ of Theorem 1 and by $d)$ of Theorem 1. Whence $U(z)$ is a linear form of $\{K(z, p_i)\}$ and $\{K(z, p_i)\}$ is linearly independent. Hence $R \in H.N.B.$ Conversely suppose $R \in H.N.B.$ Then the harmonic measure of $B - \sum p_i$ is zero, because if $\sum \mes E_i < 2\pi$, we can construct infinitely (cardinal number = $\mathfrak{Z}$) many linearly independent harmonic functions. Let $N' : N \neq N'$ be the number of singular points, then by $e)$ $R \in H.N.B. \neq H.N.B.$ Hence $N' \neq N$. Hence we have Theorem 2.

3. Harmonic functions and analytic functions in a neighbourhood of a singular point.

Let $G$ be a non compact domain in $R$ and let $U(z)$ be a positive

\[ \text{number} \]

5) In the present paper we suppose that $\partial G$ of a domain $G$ consists of an enumerably infinite number of analytic curves clustering nowhere in $R$. 

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Let $w_{n,n+i}(z)$ be a continuous superharmonic function in $R_{n}+(u_{m}(p)$\cap$R_{n+i})$ such that $w_{n,n+i}(z)$ is harmonic in $R_{n}+(u_{m}(p)$\cap$R_{n+i}),$ $w_{n,n+i}(z)$=w(p, z, R) on $(R-R_{n})\cap C_{U_{m}}(p)$ and $w_{n,n+i}(z)=0$ on $\partial R_{n+i}\cap o_{m}(p)$, where $u_{m}(p)=E[z\in\overline{R} : \delta(z, p)<\frac{1}{m}]$. Then $w_{n,n+i}(z)\uparrow w_{n}(z)$ as $i\rightarrow\infty$.

Now $w_{n}(z)$ is the lower envelope of superharmonic functions in $R$ larger than $w(p, z, R)$ on $(R-R_{n})\cap C_{U_{m}}(p)$, where $w(p, z, G)=\lim_{m}\lim_{n}w(U_{m}(p), z, G\cap R_{n})$, where $w(U_{m}(p), z, GR)=1$ on $z\in R_{n}$. Then

$$w(p, z, G)=\lim_{m}\lim_{n}w(U_{m}(p), z, G\cap R_{n}),$$

If $w(p, z, G)=0$, then $G$ contains $p$ K-approximately.

c) If $w(p, z, G)<0$, then $G$ contains $p$ K-approximately.

d) Let $U_{m}(p)$ be a neighbourhood of $p\in B$. Then $U_{m}(p)$ contains $p$ K-approximately.

Proof of a). By $3)^*$ $w_{n}(z)$ is represented by a positive mass distribution $\mu$ on $C_{U_{m}}(p)\cap(R-R_{n})$, whence $w_{B\cap C_{U_{m}}(p)}(z, R)$ is represented by a weak limit $\mu$ of $\{\mu_{n}\}$ on $B\cap C_{U_{m}}(p)$ such that $w_{B\cap C_{U_{m}}(p)}(z, R)=\int K(z, p)d\mu(p)$.

7) $C_{U_{m}}(p)$ means the complementary set of $U_{m}(p)$. 
Now \( p \in B \), and by the \( K \)-minimality of \( w(p, z, R) \), \( w_{B \cap Cv_{m}(p)}(p, z, R) = a \) \( w(p, z, R) = a' K(z, p) \) : \( a \) and \( a' \geq 0 \). Assume \( a' > 0 \). Then we can find a point \( q \in C_{m}(p) \) such that the restriction \( \mu_{t} \) of \( \mu \) on \( \delta_{i}(q) = \left( \mathbb{E} \left[ z \in R : \delta(z, q) \leq \frac{1}{l} \right] \right) \) > 0 for every \( l \). Also by the minimality of \( w(p, z, R) \), \( \int K(z, p) d\mu(p) \) is also minimal, whence \( \frac{\int K(z, p) d\mu_{l}(p)}{\int d\mu_{l}} = K(z, p) \). Now \( \left( \frac{\mu_{l}}{\int d\mu_{l}} \int \right) \) has an weak limit \( \mu^{*} \) at \( q \) as \( l \to \infty \). Hence \( K(z, q) = K(z, p) \). This is a contradiction by \( \delta(p, q) \geq \frac{1}{n} > 0 \). Hence \( a' = 0 \) and \( w_{B \cap Cv_{m}(p)}(p, z, R) = 0 \).

Proof of \( b \). Compare \( \text{inex,}Gw(p, z, R) \) and \( w(p, z, R) + w_{B \cap Cv_{m}(p)}(p, z, R) \) in \( (R \cap G) - U_{m}(p) \).

Then \( \text{inex,}Gw(p, z, R) = 0 = w(p, z, G \cap R_{n}) \) on \( \partial G \), \( \text{inex,}R_{n}w(p, z, R) = w(p, z, R) = w_{Cv_{m}(p)}(p, z, R) \) on \( \partial(R \cap G) - U_{m}(p) \) and \( \text{inex,}R_{n}w(p, z, R) = 1 = w_{Cv_{m}(p)}(p, z, R) \) on \( \partial U_{m}(p) \cap G \). Hence by the maximum principle

\[ \text{inex,}Gw(p, z, R) \leq w_{Cv_{m}(p)}(p, z, R) \leq w(p, z, R) = w_{Cv_{m}(p)}(p, z, R) \]

Let \( n \to \infty \). Then

\[ \text{inex,}Gw(p, z, R) \leq w(p, z, G) \]

Conversely, clearly \( w(p, z, R) \geq w(p, z, G) \). Hence by \( w(p, z, G) = 0 \) on \( \partial G \)

\[ \text{inex,}Gw(p, z, R) \geq w(p, z, G) = w(p, z, G) \]

Thus

\[ w(p, z, G) = \text{inex,}Gw(p, z, R) \]

Let \( w_{u_{m}(p)}(z, R) \) be H. M. of \( u_{m}(p) \) relative \( R \) i.e. \( w_{u_{m}(p)}(z, R) \) is a superharmonic function in \( R \) such that \( w_{u_{m}(p)}(z, R) \) is harmonic in \( R - u_{m}(p) \) such that \( w_{u_{m}(p)}(z, R) = 1 \) on \( u_{m}(p) \) and \( w_{u_{m}(p)}(z, R) = 0 \) on \( \partial R - u_{m}(p) \).

Let \( w_{u_{m+i}(p)}(z, R) \) be the least positive superharmonic function in \( R \) such that \( w_{u_{m+i}(p)}(z, R) = 1 \) on \( u_{m+i}(p) \) and \( w_{u_{m+i}(p)}(z, R) = 0 \) on \( \partial R - u_{m+i}(p) \).

Let \( V_{m+i, n}(u_{m}(p), z) \) be a harmonic function in \( (R \cap G) - u_{m+i}(p) \) such that \( V_{m+i, n}(u_{m}(p), z) = w_{u_{m}(p)}(z, R) \) on \( \partial G \cap R \) and \( = 0 \) on \( \partial u_{m+i}(p) + (\partial R - u_{m+i}(p)) \). Then

\[ V_{m+i, n}(u_{m}(p), z) = \frac{1}{2\pi} \int_{\partial G \cap R - u_{m+i}(p)} w_{u_{m}(p)}(z, R) \frac{\partial}{\partial n} G_{m+i, n}(z, z) ds \]

where \( G_{m+i, n}(z, z) \) is the Green's function of \( (R \cap G) - u_{m+i}(p) \).
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Now \( V_{m+i, n}(v_{m}(p), z) \uparrow V_{m+i}(v_{m}(p), z) \) as \( n \to \infty \) by \( w(v_{m}(p), z, R_{n}) \uparrow w(v_{m}(p), z, R) \) and \( \frac{\partial}{\partial n} G_{m+i, n}(\xi, z) \uparrow \frac{\partial}{\partial n} G_{m+i}(\xi, z) \) on \( \partial G \) as \( n \to \infty \).

Hence by Lebesgue's theorem

\[
V_{m+i}(v_{m}(p), z) = \frac{1}{2\pi} \int_{\partial G} w(v_{m}(p), \xi, R) \frac{\partial}{\partial n} G_{m+i}(\xi, z) ds,
\]

where \( G_{m+i}(\xi, z) \) is the Green's function of \( G-v_{m+i}(p) \).

Similarly by \( V_{m+i}(v_{m}(p), z) \uparrow V_{m+i}(v_{m}(p), z) \) as \( i \to \infty \),

\[
V_{m+i}(v_{m}(p), z) = \frac{1}{2\pi} \int_{\partial G} w(v_{m}(p), \xi, R) \frac{\partial}{\partial n} G_{m+i}(\xi, z) ds,
\]

(1)

where \( G(\xi, z) \) is the Green's function of \( G \).

Hence by (1) \( V(v_{m}(p), z) \) is the least positive harmonic function in \( G \) with value \( w(v_{m}(p), R) \) on \( \partial G \) i.e. \( V(v_{m}(p), z) = w_{CG}(v_{m}(p), z, R) \) in \( G \).

Let \( m \to \infty \). Then as above

\[
\lim_{m} V(v_{m}(p), z) = \frac{1}{2\pi} \int_{\partial G} \lim_{m=\infty} w(v_{m}(p), z, R) \frac{\partial}{\partial n} G(\xi, z) ds = w_{CG}(p, z, R).
\]

Let \( V_{m,n}(p, z) \) be a harmonic function in \( (G \cap R_{n})-v_{m}(p) \) such that \( V_{m,n}(p, z) = w(p, z, R) \) on \( (\partial G \cap R_{n})-v_{m}(p) \) and \( =0 \) on \( \partial v_{m}(p)+\partial R_{n}-v_{m}(p) \).

Then

\[
V_{m,n}(p, z) = \frac{1}{2\pi} \int_{\partial G} w(p, \xi, R) \frac{\partial}{\partial n} G_{m,n}(\xi, z) ds,
\]

\[
\lim_{m} V_{m,n}(p, z) = \lim_{n} V_{m,n}(p, z) \leq \lim_{m} V_{m,n}(v_{m}(p), z) \leq V_{m+i,n}(v_{m}(p), z).
\]

Let \( n \to \infty \). Then by \( V_{m+i}(v_{m}(p), z) \uparrow V_{m+i}(v_{m}(p), z) \) as \( i \to \infty \),

\[
V_{m}(p, z) \leq V_{m}(v_{m}(p), z) \leq V_{m+i}(v_{m}(p), z) \leq w_{CG}(v_{m}(p), z, R).
\]

(3)

Let \( m \to \infty \). Then by (2)

\[
w_{CG}(p, z, R) \leq \lim_{m} V_{m}(v_{m}(p), z) \leq w_{CG}(p, z, R).
\]

(4)

Hence \( \lim_{m} V_{m}(v_{m}(p), z) = w_{CG}(p, z, R) \).

Now \( V_{m,n}(v_{m}(p), z) = 0 \) on \( (G \cap R_{n})-v_{m}(p)+\partial v_{m}(p) \) and \( =w(v_{m}(p), z, R) \) on \( (\partial G \cap R_{n})-v_{m}(p) \). Hence \( w(v_{m}(p), z, R_{n})-V_{m,n}(v_{m}(p), z) = 0 = w(v_{m}(p), z, R_{n}) \) on \( (\partial G \cap R_{n})+(\partial R_{n}\cap G)-v_{m}(p) \) and \( w(v_{m}(p), z, R_{n})-V_{m,n}(v_{m}(p), z) = 1 = w(v_{m}\)
(p), z, G ∩ R_n) on \partial \nu_m(p) \cap G \cap R_n.

Hence by the maximum principle

\[ w(\nu_m(p), z, R_n) - V_{m,n}(\nu_m(p), z) = w(\nu_m(p), z, G \cap R_n). \]

Let \( n \to \infty \) and then \( m \to \infty \).

Then by (3) and (4)

\[ w(p, z, R) - w_{CG}(p, z, R) = w(p, z, G). \]

**Proof of c)**. Assume \( \inex_G w(p, z, R) > 0 \). Then \( \ex_G \inex_G w(p, z, R) = w(p, z, R) \). Let \( \hat{U}_n(z) \) and \( \check{U}_n(z) \) be harmonic functions in \( R_n \) such that \( \hat{U}_n(z) = w(p, z, R), \check{U}_n(z) = 0 \) on \( \partial R_n \cap G \) and \( \hat{U}_n(z) = 0, \check{U}_n(z) = w(p, z, R) \) on \( \partial R_n \cap \mathbb{R} \). Then \( \hat{U}_n(z) + \check{U}_n(z) = w(p, z, R) \).

Clearly \( \hat{U}_n(z) \leq w(p, z, R) \) and \( \hat{U}_n(z) = w(p, z, R) \geq \inex_G w(p, z, R) \) on \( \partial R_n \cap G \).

\[ \lim_n \hat{U}_n(z) = w(p, z, R) \]

\[ \lim_n \check{U}_n(z) = 0 \].

Similarly as above \( \lim_n \hat{U}_n(z) = \inex_G w(p, z, R) \). Hence \( \inex_G w(p, z, R) = 0 \).

**Proof of d)**. Assume \( w_{CG}(p, z, R) = w(p, z, R) \). Then by (3) \( w(p, z, R) \) is represented by a mass distribution \( \mu \) on \( C\nu_n(p) \) and by (4) \( w(p, z, R) = aK(z, q) : q \in C\nu_n(p) \) and \( a > 0 \). Now \( w(p, z, R) = a'K(z, p) \) and by \( K(p, q) = 1, a = a' \) and \( K(z, p) = K(z, q) \). This contradicts \( p \neq q \). Hence \( w(p, z, R) - w_{CG}(p, z, R) > 0 \) i.e. \( \nu_n(p) \) contains \( p \)-K-approximately.

**Theorem 4.**

a) Let \( G \) be a domain containing a point \( p \in B_s \). Then \( w(p, z, G) > 0 \). Map the universal covering surface \( G^\infty \) onto \( |\xi| < 1 \).

Then \( w(p, z, G) \) has angular limits \( \infty \) on a set \( E \) of positive measure and has angular limits \( 0 \) on \( CE \) almost everywhere.

Let \( U(z) \) be a Poisson's integrable harmonic function in \( G \). Then \( U(z) \) has angular limits \( \text{const} \) a.e. on \( E \).

b) Let \( G \) be a domain in a). Then there exists no non constant analytic function of bounded type in \( G \).

c) Let \( \nu_n(p) \) be a neighbourhood of \( p \in B_s \): \( \nu_n(p) = \{ z \in \overline{R} : \delta(z, p) < \frac{1}{n} \} \).

Then \( \nu_n(p) \) contains \( p \)-K-approximately and by b) there exists no non constant analytic function of bounded type.

**Proof of a)**. By \( \ex_G w(p, z, G) \subseteq w(p, z, R), \ex_G w(p, z, G) = w(p, z, R) \):

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8) Z. Kuramochi: Dirichlet problems on Riemann surfaces. 1. Proc. Japan Acad. 1954. The characteristic \( T(z) \) of an analytic function \( A(z) \) on a Riemann surface \( R \) can be defined. If \( T(z) < \infty \), \( A(z) \) has angular limits a.e. on \( |\xi| = 1 \), where \( |\xi| < 1 \) is the image of the universal covering surface \( R^\infty \) of \( R \).
$\alpha > 0$ by the $K$-minimality of $w(p, z, R)$. We show that $w(p, z, G) (> 0$ by the assumption and by Theorem 3. $b)$) is $K$-minimal in $G$. Suppose there exists a positive harmonic function $U(z)$ such that $U(z) \leq w(p, z, G)$. Then 
\[ U(z) \leq_{ex, o} w(p, z, G) = a \ w(p, z, R), \]
whence by the minimality of $w(p, z, R)$, 
\[ e_{ex, o} U(z) = b \ w(p, z, R): b > 0. \]
Hence 
\[ U(z) =_{inex, o} (ex, o) U(z) =_{inex, o} (ex, o) b \ w(p, z, R) =_{inex, o} b \ w(p, z, R). \]
Hence $w(p, z, G)$ is $K$-minimal in $G$. Similarly as $b)$ of Theorem 3, $w(p, z, G)$ has angular limits $=1$ on a set $E$ of positive measure on $|\xi| = 1$ and has angular limits $= 0$ for $a$.

**Proof of b).** Let $A(z)$ be an analytic function of bounded type. Then $ReA(z)$ and $ImA(z)$ have angular limits $= \text{const}$ a.e. on $E$. By mes $E > 0$ and by Riesz's theorem $A(z)$ must be a constant. Hence we have $b)$.

**Proof of c).** By Theorem 3. $d) \upsilon_{n}(p)$ contains $p$ $K$-approximately. Hence we have $c)$ by $b)$.

**PART II**

The present part is an application of the previous paper “Potentials on Riemann surfaces”.

**N-Martin topology.** Let $N(z, p)$ be a harmonic function in $R - R_{0}$ with one logarithmic singularity at $p \in R - R_{0}$ such that $N(z, p) = 0$ on $\partial R_{0}$ and $N(z, p)$ has the minimal Dirichlet integral over $R - R_{0}$. We use $N(z, p)$ instead of $K(z, p)$ of $K$-Martin's topology. Then we have $N$-Martin's topology. The distance between two points $p$ and $q$ of $\overline{R} - R_{0}$ is given by

\[ \delta(p, q) = \sup_{z \in \overline{R} - R_{0}} \left| \frac{N(z, p)}{1 + N(z, p)} - \frac{N(z, q)}{1 + N(z, q)} \right|. \]

We suppose that $N$-Martin's topology is defined on $\overline{R} - R_{0}$. We use the same notation as in the previous paper and refer the theorem in “Potentials on Riemann surfaces” with $P$.

4. **Theorem 5.** (Separation theorem S. 1). Let $G, G_{1}$ and $G_{2}$ be noncompact domains such that $G \supset G_{1}, G \supset G_{2}, G_{1} \cap G_{2} = 0$ and let $B'$ be a closed subset of $B$. If $C. P. \ \omega(G_{1} \cap B', z, G) > 0$ and $\omega(G_{2} \cap B', z, G - G_{1})^{9)} > 0$,

\[ \omega(G_{1} \cap B', z, G) \supset \omega(G_{2} \cap B', z, G) \]

for any domain $\tilde{G} \supset G$.

9) See the definition of $\omega(G_{2} \cap B', z, G)$ of “Potentials”.
where $\omega(B' \sim G_1, z, G) = \lim_n \omega(B_n' \cap G_1, z, G)$ and $B_n' = E[z \in \overline{R} : \delta(z, B') \leqq \frac{1}{n}]$.

Proof. For simplicity put $\omega^i(z) = \omega(G_1 \cap B', z, G)$ and $\omega^2(z) = \omega(G_2 \cap B', z, G-G_1)$. Then $\omega^i(z) (i = 1, 2)$ has properties from P.C. 1 to P.C. 7.

Put $\Omega = E[z \in G : \delta < \omega^2(z) < 1-\epsilon]$, and let $C_\delta$ and $C_{1-\epsilon}$ be regular niveau curves of $\omega^2(z)$. Since $\Omega' \cap G_1 = 0 : \Omega' = E[z \in G : \omega^2(z) > \delta]$, $\omega^i(z)$ has M.D.I. over $\Omega' \cap \Omega$ by P. 1. Then by Lemma 1. b) of $P$ (we abbreviate Potentials on Riemann surface by $P$) $\omega^1(z) \Rightarrow \omega^i(z)$ (\Rightarrow means convergence in mean and convergence), where $\omega^1(z)$ is a harmonic function in $\Omega' \cap R_n$ such that $\omega^1_n(z) = \omega^1(z)$ on $\partial \Omega' \cap R_n$ and $\frac{\partial}{\partial n} \omega^1_n(z) = 0$ on $\partial R_n \cap \Omega'$.

Let $\omega^2_n(z)$ be a harmonic function in $R_n \cap \Omega$ such that $\omega^2_n(z) = \omega^2(z)$ on $\partial \Omega \cap R_n$ and $\frac{\partial}{\partial n} \omega^2_n(z) = 0$ on $\partial R_n \cap \Omega$. Now $\omega^2(z) = \omega(\Omega_{1-\epsilon}^\prime, z, G-G_1) (1-\epsilon)$ in $C\Omega_{1-\epsilon}^\prime$; $\Omega_{1-\epsilon}^\prime = E[z \in G : \omega^2(z) > 1-\epsilon]$ by P. 4., whence by $\Omega \cap \Omega_{1-\epsilon}^\prime = 0$ $\omega^2(z)$ has M.D.I. over $\Omega$. Hence by Lemma 1. b) of $P$ $\omega^2_n(z) \Rightarrow \omega^2(z)$ as $n \rightarrow \infty$.

Now \[\int_{C_\delta \cap R_n} \frac{\partial}{\partial n} \omega^1_n(z)ds = \int_{c_{1-\epsilon} \cap R_n} \frac{\partial}{\partial n} \omega^2_n(z)ds = 0 = \int_{c_\delta \cap R_n} \frac{\partial}{\partial n} \omega^1_n(z)ds,\]

whence \[\int_{c_\delta \cap R_n} \omega^1_n(z)ds = \int_{c_1 \cap R_n} \omega^2_n(z)ds = \int_{c_{1-\epsilon} \cap R_n} \omega^2_n(z)ds = 0.\]

Hence by the Green's formula

\[\int_{c_\delta \cap R_n} \omega^1(z) \frac{\partial}{\partial n} \omega^2_n(z)ds = \int_{c_1 \cap R_n} \omega^1_n(z) \frac{\partial}{\partial n} \omega^2_n(z)ds.\]  

Since $C_{1-\epsilon}$ and $C_\delta$ are regular, $0 < \omega^1_n(z) (1-\epsilon) \rightarrow \omega^1(z)$ and $\frac{\partial}{\partial n} \omega^2_n(z) \rightarrow \frac{\partial}{\partial n} \omega^2(z)$ on $C_\delta + C_{1-\epsilon}$ imply by Theorem 3. a) of $P$

\[\int_{c_{1-\epsilon}} \omega^1(z) \frac{\partial}{\partial n} \omega^2(z)ds = \int_{c_\delta} \omega^1(z) \frac{\partial}{\partial n} \omega^2(z)ds.\]

Since $C_\delta \subset R$ and $\omega^1(z) \rightarrow 1$ in $R$, \[\int_{c_\delta} \omega^1(z) \frac{\partial}{\partial n} \omega^2(z)ds < \int_{c_\delta} \frac{\partial}{\partial n} \omega^2(z)ds = D(\omega^2(z)).\]

Hence there exists a constant $\epsilon_0 > 0$ depending only on $\omega^i(z)$ and $C_\delta$ such that

\[10) \text{See 9).}\]
Singular points of Riemann Surfaces

\[ \int_{c_{1-\epsilon}} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds < (1 - \epsilon_{0}) D(\omega^{2}(z)). \] (7)

Now \( \epsilon \) is arbitrary so long as \( C_{1-\epsilon} \) is regular. We choose \( \epsilon \) so that \( 0 < \epsilon < \frac{\epsilon_{0}}{2} \). Now \( \omega^{2}(z) = 1 - \epsilon \) on \( C_{1-\epsilon} \), whence by (6) and (7) we have

\[ \int_{c_{1-\epsilon}} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds = \int_{c_{1-\epsilon}} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds = \int_{C_{\delta}} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds < (1 - \epsilon) D(\omega^{2}(z)) < \int_{C_{\delta}} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds = \int_{1-\epsilon} \omega^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds. \]

This implies \( \omega^{1}(z) \neq \omega^{2}(z) \).

Put \( \tilde{\omega}^{1}(z) = \omega(G_{1} \cap B', z, \tilde{G}) \) and \( \tilde{\omega}^{2}(z) = \omega(G_{2} \cap B', z, \tilde{G}) \). Then by \( \tilde{G} \supset G \) and \( \tilde{G} \supset (G - G_{1}) \), \( \tilde{\omega}^{1}(z) \geq \omega^{1}(z) \) and \( D(\tilde{\omega}^{1}(z)) \leq D(\omega^{1}(z)) \). Consider the same regular niveau curve of \( \omega^{2}(z) \). Then by \( \tilde{\omega}^{1}(z) < 1 \) in \( R \), there exists a constant \( \epsilon_{0} \) such that \( \int_{C_{\delta}} \tilde{\omega}^{1}(z) \frac{\partial}{\partial n} \omega^{3}(z) ds < (1 - \epsilon_{0}) D(\omega^{2}(z)) \). Similarly as above

\[ \int_{c_{1-\epsilon}} \tilde{\omega}^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds = \int_{C_{\delta}} \tilde{\omega}^{1}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds < (1 - \epsilon_{0}) D(\omega^{2}(z)) < (1 - \epsilon) D(\omega^{2}(z)) = \int_{c_{1-\epsilon}} \omega^{2}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds \leq \int_{c_{1-\epsilon}} \tilde{\omega}^{2}(z) \frac{\partial}{\partial n} \omega^{2}(z) ds. \]

Thus \( \tilde{\omega}^{1}(z) \neq \omega^{2}(z) \).

Theorem 6. (Separation theorem. S. 2). Let \( G, G_{1} \) and \( G_{2} \) be non compact domains such that \( G \supset G_{1} \supset G_{2} \) and let \( B' \) be a closed subset of \( B \). If \( \omega(G_{1} \cap B', z, G) > \omega(G_{2} \cap B', z, G) > 0 \), we can find domains \( D_{1} \) and \( D_{2} \) in \( G \)

11) Put \( B'_{n} = E \left[ \delta(z, B') \leq \frac{1}{n} \right] \). Let \( \omega_{n,n+i}(z) \) be a harmonic function in \( G \cap R_{n+i} \) such that \( \omega_{n,n+i}(z) = 0 \) on \( \partial G \cap R_{n+i} \) and \( \omega_{n,n+i}(z) = 1 \) on \( B'_{n} \). Let \( \tilde{\omega}_{n,n+i}(z) \) be a harmonic function in \( \tilde{G} \cap R_{n+i} \) such that \( \tilde{\omega}_{n,n+i}(z) = 0 \) on \( \partial \tilde{G} \cap R_{n+i} \) and \( \tilde{\omega}_{n,n+i}(z) = 1 \) on \( B'_{n} \). Then \( D(\tilde{\omega}_{n,n+i}(z)) \leq D(\omega_{n,n+i}(z)) \). Let \( i \to \infty \) and then \( n \to \infty \). Then \( \tilde{\omega}_{n,n+i}(z) \to \tilde{\omega}^{i}(z) \) and \( \omega_{n,n+i}(z) \to \omega^{i}(z) \). Hence \( D(\tilde{\omega}^{i}(z)) \leq D(\omega^{i}(z)) \) and \( D(\tilde{\omega}(z)) \leq D(\omega(z)) \).
such that $D_1 \supset D_2$ and that $0 < \omega(D_2 \cap \tilde{G}_1 \cap B', z, G) = \omega(D_2 \cap CD_1 \cap B', z, G) > 0$ for any domain $\tilde{G} \supset G$.

Proof. For simplicity put $\omega^1(z) = \omega(G_1 \cap B', z, G)$ and $\omega^2(z) = \omega(G_2 \cap B', z, G)$. Let $D_1 = E \{ z \in G : V(z) > \frac{M}{3} \}$ and $D_2 = E \{ z \in G : V(z) > \frac{2M}{3} \}$, where $V(z) = \omega^1(z) - \omega^2(z)$ and $M = \sup_{z \in G} V(z)$.

At first we remark

$$D_1 \subset D_2^3 = E \{ z \in G : \omega^2(z) < 1 - \frac{M}{3} \},$$

because if $\omega^2(z) \geq 1 - \frac{M}{3}$, $V(z) \leq \frac{M}{3}$ by $\omega^1(z) < 1$.

Hence by P.C.2.

$\omega(G_2 \cap B' \cap D_3^3, z, G) = 0$ and by $D_1 \subset D_2^3$ we have $\omega(B' \cap D_1 \cap G_2, z, G) = 0$.

Whence by $\omega(D_1 \cap G_2 \cap B', z, G) + \omega(CD_1 \cap G_2 \cap B', z, G) \geq \omega(G_2 \cap B', z, G)$

$$\omega^2(z) = \omega(G_2 \cap B', z, G) = \omega(G_2 \cap CD_1 \cap B', z, G). \quad (8)$$

Put $\Omega_{n, n+i, n+i+j} = (D_1 \cap R_{n+i+j}) - (G_1 \cap (D_1 - D_2) \cap B_{n+i}^\prime) - (B_{n}^\prime \cap D_2)$,

where $B_{n}^\prime = E \{ z \in \overline{R} : \delta(z, B') \leq \frac{1}{n} \}$.

Then by $G_2 \cap CD_1 \cap D_1 = 0$ and by $\Omega_{n, n+i,j} \cap (CD_1 \cap G_2) = 0$ $\omega^2(z) = \omega(G_2 \cap CD_1 \cap B', z, G)$ has M. D. I. over $\Omega_{n, n+i,j} = \lim_{j} \Omega_{n, n+j, n+i+j}$ with value $\omega^2(z)$ on $\alpha_{n, n+i}(= \partial D_1 + \partial (B_{n+i}^\prime \cap (D_1 - D_2) \cap G_1) + \delta(B_{n}^\prime \cap D_2 \cap G_1))$ by P.C.2. Hence

![Fig. 3.](image_url)
\[
\omega^2_{n,n+i,n+i+j}(z) \Rightarrow \omega^2(z) \text{ as } j \to \infty,
\]
where \( \omega^2_{n,n+i,n+i+j}(z) \) is a harmonic function in \( \Omega_{n,n+i,n+i+j} \), such that
\[
\omega^2_{n,n+i,n+i+j}(z) = \omega^2(z) \text{ on } \alpha_{n,n+i} \cap \partial R_{n+i+j} \text{ and } \frac{\partial}{\partial n} \omega^2_{n,n+i,n+i+j}(z) = 0 \text{ on } \partial R_{n+i+j} \cap \Omega_{n,n+i}.
\]

Similarly \( \Omega_{n,n+i} \cap \left[ G_1 \cap \left( B'_n \cap (D_1 - D_2) + (B'_n \cap D_2) \right) \right] = 0 \) implies that \( \omega'(z) \) has M.D.I. over \( \Omega_{n,n+i} \) with value \( \omega'(z) \) on \( \alpha_{n,n+i} \). Hence
\[
\omega^1_{n,n+i,n+i+j}(z) \Rightarrow \omega^1(z) \text{ as } j \to \infty,
\]
where \( \omega^1_{n,n+i,n+i+j}(z) \) is a harmonic function in \( \Omega_{n,n+i,n+i+j} \) such that
\[
\omega^1_{n,n+i,n+i+j}(z) = \omega^1(z) \text{ on } \alpha_{n,n+i} \cap \partial R_{n+i+j} \text{ and } \frac{\partial}{\partial n} \omega^1_{n,n+i,n+i+j}(z) = 0 \text{ on } \partial R_{n+i+j} \cap \Omega_{n,n+i}.
\]

Thus
\[
\omega^1_{n,n+i,n+i+j}(z) - \omega^2_{n,n+i,n+i+j}(z) = V_{n,n+i,n+i+j}(z) \Rightarrow V(z) = \omega_1(z) - \omega_2(z) \text{ as } j \to \infty.
\]

Let \( \overline{\omega}_{n,n+i,n+i+j}(z) \) be a harmonic function in \( \Omega_{n,n+i,n+i+j} \) such that
\[
\overline{\omega}_{n,n+i,n+i+j}(z) = 0 \text{ on } \partial D_1 \cap \partial \Omega_{n,n+i,n+i+j} \text{ and } \overline{\omega}_{n,n+i,n+i+j}(z) = 1 \text{ on } \partial (D_2 \cap G_1 \cap B'_n)
\]
and
\[
\frac{\partial}{\partial n} \overline{\omega}_{n,n+i,n+i+j}(z) = 0 \text{ on } \partial R_{n+i+j} \cap \Omega_{n,n+i,n+i+j} \cap \partial B'_n \cap (D_1 - D_3) \cap G_1.
\]

Then
\[
D(\overline{\omega}_{n,n+i,n+i+j}(z)) \leq \frac{9}{M^2} D(V(z)) < \infty,
\]
because \( V'(z) = \frac{3}{M} \left( V(z) - \frac{3}{M} \right) \geq 1 \) on \( \overline{D}_2(\partial(D_2 \cap G_1 \cap B'_n)) \) and \( \overline{\omega}_{n,n+i,n+i+j}(z) \) has M.D.I. over \( \Omega_{n,n+i,n+i+j} \) among all functions with value \( \overline{\omega}_{n,n+i,n+i+j}(z) = \min(V(z), \frac{2M}{3}) \) on \( \partial(D_2 \cap G_1 \cap B'_n) \).

Hence \( \overline{\omega}_{n,n+i,n+i+j}(z) \Rightarrow \overline{\omega}_{n,n+i}(z) \) as \( i \to \infty \) and \( \overline{\omega}_{n,n+i}(z) \Rightarrow \overline{\omega}(z) \) as \( i \to \infty \) and
\[
\overline{\omega}_{n,n+i}(z) \Rightarrow \overline{\omega}(D_{2^*} \cap G_{1^*} \cap B^* \cap D_{1^*}) \text{ as } n \to \infty.
\]

Let \( \tilde{\omega}_{n,n+i,n+i+j}(z) \) be a harmonic function in \( \Omega_{n,n+i,n+i+j} \) such that
\[
\frac{\partial}{\partial n} \tilde{\omega}_{n,n+i,n+i+j}(z) = 0 \text{ on } \Omega_{n,n+i} \cap \partial R_{n+i+j} \text{ and } \tilde{\omega}_{n,n+i,n+i+j}(z) = \min(V(z), \frac{2M}{3}) \text{ on } \partial \Omega_{n,n+i,n+i+j} \cap \partial R_{n+i+j}.
\]

Put \( T(z) = V(z) \) in \( D_1 - D_2 \) and \( = \frac{2M}{3} \) in \( \Omega_{n,n+i,n+i+j^*} \).

Then
\[
D(\overline{\omega}_{n,n+i,n+i+j}(z)) \leq D(T(z)) \leq D(V(z)).
\]
Hence it can be proved by Lemma 1, b) of P that
\[
\overline{\omega}_{n,n+i,n+i+j}(z) \Rightarrow \overline{\omega}_{n,n+i}(z) \text{ as } j \to \infty.
\]

Compare \( \overline{\omega}_{n,n+i,n+i+j}(z) + \overline{\omega}_{n,n+i,n+i+j}(z) \) and \( V_{n,n+i,n+i+j}(z) \) in \( \Omega_{n,n+i,n+i+j} \).

Then
\[
\overline{\omega}_{n,n+i,n+i+j}(z) + \overline{\omega}_{n,n+i,n+i+j}(z) \geq V_{n,n+i,n+i+j}(z) \text{ on } \partial D_1.
\]
by $\tilde{\omega}_{n,n+i,n+i+j}(z)=0$ and $\tilde{\omega}_{n,n+i,n+i+j}(z)=V_{n,n+i,n+i+j}(z)=\frac{M}{3}$ on $\partial D_{1}$.

by $\tilde{\omega}_{n,n+i,n+i+j}(z)+\tilde{\omega}_{n,n+i,n+i+j}(z)\geqq V_{n,n+i,n+i+j}(z)$ on $\partial(B'_{n+i}\cap(D_{1}-D_{2})\cap G_{1})$.

Choose a subsequence $\{i'\}$ such that $\tilde{\omega}_{n,n+i}(z)\rightarrow\tilde{\omega}_{n}(z)$. Further choose a subsequence $\{n'\}$ such that $\tilde{\omega}_{n}(z)\rightarrow\tilde{\omega}(z)$.

Assume $\omega(D_{2}\cap G_{1}\cap B', z, D_{1})=0$. Then by (9) $\tilde{\omega}(z)+\omega(D_{2}\cap G_{1}\cap B', z, D_{1})\geqq V(z)$ and

$$\frac{2M}{3}\geqq \tilde{\omega}(z)\geqq V(z) : \sup_{z\in\mathbb{G}} V(z)=M.$$

This is a contradiction.

Hence

$$\omega(D_{2}\cap G_{1}\cap B', z, D_{1})>0.$$

(10)

Now by (8)

$$w(z)=\omega(G_{2}\cap CD_{1}\cap B', z, G)>0.$$

(11)

Since $(G_{2}\cap CD_{1})\cap D_{1}=0$, S. 1. is applicable to (10) and (11) and we have

$$\omega(D_{2}\cap G_{1}\cap B', z, \mathbb{G})=\omega(G_{2}\cap CD_{1}\cap B', z, \mathbb{G}).$$

Thus $D_{1}$ and $D_{2}$ are required domains.

5. Singular points. Let $p\in R-R_{0}$. Then $N(z, p)$ is $N$-minimal and its C.P. $\omega(p, z)=0$ and $N(z, p)$ likes $-\log|z-p|$ in a neighbourhood of $p$. Hence if $\omega(p, z)>0$ for a point $p\in B$, it is imagined that $N(z, p)$ and other function take queer behaviour in a neighbourhood of $p$. If $\omega(p, z)>0$, we call $p$ a singular point and denote the set of singular points by $B_{5}$ (see Theorem 7). We call a singular point $p$ a singular point of first or second kind according as H. M. (harmonic measure) of $p: w(p, z)\neq 0$ or $>0$. It is clear $\omega(p, z)\geqq w(p, z)$. It can be thought that such a large set $\Delta$ as C.P. $\omega(p, z)$ of $p>0$ is condensed so intensely by genus in a neighbourhood of $p$ that $\Delta$ may become one point $p$ in N-Martin's
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In other words, $p$ is larger, the more complicated the structure of $p$ is, and the more queer the behaviors of functions are in a neighbourhood of $p$. As an answer of problem 2, we use the notion singular point. In the present paper the discussions of singular points play the central part.

Let $G$ be a domain$^{12)}$ in $R−R_{0}$ and let $\omega(p, z)$ be C.P. of a singular point. Let $C_{G}\omega(p, z)$ be a superharmonic function in $R−R_{0}$ such that $C_{G}\omega(p, z) = \omega(p, z)$ in $G$ and $C_{G}\omega(p, z) = 0$ on $\partial R_{0}$ and harmonic in $G$ with M.D.I. over $G$. If $C_{G}\omega(p, z) < \omega(p, z)$, we say that $G$ contains $p$ $N$-approximately.

**Theorem 7.** a) Suppose that a domain $G$ contains a singular point $p$ $N$-approximately. Put $V_{M}^{c} = E[z \in R: C_{G}\omega(p, z) > M]$ ($V_{M}^{c}$ may be empty). Then there exists a constant $M_{0} < 1$ such that

$$\omega(p \cap V_{M}^{c}, z) = 0 \text{ for } M \geq M_{0},$$

where $\omega(p \cap V_{M}^{c}, z)$ is C.P. of $p \cap V_{M}^{c}$: $\omega(p \cap V_{M}^{c}, z) = \lim_{m \to \infty} \omega(\nu_{n}(p) \cap V_{M}^{c}, z)$. 

b) Let $\nu_{n}(p)$ be a neighbourhood of $p$: $p \in B_{s}$ such that $\nu_{n}(p) = E[z \in R: \delta(z, p) \leq \frac{1}{n}]$. Then $\nu_{n}(p)$ contains $p$ $N$-approximately.

By d) of Theorem 9 of P, there exists a number $n$ such that $(\nu_{n}(p) \cap R) \subset V_{M}(p)$ for any given $M < 1$, where $V_{M}(p) = E[z \in R: \omega(p, z) > M]$. But the inverse is not always true. Now as an almost inversed theorem, we shall prove the following:

c) For any given $\nu_{n}(p)$: $p \in B_{s}$

$$\omega(C_{\nu_{n}}(p) \cap V_{M}(p), z) \downarrow 0 \text{ as } M \to 1.$$ 

This means that $\{\nu_{n}(p)\}: n = 1, 2, \ldots$ is almost equivalent to $\{V_{M_{i}}(p)\}: M_{1} < M_{2} \ldots \lim M_{i} = 1$.

**Proof of a).** If $C_{G}\omega(p, z) = 0$, our assertion is clear. Suppose $C_{G}\omega(p, z) > 0$. Then $C_{G}\omega(p, z) > M$ on $V_{M}^{c} \cup \nu_{n}(p)$ and $V_{M}^{c} \cup \omega(\nu_{n}(p) \cap V_{M}^{c}, z)$ have M.D.I. over $R−R_{0}−(\nu_{n}(p) \cap V_{M}^{c})$. Hence by the maximum principle

$$C_{G}\omega(p, z) \geq C_{G}\omega(\nu_{n}(p) \cap V_{M}^{c}, z) \geq M\omega(\nu_{n}(p) \cap V_{M}^{c}, z).$$

Let $n \to \infty$. Then $C_{G}\omega(p, z) \geq M\omega(p \cap V_{M}^{c}, z)$. Now $\omega(p \cap V_{M}^{c}, z)$ has mass at only $p$ by $(p \cap V_{M}^{c}) \subset p$ by Theorem 5. b) of P. i.e.

12) See 5).

13) See “Potentials on Riemann surfaces” for the definition of $V_{M}^{c} \cup \omega(\nu_{n}(p) \cap C_{G}\omega(p, z))$. 


\[
\omega(p \cap V_{M}^{c}, z) = K \omega(p, z) : K > 0.
\]
But by P.C. 2. \( \sup_{z \in R} \omega(p \cap V_{M}^{c}, z) = 1 = \sup_{z \in R} \omega(p, z) \). Whence \( K = 1 \) and
\[
\omega(p \cap V_{M}^{c}, z) = \omega(p, z).
\]
Assume \( \omega(p \cap V_{M}^{c}, z) > 0 \) for every \( M \) such that \( M < 1 \). Then by letting \( M \to 1 \)
\[
CG \omega(p, z) \geq \lim_{M' 1} M \omega(p \cap V_{M}^{c}, z) = \omega(p, z).
\]
This contradicts the assumption. Hence we have a).

**Proof of b).** Assume \( C_{\nu_{n}(p)} \omega(p, z) = \omega(p, z) \). Then \( C_{\nu_{n}(p)} \omega(p, z) \) is \( N \)-minimal and by Theorem 5. b) of \( P_{C_{\nu_{n}(p)} \omega(p, z)} \) is represented by a mass distribution \( \mu \) over \( R - R_{0} - v_{m}(p) \) and by Theorem 9. a) of \( P_{C_{\nu_{n}(p)} \omega(p, z)} = a' N(z, q) : a' > 0 \) and \( q \notin v_{n}(p) \). On the other hand, by \( \omega(p, z) = a N(z, p) : a > 0 \) and by
\[
\sup_{z \in R} \omega(p, z) = 1 = \sup_{z \in R} \omega(p, z) \]
we have \( N(z, p) = N(z, q) \). This is a contradiction. Thus
\[
\omega(p, z) > C_{\nu_{n}(p)} \omega(p, z).
\]

**Proof of c).** At first, we remark \( \omega(p \cap CG) = 0 \), if \( G \) contains \( p \) \( N \)-approximately. In fact, assume \( \omega(p \cap CG, z) > 0 \), then by \( p \supset (p \cap CG) \), \( \omega(p \cap CG, z) \) has mass only at \( p \), i.e. \( \omega(p \cap CG, z) = a' N(z, q) : a' > 0 \) and \( q \notin v_{n}(p) \). On the other hand, by \( \omega(p, z) = a N(z, p) : a > 0 \) and by
\[
\sup_{z \in R} \omega(p, z) = 1 = \sup_{z \in R} \omega(p, z) \]
we have \( N(z, p) = N(z, q) \). This is a contradiction. Thus
\[
\omega(p, z) > C_{\nu_{n}(p)} \omega(p, z).
\]

By d) of Theorem 9 of \( P \), there exists a neighbourhood \( v_{m}(p) \) such that
\[
\omega(p, z) > 1 - \varepsilon \text{ in } v_{m}(p) \text{ for any given positive number } \varepsilon.
\]
Hence by the maximum principle
\[
(1 - \varepsilon) \omega(v_{m}(p) \cap CG, z) \leq C_{\nu_{m}(p) \cap CG} \omega(p, z) \leq \omega(p, z),
\]
because \( \omega(p, z) \geq 1 - \varepsilon \) on \( v_{m}(p) \supset (v_{m}(p) \cap CG) \) implies \( C_{\nu_{m}(p) \cap CG} \omega(p, z) = \omega(p, z) \)
\[
\geq 1 - \varepsilon \text{ on } v_{m}(p) \cap CG.
\]
Let \( v_{m}(p) \to p \) and then \( \varepsilon \to 0 \). Then \( \omega(p \cap CG, z) \leq \omega(p, z) \). Now by (a) \( C_{\nu_{m}} \omega(p, z) = \omega(p, z) \). This contradicts \( \omega(p, z) > C_{\nu_{m}} \omega(p, z) \).

Hence
\[
\omega(p \cap CG, z) = 0.
\]
By (12) and by \( \omega(p \cap CG, z) + \omega(p \cap G, z) \geq \omega(p, z) \) we have \( \omega(p \cap G, z) = \omega(p, z) \)
and by \( p \in V_{1-}(p) = E[z \in R : \omega(p, z) > 1 - \varepsilon] \), we have
\[
\omega(p, z) = \omega(p \cap V_{1-}(p) \cap G, z) > 0, \text{ whence } V_{1-}(p) \cap G \text{ is non void.}
\]
By Theorem 7. a), there exists \( V_{0}^{g} \) such that \( \omega(V_{0}^{g} \cap p, z) = 0 \), where
\[
V_{0}^{g} = E[z \in R : C_{\nu_{m}} \omega(p, z) > M_{0} \text{ and } M_{0} < 1].
\]
At present fix \( M_{0} \). Then by (13) and by (14) and by
we have
\[
\omega(p, z) = \omega(p \cap V_{1-\epsilon}(p) \cap CV_{M_{0}}^{c}, z) \\
\quad + \omega(p \cap V_{1-\epsilon}(p) \cap V_{g_{0}\cap}^\sigma G, z) \\
\geq \omega(p, z)
\]

and $CV_{M_{0}}^{c} \cap V_{1-\epsilon}(p) \cap G \cap u_{n}(p)$ is non void for every $V_{1-\epsilon}(p)$.

(15)

Let $\tilde{V} = CV_{u_{0}\cap}^{c} V_{1-\epsilon}(p) \cap o_{n}(p)$ and $G = u_{n}(p)$ and we shall prove $c)$. Since $u_{n}(p)$ contains $p$ $N$-approximately, (12), (13), (14) and (15) are valid.

Let $\tilde{\omega}(z) = \omega(\tilde{V}, z, G)$ be C.P. of $\tilde{V}$ relative $G$. Since $\tilde{\omega}(z)$ has M.D.I. $G-\tilde{V}$ among all functions with value $=1$ on $\tilde{V}$ and $=0$ on $\partial G: u_{n}(p) = G$.

\[
D(\tilde{\omega}(z)) \leq D\left(\min\left(1, \frac{\omega'(p, z)}{1-M_{0}-\epsilon}\right)\right) \leq \frac{1}{(1-M_{0}-\epsilon)^{2}} D(\omega'(p, z)),
\]

where $\omega'(p, z) = \omega(p, z) - CG\omega(p, z)$ and $\omega'(p, z) \geq 1-M_{0}-\epsilon$ on $(G \cap V_{1-\epsilon}(p)) - V_{M_{0}}^{c}$ and $=0$ on $\partial G$.

Put $\tilde{\omega}(z) = 1$ in $CG$ and $\bar{\omega}(z) = 1-\tilde{\omega}(z)$ in $G$. Then $\bar{\omega}(z) = 0$ in $\tilde{V}$ and $\tilde{\omega}(z) = 1$ in $CG$ and

\[
D(\tilde{\omega}(z)) = D(\bar{\omega}(z)) < \infty.
\]

Consider $\omega^{*}(z) = \min(\omega(V_{1-\epsilon}(p), z), \tilde{\omega}(z))$. Then $\omega^{*}(z) = 0$ on $\partial R_{0} + \tilde{V}$ and $\omega^{*}(z) = 1$ on $CG \cap V_{1-\epsilon}(p)$ and

\[
D(\omega^{*}(z)) \leq D(\omega(V_{1-\epsilon}(p), z)) + D(\tilde{\omega}(z)),
\]

because $\left| \frac{\partial \omega^{*}(z)}{\partial \bar{x}} \right| \leq \max\left( \left| \frac{\partial \omega(V_{1-\epsilon}(p), z)}{\partial \bar{x}} \right|, \left| \frac{\partial \tilde{\omega}(z)}{\partial \bar{x}} \right| \right)$.

Let $\hat{\omega}(z) = (V_{1-\epsilon}(p) \cap CG, z, R-R_{0}-\tilde{V})$. Then $\hat{\omega}(z)$ has M.D.I. over $R-R_{0}-\tilde{V}-(V_{1-\epsilon}(p) \cap CG)$ among all function with value $=1$ on $V_{1-\epsilon}(p) \cap CG$ and $=0$ on $\partial R_{0} + \partial \tilde{V}$, whence

\[
D(\hat{\omega}(z)) \leq D(\omega^{*}(z)) < \infty.
\]

Assume $\lim_{*} \omega(V_{1-\epsilon}(p) \cap CG, z) > 0$, where $D(\omega(V_{1-\epsilon}(p) \cap CG, z)) \leq D(\omega(V_{1-\epsilon}(p), z))$

$\frac{D(\omega(p, z))}{(1-\epsilon)^{2}}$. Then by the Dirichlet principle

\[
0 < D(\omega(V_{1-\epsilon}(p) \cap CG, z)) \leq D(\omega(V_{1-\epsilon}(p) \cap CG, z, R-R_{0}-\tilde{V})) \leq D(\omega^{*}(z)).
\]
Hence

$$\omega(V_{1-\epsilon}(p) \cap CG, z, R-R_0-\tilde{V}) \Rightarrow \overline{\omega}(z) > 0 \quad \text{as} \quad \epsilon \to 0,$$

(16)

where $\overline{\omega}(z)$ is C.P. defined by sequence \{\(V_{1-\epsilon_i}(p) \cap CG\): \(i=1, 2, \ldots\) and \(\lim_{i\to\infty} \epsilon_i = 0\). (see 4 of \(P\)).

Let \(C_s\) and \(C_{1-}\), be regular niveau curves of $\overline{\omega}(z)$ and put \(\Omega_s^{1-\epsilon} = E[z \in R-R_0-\tilde{V}: \delta < \overline{\omega}(z) < 1-\epsilon]\) and \(\Omega_{s'} = E[z \in R-R_0-\tilde{V}: \overline{\omega}(z) > \delta]\). Then $\omega(p, z)(=\omega(\tilde{V} \cap p, z))$ by (15)) has M.D.I. over $R-R_0-\tilde{V}$ by $(\tilde{V} \cap p) \cap (R-R_0-\tilde{V}) = 0$ by P.C.1.

Hence

$$\omega_n^1(z) \Rightarrow \omega(p, z) \quad \text{as} \quad n \to \infty,$$

where $\omega_n^1(z)$ is a harmonic function in $\Omega_s \cap (R_n-R_0)$ such that $\omega_n^1(z) = \omega(p, z)$ on $\partial\Omega_s \cap R_n$ and \(\frac{\partial}{\partial n} \omega_n^1(z) = 0\) on $\partial R_n \cap \Omega_s$.

Since by (16) $\overline{\omega}(z)$ is C.P., $\omega_n^1(z) \Rightarrow \overline{\omega}(z)$ as $n \to \infty$,

where $\omega_n^1(z)$ is a harmonic function in $\Omega_s^{1-\epsilon} \cap (R_n-R_0)$ such that $\omega_n^1(z) = \overline{\omega}(z)$ on $\partial\Omega_s^{1-\epsilon} \cap R_n$ and \(\frac{\partial}{\partial n} \omega_n^1(z) = 0\) on $\partial R_n \cap \Omega_s^{1-\epsilon}$.

Hence by the Green’s formula and by letting $n \to \infty$, we have by the regularity of $C_s$ and $C_{1-}$,

$$\int_{c_s} \omega(p, z) \frac{\partial}{\partial n} \overline{\omega}(z) = \int_{c_{1-}} \omega(p, z) \frac{\partial}{\partial n} \overline{\omega}(z) ds. \quad (17)$$

$\omega(p, z) < 1$ on $C_s$, whence there exists a constant $\epsilon_0$ such that $\int_{c_s} \omega(p, z) \frac{\partial}{\partial n} \overline{\omega}(z) ds < (1-\epsilon_0)D(\overline{\omega}(z))$. Above (17) holds so long as $C_{1-}$ is regular.

Choose $\epsilon < \frac{\epsilon_0}{2}$: Then

$$\int_{c_s} \omega(p, z) \frac{\partial}{\partial n} \overline{\omega}(z) ds = \int_{c_{1-}} \omega(p, z) \frac{\partial}{\partial n} \overline{\omega}(z) ds \leq (1-\epsilon_0)D(\overline{\omega}(z)) < (1-\epsilon)D(\overline{\omega}(z))$$

(18)

On the other hand, $\omega(p, z) > 1-\epsilon'$ on $V_{1-\epsilon'}(p)$. Hence by the maximum principle

$$\omega(p, z) > (1-\epsilon') \omega(V_{1-\epsilon'}(p), z) \geq (1-\epsilon') \omega(V_{1-\epsilon'}(p) \cap CG, z)$$

$$\geq (1-\epsilon') \omega(V_{1-\epsilon'}(p) \cap CG, z, R-R_0-\tilde{V}).$$

Let $\epsilon' \to 0$. Then $\omega(p, z) \geq \overline{\omega}(z)$. This contradicts (18).
Hence
\[ \lim_{\varepsilon \to 0} \omega(V_{1-\varepsilon}(p) \cap C_{\nu_n}(p), z) = 0. \]

Let \( F \) be a closed set of positive capacity in the \( z \)-plane and let \( \nu(F) \) be a neighbourhood of \( F \). Then \( \omega(F, z) > 0 \) implies \( \omega(F, z, \nu(F)) > 0 \) by \( \sup_{z \in \partial \nu(F)} \omega(F, z) < 1 \). We show that \( G \) has the same property, if \( G \) contains \( p \in B_s \) N-approximately.

**Theorem 8.** a) Suppose that \( G \) contains \( p \in B_s \) N-approximately. Then there exists a domain \( \tilde{V} \) in \( G \) such that \( \omega(\tilde{V} \cap p, z, G) > 0 \) and
\[ D(\omega(\tilde{V}, z, G)) < \frac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} < \infty, \]
where \( M_0 \) and \( \varepsilon \) are constants such that
\[ 1-\varepsilon > M_0 > 0 \]
and \( \omega'(p, z) = \omega(p, z) - cG\nu(p, z) > 1-M_0-\varepsilon > 0 \) in \( \tilde{V} \).

a') Conversely if there exists a domain \( \tilde{V} \) in \( G \) such that \( 0 < D(\omega(p \cap \tilde{V}, z, G)) < \infty \), then \( G \) contains \( p \) N-approximately.

b) Let \( \tilde{V} \) be the domain a). Put \( G_\delta = E[z : \omega(\tilde{V} \cap p, z, G) < \delta : 0 < \delta < 1] \).

Then
\[ \omega(\tilde{V} \cap p, z, G) = \omega(p \cap CG, z, G) = \delta \omega(CG, z, G) \] in \( G_\delta \).

c) If \( G \) contains a singular point of second kind, there exists a domain \( \tilde{V} \) in \( G \) such that
\[ \omega(\tilde{V} \cap p, z, G) : \omega(p \cap \tilde{V}, z, G) > 0 \] and \( D(\omega(\tilde{V}, z, G)) \leq \frac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} < \infty. \]

d) Let \( \nu(p) \) be a neighbourhood of a point \( p \in B_s \). Then \( \nu(p) \) contains \( p \) N-approximately by Theorem 7, b). Hence a) and c) are valid for \( \nu(p) \).

e) If \( G \) contains \( p \in B_s \) N-approximately, \( CG \) does not contain \( p \) N-approximately.

f) If \( (i=1, 2, \cdots, i_0) \) contains \( p \in B_s \) N-approximately, then \( \bigcap_{i=1}^{i_0} G_i \) contains \( p \) N-approximately.

**Proof of a).** By a) of Theorem 7, there exists a domain \( \tilde{V} = CV_{\delta_0} \cap V_{1-\varepsilon}(p) \) such that \( V_{\delta_0} = E[z \in R : cG\omega(p, z) > M_0] : 1 > M_0 > 0 \) and \( \omega(p \cap V_{\delta_0}) = 0. \) Then
\[ \omega'(p, z) = \omega(p, z) - cG\omega(p, z) > 1-M_0-\varepsilon > 0 \] in \( \tilde{V} \cap G \cap \nu_n(p) \) and = 0 on \( \partial G \) for every \( n \) by (15).

Since \( \omega(\tilde{V}, z, G) \) has M.D.I. over \( G - \tilde{V} \) among all functions with value 0 on \( \partial G \) and 1 on \( \partial \tilde{V} \),
\[ \frac{D(\omega'(p, z))}{(1-M_0-\varepsilon)^2} \geq D(\omega(\tilde{V}, z, G)) \geq D(\omega(\tilde{V} \cap \nu_n(p), z, G)) > 0, \]
because \( \omega'(p, z) > 1-M_0-\varepsilon \) on \( \tilde{V} \) and = 0 on \( \partial G \).
On the other hand, by the Dirichlet principle and by (15)

\[ D(\omega(\bar{V}, z, G)) \geq D(\omega(\bar{V} \cap \nu_n(p), z, G)) \geq D(\omega(\bar{V} \cap p, z)) = D(\omega(p, z)) > 0. \]

Let \( n \to \infty \). Then \( \omega(\bar{V} \cap p, z, G) > 0 \). Thus \( \bar{V} \) is the required domain.

**Proof of \( a' \)** Let \( C_{M_i} : M_1 < M_2 < 1 \) be a regular niveau curves of \( \omega(p \cap \bar{V}, z, G) \) such that

\[ \int_{C_{M_i}} \frac{\partial}{\partial n} \omega(p \cap \bar{V}, z, G) ds = D(\omega(p \cap \bar{V}, z, G)). \]

Now \( c_G \omega(p, z) \) is harmonic in \( G \) (\( c_G \omega(p, z) \) is harmonic in \( G \) and has M.D.I. over \( G \)). Hence

\[ c_G \omega_n(p, z) \to c_G \omega(p, z) \text{ as } n \to \infty, \]

where \( c_G \omega_n(p, z) \) is a harmonic function in \( \Omega_{M_1}^{M_2} \cap R_n \) such that \( c_G \omega_n(p, z) = c_G \omega(p, z) \) on \( \partial \Omega_{M_1}^{M_2} \cap R_n \) and \( \frac{\partial}{\partial n} c_G \omega_n(p, z) = 0 \) on \( \partial R_n \cap \Omega_{M_1}^{M_2} : \Omega_{M_1}^{M_2} = E[z \in G : M_1 < \omega(p \cap \bar{V}, z, G) < M_2]. \)

Also \( \omega(p \cap \bar{V}, z, G) \) has M.D.I. over \( \Omega_{M_2}^{M_1} \), whence \( \omega_n(p \cap \bar{V}, z, G) \to \omega(p \cap \bar{V}, z, G) \) as \( n \to \infty \), where \( \omega_n(p \cap \bar{V}, z, G) \) is a harmonic function in \( \Omega_{M_1}^{M_2} \cap R_n \) such that \( \omega_n(p \cap \bar{V}, z, G) = \omega(p \cap \bar{V}, z, G) \) on \( \partial \Omega_{M_1}^{M_2} \cap R_n \) and \( \frac{\partial}{\partial n} \omega_n(p \cap \bar{V}, z, G) = 0 \) on \( \Omega_{M_1}^{M_2} \cap \partial R_n \). Then

\[ \int_{c_{M_1}} c_G \omega_n(p, z) \frac{\partial}{\partial n} \omega_n(p \cap \bar{V}, z, G) ds = \int_{c_{M_2}} c_G \omega_n(p, z) \frac{\partial}{\partial n} \omega_n(p \cap \bar{V}, z, G) ds. \]

By letting \( n \to \infty \) and by Theorem 6 of \( P \) and since \( c_G \omega(p, z) < 1 \) on \( C_{M_1} \cap R \), there exists a constant \( \delta_0 \) such that

\[ D(\omega(p \cap \bar{V}, z, G)) - \delta_0 \geq \int_{c_{M_1}} c_G \omega(p, z) \frac{\partial}{\partial n} \omega(p \cap \bar{V}, z, G) ds \]

\[ = \int_{c_{M_2}} c_G \omega(p, z) \frac{\partial}{\partial n} \omega(p \cap \bar{V}, z, G) ds. \]

Hence

\[ \lim_{M_2 \downarrow 1} \int_{c_{M_2}} c_G \omega(p, z) \frac{\partial}{\partial n} \omega(p \cap \bar{V}, z, G) ds \leq D(\omega(p \cap \bar{V}, z, G)) - \delta_0. \] (18)

On the other hand, \( \omega(p \cap \bar{V}, z, G) \to 1 \) on \( C_{M_1} \) as \( M_2 \uparrow 1 \) and by \( \omega(p, z) \geq \omega(p \cap \bar{V}, z, G) \) we have
\[ \lim_{M_{2}\to 1} \int_{\sigma_{M_{2}}} \omega(p, z) \frac{\partial}{\partial n} \omega(p \cap \tilde{V}, z, G) ds \geq \lim_{M_{2}\to 1} \int_{\sigma_{M_{2}}} M_{2} \frac{\partial}{\partial n} \omega(p \cap \tilde{V}, z, G) ds = D(\omega(p \cap \tilde{V}, z, G)). \] (19)

(18) means by (19) that \( \omega(p, z) \leq \omega(p, z) \). Thus \( G \) contains \( p \) \( N \)-approximately.

**Proof of b.** \( D(\omega(\tilde{V} \cap p \cap G_{s}, z, G)) \leq D(\omega(\tilde{V}, z, G)) < \infty \) and by P.C.3. \( \omega(\tilde{V} \cap p \cap G_{s}, z, G) = 0 \). Now

\[ \omega(p \cap \tilde{V} \cap G_{s}, z, G) = \omega(p \cap \tilde{V} \cap CG_{s}, z, G). \]

Hence \( \omega(p \cap \tilde{V}, z, G) = 0 \) on \( \partial G_{s} \) and has M.D.I. over \( G - (v_{n}(p) \cap \tilde{V} \cap CG_{s}) \) for every \( n \).

Hence \( \omega(p \cap \tilde{V}, z, G) = 0(\omega(CG_{s}, z, G)). \)

**Proof of c.** We use the same notation as c) of Theorem 7. Put \( V_{M_{0}}^{c} = E[z \in R : c_{0} \omega(p, z) \geq M_{0}] \) and suppose that \( \omega(p \cap V_{M_{0}}^{c}, z) = 0 \). Let \( w_{c_{0}}(p, z) \) be a function such that \( w_{c_{0}}(p, z) = \omega(p, z) \) in \( CG \) and \( w_{c_{0}}(p, z) \) is the positively least harmonic function in \( G \). Then \( w_{c_{0}}(p, z) \leq c_{0} \omega(p, z) \) by \( w(p, z) \leq \omega(p, z) \). Put \( V_{1-\epsilon}^{w}(p) = E[z \in R : \omega(p, z) > 1-\epsilon] \) and \( V_{1-\epsilon}(p) = E[z \in R : \omega(p, z) > 1-\epsilon] \). Then

\[ V_{1-\epsilon}^{w}(p) \subset V_{1-\epsilon}(p). \]

By P.H.5,

\[ w(p \cap G \cap V_{1-\epsilon}^{w}(p), z) + w(p \cap G \cap CV_{1-\epsilon}^{w}(p), z) \]

\[ + w(p \cap CG, z) \leq w(p, z) \geq w(p \cap G \cap V_{1-\epsilon}^{w}(p), z). \]

Now \( w(p \cap CG, z) \leq \omega(p \cap CG, z) = 0 \) by (12) and \( w(p \cap G \cap CV_{1-\epsilon}^{w}(p), z) = 0 \) by P.H.3. Hence

\[ w(p, z) = w(p \cap G \cap V_{1-\epsilon}^{w}(p), z). \] (20)

Next \( w(p \cap V_{M_{0}}^{\infty} \cap G, z) \leq w(p \cap V_{M_{0}}^{\infty}, z) \leq \omega(p \cap V_{M_{0}}^{\infty}, z) = 0 \), whence

\[ 0 < w(p, z) = w(p \cap G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}), z). \] (20')

Thus \( v_{n}(p) \cap G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}) \) is non void for every \( v_{n}(p) \).

\[ w(p, z) - w_{c_{0}}(p, z) = 1 - M_{0} - \epsilon > 0 \] in \( G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}) \) (non void) by \( w_{c_{0}}(p, z) \leq c_{0} \omega(p, z) < M_{0} \) in \( V_{M_{0}}^{c} \). Hence

\[ w(p, z) - w_{c_{0}}(p, z) > 0. \] (22)

Put \( s_{m}(p) = v_{n}(p) \cap G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}) \). Then \( s_{m}(p) \cap G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}) \) \( \cap (G - s_{m}(p)) = 0 \). Hence by P.H.1. and by (20')

\[ w(p, z) = w(p \cap G \cap (V_{1-\epsilon}^{w}(p) - V_{M_{0}}^{\infty}), z) = \lim_{n} w_{n}^{1}(z), \]
where \( w_n^1(z) \) is a harmonic function in \((G-s_m(p))\setminus(R_n-R_0)\) such that 
\( w_n^1(z)=w(p, z) \) on \( \partial G+\partial s_m(p) \) and 
\( w_n^1(z)=0 \) on \( \partial R_n \setminus (G-s_m(p)) \).

Since \( w_{CG}(p, z) \) is the positively least harmonic function in \( G \) with \( w(p, z) \) on \( \partial G \) and \( w_{CG}(p, z) = w(p, z) \) on \( \partial G \), by P.H.1.

\[ w_{CG}(p, z) = \lim_{n} w_n^2(z) \]

where \( w_n^2(z) \) is a harmonic function in \( (G-s_m(p))\setminus R_n \) such that 
\( w_n^2(z)=w_{CG}(p, z) \) on \( \partial G+\partial s_m(p) \) and 
\( w_n^2(z)=0 \) on \( \partial R_n \). Hence

\[ 1 > w_n^1(z) - w_n^2(z) = w(p, z) - CGw(p, z) > 1-M_0- \varepsilon \]
on \( s_m(p) \) and

\[ w_n^1(z) - w_n^2(z) = 0 \] on \( \partial G+\partial s_m(p) \).

Let \( w_n^*(z) \) be a harmonic function in \( (G-s_m(p))\setminus R_n \) such that 
\( |w_n^*(z)|=0 \) on \( \partial G+\partial R_n - s_m(p) \) and 
\( w_n^*(z)=1 \) on \( \partial s_m(p) \cap R_n \).

Then by the maximum principle

\[ w_n^*(z) \leq w_n^1(z) - w_n^2(z) \]

Let \( n \to \infty \). Then
\( w(s_m(p), z, G) \geq w(p, z) - w_{CG}(p, z) \).

Let \( m \to \infty \). Then by (22)
\( w(p \cap G \cap (V_{1-\epsilon}(p) - V_{M_0}^{c}), z, G) \geq w(p, z) - w_{CG}(p, z) > 0 \).

(23)

Hence by (23) and (24) \( G \cap (V_{1-\epsilon}(p) - V_{M_0}^{c}) \) is the required domain.

**Proof of e).** Suppose \( G \) and \( CG \) contain \( p \) \( N \)-approximately. Then there exist \( V \) and \( V' \) in \( G \) and \( CG \) respectively such that 
\( \omega(p \cap V, z, G) \geq 0 \) and 
\( 0 < \omega(p \cap V', z, CG) \leq \omega(p \cap CG, z) \). On the other hand, by (12) 
\( \omega(p \cap CG, z) = 0 \). This is a contradiction. Hence we have e).

**Proof of f).** Since \( G_i \) contains \( p \) \( N \)-approximately, by a) there exists a domain \( \tilde{V}_i \) in \( G_i \) such that 
\( \tilde{V}_i = CV_{M_0,i}^{C}\cap V_{1-\epsilon_i}(p) \) and 
\( 1-M_0,i-\varepsilon_i > 0 \) and

\[ \infty > D(\omega(\tilde{V}_i, z, G_i)) \geq D(\omega(p \cap \tilde{V}_i, z, G_i)) > 0: i=1, 2, \ldots, i_0. \]

By the Dirichlet principle

\[ D(\omega(\tilde{V}_1 \cap V_{M_0,i}^{C}, z, \sum_{1}^{i_0} G_i)) \leq D(\omega(\tilde{V}_1, z, G_1)) < \infty, i=2, \ldots, i_0, \]

because \( (\tilde{V}_1 \cap V_{M_0,i}^{C}) \subset \tilde{V}_1 \) and \( \sum_{1}^{i_0} G_i \supset G_1 \). Similarly

\[ D(\omega(\tilde{V}_1 \cap CV_{1-\epsilon_i}(p), z, \sum_{1}^{i_0} G_i)) \leq D(\omega(V_1, z, G_1)) < \infty, i=2, \ldots, i_0. \]

Hence by the maximum principle

\[ \omega(p \cap \tilde{V}_1 \cap V_{M_0,i}^{C}, z, \sum_{1}^{i_0} G_i) = \omega(p \cap V_{M_0,i}^{C}, z) = 0 \] by Theorem 7, a) and
\begin{align*}
\omega(p, \overline{V}_1 \cap CV_1, z, \sum_{1}^{i_{0}} G_{i}) & \leq \omega(p, \overline{V}_1 \cap \tilde{V}_{1^\cap} CV_1, z, \sum_{1}^{i_{0}} G_{i}) \\
& \leq \omega(p, \overline{V}_{\text{levant}}, z) + \omega(p, CV_1, z) = 0, \quad i = 2, \ldots, i_0.
\end{align*}

Hence by P.C.3.

\begin{align*}
\omega(p, \overline{V}_1 \cap \tilde{V}_{i}, z, \sum_{1}^{i_{0}} G_{i}) & \geq \omega(p, \overline{V}_1 \cap \tilde{V}_{i}, z, \sum_{1}^{i_{0}} G_{i}) \\
& \geq \omega(p, \overline{V}_1 \cap \tilde{V}_{i}, z, \sum_{1}^{i_{0}} G_{i}) > 0. \quad \text{Hence}
\end{align*}

\begin{align*}
\omega(p, \overline{V}_1, z, \sum_{1}^{i_{0}} G_{i}) = 0.
\end{align*}

On the other hand, by \( \overline{V}_1 \cap \{ \bigcap_{2}^{i_{0}} \tilde{V}_{i} \} + \overline{V}_1 \cap \{ \sum_{2}^{i_{0}} C \tilde{V}_{i} \} = \overline{V}_i \)

\begin{align*}
\omega(p, \overline{V}_1 \cap (\bigcap_{i}^{i_{0}} \tilde{V}_{i}), z, \sum_{1}^{i_{0}} G_{i}) + \omega(p, \overline{V}_1 \cap (\sum_{i}^{i_{0}} C \tilde{V}_{i}), z, \sum_{1}^{i_{0}} G_{i}) \\
\geq \omega(p, \overline{V}_1, z, \sum_{1}^{i_{0}} G_{i}) > 0. \quad \text{Hence}
\end{align*}

\begin{align*}
\omega(p, \overline{V}_1, z, \sum_{1}^{i_{0}} G_{i}) > 0 \quad \text{and} \quad D(\omega(p, \overline{V}_1, z, \sum_{1}^{i_{0}} G_{i})) < \infty.
\end{align*}

(25)

Put \( \omega(z) = \min_{i} \omega(\tilde{V}_i, z, G_i) \). Then \( D(\omega(z)) \leq \sum_{1}^{i_{0}} D(\omega(\tilde{V}_i, z, G_i)) < \infty, \)

because \( \left| \frac{\partial \omega(z)}{\partial x} \right| \leq \max_{i} \left| \frac{\partial \omega(\tilde{V}_i, z, G_i)}{\partial x} \right| \) and \( \left| \frac{\partial \omega(z)}{\partial y} \right| \leq \max_{i} \left| \frac{\partial \omega(\tilde{V}_i, z, G_i)}{\partial y} \right| \).

Now \( \omega(z) \) is a continuous function in \( (\bigcap_{1}^{i_{0}} G_{i}) \) such that \( \omega(z) = 0 \) on \( \partial(\bigcap_{1}^{i_{0}} G_{i}) \) and \( \omega(z) = 1 \) on \( (\bigcap_{1}^{i_{0}} \tilde{V}_{i}) \). Hence

\begin{align*}
0 < D(\omega(\bigcap_{1}^{i_{0}} \tilde{V}_{i}, z, \bigcap_{1}^{i_{0}} G_{i})) \leq D(\omega(z)) < \infty.
\end{align*}

Whence by the Dirichlet principle and by \( (\bigcap_{1}^{i_{0}} G_{i}) \subset (\sum_{1}^{i_{0}} G_{i}) \) and by (25)

\begin{align*}
\infty > D(\omega(\bigcap_{1}^{i_{0}} \tilde{V}_{i}, z, \bigcap_{1}^{i_{0}} G_{i})) \geq D(\omega(\nu_n(p), \bigcap_{1}^{i_{0}} \tilde{V}_{i}), z, \bigcap_{1}^{i_{0}} G_{i}) \\
\geq D(\omega(\nu_n(p), \bigcap_{1}^{i_{0}} \tilde{V}_{i}), z, \sum_{1}^{i_{0}} G_{i}) > 0.
\end{align*}

Let \( n \to \infty \). Then

\begin{align*}
\infty > D(\omega(\bigcap_{1}^{i_{0}} \tilde{V}_{i}, z, \bigcap_{1}^{i_{0}} G_{i})) \geq D(\omega(p, \bigcap_{1}^{i_{0}} \tilde{V}_{i}), z, \bigcap_{1}^{i_{0}} G_{i}) \geq D(\omega(p, \bigcap_{1}^{i_{0}} \tilde{V}_{i}), \sum_{1}^{i_{0}} G_{i}) > 0
\end{align*}

and

\begin{align*}
\infty > D(\omega(p, \bigcap_{1}^{i_{0}} \tilde{V}_{i}), z, \bigcap_{1}^{i_{0}} G_{i}) > 0.
\end{align*}

Hence by \( a' \), \( \bigcap_{1}^{i_{0}} G_{i} \) contains \( p \) \( N \)-approximately.

A sufficient condition for a point \( B \in p_s \) to be a point of second kind.

**Theorem 9.** Suppose that a domain \( G \) contains \( p \) \( N \)-approximately and let \( \overline{G} \) be the closure of \( G \). If \( \overline{G} \cap (B - p) \) is an \( F_o \) set of harmonic
measure zero, \( p \) is a singular point of second kind.

**Proof.** By (16) we can find a domain \( \tilde{V} \) in \( G \) such that \( \infty > D(\omega(\tilde{V} \cap G, z, G)) \geq D(\omega(p \cup \partial \tilde{V}, z, G)) > 0. \)

Now \( (G - p) \cap B \) is an \( F_\sigma \) set of harmonic measure zero, i.e. \( F_\sigma = \sum_{i} F^i \):

\[
w(F^i, z) = 0.
\]

Let \( w(F^i, z) \) be the H.M. of \( F^i = E\{z \in R : \delta(z, F^i) \leq \frac{1}{m}\} \). Then

\[
\lim_{n \to \infty} 2^n \cdot w(F^i, z) = 0.
\]

For a given point \( z_0 \) in \( G - (V \cap u_n(p)) \), there exists a number \( m(n) \) such that \( w(F_{m(n)}^i, z_0) \leq \frac{1}{2^n} \).

Put \( w^*(F^i, z) = \sum_{n=0}^{\infty} w(F_{m(n)}^i, z) \) and \( w^*(F_\sigma, z) = \sum_{i=1}^{\infty} \frac{1}{2^i} w^*(F^i, z) \).

Then

\[
0 < \omega(p \cap \partial \tilde{V}, z, G) \leq w(p, z, G) \leq w(p, z).
\]

Thus \( p \) is a singular point of second kind.

6. Image of a singular point of second kind on the unit circle.

**Theorem 10.**

- a) Let \( V_M(p) = E\{z \in R : \omega(p, z) > M\} \). Then \( V_M(p) \downarrow \) as \( M \uparrow 1 \) and

\[
\lim_{n} w(u_n(p), z) = w(p, z) = \lim_{M \uparrow 1} w(V_M(p), z).
\]

- b) Map the universal covering surface \( (R - R_0)^\infty \) of \( (R - R_0) \) onto \( |\xi| < 1 \) by \( \xi \leftarrow f(z) \). Let \( \omega(p, z) \) be C.P. of a singular point \( p \) of second kind and let \( E \) be the set where \( \omega(p, z) \) has angular limits = 1 almost everywhere. We call \( E \) the image of \( p \). Then \( \text{mes } E > 0 \) and \( w(p, z) = w(E, \xi) \), where \( w(E, \xi) \) is the harmonic measure of \( E \) in \( |\xi| < 1 \). Let \( p_1 \) and \( p_2 \) be singular points of second kind: \( p_1 \neq p_2 \) and let \( E_i \) be its image. Then \( \text{mes } (E_1 \cap E_2) = 0 \). Hence the set of singular points of second kind is at most enumerable.

**Proof of a.** By Theorem 9. d) of \( P \), for any given \( V_M(p)(M < 1) \), there exists a number \( n_0 \) such that \( V_M(p) \cap (u_n(p) \cap R) \) for \( n > n_0 \), whence
lim \( w(V_M(p), z) \geq \lim w(u_n(p), z) = w(p, z) \). On the other hand, by Theorem 7. c) \( \omega(C_{U_n}(p) \cap V_M(p), z) \downarrow 0 \) as \( M \uparrow 1 \).

This implies \( w(C_{U_n}(p) \cap V_M(p), z) \downarrow 0 \) as \( M \uparrow 1 \). Hence by

\[
\lim_{M=1} w(V_M(p), z) \leq \lim_{M=1} w(V_M(p) \cap C_{U_n}(p), z) + w(V_M(p) \cap u_n(p), z)
\]

we have \( \lim_{M=1} w(V_M(p), z) \leq \lim_{n} w(u_n(p), z) = w(p, z) \) for every \( n \).

Hence by \( w(V_M(p), z) \leq w(C_{U_n}(p) \leftrightarrow V_M(p), z) \downarrow 0 \) as \( M \uparrow 1 \).

Thus we have a).

**Proof of b.** \( f(\partial R_0) \) consists of an enumerably infinite number of arcs on \( |\xi|=1 \). Let \( G(z, p) \) be the Green's function of \( (R-R_0) \) and let \( E_g \) be the set on \( |\xi|=1 \) where \( G(z, p) \) has angular limits=0 on \( |\xi|=1 \). Then \( \text{mes } E_g = 2\pi \).

Put \( \inf_{z \in \partial R_n} G(z, p) = \delta_n \). Then since \( \partial R_n \) is compact, \( \delta_n > 0 \).

Let \( E_0 \) be the set on \( |\xi|=1 \) such that \( E_0 \cap f(\partial R_0) = 0 \) and \( \omega(p, z) \) has angular limits<1.

Assume \( \text{mes } E_0 > 0 \). In this case, by Egoroff's theorem, for any given positive number \( \varepsilon \) we can find a closed set \( E' \subset (E_0 \cap E_g) \) (by \( \text{mes } E_g = 2\pi \)) such that \( \text{mes } (E' - E_0) < \frac{\varepsilon}{2} \) and \( \omega(p, z) \) converges uniformly in an angular domain \( A(\theta) = E [\xi : \arg \frac{\xi - e^{i\theta}}{e^{i\theta}} < \frac{2\pi}{3}] \) for every point \( e^{i\theta} \) in \( E' \) as \( \xi = f(z) \) tends to \( E' \) in \( A(\theta) \). And further we can find a closed set \( E'' \subset E' \) and a constant \( M' < 1 \) and a number \( m \) such that

\[
\text{mes } (E_0 - E'') < \varepsilon : \omega(p, z) < M' \text{ in } A(\theta) \cap \mathfrak{R}_m \text{ for } e^{i\theta} \in E'',
\]

where \( \mathfrak{R}_m \) is a ring domain: \( 1 - \frac{1}{m} < |\xi| < 1 \).

Let \( D \) be a domain containing an end part of \( A(\theta) : e^{i\theta} \in E'' \) and bounded by \( \sum_{e^{i\theta} \in E''} \partial A(\theta) + E'' \) and a circle \( \mathfrak{C}_\frac{1}{2} : |\xi| = \frac{1}{2} \). Then \( D \) may consist of at most three components. Hence there exists at least one component \( D' \) such that \( \text{mes } (\partial D' \cap E'') > 0 \). Let \( D^* \) be one of them. For simplicity denote it by \( D \) newly. Let \( w^p(\xi) \) be a harmonic function in \( D \) such that \( w^p(\xi) = 1 \) on \( \partial D - E'' \) and \( w^p(\xi) = 0 \) on \( E'' \). Then by the rectifiableness of \( \partial D \) \( w^p(\xi) < 1 \) and \( w^p(\xi) \) has angular limits=0 a.e. on \( E'' \).

Put \( V_M(p) = E [z \in R : \omega(p, z) > M] \). Then by a) \( w(p, z) = \lim_{M=1} w(V_M(p), z) \).

Let \( w_{M,n}(z) \) be a harmonic function in \( (R_n - R_0 - V_M(p)) \) such that \( w_{M,n}(z) = 0 \) on \( \partial R_n - V_M(p) + \partial R_0 \) and \( w_{M,n}(p) = 1 \) on \( \partial V_M(p) \).

Then

\[
\lim_{M=1} \lim_{n=\infty} w_{M,n}(z) = w(p, z)
\]
Consider the image of $$(R_n-R_0-V_M(p))$$ in $D$.

Since $\omega(p, z)<1$ in $R-R_0$, $f(\partial V_M(p)) \to \Gamma_1 : |\xi|=1$ as $M \to 1$. Hence there exists a number $M_2$ such that $f(V_M(p)) \subset \Re_\alpha$ for $M>M_2$. Hence by (26)

$$f(V_M(p)) \cap D = 0$$ for $M>M_3=\max(M_1, M_2)$.\n
Since \(\inf_{z \in \partial R_n} G(z, p) = \delta_n > 0\), $f(\partial R_n)$ does not tend to $E''$ in $D$, because, if it were so, there existed a sequence $\xi_k \in f(\partial R_n)$ such that $\xi_k \to E''$ as $k \to \infty$ inside of $D$. But by $E'' \subset E_\alpha$, $G(\xi, p) \to 0$. This is a contradiction. Hence $f(\partial R_n)$ separates $E''$ from $\Gamma_m : |\xi|=1-\frac{1}{m}$ in $D$ and $f(R_n-R_0) \cap E'' = 0$.

Compare $w_{M,n}(z)$ and $w^D(\xi)$ in $D \cap f(R_n-R_0-V_M(p)) : M>M_3$.

Then $w^D(\xi)=1$ and $w_{M,n}(z)<1$ on $\partial D \cap f(R_n-R_0-V_M(p))$,

$$w^D(\xi)>0$$ and $w_{M,n}(z)=0$ on $D \cap \partial(f(R_n-R_0-V_M(p))(\subset f(\partial R_n))$.

Hence by the maximum principle

$$w^D(\xi) \geq w_{M,n}(z)$$ in $D \cap f(R_n-R_0-V_M(p))$.

Let $n \to \infty$ and then $M \uparrow 1$. Then $w^D(\xi) \geq w(p, z)$.

Now $D$ is an arbitrary component and let $\varepsilon \to 0$. Then $w(p, z)$ has angular limits $=0$ a.e. on $E_0$ ($w(p, z)=0$ on $\partial f(\partial R_n)$).

Let $E_1$ be the set of $|\xi|=1$ where $\omega(p, z)$ has angular limits $=1$. Similarly as above, we can find a closed set $E'' \subset E_1$ and a domain $D$ containing an endpart of $A(\theta)=E\left[\xi : \left|\arg \frac{\xi-e^{i\theta}}{e^{i\theta}}\right| < \frac{2\pi}{3}\right]$ for $e^{i\theta} \in E'$ and bounded by $\sum_{e^{i\theta} \in E'} \partial A(\theta)+E'$ such that

$$\text{mes}(E_1-E'')<\varepsilon$$ and $\omega(p, z) \to 1$ as $\xi=\hat{f}(z) \to E'$ in $D$.\n
\[ (29) \]
Let $D$ (newly denoted) be one of the components of $D$ such that \( \operatorname{mes}(D \cap E') > 0 \). Let $w^0(\xi)$ be a harmonic function in $D$ such that $w^0(\xi) = 0$ on $\partial D - E'$ and $w^0(\xi) = 1$ on $E'$. Then $w^0(\xi) > 0$ and $w^0(\xi)$ has angular limits $= 1$ a.e. on $E'$. By (29) $f(\partial V_M(p)) : M < 1) does not tend to $E'$ in $D$. Hence

$$\operatorname{dist} (f(\partial V_M(p)) \cap D), E' = \delta_M > 0 \text{ and by (29)}$$

$$D - f(V_M(p)) \supset E'. \tag{30}$$

Since $f(\partial R_\epsilon) \to f'(1)$: \( |\xi| = 1 \) as $n \to \infty$, there exists a number $\delta_M$ such that \( \operatorname{dist} (f(\partial R_n), \xi = 0) > 1 - \delta_M \). Then by (30) $f(R_n - R_0 V_M(p)) \supset (D - f(V_M(p))$ for $n > n_M$. Hence as above we have $w^0(\xi) \leq w_{M,n}(\xi)$.

Let $n \to \infty$ and then $M \to 1$, $w^0(\xi) \leq w(p, z)$ and by letting $\epsilon \to 0$, we see that $w(p, z)$ has angular limits $= 1$ a.e. on $E$. Thus

$$w(p, z) = w(E, \xi) \text{ and mes } E > 0 \text{ by } w(p, z) > 0.$$

Assume $\operatorname{mes}(E_1 \cap E_2) > 0$. Then as above it is proved that

$$w(E_1 \cap E_2, \xi) = w(p_1 \cap p_2, z) \leq \omega(p_1 \cap p_2, z)).$$

Now $\omega(p_1 \cap p_2, z)$ has mass only at $p_1$ (i.e. $\omega(p_1 \cap p_2, z) = K \omega(p, z)$) by $(p_1 \cap p_2) \subset p_1$. By P.C.2. \( \sup \omega(p_1 \cap p_2, z) = 1 = \sup \omega(p_1, z). \) Hence $\omega(p_1 \cap p_2, z) = \omega(p_1, z)$. Similarly $\omega(p_1 \cap p_2, z) = \omega(p_2, z)$. This implies $N(z, p_1) = N(z, p_2)$. This contradicts $p_1 \neq p_2$. Hence $\operatorname{mes}(E_1 \cap E_2) = 0$. Let $p_i$ be a singular point of second kind and let $E_i$ be its image. Then $\operatorname{mes}(E_i \cap E_i) = 0$ for $i \neq j$. Hence by $\sum \operatorname{mes} E_i < 2\pi$ the set of singular points of second kind is at most enumerable.

Remark. Toplogical structure of $B_s$. Let $B_n$, the set of singular points $p_i$ such that $\operatorname{Cap}(p_i) \geq \frac{1}{n}$ and $p$ be a limit point of a sequence $\{p_i\}$ in $B_n$. Then by $\nu_m(p) \ni p_i \quad \operatorname{Cap}(p_i) \leq \operatorname{Cap}(\nu_m(p))$ for every $m$. Hence $\operatorname{Cap}(p) = \lim_{m \to \infty} \operatorname{Cap}(\nu_m(p)) \geq \frac{1}{n}$ and $B_n$ is closed. Whence by $B_s = \bigcup B_n$, $B_s$ is an $F_\sigma$ set.

By Theorem 10 the set of singular points of second kind is enumerable, but the set of singular points of first kind is not necessarily enumerable. In reality there exists a Riemann surface which has the set of singular points of first kind of the power of continuum (see the following paper "Examples of singular points").

7. Harmonic functions in a neighbourhood of a singular point of second kind. Harmonic domain and their harmonic measures. Map the universal covering surface $G^\infty$ of a domain $G$ in $R - R_0$ onto $|\xi| < 1$
by \( \xi = f(z) \) conformally. If a harmonic function \( H(z) \) in \( G \) has angular limits a.e. on \( |\xi| = 1 \), we say that \( H(z) \) is of \( F \)-type. Clearly if \( H(z) \in HB \) or \( H(z) \in HD \), \( H(z) \) is of \( F \)-type.

Consider a system of harmonic functions \( H_i(z) \) of \( F \)-type \((i = 1, 2, \ldots, l)\). If a domain \( D^H \) is defined as \( D^H = \bigcap_{i=1}^{l} E[z \in G : H_i(z) \geq a_i] \), we call \( D^H \) a harmonic domain in \( G \). Let \( B_i \) be the set on \( |\xi| = 1 \) where \( H_i(z) \) has angular limits \( \geq a_i \). Put \( B(D^H) = \cap^{l} B_{i} \). We call \( B(D^H) \) the image of the ideal boundary of \( D^H \).

Let \( D_{k}^{H} = \cap^{l} E[z \in G; H_{i}(z) \geq a_{i} - \frac{1}{k}] \). Then \( D^{H} = \bigcap_{k} D_{k}^{H} \). Put \( w(p \cap [D^{H}], z, G) = \lim_{n} \lim_{k} w(u_{n}(p) \cap D_{k}^{H}, z, G) \), where \( w(u_{n}(p) \cap D_{k}^{H}, z, G) \) is the harmonic measure of \( u_{n}(p) \cap D_{k}^{H} \) relative to \( G \), i.e. \( w(u_{n}(p) \leftrightarrow D_{k}^{H}, z, G) \) is the least positive superharmonic function in \( G \) larger than 1 on \( v_{n}(p) \cap D_{k}^{H} \).

In the following we distinguish it from \( w(p \cap D^{H}, z, G) = \lim_{n} w(U_{n}(p) \cap D'', z, G) \) by \([ \] \). Then it is clear

\[
  w(p \cap D_{k}^{H}, z, G) \geq w(p \cap [D^{H}], z, G) = w(p \cap D^{H}, z, G).
\]

**Theorem 11.**

**a)** Let \( D_{k}^{H} \) be a harmonic domain in \( G \subset (R - R_{0}) \). Put \( \tilde{V}_{M}^{W} = E[z \in G : \omega(p \cap [D^{H}], z, G) > M] \). Then \( \tilde{V}_{M}^{W} \subset V_{M}^{W}(p) = E[z \in R : w(p, z) > M] \subset V_{M}(p) = E[z \in R : \omega(p, z) > M] \) by \( w(p \cap [D^{H}], z, G) \leq w(p, z) \leq \omega(p, z) \). Then for any domain \( G' \) and \( G'' \) such that \( G' \subset G \) and \( G'' \subset G \).

\[
  \lim_{M=1} \lim_{k=\infty} w(\tilde{V}_{M}^{W} \cap G'' \cap D_{k}^{H}, z, G') = \lim_{M=1} w(\tilde{V}_{M}^{W}, z, G) = \lim_{M=1} w(\tilde{V}_{M}^{W} \cap D_{k}^{H}, z, G) = w(E, \xi),
\]

where \( w(E, \xi) \) is the harmonic measure of \( E \) with respect to \( |\xi| < 1 \).

**b)** Let \( G \) be a domain in \( R - R_{0} \) and \( D^{H} \) be a harmonic domain in \( G \). Let \( E \) be the set of positive measure where \( w(p \cap [D^{H}], z, G) \) has angular limits \( = 1 \) on \( |\xi| = 1 \) and let \( B(D^{H}) \) be the image of the ideal boundary of \( D^{H} \). Then

\[
  \text{mes}(E - B(D)) = 0,
\]

and

\[
  w(p \cap [D^{H}], z, G) = \lim_{M=1} w(\tilde{V}_{M}^{W}, z, G) = \lim_{M=1} w(\tilde{V}_{M}^{W} \cap [D^{H}], z, G) = (\lim_{M=1} \lim_{k} w(\tilde{V}_{M}^{W} \cap D_{k}^{H}, z, G)) = w(E, \xi),
\]

where \( w(E, \xi) \) is the harmonic measure of \( E \) with respect to \( |\xi| < 1 \).

**c)** Let \( D^{H} \) be a harmonic domain in \( G \subset R - R_{0} \). Let \( U(z) \) be a harmonic function of \( F \)-type in \( G \) such that \( U(z) \) has angular limit \( U(e^{i\theta}) \) at \( e^{i\theta} \) which is not a constant a.e. on \( E \), where \( \text{mes} E > 0 \) and \( w(p \cap [D^{H}], z, G) \) has angular limits \( = 1 \) a.e. on \( E \). Then we can find two constants
L and \( \delta \) such that

\[
 w(p \cap [D^u] \cap G^u_{L-\delta}, z, G^u_{L} \cap G) > 0 \text{ and } w(p \cap [D^u] \cap G^u_{L+\delta}, z, G^u_{L} \cap G) > 0,
\]

where \( G^u_a = E[z \in G : U(z) > a] \) and \( G^u_a = E[z \in G : U(z) < a] \).

Whence \( w(p \cap [D^H] \cap G^u_{L+\delta}, z, G^u_{L} \cap G) > 0 \) and \( w(p \cap [D^H] \cap G^u_{L-\delta}, z, G^u_{L} \cap G) > 0 \) for \( k < \infty \).

**Proof of a.** \( w(\tilde{V}_M^W \cap C_{U_n}(p) \cap [D^H] \cap G^f, z, G^f) \leq w(V_M^{W}(p) \cap C_{U_n}(p), z) \leq \omega(V_M(p) \cap C_{U_n}(p), z) \rightarrow 0 \) as \( M \rightarrow 1 \) by (c) of Theorem 5.

By \( w(\tilde{V}_M^W \cap C_{U_n}(p) \cap [D^H] \cap G^f, z, G^f) \geq w(\tilde{V}_M^W \cap [D^H] \cap G^f, z, G^f) \), we have by letting \( M \rightarrow 1 \)

\[
 \lim_{M \rightarrow 1} w(\tilde{V}_M^W \cap [D^H] \cap G^f, z, G^f) = \lim_{M \rightarrow 1} w(\tilde{V}_M^W \cap [D^H] \cap G^f, z, G^f) \leq w(p \cap [D^H] \cap G^f, z, G^f) \text{ as } f(z) = \xi \rightarrow E' \text{ along Stolz's path.}
\]

Further we can find two closed sets \( E^{(1)} \) and \( E^{(2)} \) in \( E' \) such that \( E^{(1)} \cap E^{(2)} = 0 \), \( mes\ (E^{(1)} - E^{(2)}) < \varepsilon \) and domains \( D^{(1)} \) and \( D^{(2)} \) such that \( D^{(i)}(i=1,2) \) containing an endpart of

\[
 A(\theta) = E\left[ \xi : \frac{1}{3} \leq \frac{\xi - e^{i\theta}}{e^{i\theta}} < \frac{2\pi}{3} \right] \text{ for } e^{i\theta} \in E^{(1)} \text{ and bounded by } \sum_{e^{i\theta} \in E^{(1)}} \partial A(\theta) + E^{(1)}
\]

and a ring \( R_m : 1 - \frac{1}{m} < |\xi| < 1 \) such that \( w(p \cap [D^H], G) < 1 - \frac{\delta}{2} \) in \( D^{(1)} \int R_m \) and \( H_i(z) < a_i - \frac{\delta}{2} \) in \( D^{(2)} \int R_m \) for a number \( i_0 \) respectively. Hence

\[
 f(\tilde{V}_M^W) \cap D^{(1)} \cap R_m = 0 \text{ and } f(D_H^M) \cap D^{(2)} \cap R_m = 0 \text{ for } M > 1 - \frac{\delta}{2} \text{ and } \frac{1}{k} < \frac{\delta}{2},
\]

where \( D^H_k = \bigcap_{i=1}^{l} E[z \in G : H_i(z) \geq a_i - \frac{1}{k}] \). (a)

Now \( D^{(1)} \) is composed of a finite number of components. Let \( D' \) be one of them such that \( mes(D' \cap E^{(1)}) > 0 \). Now \( w(p \cap [D^H], z, G) = \lim M w(\tilde{V}_M^W, z, G) \) (by P.H.4.) and \( = \lim \lim_{n \rightarrow k} (\nu_n(p) \cap D_H^M, z, G) \). Put \( G_f \),
$E \left[ z \in G : \delta(\partial G_j, z) > \frac{1}{j} \right]$. Let $w_{M,j,n}(z)$ be a harmonic function in $(R_n \cap G_j) - \tilde{V}_M^w$ such that $w_{M,n,j}(z) = 0$ on $\partial G_j + \partial R_n - \tilde{V}_M^w$ and $w_{M,j,n}(z) = 1$ on $\partial \tilde{V}_M^w$. Consider the image of $(G_j \cap R_n) - \tilde{V}_M^W$. Then by $E^{(1)} \subset E_q$, image of $(\partial G_j + \partial R_n)$ separates $\Gamma_{1-\frac{1}{m}}$ from $E^{(1)}$ in $D'$, (b) because $\Gamma_{1-\frac{1}{m}}$.

Hence as usual (see the proof of Theorem 8) by a) and b) and by the maximum principle $w_{M,j,n}(z) \leq w^D(\xi)$, where $w^D(\xi)$ is a harmonic function in $D'$ such that $w^D(\xi) = 1$ on $\partial D^{(2)} - E^{(2)}$ and $w^D(\xi) = 0$ on $\partial D^{(2)} \cap E^{(2)}$.

Let $i \to \infty$ and then $n \to \infty$ and then $M \to 1$. Then $w(p \cap [D^H], z, G) = \lim_{M} w(\tilde{V}_M^w, z, G) \leq w^D(\xi)$.

(c) Since $w^D(\xi)$ has angular limits $= 0$ a.e. on $E^{(1)}$ and $D'$ is arbitrary, $w(p \cap [D^H], z, G)$ has angular limits $= 0$ a.e. on $E^{(1)}$ by letting $\varepsilon \to 0$.

Let $w_{k,j,n}(z)$ be a harmonic function in $(R_n \cap G_j) - D_k^H$ such that $w_{k,j,n}(z) = 0$ on $\partial R_n + \partial G_j - D_k^H$ and $w_{k,j,n}(z) = 1$ on $\partial D_k^H$. Then the image of $(G_j \cap R_n) - \partial D_k^H$ separates $|\xi| = 1 - \frac{1}{m}$ from $E^{(2)}$ and as above

$w_{k,j,n}(z) \leq w^D(\xi)$ and $w(p \cap [D^H], z, G) \leq w^D(\xi)$, (d) where $w^D(\xi)$ is a harmonic function in $D^{(2)}$ such that $w^D(\xi) = 1$ on $\partial D^{(2)} - E^{(2)}$ and $w^D(\xi) = 0$ on $\partial D^{(2)} \cap E^{(2)}$. Let $\varepsilon \to 0$. Then by (c) and (d) $w(p \cap [D^H], z, G)$ has angular limits $= 0$ a.e. on $E_c$. Hence $\text{mes} (E - B(D^H)) = 0$, because if $\text{mes} (E - B(D^H)) > 0$, we can find a subset of $E - B(D^H)$ of positive measure on which $w(p \cap [D^H], z, G)$ has angular limits $= 0$. This contradicts the definition of $E$ on which $w(p \cap [D^H], z, G)$ has angular limits $= 1$ almost everywhere.

Now by $\text{mes} E_c + \text{mes} E = 2\pi$ $w(p \cap [D^H], z, G)$ has angular limits $= 1$ on $E$. Hence

$w(p \cap [D^H], z, G) = w(E, \xi) = \lim_{M \to 1} w(\tilde{V}_M^w, z, G)$.

$\lim_{M=1} w(\tilde{V}_M^w, z, G) \geq \lim_{M=1} w(\tilde{V}_M^w \cap [D^H], z, G)$ is clear. We show $\lim_{M=1} w(\tilde{V}_M^w, z, G)$ $= \lim_{M=1} w(\tilde{V}_M^w \cap [D^H], z, G) = \lim_{M=1} w(\tilde{V}_M^w \cap [D^H], z, G)$. By $\text{mes} (E - B(D^H)) = 0$, $\text{mes} E = \text{mes} (E \cap B(D^H))$. If $w(E, \xi) = 0$, $\text{mes} (E \cap B(D^H)) = 0$.

We have $0 = \lim_{M=1} w(\tilde{V}_M^w, z, G) \geq \lim_{M=1} w(\tilde{V}_M^w \cap [D^H], z, G)$ and we have our
assertion. Hence we assume \( \text{mes} (E \cap B(D^H)) > 0 \). In this case, for any given positive number \( \varepsilon \), we can find a closed set \( E' \subset (E \cap B(D^H)) \) such that \( \text{mes} ((E \cap B(D^H) - E') < \varepsilon \) and a domain \( D \) (\( D \) is containing and endpart of \( A(\theta) : e^{i\theta} \in E' \) and bounded by \( \sum_{e^{i\theta} \in E'} \partial A(\theta) + E' \) and a circle: \( |\xi| = \frac{1}{2} \)) and bounded a number \( m_0 \) such that

\[
\text{dist} (f(\tilde{V}_{1-k}^{W} + \partial D_{k}^{H}), \xi) = 0) > 1 - \delta \quad \text{for} \quad n \geq n_0 \quad \text{and} \quad j \geq j_0.
\]

Let \( D' \) be a component of \( D \cap \Re_{m} \) such that \( \text{mes} (\partial D' \cap E') > 0 \). Then by \( (b) \) \( f(\partial R_n + \partial G_j), \xi = 0 \) for \( n \geq n_0 \) and \( j \geq j_0 \).

Let \( w^{D}(\xi) \) be a harmonic function in \( D' \) such that \( w^{D}(\xi) = 0 \) on \( \partial D' - E' \) and \( w^{D}(\xi) = 1 \) on \( E' \). Then by \( (a) \) and by the maximum principle

\[
\sup_{\xi} w^{D}(\xi) \leq \sup_{M=1} \lim_{k=\infty} w(\tilde{V}_{M}^{W} \cap D_{k}^{H}, z, G)
\]

But \( D' \) is arbitrary and by letting \( \varepsilon \to 0 \),

\[
\lim_{M=1} w(\tilde{V}_{M}^{W} \cap [D^H], z, G) \text{ has angular limits} = 1 \text{ a.e. on } (E \cap B(D^H)).
\]

On the other hand, \( w(E, \xi) = \lim_{M=1} w(\tilde{V}_{M}^{W}, z, G) \geq \lim_{M=1} w(\tilde{V}_{M}^{W} \cap [D^H], z, G) \) has angular limits = 0 a.e. on \( CE \). Thus

\[
\lim_{M=1} w(\tilde{V}_{M}^{W}, z, G) = \lim_{M=1} w(\tilde{V}_{M}^{W} \cap [D^H], z, G).
\]

**Proof of c.** Since \( U(e^{i\theta}) \neq \text{const} \) a.e. on \( E \), we can find two constants \( L \) and \( \delta > 0 \) and two sets \( E_1 \) and \( E_2 \) of positive measure in \( E \) such that the angular limits \( U(e^{i\theta}) \geq L + 2\delta \) on \( E_1 \) and \( U(e^{i\theta}) < L - 2\delta \) on \( E_2 \) respectively. For any given positive number \( \varepsilon \) we can find a closed subset \( E' \)
of $E_1$ and a domain $D$ such that $\text{mes}(E_1 - E') < \epsilon$ and $D$ is containing an endpart of $A(\theta) : e^{i\theta} \in E'$ and bounded by $E' + \sum_{e^{i\theta} \in E'} \partial A(\theta)$ with the property as follows:

$$U(\xi) \rightarrow \geqq L + 2\delta \text{ as } \xi \rightarrow E' \text{ in } D. \quad (a)$$

$$w(p \cap [D^H], z, G) \rightarrow 1 \text{ and } H_i(z) \rightarrow \geqq a_i (i = 1, 2, \cdots, l) \text{ as } \xi \rightarrow E' \text{ in } D. \quad (b)$$

Put $G^U_L = E[z \in G : U(z) > L]$. Since $U(z) \rightarrow \geqq L + 2\delta$ as $\xi \rightarrow E'$, $f(\partial G^U_L)$ does not tend to $E'$ in $D$, whence $\text{dist}(f(\partial G^U_L) \cap D, E') > \delta_U > 0$. Let $R_m$ be a ring domain: $1 > |\xi| > 1 - \frac{1}{m}$. Then

$$U(z) \geqq L \text{ in } D \cap R_m \text{ for } \frac{1}{m} < \delta_U. \quad (c)$$

$D \cap R_m$ is composed of at most $n(m)$ number of components. Let $D'$ be a component such that $\text{mes}(E' \cap D') > 0$ and let $G^U_L$ be the component of $G^U_L$ such that there exists a point $\xi$ in $D'$ with $f(\xi) = \xi : z \in G^U_L$. We denote it also by $G^U_L$.

By $\text{mes}(E - B(D^H)) = 0$, $w(p \cap [D^H], z, G) \rightarrow 1$ and $H_i(z) \rightarrow \geqq a_i$ as $\xi \rightarrow E'$ in $D'$, dist $(f(\partial V_{1-\frac{1}{k}}^W + \partial D^H_k \cap D', E') = \delta_k > 0$ for $k > 0$ and since $U(z) \rightarrow \geqq L + 2\delta$ as $\xi \rightarrow E'$,

$$\text{dist}(f(\partial G^U_L) \cap D', E') = \delta_L > 0. \quad (d)$$

Let $G_j = E[z \in G : \delta(z, \partial G) > \frac{1}{j}]$. Then since $f(\partial R_n + \partial G_j) \rightarrow |\xi| = 1$ as $n \rightarrow \infty$ and $j \rightarrow \infty$, there exist numbers $n_0$ and $j_0$ such that

$$\text{dist}(f(\partial R_n + \partial G_j) \cap D', \xi = 0) > 1 - \delta : \delta = \min(\delta_U, \delta_L', \delta_k) \text{ for } n \geqq n_0 \text{ and } j \geqq j_0. \quad (e)$$

Let $w_{j,n,k}(z)$ be a harmonic function in $(R_n \cap G_j \cap G^U_L \cap \tilde{V}_{1-\frac{1}{k}} - (\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k))$ such that $w_{j,n,k}(z) = 0$ on $\partial(G_k \cap R_n \cap G^U_L) - (\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k)$ and $w_{j,n,k}(z) = 1$ on $\partial(\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k)$. Then by $(e)$ for $n \geqq n_0$ and $j \geqq j_0$ the image of $(\partial R_n + \partial G_j)$ does not fall in $D'$ and by $(c) \partial G^U_L$ does not fall in $D'$ and also by $(d) \partial(\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k)$ separates $E'$ from $|\xi| = 1 - \frac{1}{m}$.

Let $w^D(\xi)$ be a harmonic function in $D'$ such that $w^D(\xi) = 0$ on $\partial D' - E'$ and $w^D(\xi) = 1$ on $E' \cap \partial D'$. Consider $w^D(\xi)$ and $w_{j,n,k}(z)$ in $D' \cap f(R_n \cap G_j \cap G^U_L = (\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k))$. Then $0 = w^D(\xi) \leqq w_{j,n,k}(z)$ on $\partial D' - E'$ and $w^D(\xi) < 1 = w_{j,n,k}(z)$ on $D \cap f(\tilde{V}_{1-\frac{1}{k}} \cap G^U_L \cap D^H_k)$. Hence by the maximum principle

$$w^D(\xi) \leqq w_{j,n,k}(z).$$
Let $j \to \infty$ and $n \to \infty$ then

$$w^{D}(\xi) \leq \omega(\tilde{V}_{1-\frac{1}{k}}^{W} \cap D_{k}^{U}, z, G_{L}^{U} \cap G) = \lim_{j} \lim_{n} w_{j,n,k}(z).$$

Hence by letting $k \to \infty$ and letting $\varepsilon \to 0$, \( \lim_{k=\infty} w(\tilde{V}_{1-\frac{1}{k}}^{W} \cap D_{k}^{H} \cap G_{L}^{U}, z, G_{L}^{U} \cap G) \) has angular limits $= 1$ a.e. on \( E_{1} \), since \( w^{D}(\xi) \) has angular limits $= 1$ a.e. on \( E_{1} \). Thus by Theorem 11. a)

$$0 < \lim_{M=1} \lim_{k} w(p \cap D_{k}^{H} \cap G_{L}^{U}, z, G_{L}^{U} \cap G) = \omega(p \cap D_{k}^{H}, z, G_{L}^{U} \cap G).$$

Similarly

$$\omega(p \cap D_{k}^{H}, z, G_{L}^{U} \cap G) \geq 0,$$ \( \lim_{k=\infty} w(p \cap D_{k}^{H}, z, G_{L}^{U} \cap G) \) has angular limits $= 1$ a.e. on \( E_{1} \).

Thus by Theorem 12. a) Let \( p \) be a singular point of second kind and suppose a domain \( G \) contains \( p \) N-approximately. Then there exists by Theorem 8. c) a harmonic domain \( D_{k}^{H} = \omega_{m,n.1}^{f}(z) \) such that $w(p \cap D_{k}^{H}, G \cap G_{L}^{U} \cap R_{n+i}(p)) = 0$ and

$$\omega(p \cap D_{k}^{H}, G) < \frac{D(\omega(p,z) - c\omega(p,z))}{(1-M_{0} - \frac{2}{k})^{2}} < \infty \text{ for } 1-M_{0} - \frac{2}{k} > 0.$$
\[
\frac{U(z) - L}{\delta} \leq 0 = \omega_{m,n,n+i}^f(z) \quad \text{and} \quad \frac{\partial}{\partial n}\omega_{m,n,n+i}^f(z) \leq 0 \quad \text{by} \quad \omega_{m,n,n+i}^f(z) = \min_{z \in G} \omega_{m,n,n+i}^f(z) \quad \text{on} \quad \partial G_L \cap G \cap R_{n+i}.
\]

Hence
\[
D\left(\frac{U(z) - L}{\delta} - \omega_{m,n,n+i}^f(z), \omega_{m,n,n+i}^f(z)\right) \geq 0,
\]
whence
\[
D(\omega_{m,n,n+i}^f(z)) \leq \frac{1}{\delta^2} D(U(z)) < \infty.
\]

(31)

Let \( \omega_{m,n,n+i}''(z) \) be a harmonic function in \((G_{L}^{U} \cap G_{L+\delta}^{U}) - (D_{k}^{H} \cap D_{k}^{H}) \) such that \( \omega_{m,n,n+i}''(z) = 0 \) on \( \partial G \cap \partial G_{L}^{U} \cap G_{L+\delta}^{U} \cap (R_{n+i} - R_{n}) \) and \( \omega_{m,n,n+i}''(z) = 1 \) on \( (D_{k}^{H} \cap D_{k}^{H}) \cap (R_{n+i} - R_{n}) \cap G_{L+\delta}^{U} \). Since \( \omega_{m,n,n+i}''(z) \) has M.D.I. among all functions with value 1 on \( G_{L+\delta}^{U} \cap D_{k}^{H} \cap U_{m}(p) \cap (R_{n+i} - R_{n}) \cap G_{L+\delta}^{U} \) and \( = 0 \) on \( \partial G \cap \partial G_{L}^{U} \cap G \),
\[
D(\omega_{m,n,n+i}''(z)) \leq D(\omega_{m,n,n+i}^f(z)) \leq M. \frac{1}{\delta^2} D(U(z)) < \infty.
\]

(32)

As above we have also the existence of \( \omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G_{L}^{U} \cap G) \). Suppose \( w(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G_{L}^{U} \cap G) > 0 \) and \( w(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z, G_{L}^{U} \cap G) > 0 \). Clearly
\[
\omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G_{L}^{U} \cap G) \geq w(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G_{L}^{U} \cap G) > 0
\]
\[
\omega(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z, G_{L}^{U} \cap G) \geq w(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z, G_{L}^{U} \cap G) > 0.
\]

(33)

Now \( G_{L}^{U} \cap G_{L}^{U} = 0 \) and \( w(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G_{L}^{U} \cap G) \) and \( w(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z, G_{L}^{U} \cap G) \) can be consiered as C.P.s. defined by sequences \( \{u_{m}(p) \cap D_{k}^{H} \cap G_{L+\delta}^{U}\} \) and \( \{u_{m}(p) \cap D_{k}^{H} \cap G_{L-\delta}^{U}\} \) respectively. Hence these have properties from C.P.1. to C.P.6. Hence the Separation Theorem S. 1 is applicable by putting \( \tilde{G} = R - R_{0} \) and \( B' = p \), whence
\[
\omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z) = \omega(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z).
\]

(34)

On the ther hand, \( \omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z) \) and \( \omega(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z) \) have their masses only on \( \bigcap_{n=1}^{\infty} U_{n}(p) = 0 \) by \( \nu_{n}(p) \supseteq (\nu_{n}(p) \cap D_{k}^{H} \cap G_{L+\delta}^{U}) \). But by
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\[ \sup \omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z) = 1 = \sup \omega(p \cap D_{k}^{H} \cap G_{L-\delta}^{U}, z) \], these are equal to \( \omega(p, z) \). This contradicts (34). Hence we have the theorem.

Proof of b). Assume angular limits \( U(e^{i\theta}) \equiv \text{const} \) a.e. on \( E \), by Theorem 11. c) we can find constants \( \lambda \) and \( \delta \) and domains \( G_{L+\delta}^{U}, G_{L}^{U}, G_{L-\delta}^{U} \) and \( G_{L+\delta}^{L} \) such that

\[ w(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G \cap G_{L+\delta}^{U}) > w(p \cap D_{k}^{H} \cap G_{L}^{U}, z, G \cap G_{L}^{U}) > 0 \] and \( w(p \cap D_{k}^{H} \cap G_{L-\delta}^{L}, z, G \cap G_{L-\delta}^{L}) > w(p \cap D_{k}^{H} \cap G_{L}^{L}, z, G \cap G_{L}^{L}) > 0 \). By \( D(\omega(D_{k}^{H}, z, G)) < \infty \) by a) and by (32') and (33), \( 0 < D(\omega(p \cap D_{k}^{H} \cap G_{L+\delta}^{U}, z, G \cap G_{L+\delta}^{U})) < \infty \) and \( 0 < D(\omega(p \cap D_{k}^{H} \cap G_{L-\delta}^{L}, z, G \cap G_{L-\delta}^{L})) < \infty \). Hence we have the theorem, because such constants \( L \) and \( \delta \) do not exist by Theorem 12. b).

Proof of c). Put \( G = D_{k} = R - R_{0} \). Then by b) we have at once c).

8. Analytic functions in a neighbourhood of a singular point.

Theorem 12. a) Let \( p \) be a singular point of second kind and suppose that a domain \( G \) contains \( p \) N-approximately. Then there exists no non constant analytic function \( T(z) \) in \( G \) with \( D(T(z)) < \infty \).

b) Let \( p \) be a singular point of second kind and let \( G \) be a domain containing \( p \) N-approximately. Then there exists no non constant analytic function \( T(z) \) in \( G \) such that the spherical area \( A(T(z)) \) of \( T(z) \) is finite.

c) Suppose a domain \( G \) contains a singular point \( p \) (of first or second kind) N-approximately. Then there exists no non constant analytic function \( T(z) \) in \( G \) such that \( n(w) < M < \infty \), where \( n(w) \) is the number of zero points of \( T(z) - w \) in \( G \).

Remark. Theorem 12. c) is Theorem 8 of the previous paper mentioned at the top of the present paper. Since I thought that \( M_{r} \rightarrow 0 \) as \( r \rightarrow 0 \), the condition \( n(w) < M < \infty \) was left unnoticed, where \( M_{r} = \max \left( \left| \frac{\partial U_{\lambda}(w)}{\partial u} \right|, \left| \frac{\partial U_{\lambda}(w)}{\partial v} \right| \right) \) with \( w = v + iv \) and \( U_{\lambda}(w) \) is a harmonic function in \( r < |w| < 1 \) such that \( U_{\lambda}(w) = 1 \) on \( |w| = r \) and \( U_{\lambda}(w) = 0 \) on \( |w| = 1 \). But the condition that \( n(w) < M < \infty \) is necessary. In reality, there exists a Riemann surface \( R \) such that the area of \( R \) is finite and \( R \) has a singular point of first kind. (See the example 4 of the following paper "Examples of singular points").

Proof of a). Map the universal covering surface \( G^{\infty} \) of \( G \) onto \( |\xi| < 1 \). Then by Theorem 12. b) there exists a set \( E \) of positive measure on \( |\xi| = 1 \) such that \( ReT(z) \) and \( ImT(z) \) have angular limits \( = a \) and \( b \) respectively a.e. on \( E \) by \( D(T(z)) < \infty \). Hence by Riesz's theorem \( T(z) \equiv a + ib \). This is a contradiction. Hence we have a).
Proof of $b)$. Put $D^H_k = V^W_{1-\varepsilon-\frac{1}{k}} \cap CV^*_M$, where $V^W_{1-\varepsilon-\frac{1}{k}} = E\{z \in R: w(p, z) > 1-\varepsilon-\frac{1}{k}\}$ and $V^*_M = E\{z \in R: -\omega(p, z) \leq -M+\frac{1}{k}\}$. Then $D^H = (V^W_{1-\varepsilon} \cap CV^*_M) \subset D^H_k$ and $D^H$ and $D^H_k$ are harmonic domains. Then by Theorem 8. $c)$

$$D(\omega(D^H_k, z, G)) \leq \frac{D(\omega(D^H, z, G))}{(1-M-\frac{2}{k})} \leq M^*: 1-\varepsilon > 0$$

(35)

and $w(p \cap D^H_k, z, G) \geq \lim_{m=\infty} \lim_{h=\infty} w(u_m(p) \cap D^H_k, z, G) = w(p \cap D^H, z, G) \geq \lim_{m=\infty} \lim_{h=\infty} w(u_m(p) \cap D^H, z, G)$.

(36)

Without loss of generality, we can suppose that $\Delta$ is a component of $E[z \in G: |Re w| \leq 1$ and $|Im w| \leq 1]$. Let $\Delta_k$ be a component of $E[z \in G: |Re w| \leq 1+\frac{1}{k}$ and $|Im w| \leq 1+\frac{1}{k}]$ containing $\Delta$. Then $\Delta = \bigcap_{k=1}^{\infty} \Delta_k$ and $\Delta_k \cap D^H_k$ and $\Delta \cap D^H$ are harmonic domains. Hence

$$0 < w(p \cap \Delta \cap D^H, z, G) = w(E, \xi),$$

whence $\text{mes } E > 0$,

where $E$ is the set on which $w(p \cap \Delta \cap D^H, z, G) = 1$ almost everywhere.

Also by $\text{mes}(E-B(D^H \cap \Delta)) = 0$ by Theorem 11. $b)$. Hence

$$w(p \cap \Delta \cap D^H, z, G) \rightarrow 1 \text{ as } \xi \rightarrow E \text{ along Stolz's path and}$$

$$T(z)(\xi=f(z)) \rightarrow \Delta_k \text{ as } \xi \rightarrow E \text{ along Stolz's path.}$$

(a) Hence for any given positive number $\varepsilon$, we can find a closed set $E'$ in $E$ such that $\text{mes}(E-E') < \frac{\varepsilon}{2}$ and the above functions converge uniformly

$$w(p \cap \Delta \cap D^H, z, G) \rightarrow 1$$

as $\xi \rightarrow E'$ along Stolz’s path.
in angular domains $A(\theta): e^{i\theta} \in E'$. Then we can find a closed set $E'' \subset E'$ such that $\text{mes}(E' - E'') < \frac{\varepsilon}{2}$ and a domain $D$ (containing an endpart of $A(\theta): e^{i\theta} \in E''$ and bounded by $\sum_{e^{i\theta} \in E'} \partial A(\theta) + E''$ and $|\xi| = 1 - \frac{1}{m}$) and a number $k_0$ such that

$$w(p \cap [A \sim D''], z, G) > 1 - \frac{1}{k_0} \quad \text{and}$$

$$T(z) \in \mathcal{A}_{k_0} = E\left[w : |\text{Re } w| < 1 + \frac{1}{k_0}, \ |\text{Im } w| < 1 + \frac{1}{k_0}\right] \text{ in } D \cap R_m \quad \text{(b')}$$

for $f(z) \in D \sim \mathfrak{M}_m : \mathfrak{M}_m = E\left[\xi : 1 > |\xi| > 1 - \frac{1}{m}\right]$. 

Let $\mathcal{A}_0 = E\{|\text{Re } w| \leq 3, \ |\text{Im } w| \leq 3\}$ and $\mathcal{J}$ be the component of $T^{-1}(\mathcal{A}_0)$ containing $A(\supset \mathcal{J})$. Put $G_j = E\{z \in G : \delta(z, \partial G) > \frac{1}{j}\}$. Then $f(\partial G_j + (\partial R_n \cap G))$ tends to $|\xi| = 1$ as $n \to \infty$ and $j \to \infty$, i.e.

$$\text{dist}(\partial f(\partial G_j + \partial R_n), \xi = 0) > 1 - \frac{1}{m} \text{ for } n \geq n_0 \text{ and } j \geq j_0 \quad \text{(c)}$$

Now by (b') $f(\partial \mathcal{J} \cap G)$ does not fall in $D$. \quad \text{(d)}

By (a')

$$f(\mathcal{V}_{1 - \frac{1}{k_0}}^{w} \sim D''_k) \text{ covers } D \sim \mathfrak{M}_m \text{ and } f(\partial(\mathcal{V}_{1 - \frac{1}{k_0}}^{w} \sim A'')) \text{ separates } E'' \text{ in } D \sim \mathfrak{M}_m,$$

where $\mathcal{V}_{1 - \frac{1}{k_0}}^{w} = E\{z \in G : w(p \cap [A \sim D''], z, G) > 1 - \frac{1}{k_0}\}. \quad \text{(e)}$

Let $w_{j,n,k}(z)$ be a harmonic function in $(\mathcal{J}_0 \cap G \sim R_n \cap G_j) - (\mathcal{V}_{1 - \frac{1}{k_0}}^{w} \sim \mathcal{J}_k \sim D''_k)$ such that $w_{j,n,k}(z) = 0 \partial(\mathcal{J}_0 \cap R_n \cap G_j) - (\mathcal{V}_{1 - \frac{1}{k}}^{w} \sim \mathcal{J}_k \sim D''_k)$ and $w_{j,n,k}(z) = 1$ on $\partial(\mathcal{V}_{1 - \frac{1}{k}}^{w} \sim \mathcal{J}_k \sim D_j)$. Let $w''(\xi)$ be a harmonic function in $D \sim \mathfrak{M}_m$ such that $w''(\xi) = 0$ on $\partial(D \sim \mathfrak{M}_m) - E''$ and $w''(\xi) = 1$ on $E''$. Then by the rectifiability of $\partial D$ and by $\text{mes } E'' > 0$, $w''(\xi) > 0$.

Then by (c) and (d) $\partial(\mathcal{J}_0 \cap G \sim R_n \cap G_j) - (\mathcal{V}_{1 - \frac{1}{k_0}}^{w} \sim \mathcal{J}_k \sim D''_k)$ does not fall in $D \sim \mathfrak{M}_m$ for $n \geq n_0$, $k \geq k_0$ and $j \geq j_0$ and by (e) $f(\mathcal{V}_{1 - \frac{1}{k_0}}^{w} \sim \mathcal{J}_k \sim D''_k)$ covers $E'' \sim \partial D$. Hence by the maximum principle

$$w_{j,n,k}(z) \geq w''(\xi) \text{ for } n \geq n_0, k \geq k_0 \text{ and } j \geq j_0.$$

Let $j \to \infty$ and $n \to \infty$ and then $k \to \infty$. Then by Theorem 12. a)

$$w(p \cap \mathcal{J}_0 \cap \mathcal{J}_k \sim D''_k, z, \mathcal{J}_0 \cap G) \geq \lim_{k \to \infty} w(p \cap \mathcal{J}_0 \cap \mathcal{J}_k \sim D''_k, z, \mathcal{J}_0 \cap G)$$

$$\geq \lim_{k \to \infty} w(\mathcal{V}_{1 - \frac{1}{k}}^{w} \sim \mathcal{J}_k \sim D''_k, z, \mathcal{J}_0 \cap G) \geq w''(\xi). \quad \text{Hence}$$

$$w(p \cap \mathcal{J}_k \sim D''_k, z, \mathcal{J}_0 \cap G) > 0 \text{ for } k_0 < \infty. \quad \text{(37)}$$
Let \( \alpha(w) \) be a continuous function in the \( w \)-plane such that \( \alpha(w) = 1 \) in \( |w| \leq 2 + \frac{1}{k_0} \), harmonic in \( 2 + \frac{1}{k_0} < |w| < 3 \) and \( \alpha(w) = 0 \) in \( |w| \geq 3 \).

Put \( \alpha(z) = \alpha(T(z)) \). Then \( \alpha(z) \) is a continuous function in \( G \) and \( \alpha(z) = 1 \) in \( \Delta_{k_0} \) and \( \alpha(z) = 0 \) on \( \partial \Delta_0 \cap G \), because \( |T(z)| < \sqrt{2} + \frac{1}{k_0} \) for \( z \in \Delta_{k_0} \) and \( |T(z)| \geq 3 \) for \( z \in \partial \Delta_0 \cap G \).

Clearly
\[
D(\alpha(z)) < M^2 KA(T(z)) < M^{**} < \infty ,
\]
where \( M = \max \left( \left| \frac{\partial \alpha(w)}{\partial u} \right|^2, \left| \frac{\partial \alpha(w)}{\partial v} \right|^2 \right) \) and \( K = \max \left( \frac{\text{area element}}{\text{spherical area element}} \right) \)

in \( \Delta_0 \) and \( w = u + iv \).

Put \( \omega'(z) = \min(\omega(\nu_n(p) \cap D^w_{k_0}, z, G), \alpha(z)) \). Then \( \omega'(z) = 0 \) on \( \partial \Delta_0 + \partial G \) and \( \omega'(z) = 1 \) on \( \Delta_{k_0} \cap D^w_{k_0} \cap \nu_n(p) \), where \( \omega(\nu_n(p) \cap D^w_{k_0}, z, G) \) is C.P. of \( \nu_n(p) \cap D^w_{k_0} \) relative \( G \).

By (35) and (38) and as in case of the proof of Theorem 7. c)
\[
D(\omega'(z)) \leq D(\omega(\nu_n(p) \cap D^w_{k_0}, z, G)) + D(\omega'(z)) \leq M^* + M^{**}
\]
Let \( \omega(\nu_n(p) \cap D^w_{k_0} \cap \Delta_{k_0}, z, G \cap \Delta_0) \) be C.P. of \( (\nu_n(p) \cap D^w_{k_0} \cap \Delta_{k_0}) \) relative \( \Delta_0 \cap G \). Then it has M.D.I. among all functions having value 1 on \( (\nu_n(p) \cap D^w_{k_0} \cap \Delta_{k_0}) \) and 0 on \( \partial G \cap \Delta_0 \). Hence by (39)
\[
D(\omega(\nu_n(p) \cap D^w_{k_0} \cap \Delta_{k_0}, z, G \cap \Delta_0)) \leq D(\omega'(z)) < \infty .
\]

Now by (37)
\[
\omega(p \cap D^w_{k_0} \cap \Delta_{k_0}, z, G \cap \Delta_0) > 0.
\]

Hence by Theorem 8, a) \( G \cap \Delta_0 \) contains \( p \) \( N \)-approximately. On the other hand, clearly \( D_{\Delta_0}(T(z)) < KA(T(z)) < \infty \). Hence by Theorem 12. a) \( T(z) \)

must reduce to be a constant. This is a contradiction. Hence we have b).

Proof of c).
By Theorem 8. a) there exists a harmonic domain \( D^w = E \{ z \in R : \omega(p, z) > 1 - \varepsilon \} \cap E \{ z \in R : c \omega(p, z) < M \} \) such that \( \omega(D^w \cap p, z, G) > 0 \) and \( D(\omega(D^w, z, G)) < \frac{D(\omega'(p, z))}{(1-M-\varepsilon)^2} < \infty \).

Let \( \mathfrak{S}_n(\Delta_n^{\nu, n}) \) be a triangulation of the \( w \)-plane and \( \mathfrak{S}_{n+1}(\Delta_n^{\nu, n+1}) \) be a subdivision of \( \mathfrak{S}_n \) and becomes as fine as we please as \( n \to \infty \). Put \( \Delta_n^w = E \{ z \in G : T(z) \in \Delta_n^{\nu, n} \} \). Then \( \Delta_n^{w, j} \) may consist of at most enumerably infinite number of components \( \Delta_n^{w, j, l} \). By \( \omega(D^w \cap p, z, G) \leq \sum \omega(D^w \cap \Delta_n^{w, j, l} \cap p, z, G) \) there exists at least one \( \Delta_n^{w, j, l} \) such that \( \omega(D^w \cap \Delta_n^{w, j, l} \cap p, z, G) > 0 \). Similarly there exists at least one \( (\Delta_n^{w, j, l} \cap \Delta_n^{w, j, l+1}) \) such that \( \omega(\Delta_n^{w, j, l+1} \cap p, D^w, z, G) > 0 \). In this fashion we can construct a sequence \( \Delta_n^{w, j, l} \subset \Delta_n^{w, j, l+1} \subset \cdots \) such that \( \omega(p \cap D^w \cap \Delta_n^{w, j, l}, z, G) > 0 \) for every \( n \) and \( i(n) \) and \( j(n) \). We denote them by \( \Delta_1 \supset \Delta_2 \supset \cdots \) and suppose diameter of \( T(\Delta_n^w) < \frac{1}{n} \) and put \( q = \bigcup_n T(\Delta_n^w) \).
Let $L$ be a compact curve on $\partial G$ such that $\text{dist}(T(L), T(\Delta^n)) > \delta > 0$ for $n \geq n_0$. Such a curve $L$ can be chosen by the fact that $T(\partial G)$ is not a point.

Let $C_s$ and $C_n$ be circles with radii $\delta$ and $\frac{2}{n} \left( \frac{\delta}{n_0} \right)$ with centres at $q$ such that $C_n$ encloses $T(\Delta_n)$ for $n > n_0$ and $T(L)$ lies outside of $C_s$.

Let $\alpha_n(w)$ be a continuous function in the $w$-plane such that $\alpha_n(w) = 0$ outside of $C_n$, $\alpha_n(w)$ is harmonic in $C_n - C_{\delta}$ and $\alpha_n(w) = 1$ in $C_n$. Then $D(\alpha_n(w)) \downarrow 0$ as $n \rightarrow \infty$.

Put $\alpha_n(z) = \alpha_n(T(z))$. Then $\alpha_n(z)$ is a function in $G$ such that $\alpha_n(z) = 1$ in $\Delta_n$ and $= 0$ on $L$. And by $n(w) \leq M$, $D(\alpha_n(z)) \leq MD(\alpha_n(w)) < \infty$.

Let $U_{n,m}(z)$ be a harmonic function in $(R_m \cap G) - (D^H \cup U_n(p) \cap \Delta_n)$ such that $U_{n,m}(z) = 0$ on $\partial G \cap R_m$, $\frac{\partial}{\partial n} U_{n,m}(z) = 0$ on $\partial R_m - (D^H \cup U_n(p))$ and $U_{n,m}(z) = 1$ on $(D^H \cup U_n(p) \cap \Delta_n)$.

Let $U_{n,m}^*(z)$ be a harmonic function in $(R_m \cap G) - (D^H \cup U_n(p) \cap \Delta_n)$ such that $U_{n,m}^*(z) = 0$ on $L$, $\frac{\partial}{\partial n} U_{n,m}^*(z) = 0$ on $(\partial G \cap R_m - L)$ and $U_{n,m}^*(z) = 1$ on $(D^H \cup U_n(p) \cap \Delta_n)$. Then by the maximum principle

$$ U_{n,m}(z) \leq U_{n,m}^*(z). $$

Clearly $D(U_{n,m}^*(z)) = \int_{\partial(D^H \cup U_n(p))} U_{n,m}^*(z) \frac{\partial}{\partial n} U_{n,m}^*(z) ds = \int_{L} \frac{\partial}{\partial n} U_{n,m}^*(z) ds$.

On the other hand, $\alpha_n(z) = 0$ on $L$ and $\alpha_n(z) = 1$ on $\Delta_n (\cap (D^H \cup U_n(p)))$.

Since $U_{n,m}^*(z)$ has M.D.I. among all functions with value $= 0$ on $L$ and $= 1$ on $(D^H \cup U_n(p) \cap \Delta_n)$,

$$ D(U_{n,m}^*(z)) \leq D(\alpha_n(z)) \leq MD(\alpha_n(w)). $$

Now $U_{n,m}^*(z) \geq \omega(U_n(p) \cap D^H \cap \Delta_n \cap z, G)$ as $m \rightarrow \infty$ and $U_{n,m}^*(z) \rightarrow U_n^*(z)$ as $m \rightarrow \infty$ by $D(U_{n,m}^*(z)) \leq D(\alpha_n(z)) < \infty$ and $D(U_{n,m}^*(z)) \leq D(\alpha_n(z))$ respectively for every $m$.

Because $\omega(p \cap D^H \cap \Delta_n, z, G) > 0$ by the assumption.

Since $MD(\alpha_n(w)) \downarrow 0$ as $n \rightarrow \infty$, $\frac{\partial}{\partial n} U_{n,m}^*(z) \downarrow 0$ on $L$ uniformly by the compactness of $L$ as $n \rightarrow \infty$. Hence there exists a number $n_0$ and a point $z_0$ in a neighbourhood of $L$ such that

$$ 0 < \omega(p \cap D^H \cap \Delta_{n_0}, z_0, G) < U_{n_0}^*(z_0) < \omega(p \cap D^H, z_0, G). $$

In Separation Theorem S. 2 put $B' = p$, $G_1 = D^H$, $G_2 = D^H \cap \Delta_{n_0}$ and $\mathcal{G} = R$. 

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$-R_{0}$. Then there exist domains $D_{1}$ and $D_{2}$ such that

$$0 = \omega(p \cap D^{H} \cap D_{2}, z) = \omega(p \cap CD_{1} \cap D^{H} \cap \Delta_{n_{0}}, z) \geq 0$$

(40)

But these have their masses only at $p$, whence as usual they are equal to $\omega(p, z)$. This contradicts (40). Hence we have $c$.

Corollary 1. Let $v_{n}(p)$ be a neighbourhood of a singular point of $p$, then there exists no analytic function in $v_{n}(p)$ such that $n(w) \leq M$.

Corollary 2. Let $R$ be a Riemann surface of finite genus. Then there exists no singular point.

Corollary 1) is evident, since $v_{n}(p)$ contains $p$ $N$-approximately. And the Riemann surface of finite genus $g$ can be represented as a covering surface of at most $2g$ number of sheets over the $w$-plane. Hence by Theorem 12, $c$ we have Corollary 2.

9. Relation between HD untelbare Menge and N-minimal points.

Let $R$ be a Riemann surface with positive boundary. Map the universal covering surface $(R-R_{0})^{\infty}$ onto $|\xi|<1$, where $R_{0}$ is a compact set of $R$. Constantinescu and Cornea\textsuperscript{14} introduced the $\underline{HD}$ class and the maximal $HD$ indivisible sets to extend Theorem 1 in some way. We shall consider the relations between theirs and ours.

Let $U(z)$ be a positive harmonic function. If $U(z) \geq V(z) > 0$ implies $V(z) = KU(z)$, $U(z)$ is called a minimal function.

If there exist decreasing positive harmonic functions $U_{n}(z)$: $U_{n}(z) \in HD\textsuperscript{15}$ such that $U(z) = \lim U_{n}(z)$, $U(z)$ is called a function in $\underline{HD}$ class.

Let $E$ be a set on $|\xi| = 1$ such that every $U(z) \in HD$ has angular limits $= \text{const}$. a.e. on $E$. Then $E$ is called a $HD$ indivisible set. If there exists no set $E^{*}$ such that $E^{*} \supset E$ and mes $(E^{*} - E) > 0$ and $E^{*}$ is $HD$ indivisible, $E$ is called a maximal indivisible set. They proved the following

Theorem 13. If $E$ is a maximal indivisible set, then H.M. (harmonic measure of $E$) $w(E, \xi)$ is minimal in the class $\underline{HD}$. If $U(z)$ is minimal in $\underline{HD}$ class, $U(z) = K w(E, \xi)$, where $E$ is a maximal indivisible set.

We show that a maximal indivisible set is the image of a singular point of second kind.

Theorem 14. a) Let $\omega(p, z)$ be C.P. of a singular point $p$ of second kind and let $E$ be the image of $p$, i.e. the set on which $\omega(p, z)$ has angular limits $= 1$ almost everywhere. Then $E$ is maximal indivisible and


\textsuperscript{15} HD means the class of Dirichlet bounded harmonic functions.
w(p, z) (H.M. of p) is minimal in HD class.

b) If E is maximal indivisible, there exists a singular point p of second kind such that w(p, z) = w(E, \xi) and E is the set where \omega(p, z) has angular limits = 1 almost everywhere.

Proof of a). At first we show that \omega(p, z) is in HD class. Let p be a singular point of second kind. Let \( V_{1 - \frac{1}{m}} = \left\{ z \in R : \omega(p, z) > 1 - \frac{1}{m} \right\} \).

Let \( T_{m, n, n+i, n+i+j}(z) \) be a harmonic function in \( R_{n+i+j} - ((R_{n+i+j} - R_{n+i}) \cap (V_{1 - \frac{2}{m}} - V_{1 - \frac{1}{m}})) - \Omega_{m, n, n+i, n+i+j} \) such that

\[
\begin{align*}
T_{m, n, n+i, n+i+j}(z) &= 0 \text{ on } \partial R_0 + (\partial R_{n+i+j} - V_{1 - \frac{2}{m}}) + (\partial V_{1 - \frac{2}{m}} - (R_{n+i+j} - R_{n+i})), \\
T_{m, n, n+i, n+i+j}(z) &= 1 \text{ on } (R_{n+i} - R_n) \cup V_{1 - \frac{1}{m}} \text{ and } \frac{\partial}{\partial n} T_{m, n, n+i, n+i+j}(z) = 0 \text{ on } \partial R_{n+i} \cup (V_{1 - \frac{2}{m}} - V_{1 - \frac{1}{m}}). 
\end{align*}
\]

Put \( S(z) = \frac{\omega(p, z) - (1 - \frac{2}{m})}{\frac{1}{m}} \). Then \( S(z) = 0 \) on \( \partial V_{1 - \frac{2}{m}} \) and \( = 1 \) on \( \partial V_{1 - \frac{1}{m}} \).

Let \( S'(z) \) be a function in \( \Omega_{m, n, n+i, n+i+j} \) such that \( S'(z) = \min(1, S(z)) \) in \( V_{1 - \frac{2}{m}} \) and \( S'(z) = 0 \) in \( R - R_0 - V_{1 - \frac{1}{m}} \). Then by the Dirichlet principle

\[
D(S'(z)) < D(S(z)) \leq \frac{1}{(\frac{1}{m})^2} D(\omega(p, z)) = M_m < \infty.
\]  (41)

Let \( S^2(z) \) be a harmonic function in \( \Omega_{m, n, n+i, n+i+j} \) such that \( S^2(z) = S'(z) \) on \( \partial R_0 + (\partial R_{n+i+j} - V_{1 - \frac{2}{m}}) + (\partial V_{1 - \frac{2}{m}} - (R_{n+i+j} - R_{n+i})) \) and \( = 0 \) in \( R - R_0 - V_{1 - \frac{1}{m}} \). Then by the Dirichlet principle \( D(S^2(z)) \leq D(S(z)) \). Now

\[
D(S^2(z) - T_{m, n, n+i, n+i+j}(z), T_{m, n, n+i, n+i+j}(z)) = 0, \text{ whence}
\]

\[
D(S^2(z) - T_{m, n, n+i, n+i+j}(z)) = D(S^2(z)) - D(T_{m, n, n+i, n+i+j}(z)), \text{ hence}
\]

\[
D(T_{m, n, n+i, n+i+j}(z)) \leq M_m < \infty.
\]  (42)

Clearly \( T_{m, n, n+i, n+i+j}(z) \uparrow T_{m, n, n+i}(z) \) as \( j \to \infty \) and by (41) and by Lemma 1. b) of P \( T_{m, n, n+i}(z) \to T_{m, n}(z) \) as \( i \to \infty \) and \( T_{m, n}(z) \downarrow T_{m}(z) \) as \( n \to \infty \).

Let \( E' \) be the set on which \( \omega(p, z) \) has angular limits \( < 1 - \frac{2}{m} \) almost everywhere and let \( E_\theta \) be the set where the Green's function \( G(z, p) \) has angular limits = 0. Then we can find for any given positive number \( \epsilon \) a closed set \( E'' \subset (E' \cap E_\theta) \) such that \( \text{mes} (E' - E'') < \epsilon \) and a domain \( D \) containing an endpart \( A(\theta) : e^{i\theta} \in E'' \) and bounded by \( \sum_{e^{i\theta} \in E''} \partial A(\theta) + E'' \) and a ring \( \Re_i : 1 > |\xi| > 1 - \frac{1}{l} \) with the following property
\( \omega(p, z) < 1 - \frac{2}{m} - \delta \) in \( D \setminus \Re \) for a positive constant \( \delta \).

Hence the image of \( \partial V_{1 - \frac{2}{m}} \) does not fall in \( D \setminus \Re \).

(a) Since \( E' \subset \Re \) and \( G(z, p) > \delta_{n+i+j} > 0 \) on \( \partial R_{n+i+j} \), \( \partial R_{n+i+j} \) does not tend to \( |\xi| = 1 \) and by

\[ \partial R_{n+i+j} \to |\xi| = 1 \text{ as } j \to \infty, \quad \partial R_{n+i+j} \text{ separates } E'' \text{ from } |\xi| = 1 - \frac{1}{l}. \]

(b) Hence by the maximum principle

\[ T_{m, n+i, n+i+j}(z) \leq w^{D}(\xi) \]

where \( w^{D}(\xi) \) is a harmonic function in \( D \cup \Re \) such that \( w^{D}(\xi) = 0 \) on \( E' \) and \( = 1 \) on \( \partial (D \cup \Re) - E'' \).

On the other hand, by (42) \( T_{m}(z) \in HD \).

By Theorem 12. (a) every \( U(z) \in HD \) has angular limits \(-\text{const} \) a.e. on \( E \). Hence \( E \) is an indivisible set.

Let \( E \) be a maximal indivisible set. Then \( w(E, z) \) can be considered as a function \( w(E, z) \) in \( R - R_0 \). Put \( V_{M_i} = \{ z \in R : w(E, z) > M_i \} \). Then \( \{ V_{M_i} \} : M_1 < M_2 < \ldots \) is a sequence of decreasing domains. Let \( \omega(V_{M_i}, z) \) be C.P. of \( V_{M_i} \). Then \( \omega(V_{M_i}, z) \) and \( \lim \omega(V_{u_i}, z) \) (\( = \omega(V, z) \)) is C.P. defined by a sequence of decreasing domains) are superharmonic in \( \overline{R} - R_0 \) by Theorem 5. (a) of P.

Let \( w(V_{M_i}, z) \) be H.M. of \( V_{M_i} \). Then by P.H. 4 \( M_i w(V_{M_i}, z) = w(E, z) \) whence \( \lim w(V_{M_i}, z) = w(V, z) = w(E, z) \). Clearly \( \omega(V_{M_i}, z) \geq w(V_{M_i}, z) \).

Hence

\[ 1 \geq \omega(V, z) \geq w(V, z). \]

(43)

On the other hand, \( v_{M_i} \omega(V, z) = \omega(V, z) \) for every \( M_i \) by P.C.4.

Let \( U(z) \) be a positive superharmonic function in \( \overline{R} - R_0 \) such that \( \omega(V, z) - U(z) = T(z) \) is also superharmonic in \( \overline{R} - R_0 \). Then

\[ v_{M_i} U(z) \leq U(z) \text{ and } v_{M_i} T(z) \leq T(z). \]

Now \( \omega(V, z) = U(z) + T(z) \geq v_{M_i} U(z) + v_{M_i} T(z) = v_{M_i} \omega(V, z) = \omega(V, z), \)
whence \( v_{M_{i}} U(z) = U(z) \) and \( v_{M_{i}} T(z) = T(z) \).

Let \( U(z)(T(z)) \) be positive superhearmomic in \( \overline{R} - R_{0} \) and harmonic in \( R - R_{0} \). Then by Theorem 5. a) and b) of P \( U(z) = \int_{R} N(z, p) d\mu(p) \) and \( D(\min (M, U(z)) \leq MK < \infty \). But sup \( U(z) \leq 1 \) and sup \( T(z) \leq 1 \), whence \( U(z) \) and \( T(z) \) are contained in HD.

Since \( E \) is indivisible, \( U(z) \) and \( T(z) \) have angular limits \(-c_{1} \) and \( c_{2} \) a.e. on \( E \). By (43) and \( c_{1} + c_{2} = 1 \) a.e. on \( E \) we have \( c_{1} + c_{2} = 1 \). \( U(z) \) has angular limits \( \leq 0 \) on \( CE \) and \( c_{1} \) on \( E \) almost everywhere. Hence \( U(z) \geq c_{1} w(E, z) \). This implies \( U(z) \geq c_{1} M_{i} \) on \( V_{M_{i}} \). Hence

\[
U(z) = v_{M_{i}} U(z) \geq c_{1} \omega(V_{M_{i}}, z) \text{ and } U(z) \geq c_{1} \omega([V], z).
\]

Similarly \( T(z) \geq c_{2} \omega([V], z) \). Hence

\[
\omega([V], z) = U(z) + T(z) \geq c_{1} \omega([V], z) + c_{2} \omega([V], z) = \omega([V], z).
\]

Thus \( U(z) = c_{1} \omega([V], z) \) and \( \omega([V], z) \) is N-minimal, hence \( \omega([V], z) = K N(z, p) \).

Let \( \omega(p, z) \) be C.P. of \( p \). Then \( \omega(p, z) = K' N(z, p) \). By sup \( \omega(p, z) = 1 \) \( = \sup \omega([V], z) \), we have \( K = K' \) and \( \omega(p, z) = \omega([V], z) \). By sup \( N(z, p) < \infty \), \( p \in B_{S} \). Let \( V_{M_{i}}(p) = E[z \in R: \omega(p, z) > M_{i}] \). Then by \( \omega([V], z) \geq w([V], z) = w(E, z) \omega([V], z) > M_{i} \) on \( V_{M_{i}}(p) \supset V_{M_{i}} \). Hence \( w([V_{M_{i}}(p)], z) \) (H.M. determined by a decreasing sequence \( V_{M_{i}}(p) : \lim_{i=\infty} M_{i} = 1 \) is larger than \( w([V], z) = w(E, z) > 0 \). Now by Theorem 10, a)

\[
w(p, z) = \lim_{i=\infty} w(V_{M_{i}}(p), z) \geq w([V], z) = w(E, z) > 0.
\]

Whence \( p \) is a singular point of second kind. Hence a singular point of second kind corresponds to a maximal indivisible set.

Let \( \tilde{E} \) be the image of \( p \) (\( \tilde{E} \) is the set on which \( \omega(p, z) = 1 \) almost everywhere). Then by \( \omega(p, z) \geq w(p, z) \geq w(E, \xi) > \tilde{E} \subseteq E \). Now \( \tilde{E} \) is maximal by a). By the assumption \( E \) is maximal and we have \( \tilde{E} = E \). Next by Theorem 10. b) \( \text{mes} (E \sim E_{i}) = 0 \), where \( E_{i} \) is the image of a singular point \( p_{i} \) of second kind such that \( p \equiv p_{i} \). Hence a uniquely determined singular point of second kind corresponds to a maximal indivisible set. Thus we have b).

10. Class H.N.D. Let H.N.D. be the class of Riemann surfaces on which \( N \) number of linearly independent harmonic functions \( \epsilon HD \) exist, where \( N \leq \infty \) and the cardinal number of \( N \) is \( \aleph_{0} \).

**Theorem 14.** The set of HD functions in \( R \) is isomorphic to the set of HD functions in \( R - R_{0} \) vanishing on \( \partial R_{0} \). Hence without loss of generality, we can consider the set of HD functions vanishing on \( \partial R_{0} \).
instead of HD function in $R$.

Proof. Let $U(z)$ be a Dirichlet bounded harmonic function in $R$ with positive boundary. Let $U_n(z)$ be a harmonic function in $R_n - R_0$ such that $U_n(z) = U(z)$ on $\partial R_0$ and $U_n(z) = 0$ on $\partial R_n$. Then $U_n(z)
=rac{1}{2\pi} \int_{\partial R_0} U(\xi) \frac{\partial}{\partial n} G_n(\xi, z) ds$, where $G_n(\xi, z)$ is the Green's function of $(R_n - R_0)$.

Now $G_n(\xi, z) \to G(\xi, z)$ as $n \to \infty$ and $\frac{\partial}{\partial n} G(\xi, z)$ is continuous on $\partial R_0$ by the compactness of $\partial R_0$. Hence $\lim_n U_n(z)$ exists and $\lim_n U_n(z)
=\frac{1}{2\pi} \int_{\partial R_0} U(\xi) \frac{\partial}{\partial n} G(\xi, z) ds$, where $G(\xi, z)$ is the Green's function of $R - R_0$.

Put $\lim U_n(z) = U'(z)$. Since $U_n(z) - U'(z) = 0$ on $\partial R_0$, $U_n(z) - U'(z)$ is harmonic in a neighbourhood of $\partial R_0$. Now $U_n(z) \to U'(z)$ implies $\frac{\partial}{\partial n} U_n(z) \to \frac{\partial}{\partial n} U'(z)$ uniformly on $\partial R_0$ and $\frac{\partial}{\partial n} U'(z)$ is continuous on $\partial R_0$. Hence

$D(U_n(z)) = \int_{\partial R_0} U_n(z) \frac{\partial}{\partial n} U_n(z) ds \to \int_{\partial R_0} U'(z) \frac{\partial}{\partial n} U'(z) ds = D(U'(z)) < \infty$,

whence $U_n(z) \Rightarrow U'(z)$. Put $U^*(z) = U(z) - U'(z)$. Then $U^*(z)$ is uniquely determined by $U(z)$ and $D(U^*(z)) = D(U(z)) + D(U'(z)) - 2D(U(z), U'(z)) < \infty$.

Put $B_n = R_n - R_0$ and let $w(B_n, z, R_n - R_0)$ be a harmonic function in $R_n - R_0$ such that $w(B_n, z, R_n - R_0) = 1$ on $\partial R_n$ and $w(B_n, z, R_n - R) = 0$ on $\partial R_0$. Then by the maximum principle

$|U_n(z)| \leqq M(1 - w(B_n, z, R_n - R_0))$, where $M = \max |U(z)|$ on $\partial R_0$. Let $n \to \infty$.

Then $w(B_n, z, R_n - R_0) \to w(B, z, R - R_0)$ (H.M. of $B$ relative $R - R_0$). (44)

Put $G_\delta = E[z \in R : w(B, z, R - R_0) < \delta]$. Then by P.H.2. $w(G_\delta \cap B, z, R - R_0) = \lim_n w(G_\delta \cap B_n, z, R - R_0) = 0$ for $\delta < 1$, where $w(G_\delta \cap B_n, z, R - R_0)$ is H.M. of $G_\delta \cap B_n$ relative $(R - R_0)$. By (44) we have

$|U'(z)| < M(1 - w(B, z, R - R_0))$. (45)

Let $w(G_\delta \cap B_n, z, R)$ be H.M. of $G_\delta \cap B_n$ relative $R$, i.e. $w(G_\delta \cap B_n, z, R)$ is the least positive superharmonic function in $R$ larger than 1 in $G_\delta \cap B_n$. We show $w(G_\delta \cap B, z, R) = \lim_n w(G_\delta \cap B_n, z, R) = 0$.

In fact, $\frac{1 - w(B, z, R - R_0)}{1 - \delta} \geqq 1$ in $G_\delta$ and $\geqq 1$ on $\partial R_0$. Hence by the maximum principle $\frac{1 - w(B, z, R - R_0)}{1 - \delta} \geqq w(G_\delta \cap B, z, R)$.

By P.H.2. $\sup w(B, z, R - R_0) = 1$, whence $\inf w(G_\delta \cap B, z, R) < 1$ and
Let $U''(z)$ be a harmonic function in $R_n$ such that $U''(z) = U'(z)$ on $\partial R_n$. Then by (45)

$$U'(z) \leq M(1-\delta) + Mw(B_n \cap G, z, R),$$

because $|U'(z)| < 1-\delta$ in $C G$ and $|U'(z)| \leq M$ in $R \cap G$.

Let $n \to \infty$ and then $\delta \to 1$. Then $\lim_{n} U''(z) = 0$, by $w(B \cap G, z, R) = 0$. 

Let $U'''(z)$ be a harmonic function in $R_n$ such that $U'''(z) = U^*(z) = U(z) - U'(z)$ on $\partial R_n$. Then $U'''(z) = U(z) - U''(z)$. Hence by letting $n \to \infty$

$$U'''(z) \to U(z).$$

Thus $U(z)$ is uniquely determined by $U^*(z)$ and the sets $\{U(z)\}$ and $\{U^*(z)\}$ are isomorphic hence we have the theorem.

**Green's function and generalized Green's function.**

Let $q$ be a point of $B$ and let $G(z, p)$ be the Green's function. We call $q$ a regular or irregular point for the Green's function according as $\lim_{z \to q} G(z, p) = 0$ or $\lim_{z \to q} G(z, p) > 0$, where $z \to q$ means that $z$ converges with respect to the $\delta$-metric. Let $B_{\delta, i}$ be the set of irregular points for the Green's function. Then by Theorem 11. b) of $P B_{\delta, i}$ is an $F_\sigma$ set of capacity zero. Map the universal covering surface $(R - R_0)^\infty$ conformally onto $|\xi| < 1$. If a harmonic function $G(z)$ which is positive and $D(\min(M, G(z))) \leq MK: K < \infty$ and $G(z)$ has angular limits $= 0$ a.e. on $|\xi| = 1$, we call $G(z)$ a generalized Green's function.

We proved in the previous paper the following

**Theorem.** Let $W(z)$ be a positive superharmonic in $R - R_0$ and harmonic in $R - R_0$. Then $W(z) = P(z) + G(z)$, where $G(z)$ is a generalized Green's function and $P(z)(\leq W(z))$ is a harmonic function representable by Poisson's integral.

Now we shall prove

**Theorem 15.** Let $p \in B_{\delta, i} \cap (B_1 - B_S)$ and let $G(z)$ be a generalized
Green's function such that $G(z) \leqq N(z, p)$. Then $N(z, p) = P(z)$, i.e. $G(z) = 0$ and $N(z, p)$ is representable by Poisson's integral, where $B_{q,r}$ is the set of regular points for the Green's function.

**Proof.** Assume $G(z) > 0$. Put $R_{\frac{\delta}{2}} = E[z \in R - R_{0} : G(z) > \frac{\delta}{2}]$. Map the universal covering surface $(R - R_{0})^{\infty}$ of $(R - R_{0})$ onto $|\xi| < 1$. Let $w(B \cap R_{\frac{\delta}{2}}, z)$ be H.M. of $(B \cap R_{\delta})$ (H.M. of the boundary determined by $R_{\frac{\delta}{2}}$). Then $w(B \cap R_{\frac{\delta}{2}}, z)$ has angular limits $= 0$ a.e. on the set $E$ where $G(z)$ has angular limits $< \frac{\delta}{2}$. In fact for any given positive number $\varepsilon$ we can find a closed set $E' \subset E$ such that $\text{mes}(E - E') < \varepsilon$ and a domain $D$ containing an endpart of $A(\theta) : e^{i\theta} \in E'$ and bounded by $\sum_{e^{i\theta} \in E'} \partial A(\theta) + E'$ and a ring $\Re_{m} : 1 > |\xi| > 1 - \frac{1}{m}$ and a number $k$ such that

$$ G(z) > \frac{\delta}{2} - \frac{1}{k} \text{ in } D \setminus \Re_{m}. $$

Hence usually it can be proved that $w(B \cap R_{\frac{\delta}{2}}, z) = 0$ on $E'$. But $G(z)$ has angular limits $= 0$ a.e. on $|\xi| = 1$, whence $\text{mes} E = 2\pi$. Hence $w(B \cap R_{\frac{\delta}{2}}, z) = 0$ and

$$ 0 = w(B \cap R_{\frac{\delta}{2}}, z) \geqq w(B \cap R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}}) = 0. \quad (46) $$

Let $\omega_{n,n+i}(z)$ be a harmonic function in $(R_{\frac{\delta}{2}} \cap R_{n+i}) - ((R_{n+i} - R_{n}) \cap R_{\delta})$ such that $\omega_{n,n+i}(z) = 0$ on $\partial R_{\frac{\delta}{2}}$, $\omega_{n,n+i}(z) = 1$ on $((R_{n+i} - R_{n}) \cap R_{\delta})$ and

$$ \frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0 \text{ on } \partial R_{n+i} - R_{\delta}. $$

Then $D(\omega_{n,n+i}(z)) \leqq \frac{4D_{R_{\frac{\delta}{2}}}}{(\frac{1}{\delta})^{2}} < \infty$. Hence $\omega_{n,n+i}(z) \Rightarrow \omega_{n}(z)$ as $i \rightarrow \infty$ and $\omega_{n}(z) \Rightarrow \omega(B \cap R_{\delta}, z, R_{\frac{\delta}{2}})$ as $n \rightarrow \infty$. On the other hand, $w(B \cap R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}}) = \lim_{n} w_{n}(z)$, where $w_{n}(z)$ is a harmonic function in $R_{n} \cap R_{\frac{\delta}{2}}$ such that $w_{n}(z) = 0$ on $\partial R_{\frac{\delta}{2}}$, and $w_{n}(z) = 1$ on $\partial R_{\delta} \cap R_{n}$. By the maximum principle $w_{n}(z) \geqq \omega_{n,n+i}(z)$ and $w_{n}(z) \geqq \omega_{n}(z)$. Whence by letting $n \rightarrow \infty$, $0 = w(B \cap R_{\frac{\delta}{2}}, z, R_{\frac{\delta}{2}}) \geqq \omega(B \cap R_{\delta}, z, R_{\frac{\delta}{2}})$. Whence

$$ D(\omega_{n}(z)) \downarrow 0 \text{ as } n \rightarrow \infty. \quad (47) $$

Put $R^{*} = (R_{\frac{\delta}{2}} \cap R_{n_{0}}) + R_{\delta}$, where $n_{0}$ is a certain number. Let $F$ be a closed compact arc on $\partial R_{\frac{\delta}{2}} \cap R_{n_{0}}$. Let $\omega'_{n,n+i}(z)$ be a harmonic function in $R^{*} \cap R_{n}$ such that $\omega'_{n}(z) = 0$ on $F$, $\omega'_{n}(z) = 1$ on $R_{\delta} - \partial R_{n}$ and $\frac{\partial}{\partial n} \omega'_{n}(z) = 0$ on
Then $D(\omega_{n}'(z)) \leq D(\omega_{n}(z))$. Hence $\omega_{n}'(z) \downarrow 0$ as $n \to \infty$ and

$$\lim_{n}D(\omega_{n}'(z))=0.$$ 

Hence there exists a sequence of harmonic functions $\omega_{n}^{*}(z)$ in $R^{*} \cap R_{n}$ such that $\omega_{n}^{*}(z)=0$ on $F$, $\frac{\partial}{\partial n} \omega_{n}^{*}(z)=0$ on $(\partial R^{*} \cap R_{n})-F$ and $\omega_{n}^{*}(z)=M_{n}$ on $\partial R_{n} \cap R^{*}$ and $\int_{F}\frac{\partial}{\partial n}\omega_{n}^{*}(z)ds=2\pi$ and $M_{n} \uparrow \infty$ as $n \to \infty$. For any given number $n'$ there exists a number $n''$ such that $\omega_{n}^{*}(z)<\frac{M_{n}}{2}$ in $R^{*} \cap R_{n'}$ for $n>n''$, i.e. every niveau curve of $\omega_{n}^{*}(z)$ with height $\frac{M_{n}}{2}$ is contained in $R^{*} \cap (R-R_{n'}),$ because $\omega_{n}^{*}(z) \to 0$ as $n \to \infty$.

**Lemma a.** Let $G_{i}$ be a domain in $R^{*}$ and let $U_{i}(z)$ be a function in $R^{*}-G_{i}$ which is harmonic in $R^{*}-G_{i}$ and on $\partial G_{i}$ such that the Dirichlet integral of $U_{i}(z)$ is finite $(i=1,2, \cdots i_{0})$. Then there exists a sequence of compact curves $\{\gamma_{j}\}$ such that $\gamma_{j}$ separates $B$ from $F$, $\{\gamma_{j}\}$ clusters at $B$ and that $\int_{\gamma_{j} \cap R^{*}-G_{i}}|\frac{\partial}{\partial n}U_{i}(z)|ds \to 0$ as $j \to \infty$ for every $i$.

**Proof.** Let $U(z)$ be one of $\{U_{i}(z)\}$ and put

$$L(r)=\int_{C_{r}}\frac{\partial}{\partial n}|U(z)|r\,d\theta=\int_{C_{\gamma}}\frac{\partial}{\partial n}U(z)ds, \quad C_{r}=\{z \in R^{*} : \omega_{n}^{*}(z)=\log r\}$$

where $r \cdot e^{i\theta}=e^{\alpha^{*}(z)+i\tilde{\omega}_{n}^{*}(z)}$ and $\tilde{\omega}_{n}^{*}(z)$ is the conjugate function of $\omega_{n}^{*}(z)$.

Suppose that there exist two positive constants $\eta$ and $\delta$ and infinitely many numbers $n'$ with the following properties:

There exists a closed set $H_{n}$ in the interval $(e^{M_{n}}, e^{\frac{M_{n}}{2}})$ such that

$$\frac{\text{mes}H_{n}}{(e^{M_{n}}-e^{\frac{M_{n}}{2}})} \geq \eta \quad \text{and} \quad L(r) \geq \delta > 0$$

for any $r \in H_{n}$.

Since $\int_{C_{r}}d\theta \leq 2\pi$, we have by Schwarz's inequality

$$D_{R-G}(U(z))=\int_{e^{M_{n}}/2}^{e^{M_{n}}}\left(\frac{\partial U(z)}{\partial r}\right)^{2}+\frac{1}{r^{2}}\left(\frac{\partial U(z)}{\partial \theta}\right)^{2}r\,dr \geq \frac{1}{2\pi}\int_{e^{M_{n}}/2}^{e^{M_{n}}}\frac{L^{2}(r)}{r}dr \geq \frac{1}{2\pi}\int_{e^{M_{n}}/2}^{e^{M_{n}}}\frac{1}{r}dr=\frac{M_{n}}{4\pi}(1-\eta)\delta^{2}.$$ 

Let $n \to \infty$. Then $D_{R-G}(U(z)) \uparrow \infty$. This is a contradiction. Hence there exists a sequence of exceptional sets $H_{n}$ in the interval $(e^{M_{n}}, e^{\frac{M_{n}}{2}})$ such that $\lim_{n} \frac{\text{mes}H_{n}}{(e^{M_{n}}-e^{\frac{M_{n}}{2}})} \leq \frac{1}{4\tilde{\eta}_{0}}$ and that $r \notin H_{n}$ implies $L(r) < \delta_{n}$, where $\lim_{n} \delta_{n} = 0.$
Returning to case of $U_i(z)$. Let $\{H_{i,n}\}$ be a sequence of exceptional sets corresponding to $U_i(z)$ and $\delta_{i,n}$ be the corresponding quantities of $H_{i,n}$. Then we see that
\[
\lim_{n} \frac{\text{men}(\sum_{1}^{i_0} H_{i,n})}{(e^{u_n}-e^{\frac{M_n}{2}})} \leq \frac{1}{4} \quad \text{and} \quad \max \delta_{i,n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
On the other hand, the niveau curves with height $\geq \frac{M_n}{2}$ are contained in $R^* - R_{\alpha'}$, since $\omega_n^*(z) < \frac{M_n}{2}$ in $(R_{\alpha'} - R_\delta) - R_0$. It follows that
\[
\sum_{i} H_{i,n} \text{ clusters at } B \quad \text{as} \quad n \rightarrow \infty.
\]
Consider niveau curves $C_{r}$ mentioned above as $r_j$. Then we have the lemma.

**Lemma b.** Let $V_{\infty}(p) = E[z \in R : N(z, p) > M] : p \in B_1 - B_{\delta'}$ and $\nu_{m}(p) = E[z \in \overline{R} : \delta(z, p) < \frac{1}{m}]$. Then
\[
\lim_{M=\infty} V_{M}(p) \cap \nu_{m}(p) = 0.
\]
Hence by Theorem 6. b) of $P$, $N(z, p) - N'(z, p)$ is superharmonic in $\overline{R} - R_{\alpha'}$. Hence $N'(z, p) = a N(z, p)$: $a \geq 0$ by the $N$-minimality of $N(z, p)$. By Theorem 9. a) of $P$ $N'(z, p) = a' N(z, q)$: $a' \geq 0$ and $q \in C_{U_{m}}(p) \cap B_1$. But by
\[
\int_{\partial B_{n}} \frac{\partial}{\partial n} N(z, p) ds = \int_{\partial B_{n}} \frac{\partial}{\partial n} N(z, q) ds
\]
we have $N(z, p) = N(z, q)$. This is a contradiction. Hence $\lim_{M=\infty} V_{M}(p) \cap \nu_{m}(p) = 0$.

Let $U_{m,M}(z)$ be a harmonic function in $R_{\delta} - (R_{M} \cap C_{U_{m}}(p))$ such that $U_{m,M}(z) = 0$ on $\partial R_{\delta}$ and $U_{m,M}(z) = M - \delta$ on $\partial (R_{M} \cap C_{U_{m}}(p))$ and has M.D.I., where $R_{M} = E[z \in R_{\delta} : G(z) > M]$. Then by the Dirichlet principle
\[
D(U_{m,M}(z)) \leq CR_{M} D(G(z)) \leq MK.
\]
Let $U'_{m,M}(z)$ be a harmonic function in $R_{\delta} - (R_{M} \cap C_{U_{m}}(p))$ such that $U'_{m,M}(z) = 0$ on $\partial R_{\delta}$ and $U'_{m,M}(z) = G(z) - \delta$ on $\partial (R_{M} \cap C_{U_{m}}(p))$ and has M.D.I. over $R_{\delta} - (R_{M} \cap C_{U_{m}}(p))$. Consider $(U_{m,M}(z) + U'_{m,M}(z))$ and $G(z) - \delta$ in $(R_{\delta} - R_{M} \cap R_{\delta})$. Then by the maximum principle
\[
U_{m,M}(z) + U'_{m,M}(z) + M w(B_n \cap R_{\delta}, z, R_{\delta}) \geq G(z) - \delta \geq U_{m,M}(z) - M w(B_n \cap R_{\delta}, z, R_{\delta}),
\]
where $B_n = R - R_n$ and $w(B_n \cap R_{\delta}, z, R_{\delta})$ is H.M. of $B_n \cap R_{\delta}$ relative to $R_{\delta}$, i.e. $w(B_n \cap R_{\delta}, z, R_{\delta})$ is a harmonic function in $R_{\delta} - (B_n \cap R_{\delta})$ such that $w(B_n$
\( \sim R_3, z, R_3) = 0 \) \( \partial R_3 - (B_n \cap R_3) \) and \( = 1 \) on \( \partial(B_n \cap R_3) \). Let \( n \to \infty \).

Then \( w(B_n \cap R_3, z, R_3) \to w(B \cap R_3, z, R_3) \leq w(B \cap R_{\delta}, z, R_{\delta}^e) = 0 \) by (46). Hence

\[
U_{m,M}(z) + U'_{m,M}(z) \geq G(z) - \delta \geq U_{m,M}(z).
\]

Now by \( G(z) \leq N(z, p) \), \( E[z: G(z) > M] = R_M \subset V_M(p) \). \( U'_{m,M}(z) \) and \( N(z, p) \) have M.D.I. over \( R_3 - (R_M \cap C_{U_2}(p)) \). \( U'_{m,M}(z) \leq G(z) \leq N(z, p) \) on \( \partial R_3 + \partial(R_M \cap C_{U_2}(p)) \), whence by the maximum principle

\[
U'_{m,M}(z) \leq N(z, p).
\]

Hence by Lemma b

\[
\lim_{M \to \infty} U'_{m,M}(z) \leq \lim_{M \to \infty} v_{M - C_{U_2}(p)} N(z, p) = 0.
\]

Hence

\[
U_{m,M}(z) = G(z) - \delta.
\]

Let \( G(z, q) \) be the Green's function of \( R \), and let \( v(q) \) be a neighbourhood of \( q \). Then \( D(G(z, q)) < \infty \). By \( D(U_{m,M}(z)) \leq MK \), hence by Lemma a) there exists a sequence \( \{r_j\} \) such that

\[
\int_{r_j} \left| \frac{\partial}{\partial n} U_{m,M}(z) \right| ds \to 0 \quad \text{and} \quad \int_{r_j} \left| \frac{\partial}{\partial n} G(z, q) \right| ds \to 0
\]

and \( r_j \) clusters at \( B \) as \( j \to \infty \).

Let \( R_j \) be the compact part of \( R_3 \) divided by \( r_j \). Then \( R_3 = \bigcup_{j}^\infty R_j \).

Put \( \Omega_M^C = E[z \in R_3 : M_0 < U_{m,M}(z) < M_1] \) and \( C_M = E[z \in R : U_{m,M}(z) = M_i] \). Then

\[
\int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds = \int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds + \int_{r_j^c \cap \Omega_M^C} \frac{\partial}{\partial n} U_{m,M}(z) ds.
\]

Since \( U_{m,M}(z) \geq 0 \) on \( C_M \) and \( C_M \), we have by (49)

\[
\int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds = \lim_{j \to \infty} \int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds = \lim_{j \to \infty} \int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds
\]

\[
= \int_{C_M} \frac{\partial}{\partial n} U_{m,M}(ds).
\]

Since \( D(U_{m,M}(z)) = \int U_{m,M}(z) \frac{\partial}{\partial n} U_{m,M}(z) ds + \int U_{m,M}(z) \frac{\partial}{\partial n} U_{m,M}(z) ds \)

and \( U_{m,M}(z) = M - \delta \) on \( \partial(R_M \cap C_{U_2}(p)) \), we have by letting \( j \to \infty \), \( D(U_{m,M}(z)) \)

\[
= \int U_{m,M}(z) \frac{\partial}{\partial n} U_{m,M}(z) ds (\leq MK). \quad \text{Since} \quad U_{m,M}(z) = M - \delta \quad \text{on} \quad \partial(R_M \cap C_{U_2}(p))
\]

we have by (50)

\[
\int_{C_M} \frac{\partial}{\partial n} U_{m,M}(z) ds \leq 2K \quad \text{for} \quad \frac{M}{2} > \delta, \quad \text{for every} \quad L.
\]
By the Green's formula
\[ \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} U_{m,M}(z) \frac{\partial}{\partial n} G(z, q) ds = \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds. \]

Now \( U_{m,M}(z) = M - \delta \) on \( (R_{M} \cap v_{m}(p)) \), whence
\[ \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} U_{m,M}(z) \frac{\partial}{\partial n} G(z, q) ds = (M - \delta) \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds \leq (M - \delta) \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \]

By \( U_{m,M}(z) \leq M - \delta, \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds \leq M \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \]

Similarly by \( G(z, q) < L_{0} \) in \( Cv(q) \), \( \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty. \]

Hence by letting \( j \rightarrow \infty \), we have
\[ U_{m,M}(q) = \frac{1}{2\pi} \int_{\partial U(q) + (\gamma_j - (R_{M} \cap v_{m}(p)) + (\partial(R_{M} - v_{m}(p))))} G(z, q) \frac{\partial}{\partial n} U_{m,M}(z) ds. \] (52)

\( p \) is regular by the assumption. Hence for any given positive number \( \varepsilon \) there exists a neighbourhood \( v_{m}(p) \) such that \( G(z, q) < \varepsilon \) in \( v_{m}(p) \). Hence by (51) and (52)
\[ U_{m,M}(q) \leq \frac{1}{\pi} K \varepsilon. \]

Let \( M \rightarrow \infty \). Then \( G(q) = \lim_{M} U_{m,M}(q) + \delta \) by (48.) But \( \varepsilon \) and \( \delta \) are arbitrary.

Hence \( G(z) = 0 \). Thus \( N(z, p) = P(z) \) and \( N(z, p) \) is Poisson's integrable.

Let \( U_{i}(z) \) \((i = 1, 2, \cdots, N)\) and the cardinal number of \( N \leq \sigma \) be a harmonic function in \( R - R_{0} \) vanishing on \( \partial R_{0} \). If any finite number of \( \{U_{i}(z)\} \) are linearly independent, we say that \( \{U_{i}(z)\} \) is linearly independent.

**Theorem 16.**

a) \( \{N(z, p_{i})\} : p_{i} \in B_{1} \) is linearly independent.

b) If there exists a point in \( B_{g,r} \cap (B_{1} - B_{3}) \), then there exist infinitely many linearly independent H.D. functions vanishing on \( \partial R_{0} \) where \( B_{g,r} \) is the set of regular points for the Green's function.

c) \( R \in H.N.D. (N = \infty \text{ and } \sigma) \) if and only if both \( B_{g,r} \cap (B_{1} - B_{3}) \) and \( B_{S,1} \) (set of singular points of first kind) are at most enumerable and the number of points of \( B_{S,2} \) (set of singular points of second kind) = \( \infty \) and \( \sigma \).

d) \( R \in H.N.D. (N \leq \infty) \) if and only if \( B_{g,r} \cap (B_{1} - B_{3}) = 0 = B_{S,1} \) and the number of points of \( B_{S,2} \) is \( N \).

**Proof of a.** Assume \( \{N(z, p_{i})\} \) is not linearly independent. Then
there exists a linear form $\sum_{j=1}^{l} c_{i} N(z, p_{i}) = 0$. Suppose $c_{1}, c_{2}, \ldots, c_{m} > 0$ and $c_{m+1}, \ldots, c_{l} < 0$.

Then $$\sum_{1}^{m} c_{i} N(z, p_{i}) = \sum_{m+1}^{l} c_{i} N(z, p_{i}).$$

Put $U(z) = \sum_{1}^{m} c_{i} N(z, p_{i})$ and $F = \sum_{1}^{m} p_{i}$, $F' = \sum_{m+1}^{l} p_{i}$. Then $F$ and $F'$ are closed and $\text{dist}(F, F') > 0$. By Theorem 12. a) and b) of $P$

$$U(z) =_{F} U(z) > F' U(z). \quad \text{(53)}$$

On the other hand, $U(z)$ is also $\sum_{m+1}^{l} c_{i} N(z, p_{i})$, whence as above $U(z) =_{F} U(z) > F' U(z)$.

This is a contradiction. Hence $\{N(z, p_{i})\}$ is linearly independent.

\textbf{Proof of b).} Map the universal covering surface $(R-R_{0})^\infty$ of $R-R_{0}$ onto $|\xi| < 1$. Then $E_{i} \supset E_{i+1} \supset E_{i+2} \cdots$, whence such $\{\omega(B \cap V_{M_{i+1}}(p), z)\}$ is linearly independent, where $E_{i}$ is the set on which $N(z, p)$ has angular limits $= M_{i}$.

Hence we have b).

\textbf{Proof of c).} By Theorem 14 it is sufficient to consider $H_{0}D$ functions vanishing on $\partial R_{0}$ instead of $HD$ function in $R$. \textit{Necessity.} Suppose $R \in H_{0}D (N= \infty$ and $N= \mathcal{K}_{0})$. Suppose the cardinal number of points of $B_{g,r} \cap (B_{1} - B_{S})$ is $\mathcal{K}$. Let $p \in B_{g,r} \cap (B_{1} - B_{S})$. Then $N(z, p)$ is Poisson’s integrable. Let $U_{n}(z)$ be a harmonic function in $R_{n} - R_{0}$ such that $U_{n}(z)$ is a harmonic function in $R_{n} - R_{0}$ such that $U_{n}(z) = \min(M, N(z, p))$ on $\partial R_{0} + \partial R_{n}$. Then $U_{n}(z) \to U(z, M, p)$ as $n \to \infty$, where

$$U(z, M, p) = \frac{1}{2\pi} \int \varphi^{M}(\omega^{is}) \frac{1-r^{2}}{1-2r \cos(\varphi - \theta) + r^{2}} d(\theta),$$

where $\varphi^{M}(\omega^{is}) = \min(M, \varphi^{is})$ and $\varphi^{is}$ is angular limits of $N(z, p)$ at $e^{is}$. 

On the other hand, by the Dirichlet principle $D(U(z, M, p)) \leq D(\min(N, N(z, p))) = 2\pi M$. Hence $U(z, M, p) \in H_{0}D$. 

\textbf{Proof of a).}
Let $\alpha$ and $\alpha_n$ be the cardinal numbers of linearly independent functions \{\(N(z, p_i)\)} and \{\(U(z, M, p_i)\)}: \(p_i \in B_{g, r} \cap (B_1 - B_S)\) respectively. Assume $\alpha = \aleph$. Since $\alpha = \lim_{M=\infty} \alpha_n$, there exists a constant $M_0$ such that $\alpha_{M_0} \geq \aleph$. Hence there exist an $\aleph$ number of linearly independent $H_0D$ functions in $R-R_0$ vanishing on $\partial R_0$, whence $R \in H.N.D. (N = \aleph)$. This is a contradiction. Hence $B_{g, r} \cap (B_1 - B_S)$ is at most enumerable.

Next $N(z, p) = \omega(p, z)$: $a > 0$ for $p \in B_{S,1} (\subset B_1)$ (whence $N(z, p) \in HD$). Hence by a) $\{\omega(p, z)\}$ is linearly independent, whence $B_{S,1}$ is also at most enumerable. We show $B_{S,2}$ consists of infinitely many points (clearly $\leq \aleph_0$ by Theorem 10. b). Assume that $B_{S,2}$ consists of $n_0 < \infty$ number of points $p_1, p_2, \cdots, p_{n_0}$. If $B_{S,1} = 0$, let $q \in B_{S,1}$. Now $B_{S,1} + (B_{g, r} \cap (B_1 - B_S))$ is at most enumerable, $B_{g,1}$ (set of irregular points) is an $F_\sigma$ set of capacity zero by Theorem 11. d) of $P$, $B_0$ is also an $F_\sigma$ set of capacity zero. Whence H.M. of $B_{S,1} + B_{g,1} + B_0 + B_{g, r} \cap (B_1 - B_S)$ is zero. Let $\nu_0(q) = E[z \in R: \delta(z, q) < \delta_0]$: $2\delta_0 = 2 \min (p_i, q)$. Then $B \cap (\nu_0(q) - q)$ is an $F_\sigma$ set of harmonic measure zero. Hence by Theorem 9 $q \in B_{S,2}$. This is a contradiction. Hence if $B_{S,2} = p_1 + \cdots + p_{n_0}$, $B_{S,1} = 0$. Assume $B_{S,2} = p_1 + p + \cdots + p_{n_0}$. Then $B_0 + B_{g,1} + B_{S,1} + (B_{g, r} \cap (B_1 - B_S))$ is an $F_\sigma$ set of capacity zero (clearly of harmonic measure zero). Hence it can be proved similarly as in Theorem 1. d)

\[\sum_{i=1}^{n_0} \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0),\]
where $E_i$ is the image of $p_i \in B_{S,2}$ on $|\xi| = 1$.

By Theorem 12. c) every $U(z) \in H.D.$ has angular limits $= \text{const}$ a.e. on $E_i$. Whence $R \in H.N.D. (N = n_0)$. This is a contradiction. Hence $B_{S,2}$ consists of infinitely many points and we have also $\sum \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0)$.

Sufficiency is evident. Thus we have c).

Proof of d). Suppose $R \in H.N.D. N < \infty$. Then by a) $B_{g, r} \cap (B_1 - B_S) = 0$ and $B_{S,2}$ consists of at most $n_0 \leq N$ number of points. Whence as in c) $B_{S,1} = 0$ and

\[\sum_{i=1}^{n_0} \text{mes } E_i = 2\pi - \text{mes (image of } \partial R_0),\]
where $E_i$ is the image of $p_i \in B_{S,2}$.

Whence $R \in H.N.D. n_0D$. Hence $n_0 = N$, i.e. $B_{S,2}$ consists of $N$ number of points. Thus we have d).

Remark. If $R \in H.N.D. (N < \infty)$, $\omega(p, z)$ of $p \in B_{S,2}$ is an $H.D.$ function. In fact, $\omega(p, z) = \sum \alpha_{ij} w(p_i, z)$. Now $\{\omega(p, z)\}$ is linearly independent,
whence the determinant $|\alpha_{ij}| \equiv 0$ and $w(p_j, z) = \sum \beta_{jj} w(p_i, z)$, where $|\beta_{ij}|$ is the inverted matrix of $|\alpha_{ji}|$. Hence $w(p_j, z) \in H.D.$

**PART III. On Subdomains.**

Let $G$ be a domain\(^{18}\) in $R - R_n$. Let $w(B \cap G, z, G)$ be H.M. of $(B \cap G)$ relative to $G$. i.e. $w(B \cap G, z, G) = \lim w_n(z)$, where $w_n(z)$ is a superharmonic function in $G$ which is harmonic in $G \cap R_n$. $w_n(z) = 0$ on $\partial G \cap R_n$ and $w_n(z) = 1$ on $(R - R_n) \cap G$. Map the universal covering surface $G^\omega$ of $G$ onto $|\xi| < 1$. Then $w(B \cap G, z, G)$ has angular limits $= 1$ or $0$ a.e. on $|\xi| = 1$.

**Proof.** The image of $\partial G$ is composed of enumerably infinite number of arcs on $|\xi| = 1$. Clearly $w(B \cap G, z, G) = 0$ on the image of $\partial G$. For simplicity put $w(z) = w(B \cap G, z, G)$. If $w(z) = 0$, our assertion is trivial. Suppose $w(z) > 0$. Then by P.H.2. sup $w(z) = 1$. Let $E_o$ be the set where the Green's function $G(z, p)$ of $G$ has angular limits = 0. Then mes $E_o = 2\pi$. Let $E$ be the set where $w(z)$ has angular limits between 0 and 1. Assume mes $E > 0$. Then for any given positive number $\varepsilon$, we can find a closed set $E' \subset (E \cap E_o)$, a number $\delta > 0$, a domain $D$ and a ring $\Re_m : 1 > |\xi| > 1 - \frac{1}{m}$, such that mes $(E - E') < \varepsilon$, $D$ is containing an end part of $A(\theta)$ of $e^{i \theta} \in E'$ and bounded by $\sum_{e^{i \theta} \in E'} \partial A(\theta) + E'$ and a circle $|\xi| = 1 - \frac{1}{m}$ with the following property: $1 - \delta > w(z) > \delta$ in $D \cap \Re_m$. i.e. $D \cap \Re_m$ is contained in the image of $\Omega_{1 - \delta} = E\{z \in G : \delta < w(z) < 1 - \delta\}$. Since $\partial G$ is composed of analytic curves and $w(z) = 0$ on $\partial G$, the image of $\partial G$ does not fall in $D \cap \Re_m$, image of $(\partial R_n \cap G) \to |\xi| = 1$ as $n \to \infty$ and $\partial R_n \cap G$ separates $E'$ from $|\xi| = 1 - \frac{1}{m}$ by $G(z, p) > 0$ on $\partial R_n \cap G$. Let $w^o(\xi)$ be a harmonic function in $D$ such that $w^o(\xi) = 0$ on $E'$ and $= 0$ on $\partial D - E'$. Then as usual $w(B \cap \Omega_{1 - \delta}, z, G) \geq w^o(\xi) > 0$.

On the other hand, $w(Q_{1 - \delta} \cap B, z, G) \leq w(B \cap G, z, G)$ and sup $w(G \cap \Omega_{1 - \delta}, z, G) \leq 1 - \delta$. Hence by P.H.2. $w(Q_{1 - \delta} \cap B, z, G) = 0$. This is a contradiction. Hence mes $E = 0$. Thus $w(B \cap G, z, G)$ has angular limits $= 0$ or 1 a.e. on $|\xi| = 1$.

Let $\{z_i\}$ be a sequence such that $z_i \to B$ and $\lim w(z_i) < 1$, we say that $\{z_i\}$ converges to the annexed relative boundary. Let $E_{K, A}$ be the set where $w(z)$ has angular limits $= 0$ and is not the image of $\partial G$. We call

18) See 5.)
$E_{K,\lambda}$ K-annexed relative boundary. We call $E_B$ where $w(z)$ has angular limits = 1 the image of the ideal boundary. Then by the above fact $w(B \setminus G, z, G) = w(E_B, \xi)$, where $w(E_B, \xi)$ is H.M. of $E_B$ with respect to $|\xi| < 1$. We denote by $H_0.B. (H_0.D)$ the class of bounded (Dirichlet bounded) harmonic functions in a domain $G$ vanishing on $\partial G$. Also we denote by $O_{H_0,B} (O_{H_0,D})$ and $H_0.N.B. (H_0.N.D)$ the class of domain in which there exist no non constant $H_0.B. (H_0.D.)$ function and there exist an $N$ number of linearly independent $H_0.B. (H_0.D.)$ functions respectively.

11. Class $O_{H_0,B}$ and $H_0.N.B$. Since $G$ is a Riemann surface, we can define K-Martin’s topology in $G$ and we have at once Theorem 2. But it is more interesting to characterize the class $H_0.N.B.$ by K-Martin’s topology of $R \supset G$.

Theorem 17. a) Every $H_0.B.$ function has angular limits $= 0$ a.e. on $E_{K,\lambda}$.

b) $G \in O_{H_0,B}$ if and only if mes $E_B = 0$.

c) $G \in H_0.N.B.$ if and only if $G$ contains K-approximately $N$ number of singular points $p_i$ of $R$ and $w((B - \sum p_i) \setminus G, z, G) = 0$.

d) If $R \in H.N.B., G \in H_0.N'.B.$ and $N' \leq N$.

Proof of a). Let $U(z) \in H_0.B.$ and let $U_n(z)$ be a harmonic function in $G \setminus R_n$ such that $U_n(z) = |U(z)|$ on $(\partial R_n \setminus G) = (\partial G \setminus R_n)$. Then $|U(z)| \leq |U_n(z)| \leq M w_n(z)$, where $M = \sup_{z \in G} U(z)$ and $w_n(z)$ is a harmonic function in $G \setminus R_n$ such that $w_n(z) = 0$ on $\partial G \setminus R_n$ and $w_n(z) = 1$ on $G \setminus \partial R_n$. Let $n \rightarrow \infty$. Then $|U(z)| \leq M w(B \setminus G, z, G)$. (54)

Hence we have a).

Proof of b). If mes $E_B = 0$, $w(B \setminus G, z, G) = 0$ and every harmonic function $U(z)$ has angular limits $= 0$ by (54) and $G \in O_{H_0,B}$. Suppose mes $E_B > 0$. Then $w(B \setminus G, z, G) > 0$ and $G \not\in 0_{H_0,B}$. Thus we have b).

Proof of c). Suppose $G$ contains $p_i \in B_s$ K-approximately. Then $w(p \setminus G, z, G) = 0$ and $K$-minimal in $G$ by Theorem 4. a). Let $E_i$ be the set on which $w(p_i \setminus G, z, G)$ has angular limits $= 1$ almost everywhere. Then by Theorem 1 $w(p_i \setminus G, z, G) = w(E_i, \xi)$ and every $U(z) \in H_0.B.$ has angular limits $= a_i$ a.e. on $E_i$ and mes $E_i \cap E_j = 0$ for $p_i \neq p_j$. Hence the only one uniquely determined bounded $K$-minimal function in $G$ corresponds to every $E_i$. If $w(B \setminus G, z, G) > w(\sum p_i \setminus G, z, G)$, we can construct an $\aleph$ number of linearly independent $H_0.B.$ functions. Hence if $G \in H.N.B. (N = \aleph_0)$, mes $(E_B - \sum E_i) = 0$. Conversely suppose mes $(E_B - \sum E_i) = 0$. Then every function $\in H_0.B.$ is a linear form of $\{w(G \setminus p_i, z, G)\}$. Hence we have b).
Proof of d). This is clear by c).

12. Class $O_{H_{0}D}$ and $H_{0}.N.D$. Let $N'(z, p)$ be a harmonic function with one logarithmic singularity at $p$, $N'(z, p)=0$ on $\partial G$ and $N'(z, p)$ has minimal $D_{*}\text{irichlet}$ integral. Then $N'(z, p)$ is uniquely determined. We call $N'(z, p)$ $N$-Green’s function of $G$. Let $G(z, p)$ be the Green’s function of $G$. If $N'(z, p)>G(z, p)$ for at least one point $p$, we say that $G$ has the ideal boundary of positive capacity.

Theorem 18. a) If $N'(z_{0}, p)>G(z_{0}, p)$ for two points $p$ and $z_{0}$ then $N'(z, p)>G(z, p)$ for any points $q$ and $z$ in $G$.

b) Let $E_{N,A}$ be the set where $N'(z, p)$ has angular limits $=0$ (which is not the image of $\partial G$), we call $E_{N,A}$ $N$-annexed relative boundary. Then every $U(z)\in H_{0}.D$ function in $G$ has angular limits $=0$ a.e. on $E_{N,A}$ and $E_{N,A}\supset E_{K,A}$.

c) The following conditions are equivalent.

1) $G$ has the ideal boundary of positive capacity, i.e. $N'(z, p)>G(z, p)$

2) $G\notin 0_{H_{0}D}$.

3) $\text{mes} (E_{B}-E_{N,A})>0$.

Proof of a). $N'(z, p)\geq G(z, p)$. If for a point $z_{0}$ $N'(z_{0}, p)=G(z_{0}, p)$, $N'(z, p)\equiv G(z, p)$. This is a contradiction. Hence $N'(z, p)>G(z, p)$ for any points $p$ and $z$. Next by the Green’s formula $N'(p, q)\equiv N'(q, p)$ and $G(p, q)=G(q, p)$, whence $N'(p, q)-G(p, q)>0$ and $N'(z, q)>G(z, q)$. Hence we have a).

Proof of b). If $\text{mes} E_{N,A}=0$, our assertion is trivial. Suppose $\text{mes} E_{N,A}>0$. Let $U(z)\in H_{0}.D$. Then $U(z)$ has angular limits a.e. on $|z|=1$. Suppose $U(z)$ has angular limits $=0$ a.e. on $E_{N,A}$. Then we can find a closed set $E'$ in $E_{N,A}$ and a number $a>0$ such that $U(z)$ has angular limits $>a$ or $<-a$ a.e. on $E'$. Without loss of generality, we can suppose that $U(z)$ has angular limits $=a>0$ a.e. on $E'$. Put $\Omega_{0}=E[z\in G: U(z)>a]$ and $\Omega_{a}=E[z\in G: U(z)>0]$. Let $\omega(G\cap \Omega_{a}, z, \Omega_{0})$ be C.P. of $(G\cap \Omega_{a})$ relative $\Omega_{0}$. Then by the Dirichlet priciple

$$D(\omega(G\cap \Omega_{a}, z, G)) \leq D(\omega(G\cap \Omega_{a}, z, \Omega_{0})) \leq \frac{D(U(z))}{\alpha^{2}} \leq M < \infty.$$  

Let $N\Omega_{a}=E[z\in G: N'(z, p)\in \varepsilon]$. Then by $(N\Omega_{a}\cap \Omega_{a})\subset \Omega_{a}$

$$D(\omega(N\Omega_{a}\cap \Omega_{a}, z, G)) \leq D(\omega(G\cap \Omega_{a}, z, G)) \leq M.$$  

19) This condition is a far simpler for $G\notin 0_{H_{0}D}$ than the condition given by A. Mori: On the existence of harmonic functions on a Riemann surface. Journ of the Faculty of Science University Tokyo. Vol. VI, 1951.
Now as usual it can be proved that H.M. $w(\mathcal{N}_\Omega \cap \Omega_a \cap B, z, G)$ of 
$(\mathcal{N}_\Omega \cap \Omega_a \cap B)\subseteq w(E', \xi)>0$, because $N'(z, p)$ has angular limits $=0<\varepsilon$ on $E'$ and $U(z)>a$ a.e. on $E'$. Hence

$0<w(\mathcal{N}_\Omega \cap \Omega_a \cap B, z, G)\leq w(\mathcal{N}_\Omega \cap \Omega_a \cap B, z, G)$.

Let $F_n$ be a closed set in $G$ such that $F_n \uparrow F$. If $D(\omega(F, z, G))<\infty$, then $D(\omega(F_n, z, G)) \leq D(\omega(F, z, G))$. Hence $D(\lim \omega(F_n, z, G)) \leq D(\omega(F, z, G))$. Now $\lim \omega(F_n, z, G)=1$ on $F$ except at most a set of capacity zero and has M.D.I. On the other hand, $\omega(F, z, G)$ has also M.D.I. Hence by Lemma 1. b) of $P$

$\omega(F', z, G)=\lim_n \omega(F_n, z, G)$.

Put $G^*=E[\delta(z, \partial G)\geq \frac{1}{n}] \cap R_n$. Then $\mathcal{N}_\Omega \cap \Omega_a = \lim_n (G^* \cap \mathcal{N}_\Omega \cap \Omega_a)$. Hence for the same $\varepsilon$ used for $\mathcal{N}_\Omega$, and a point $z_0 \notin (\mathcal{N}_\Omega \cap \Omega_a)$, there exists a number $n_0$ such that

$\omega(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0}, z_0, G)>\omega(\mathcal{N}_\Omega \cap \Omega_a, z_0, G)-\varepsilon$, \hspace{1cm} (55)

Then also by the Dirichlet principle

$D(\omega(F_n, \mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0}, z, G)) \leq D(\omega(F, \mathcal{N}_\Omega \cap \Omega_a, z, G))\leq M$.

Let $N_n'(z, z_0)$ be a harmonic function in $G \cap R_n$ with a logarithmic singularity at $z_0 \notin (\mathcal{N}_\Omega \cap \Omega_a)$ such that $N_n'(z, z_0)=0$ on $\partial G \cap R_n$ and $\frac{\partial}{\partial n}N_n'(z, z_0)$.

$N_n'(z, z_0)=0$ on $\partial R_n \cap \mathcal{N}_\Omega \cap \Omega_a$. Then $N_n'(z, z_0)=N'(z, z_0)$. Let $\omega_n(z)$ be a harmonic function in $(G \cap R_n) - (\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})$ such that $\omega_n(z)=0$ on $\partial G \cap R_n$, $\omega_n(z)=1$ on $(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\partial R_n \cap G$. Then $\omega_n(z) \Rightarrow \omega(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0}, z, G)$ and by the compactness of $\partial(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})$

$\int_{\partial(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})} \frac{\partial}{\partial n} \omega_n(z)ds = D(\omega_n(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0}, z, G)) \leq M$.

By the Green's formula

$\int N_n'(z, z_0) \frac{\partial}{\partial n}\omega_n(z)ds = \int \omega_n(z) \frac{\partial}{\partial n}N_n'(z, z_0)ds$.

By $\int_{\partial(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})} \omega_n(z) \frac{\partial}{\partial n}N_n'(z, z_0)ds = 0$ we have

$\omega_n(z) = \frac{1}{2\pi} \int_{\partial(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})} N_n'(z, z_0) \frac{\partial}{\partial n}\omega_n(z)ds \leq \frac{1}{2\pi} \max N_n'(z, z_0) \int_{\partial(\mathcal{N}_\Omega \cap \Omega_a \cap G^*_{n_0})} \omega(z)ds$

$\leq \frac{1}{2\pi} \max N_n'(z, z_0) M$. Let $n \rightarrow \infty$. Then
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\[ \omega(\omega \Omega, \Omega_d \cdot G^*, z_0, G) \leq \frac{\varepsilon}{2\pi} M \] by \( N'(z, z_0) < \varepsilon \) in \( \omega \Omega \).

Take \( \varepsilon < \min \left( \frac{1}{M}, \frac{w(\omega \Omega \cdot \Omega_d \cdot B, z_0, G)}{4} \right) \). Then by (55) and (56)
\[
0 < w(\omega \Omega \cdot \Omega_d \cdot B, z_0, G) \leq w(\omega \Omega \cdot \Omega_d \cdot G^*, z_0, G) + \varepsilon \leq \varepsilon + \frac{\varepsilon}{2\pi} M
\]

This is a contradiction. Hence \( \text{mes } E' = 0 \) and every \( U(z) \in H_oD \) has angular limits = 0 a.e. on \( E_{N,A} \). There exists a number \( M \) such that \( N(z, p) \leq M \) in \( G \cap (R - R_n) \) for a number \( n_0 \), whence \( N(z, p) \leq M w(G \cap B, z, G) \) in \( (R - R_n) \). Hence \( E_{K,A} \subseteq E_{N,A} \). Thus we have b).

Proof of c). Clearly by \( D(N(z, p) - G(z, p)) < \infty \), \( D(N(z, p)) < \infty \) and \( D(G(z, p)) < \infty \) we have \( D(N(z, p) - G(z, p)) < \infty \), where \( v(p) \) is a neighbourhood of \( p \). Hence \( N(z, p) - G(z, p) \in H_oD \) and 1) implies 2). Assume \( \text{mes } (E_{B} - E_{N,A}) = 0 \). Then by b) \( U(z) \in H_oD \) has angular limits = 0 a.e. on \( |\xi| = 1 \), whence \( G \in 0_{H_o,D} \). Hence (2) implies (3).

Assume \( N'(z, p) = G(z, p) \). Then \( N'(z, p) \) has angular limits = 0 a.e. on \( |\xi| = 1 \), whence \( \text{mes } (E_{B} - E_{N,A}) = 0 \). Hence (3) implies (1).

Suppose that \( G \) has the ideal boundary of positive capacity, i.e. \( N'(z, p) = G(z, p) > 0 \). In \( G \) we can introduce \( N \)-Martin's topology by use of \( N \)-Green's function of \( G \), but \( \int_{\partial G} \frac{\partial}{\partial n} N(z, p) ds \) is not necessarily = 2\( \pi \) but \( \leq 2\pi \). Then we have Theorem 16. Now as in case of H.N.B. it is more useful to characterize the class \( H_oN_D \) by the \( N \)-Martin's topology of \( R - R_o \). Even when \( \sup N(z, p) = \infty \) : \( p \in R - R_o + B \), we say that \( G \) contains \( p \) \( N \)-approximately, if \( N(z, p) > cG N(z, p) \) and if \( \rho_{cG} N(z, p) = 0 \). We denote by \( B_{\varepsilon} \) and \( B_{\varepsilon',r} \) the sets of singular points, of regular points for the \( N \)-Green's function of \( R - R_o \).

Superharmonicity in a domain \( G \). Let \( U(z) \) be a function of \( C_1 \)-class such that \( U(z) = 0 \) on \( \partial G \), \( U(z) > 0 \) in \( G \) and \( D(\min (M, U(z))) < \infty \) for \( M < \infty \). Let \( D \) be a domain with compact \( \partial D \). Let \( \bar{U}'(z) \) be a function in \( G \) such that \( \bar{U}'(z) = \min (M, U(z)) \) on \( D \), \( \bar{U}'(z) = 0 \) on \( \partial G \) and \( \bar{U}'(z) \) has M.D.I. over \( G - D \). Put \( \bar{U}(z) = \lim_{M \to \infty} \bar{U}'(z) \). If \( \bar{U}(z) \leq U(z) \) for any domain \( D \) with compact \( \partial D \), we say that \( U(z) \) is superharmonic in \( G \).

Then it is easily seen that all theorems about superharmonic functions in \( R - R_o \) are valid for superharmonic functions in \( \bar{G} \). Hence for a non compact
domain $D \subset \mathbb{U}(z)$ is also defined.

**Theorem 19.**

a) Let $\nu_n(p) = E \left[ z \in R : \delta(z, p) < \frac{1}{n} \right] : p \in R - R_0 + B_1 - B_s$ with respect to $N$-Martin's topology of $R - R_0$. Then $\nu_n(p)$ contains $p$ $N$-approximately.

b) Let $G$ be a domain in $R - R_0$ such that $G$ contains $p \in (R - R_0 + B_1 - B_s)$ $N$-approximately. Then $N(z, p)$ is superharmonic in $G$ and $\tilde{N}(z, p) = \tilde{N}(z, p)$, where $\tilde{N}(z, p) = N(z, p) - c_0 N(z, p)$.

c) If $G$ contains $p \in (B_1 - B_s) \cap B_{g,r}$ $N$-approximately. Then $\tilde{N}(z, p)$ is representable by Poisson's integral and $\sup \tilde{N}(z, p) = \infty$. Hence there exist infinitely many linearly independent $H_0.D$ functions in $G$.

**Proof of a.** $p$ is the kernel of the canonical mass distribution of $N(z, p)$ and $\text{dist}(p, C_{U_n}(p)) \geq \frac{1}{n}$. Hence by Theorem 13, c) of $P\ N(z, p)$ > $c_{1n,p}N(z, p)$. Since $\text{Cap}(p) = 0$, by Theorem 6) a) of $P\ c_{1n,p}N(z, p)$ $- p(c_{1n,p}N(z, p)) = U(z)$ is superharmonic in $R - R_0$. Let $F$ be the kernel of the canonical mass distribution of $U(z)$. Assume $p(c_{1n,p}N(z, p)) > 0$. Then $F + p$ is the kernel of the canonical mass distribution of $c_{1n,p}N(z, p)$ ($= U(z) + p(c_{1n,p}N(z, p)) = U(z) + a N(z, p) : a > 0$). On the other hand, $F'$ (the kernel of the canonical mass distribution of $c_{1n,p}N(z, p)$) is contained in $C_{U_n}(p)$ by Theorem 13 d) of $P$. But by the same theorem $F' = F + p$. This is a contradiction. Hence $p(c_{1n,p}N(z, p)) = 0$. Thus $\nu_n(p)$ contains $p$ $N$-approximately.

**Proof of b.** Put $\tilde{N}(z, p) = N(z, p) - c_0 N(z, p)$. We show that $\tilde{N}(z, p)$ is superharmonic in $\overline{G}$. Put $V_M = E \left[ z \in G : \tilde{N}(z, p) > M \right]$. $\nu_n(p) - c_{1n,p} N(z, p)$ and $\nu_n(p) - c_{1n,p} N(z, p) + M \omega(V_M, z)$ are superharmonic in $R - R_0$ and harmonic in $R - R_0 - (\nu_n(p) - cV_M)$ and $\nu_n(p) - c_{1n,p} N(z, p) + M \omega(V_M, z)$ on $\partial(\nu_n(p) - cV_M)$ and $\partial R_0$, where $\nu_n(p) = E \left[ z \in R : \delta(z, p) < \frac{1}{n} \right]$. Whence by the maximum principle

$$\nu_n(p) - c_{1n,p} N(z, p) \leq \nu_n(p) - c_{1n,p} N(z, p) + M \omega(V_M, z) \leq \nu_n(p) - c_{1n,p} N(z, p) + M \omega(\nu_n(p), z).$$

Let $n \to \infty$. Then by the assumption that $\nu_n(p) = 0$, we have by $\text{Cap}(p) = 0$$$

$$\nu_n(p) - c_{1n,p} N(z, p) + M \omega(\nu_n(p), z) \downarrow 0 \text{ as } n \to \infty \text{ and } \lim_{n} \nu_n(p) - c_{1n,p} N(z, p) = 0.
Hence
\[ N(z, p) \geq V_{\mu} N(z, p) \geq v_{\nu}(p) \geq V_{\mu} N(z, p) \geq N(z, p). \]  \hspace{1cm} (56)

By Theorem 4. c) of $P$, $v_{\mu} N(z, p) \uparrow v_{\mu} N(z, p) = N(z, p)$ as $n \to \infty$.

$v_{\mu} N(z, p)$ and $CG(v_{\mu} N(z, p))$ are superharmonic in $\overline{R} - R_{0}$ by Theorem 4. a) of $P$. Now $v_{\mu} N(z, p) = N(z, p)$ on $V_{\mu} - R_{n}$ and $CG(v_{\mu} N(z, p)) \leq CG N(z, p)$, whence
\[ v_{\mu} N(z, p) - CG(v_{\mu} N(z, p)) \geq N(z, p) - CG N(z, p) = \tilde{N}(z, p) \geq M \text{ on } V_{\mu} - R_{n}. \]

On the other hand, $v_{\mu} N(z, p) - CG(v_{\mu} N(z, p)) = 0 = \tilde{N}(z, p)$ on $\partial G$. Hence
\[ E[z \in G : 0 < v_{\mu} N(z, p) - CG(v_{\mu} N(z, p)) < M] = \Omega_{n} M \subset \Omega_{M}. \]

Since $V_{\mu} - R_{n}$ is compact, $v_{\mu} N(z, p) < \max_{z \in V_{\mu} - R_{n}} N(z, p) < M < \infty$ in $R - R_{0} - (V_{\mu} - R_{n})$, whence
\[ D_{R - R_{0} - (V_{\mu} - R_{n})}(v_{\mu} N(z, p)) \leq 2\pi M \text{ and } D_{R - R_{0} - (V_{\mu} - R_{n})}(CG(v_{\mu} N(z, p))) \leq 2\pi M \]
by Lemma 2. c) of $P$.

Hence
\[ D_{G - (V_{\mu} - R_{n})}(v_{\mu} N(z, p) - CG(v_{\mu} N(z, p))) < \infty. \]

Let $A_{m}(z)$ be a harmonic function in $R_{m} - R_{0} - (V_{\mu} - R_{n})$ such that $A_{m}(z) = v_{\mu} N(z, p)$ on $\partial(V_{\mu} - R_{n}) + \partial R_{0}$ and $\frac{\partial}{\partial n} A_{m}(z) = 0$ on $\partial R_{m}$. Then $A_{m}(z) \Rightarrow v_{\mu} N(z, p)$ as $m \to \infty$.

Let $B_{m}(z)$ be a harmonic function in $G - R_{m}$ such that $B_{m}(z) = CG(v_{\mu} N(z, p))$ on $\partial G - R_{m}$ and $\frac{\partial}{\partial n} B_{m}(z) = 0$ on $\partial R_{m} - G$.

Then $B_{m}(z) \Rightarrow CG(v_{\mu} N(z, p))$ as $m \to \infty$.

Put
\[ \Omega_{n,m} M = E[z \in G : 0 < A_{m}(z) - B_{m}(z) < M] \cap R_{m}. \]

Consider the Dirichlet integral of $(A_{m}(z) - B_{m}(z))$ on $\Omega_{n,m} M$.

Then
\[ D_{\Omega_{n,m} M}(A_{m}(z) - B_{m}(z)) = M \int_{\frac{\partial}{\partial n} A_{m}(z)} B_{m}(z) ds = M \int_{\frac{\partial}{\partial n} A_{m}(z)} B_{m}(z) ds = 0. \]

By $A_{m}(z) \Rightarrow v_{\mu} - R_{n} N(z, p) \leq N(z, p)$ as $m \to \infty$. Hence for any given positive number $\varepsilon$, there exists a number $m_{0}$ such that
\[ \int_{\frac{\partial}{\partial n} A_{m}(z)} B_{m}(z) ds = 0. \]

By $A_{m}(z) \Rightarrow v_{\mu} - R_{n} N(z, p)$ and $B_{m}(z) \Rightarrow CG(v_{\mu} - R_{n} N(z, p))$ as $m \to \infty$,
\[ \Omega_{n,m} M \to \Omega_{M} M = E[z \in G : 0 < v_{\mu} - R_{n} N(z, p) - CG(v_{\mu} - R_{n} N(z, p)) \leq M] \text{ as } m \to \infty. \]
Let $\Omega$ be a domain completely contained in $\Omega^M_n$. Then for any given number $l$ there exists a number $m'$ such that $(\Omega \cap R_l) \subset \Omega^M_{n,m}$ for $m \geq m'$. Then by Fatou's lemma

$$D_{\Omega \cap R_l}(V_{M^\cap} R_{n}N(z, p) - CG(V_{M^\cap} R_{n}N(z, p))) \leq \varliminf_{m=\infty} D_{\Omega^M_{n,m}}(A_m(z) - B_m(z)) \leq 2\pi M.$$  

Let $l \to \infty$ and then $\Omega \uparrow \Omega^M_n$. Then

$$D_{\Omega^M_n}(V_{M^\cap} R_{n}N(z, p) - CG(V_{M^\cap} R_{n}N(z, p))) \leq 2\pi M.$$  

Next $v_{M^\cap} R_{n}N(z, p) \uparrow v_{M} N(z, p)$ as $n \to \infty$ and by Theorem 4. h) of $P$

$$D(\min(M, N(z, p)) \to \varlimsup_{n=\infty} D(G) \leq 2\pi M).$$

Let $D$ be a domain in $G$ with compact $\partial D$. Then there exists a number $n_0$ such that $\partial D \subset R_{n_0}$. Let $J_n(z)$ be a harmonic function in $(G \cap R_n) - D$ such that $J_n(z) = \min(M, N(z, p))$ on $\partial D \cap R_n$ and $\frac{\partial}{\partial n} J_n(z) = 0$ on $\partial R_n \cap (G - D)$, where $M > \sup_{z \in \partial D} N(z, p)$. Then $D_{(G \cap R_n) - D}(J_n(z)) \leq 2\pi M$ by Lemma 2. c) of $P$ and $J_n(z) \to CG \uparrow N^M(z, p)$ in $G - D$ as $n \to \infty$, where $CG \uparrow N^M(z, p)$ has M.D.I. over $(R - R_0 - (CG + D))$ and $CG \uparrow N^M(z, p) = \min(M, N(z, p)) = N(z, p) - CG N(z, p)$ on $\partial D$.

Let $H_n(z)$ be a harmonic function in $(G \cap R_n) - D$ such that $H_n(z) = \min(M, CG \uparrow N^M(z, p))$ on $\partial D \cap R_n$ and $\frac{\partial}{\partial n} H_n(z) = 0$ on $\partial R_n \cap (G - D)$. Then $D_{(G \cap R_n) - D}(H_n(z)) \leq 2\pi M$ and $H_n(z) \to CG \uparrow N^M(z, p)$ in $G - D$ as $n \to \infty$. Then $J_n(z) - H_n(z)$ is a harmonic function in $(G \cap R_n) - D$ such that $J_n(z) - H_n(z) = 0$ on $\partial G \cap R_n$ (because $N(z, p) = CG \uparrow N(z, p)$ on $\partial G$), $J_n(z) - H_n(z) = \min(M, N(z, p))$ on $\partial D \cap R_n$ and $\partial D \cap R_n$ by $M > \sup_{z \in \partial D} N(z, p) - CG N(z, p)$.

Hence $J_n(z) - H_n(z)$ has M.D.I. over $(G \cap R_n) - D$ among all functions with value 0 on $\partial G \cap R_n$ and $\partial D \cap R_n$ for every $M$. Now $(J_n(z) - H_n(z))$ converges in mean as $n \to \infty$.

Put $\tilde{N}(z, p) = \lim_n (J_n(z) - H_n(z))$. Then $\tilde{N}(z, p)$ has M.D.I. over $G - D$ with value $N(z, p) - CG N(z, p)$ on $\partial G + \partial D$ and $\tilde{N}(z, p) = \lim_n (J_n(z) - H_n(z))$ for every $M$.

Let $M \to \infty$. Then $\tilde{N}(z, p) = CG \uparrow N(z, p) - CG \uparrow D(CG \uparrow N(z, p))$. On the other hand, by Theorem 4. b) of $P_{CG \uparrow N(z, p) = CG \uparrow D(CG \uparrow N(z, p))}$. Hence
\[ \tilde{\mathcal{N}}(z, p) = c_0 + bN(z, p) - c_0 \cdot b(c_0 N(z, p)) \leq N(z, p) - c_0 N(z, p) = \tilde{N}(z, p). \quad (58) \]

Put \( D = v_m(p) \cap R_n \cap G \). Then \( \tilde{\mathcal{N}}(z, p) = c_{o} + D N(z, p) - c_{o} N(z, p) \). Let \( n \to \infty \). Then by \( (v_m(p) + CG) \cdot p \) we have \( s_{m}(p) + c_{o} N(z, p) = N(z, p) \). Whence by letting \( m \to \infty \)

\[ \tilde{\mathcal{N}}(z, p) = N(z, p) - c_{o} N(z, p) = \tilde{N}(z, p). \quad (59) \]

Assume \( \tilde{N}(z, p) \leq M < \infty \). Then \( \tilde{\mathcal{N}}(z, p) \leq M \omega(p, z) = 0 \). This is a contradiction. Hence

\[ \sup \tilde{\mathcal{N}}(z, p) = \infty. \quad (60) \]

By (57), (58), (59) and (60) we have \( b \).

**Proof of c.** Put \( \tilde{\mathcal{N}}(z, p) = N(z, p) - c_{o} N(z, p) \). Suppose \( p \in B_{s, r} \cap (B_{1} - B_{s}) \). Then there exists no non constant generalized Green’s function in \( R - R_{0} \) smaller than \( N(z, p) \) by Theorem 15. Assume that there exists a non constant generalized Green’s function \( \tilde{G}(z) \) in \( G \) smaller than \( \tilde{N}(z, p) \).

Put \( V_{M} = E[z \in G : \tilde{G}(z) > M] \). Let \( G_{M}^{n}(z) \) be a harmonic function in \( R_{n} - R_{0} - V_{M} \) such that \( G_{M}^{n}(z) = \min(M, G(z)) \) on \( V_{M} + \partial R_{n}, G_{M}^{n}(z) = 0 \) on \( \partial R_{n} + (\partial R_{n} - G) \) and has M.D.I. over \( R_{n} - R_{0} - V_{M} \). Then by the Dirichlet principle \( D(G_{M}^{n}(z)) \leq D(\min(M, \tilde{G}(z)) < \infty \). Put \( G(z) = \lim \lim G_{M}^{n}(z) \). Then \( G(z) \) is a generalized Green’s function in \( R - R_{0} \) smaller than \( N(z, p) \). This is a contradiction. Hence there exists no generalized Green’s function \( G(z) \) smaller than \( \tilde{N}(z, p) \). Hence \( \tilde{N}(z, p) \) is Poisson’s integrable and by (60) \( \sup \tilde{\mathcal{N}}(z, p) = \infty \). Thus by Theorem 16. b) we have c).

**Theorem 20.** a) If \( G \) contains \( p \in B_{s, z} \) N-approximately, \( G \) contains \( p \) K-approximately by Theorem 12. b) and there exists a harmonic domain \( V \) such that \( D(\omega(V, z, G)) < \infty \) and \( \omega(p \setminus V, z, G) > 0 \). Map the universal covering surface \( G^{\omega} \) of \( G \) onto \( |\xi| < 1 \). Let \( E \) be the set on which \( \omega(p \setminus V, z, G) \) has angular limits = 1 almost everywhere. Then \( \mes E > 0 \), \( \mes (E \setminus E_{K, \omega}) = 0 \) and \( E \) does not depend on the domain \( V \), in other words, let \( V_{1} \) and \( V_{2} \) be domains such that \( \infty > D(\omega(V_{i}, z, G)) \) and \( \omega(p \setminus V_{i}, z, G) \geq \omega(p \setminus V_{i}, z, G) > 0 \) \( (i = 1, 2) \) and let \( E_{i} \) be the set where \( \omega(p \setminus V_{i}, z, G) \) has angular limits = 1. Then \( E_{1} = E_{2} \). We call \( E \) the image of \( p \) relative to \( G \). Every \( U(z) \in H_{0} \cdot D \) has angular limits = const a.e. on \( E \).

b) Suppose \( G \) contains \( p_{i} \in B_{s, z} \) N-appoximately. Then \( \omega(p_{i} \setminus V, z, G) \) has angular limits = const < 1 a.e. on \( E_{i} \), where \( E_{j} \) is the image of \( p_{j} : p_{i} \not\equiv p_{j} \) and \( p_{i} + p_{j} \in B_{s, 2} \) and \( G \) contains \( p_{j} \) N-approximately.

c) If \( G \) contains \( p \) K-approximately and not contains \( p \) N-appro-
approximately, then the set $E_{W}$ where $w(p \cap G, z, G)$ has angular limits $= 1$ is contained in $E_{N,A}$.

Proof of a). By Theorem 12. b) mes $E > 0$ and $U(z) \in H_{0,D}$. has angular limits $= a$ a.e. on $E$. Next $U(z) \in H_{0,D}$ must have angular limits $= 0$ a.e. on $E_{N,A}$, whence mes $(E \cap E_{N,A}) = 0$, because $w(p \cap V, z, G) \in H_{0,D}$.

Assume mes $(E_{1} \cap E_{2}) = 0$. Then $w(p \cap V_{1}, z, G) = 1$ a.e. on $E_{1}$ and $= a < 1$ a.e. on $E_{2}$. Put $\Omega_{2} = E[z \in G: w(p \cap V_{1}, z, G) > 1 - 2\delta]$ and $\Omega_{1} = E[z \in G: w(p \cap V_{1}, z, G) > 1 - \delta]$, where $\delta = \frac{1 - a}{3}$. Then

$$\lim_{\epsilon \to 0} D(\omega(\Omega_{1}, z, \Omega_{2})) = \lim_{\epsilon \to 0} D(\omega(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2})) = 0.$$

It follows that $\omega(p \cap \Omega_{1}, z, \Omega_{2}) = \lim_{\epsilon \to 0} \omega(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2}) = 0$ a.e. on $E_{N,A}$.

On the other hand, $U(z) \in H_{0,D}$ has angular limits $a_{1} \leq a_{2}$ a.e. on $E_{1}$ and $a_{2}$ on $E_{2}$, but by mes $(E_{1} \cap E_{2}) > 0$. Therefore, the image of $\partial \Omega_{2}$ does not fall in $D$ and the image of $\partial(V_{1-\epsilon})$ separates $E'$. Hence as usual, we have $w(p \cap \Omega_{1}, z, \Omega_{2}) = \lim_{\epsilon \to 0} w(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2}) = 0$. Hence $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) > 0$ and $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) = D(\omega(\Omega_{1}, z, \Omega_{2})) < M < \infty$.

Similarly we have $\omega(p \cap C \Omega_{2}, z, G) \in H_{0,D}$ has angular limits $a_{1} \leq a_{2}$ a.e. on $E_{1}$ and $a_{2}$ on $E_{2}$, but by mes $(E_{1} \cap E_{2}) > 0$. Therefore, the image of $\partial \Omega_{2}$ does not fall in $D$ and the image of $\partial(V_{1-\epsilon})$ separates $E''$. Hence as usual, we have $w(p \cap \Omega_{1}, z, \Omega_{2}) = \lim_{\epsilon \to 0} w(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2}) = 0$. Hence $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) > 0$ and $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) = D(\omega(\Omega_{1}, z, \Omega_{2})) < M < \infty$.

Similarly we have $\omega(p \cap C \Omega_{2}, z, G) \in H_{0,D}$ has angular limits $a_{1} \leq a_{2}$ a.e. on $E_{1}$ and $a_{2}$ on $E_{2}$, but by mes $(E_{1} \cap E_{2}) > 0$. Therefore, the image of $\partial \Omega_{2}$ does not fall in $D$ and the image of $\partial(V_{1-\epsilon})$ separates $E''$. Hence as usual, we have $w(p \cap \Omega_{1}, z, \Omega_{2}) = \lim_{\epsilon \to 0} w(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2}) = 0$. Hence $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) > 0$ and $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) = D(\omega(\Omega_{1}, z, \Omega_{2})) < M < \infty$.

Similarly we have $\omega(p \cap C \Omega_{2}, z, G) \in H_{0,D}$ has angular limits $a_{1} \leq a_{2}$ a.e. on $E_{1}$ and $a_{2}$ on $E_{2}$, but by mes $(E_{1} \cap E_{2}) > 0$. Therefore, the image of $\partial \Omega_{2}$ does not fall in $D$ and the image of $\partial(V_{1-\epsilon})$ separates $E''$. Hence as usual, we have $w(p \cap \Omega_{1}, z, \Omega_{2}) = \lim_{\epsilon \to 0} w(V_{1-\epsilon} \cap \Omega_{1}, z, \Omega_{2}) = 0$. Hence $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) > 0$ and $D(\omega(p \cap \Omega_{1}, z, \Omega_{2})) = D(\omega(\Omega_{1}, z, \Omega_{2})) < M < \infty$.
$a_{1} = a_{2}$ and $E_{1} = E_{2}$. Thus $E$ does not depend on $V$.

Proof of b). Map the universal covering surface $(R - R_{0})^{\infty}$ onto $|\xi| < 1$. Let $E_{i}$ be the set where $\omega(p_{i}, z)$ has angular limits $= 1$. Suppose $\omega(p_{i}, z)$ has angular limits $= a$ a.e. on $E_{j}$ $(i \neq j)$. Assume $a = 1$. Put $V_{1 - \epsilon}^{j} = \{z \in R : w(p_{j}, z) > 1 - \epsilon\}$.

By Theorem 10, b) $w(p_{i}, z) \geq w(E_{i}, \xi)$. Since $a = 1$, $w(p_{i}, z) \geq w(E_{j}, \xi)$, whence $\omega(p_{i}, z) \geq w(E_{j}, \xi)$, $\omega(E_{i}, \xi) \geq w(V_{1 - \epsilon}^{j}, z) > 0$. Now $\omega(p_{i}, z) = \lim_{n} \omega(V_{1 - \epsilon}^{j} \cap \rho_{n}(p_{j}), z)$ has its mass only at $p_{j}$ and by P.C.3. $\sup \omega(p_{i} \cap V_{1 - \epsilon}^{j}, z) = 1$. Hence $\omega(p_{i}, z) = \omega(p_{j}, z)$. Let $\epsilon \to 0$. Then $\omega(p_{i}, z) \geq \omega(p_{j}, z)$.

(63)

But by Theorem 12, c) of $P$ $\omega(p_{i}, z) < 1$ in $R - R_{0} - p_{i}$ except at most a set of capacity zero. (63) means that $\omega(p_{i}, z) = 1$ on a set $p_{j}$ of positive capacity. This is a contradiction. Hence $a < 1$.

Proof of c). Assume $E_{w} \subset E_{NA}$. Then there exists a function $U(z) \in H_{0}.D$ such that $U(z)$ has angular limits $\neq 0$ a.e. on $E_{w}$. Then we can suppose without loss of generality that $U(z)$ has angular limits $> a > 0$ on a set $E' \subset E_{w}$ of positive measure. Put $\Omega_{0} = E[z \in G : U(z) > 0]$ and $\Omega_{a} = E[z \in G : U(z) > \frac{a}{2}]$. Then

$$D(\omega(p \cap \Omega, z, G)) \leq D(\omega(\Omega_{a}, z, \Omega_{0})) \leq \frac{D(U(z))}{(\frac{a}{2})^{2}} < \infty.$$  

As usual we can find a closed set $E'' \subset E'$ of positive measure such that both $U(z)$ and $w(p, z, G)$ converge uniformly along Stolz's path terminating at $E''$ and it can be proved that

$$w(p \cap \Omega_{a}, z, G) = \lim_{\epsilon \to 0} w(V_{1 - \epsilon}^{w} \cap \Omega_{a}, z \in G : w(p, z, G) > 0,$n

where $V_{1 - \epsilon}^{w} = E[z \in G : w(p, z, G) > [1 - \epsilon]]$. Hence

$$\infty > D(\omega(p \cap \Omega_{a}, z, G)) > 0.$$  

This means that $G$ contains $p$ $N$-approximately. This contradicts the assumption. Hence we have c).

It is necessary to make many preparations to study the class $H.N.D.$ $(N = \infty$ and $= \mathcal{X}_{\theta})$. Hence we consider only the class $H_{0}.N.D.$ $(N < \infty)$. 


Theorem 21. $G \in H_{0}.N.D.(N < \infty)$ if and only if $G$ contains $N$ number of points $\in B_{S_{2}}$ of $R-R_{0}$ $N$-approximately and does not contain any point of $B_{S_{r}} \cap (B_{1}-B_{S})+B_{S_{1}}$ $N$-approximately. Map the universal covering surface $G^{\infty}$ onto $|\xi|<1$. Then the set $\Gamma: |\xi|=1$ is divided into three kinds of sets, $1^{o}$ image of $\partial G$, $2^{o}$ image of $p(\in B_{S_{2}})$ contained $N$-approximately and does not contain any point of $B_{g,r} \cap (B_{1}-B_{S})+B_{S_{1}}$ $N$-approximately.

Map the universal covering surface $(R-R_{0})^{\infty}$ onto $|\xi|<1$. Then the set $\Gamma: |\xi|=1$ is divided into three kinds of sets, $1^{o}$ image of $\partial G$, $2^{o}$ image of $p(\in B_{S_{2}})$ contained $N$-approximately and does not contain any point of $B_{g,r} \cap (B_{1}-B_{S})+B_{S_{1}}$ $N$-approximately.

Proof. Map the universal covering surface $(R-R_{0})^{\infty}$ onto $|\xi|<1$. Let $q \in B-B_{S_{2}}$ and let $u_{n}(q)=E[\in R: \delta(z, q)<\frac{1}{n}]$. Let $\omega(u_{n}(p) \cap B, z)$ be $C.P$. of $(u_{n}(q) \cap B)$ determined by $u_{n}(q)$. We prove the following

Lemma. For any given point $q \in B-B_{S_{2}}$ with $u_{n}(q)$, for any given positive number $\varepsilon>0$, we can find a number $n(q)$ such that the measure of the set $E_{n}<\varepsilon$, where $E_{n}$ is the set on which $\omega(u_{n}(q) \cap B, z)$ has angular limits $=1$ almost everywhere.

Assume $\omega(u_{n}(q) \cap B, z)$ has angular limits $=1$ on a set $E$ of positive measure $\delta>0$ for every $n$. Then by letting $n \rightarrow \infty$ $\omega(q, z) \geq w(E, \zeta)>0$, (64)

where $w(E, \zeta)$ is H.M. of $E$ with respect to $|\zeta|<1$.

On the other hand, by Theorem 10. b) for $q \in B_{S_{1}}$, $w(q, z)=\lim_{\epsilon=0} w(V_{1-\epsilon}(q), z)$, where $w(V_{1-\epsilon}(q))=E[\in R: \omega(q, z)>1-\epsilon]$, i.e. $w(q, z)$ is H.M. of the set on which $\omega(q, z)$ has angular limits $=1$. But by $q \in B_{S_{1}}$, $w(q, z)=0$, i.e. $\omega(q, z)$ has angular limits $<1$ a.e. on $|\xi|=1$. For $q \in B-B_{S_{1}}$, we have $\omega(q, z)=0$. This contradicts (64). Hence the measure of the set $E_{n}$ (where $\omega(u_{n}(q) \cap B, z)$ has angular limits $=1$) $\downarrow 0$ as $n \rightarrow \infty$.

We show

$$\text{mes}(E_{B}-E_{N_{A}}-\sum E_{i})>0 \text{ implies } G \in H_{0}.N.D. (N=\infty).$$

(65)

Put $E^{*}=E_{B}-E_{N_{A}}-\sum E_{i}$. We cover $B-\sum_{i} p_{i}(p \in B_{S_{2}})$ by a system of neighbourhoods $u_{n}(q_{j})$ of $q_{j} \in B-B_{S_{2}}$ such that $\omega(u_{n}(q_{j}) \cap B, z)$ has angular limits $=1$ on the set whose measure $<\frac{1}{2}(u_{n}(q_{j})$ depends on $q_{j})$.

By

$$\sum_{j}w(B \cap u_{n}(q_{j}), z, G)+\sum_{i}w(p_{i}, z, G) \leq w(B, z, G)$$

there exists at least one $u_{n}(q_{j})$ such that $\text{mes}(E_{j}^{w} \cap E^{*})>0$, where $E_{j}^{w}$ is the set on which $w(u_{n}(q_{j}) \cap B, z, G)$ has angular limits $=1$. Denote the
above \( u_{n}(q) \) by \( \mathfrak{V}(q_{1}) \). Next cover \( \mathfrak{V}(q_{1}) \) by \( u_{n}(q_{j}) \): \( q_{j} \in B - B_{S,2} \) such that the measure of the set (where \( \omega(u_{n}(q_{j}) \cap B, z) \) has angular limits \( = 1 \)) \( < \frac{1}{2^{n}} \).

Then there exists at least one \( u_{n}(q_{j}) \) such that \( \text{mes} (E_{j}^{w} \cap E^{*}) > 0 \), where \( E_{j}^{w} \) is the set where \( w(u_{n}(q_{j}) \cap \mathfrak{V}(q_{1}) \cap B, z, G) \) has angular limits \( = 1 \). Put \( \mathfrak{V}_{1}(q) = \mathfrak{V}(q) \cap u_{n}(q_{j}) \). In this way we can find a sequence

\[ \mathfrak{V}_{1}(q) \supset \mathfrak{V}_{2}(q) \supset \cdots. \]

This sequence has the following properties:

1°. The measure of the set where \( \omega(\mathfrak{V}_{n}(q_{n}) \cap B, z) \) has angular limits \( = 1 \)< \( \frac{1}{2^{n}} \).

2°. Let \( E_{n}^{w} \) be the set where \( w(\mathfrak{V}_{n}(q_{n}) \cap B, z, G) \) has angular limits \( = 1 \). Then \( E_{1}^{w} \supset E_{2}^{w} \supset \cdots \), \( \text{mes} (E_{n}^{w} \cap E^{*}) = \alpha_{n} > 0 \) and \( \alpha_{n} < \frac{1}{2^{n}} \) we have \( \lim \alpha_{n} = 0 \).

From the above sequence we extract a subsequence as follows:

Put \( \mathfrak{V}_{1}(q_{1}) = \mathfrak{V}_{1}(q_{1}) \). Let \( n_{1} \) be the least integer such that \( \frac{1}{2^{n_{1}}} < \alpha_{1} \).

Put \( \mathfrak{V}_{2}(q_{2}) = \mathfrak{V}_{w}(q_{w}) \). Suppose \( \mathfrak{V}_{n}(q_{n}) \) is defined. Let \( n'' \) be the least integer such that \( \frac{1}{2^{n''}} < \alpha_{n} \) and put \( \mathfrak{V}_{n+1}(q_{n+1}) = \mathfrak{V}_{w}(q_{w'}) \). Then we have a sequence

\[ \mathfrak{V}_{1}(q_{1}) \supset \mathfrak{V}_{2}(q_{2}) \supset \cdots. \]

The above sequence has the following properties:

1°. The measure of the set where \( \omega(\mathfrak{V}_{n}(q_{n}) \cap B, z) \) has angular limits \( = 1 \)< \( \frac{1}{2^{n_{1}}} \).

2°. The measure of the set \( (E_{n}^{w} \cap E^{*}) \) (where \( w(\mathfrak{V}_{n}(q_{n}) \cap B, z, G) \) has angular limits \( = 1 \) on \( E_{n}^{w} \)) \( = \alpha_{n} \), \( \alpha_{n} > 0 \) and \( \alpha_{1} > \alpha_{2} > \alpha_{3} > \cdots \): \( \lim \alpha_{n} = 0 \).

Let \( l \) be a given integer, we shall construct \( l \) number of linearly independent \( H_{0}.D \). functions in \( G \). \( N(z, p) - G(z, p) > 0 \) on \( E^{*} \). We can find a closed set \( \hat{E} \) in \( E^{*} \cap E_{1}^{w} \) such that \( \text{mes} ((E^{*} \cap E_{1}^{w}) - \hat{E}) < \epsilon_{0} \) and \( N(z, p) - G(z, p) > \delta > 0 \) on \( \hat{E} \), where \( E_{1}^{w} \) is the set on which \( w(\mathfrak{V}_{1}(q_{1}) \cap B, z, G) \) has angular limits \( = 1 \) and \( \epsilon_{0} = \min (\alpha_{n} - \alpha_{n+1}) \).

Put \( \Omega = E \left[ z \in G : N(z, p) - G(z, p) > \frac{\delta}{2} \right] \). Then

\[ D(\omega(\Omega, z, G)) = \frac{D(N(z, p) - G(z, p))}{(\frac{\delta}{2})^{2}} < M < \infty. \]

Since \( w(B \cap \mathfrak{V}_{n}(q_{n}), z, G) = 1 \) a.e. on \( E_{n}^{w} \cap \hat{E} (n = 1, 2, \cdots l) \), it can be proved as
usual \( M > D(\omega(Q \cap B \setminus \mathcal{W}(q), z, G)) \), \( \omega(Q \cap B \setminus \mathcal{W}(q), z, G) \geq w(Q \cap B \setminus \mathcal{W}(q), z, G) \) and \( w(Q \cap B \setminus \mathcal{W}(q), z, G) = 1 \) a.e. on \( E'_{n}^{w} \), where \( E'_{n}^{w} \) is the set where \( w(B \setminus \mathcal{W}(q), z, G) \) has angular limits = 1.

Let \( E_{n}^{*} \) be the subset of \( E^{*} \) where \( \omega(Q \cap B \setminus \mathcal{W}(q), z, G) = 1 \) and let \( \hat{\alpha}_{n} \) be its measure. Then by \( \omega(Q \cap B \setminus \mathcal{W}(q), z, G) \leq \omega(B \setminus \mathcal{W}(q), z) \alpha_{n-1} \geq \hat{\alpha}_{n} \).

Next by \( \omega(Q \cap B \setminus \mathcal{W}(q), z, G) \geq w(Q \cap B \setminus \mathcal{W}(q), z, G) \) and (\( E_{n}^{*} \setminus \hat{E} \) \( \supset \) \( E_{n}^{w} \setminus \hat{E} \)) we have \( \hat{\alpha}_{n} \geq \alpha_{n-1} - \varepsilon_{0} \). Hence \( \alpha_{n-1} = \hat{\alpha}_{n} \geq \alpha_{n} - \varepsilon_{0} \). Whence by the property of \( \varepsilon_{0} \) we have \( \hat{\alpha}_{1} > \hat{\alpha}_{2} > \cdots > \hat{\alpha}_{i} \).

Clearly by \( (\Omega \cap B \setminus \mathcal{W}(q)) \supset (\Omega \cap B \setminus \mathcal{W}(q+1)) \) and by (66) \{\( \omega(Q \cap B \setminus \mathcal{W}(q), z, G) \)\} is linearly independent and \( \varepsilon \in H_{0}.D. \) Now \( l \) is any integer. Hence \( G \in H_{0}.D. (N = \infty) \).

We show if \( G \) contains \( p_{k} \in B_{S_{2}} \) (\( k = 1, 2, \cdots n_{0} \)), then we can construct \( n_{0} \) number of linearly indepent \( H_{0}.D. \) functions.

Since C.P. of \( p_{k} : \omega(p_{k} \cap V, z, G) \) does not depend on \( V \) by Theorem 20. a), we denote it by \( \omega(p_{k}, z, G) \) for simplicity. Let \( E_{k} \) be the image of \( p_{k} \in B_{S_{2}} \) relative to \( G \). Then by Theorem 20. b) \( \omega(p_{k}, z, G) = a_{j} \) on \( E_{j} \) and \( a_{j} < 1: k \neq j \).

Put \( \alpha = \max a_{j} \) and put \( a_{j} \in \omega_{n}(z) \downarrow \omega_{1}(z) \). Let \( V = V[z \in G : \omega(p_{k}, z, G) > \alpha + \varepsilon] \). Let \( V = V[z \in G \) be a harmonic function in \( (G \setminus R_{n+i+j}) \supset \Omega \cap B_{(R_{n+i+j} - R_{n+i})} - (V \setminus (R_{n+i+j} - R_{n+i})) \) such that \( \omega_{n}(z) \downarrow \omega(z) \) and \( \omega_{n,n+i,n+i+j}^{k}(z) \downarrow \omega_{n}(z) \) on \( \partial R_{n+i+j} - \Omega + \partial G \). Hence for \( j = 1, \cdots n_{0} \) we have \( \omega_{n,n+i,n+i+j}^{k}(z) \downarrow \omega(z) \). It can be easily proved that \( \omega(z) = 1 \) a.e. on \( E_{k} \) and \( = 0 \) a.e. on \( \sum_{j \neq k} E_{j} \) and \( D(\omega(z)) \leq D(\omega(p_{k}, z, G)) \). Whence \( \omega(z) \) is linearly independent and \( \varepsilon H_{0}.D. \).

Proof of the theorem. Suppose \( G \in H_{0}.D. (N < \infty) \). Then by Theorem 19. c) \( G \) does not contain any point of \( B_{v,\tau} \cap (B_{1} - B_{0}) \) \( N \)-approximately and by (65) \( \operatorname{mes}(E_{B} - E_{N,A}) = \operatorname{mes} \sum E_{i} \). Let \( N' \) be the number of points \( \in B_{S_{2}} \) contained in \( G \) \( N \)-approximately. Then we can construct \( N' \) number of linearly independent \( H_{0}.D. \) functions. On the other hand, every \( H_{0}.D. \) function has angular limits = \( \alpha_{i} \) on \( E_{i} \), whence \( G \in H_{0}.N'.D. \) and \( N = N' \). Conversely if \( G \) contains \( p_{k} \in B_{S_{2}} \) (\( i = 1, 2, \cdots N \)) \( N \)-approximately and \( \operatorname{mes}(E_{B} - E_{N,A}) = \operatorname{mes} \sum E_{i} \), we have \( G \in H_{0}.N. \) Thus we have the theorem.

By Theorems 16 and 20 we have easily the following

Corollary. If \( R - R_{0} \in H_{0}.D. (N \leq \infty \) and \( N = \mathcal{X}_{0} \), \( G \in H_{0}.N'.D. \) and \( N' \leq N \).
In fact, \( w(B-B_{s,2}, z)=0 \) implies \( \text{mes}(E_{B}-E_{x,4})=\text{mes} \sum E_{i} \), where \( E_{i} \) is the image of \( p_{i} \in B_{s,2} \) contained in \( G \) \( N \)-approximately. Hence we have by Theorem 12. \( \text{b}) \), \( G \in H_{0}.N.D. \) and \( N' \leq N \).

13. Covering property of Riemann surfaces.

Consider an analytic function \( w= f(z): z \in R \). Let \( w_{o} \) be a point of the \( w \)-plane. Then the part of \( R \) over \( |w-w_{o}|<r \) consists of at most an enumerably infinite number of components. Such one component is called a connected piece on \( |w-w_{o}|<r \). Then

**Theorem 22.** \( \text{a}) \) Let \( R \in H.N.B. (N \leq \infty, N=\mathfrak{X}_{o}) \). Then every connected piece \( G \) over \( |w-w_{o}|<r \) covers \( |w-w_{o}|<r \) except at most a set of capacity zero.

\( \text{b}) \) Let \( R \in H.N.D. (N \leq \infty, N=\mathfrak{X}_{o}) \). Let \( G \) be a connected piece over \( |w-w_{o}|<r \). If the area of \( G \) is finite, \( G \) covers \( |w-w_{o}|<r \) except at most a set of capacity zero.

**Proof of a).** Suppose that \( G \) does not cover a set \( F \) (clearly closed) of positive capacity. Then there exists a subset \( F_{r_{o}} \) of \( F \) of positive capacity such that \( F_{r_{o}} \) is contained in the circle \( |w-w_{o}|<r_{o}<r \). Let \( T(w) \) be a bounded harmonic function in \( E[w: |w-w|<r_{o}|=F_{r_{o}} \) such that \( T(w)=0 \) on \( |w-w_{o}|=r \) and \( T(w)=1 \) on \( F_{r_{o}} \). Then \( T(z)=w(T(z))>0 \) in \( G \) and=0 on \( \partial G \). Hence \( w(B \cap G, z, G) \geq T(z)>0 \).

\( w(B \cap G, z, G) \leq w(B-B_{s}, z, G)+w(B-B_{s}, z, G) \leq \sum w(B \cap p_{i}, z, G) \), where \( p_{i} \in B_{s} \) and \( w(B-B_{s}, z, G)=0 \) by \( R \in H.N.B. \). Hence there exists at least one point \( p \in B_{s} \) such that \( w(p \cap G, z, G)>0 \). This means that \( G \) contains \( p \) \( K \)-approximately. Hence by Theorem 4. \( \text{b}) \) there exists no non constant bounded analytic function. On the other hand, \( f(z) \) is bounded in \( G \). This is a contradiction. Hence we have \( \text{a}) \).

**Proof of b).** Suppose that \( G \) does not cover a set of positive capacity in \( |w-w_{o}|<r \). Then there exists a number \( r_{o} \) such that \( F_{r_{o}} \) of \( F \) in \( |w-w_{o}|<r_{o}<r \) is of positive capacity. Put \( Q=E[z \in G: |f(z)-w_{o}|<r_{o}] \). Then \( w(Q \cap B, z, G) \geq T(z) \), where \( T(z) \) is the function used in \( \text{a}) \). Since \( R \in H.N.D., w(B-B_{s,2}, z)=0 \), hence there exists at least one point \( p \in B_{s,2} \) such that \( w(Q \cap p, z, G)>0 \). Let \( S(w) \) be a continuous function in \( |w-w_{o}|<r \) such that \( S(w) \) is harmonic in \( r_{o}<|w-w_{o}|<r \), \( S(w)=0 \) on \( |w-w_{o}|=r \) and \( S(w)=1 \) on \( |w-w_{o}| \leq r_{o} \). Put \( S(z)=S(f(z)) \). Then

\[
D(S(z)) \leq M^{2} A,
\]

where \( A \) is the area of \( G \) and \( M=\max \left\{ \left| \frac{\partial S(w)}{\partial u} \right|, \left| \frac{\partial S(w)}{\partial v} \right| \right\} : w=u+iv \).
Hence $0 < w(\Omega \cap p, z, G) \leq w(\Omega \cap p, z, G)$ and $D(w(\Omega \cap p, z, G)) \leq D(S(z)) < \infty$. This means $G$ contains $p \in B_{s, \zeta}$ $N$-approximately. Hence by Theorem 12. 

a) there exists no non constant analytic function with finite area in $G$. But $f(z)$ has finite area. This is a contradiction. Hence we have b).

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