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ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATION

By

Hidemaro KÔJYÔ

§0. Introduction. Recently, T. Ôtsuki and Y. Tashiro [1] have studied holomorphically projective correspondences of Kählerian manifolds.

On the other hand, K. Yano and T. Nagano [2] and T. Sumitomo [3] have studied infinitesimal projective transformations in a Riemannian manifold and obtained valuable results. Further, S. Tachibana and S. Ishihara [4] have considered analogous problems concerning the infinitesimal holomorphically projective transformations, which will be briefly called an HP-transformation, and obtained that a Kählerian manifold satisfying $R_{ijkl}=0$, which admits a non-trivial analytic HP-transformation reduces to an Einstein one.

The purpose of the present paper is to generalize more the above result of S. Tachibana and S. Ishihara, that is, we shall give a theorem about a Ricci-recurrent Kählerian manifold in §1 and one in a Ricci-recurrent K-space in §2, which is one of the generalization of the theorem in §1.

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§1. An analytic HP-transformation on a Kählerian manifold.

A vector field $v^{i}$ is called an HP-transformation, if it satisfies

$$(1.1) \quad \mathfrak{L}_{v^{i}}\varphi^{i} = P_{h}(\partial_{j}\varphi_{k}^{i} - \varphi_{j}^{h}\varphi_{k}^{i}) + P_{h}(\partial_{k}\varphi_{j}^{i} - \varphi_{k}^{h}\varphi_{j}^{i})$$

where $P_{h}$ is a certain vector and $\varphi_{j}^{i}$ is the complex structure, and semi-colon and $\mathfrak{L}$ denote the covariant differentiation with respect to $v^{i}$ and Lie differentiation with respect to $v^{i}$, respectively. We shall call $P_{h}$ in (1.1) the associated vector of the HP-transformation. Contracting (1.1)

1) Numbers in brackets refer to the references at the end of the paper.
with respect to $i$ and $k$, we get $P_h = \frac{1}{n+2} v_{;i} v^i$, which shows that $P_h$ is gradient.

An infinitesimal affine transformation $v^i$ is defined by
$$\mathfrak{L}\{jik\} \equiv v^i_{;j;k} + R^i_{jkl} v^l = 0.$$ 
If $P_h = 0$, then the HP-transformation reduces to an affine one.

A vector field $v^i$ is called analytic on a Kählerian manifold, if it satisfies
(1.2) $$\mathfrak{L} \varphi_j^i = v^i - \varphi_j^k v^i_{;k} + \varphi_k^i v^i_{;j} = 0.$$ 

We shall give here preliminary formulas on Kählerian manifold. Let our manifold be a real $n(=2m+2)$ dimensional Kählerian manifold with local coordinates $\{x^i\}$. Then the Riemannian metric $g_{ij}$ and the complex structure $\varphi_{i}^{j}$ satisfy
$$\varphi_{i}^{k} \varphi_{k}^{j} = -\delta_{i}^{j}, \quad g_{hk} \varphi_{i}^{h} \varphi_{j}^{k} = g_{ij}, \quad \varphi_{i;k}^{j} = 0, \quad g_{ij;k} = 0.$$ 

Then the following equation holds:
(1.3) $$R^i_{jkl} \varphi_{i}^{h} \varphi_{j}^{k} = R_{i;j},$$
where $R^i_{jkl}$ is the Riemannian curvature tensor, and
$$R^h_{jkl} = R_{jkl}, \quad R^h_{jkl} g_{hi} = R_{ijkl}.$$
If $P_h$ is the associated vector of an analytic HP-transformation, then we get
(1.4) $$P_h;_{k} \varphi_{i}^{h} \varphi_{j}^{k} = P_{i;j}.$$ 
Moreover, if $v^i$ be an analytic HP-transformation, then we have

**Lemma.** Let $v^i$ be an analytic HP-transformation, then the following relation holds:
(1.5) $$\mathfrak{L} g_{ik} R^i_{jk} = \mathfrak{L} g_{jk} R^i_{ik}.$$

**Proof.** From the assumptions, it follows that
$$\mathfrak{L} \varphi_{i}^{j} = 0,$$
$$\mathfrak{L}\{jik\} = v^i_{;j;k} + R^i_{jkl} v^l = P_h (\delta_{i}^{j} \delta_{k}^{l} - \varphi_{j}^{h} \varphi_{k}^{l}) + P_h (\delta_{k}^{j} \delta_{i}^{l} - \varphi_{k}^{h} \varphi_{j}^{l}).$$ 

Since $R_{ijkl}$ is anti-symmetric with respect to $i$ and $j$, we get
(1.6) $$(\mathfrak{L} g_{ij})_{,k} = (v_{i;\ j} + v_{j;\ i})_{,k} = 2P_{k} g_{ij} + P_{j} g_{ik} + P_{i} g_{jk} - P_{\ a} \varphi_{j}^{a} \varphi_{k}^{l} - P_{a} \varphi_{i}^{a} \varphi_{k}^{l}.$$

The integrable condition of the above equation is that
$$(\mathfrak{L} g_{ij})_{,k} + (\mathfrak{L} g_{ik})_{,j} = P_{j;k} g_{il} + P_{i;k} g_{jl} - P_{j;i} g_{ik} - P_{i;j} g_{jk} + \varphi_{j}^{a} (P_{a;} \varphi_{k}^{l} - P_{a;k} \varphi_{i}) + \varphi_{i}^{a} (P_{a;} \varphi_{k}^{l} - P_{a;k} \varphi_{j}).$$
If we contract $g^{il}$ to this equation and take account of (1.4), then we have

$$(
abla \bar{g}_{ai}) R^{a}_{ik} - (\nabla \bar{g}_{ai}) R^{2}_{i} = n P_{i}^{k} - P_{a}^{;a} g_{ik}. $$

Since $P_{h}$ is gradient and $(\nabla \bar{g}_{af}) R^{a}_{ik}$ is symmetric with respect to $i$ and $k$, we obtain the conclusion.

Recently S. Tachibana and S. Ishihara [4] obtained the following

**Theorem.** If a Kählerian manifold satisfying $R_{ij;k} = 0$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.

We shall now consider a Ricci-recurrent Kählerian manifold, i.e., a Kählerian manifold such that $R_{ij;k} = R_{ij} v_{k}$, and we obtain the following

**Theorem.** If a Kählerian manifold satisfying $R_{ij;k} = R_{ij} v_{k}$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.

**Proof.** Covariantly differentiating (1.5) with respect to $x^\iota$ and making use of (1.5), we find

$$(\nabla \bar{g}_{ai}) \iota R^{a}_{ik} = (\nabla \bar{g}_{ka}) \iota R^{i}_{a}. $$

Substituting (1.6) into the last equation, we have easily

$$( Pa g_{ii} + P_{i} g_{ia} + 2 P_{i} g_{ia} - \varphi_{ia}^{b} \varphi_{i}^{b} P_{b} - \varphi_{ia}^{b} \varphi_{la}^{b} P_{b} ) R^{a}_{i} = ( Pa g_{ki} + P_{k} g_{ai} + 2 P_{k} g_{ai} - \varphi_{ki}^{b} \varphi_{i}^{b} P_{b} - \varphi_{ki}^{b} \varphi_{la}^{b} P_{b} ) R^{i}_{a}. $$

Contracting this equation with $g^{il}$ and $R^{il}$, and taking account of (1.3) and (1.4), we have

$$nP_{a} R^{a}_{i} = \dot{R} P_{k}.$$

$RR^{a}_{k} P_{a} = R_{ij} R^{ij} P_{k}.$

From the above equations, we get

$$\left( R_{ij} R^{ij} - \frac{R^{2}}{n} \right) P_{k} = 0.$$

Since $P_{k} \neq 0$, we must have

$$R_{ij} R^{ij} - \frac{R^{2}}{n} = 0.$$

On the other hand, according to the theorem obtained by T. Sumitomo [3], a Riemannian manifold satisfying the relation $R_{ij} R^{ij} = \frac{R^{2}}{n}$ is an Einstein manifold. Therefore, we get the conclusion.

In this section, we shall consider only a K-space, which is another generalization of a Kählerian manifold.

If \( \varphi_{ij} \) (\( \varphi_{ij} = \varphi_{ik}g_{kj} \)) is a Killing tensor, i.e., it satisfies the equation
\[
\varphi_{ij;k} + \varphi_{ik;j} = 0,
\]
an almost-Hermitian space is called a K-space. After some calculations we get also the following identities in a K-space:
\[
\begin{align*}
R_{hik} \varphi_{i}^{h} \varphi_{j}^{k} &= R_{ij}, \\
P_{h;k} \varphi_{i}^{h} \varphi_{j}^{k} &= P_{i;j}.
\end{align*}
\]

Thus, by virtue of (2.1), (2.2), and Lemma, we have the following

**Theorem.** If a K-space satisfying \( R_{ij;k} = R_{i;j}v_{k} \) admits an analytic non-affine HP-transformation, it is an Einstein K-space.

The method of the proof is analogous to that in Kählerian manifold.

References


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