ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATION

By

Hidemaro KÔJYÔ

§0. **Introduction.** Recently, T. Ôtsuki and Y. Tashiro [1][1] have studied holomorphically projective correspondences of Kählerian manifolds.

On the other hand, K. Yano and T. Nagano [2] and T. Sumitomo [3] have studied infinitesimal projective transformations in a Riemannian manifold and obtained valuable results. Further, S. Tachibana and S. Ishihara [4] have considered analogous problems concerning the infinitesimal holomorphically projective transformations, which will be briefly called an *HP-transformation*, and obtained that a Kählerian manifold satisfying $R_{ij;k}=0$, which admits a non-trivial analytic HP-transformation reduces to an Einstein one.

The purpose of the present paper is to generalize more the above result of S. Tachibana and S. Ishihara, that is, we shall give a theorem about a Ricci-recurrent Kählerian manifold in §1 and one in a Ricci-recurrent K-space in §2, which is one of the generalization of the theorem in §1.

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§1. An analytic HP-transformation on a Kählerian manifold.

A vector field v^i is called an HP-transformation, if it satisfies

$$(1.1) \qquad \underset{v}{\mathfrak{L}}\left\{{}^{\,i}_{jk}\right\} = P_h(\delta^h_j\delta^i_k - \varphi^h_j\varphi^i_k) + P_h(\delta^h_k\delta^i_j - \varphi^h_k\varphi^i_j),$$

where P_h is a certain vector and φ^i_j is the complex structure, and semicolon and $\underset{v}{\mathfrak{L}}$ denote the covariant differentiation with respect to v^i and Lie differentiation with respect to v^i, respectively. We shall call P_h in (1.1) the associated vector of the HP-transformation. Contracting (1.1)

1) Numbers in brackets refer to the references at the end of the paper.

with respect to i and k, we get $P_h = \frac{1}{n+2} v^i{}_{;i;h}$, which shows that P_h is gradient.

An infinitesimal affine transformation v^i is defined by

$$\underset{v}{\mathfrak{L}}\left\{{}^{\,i}_{jk}\right\} \equiv v^i{}_{;j;k} + R^i_{jkl} v^l = 0.$$

If $P_h = 0$, then the HP-transformation reduces to an affine one.

A vector field v^i is called analytic on a Kählerian manifold, if it satisfies

$$(1.2) \qquad \underset{v}{\mathfrak{L}} \varphi^i_j = -\varphi^k_j v^i{}_{;k} + \varphi^i_k v^k{}_{;j} = 0.$$

We shall give here preliminary formulas on Kählerian manifold. Let our manifold be a real $n(=2m>2)$ dimensional Kählerian manifold with local coordinates $\{x^i\}$. Then the Riemannian metric g_{ij} and the complex structure φ^j_i satisfy

$$\varphi^k_i \varphi^j_k = -\delta^j_i, \qquad g_{hk}\varphi^h_i \varphi^k_j = g_{ij},$$
$$\varphi^j_{i;k} = 0, \qquad g_{ij;k} = 0.$$

Then the following equation holds:

$$(1.3) \qquad R_{hk}\varphi^h_i \varphi^k_j = R_{ij},$$

where R^i_{jkl} is the Riemannian curvature tensor, and

$$R^i_{jki} = R_{jk}, \qquad R^h_{jkl} g_{hi} = R_{ijkl}.$$

If P_h is the associated vector of an analytic HP-transformation, then we get

$$(1.4) \qquad P_{h;k}\varphi^h_i \varphi^k_j = P_{i;j}.$$

Moreover, if v^i be an analytic HP-transformation, then we have

Lemma. *Let v^i be an analytic HP-transformation, then the following relation holds:*

$$(1.5) \qquad (\underset{v}{\mathfrak{L}} g_{ik}) R^k_j = (\underset{v}{\mathfrak{L}} g_{jk}) R^k_i.$$

Proof. From the assumptions, it follows that

$$\underset{v}{\mathfrak{L}} \varphi^j_i = 0,$$

$$\underset{v}{\mathfrak{L}}\left\{{}^{\,i}_{jk}\right\} \equiv v^i{}_{;j;k} + R^i_{jkl} v^l = P_h(\delta^h_j \delta^i_k - \varphi^h_j \varphi^i_k) + P_h(\delta^h_k \delta^i_j - \varphi^h_k \varphi^i_j).$$

Since R_{ijkl} is anti-symmetric with respect to i and j, we get

$$(1.6) \qquad (\underset{v}{\mathfrak{L}} g_{ij})_{,k} \equiv (v_{i;j} + v_{j;i})_{;k} = 2P_k g_{ij} + P_j g_{ik} + P_i g_{jk} - P_a \varphi^a_j \varphi_{ki} - P_a \varphi^a_i \varphi_{kj}.$$

The integrable condition of the above equation is that

$$(\underset{v}{\mathfrak{L}} g_{aj}) R^a_{ikl} + (\underset{v}{\mathfrak{L}} g_{ia}) R^a_{jkl} = P_{j;k} g_{il} + P_{i;k} g_{jl} - P_{j;l} g_{ik} - P_{i;l} g_{jk}$$
$$+ \varphi^a_j (P_{a;l}\varphi_{ki} - P_{a;k}\varphi_{li}) + \varphi^a_i (P_{a;l}\varphi_{kj} - P_{a;k}\varphi_{lj}).$$

If we contract g^{jl} to this equation and take account of (1.4), then we have

$$(\underset{v}{\mathfrak{L}} g_{aj})R_{ik}^{a\ j} - (\underset{v}{\mathfrak{L}} g_{ia})R_k^a = nP_{i;k} - P_a{}^{;a} g_{ik}.$$

Since P_h is gradient and $(\underset{v}{\mathfrak{L}} g_{aj})R_{ik}^{a\ j}$ is symmetric with respect to i and k, we obtain the conclusion.

Recently S. Tachibana and S. Ishihara [4] obtained the following

Theorem. *If a Kählerian manifold satisfying $R_{ij;k}=0$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

We shall now consider a Ricci-recurrent Kählerian manifold, i.e., a Kählerian manifold such that $R_{ij;k}=R_{ij}v_k$, and we obtain the following

Theorem. *If a Kählerian manifold satisfying $R_{ij;k}=R_{ij}v_k$ admits an analytic non-affine HP-transformation, it is a Kähler-Einstein manifold.*

Proof. Covariantly differentiating (1.5) with respect to x^l and making use of (1.5), we find

$$(\underset{v}{\mathfrak{L}} g_{ia})_{;l} R_k^a = (\underset{v}{\mathfrak{L}} g_{ka})_{;l} R_i^a.$$

Substituting (1.6) into the last equation, we have easily

$$\begin{aligned}(P_a g_{il} + P_i g_{al} + 2P_l g_{ia} - \varphi_a^b \varphi_{li} P_b - \varphi_i^b \varphi_{la} P_b)R_k^a \\ = (P_a g_{kl} + P_k g_{al} + 2P_l g_{ka} - \varphi_a^b \varphi_{lk} P_b - \varphi_k^b \varphi_{la} P_b)R_i^a.\end{aligned}$$

Contracting this equation with g^{il} and R^{il}, and taking account of (1.3) and (1.4), we have

$$nP_a R_k^a = RP_k,$$
$$RR_k^a P_a = R_{ij}R^{ij}P_k.$$

From the above equations, we get

$$\left(R_{ij}R^{ij} - \frac{R^2}{n}\right)P_k = 0.$$

Since $P_k \neq 0$, we must have

$$R_{ij}R^{ij} - \frac{R^2}{n} = 0.$$

On the other hand, according to the theorem obtained by T. Sumitomo [3], a Riemannian manifold satisfying the relation $R_{ij}R^{ij}=\frac{R^2}{n}$ is an Einstein manifold. Therefore, we get the conclusion.

§2. An analytic HP-transformation in a K-space.

In this section, we shall consider only a K-space, which is another generalization of a Kählerian manifold.

If φ_{ij} ($\varphi_{ij} \overset{\text{def}}{=} \varphi_i^k g_{kj}$) is a Killing tensor, i.e., it satisfies the equation

$$\varphi_{ij;k} + \varphi_{ik;j} = 0,$$

an almost-Hermitian space is called a K-space. After some calculations we get also the following identities in a K-space:

$$R_{hk}\varphi_i^h \varphi_j^k = R_{ij}, \tag{2.1}$$

$$P_{h;k}\varphi_i^h \varphi_j^k = P_{i;j}. \tag{2.2}$$

Thus, by virtue of (2.1), (2.2), and Lemma, we have the following

Theorem. *If a K-space satisfying $R_{ij;k} = R_{ij}v_k$ admits an analytic non-affine HP-transformation, it is an Einstein K-space.*

The method of the proof is analogous to that in Kählerian manifold.

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Department of Mathematics,
Faculty of Science,
Hokkaido University, Japan.

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