POTENTIALS ON RIEMANN SURFACES

By
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The purpose of the present paper is to simplify and to extend the almost all theorems contained in the previous papers. The potential theory has been discussed in euclidean space. Recently it is discussed by many authors under weaker conditions of the kernel $K(x, y)$ of the potential $U(x) = \int K(x, y) d\mu(y)$ in more general space $S$ which is locally compact and homogeneous and $K(x, y)$ has not necessarily symmetry property, superharmonicity. On the other hand, the space in which we shall consider the potential $U(z) = \int N(z, p) d\mu(p)$ is a Riemann surface with ideal boundary $B$, which is locally euclidean in $R$ and locally compact in $R+B$ and $R+B$ is not homogeneous by the existence of $B$. The kernel $N(z, p)$ which will be defined is harmonic in $R$ and superharmonic in some sense in $R+R_1$ ($R_1$ is the part of $B$), $N(z, p)$ has symmetry, lower semicontinuity in $R+B$, $N(p, p) \leq \infty$. Further there may exist $B_0 = B - R_1$ where we cannot distribute any true mass. In the above sense our space is not so restricted. To construct potentials we make some preparations.

1. Let $R$ be a Riemann surface with positive boundary and let $R_n$ ($n=0,1,2,\cdots$) be its exhaustion with compact relative boundaries $\partial R_n$. Let $N_n(z, p)$ be a positive function in $R_n-R_0$ such that $N_n(z, p)$ is harmonic in $R_n-R_0$ except one point $p \in R_n-R_0$, $N_n(z, p) = 0$ on $\partial R_0$, $\frac{\partial}{\partial n} N_n(z, p) = 0$ on $\partial R_n$ and $N_n(z, p) + \log |z-p|$ is harmonic in a neighbourhood of $p$. We define the D-integral $D^*(N_{n+1}(z, p), N_n(z, p))$ of $N_n(z, p)$ over $R_n-R_0$ as follows:

Let $\nu_r(p)$ be a circular neighbourhood of $p$ with respect to the local parameter at $p$: $\nu_r(p) = E[z \in R : |z-p| < r]$. Put

$$D^*_{R_n-R_0-\nu_r(p)}(N_{n+1}(z, p), N_n(z, p)) = \int_{\partial R_n+\partial R_0} N_{n+1}(z, p) \frac{\partial}{\partial n} N_n(z, p) ds + \int_{\nu_r(p)} (N_{n+1}(z, p)$$

\[ + \log |z-p| \frac{\partial}{\partial n} N_n(z, p) \, ds = \int (N_{n+i}(z, p) + \log |z-p|) \frac{\partial}{\partial n} N_n(z, p) \, ds. \]

We define \( D_{R_n-R_0}^{*}(N_{n+i}(z, p), N_n(z, p)) \) as
\[
\lim_{r \to 0} D_{R_n-R_0-v_r(p)}^{*}(N_{n+i}(z, p), N_n(z, p)) = 2\pi U_{n+i}(p),
\]
where \( U_{n+i}(p) = \lim_{r \to 0} (N_{n+i}(z, p) + \log |z-p|) \).

Similarly
\[
D_{R_n-R_0}^{*}(N_n(z, p)) = 2\pi U_n(p).
\]

By the Green's formula
\[
D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p)) = \int_{\partial R_n} N_n(z, p) \frac{\partial}{\partial n} N_{n+i}(z, p) \, ds - \int_{\partial v_r(p)} (N_{n+i}(z, p) + \log |z-p|) \frac{\partial}{\partial n} N_{n+i}(z, p) \, ds
\]
\[+\int_{\partial R_n} (N_n(z, p) + \log |z-p|) \frac{\partial}{\partial n} N_{n+i}(z, p) \, ds \]
\[= D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p)) = 2\pi (U_n(p) - U_{n+i}(p)).
\]

Since \( N_n(z, p) - N_{n+i}(z, p) \) is harmonic at \( p \), \( \int \log |z-p| \frac{\partial}{\partial n} N_n(z, p) \, ds \to 0 \) as \( r \to 0 \). Hence
\[
D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p)) = \lim_{r \to 0} D_{R_n-R_0-v_r(p)}^{*}(N_n(z, p), N_{n+i}(z, p))
\]
\[= D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p)).
\]

By (1), (2), (3) and by \( D_{R_n-R_0}^{*}(N_{n+i}(z, p)) \geq D_{R_n-R_0}^{*}(N_n(z, p)) \) the \( D \)-integral \( D_{R_n-R_0}^{*}(N_{n+i}(z, p)) \) is given as follows:
\[
0 \leq D_{R_n-R_0}^{*}(N_n(z, p) - N_{n+i}(z, p)) = D_{R_n-R_0}^{*}(N_n(z, p)) - 2D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p))
\]
\[+ D_{R_n-R_0}^{*}(N_n(z, p)) < D_{R_n-R_0}^{*}(N_n(z, p)) - 2D_{R_n-R_0}^{*}(N_n(z, p), N_{n+i}(z, p))
\]
\[+ D_{R_n-R_0}^{*}(N_{n+i}(z, p)) = 2\pi (U_n(p) - U_{n+i}(p)).
\]

On the other hand, let \( G_n(z, p) \) be the Green's function of \( R_n - R_0 \) with pole at \( p \in R_0 - R_0 \). Then
\[
G_{n+1}(z, p) < N_{n+1}(z, p) \quad (j = 0, 1, 2, \cdots) \text{ for } n \geq n_0.
\]

This implies
\[ \lim_{j \to \infty} U_{n+j}(p) \geq \lim_{j \to \infty} (G_{n_0}(z, p) + \log |z - p|) = L > -\infty \ (j = 0, 1, 2, \cdots). \]

Hence \( U_n(p) \) is decreasing with respect to \( n \) and \( \lim_{n} U_n(p) \geq L \). Whence \( \{U_n(p)\} \) converges. Therefore \( D(N_{n+i}(z, p) - N_n(z, p)) \to 0 \), if \( n \) and \( i \to \infty \) or only \( n \to \infty \), which implies that \( N_n(z, p) \) converges in mean. Further \( N_n(z, p) = 0 \) on \( \partial R_0 \) yields that \( \{N_n(z, p)\} \) converges uniformly to a function \( N(z, p) \) as \( n \to \infty \). Clearly by the compactness of \( \partial R_0 \), \( \int_{\partial R_0} \frac{\partial}{\partial n} N(z, p) \, ds = 2\pi \). We call \( N(z, p) \) the \( N \)-Green's function of \( R - R_0 \) with pole at \( p \).

**Remark.** If \( R \) is a Riemann surface with null-boundary, we see that \( N(z, p) \) reduces to be the Green's function of \( R - R_0 \).

After R. S. Martin we shall define the ideal boundary points as follows. Let \( N(z, p) \) be the \( N \)-Green's function of \( R - R_0 \) with pole at \( p \). Consider now a sequence of points \( \{p_i\} \) of \( R - R_0 \) having no points of accumulation in \( R - R_0 + \partial R_0 \). Since the function \( N(z, p_i) \) \((i = 1, 2, \cdots)\) forms, from some \( i \) on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of \( R - R_0 \) to a positive harmonic function. A sequence \( \{p_i\} \) of \( R - R_0 \) having no point of accumulation in \( R - R_0 + \partial R_0 \), for which the corresponding \( N(z, p) \)'s have the property just mentioned, that is, \( \{N(z, p)\} \) converges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function \( N(z, p) \), we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of \( R \). The set of all the ideal boundary points of \( R \) will be denoted by \( B \) and the set \( R - R_0 + B \) by \( \overline{R} - R_0 \). The domain of definition of \( N(z, p) \) may now be extended by writing \( N(z, p) = \lim_{i \to \infty} N(z, p_i) \) \((z \in R - R_0, p \in B)\), where \( \{p_i\} \) is any fundamental sequence determining \( p \). The function \( N(z, p) \) is characteristic of the point \( p \) of their corresponding \( N(z, p) \) as a function of \( z \). The function \( \delta(p_1, p_2) \) of two points \( p_1 \) and \( p_2 \) in \( \overline{R} - R_0 \) is defined as

\[ \delta(p_1, p_2) = \sup_{z \in \overline{R} - R_0} \left| \frac{N(z, p_1)}{1 + N(z, p_1)} - \frac{N(z, p_2)}{1 + N(z, p_2)} \right|. \]

Evidently, \( \delta(p_1, p_2) = 0 \) is equivalent to \( N(z, p_1) = N(z, p_2) \) for all points \( z \) in \( R_1 - R_0 \). Therefore we have \( N(z, p) = N(z, p_2) \) in \( R - R_0 \), i.e. \( \delta(p_1, p_2) = 0 \) implies \( p_1 = p_2 \) and it is clear that \( \delta(p_1, p_2) \) satisfies the axioms of distance. Therefore \( \delta(p_1, p_2) \) can be considered as the distance between two points \( p_1 \) and \( p_2 \) of \( \overline{R} - R_0 \).
The topology (we call N-Martin’s topology) induced by this metric is homeomorphic to the original topology, when it is restricted in $R-R_0$. Since $N(z, p): p \in \overline{R-R_0}$ is also a normal family, both $(R-R_1)+\partial R_1+B$ and $B$ are closed and compact. For a fixed point $z$, $N(z, p)$ is continuous with respect to this metric (we denote it shortly by $\delta$-continuous) as a function of $p$ in $\overline{R-R_0}$ except at $p$.

2. Properties of the function $N(z, p)$.

Lemma 1. a). Let $G_1$ be a compact or non compact domain in $R-R_0$ containing another domain $G_2$. Let $U(z)$ be a function of $C_1$-class such that $D(U(z))$ is finitely minimal Dirichlet integral (we abbreviate it by M.D.I.) among all functions of $C_1$-class with the same boundary value on $\partial G_1$. Then $U(z)$ is also M.D.I. function in $G_2$ among all functions with the same boundary value as $U(z)$ on $\partial G_2$.

b). Let $G$ be a domain as a) and let $U(z)$ be a harmonic function with M.D.I. over $G$ with the boundary value $\varphi(z)$ on $\partial G$. Then $U(z)$ is uniquely determined and $U_n(z) \to U(z)$ (we denote by $\Rightarrow$ that $U_n(z)$ converges and converges in mean to $U(z)$), where $U_n(z)$ is a harmonic function in $R_n \cap G$ such that $U_n(z)=U(z)$ on $\partial G \cap R_n$ and $\frac{\partial}{\partial n} U_n(z)=0$ on $\partial R_n$. Whence

\[ \inf_{z \in \partial G} U(z) \leq U(z) \leq \sup_{z \in \partial G} U(z). \]

If $U(z) \neq \text{const}$, $\inf_{z \in \partial G} U(z) < U(z) < \sup_{z \in \partial G} U(z)$.

c). Let $G$ be a domain. The necessary and sufficient condition for a harmonic function $U(z)$ to have M.D.I. over $G$ among all functions with the value $U(z)$ on $\partial G$ is that $D(U(z), C(z))=0$ for every harmonic function $C(z)$ such that $C(z)=0$ on $\partial G$ and $D(C(z))<\infty$.

d). Let $U_n(z)$ $(n=1, 2, \cdots)$ be a harmonic function in $G \cap R_n$ with boundary value $\varphi_n(z)$ on $\partial G \cap R_n$ such that $U_n(z)$ has M.D.I. over $R_n \cap G$. If $U_n(z) \Rightarrow U(z)$, then $U(z)$ has M.D.I. over $G$ with boundary value $\varphi(z)=\lim_{n} \varphi_n(z)$ on $\partial G$.

Similarly let $U_n(z)$ be a harmonic function in $G$ with boundary value $\varphi_n(z)$ on $\partial G$ such that $U_n(z)$ has M.D.I. over $G$. If $U_n(z) \Rightarrow U(z)$, then $U(z)$ has M.D.I. over $G$ with boundary value $\varphi(z)=\lim_{n} \varphi_n(z)$ on $\partial G$.

Proof of a). Assume that there exists another function $B(z)$ of $C_1$-class such that $B(z)=U(z)$ on $\partial G_2$ and $D_{\partial_1}(B(z))<D_{\partial_1}(U(z))$. Put $U^*(z)=B(z)$

2) In the present paper, we suppose that $\partial G$ of a domain $G$ consists of at most enumerably infinite number of analytic curves clustering nowhere in $R$.

3) If $U(z)$ is continuous and has partial derivatives almost everywhere, we say that $U(z) \in C_1$-class.
in $G_{2}$ and $U^{*}(z)=U(z)$ in $G_{1}-G_{2}$. Then $D(U^{*}(z))<D(U(z))$. This contradicts that $U(z)$ has M.D.I. Hence we have a).

Proof of b). Let $U_{i}(z)$ ($i=1,2$) be a function of $C_{1}$-class such that $U_{i}(z)=\varphi(z)$ on $\partial G$ and has M.D.I. Then

$$D(U_{i}(z)+\varepsilon(U_{i}(z)-U_{z}(z))) \geq D(U(z))$$
for any $\varepsilon$.

By considering $\pm \varepsilon$ such that $|\varepsilon|$ is sufficiently small, we have

$$D(U_{i}(z), U_{i}(z)-U_{z}(z)) = 0 : i=1,2.$$ 
Hence $D(U_{i}(z)-U_{z}(z))=0$, i.e. $U_{1}(z)=U_{z}(z)$.

Let $U_{n}(z)$ ($n=1,2, \cdots$) be a harmonic function in b). Then

$D_{G-R_{n}}(U(z)-U_{n}(z), U_{n}(z)) = 0$,

whence

$D_{G-R_{n}}(U(z)) - D_{G-R_{n}}(U_{n}(z)) = D_{G-R_{n}}(U(z)-U_{n}(z)) \geq 0$.

and $D_{G-R_{n}}(U_{n}(z)) \uparrow L \leq D(U(z))$.

Similarly

$$0 \leq D_{G-R_{n+1}}(U_{n+1}(z)-U_{n}(z)) = D_{G-R_{n}}(U_{n+1}(z))-D_{G-R_{n}}(U_{n}(z)) \leq D_{G-R_{n+1}}(U_{n+1}(z))-D_{G-R_{n}}(U_{n}(z)) \rightarrow 0$$
as $n \rightarrow \infty$ and $i \rightarrow \infty$ by (4). This implies $U_{n}(z) \Rightarrow U^{*}(z)$.

Now by (4) and by Fatou's lemma

$$D_{G-R_{n}}(U^{*}(z)) \leq \lim_{m=\infty} D_{G-R_{n}}(U_{n}(z)) \leq D_{G}(U(z))$$
for $n \geq m$.

Let $m \rightarrow \infty$. Then $D_{G}(U^{*}(z)) \leq D_{G}(U(z))$. Thus $U^{*}(z)$ has M.D.I. By the assumption that $U(z)$ has M.D.I., we have $U^{*}(z)=U(z)$.

Let $U(z)$ be the function in b). Then by $\frac{\partial}{\partial n} U_{n}(z)=0$ on $\partial R_{n}$, $\inf_{z \in \partial G} U_{n}(z) \leq U(z) \leq \sup_{z \in \partial G} U(z)$ is clear by the maximum principle, whence $\inf_{z \in \partial G} U(z) \leq U(z) \leq \sup_{z \in \partial G} U(z)$. Suppose $U(z) \neq \text{const}$.

Then also by the maximum principle we have

$$\inf_{z \in \partial G} U(z) < U(z) < \sup_{z \in \partial G} U(z).$$

Proof of c). Suppose, $U(z)$ has M.D.I. Since $U(z)+\varepsilon C(z)=U(z)$ on $\partial G$,

$$D(U(z)+\varepsilon C(z)) = D(U(z)) + 2\varepsilon D(U(z), C(z)) + \varepsilon^{2} D(C(z)) \geq D(U(z))$$
for any $\varepsilon$. We see that $D(U(z), C(z))=0$ in considering $\varepsilon=\pm \gamma$ for $\varepsilon$ such that $|\gamma|$ is sufficiently small.

Conversely, assume $D(C(z), U(z))=0$. Let $U'(z)$ be a harmonic function such that $U'(z)=U(z)$ on $\partial G$ and $D(U'(z))<\infty$. Then by putting $C(z)=U(z)-U'(z)$, we have $D(U(z))=D(U(z), U'(z))$ and $D(U'(z)) \geq D(U(z))$. Now $U'(z)$
is any function. Hence $U(z)$ has M.D.I. Thus we have c).

**Proof of d).** At first we remark, by $U_{n}(z) \Rightarrow U(z)$,

$$D_{G}(U(z)) = \lim_{m} D_{G-R_{m}}(U(z)) \leqq \lim_{m} \left( \lim_{n} D_{R_{m}-R_{n}}(U_{n}(z)) \right) \leqq M.$$ 

Let $C(z)$ be a harmonic function in $G$ such that $D(C(z)) < \infty$ and $C(z) = 0$ on $\partial G$. Then

$$\left| D_{G-(R_{n}-R_{m})}(U_{n}(z), C(z)) \right| \leqq \sqrt{D_{G-(R_{n}-R_{m})}(U_{n}(z)) D_{G-(R_{n}-R_{m})}(C(z))} \leqq \sqrt{M} \sqrt{D_{G-(R-R_{m})}(C(z))} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \quad (5)$$

Hence for any given positive number $\varepsilon$, there exists a number $m_{0}$ such that

$$\left| D_{G-(R_{n}-R_{m})}(U_{n}(z), C(z)) \right| < \varepsilon \quad \text{for} \quad m \geqq m_{0}, \quad n \geqq m_{0}. \quad (6)$$

Since $U_{n}(z)$ has M.D.I. over $G \cap R_{n}$, by c) $D_{G-R_{n}}(U_{n}(z), C(z)) = 0$, we have by (6)

$$\left| D_{G-R_{m}}(U_{n}(z), C(z)) \right| < \varepsilon. \quad (7)$$

On the other hand, by $U_{n}(z) \Rightarrow U(z)$, for the same $\varepsilon$ and the above number $m$, there exists a number $n_{0} = n_{0}(m)$ such that

$$\left| D_{G-R_{m}}(U(z) - U_{n}(z), C(z)) \right| < \varepsilon, \quad \text{for} \quad n \geqq n_{0}. \quad (8)$$

because

$$\left| D_{G-R_{m}}(U(z) - U_{n}(z), C(z)) \right| \leqq \sqrt{D_{G-R_{m}}(U(z) - U_{n}(z)) D(C(z))}.$$

Thus

$$\left| D_{G-R_{m}}(U(z), C(z)) \right| \leqq \left| D_{G-R_{m}}(U(z) - U_{n}(z), C(z)) \right| + \left| D_{G-R_{m}}(U_{n}(z), C(z)) \right| < 2\varepsilon, \quad m > m_{0}, \quad n > n_{0}(m).$$

Hence

$$D_{G-R_{m}}(U(z), C(z)) \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.$$ 

Whence by c) $U(z)$ has M.D.I. over $G$ with value $\varphi(z) = \lim_{n} \varphi_{n}(z)$ on $\partial G$. The latter part is proved similarly.

**Theorem 1.** a). Let $N(z, p)$ be the $N$-Green's function of $R-R_{0}$ with pole $p$ in $R-R_{0}$. Let $G$ be a compact or non compact domain containing $p$. Then $N(z, p)$ has M.D.I. over $R-R_{0} - G$ among all functions with the same value as $N(z, p)$ on $\partial G + \partial R_{0}$, whence by b) of Lemma 1 $N(z, p)$

$$< \sup_{z \in \partial G} N(z, p) \quad \text{and} \quad \lim_{M=\infty} V_{M}(p) = p,$$

where $V_{M}(p) = E[z \in R : N(z, p) > M]$.

b). $N(z, p)$ satisfies

$$D(\min (M, N(z, p))) \leqq 2\pi M \quad \text{for} \quad p \in \overline{R-R_{0}}.$$ 

**Proof of a).** Let $G'$ be a compact domain with smooth boundary such that $G \supset G'$ and that $\partial G'$ is rectifiable. Since $N(z, p)$ is harmonic in $R-R_{0}$
$G, \max_{z \in G'} N(z, p) \leq L < \infty$. Let $N_n(z, p)$ be a harmonic function in $R_n - R_0$ such that $N_n(z, p) = 0$ on $\partial R_0$, $\frac{\partial}{\partial n} N_n(z, p) = 0$ on $\partial R_n$ and $N_n(z, p)$ has a logarithmic singularity at $p$. Then $N_n(z, p) \Rightarrow N(z, p)$. Hence by the compactness of $\partial G'$ there exists a number $n_0$ such that $N_n(z, p) < L + \varepsilon$ on $\partial G'$ for $n \geq n_0$, for any given positive number $\varepsilon$. Let $V_{L+\varepsilon}^{n}(p) = E[\{z \in R: N_n(z, p) > L + \varepsilon\}]$. Then by the maximum principle $G' \supset V_{L+\varepsilon}^{n}(p)$ for $n \geq n_0$, because $\frac{\partial}{\partial n} N_n(z, p) = 0$ on $\partial R_n$. Since $N_n(z, p)$ is harmonic in $R_n - R_0 - V_{L+\varepsilon}^{n}(p)$ with continuous normal derivative on $\partial V_{L+\varepsilon}^{n}(p)$, Dirichlet integral of $N_n(z, p)$ over $R_n - R_0 - V_{L+\varepsilon}^{n}(p)$ is finite. Hence there exists a number $n_0$ such that $N_n(z, p) < L + \varepsilon$ on $\partial G'$ for $n \geq n_0$. Let $U(z)$ be a harmonic function in $R_n - R_0 - V_{L+\varepsilon}^{n}(p)$ such that $U(z) = N_n(z, p)$ on $\partial R_0 + \partial V_{L+\varepsilon}^{n}(p) + \partial R_n$ and the Dirichlet integral of $U(z)$ is finite. Now

$$D_{R_n - R_0 - V_{L+\varepsilon}^{n}(p)}(N_n(z, p), N_n(z, p) - U(z)) = 0.$$ 

But $U(z)$ is arbitrary, hence $N_n(z, p)$ has M.D.I. over $R_n - R_0 - V_{L+\varepsilon}^{n}(p)$ and

$$D_{R_n - R_0 - V_{L+\varepsilon}^{n}(p)}(N_n(z, p)) = \int_{\partial V_{L+\varepsilon}^{n}(p)} N_n(z, p) \frac{\partial}{\partial n} N_n(z, p) ds = (L + \varepsilon) \int_{\partial R_0} \frac{\partial}{\partial n} N_n(z, p) ds = 2\pi(L + \varepsilon).$$

Hence by Lemma 1. a) $N_n(z, p)$ has M.D.I. ($\leq 2\pi(L + \varepsilon)$) over $R_n - R_0 - G' \subset R_n - R_0 - G'$ and over $R - R_0 - V_{L+\varepsilon}^{n}(p)$ for $n \geq n_0$ by $G' \supset G' \supset V_{L+\varepsilon}^{n}(p)$. On the other hand, $N_n(z, p) \Rightarrow N(z, p)$. This implies by Lemma 1. c) that $N(z, p)$ has also M.D.I. over $R - R_0 - G'$ and over $R - R_0 - G$, which is clearly $\leq 2\pi(L + \varepsilon)$. Next by Lemma 1. b)

$$N(z, p) \leq \sup_{z \in R - R_0 - G'} N(z, p) \leq L,$$

i.e. $E[\{z \in R: N(z, p) > L\}] = V_{L}(p) \subset G'$. Now $G'$ is arbitrary. Hence

$$\lim_{M \to \infty} V_M(p) = p.$$ 

Proof of b). Case 1. $p \in R - R_0$. Since $R_m - V_M(p) - R_0$ is compact for sufficiently large number $M$, by $\lim_{M \to \infty} V_M(p) = p$, for any given positive number $\varepsilon$ and a number $m$, we can find a number $n_0 = n_0(\varepsilon, m)$ such that

$$R_m - R_0 - V_M(p) \subset E[\{z \in R: N_n(z, p) < M + \varepsilon\}] \quad \text{for } n \geq n_0.$$ 

On the other hand, by Factou's lemma

$$D_{R_m - R_0}(\min(M, N(z, p))) \leq \lim_{n \to \infty} D(\min(M + \varepsilon, N_n(z, p))) \leq 2\pi(M + \varepsilon).$$

Let $\varepsilon \to 0$ and then $m \to \infty$. Then

$$D_{R - R_0}(\min(M, N(z, p))) \leq 2\pi M.$$
Case 2. \( p \in B \). Let \( \{p_i\} \) be a fundamental sequence determining \( p \) and let \( V_M(p_i) = E[z \in R : N(z, p_i) > M] \). Then by case 1, \( D_{R - R_0 - V_M(p_i)}(N(z, p_i)) \leq 2\pi M \) for every \( M \). On the other hand, \( N(z, p_i) \to N(z, p) \). Hence by the same manner as in case 1, we have by Fatou's lemma

\[
D_{R - R_0}(\min(M, N(z, p))) \leq 2\pi M.
\]

3. Harmonic measure (H.M.) and capacities (C.P.) of the ideal boundary \((B \cap G_2)\) determined by a domain \( G_2 \) with respect to a domain \( G_1, G_2 \subset G_1 \).

So far as we discuss the ideal boundary, without loss of generality, we can suppose that non compact domains have no intersection with \( R_0 \). In the following we assume \( G_1 \cap R_0 = \emptyset \). Let \( w_{n, n+i}(z) = \omega_{n, n+i}(z) = 1 \) in \( G_2 \cap (R_{n+i} - R_n) \) and is harmonic in \( \Omega_{n, n+i} = (G_1 \cap R_{n+i}) - (G_2 \cap (R_{n+i} - R_n)) \), \( w_{n, n+i}(z) = \omega_{n, n+i}(z) = 0 \) on \( \partial G_1 \cap R_{n+i}, w_{n, n+i}(z) = \frac{\partial}{\partial n} \omega_{n, n+i}(z) = 0 \) on \( \partial R_{n+i} \cap (G_1 - G_2) \). Then by the maximum principle \( w_{n, n+i}(z) \uparrow w_n(z) \) as \( i \to \infty \) and \( w_n(z) \downarrow w(z) \) as \( n \to \infty \). We call \( w(z) \) the harmonic measure H.M. of the ideal boundary \((G_2 \cap B)\) determined by \( G_2 \) relative \( G_1 \). We denote it by \( w(G_2 \cap B, z, G_1) \).

If there exists a constant \( M \) and a number \( n_0 \) such that

\[
D_{\Omega_{n, n+i}}(w_{n, n+i}(z)) \leq M
\]

for every \( n \geq n_0 \) and \( i \geq 0 \), then \( w_{n, n+i}(z) \Rightarrow w_n(z) \) as \( i \to \infty \) and \( w_n(z) \Rightarrow w(z) \) as \( n \to \infty \).

In fact, \( D_{\Omega_{n, n+i}}(w_{n, n+i}(z), w_{n, n+i+j}(z)) = \int_{\partial \Omega_{n, n+i}} \omega_{n, n+i+j}(z) \frac{\partial}{\partial n} \omega_{n, n+i}(z) ds \)

\[
= \int_{\partial \Omega_{n, n+i}} \frac{\partial}{\partial n} \omega_{n, n+i}(z) ds = D_{\Omega_{n, n+i}}(w_{n, n+i}(z)),
\]

whence

\[
0 \leq D_{\Omega_{n, n+i}}(w_{n, n+i}(z), w_{n, n+i+j}(z)) = D_{\Omega_{n, n+i}}(w_{n, n+i}(z)) - D_{\Omega_{n, n+i}}(w_{n, n+i+j}(z)) \\
\leq D_{\Omega_{n, n+i}}(w_{n, n+i+j}(z), w_{n, n+i}(z)) - D_{\Omega_{n, n+i}}(w_{n, n+i}(z)) = 0
\]

Whence \( D_{\Omega_{n, n+i}}(w_{n, n+i}(z)) \uparrow \) as \( i \to \infty \). But \( \leq M \). Hence

\[
D_{\Omega_{n, n+i}}(w_{n, n+i+j}(z), w_{n, n+i}(z)) \leq D_{\Omega_{n, n+i}}(w_{n, n+i+j}(z)) - D_{\Omega_{n, n+i}}(w_{n, n+i}(z)) \downarrow 0 \text{ as } i \to \infty.
\]

Thus \( w_{n, n+i}(z) \Rightarrow w_n(z) \) as \( i \to \infty \).

Next similarly,

\[
D_{\Omega_{n, n+i+j}}(w_{n, n+i+j}(z), w_{n+i, n+i+j}(z)) = \int_{\partial \Omega_{n, n+i+j}} \omega_{n+i, n+i+j}(z) \frac{\partial}{\partial n} \omega_{n+i, n+i+j}(z) ds
\]
\[= \int_{\partial((R_{n+i} - R_n) \cap G_2)} \frac{\partial}{\partial n} \omega_{n+i, n+i+j}(z) \, ds \]
\[= D_{\Omega_{n+i, n+i+j}}(\omega_{n+i, n+i+j}(z)) . \tag{10} \]

Whence
\[
D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z)-\omega_{n+i,n+i+j}(z)) = D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z)) + D_{\Omega_{n,n+i+j}}(\omega_{n+i,n+i+j}(z)) - 2D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z), \omega_{n+i,n+i+j}(z)).
\]

Hence by (10)
\[
0 \leq D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z)-\omega_{n+i,n+i+j}(z)) \leq D_{\Omega_{n,n+i+j}}(\omega_{n,n+i+j}(z)) - D_{\Omega_{n+i,n+i+j}}(\omega_{n+i,n+i+j}(z)) .
\]

Let \( j \to \infty \). Then by (9)
\[
0 \leq D_{\Omega_{n}}(\omega_{n}(z)-\omega_{n+i}(z)) \leq D_{\Omega_{n}}(\omega_{n}(z)) - D_{\Omega_{n+i}}(\omega_{n+i}(z)) ,
\]
where \( \Omega_{n} = \lim_{i} \Omega_{n,n+i} = G_1 - (G_2 \cap (R-R_n)) \).

Hence
\[
D_{\Omega_{n}}(\omega_{n}(z)) \downarrow \geq 0 \quad \text{and} \quad D(\omega_{n}(z)-\omega_{n+i}(z)) \to 0 \quad \text{as} \quad n \to \infty .
\]

Thus \( \omega_{n}(z) \to \omega(z) \). We call \( D(\omega(z)) \) and \( \omega(z) \) the capacity and the capacitary potential C.P. of \((B \cap G_2)\) relative \(G_1\), and denote it by \( \omega(B \cap G_2, z, G_1) \).

Let \( G_i \) \((i=3,4,\cdots)\) be non compact domains \( \subset G_1 \). We consider H.M. and C.P. of \( G_{i_1} \cap G_{i_2} \cdots \cap B \). If \( G_{i_1} \cap G_{i_2} = 0 \), we define \( w(G_{i_1} \cap G_{i_2} \cap B, z, G_1) = w(G_{i_1} \cap G_{i_2} \cap B, z, G_1) = 0 \) \( i \neq j \). We shall prove the following

**Theorem 2.** Let \( G' \) be a compact or non compact domain such that \( G' \subset G_1 \), and \( G' \cap G_2 \cap (R-R_n) = 0 \) for a certain number \( n \).

**P.H.1.** Let \( V(z) \) be the non negatively least harmonic function in \( G' \) such that \( V(z) = w(B \cap G_2, z, G_1) \) on \( \partial G' \). Then \( V(z) = w(B \cap G_2, z, G_1) \) in \( G' \).

**P.C.1.** Let \( V(z) \) be a harmonic function in \( G' \) such that \( V(z) = w(B \cap G_2, z, G_1) \) on \( \partial G' \) and \( V(z) \) has M.D.I. over \( G' \). Then \( V(z) = w(B \cap G_2, z, G_1) \) in \( G' \).

**P.H.2.** \( w(B \cap G_2, z, G_1) > 0 \) implies \( \sup_{z \in (G_2 \cap (R-R_n))} w(B \cap G_2, z, G_1) = 1 \) for every \( n \).

**P.C.2.** \( w(B \cap G_2, z, G_1) > 0 \) implies \( \sup_{z \in (G_2 \cap (R-R_n))} w(B \cap G_2, z, G_1) = 1 \) for every \( n \).

**P.H.3.** \( w(B \cap G_2 \cap G_3, z, G_1) = 0 \) for \( G_3 = E[\{z \in R: w(B \cap G_2, z, G_1) < 1-\delta\}] \), \( 1 > \delta > 0 \).

**P.C.3.** \( w(B \cap G_2 \cap G_3, z, G_1) = 0 \) for \( G_3 = E[\{z \in R: w(B \cap G_2, z, G_1) < 1-\delta\}] \), \( 1 > \delta > 0 \).

We define H.M.(C.P.) for a set \( K \) in \( G_1 \) denoted by \( w(K, z, G_1) (\omega(K, z, G_1)) \) such that \( w(K, z, G_1) (\omega(K, z, G_1)) \) is harmonic in \( G_1 - K \) and \( w(K, z, G_1) = \omega(K, z, G_1) = 0 \) on \( \partial G_1 \) and \( w(K, z, G_1) = \omega(K, z, G_1) = 1 \) on \( K \) except a set of
capacity zero and non negatively least (has finitely M.D.I.). Then

P.H.4. \((1-\delta)w(G'_{i}, z, G)=w(B \cap G_{2}, z, G_{1}) \text{ in } G_{1}-G'_{i} \text{ for } G'_{i}=E[z \in G_{1}: w(B \cap G_{2}, z, G_{1})>1-\delta].\)

P.C.4. \((1-\delta)\omega(G'_{i}, z, G_{i})=\omega(B \cap G_{2}, z, G_{1}) \text{ in } G_{1}-G'_{i} \text{ for } G'_{i}=E[z \in G_{1}: \omega(B \cap G_{2}, z, G_{1})>1-\delta].\)

Let \(G_{k} (k=3,4, \cdots)\) be compact or non compact domains in \(G_{1}.\) Then

P.H.5. \(\sum_{k}w(B \cap G_{k}, z, G_{1}) \geqq w(B \cap \sum_{k}G_{k}, z, G_{1}).\)

P.C.5. \(\sum_{k}\omega(B \cap G_{k}, z, G_{1}) \geqq \omega(B \cap \sum_{k}G_{k}, z, G_{1}).\)

P.H.6. \(w(B \cap G_{2} \cap G'_{i}, z, G_{1})=w(B \cap G_{2}, z, G_{1}) ; G'_{i} \text{ is the domain in P.H.4.}\)

P.C.6. \(\omega(B \cap G_{2} \cap G'_{i}, z, G_{1})=\omega(B \cap G_{2}, z, G_{1}) ; G'_{i} \text{ is the domain in P.C.4.}\)

P.C.7. \(w(B \cap G_{2}, z, G_{1})>0, \text{ there exists an exceptional set } E \text{ of measure zero in the interval } (0,1) \text{ such that if } L \in E, \text{ then the niveau curve}^{4)} C_{L}=E[z \in G_{1} : \omega(B \cap G_{2}, z, G_{1})=L] \text{ has the following property}\)

\[
\int_{c_{L}}\frac{\partial}{\partial n}\omega(B \cap G_{2}, z, G_{1}) \, ds = D_{G_{1}}(\omega(B \cap G_{2}, z, G_{1})).
\]

**Proof of P.H.1.** Since \(w_{n, n+i}(z) = 0 \text{ on } G' \cap \partial R_{n+i},\)

\[
w_{n, n+i}(z) = \frac{1}{2\pi} \int_{\partial G' \cap R_{n+i}} w_{n, n+i}(\zeta) \frac{\partial}{\partial n} G_{n+i}(\zeta, z) \, ds,
\]

where \(G_{n+i}(\zeta, z)\) is the Green’s function of \(G' \cap R_{n+i}.\)

Since

\[
0 \leqq \frac{\partial}{\partial n} G_{n+i}(\zeta, z) \uparrow \frac{\partial}{\partial n} G(\zeta, z) \text{ and } w_{n, n+i}(z) \uparrow w_{n}(z) \text{ on } \partial G', \text{ we have by Lebesgue's theorem}\]

\[
w_{n}(z) = \frac{1}{2\pi} \int_{\partial G'} w_{n}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) \, ds,
\]

where \(G(\zeta, z)\) is the Green’s function of \(G'.\)

Next similarly \(w_{n}(z) \downarrow w(z)\) and

\[
w(z) = \lim_{n \to \infty} w_{n}(z) = \lim_{n \to \infty} \frac{1}{2\pi} \int_{\partial G'} w_{n}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) \, ds = \frac{1}{2\pi} \int_{\partial G'} \lim_{n \to \infty} w_{n}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) \, ds
\]

\[
= \frac{1}{2\pi} \int_{\partial G'} w(\zeta) \frac{\partial}{\partial n} G(\zeta, z) \, ds. \quad (11)
\]

On the other hand, clearly \(V(z) = \lim_{i \to \infty} V_i(z), \text{ where } V_i(z) \text{ is a harmonic function in } G' \cap R_{n+i} \text{ such that } V_i(z) = w(z) \text{ on } \partial G' \cap R_{n+i} \text{ and } V_i(z) = 0 \text{ on } \partial R_{n+i} \cap G'. \text{ Thus as above by (11)}

^{4)} \text{ We call a niveau curve with the property: } \int_{c} \frac{\partial}{\partial n} \omega(z) \, ds = D(\omega(z)) \text{ a regular niveau curve.}
\[
V(z) = \lim_{i=\infty} V_i(z) = \frac{1}{2\pi i} \lim_{i=\infty} \int_{\partial G_i} w(\zeta) \frac{\partial}{\partial n} G_{n+i}(\zeta, z) d\zeta = \frac{1}{2\pi} \int_{\partial G'} w(\zeta) \frac{\partial}{\partial n} G(\zeta, z) d\zeta = w(z).
\]

Hence we have P.H.1.

**Proof of P.C.1.** Let \( \omega_{n,n+i}(z) \) be the function in the definition of C.P.

Now
\[
\Omega_{n,n+i} \supset G' \quad \text{for} \quad n \geq n_0, \quad \text{whence} \quad G' \cap \Omega_{n,n+i} = G' \cap R_{n+i} \quad \text{for} \quad n \geq n_0.
\]

Since \( \frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0 \) on \( \partial R_{n+i} \cap G' \) and \( D(\omega_{n,n+i}(z)) \leq M \),
\[
D_{G' \cap R_{n+i}}(\omega_{n,n+i}(z), C(z)) = 0,
\]
for any harmonic function \( C(z) \) such that \( C(z) = 0 \) on \( \partial G' \) and \( D(C(z)) < \infty \).

Hence by Lemma 1.e), \( \omega_{n,n+i}(z) \) has M.D.I. over \( G' \cap R_{n+i} \). Now \( \omega_{n,n+i}(z) \implies \omega_n(z) \) implies by Lemma 1.d) that \( \omega_n(z) \) has M.D.I. over \( G' \). Also by \( \omega_n(z) \implies \omega(z) \) and by the latter part of Lemma 1.d) \( \omega(z) \) has M.D.I. over \( G' \). Thus we have P.C.1.

**Proof of P.H.2.** As \( G' \) in the proof of P.H.1 take \( G' = G_1 - ((G_2 \cap R_n) \cap R_n) = 0. \) Put \( w(z) = (G_2 \cap B, z, G_1) \). Fix \( n \) at present. Then by P.H.1.
\[
w(z) = \lim F_i(z),
\]
where \( V_i(z) \) is a harmonic function in \( G' \cap R_{n+i} \) such that \( V_i(z) = 0 = w_{n,n+i}(z) \) on \( \partial G_1 \cap R_{n+i} \), \( V_i(z) = \omega_{n,n+i}(z) = 0 \) on \( \partial R_{n+i} \cap (G_1 - G') \) and \( V_i(z) = w(z) \leq w_{n,n+i}(z) = 1 \) on \( \partial ((R_{n+i} - R_n) \cap G_2) \).

Assume \( \sup w(z) \leq K < 1 \) in \( G_1 \cap (R - R_n) \). Then by the maximum principle \( V_i(z) \leq Kw_{n,n+i}(z) \). Let \( i \to \infty \) and then \( n \to \infty \). Then
\[
w(z) \leq Kw(z).
\]
Hence \( w(z) = 0 \). This is a contradiction. Thus we have P.H.2.

**Proof of P.C.2.** Put \( \omega(z) = \omega(G_2 \cap B, z, G_1) \). Assume \( \omega(z) \leq K < 1 \) in \( G_2 \cap (R - R_n) \). \( \omega(z) \) has M.D.I. over \( G' = G_1 - (G_2 \cap R_n) \). Hence by Lemma 1.b)
\[
\omega(z) = \lim_{i=\infty} U_i(z),
\]
where \( U_i(z) \) is a harmonic function in \( G' \cap R_{n+i} \) such that \( U_i(z) = 0 = \omega_{n,n+i}(z) \) on \( \partial G_1 \cap R_{n+i} \), \( U_i(z) = \omega(z) \leq \omega_{n,n+i}(z) = 1 \) on \( \partial ((R_{n+i} - R_n) \cap G_2) \) and \( \frac{\partial}{\partial n} U_i(z) = 0 \) on \( \partial R_{n+i} \cap (G_1 - G') \). Hence by the maximum principle and by the assumption \( U_i(z) \leq K\omega_{n,n+i}(z) \).

Let \( i \to \infty \) and then \( n \to \infty \). Then \( \omega(z) \leq Kw(z) \).
Hence we have P.C.2.
Proof of P.H.3. (P.C.3). H.M. (C.P.) of \((G_2 \cap B) \uparrow \text{as} \ G_2 \uparrow\). Hence
\[
\omega(G_2 \cap G_3 \cap B, z, G_1) \leq \omega(G_2 \cap B, z, G_1) \leq \omega(G_2 \cap B, z, G_1)
\]
and
\[
\sup_{x \in (G_2 \cap G_3)} \omega(G_2 \cap B, z, G_1) \leq \sup_{x \in G_3} \omega(G_2 \cap B, z, G_1) \leq 1 - \delta.
\]
Hence
\[
\omega(G_2 \cap G_3 \cap B, z, G_1) \leq \omega(G_2 \cap B, z, G_1) \leq \omega(G_2 \cap G_3 \cap B, z, G_1) \leq 1 - \delta.
\]
Whence by P.H.2 (P.C.2)
\[
w(G_2 \cap G_3 \cap B, z, G_1) = 0 \quad \text{and} \quad \omega(G_2 \cap G_3 \cap B, z, G_1) = 0.
\]

Proof of P.H.5 (P.C.5). Let \(\omega_n, n+i(z)\) be a harmonic function in
\[
G_1 \setminus \left(G_k \cap (R_{n+i} - R_n)\right)
\]
such that \(\omega_n, n+i(z) = 0\) on \(\partial G_1\), \(\omega_n, n+i(z) = 1\) on \(\partial((G_1 \cap G_k) \cap (R_{n+i} - R_n))\) and \(\frac{\partial}{\partial n} \omega_n, n+i(z) = 0\) on \(\partial R_{n+i} \cap (G_1 - G_k)\). Let \(\omega_{n, n+i}(z)\) be a harmonic function in
\[
G_1 \setminus \left(\sum_k G_k \cap (R_{n+i} - R_n)\right)
\]
such that \(\omega_{n, n+i}(z) = 0\) on \(\partial G_1\), \(\omega_{n, n+i}(z) = 1\) on \(\partial(G_1 - \sum_k G_k) \cap (R_{n+i} - R_n)\) and \(\frac{\partial}{\partial n} \omega_{n, n+i}(z) = 0\) on \(\partial R_n \setminus (G_1 - \sum_k G_k)\).

Then by the maximum principle
\[
\sum_k \omega_{n, n+i}(z) \geq \omega_{n, n+i}(z).
\]
Let \(i \to \infty\) and then \(n \to \infty\). Then
\[
\sum_k \omega(G_k \cap B, z, G_1) \geq \omega(G_k \cap G_3 \cap B, z, G_1).
\]
Similarly we have P.H.5.

Proof of 6). \(G_2 \supset (G_2 \cap G_3)\) and \(G_2 = (G_2 \cap \overline{G_3})^{5)} + (G_2 \cap G_3)\). Hence by P.C.5 \(\omega(B \cap G_2, z, G_1) + \omega(B \cap G_2 \cap \overline{G_3}, z, G_1) = \omega(B \cap G_2, z, G_1) \geq \omega(B \cap G_2 \cap G_3, z, G_1)\). But by P.C.2
\[
\omega(B \cap G_2 \cap \overline{G_3}, z, G_1) = 0, \quad \text{whence} \quad \omega(B \cap G_2 \cap G_3, z, G_1) = 0.
\]
Similarly we have P.H.6.

Proof of P.C.7. Let \(\omega_n, n+i(z)\) be the function in the definition of \(\omega(G_2 \cap B, z, G_1)\). Put \(\Omega^L = E[z \in G_1 : \omega(B \cap G_2, z, G_1) < L]\) and \(\Omega^L_{n, n+i} = E[z \in G_1 : \omega_n, n+i(z) < L] : 0 < L < 1\) respectively. Let \(\Omega\) be a domain completely contained in \(\Omega^L\). Since \(\omega_n, n+i(z) \Rightarrow \omega_n(z)\) and \(\omega_n(z) \Rightarrow \omega(B \cap G_2, z, G_1)\), there exist numbers \(n_0\) and \(i_0\) for any given number \(m\) such that
\[
(R_m \cap \Omega') \subset \Omega^L_{n_0, n+i_0} \quad \text{for} \quad n \geq n_0 \quad \text{and} \quad i \geq i_0(i_0(n_0))
\]
• Then by Fatou’s lemma
\[
D_{\Omega^L} \omega_n, n+i(z) \leq \liminf_{n \to \infty} D_{\Omega^L} \omega_n, n+i(z) \leq \liminf_{n \to \infty} D_{\Omega^L} \omega_n, n+i(z).
\]
5) \(\overline{G}\) means the closure of \(G\) with respect to N-Martin’s topology.
\begin{align*}
&\leq \lim_{n} \lim_{t} D_{\Omega^{t}}(\omega_{n,n+i}(z)) = \lim_{n} \lim_{t} \int_{\partial \Omega^{t}} \omega_{n,n+i}(z) \frac{\partial}{\partial n} \omega_{n,n+i}(z) ds = \lim_{n} \lim_{t} L \int_{\partial \Omega^{t}} \omega_{n,n+i}(z) ds = L \lim_{n} \lim_{t} D_{\Omega^{t}}(\omega_{n,n+i}(z)) \\
&= LD_{G_{1}}(\omega(B \cap G_{2}, z, G_{1})),
\end{align*}

because \( \omega_{n,n+i}(z) \to \omega_{n}(z) \) and \( \omega_{n}(z) \to \omega(B \cap G_{2}, z, G_{1}) \).

Let \( m \to \infty \) and then \( n \to \infty \). Then let \( \Omega^{t} \uparrow \Omega^{L} \), then
\[ D_{\Omega^{L}}(\omega(B \cap G_{2}, z, G_{1})) \leq LD_{G_{1}}(\omega(G_{2} \cap B, z, G_{1})). \]

Similarly
\[ D_{G_{1} \cap \Omega^{L}}(\omega(B \cap G_{2}, z, G_{1})) \leq (1-L)D_{G_{1}}(\omega(B \cap G_{2}, z, G_{1})). \]

On the other hand,
\begin{align*}
D_{G_{1}}(\omega(B \cap G_{2}, z, G_{1})) &= D_{\Omega^{L}}(\omega(B \cap G_{2}, z, G_{1})) + D_{G_{1} \cap \Omega^{L}}(\omega(B \cap G_{2}, z, G_{1})).
\end{align*}

Hence
\[ D_{\Omega^{L}}(\omega(B \cap G_{2}, z, G_{1})) = LD_{G_{1}}(\omega(B \cap G_{2}, z, G_{1})) \quad \text{for} \quad 1 > L > 0. \quad (12) \]

Let \( \omega'(z) \) be a harmonic function in \( \Omega^{L} \cap R_{n} \) such that \( \omega'(z) = L \) on \( C_{L} = E[z \in G_{1} : \omega(B \cap G_{2}, z, G_{1}) = L] \), \( \frac{\partial}{\partial n} \omega'(z) = 0 \) on \( \Omega^{L} \cap \partial R_{n} \) and \( \omega'(z) = 0 \) on \( \partial G_{1} \). Since \( \omega(B \cap G_{2}, z, G_{1}) \) has M.D.I. over \( \Omega^{L} \), by lemma 1.b) \( \omega'(z) \to \omega(z) \) on \( \partial G_{1} \) and by (12)
\[ \lim_{n} \int_{C_{L}} \frac{\partial}{\partial n} \omega(z) ds \leq \lim_{n} \int_{C_{L} \cap R_{n}} \frac{\partial}{\partial n} \omega'(z) ds = \frac{1}{L} \lim_{n} D_{\Omega^{L} \cap R_{n}}(\omega'(z)) = \frac{1}{L} D_{G_{1}}(\omega(z)) = D_{G_{1}}(\omega(z)). \]

Thus
\[ A_{L} = \int_{C_{L}} \frac{\partial}{\partial n} \omega(z) ds \leq D_{G_{1}}(\omega(z)) \quad \text{for every niveau curve} \ C_{L}. \quad (13) \]

Now we can take \( p+iq = \omega(z) + i\tilde{\omega}(z) \) as the local parameter of every point \( z \) in \( G_{1} \) except at most enumerably infinite number of branch points of \( p+iq \), where \( \tilde{\omega}(z) \) is the conjugate function of \( \omega(z) \). Then \( \frac{\partial}{\partial p} \omega(z) = 1, \quad \frac{\partial}{\partial q} \omega(z) = 0 \) and
$$D_{G_{1}}(\omega(z))=\int_{G_{1}}\int\left\{ \left(\frac{\partial}{\partial p}\omega(z)\right)^{2}+\left(\frac{\partial}{\partial q}\omega(z)\right)^{2}\right\} dp\, dq=\int_{0}^{1}\left[\int_{c_{p}}dq\right]dp=\int_{0}^{1}A_{p}dp,$$

(14)
because $dq=d\tilde{\omega}=\frac{\partial}{\hat{\partial}os}\tilde{\omega}ds=\frac{\partial}{\partial n}\omega ds$,
where $ds$ is the line element along $C_{p}$.

Hence by (14) and (13)

$$A_{L}=D_{G_{1}}(\omega(B\cap G_{2}, z, G_{1}))$$

for almost $L$.

Thus we have P.C.6.

Theorem 3. a). Let $C_{j}(j=1,2)$ be regular niveau curve of C.P. $\omega(z)$ ($=\omega(B\cap G_{2}, z, G_{1})$) such that $\omega(z)=L_{j}$; $0<L_{1}<L_{2}<1$ and $\int_{\partial G_{1}}\frac{\partial}{\partial n}\omega(z)ds=D(\omega(z))$.

Since $\omega(z)$ has M.D.I. over $\Omega=E[z\in G_{1}: L_{1}<\omega(z)<L_{2}]$, $\omega_{n}^\prime(z)\rightarrow\omega(z)$ as $n\rightarrow\infty$, where $\omega_{n}^\prime(z)$ is a harmonic function in $\Omega\cap R_{n}$ such that $\omega_{n}^\prime(z)=\omega(z)$ on $\partial\Omega\cap R_{n}$ and $\frac{\partial}{\partial n}\omega_{n}^\prime(z)=0$ on $\Omega\cap\partial R_{n}$. Let $A_{n}(z)$ be a continuos function on $C_{L_{j}}$ such that $A_{n}(z)\rightarrow A(z)$ as $n\rightarrow\infty$ and $M\geqq A_{n}(z)\geqq 0$ for every $n$. Then

$$\int_{C_{L_{j}}}A(z)\frac{\partial}{\partial n}\omega(z)ds=\lim_{n\rightarrow\infty}\int_{C_{L_{j}}}A_{n}(z)\frac{\partial}{\partial n}\omega_{n}^\prime(z)ds.$$

b). Let $U_{n}(z)$ be a harmonic function in $\Omega\cap R_{n}$ such that $U_{n}(z)=0$ on $C_{L_{1}}$, $0\leqq U_{n}(z)\leqq N$ on $C_{L_{2}}$ and $\frac{\partial}{\partial n}U_{n}(z)=0$ on $\Omega\cap\partial R_{n}$. If $U_{n}(z)\rightarrow U(z)$, then

$$\int_{C_{L_{1}}}A(z)\frac{\partial}{\partial n}U(z)ds=\lim_{n\rightarrow\infty}\int_{C_{L_{1}}}A_{n}(z)\frac{\partial}{\partial n}U_{n}(z)ds.$$

Proof of a). Assume that there exist a positive number $\delta$ and infinitely many numbers $n$ and 1 such that

$$\int_{C_{L_{j}}}\frac{\partial}{\partial n}\omega_{n}^\prime(z)ds>\delta>0.$$

Then by $(L_{2}-L_{1})$ \int_{C_{L_{j}}}\frac{\partial}{\partial n}\omega_{n+1}^\prime(z)ds=D_{\partial \cap R_{n+1}}(\omega_{n+1}^\prime(z)),$

$$\int_{C_{L_{j}}}\frac{\partial}{\partial n}\omega_{n+1}^\prime(z)ds=\int_{C_{L_{j}}}\frac{\partial}{\partial n}\omega_{n+1}^\prime(z)ds-\int_{C_{L_{j}}-(R_{n+1}\cap R_{n})}\frac{\partial}{\partial n}\omega_{n+1}^\prime(z)ds\leqq D_{\partial \cap R_{n+1}}(\omega_{n+1}^\prime(z))/(L_{2}-L_{1})-\delta.$$

Then by Fatou’s lemma and by (15)

$$D_{G_{1}}(\omega(z))=D_{G_{1}}(\omega(z))(L_{2}-L_{1}).$$

Then by Fatou’s lemma and by (15)
\[ \int_{c_{L_j} \cap R_n} \frac{\partial}{\partial n} \omega(z) \, ds = \int_{c_{L_j} \cap R_n} \left( \lim_{i \to \infty} \frac{\partial}{\partial n} \omega'_{n+i}(z) \right) \, ds \leq \lim_{i \to \infty} \int_{c_{L_j} \cap R_n+i} \frac{\partial}{\partial n} \omega'_{n+i}(z) \, ds - \delta = D_{G_1}(\omega(z)) - \delta. \]

Let \( n \to \infty \). Then \( \int_{c_{L_j}} \frac{\partial}{\partial n} \omega(z) \, ds \leq D_{G_1}(\omega(z)) - \delta \).

This contradicts the regularity of \( C_{L_j} \). Hence for any given positive number \( \epsilon \), there exists a number \( n_0 \) such that
\[
0 \leq \int_{c_{L_j} \cap (R_{n+i} - R_n)} \frac{\partial}{\partial n} \omega'_{n+i}(z) \, ds < \frac{\epsilon}{M} \quad \text{for} \quad i \geq 0 \quad \text{and} \quad n \geq n_0. \tag{16}
\]

At present fix \( m(\geq n_0) \). Then by \( \frac{\partial}{\partial n} \omega'_{n+i}(z) \to \frac{\partial}{\partial n} \omega(z) \) and \( A_n(z) \to A(z) \) on \( C_{L_j} \setminus R_n \), there exists a number \( i_0 \) such that
\[
\int_{c_{L_j} \cap R_n} A_{m+i}(z) \frac{\partial}{\partial n} \omega'_{m+i}(z) \, ds - \epsilon \leq \int_{c_{L_j}} A(z) \frac{\partial}{\partial n} \omega(z) \, ds \quad \text{for} \quad i \geq i_0. \tag{17}
\]

By (16) and (17)
\[
\int_{c_{L_j}} A(z) \frac{\partial}{\partial n} \omega(z) \, ds \geq \int_{c_{L_j} \cap R_n} A(z) \frac{\partial}{\partial n} \omega(z) \, ds \geq \int_{c_{L_j} \cap R_n} A_{m+i}(z) \frac{\partial}{\partial n} \omega'_{m+i}(z) \, ds - \epsilon
\]
\[
= \int_{c_{L_j} \cap (R_{m+i} - R_n)} A_{m+i}(z) \frac{\partial}{\partial n} \omega'_{m+i}(z) \, ds - \int_{c_{L_j} \cap (R_{m+i} - R_n)} A_{m+i}(z) \frac{\partial}{\partial n} \omega_{m+i}(z) \, ds - \epsilon
\]
\[
\geq \int_{c_{L_j} \cap (R_{m+i} - R_n)} A_{m+i}(z) \frac{\partial}{\partial n} \omega_{m+i}(z) \, ds - 2\epsilon.
\]

Let \( \epsilon \to 0 \). Then
\[
\int_{c_{L_j}} A(z) \frac{\partial}{\partial n} \omega(z) \, ds \geq \lim_{n \to \infty} \int_{c_{L_j} \cap R_n} A_n(z) \frac{\partial}{\partial n} \omega_n'(z) \, ds.
\]

On the other hand, by Fatou’s Lemma
\[
\int_{c_{L_j}} A(z) \frac{\partial}{\partial n} \omega(z) \, ds = \int_{c_{L_j}} \left( \lim_{n} A_n(z) \frac{\partial}{\partial n} \omega_n'(z) \right) \, ds \leq \lim_{n} \int_{c_{L_j}} A_n(z) \frac{\partial}{\partial n} \omega_n'(z) \, ds.
\]

Hence
\[
\int_{c_{L_j}} A(z) \frac{\partial}{\partial n} \omega(z) \, ds = \lim_{n \to \infty} \int_{c_{L_j} \cap R_n} A_n(z) \frac{\partial}{\partial n} \omega_n'(z) \, ds.
\]

**Proof of b).** By the maximum principle \( U_n(z) \leq N \frac{\omega_n'(z) - L_1}{L_2 - L_1} \). Hence
\[
0 \leq \frac{\partial}{\partial n} U_n(z) \leq \frac{N}{L_2 - L_1} \frac{\partial}{\partial n} \omega_n'(z) \quad \text{on} \quad C_{L_j}, \quad \text{whence there exists a number} \quad n_0 \quad \text{for}
\]
any given positive number $\varepsilon$ such that

$$0 \leq \int_{c_{L_{1}- \partial R_{n+1}}(z)} \frac{\partial}{\partial n} U_{n+i}(z) \, ds \leq \frac{L_{2}-L_{1}}{MN} \varepsilon$$

for $n \geq n_0$ and $i \geq 0$. \hspace{1cm} (18)

Thus similarly as a) we have b).

4. Harmonic measures and capacity potentials of a closed set $F$ and of a decreasing sequence of compact or non compact domains.

Let $F$ be a closed set in $\overline{R} - R_0$ with respect to $\delta$-metric. Put $F_m = \{ z \in R : \delta(z, F) \leq \frac{1}{m} \}$. Let $w_{m,n}(z)(\omega_{m,n}(z))$ be a function in $(R_n \cap G_1)$ such that $w_{m,n}(z)(\omega_{m,n}(z))$ is harmonic in $(R_n \cap G_1) - (F_m \cap G_2)$, $w_{m,n}(z) = \omega_{m,n}(z) = 1$ on $F_m \cap G_2$, $w_{m,n}(z) = \omega_{m,n}(z) = 0$ on $\partial G_1 \cap R_n$ and $w_{m,n}(z) = \frac{\partial}{\partial n} \omega_{m,n} = 0$ on $\partial R_n - (F_m \cap G_2)$. Then $w_{m,n}(z) \uparrow w_m(z)$ as $n \to \infty$ and $w_m(z) \downarrow w(z)$ as $m \to \infty$. If $D(\omega_{m,n}(z)) < M$ for a certain number $m_0$ and for every $n \geq 0$, $\omega_{m,n}(z) \Rightarrow \omega(z)$ as $n \to \infty$ and $\omega_m(z) \Rightarrow \omega(z)$ as $m \to \infty$. We denote $w(z)$ and $\omega(z)$ by $w(F \cap G_2, z, G_1)$ and $\omega(F \cap G_2, z, G_1)$ respectively. Let $\{V_m\} (m=1,2,\cdots)$ be a decreasing sequence of compact or non compact domains. We define H.M. $w(\{V_m\} \cap G_2, z, G_1)$ and C.P. $\omega(\{V_m\} \cap G_2, z, G_1)$ of $\{V_m\} \cap G_2$ as H.M. and C.P. of $F \cap G_2$ by replacing $V_m \cap G_2$ instead of $F_m \cap G_2$. We proved the properties of H.M.(C.P.) only by the fact that $w_{n,n+i}(z) \uparrow w_n(z)$ and $w_n(z) \downarrow w(z)$ ($\omega_{n,n+i}(z) \Rightarrow \omega_n(z)$ and $\omega_n(z) \Rightarrow \omega(z)$). Hence these H.M.'s and C.P.'s have all the properties stated before. In this paper we denote by P.H.N.(C.P.N.) $(N=1,2,\cdots 7)$ the properties of the above H.M.'s and C.P.'s respectively.

If $G_1 = R - R_0$ and $G_2 \cap F_{m_0} \cap R_1 = 0$ or $G_2 \cap V_{m_0} \cap R_1 = 0$, by the Dirichlet principle

$$D(\omega_{m,n}(z)) < D(\hat{\omega}(z)) \quad \text{for} \quad m \geq m_0 \quad \text{and} \quad n \geq 0,$$

where $\hat{\omega}(z)$ is a harmonic function in $R_1 - R_0$ such that $\hat{\omega}(z) = 0$ on $\partial R_0$ and $\omega(z) = 1$ on $\partial R_1$. In this case, we omit $R - R_0$ and denote it by $w(G_2 \cap F, z) (\omega(G_2 \cap F, z))$ simply.

5. Superharmonic function in $\overline{R} - R_0$.

Let $G$ be a compact or non compact domain in $R - R_0$. If $U(z)$ is continuous in $G$ except a closed set of capacity zero and $U(z)$ has partial derivatives almost everywhere in $G$, we call $U(z)$ a $C_1$-class function.

Let $U(z)$ be a positive function of $C_1$-class in $G$ and continuous on $\partial G$ except a set of capacity zero such that $U(z) = 0$ on $\partial R_0 \cap G$ (may be
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Let $c_0 U^M(z)$ be a harmonic function in $G$ such that $c_0 U^M(z) = \min(M, U(z))$ on $\partial G$ and $c_0 U^M(z)$ has M.D.I. $\leqq D(\min(M, U(z)))$ over $G$. Then $c_0 U^M(z)$ is uniquely determined by Lemma 1. a). $c_0 U^M(z) \uparrow$ as $M \uparrow \infty$. Put $c_0 U(z) = \lim_{M=\infty} c_0 U^M(z)$. If $c_0 U(z) = \overline{U(z)}$, we call $U(z)$ a \textit{harmonic function} in $G$ with boundary value $U(z)$ on $\partial G$. Let $U(z)$ be a positive function of $C_1$-class in $R - R_0$ such that $U(z) = 0$ on $\partial R_0$ and $D(\min(M, U(z))) < \infty$ for $M < \infty$. Let $D$ be a compact or non compact domain in $R - R_0$. Let $\partial U^M(z)$ be a function such that $\partial U^M(z) = \min(M, U(z))$ on $\partial D + D$ and $\partial U^M(z)$ is harmonic in $R - R_0 - D$ with $\partial U^M(z) = 0$ on $\partial R_0$ (may be void). Put $\partial U(z) = \lim_{M=\infty} \partial U^M(z)$. If $\partial U(z) \leqq U(z)$ for every domain $D$ such that $\partial D$ is compact, we say that $U(z)$ is \textit{superharmonic} in $\overline{R} - R_0$. From this definition, if $U(z)$ is superharmonic in $\overline{R} - R_0$ and $U(z)$ is continuous in an open compact set $G$ in $R - R_0$, $U(z)$ is superharmonic in $G$ (in ordinary sense).

\textbf{Lemma 2. a). Maximum principle.} Let $U_i(z)$ $(i = 1, 2)$ be a \textit{harmonic} function in a compact or non compact domain $G$ such that $U_1(z) \geqq U_2(z)$ on $\partial G$. Then

$$U_1(z) \geqq U_2(z).$$

\textbf{b).} Let $U(z)$ be a \textit{harmonic function} in $G$ such that $M \geqq U(z)$ on $\partial G$. Then

$$U(z) \leqq M \text{ in } G.$$

\textbf{c).} Let $D$ be a compact or non compact domain in $R - R_0$. Let $U(z)$ be a positive function of $C_1$-class in $R - R_0$ such that $U(z) = 0$ on $\partial R_0$ and $D(\min(M, U(z))) < \infty$ for $M < \infty$ and $\partial U(z) \leqq U(z)$. Then

$$D(\min(M, \partial U(z))) \leqq D(\min(M, U(z))).$$

\textbf{Proof of a).} Let $U_n^M(z)$ be a harmonic function in $G - R_n$ such that $U_n^M(z) = \min(M, U_1(z))$ on $\partial G - R_n$ and $\frac{\partial}{\partial n} U_n^M(z) = 0$ on $\partial R_n - G$. Then $U_n^M(z) \geqq U_n^M(z)$. Let $n \rightarrow \infty$ and then $M \rightarrow \infty$. Then

$$\lim_{M} \lim_{n} U_n^M(z) = U_1(z) \geqq U_2(z) = \lim_{M} \lim_{n} U_n^M(z).$$

Similarly we have $\textbf{b).}$

\textbf{Proof of c).} \[E[z \in R : \partial U^M(z) < M] = \partial Q^M \downarrow \partial Q^M = E[z \in R : U(z) < M] \supset \partial Q^M = E[z \in R : U(z) < M] \text{ as } L \rightarrow \infty.\]

Suppose $L \geqq M$. $\partial U^L(z)$ has M.D.I. over $R - R_0 - D$ with value $\min(U(z), L)$
on \(\partial R_0 + \partial D\). This implies by Lemma 1.a) that \(\partial U^L(z)\) has also M.D.I. over \(\mathcal{Q}^\omega - D\) with value \(\partial U^L(z) = M = \min (M, U(z))\) on \(\partial \mathcal{Q}^\omega - D\) and with value \(\partial U^L(z) = \min (M, \partial U(z)) = U(z)\) on \(\partial D - \mathcal{Q}^\omega\), by \(\partial U^L(z) \leq U(z)\), i.e. \(\partial U^L(z) = \min (M, U(z))\) on \(\partial (R - R_0 - \mathcal{Q}^\omega - D)\).

Hence \(\partial L \Omega^M - D\left(\min (M, U(z))\right) \leq \partial \Omega^M - D\left(\min (M, U(z))\right)\).

On the other hand, \(\partial U^L(z) = \min (L, U(z))\) in \(D\) and by \(M \leq L\)
\[D_{D}(\min (M, U(z))) = D_{D}(\min (M, L, U(z))) = D_{D}(\min (M, U(z)))\]
and \(M \leq DU^L(z) \leq U(z)\) in \(\partial R - R_0 - \mathcal{Q}^\omega\), whence \(\partial U^L(z) \leq \min (M, U(z)) = M\) in \(\partial (R - R_0 - \mathcal{Q}^\omega)\) and
\[DL R M (\min (M, U(z))) = DL R M (\min (M, U(z))) = 0.\]
Thus \(DU^L(z) \leq U(z)\) and \(GU^L(z) \leq U(z)\), then \(\partial (\min (M, U(z))) \leq DU^L(z)\).

Let \(U(z)\) be a positive function of \(C_1\)-class with \(U(z) = 0\) on \(\partial R_0\). If \(\partial U(z)\) exists \(\partial U(z)\) has M.D.I. \(\geq 0\) over \(\partial R - R_0 - D\) and \(\partial U(z) = U(z)\) on \(\overline{D}\) for any compact or non compact domain such that \(\partial D\) is compact and \(\sup_{z \in \partial D} U(z) < \infty\) and if \(\partial U(z) \leq U(z)\), we say that \(U(z)\) is \(\overline{\sup}erharmonic\) in \(\overline{R} - R_0\) in the weak sense.

**Theorem 4. a).** Let \(U(z)\) be a positive function of \(C_1\)-class with \(U(z) = 0\) on \(\partial R_0\) and \(D(\min (M, U(z))) < \infty\). Let \(D\) be a domain (compact or non compact) and \(G\) be a domain with compact \(\partial G\). If \(\partial U(z) \leq U(z)\) and \(\partial U(z) \leq U(z)\), then
\[\partial (\min (M, U(z))) \leq DU^L(z).\]

Therefore if \(U(z)\) is \(\overline{\sup}erharmonic\) \(\partial U(z) \leq U(z)\) for any domain \(G\) with compact \(\partial G\) and \(\partial U(z) \leq U(z)\), \(\partial (\min (M, U(z))) \leq DU^L(z),\) i.e. \(\partial U(z)\) is superharmonic in \(\overline{R} - R_0\).

If \(U(z)\) is \(\overline{\sup}erharmonic\) in \(\overline{R} - R_0\) \(\partial U(z) \leq U(z)\) and \(\partial U(z) \leq U(z)\) for any domains \(D\) and \(G\) with compact \(\partial D\) and \(\partial G\), \(\partial U(z)\) is also superharmonic in \(\overline{R} - R_0\).

\(a')\). If \(U(z)\) is \(\overline{\sup}erharmonic\) in \(\overline{R} - R_0\) in the weak sense and if \(\partial U(z)\) and \(\partial U(z)\) are defined \(\sup_{z \in \partial (D + G)} U(z) < \infty\), then \(\partial (\min (M, U(z))) \leq DU^L(z)\) and
\[\partial (\min (M, U(z))) \leq DU^L(z),\]
where \(\partial D\) and \(\partial G\) are compact.

\(b)\). Let \(U(z)\) be superharmonic in \(\overline{R} - R_0\), then for any domains \(D_1\) and \(D_2\) with compact relative boundaries
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$b'$. Let \( U(z) \) be superharmonic in \( \overline{R} - R_0 \) in the weak sense. Then for any domains \( D_1 \) and \( D_2 \) with compact relative boundary such that \( \sup_{z \in \partial D_i} U(z) < \infty \) \( (i=1,2) \), we have

\[
D_{2}(_{D_{1}}U(z))=_{D_{1}}U(z) \leq D_{2}U(z); D_{1} \subset D_{2}.
\]

$c)$. Let \( U(z) \) be \( \overline{\sup}erharmonic \) in \( \overline{R}-R_0 \) in the weak sense. Then for any domains \( D_1 \) and \( D_2 \) with compact relative boundary such that \( \sup_{z \in \partial D_i} U(z) < \infty \) \( (i=1,2) \), we have

\[
D_{2}(_{D_{1}}U(z))=_{D_{1}}U(z) \leq D_{2}U(z); D_{1} \subset D_{2}.
\]

$c'$. Let \( U(z) \) be \( \overline{\sup}erharmonic \) \( \overline{R}-R_0 \) in the weak sense. Suppose \( D_n \supset D'_n \supset D_{n+1}, D_n = D \cap R_n \) \( (n=1,2, \cdots) \) and \( \sup U(z) < \infty \). Then \( dU_n(z) \uparrow U^*(z) \leq U(z) \) if \( dU_M(z) \) exists for every \( M<\infty \), then \( U^*(z) = \lim_{M=\infty} dU_M(z) \).

$d)$. Let \( U(z) \) be \( \overline{\sup}erharmonic \) in \( \overline{R}-R_0 \). Then \( dU(z) \leq U(z) \) for compact or non compact domain \( D \).

$e)$. Let \( U(z) \) be \( \overline{\sup}erharmonic \) in \( \overline{R}-R_0 \). Then \( dU(z) \leq U(z) \) for compact or non compact domain \( D \).

$f)$. Let \( U(z) \) be \( \overline{\sup}erharmonic \) in \( \overline{R}-R_0 \). Then for compact or non compact domains \( D_1 \) and \( D_2 : D_2 \supset D_1 \)

\[
d_4(D_1 U(z)) = d_1 U(z) \leq d_4 U(z).
\]

$g)$. Let \( U_n(z) \) \( (n=1,2, \cdots) \) be a superharmonic function in \( \overline{R}-R_0 \) and \( U_n(z) \rightarrow U(z) \) in every compact domain in \( R-R_0 \). If \( D(\min(M, U(z))) < \infty \) for \( M<\infty \), then \( U(z) \) is superharmonic in \( \overline{R}-R_0 \).

$g')$. Let \( U_n(z) \) \( (n=1,2, \cdots) \) be superharmonic in \( \overline{R}-R_0 \) in the weak sense such that \( \sup_{z \in \partial D} U_n(z) < \infty \). If \( U_n(z) \rightarrow U(z) \) in \( R-R_0 \) and if \( dU(z) \) exists, \( U(z) \) is superharmonic in the weak sense.

$h)$. Let \( U_n(z) \) \( (n=1,2, \cdots) \) be superharmonic in \( \overline{R}-R_0 \) such that \( U_n(z) \) is continuous in \( R-R_0 \) and \( U_n(z) \uparrow U(z) \). If \( U(z) \) is finite in \( R-R_0 \) and \( D(\min(M, U(z))) < \infty \) for \( M<\infty \), then for compact or non compact domain \( D \)

\[
d(\lim_{n} U_n(z)) = dU(z) = \lim_{n} dU_n(z).
\]

$i)$. Let \( U(z) \) and \( V(z) \) be \( \overline{\sup}erharmonic \) in \( \overline{R}-R_0 \). Then for a compact or non compact domain \( D \)

\[
dU(z) + dV(z) = d(U(z) + V(z)), \quad V(z) \geq U(z) \text{ implies } dV(z) \geq dU(z),
\]

\[
C_d(U(z)) = d(CU(z)) \text{ for } C \geq 0.
\]

$j)$. Let \( D_1 \) and \( D_2 \) be two compact or non compact domains. Then

\[
d_1 + d_4 U(z) \leq d_1 U(z) + d_4 U(z).
\]
k). \( \min (M, U(z)) \) and \( \min (U(z), V(z)) \) are superharmonic in \( \overline{R} - R_0 \).

Proof of a) Since \( \lim_{L=\infty} LDU(z) = D U(z) \) on \( \partial G \), for any given numbers \( M \) and \( \epsilon \) there exists a number \( L_0 \) such that \( L_0 > M \) and \( \epsilon + \frac{L_0}{r} U(z) \geq \min (M, DU(z)) \) on \( \partial G \).

In fact, assume that there exists a sequence \( \{z_i\} \) on \( \partial G \) such that \( \frac{L_i}{r} U(z_i) \leq \min (M, DU(z_i)) - \delta_0 \): \( \delta_0 > 0 \) for infinitely many numbers \( L_i \) such that \( \lim_{i} L_i = \infty \).

Since \( \partial G \) is compact, there exists a point \( z^* \) such that \( z_i \rightarrow z^* \), where \( \{z_i^*\} \) is a subsequence of \( \{z_i\} \). Then two cases occur.

Case 1. \( z \in \partial D \). In this case \( \partial D \) is composed of analytic curves and every point of \( \partial D \) is regular for Dirichlet problem. Now \( D U^M(z) = \min (M, U(z)) \) on \( \partial D \). Hence

\[
\lim_{i=\infty} U(z_i) \geq \lim_{i=\infty} (\min (L_0, U(z_i))) \geq \lim_{i} DU^M(z_i) = \min (M, U(z_i)) \text{ for } L_0 \geq M.
\]

Case 2. \( z \in \partial G - D \). In this case, there exists a neighbourhood \( u(z^*) \) of \( z^* \) such that \( u(z^*) \cap \overline{D} = \emptyset \). \( D U^L(z) : i = 1, 2, \ldots \) are harmonic in \( u(z^*) \) and \( \frac{L_i}{r} U(z) \uparrow DU(z) \) uniformly in \( u(z^*) \), whence \( \lim_{i} DU^L(z_i) \geq \min (M, DU(z^*)) \).

Cases 1 and 2 are contradictions. Hence

\[
\epsilon + \frac{L_0}{r} U(z) \geq \min (M, DU(z)) \text{ on } \partial G.
\]

Let \( _o V^M(z) \) be a harmonic function in \( R - R_0 - G \) such that \( _o V^M(z) = \min (M, D U(z)) \) on \( G + \partial R_0 \). This can be defined by \( D(\min (M, U_D(z))) \leq D(\min (M, U(z))) < \infty \) by Lemma 2. c). By the assumption: \( U(z) \geq _o U(z) \) and \( U(z) \geq U_D(z) \), which imply

\[
_o U(z) = \min (M, U(z)) \geq \min (M, D U(z)) = _o V^M(z) \text{ on } \partial G.
\]

Both \( _o U^M(z) \) and \( _o V^M(z) \) have M.D.I. over \( R - R_0 - G \). Hence by the maximum principle

\[
U(z) \geq _o U^M(z) \geq _o V^M(z) \text{ in } R - R_0 - G.
\]

Whence

\[
D U^L(z) = \min (L, D U(z)) = \min (L, U(z)) \geq \min (L, _o U^M(z)) \geq _o V^M(z) = _o V^M(z) - \epsilon \text{ on } \overline{D} - G.
\]

On the other hand, \( \epsilon + \frac{L}{r} U(z) \geq \min (M, D U(z) - \epsilon) \) on \( \partial G \) for \( L \geq L_0 \), and both \( _o V^M(z) \) and \( D U^L(z) \) have M.D.I. over \( R - R_0 - D - G \).

Hence by the maximum principle

\[
D U^L(z) \geq _o V^M(z) - \epsilon \text{ in } R - R_0 - G - D \text{ for } L \geq L_0.
\]

Let \( L \rightarrow \infty \) and then \( M \rightarrow \infty \) and then \( \epsilon \rightarrow 0 \). Then
\[ DU(z) = \lim_{M} GV^{M}(z) = \lim_{M} MG(DU(z)) =_{G}(DU(z)) \]

in \( R - R_{0} - D - G \).

Let \( L \rightarrow \infty \) and then \( M \rightarrow \infty \). Then \( DU(z) \geqq_{G} DU(z) \) on \( D \)

Thus \( DU(z) \geqq_{G} DU(z) \).

Next \( DU(z) \geqq \min(M, DU(z)) =_{G} V^{M}(z) \) on \( G \).

Thus \( DU(z) \geqq_{G} DU(z) \).

The latter part of \( a \) is proved at once by the definition of \( \overline{D} \geqq R_{0} \).

Proof of \( a' \). By the assumption \( DU(z) \) and \( GU(z) \) exist and

\[ D_{R - R_{0} - D}(DU(z)) < \infty \]

and \( D_{R - R_{0} - G}(GU(z)) < \infty \).

Put \( T(z) = \min(DU(z), GU(z)) \).

Then

\[ \left| \frac{\partial T(z)}{\partial x} \right| \leq \max \left( \left| \frac{\partial_{D} U(z)}{\partial x} \right|, \left| \frac{\partial_{G} U(z)}{\partial x} \right| \right), \quad \left| \frac{\partial T(z)}{\partial y} \right| \leq \max \left( \left| \frac{\partial_{D} U(z)}{\partial y} \right|, \left| \frac{\partial_{G} U(z)}{\partial y} \right| \right) \]

whence \( D_{R - R_{0} - D}(T(z)) \leq D_{R - R_{0} - D}(DU(z)) + D_{R - R_{0} - G}(GU(z)) < \infty \).

By \( T(z) = DU(z) \) in \( D - G \), because \( DU(z) \geqq_{G} DU(z) \) in \( D \),

\[ D_{R - R_{0} - G}(T(z)) = D_{D}(DU(z)) < \infty \]

Hence \( D_{R - R_{0} - G}(T(z)) < \infty \).

On the other hand,

\[ T(z) = DU(z) = DU(z) \]

on \( D \cap \partial G \) by \( DU(z) \geqq_{G} DU(z) \) in \( D - \partial G \),

\[ T(z) = DU(z) \]

on \( \partial G - D \) by \( DU(z) \leqq_{G} DU(z) \) on \( \partial G \),

hence \( T(z) = DU(z) \) on \( \partial R_{0} + \partial G \) and \( D_{R - R_{0} - G}(T(z)) < \infty \).

Hence there exists a harmonic function \( H(z) \) in \( R - R_{0} - G \) such that \( H(z) \) has M.D.I. \( (\leq D_{R - R_{0} - D}(T(z)) < \infty ) \) over \( R - R_{0} - G \) and \( H(z) = DU(z) \) on \( \partial G + \partial R_{0} \). Thus \( DU(z) \) \( (= H(z) \) in \( R - R_{0} - G \) and \( = DU(z) \) in \( G \)) is defined.

Now as above

\[ DU(z) \leqq_{D} DU(z) \]

on \( \partial R_{0} + \partial D + \partial G \)

and by the maximum principle

\[ DU(z) \leqq_{D} DU(z) \] in \( R - R_{0} - G - D \).

\[ DU(z) \leqq_{D} DU(z) \]

in \( G \) and \( DU(z) \leqq_{D} DU(z) \)

in \( D \).

Thus \( DU(z) \leqq_{D} DU(z) \).

Proof of \( b \). \( DU(z) \) has M.D.I. over \( R - R_{0} - D_{1} \), whence by Lemma 1. a) \( DU(z) \) has M.D.I. over \( R - R_{0} - D_{2} \) with value \( DU(z) \) in \( \overline{D}_{2} + \partial R_{0} \),
i.e. $D_{1}U^{L}(z)$ is harmonic in $R-R_{0}-D_{2}$.

Let $V^{L}(z)$ be a harmonic function in $R-R_{0}-D_{2}$ such that $V^{L}(z) = D_{1}U^{L}(z)$ on $\overline{D_{2}}+\partial R_{0}$. Then by the maximum principle $V^{L}(z) = D_{1}U^{L}(z)$.

Hence
\[
\lim_{L=\infty} V^{L}(z) = \lim_{L=\infty} D_{1}U^{L}(z) = p_{1}(z).
\]

On the other hand, by $a)$ $D_{1}(U(z))$ is superharmonic, whence $D_{1}(D_{1}U(z)) \leq D_{1}(U(z))$.

Next from $U(z) \geq D_{1}(U(z))$,
\[
D_{1}(U(z)) = D_{1}(D_{1}U(z)).
\]

Proof of $b^\prime)$. $D_{R-R_{0}-D_{2}}(D_{1}U(z)) \leq D_{R-R_{0}-D_{2}}(D_{1}U(z)) < \infty$, whence as above we have
\[
D_{1}(D_{1}U(z)) = D_{1}(D_{1}U(z)).
\]

Proof of $c)$. $D_{n} = D \cap R_{n}$ is compact, $D_{n}U(z)$ increases to a function $U^{*}(z)(\leq U(z))$ as $n \to \infty$ by $b)$. By lemma 2. c) $D(\min(M, D_{n}U(z))) \leq D(\min(M, U(z))) < \infty$, $\min(M, D_{n}U(z))$ is harmonic or a constant $M$ in $R-R_{0}-D$ and $=\min(M, U(z))$ in $D$. Hence by Fatou's lemma
\[
D(\min(M, U^{*}(z))) \leq \lim_{n=\infty} D(\min(M, D_{n}U(z))) \leq D(\min(M, U(z))) < \infty.
\]

By the superharmonicity of $U(z) \leq U(z)$ on $\partial D$ and has M.D.I. over $R-R_{0}-D$ by $(R-R_{0}-D) \subset (R-R_{0}-D_{n})$, whence by the maximum principle
\[
D_{n}U^{M}(z) \leq \sup_{\partial D_{n}} D_{n}U^{M}(z) = M \quad \text{in} \quad R-R_{0}-D_{n}.
\]

By Lemma 1. a) $D_{n}U^{M}(z)$ has M.D.I. over $R-R_{0}-D$ with value $\leq \min(M, U(z))$ on $\overline{D}+\partial R_{0}$. On the other hand, $D_{n}U^{M}(z)$ has M.D.I. over $R-R_{0}-D$ with value $\min(M, U(z))$ on $\overline{D}$. Hence by the maximum principle
\[
D_{n}U^{M}(z) \leq D_{n}U^{M}(z) \quad \text{in} \quad R-R_{0}-D.
\]

Clearly $\min(M, U(z)) = D_{n}U^{M}(z) \geq D_{n}U^{M}(z)$ in $D$. Let $M \to \infty$ and then $n \to \infty$. Then
\[
D_{n}U(z) \geq U^{*}(z) = \lim_{n} D_{n}U(z).
\]

Since $D_{R-R_{0}}(D_{n}U^{M}(z)) < \infty$, for any given positive number $\varepsilon > 0$, there exists a number $n_{0}$ such that $D_{D_{n}}(D_{n}U^{M}(z)) < \varepsilon$ for $n \geq n_{0}$. Now $D_{n}U^{M}(z)$
$\min(M, U(z)) = \min(D_n + \partial R_0, U(z)) = D_n U^M(z)$ on $D_n + \partial R_0$ and $D_n U^M(z)$ has M.D.I. over $R - R_0 - D_n$.

Hence

$$D_{R - R_0 - D_n}(D_n U^M(z)) \leq \lim_{n=\infty} D_n U^M(z) = \lim_{n=\infty} D U^M(z) + \epsilon.$$ 

Let $\epsilon \to 0$. Then $V^M(z) = \min(M, U(z))$ has M.D.I. over $R - R_0 - D$, because $D U^M(z) = \min(M, U(z))$ on $\partial D$ has M.D.I. Hence by Lemma 1. b)

$$V^M(z) = D U^M(z) \text{ in } R - R_0.$$ 

By $D_n U^M(z) \leq D U(z)$ and by (20), (19) and (21)

$$D U(z) = \lim_{M=\infty} D U^M(z) = \lim_{M=\infty} V^M(z) = \lim_{M=\infty} \lim_{n=\infty} D_n U^M(z) = \lim_{n=\infty} D_n U(z) = \lim_{n=\infty} U(z).$$ 

Thus we have c). c') is proved similarly.

Proof of d). If $\partial D$ is compact, it is clear by definition. If $\partial D$ is not compact, put $D_n = \overline{D} - R_0 - D_n$. Then by c)

$$D U(z) = \lim_{n=\infty} D_n U(z) \leq D U(z).$$ 

Thus $\overline{\text{superharmonic}}$ in $R - R_0$.

Proof of e). If $\partial D$ is compact, this case reduces to the case of Theorem 4. a). Suppose that $\partial D$ is non compact. Let $G$ be a domain such that $\partial G$ is compact. Then by Lemma 2. b) $D (\min(M, U(z))) \leq D (\min(M, U(z))) < \infty$ and by d) $D U(z) \leq D U(z)$.

Thus we have f).

Proof of g). It is sufficient to show $D U(z) \leq U(z)$ for domain whose relative boundary $\partial D$ is compact. Suppose that $\partial D$ is compact. Since $D U(z) = \lim_{M=\infty} D U^M(z)$, for any given positive number $\epsilon$, there exists a number $M_0$ such that (see a))

$$D U(z) \leq D U^M(z) + \epsilon \text{ for } M \geq M_0.$$ 

Then $V^M(z) = \min(M, U(z))$ on $\overline{D}$ and $D_n U^M(z)$ has M.D.I. over $R - R_0 - D_n$.

Hence

$$D_{R - R_0 - D_n}(D_n U^M(z)) \leq \lim_{n=\infty} D_n U^M(z) = \lim_{n=\infty} D U^M(z) + \epsilon.$$ 

Let $\epsilon \to 0$. Then $V^M(z) = \min(M, U(z))$ has M.D.I. over $R - R_0 - D$, because $D U^M(z) = \min(M, U(z))$ on $\partial D$ has M.D.I. Hence by Lemma 1. b)

$$V^M(z) = D U^M(z) \text{ in } R - R_0.$$ 

By $D_n U^M(z) \leq D U(z)$ and by (20), (19) and (21)

$$D U(z) = \lim_{M=\infty} D U^M(z) = \lim_{M=\infty} V^M(z) = \lim_{M=\infty} \lim_{n=\infty} D_n U^M(z) = \lim_{n=\infty} D_n U(z) = \lim_{n=\infty} U(z).$$ 

Thus we have c). c') is proved similarly.

Proof of d). If $\partial D$ is compact, it is clear by definition. If $\partial D$ is not compact, put $D_n = D - R_0 - D_n$. Then by c)

$$D U(z) = \lim_{n=\infty} D_n U(z) \leq U(z).$$ 

Thus $\overline{\text{superharmonic}}$ in $R - R_0$.

Proof of e). If $\partial D$ is compact, this case reduces to the case of Theorem 4. a). Suppose that $\partial D$ is non compact. Let $G$ be a domain such that $\partial G$ is compact. Then by Lemma 2. b) $D (\min(M, U(z))) \leq D (\min(M, U(z))) < \infty$ and by d) $D U(z) \leq D U(z)$.

Thus we have f).

Proof of g). It is sufficient to show $D U(z) \leq U(z)$ for domain whose relative boundary $\partial D$ is compact. Suppose that $\partial D$ is compact. Since $D U(z) = \lim_{M=\infty} D U^M(z)$, for any given positive number $\epsilon$, there exists a number $M_0$ such that (see a))

$$D U(z) \leq D U^M(z) + \epsilon \text{ for } M \geq M_0.$$
Now by $U_{n}^{M}(z) \rightarrow U^{M}(z)$ on $\partial D$, there exists a number $n_{0}(M)$ by the maximum principle such that

$$|D_{n}U_{n}^{M}(z)-D_{n}U^{M}(z)| < \varepsilon \quad \text{in} \quad R-R_{0}-D; \quad n \geqq n_{0}(M).$$

Hence $D_{n}U(z) \leqq D_{n}U^{M}(z) + \varepsilon \leqq D_{n}U^{M}(z) + 2\varepsilon \leqq \min (M, U_{n}(z)) + 2\varepsilon$ in $R-R_{0}-D$. Let $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$. Then $D_{n}U(z) \leqq U(z)$ in $R-R_{0}-D$ and $D_{n}U(z) = U(z)$ in $\overline{D}$. Thus $U(z)$ is superharmonic in $\overline{R-R_{0}}$. $g'$ is proved similarly.

Proof of h): Put $D_{m} = D \cap R_{m}$. Then $\partial D_{m}$ is compact. By $U(z) \geqq U_{n}(z)$,

$$D_{n}U(z) \geqq D_{n}U_{n}(z) \geqq D_{m}U(z).$$

Let $m \rightarrow \infty$. Then $D_{n}U(z) \geqq D_{n}U_{n}(z)$ and $D_{n}U(z) \geqq \lim_{n=\infty} D_{n}U_{n}(z)$.

Conversely, since $U_{D}(z) = \lim D_{m}U(z)$, there exists a number $m$ such that $U_{D}(z) \leqq D_{m}U(z) + \varepsilon$. Next by c) for any given positive number $\varepsilon$, since $\partial D$ is compact and $U_{n}(z) \rightarrow U(z)$ on $\partial D_{m}$, there exists a number $n(m)$ such that

$$D_{n}U(z) \leqq D_{m}U(z) + \varepsilon \leqq D_{n}U(z) + 2\varepsilon \leqq D_{n}U_{n}(z) + 2\varepsilon$$

in $R-R_{0}-D_{m}(\supset (R-R_{0}-D))$ for $n \geqq n(m)$.

In $D_{m}$, $D_{n}U(z) = U(z) = \lim U_{n}(z) = \lim D_{m}U_{n}(z) \leqq \lim D_{n}U_{n}(z)$.

Let $\varepsilon \rightarrow 0$. Then $D_{n}U(z) \leqq \lim D_{n}U_{n}(z)$.

Thus we have h).

Proof of i) and j) are clear by the definition and by the maximum principle.

Proof of k). Put $T(z) = \min (U(z), V(z))$. Then

$$\frac{\partial T(z)}{\partial x} \leqq \max \left( \frac{\partial U(z)}{\partial x}, \frac{\partial V(z)}{\partial x} \right), \quad \frac{\partial T(z)}{\partial y} \leqq \max \left( \frac{\partial U(z)}{\partial y}, \frac{\partial V(z)}{\partial y} \right).$$

$$E[z \in R: T(z) < M] = E[z \in R: U(z) < M] + E[z \in R: V(z) < M].$$

Hence $D(\min (M, T(z))) \leqq D(\min (M, U(z))) + D(\min (M, V(z))) < \infty$ for every $M < \infty$.

Let $G$ be a compact or non compact domain in $R-R_{0}$. Then $T(z) \leqq U(z)$ and $\leqq V(z)$ on $\partial G$. Hence by the maximum principle

$$ \min (\min (U(z), V(z))) \leqq \min (U(z), V(z)) = T(z).$$

Thus $T(z)$ is superharmonic in $\overline{R-R_{0}}$. The latter part is proved similarly.

6. Integral representation of superharmonic functions.

Theorem 5. a). C.P.'s $\omega(B \cap G, z)$, $\omega(F, z)$, $\omega([V], z)$ and $N(z, p): p \in \overline{R-R_{0}}$ and $\int N(z, p) d\mu(p): d\mu(p) \geqq 0$ are superharmonic in $\overline{R-R_{0}}$. 
b). Let $U(z)$ be a positive superharmonic function in $\overline{R}-R_0$ with $U(z)=0$ on $\partial R_0$. Let $F$ be a closed set in $\overline{R}-R_0$. Put $\overline{F}_m=E\left\{z \in \overline{R}: \delta(z, F) \leq \frac{1}{m}\right\}$. Then $\overline{F}_m$ is a compact or non compact closed domain. By Theorem 4, $\lim_{m} U(z) \downarrow$. Let $F_{m,n}=E\left\{z \in \overline{R} : \delta(z, F) \leq \frac{1}{m}\right\}$. Then $F_{m,n}$ is a compact or non compact closed domain.

$\omega_{m,n}(z)$ has M.D.I. over $R-R-F_{m,n}-D$.

Hence by the Dirichlet principle

$$D(\omega_{m,n}(z)) \leq D(\omega(z)) \text{ for } m \geq m_0 \text{ and } n \geq 0,$$

whence

$$D(\min(M, \omega_{m,n}(z))) < \infty \text{ for every } M.$$  \hspace{1cm} (22)

Let $D$ be a domain in $R-R_0$ with compact relative boundary. Then $D(\omega_{m,n}(z))=\omega_{m,n}(z)$ on $(\partial D \cap CF_{m,n})+\partial R_0$ and $D\omega_{m,n}(z)<1=\omega_{m,n}(z)$ on $F_{m,n} \cap CD$. Now both $D(\omega_{m,n}(z))$ and $\omega_{m,n}(z)$ have M.D.I. over $R-R-F_{m,n}-D$. Hence by the maximum principle

$$D\omega_{m,n}(z) \leq \omega_{m,n}(z).$$  \hspace{1cm} (23)
Hence by (22) and (23) \( \omega_{m,n}(z) \) is superharmonic in \( \overline{R}-R_{0} \). Now \( \omega_{m,n}(z) \Rightarrow \omega_{m}(z) \Rightarrow \omega(z) \), whence \( \omega(F,z) \Rightarrow \omega(z) \) is superharmonic in \( \overline{R}-R_{0} \) by Theorem 4. g). For other C.P.'s we can prove similarly. We show that \( N(z, p) \) is superharmonic in \( \overline{R}-R_{0} \). \( D(\min(M,N(z,p))) \leqq 2\pi M \) by Theorem 1. b). Next let \( D \) be a domain with compact relative boundary \( \partial D \). Put \( V_{n}(p)=E \left[ z \in R: N(z, p)>M \right] \).

Case 1. \( p \in D \). In this case \( V_{n}(p) \subset D \) for sufficiently large \( M \) by Theorem 1. a). Since \( N(z, p) \) has M.D.I. over \( R-R_{0}-V_{n}(p) \supset (R-R_{0}-D) \), \( pN(z, p)=N(z, p) \) is superharmonic in \( \overline{R}-R_{0} \) by Theorem 4. g).

Case 2. \( p \in \overline{D} \): \( N(z, p)=D_{+V_{n}(p)}N(z, p) \) by case 1. Let \( M>\sup_{z \in \partial D}N(z, p) \). Then \( pN(z, p) \) has M.D.I. over \( R-R_{0}-D \), whence by the maximum principle \( pN(z, p)<M \) in \( R-R_{0}-D-V_{n}(p) \). \( pN(z, p) \) and \( D_{+V_{n}(p)}N(z, p) \) have M.D.I. over \( R-R-V_{n}(p)-D \). Hence by the maximum principle

\[
D_{n}N(z, p) \leq D_{n}+V_{n}(p)N(z, p) \leq N(z, p) \quad \text{in} \quad R-R_{0}-V_{n}(p)-D,
\]

because \( D_{n}N(z, p) \leq D_{n}+V_{n}(p)N(z, p) \) on \( \partial D+\partial V_{n}(p)+\partial R_{0} \). And \( N(z, p) \geq M \geq D_{n}N(z, p) \) in \( V_{n}(p) \) and \( pN(z, p)=N(z, p) \) in \( D \). Hence \( N(z, p) \geq D_{n}N(z, p) \).

Case 3. \( p \in \partial D \). In this case \( D_{n}N(z, p) \leq D_{n}M \) on \( V_{n}(p) \cap CD \). \(^{5}\) Hence as in case 2

\[
D_{n}N(z, p) \leq D_{n}+V_{n}(p)N(z, p) \leq N(z, p) \quad \text{in} \quad R-R_{0}-D-V_{n}(p).
\]

Let \( M \to \infty \). Then \( V_{n}(p) \to p \in \partial D \) and \( pN(z, p)=\lim_{M \to \infty}D_{n}N(z, p) \leq N(z, p) \) in \( R-R_{0}-D \). Now \( pN(z, p)=N(z, p) \) in \( \overline{D} \). Hence \( pN(z, p) \leq N(z, p) \) in \( D \). Thus by case 1, 2 and 3 \( N(z, p) \) is superharmonic in \( \overline{R}-R_{0} \) for \( p \in R-R_{0} \).

Next suppose \( N(z, p)=\lim_{t}N(z, p_{t}) \): \( p \in B \) and \( p_{t} \in R-R_{0} \), where \( \{p_{t}\} \) is a fundamental sequence. Then by the superharmonicity of \( N(z, p_{t}) \), \( N(z, p) \) is superharmonic in \( \overline{R}-R_{0} \) by Theorem 4. g) and by \( D(\min(M,N(z,p))) \leq 2\pi M \).

Let \( V(z)=\int N(z, p) \, d\mu(p) \). Since \( N(z, p) \) is continuous (for fixed \( z \)) with respect to \( p \), the approximation to \( V(z) \) is done in every compact domain in \( R-R_{0} \) by \( V_{n}(z)=\sum_{i=1}^{n}c_{i}N(z, p):c_{i}>0, \sum c_{i}=\int d\mu(p), p_{t} \in R-R_{0} \) \((n=1,2,\cdots)\).

Since \( N(z, p_{t})=\infty \) at \( p_{t} \), there exists a neighbourhood \( \nu \) of \( \sum p_{t} \) such

\(^{5}\) \( CD \) means the complementary set of \( D \).
that \( \sum c_i N(z, p_i) > N \) in \( \nu \) for any given large number \( N \). Now \( N(z, p_i) \) has M.D.I. over \( R - R_0 - \nu \). Hence \( V_n(z) \) has M.D.I. over \( \Omega_M (\subset R - R_0 - \nu) \) for \( M < N : \Omega_M = E \{ z \in R : V_n(z) > M \} \), whence \( V_n^m(z) \to V_n(z) \) as \( m \to \infty \), where \( V_n^m(z) \) is a harmonic function in \( (R_m - R_0) \cap \Omega_M \) such that \( V_n^m(z) = 0 \) on \( \partial R_0 \), \( V_n^m(z) = M \) on \( \partial \Omega_N \cap R_m \) and \( \frac{\partial}{\partial n} V_n^m(z) = 0 \) on \( \partial R_m \).

Hence

\[
D_{\Omega_M}(V_n(z)) = \lim_{m} D_{\Omega_M}(V_n^m(z)) = \lim_{m} M \int_{\partial \Omega_M \cap R_m} \frac{\partial}{\partial n} V_n^m(z) \text{d}s = M \int_{\partial R_0} \frac{\partial}{\partial n} \sum c_i N(z, p_i) \text{d}s = 2\pi M \sum c_i.
\]

Now \( V_n(z) \to V(z) \), whence

\[
D(\min (M, V(z))) \leq \lim_n D(\min (V_n(z), M)) = 2\pi M \int \text{d}\mu(p). \tag{24}
\]

Clearly \( V_n(z) \geq (V(z)) \) for any domain \( D \) with compact \( \partial D \) in \( R - R_0 \). Hence \( V_n(z) \) is superharmonic in \( \overline{R} - R_0 \). Now \( V_n(z) \to V(z) \). Hence by (24) and by the superharmonicity of \( V_n(z) \), \( V(z) \) is superharmonic in \( \overline{R} - R_0 \) by Theorem 4.9.

**Proof of b).** Let \( F_m' \) be a closed set such that every point of \( \partial F_m' \) is regular for Dirichlet problem, \( U(z) \) is continuous on \( \partial F_m' \) and \( F_m' \subset F_m' \subset F_m' \) (\( m = 1, 2, \cdots \)). Put \( F_{m,n}' = F_m' \cap R_n \). Now \( U(z) \) is superharmonic (in ordinary sense) at every point of \( F_m' \). Hence it can be proved by the method of F. Riesz-Frostmann that the functional

\[
J(\mu) = \frac{1}{2} \frac{1}{4\pi^2} \int \int N(z, p) \text{d}\mu(p) \text{d}\mu(z) - \frac{1}{2\pi} \int U(z) \text{d}\mu(z)
\]

is minimized by a unique mass distribution \( \mu_{m,n} \) on \( F_{m,n}' \) among all non-negative mass distributions. The function \( V(z) \) given by

\[
\frac{1}{2\pi} \int N(z, p) \text{d}\mu_{m,n}(p)
\]

is equal to \( U(z) \) on \( F_{m,n}' \) and \( U(z) = V(z) \) on \( \partial F_{m,n}' \) by the regularity of \( \partial F_{m,n}' \). \( V(z) \) is continuous (= \( U(z) \)) on \( \partial F_{m,n}' \). Since \( F_{m,n}' \) is compact, the continuity principle of the potential in euclidean space is valid, whence \( V(z) \) is continuous in \( R - R_0 - F_{m,n}' + \partial F_{m,n}' \). Put \( K = F_{m,n}' \) and \( K_i = E \{ z \in R : \delta(z, K) \leq \frac{1}{l} \} \). Since \( K \) is closed and compact, \( U(z) - V(z) \) and \( U(z) - \kappa U(z) \) are uniformly continuous in \( R - R_0 - K \). Hence for any given positive number \( \epsilon \), there exists a number \( l_0 \) such that

\[
|U(z) - V(z)| < \epsilon \quad \text{and} \quad |U(z) - \kappa U(z)| < \epsilon \quad \text{on} \quad \partial K_{l_0},
\]

by \( U(z) - V(z) = 0 = U(z) - \kappa U(z) \) on \( \partial K \).
We can find a sequence $V_m(z) = \sum_{i=1}^{m} c_i N(z, p_i)$ \((m=1, 2, \cdots)\) such that the total mass of $V_m(z) = \sum_{i=1}^{m} c_i$ and $V_m(z) \to V(z)$ in $R - R_0 - K_l$ and every pole $p_i$ of $V_m(z)$ is contained in $K_l(l>2l_0)$. Since $V_m(z) \to V(z)$, there exists a number $m_0$ such that

$$|V_m(z) - V(z)| < \varepsilon$$

on $\partial K_{l_0}$ for $m \geqq m_0$.

Hence

$$|\kappa U(z) - V_m(z)| < |\kappa U(z) - U(z)| + |U(z) - V(z)| + |V(z) - V_m(z)| < 3\varepsilon$$

on $\partial K_{l_0}$.

Let $m \to \infty$ and then $\varepsilon \to 0$. Then $\kappa U(z) = V(z)$ in $R - R_0 - K_{l_0}$, whence $\kappa U(z) = V(z)$ in $R - R_0 - F_{m,n}'$, where the total mass of $F_{m,n}' U(z)$ is given by

$$\frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} F_{m,n}' U(z) ds \leqq \frac{1}{2\pi} \int_{\partial K_{l_0}} \frac{\partial}{\partial n} U(z) ds$$

for every $n$ and $m$. Since $N(z, p)$ is a continuous function of $p \in \overline{R} - R_0$ for fixed $z$ and the total mass of $\mu_{m,n}$ is less than $\frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} U(z) ds$ by $F_{m,n}' U(z) \leqq F_{m,n} U(z) \leqq U(z)$, $\{\mu_{m,n}\}$ has an weak limit $\mu_m$ on $F_{m,n}'$ as $n \to \infty$. Hence $F_{m,n}' U(z) = \frac{1}{2\pi} \int N(z, p) d\mu_m(p)$ and by letting $m \to \infty$, $F U(z) = \frac{1}{2\pi} \int N(z, p) d\mu(p)$.

Proof of b'). The former part is proved similarly, and the latter part is easily proved by taking account of the fact that $U(z) = \kappa U(z)$, because $R - R_0 - K_m$ is compact, where $K_m = E_{z \in \overline{R}}: \delta(z, (F+B)) \leqq \frac{1}{m}$.

**Theorem 6.** a). Let $A$ be a closed set of capacity zero. If $U(z)$ is positively superharmonic in $\overline{R} - R_0$ and harmonic in $R - R_0 - A$ with $U(z) = 0$ on $\partial R_0$. Then $U(z) - A U(z)$ is superharmonic and

$$A U(z) = \Delta(A U(z))$$

b). Let $\{A_m\} (m=1, 2, \cdots)$ be a sequence of decreasing domain such that $\omega(\{A_m\}, z) = 0$. If $U(z)$ is positively superharmonic in $\overline{R} - R_0$ and harmonic in $R - R_0 - A_m$ with $U(z) = 0$ on $\partial R_0$. Then $U(z) - \lim A_m U(z)$ is superharmonic in $\overline{R} - R_0$ and

$$\lim m A_m U(z) = \lim m \Delta m U(z) = \lim A_m U(z)$$

Proof. Let $G$ be a domain in $R - R_0$ such that $\partial G$ is compact and $\sup_{z \in \partial G} U(z) < N < \infty$. Put

$$U(z) = \partial U(z) + V(z).$$
Then $V(z) = 0$ on $\partial R_0 + \partial G$ and $V(z)$ is superharmonic in $\overline{R} - R_0 - G$.

In fact, $V(z) = 0$ on $\partial R_0 + \partial G$. $V(z) \leq M$ implies $U(z) \leq M + N$ in $R - R_0 - G$, because $\partial U(z) \leq N$ in $R - R_0 - G$ by the maximum principle. Hence

$$E[z \in R - R_0 - G : V(z) < M] = \Omega_{\partial R_0 + \partial G}^M = E[z \in R - R_0 - G : U(z) < M + N].$$

Hence $D(\min (M, V(z))) = D_{\Omega_{\partial R_0 + \partial G}^M}(V(z)) \leq D_{\Omega_{\partial R_0 + \partial G}^{M+N}}(U(z) - \partial U(z)) \leq D_{\Omega_{\partial R_0 + \partial G}^{M+N}}(U(z))$.

Therefore, $V(z) = 0$ on $\partial R_0 + \partial G$. $V(z) \leq M$ implies $U(z) \leq M + N$ in $R - R_0 - G - \Omega$, because $\partial U(z) \leq N$ in $R - R_0 - G - \Omega$ by the maximum principle.

Hence $\partial U(z) \leq N$ on $\partial R_0 + \partial G$, whence

$$\partial U(z) \leq N \text{ in } R - R_0 - G - \Omega.$$ (25)

Let $\Omega$ be a domain in $R - R_0$ (not necessarily $R - \Omega = \emptyset$). Let $\partial V^M(z, G)$ be a function in $R - R_0 - G - \Omega$ such that $\partial V^M(z, G) = \partial V^M(z) = 0$ on $\partial R_0 + \partial G$, $\partial V^M(z, G) = \min (M, V(z))$ on $\partial \Omega$, $\partial V^M(z, G)$ is harmonic in $R - R_0 - G - \Omega$ and $\partial V^M(z, G)$ has M.D.I. over $R - R_0 - G - \Omega$, which can be defined by (25).

Put $Q' = E[z \in \Omega : U(z) > M + N]$. Then $U(z) \geq M + N$ on $\partial Q'$ and $U(z)$ $\leq M + N$ on $\partial \Omega - Q'$. Now $\partial U(z) \leq N$ on $\partial G$, whence by the maximum principle

$$\partial U(z) \leq N \text{ in } R - R_0 - G.$$ (26)

Put $\partial U(z) = \partial V^M(z, G)$.

Now $\partial U(z) = \partial V^M(z, G)$ in $R - R_0 - G$. Hence by the maximum principle

$$\partial U(z) \leq \partial V^M(z, G) \text{ in } R - R_0 - G.$$ (25)

Thus by (25) $V(z)$ is superharmonic in $\overline{R} - R_0 - G$. 

Let \( D \) be a domain with compact \( \partial D \) in \( \mathbb{R}^n \) such that \( \sup_{z \in \partial D} U(z) < L < \infty \). Then \( D U(z) \) has M.D.I. over \( R - R_0 - D \) \( ( < D(\min (L, U(z))) \) ), whence \( D U(z) \) has also M.D.I. over \( R - R_0 - G - D \). Put \( V(z, G) = U(z) - G U(z) \) and \( \partial V(z, G) = \lim_{M \to \infty} \partial^M V(z, G) \).

Since \( D U(z) - G (D U(z)) = 0 = D V(z, G) \) on \( \partial D - G + \partial R \), \( D U(z) - G (D U(z)) - D V(z, G) = 0 \) on \( \partial D - G + \partial R \).

(27)

Since \( D U(z) - G (D U(z)) \) is harmonic in \( R - R_0 - G - D \) and has M.D.I. \( < \infty \) over \( R - R_0 - G - D \) (because \( D U(z) \), \( G (D U(z)) \) and \( D V(z, G) \) have M.D.I. \( < \infty \) over \( R - R_0 - G - D \)), we have by (27) and (28)

\[
T(z, D, G) = D U(z) - G (D U(z)) - D V(z, G) \text{ in } R - R_0 - G - D,
\]

where \( T(z, D, G) \) is a harmonic function in \( R - R_0 - G - D \) such that \( T(z, D, G) = 0 \) on \( \partial R + \partial D - G \) and \( T(z, D, G) = G (U(z) - G (D U(z))) \leq G U(z) \) on \( \partial D - G \).

(28)

Then \( T(z, D, G) \) and \( \omega (D, z) \) have M.D.I. over \( R - R_0 - G - D \), whence by the maximum principle

\[
T(z, D, G) \leq N \omega (D, z).
\]

(29)

Put \( D = A_{m,n}^\prime = A_{m}^\prime \cap R_n \), where \( A_{m}^\prime \) is a domain such that \( A_{m+1} \subset A_{m}^\prime \subset A_{m} \), \( \sup_{z \in \partial A_{m,n}^\prime} U(z) < \infty \) for every \( m \) and \( n \).

Then \( A_{m,n}^\prime \uparrow A_{m}^\prime \cap R_n \) and \( A_{m}^\prime \downarrow A_{m} \). Since \( V(z) \) is superharmonic in \( R - R_0 - G \), \( A_{m,n}^\prime U(z) \uparrow A_{m}^\prime U(z) \) and \( A_{m}^\prime U(z) \downarrow A_{m} U(z) \). Whence \( T(z, A_{m,n}^\prime, G) \to T(z, A_{m}^\prime, G) \) and \( T(z, A_{m}^\prime, G) \to T(z, A, G) \). Now by the assumption

\[
0 \leq T(z, A, G) \leq N \omega (A, z) = 0.
\]

(30)

Thus \( \lim_{A_{m,n}^\prime} U(z) = \lim_{A_{m}^\prime} U(z) \) and \( \lim_{A_{m}^\prime} U(z) = A U(z) \). Since \( V(z) \) is superharmonic in \( R - R_0 - G \) and \( \lim_{A} U(z) \) \( \leq \sup_{z \in \partial A} U(z) \) (because \( \lim_{A} U(z) \leq U(z) \)), and by \( U(z) = \sup_{z \in \partial A} U(z) + V(z) \), we have by (30)

\[
U(z) - A U(z) \leq G (U(z) - G (A U(z))) + V(z) - A V(z, G) \leq G U(z) - G (A U(z)).
\]

(31)

Now \( \lim_{A} U(z) - \lim_{A} U(z) \), \( \lim_{A} U(z) \) and \( \lim_{A} U(z) \) have M.D.I. \( \leq 4 D(\min (N, U(z))) \) if \( N > \sup_{z \in \partial A} U(z) \) over \( R - R_0 - G \). Hence by the maximum principle

\[
\lim_{A} U(z) = \lim_{A} U(z) = \lim_{A} U(z).
\]
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\[ o(U(z)-G(U(z))) = G(U(z)-A(U(z))) \]

By (31) \( U(z)-A(U(z)) \geq G(U(z)-A(U(z))) \) in \( R-R_0-G-A \). (32)

Put \( U(z)-A(U(z)) = G(U(z)-A(U(z)) \) in \( G \). Then

\[ U(z)-A(U(z)) \geq G(U(z)-A(U(z)) \) in \( R-R_0-A \). (33)

But \( G \) is any compact domain such that \( \sup U(z) < \infty \). Hence \( K(z)=U(z)-A(U(z)) \) is superharmonic in the weak sense by (33). Whence \( K(z) \) is superharmonic in \( \overline{R}-R_0 \) by Theorem 5. b).

By (30) and (29) and by putting \( D=A_{m,n} \) and by letting \( n \to \infty \),

\[ A_{m}^\prime U(z)-A(U(z))=o(A_{m}^\prime U(z))-o(A(U(z))+(A_{m}^\prime V(z,G)-A(V(z,G))+T(z,A_{m}^\prime,G)) \geq G(A_{m}^\prime U(z))-G(A(U(z))) \] (similarly as (31)),

because \( A_{m}^\prime V(z,G) \geq V(z,G) \) and \( T(z,A_{m}^\prime,G) \geq 0 \).

Put \( G=A_{m,n}^\prime=A_{m}^\prime \cap R_n \) such that \( A_{m}^\prime \supseteq A_{m}^\prime \) (i.e. \( m' < m \)) and \( \sup U(z) \in R-R_0 \).

Then

\[ A_{m}^\prime U(z)-A(U(z)) \geq A_{m,n}^\prime U(z)-A(U(z)) \leq A_{m,n}^\prime (A_{m}^\prime U(z)) \]

By the superharmonicity of \( A_{m}^\prime U(z) \) and \( A U(z) \), because \( A U(z) \) is limit of \( A_{m}^\prime U(z) \),

\[ A_{m,n}^\prime (A_{m}^\prime U(z)) \uparrow A_{m}^\prime (A_{m}^\prime U(z)) \]

On the other hand, since \( A_{m}^\prime \) and \( A_{m}^\prime \) can be considered as domains. Then by Theorem 4. f) \( A_{m}^\prime (A_{m}^\prime U(z)) = A_{m}^\prime U(z) \). Hence by (33') we have by letting \( n \to \infty \)

\[ A U(z) = A_{m}^\prime (A_{m}^\prime U(z)) \]

Let \( m' \to \infty \). Then \( A_{m}^\prime U(z) = A(U(z)) \). On the other hand, \( A U(z) \) is superharmonic (because \( A U(z) \) is the limit of superharmonic functions \( A_{m}^\prime U(z) \)), whence \( A(A U(z)) \leq A(U(z)) \).

Thus

\[ A(A U(z)) = A U(z) \].

b) is proved similarly.

Theorem 7. a). Let \( A \) be a closed set in \( \overline{R}-R_0 \). Then \( \omega(A,z)=\omega(A,z) = \int N(z,p) d\mu(p) \) for \( \omega(A,z) \geq 0 \).

b). \( \omega(p,z) = 0 \) for \( p \in R-R_0 \). If \( p \) is an ideal boundary point such that \( \omega(p,z) > 0 \), then

\[ \omega(p,z) = KN(z,p), K > 0 \].

We call such a point a singular boundary point and denote by \( B_z \) the
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set of singular boundary points.

c). $v_{n}(p)=E\left[z\in \overline{R} : \delta(z, p) < \frac{1}{n} \right]$. Then $v_{n}(p)N(z, p) \downarrow$ and has limit ($=_{p}N(z, p)$) as $n \to \infty$. Put $\phi(v_{n}(p))=\int_{z\in \overline{R}} \frac{\partial}{\partial n} v_{n}(p)N(z, p) ds$ and $\phi(p)=\lim_{n=\infty} \phi(v_{n}(p))$. Then $\phi(p)=\int_{z\in \overline{R}} \frac{\partial}{\partial n} v_{n}(p)N(z, p) ds$ and $\phi(p)=1$ for $p\in R-R_{0}+B_{s}$

dan further $\phi(p)=1$ or $0$ for $p\in \overline{R}-R_{0}$.

$\phi(v_{n}(p))$ is lower semicontinuous with respect to $\delta$-metric. Denote by $B_{0}$ and $B_{1}$ the set of points $p$ of $B$ for which $\phi(p)=0$ and $\phi(p)=1$ respectively. Then by b) $B_{s}\subseteq B$, $B=B_{0}+B_{1}$ and $B_{0}$ is an $F_{\sigma}$ set or void.

d). $B_{0}$ is an $F_{\sigma}$ set of capacity zero, whence $B_{s}\subseteq B$.

e). If $U(z)$ is given by $\int_{z\in \overline{R}} N(z, p) d\mu(p)$ ($\mu(p) \geq 0$), then $\mu_{0} U(z)=0$.

f). If $U(z)$ is positively harmonic in $R-R_{0}-F$ with $U(z)=0$ on $\partial R_{0}$ and superharmonic in $\overline{R}-R_{0}$,

$U(z)=\int_{z\in \overline{R}} N(z, p) d\mu(p)$,

where $F$ is a closed in $\overline{R}-R_{0}$.

Proof of a). Put $A_{n}=E\left[z\in \overline{R} : \delta(z, A) \leq \frac{1}{n} \right]$. Then by Theorem 2, P.C.1 $\omega(A, z)=\lim_{n=\infty} \omega(A, z)$, whence $\omega(A, z)=_{\partial} \omega(A, z)$. Next by Theorem 5. b) $\omega(A, z)=\int_{\partial} N(z, p) d\mu(p)$.

Proof of b). By Theorem 1, a) $\lim_{M=\infty} V_{M}(p)=p$ and $N(z, p)$ has M.D.I. over $R-R_{0}-V_{M}(p)$ for $p\in R-R_{0}$. Hence by the maximum principle

$\omega(p, z) \leq \omega(V_{M}(p), z) \leq \frac{N(z, p)}{M}$.

Let $M=\infty$. Then $\omega(p, z)=0$.

Put $p=A$ in a). Then $d\mu(p)$ is a point mass, whence we have at once b).

Proof of c). $v_{n}(p)N(z, p)$ is superharmonic in $\overline{R}-R_{0}$, whence $v_{n}(p)N(z, p)$

$=\lim_{n=\infty} v_{n}(p)N(z, p)$ and since $\partial R_{0}$ is compact $\phi(p)=\int_{\partial} \lim_{n=\infty} \frac{\partial}{\partial n} v_{n}(p)N(z, p) ds$ $=\lim_{n=\infty} \int_{\partial} \frac{\partial}{\partial n} v_{n}(p)N(z, p) ds=\lim_{n=\infty} \phi(v_{n}(p))$. $N(z, p)$: $p\in R-R_{0}$ has M.D.I. over $R-R_{0}-v_{n}(p)$, because $N$-Martin's topology is homeomorphic in $R-R_{0}$ to
the original topology, \( \nu_n(p) \ni p \) and \( \sup \nabla(z, p) < \infty \), whence \( \nu_m(p) \ni \nabla(z, p) \). Hence \( \nu_m(p) = \nabla(z, p) \) and \( \phi(p) = 1 \). For \( p \in B_S \), \( \nabla(z, p) = K \omega(p, z) \), \( K \omega(p, z) = \frac{\pi}{3} \omega(p, z) \): \( K > 0 \) by (b). Hence \( \phi(p) = 1 \). We consider the case: \( \omega(p, z) = 0 \) and \( p \in B \). In this case \( p \) is closed and of capacity zero. Hence \( \phi(p) = 1 \). For \( p \in B \), \( \nabla(z, p) = K \omega(p, z) \) \( K_{p} \omega(p, z) = K(p) \omega(p, z) \). Hence \( \phi(p) = 1 \). We consider the case: \( \omega(p, z) = 0 \) and \( p \in B \). In this case \( p \) is closed and of capacity zero. Hence \( \phi(p) = 1 \). The set \( \Gamma_0 \) is defined as the set (possible void) of all points \( R - R_0 \) such that \( \phi(x, p) = \frac{1}{2 \pi} \int_{\partial R_0} \frac{\partial}{\partial n} \omega(x, p) \mu(p) ds \leq \frac{1}{2} \). This means \( \phi(p) = 0 \). Then \( B_0 = \bigcup_{m \geq 1} \Gamma_0 \). By definition \( \nu_m(p) \nabla(z, p) = \lim_{n = \infty} \nu_m(p) \cap \nabla(z, p) \). Hence for any given positive number \( \varepsilon \), there exists a number \( n_0 \) such that \( \phi'(p) = \frac{1}{2 \pi} \int_{\partial \nabla(z, p)} \frac{\partial}{\partial n} \omega(p, z) \mu(p) ds - \varepsilon = \phi(p) - \varepsilon \) for \( n \geq n_0 \). Suppose \( p_i \rightarrow p \). Then \( \nabla(z, p_i) \rightarrow \nabla(z, p) \) uniformly on compact \( \partial(\nu_m(p) \cap R_n) \) and \( \nu_m(p) \cap R_n \rightarrow \nu_m(p) \cap R_n \). Now \( \nu_m(p) \cap R_n \nabla(z, p_i) \) \( \nu_m(p) \cap R_n \nabla(z, p) \) and are determined by the values of \( \nabla(z, p_i) \) and of \( \nabla(z, p) \) on \( \partial(\nu_m(p) \cap R_n) \) and on \( \partial(\nu_m(p) \cap R_n) \) respectively. Hence \( \lim_{i = \infty} \nu_m(p) \nabla(z, p) = \lim_{i = \infty} \nu_m(p) \nabla(z, p) \cap R_n \nabla(z, p) \cap R_n \). Thus \( \lim_{i = \infty} \nu_m(p) \nabla(z, p) \leq \nu_m(p) \nabla(z, p) \cap R_n \). Therefore \( \nu_m(p) \nabla(z, p) \) is lower semicontinuous with respect to \( p \) and by \( \phi'(p) = \nu_m(p) \nabla(z, p) \), \( \phi(p) \) is also lower semicontinuous, whence \( \Gamma_0 \) is closed and \( B_0 \) is an \( F_\sigma \) set.

**Proof of d).** The set \( \Gamma_0 \), being closed and compact, may be covered by a finite number of its closed subsets whose diameters are less than \( \frac{1}{m} \). It is sufficient by P.C.5 to prove (d) for any closed subset \( A \) of \( \Gamma_0 \) whose diameter is less than \( \frac{1}{m} \). Assume \( \text{Cap}(A) > 0 \). Then \( \omega(A, z) = A \omega(A, z) = \int_{\partial A} \nabla(z, p) d\mu(p) \). On the other hand, since \( \omega(A, z) = \lim_{m \to \infty} \omega(A_{2m,n}(A, z)) \), \( A_{2m,n} = A_{2m} \cap R_n \) and \( A_{2m} = E \left\{ \delta(z, A) \leq \frac{1}{2m} \right\} \), for any given positive number \( \varepsilon \), there exist numbers \( m \) and \( n \) such that \( \text{Cap}(A) = \int_{\partial A} \omega(A, z) ds \leq \int_{\partial A} \omega(A_{2m,n}, \omega(A, z) ds + \varepsilon \).
Now $\omega(A, z)$ can be approximated on $A_{2m, n}$ by a sequence of functions $V_{l}(z) = \sum_{i}^{l} c_{i} N(z, p_{i}) : p_{i} \in A (l=1, 2, \cdots)$. Then by Fatou's lemma

$$\text{Cap} (A) - \epsilon \leq \int_{\partial R_{0}} \frac{\partial}{\partial n} A(A_{2m, n}) \omega(A, z) ds \leq \lim_{l \to \infty} \int_{\partial R_{0}} \frac{\partial}{\partial n} A_{2m, n} V_{l}(z) ds \leq \frac{1}{2} \int_{dR_{0}} \frac{\partial}{\partial n} A U(z) ds (34)$$

On the other hand, by Theorem 6. a) $A U(z) = \int_{A} N(z, p) d\mu(p)$. Hence $A U(z)$ can be expressed by a limit of linear forms $V_{l}(z) = \sum_{i}^{l} c_{i} N(z, p_{i}) : p_{i} \in A (l=1, 2, \cdots)$. Hence as above

$$\int_{\partial R_{0}} \frac{\partial}{\partial n} A U(z) ds \leq \int_{\partial R_{0}} \frac{\partial}{\partial n} A_{2m} U(z) ds \leq \lim_{l \to \infty} \int_{\partial R_{0}} \frac{\partial}{\partial n} A_{2m} V_{l}(z) ds \leq \frac{1}{2} \int_{\partial R_{0}} \frac{\partial}{\partial n} A U(z) ds.$$

Let $\epsilon \to 0$. Then Cap $(A) \leq \frac{1}{2} \text{Cap} (A)$. Hence Cap $(A) = 0$, Cap $(\Gamma_{m}) = 0$ and $\omega(B_{0}, z) = 0$ by P.C.5. Since $\omega(p, z) > 0$ for $p \in B_{S}$. Hence $B_{S} \subset B_{1}$.

Proof of e). Let $A$ be a closed subset of $\Gamma_{m}$ whose diameter $\leq \frac{1}{m}$. By Theorem 5, b) $A U(z) = \int_{A} N(z, p) d\mu(p)$. Hence $A U(z)$ can be expressed by a limit of linear forms $V_{l}(z) = \sum_{i}^{l} c_{i} N(z, p_{i}) : p_{i} \in A (l=1, 2, \cdots)$. Hence above

$$\int_{\partial R_{0}} \frac{\partial}{\partial n} A U(z) ds \leq \int_{\partial R_{0}} \frac{\partial}{\partial n} A_{2m} U(z) ds \leq \lim_{l \to \infty} \int_{\partial R_{0}} \frac{\partial}{\partial n} A_{2m} V_{l}(z) ds \leq \frac{1}{2} \int_{\partial R_{0}} \frac{\partial}{\partial n} A U(z) ds.$$

On the other hand, by Theorem 6. a) $A U(z) = \int_{A} N(z, p) d\mu(p)$. Hence by (34) $A U(z) = 0$, $r_{m} U(z) = 0$ and $p_{m} U(z) = 0$.

Proof of f). Since $U(z)$ is harmonic in $R-R_{0}-F_{m}$, where $F_{m} = \{ z \in \overline{R} : \delta(z, F+B) \leq \frac{1}{m} \}$, $U(z) = F_{m} U(z)$ and $F_{m} U(z)$ is harmonic in $R-R_{0}$. Hence by Theorem 5. b) $U(z) = \int_{F_{m}} N(z, p) d\mu(p)$.

7. Canonical mass distributions. Let $U(z)$ the superharmonic function in $\overline{R}-R_{0}$. Let $U_{n}^{*}(z)$ be a function in $\overline{R}-R_{0}$ such that $U_{n}^{*}(z) = U(z)$ in $R_{n}-R_{0}$ and $U_{n}^{*}(z)$ is harmonic in $R-R_{n}$. Then $U(z) = \lim_{n} U_{n}^{*}(z)$ in $R-R_{0}$. Clearly $U_{n}^{*}(z) = r_{n} U(z)$. Hence $U_{n}^{*}(z)$ is representable by a uniquely determined mass distribution $\mu_{n}(p)$ on $\overline{R}_{n}-R_{0}$, because $\overline{R}_{n}-R_{0}$ is compact.

Operation $p[U(z)]^{*}$. Let $D$ be a compact or non compact domain in $R-R_{0}$. Let $D_{n}[U(z)]^{*}$ be a function in $R-R_{0}$ such that $U_{n}^{*}(z) = D_{n}[U(z)]^{*}$
is harmonic in \( D_n = D \cap R_n \) and superhemonic in \( \overline{R} - R_0 \) and further \( p_n[U(z)]^* \) is harmonic in \( R - R_0 - D_n \). \( p_n[U(z)]^* = 0 \) on \( \partial R_0 \) and superharmonic in \( \overline{R} - R_0 \). Such \( p_n[U(z)]^* \) is uniquely determined. In fact, let \( 1\mu_n(p) \) be the restriction of \( \mu_n(p) \) on \( \overline{D}_\cap \overline{R}_n \). Then

\[
D_n[U(z)]^* = \int N(z, p) d_1\mu_n(p).
\]

Now \( 2\mu_n(p) = \mu_n(p) - 1\mu_n(p) \) is also a positive mass distribution, which implies that \( U^*_n(z) - D_n[U(z)]^* \) is superharmonic in \( \overline{R} - R_0 \). Let \( \{n'\} \) be a subsequence of \( \{n\} \) such that \( p_{n'}[U(z)]^* \) converges uniformly in \( \overline{R} - R_0 \). Put \( p[U(z)]^* = \lim_{n' \to n} p_{n'}[U(z)]^* \). \( p[U(z)]^* \) depends on \( D \) and the subsequence \( \{n'\} \).

**Theorem 7.** Let \( D_1 \) and \( D_2 \) be two domains and \( \{n'\} \) be a subsequence such that both \( D_{1,n'}[U(z)]^* \) and \( D_{2,n'}[U(z)]^* \) converge uniformly in \( \overline{R} - R_0 \). Then

a) \( D_{1+}D_{2}[U(z)]^* \leqq D_{1}[U(z)]^* + D_{2}[U(z)] \).

b) \( D[CU(z)]^* = C_D[U(z)]^* \) for any constant \( C \geqq 0 \).

c) \( D[U(z)]^* \leqq DU(z) \leqq U(z) \).

d) Both \( D[U(z)]^* \) and \( U(z) - D[U(z)]^* \) are superharmonic in \( \overline{R} - R_0 \).

e) \( D[U(z)]^* \) is representable by a mass distribution on \( \overline{D} \), where \( \overline{D} \) is the closure of \( D \).

g) Let \( p \in \overline{R} - R_0 \). Then \( N(z, p) = \lim_{n \to \infty} v_{n}(p)[N(z, p)]^* \) for every \( v_{n}(p) \).

Let \( B_0^* \) be the set of points of \( \overline{R} - R_0 \) such that \( \lim_{n \to \infty} v_{n}(p)[N(z, p)]^* = 0 \) for every sequence: \( n_1 < n_2 < n_3 \cdots \). Then by the above fact \( B_0^* \cap (\overline{R} - R_0) = \emptyset \), and by \( c) \) \( v_{n}(p)[N(z, p)]^* \geqq v_{n}(p)[N(z, p)]^* \), whence \( B_0^* \supset B_0 \).

\( h) \) \( B_0[\{z\}]^* = 0 \) for \( U(z) = \int_{B_0} N(z, p) d\mu(p) \).

**Proof of a), b), d) and e)** is clear by the definition.

**Proof of c)** \( D[U(z)] = \lim_{n \to \infty} p_n U(z) : D_n = D \cap R_n \). Now \( U(z) = p_n U(z) \geqq p_n[U(z)]^* \) on \( D_n \) and both \( p_n U(z) \) and \( p_n[U(z)]^* \) are harmonic in \( R - R_0 - D \), whence by the maximum principle \( p_n[U(z)]^* \leqq p_n U(z) \). Hence

\[
P[U(z)]^* = \lim_{n \to \infty} p_n[U(z)]^* = \lim_{n \to \infty} p_n U(z) = D[U(z)].
\]

**Proof of f)** \( p_n[U(z)]^* \) and \( U_n^*(z) - p_n[U(z)]^* \) are representable by

6) \( B_0 = B_0^* \) will be proved in Theorem 9.
positive mass distributions $\mu_n^1$ and $\mu_n^2 = \mu_n^1 - \mu_n^2$ on $R_n \cap \overline{D}$ and $R_n \cap \overline{CD}$ respectively. But the total masses of $\mu_n^1$ and $\mu_n^2$ are bounded $\leq \int_{\partial R_0} \frac{\partial}{\partial n} U(z) ds$.

We can find a subsequence $\{n'_k\} \subset \{n\}$ such that both $\mu_n^1$ and $\mu_n^2$ have weak limits $\mu_1$ on $\overline{D} \cap \overline{R}$ and $\mu_2$ on $\overline{CD} \cap \overline{R}$ respectively. Clearly by $\{n'_k\} \subset \{n\}$, $U(z) = \int N(z, p) d\mu(p) : \mu = \mu_1 + \mu_2$, $\partial [U(z)]^* = \int N(z, p) d\mu(p)$ and $U(z) - \partial [U(z)]^* = \int N(z, p) d\mu_2(p)$. Hence $\partial [U(z)]^*$ and $U(z) - \partial [U(z)]^*$ are superharmonic in $\overline{R} - R_0$.

Proof of $g)$. Since $p \in R - R_0$, there exists a number $n_0$ for any given neighbourhood $\nu_{n_0}(p)$ of $p$ such that $\nu_{n_0}(p) \subset R_n - R_0$ for $n \geq n_0$. Then $N(z, p)$ has M.D.I. over $R - R_n$, whence $N(z, p)$ is harmonic in $\overline{R} - R_n$ and $N^*_n(z, p) = N(z, p)$ in $R - R_n$. In this case $N^*_n(z, p) = \int N(z, p) d\mu(p)$ and $\nu_{n_0}(p) [N(z, p)]^* = N^*_n(z, p) = N(z, p)$ for $n \geq n_0$. Hence $p [N(z, p)]^* = N(z, p)$ and $B^*_0 \cap (R - R_0)$.

Proof of $h)$. $n_0(U(z) = 0$ implies $h)$ by $c)$.

**Theorem 8.** Every positive superharmonic function in $\overline{R} - R_0$ such that $U(z) = 0$ on $\partial R_0$ is representable by a positive mass distribution $\mu$ on $\overline{R} - R_0 + B_1$ such that

$$U(z) = \int N(z, p) d\mu(p) \quad \text{for } z \in R - R_0.$$  

We call such a canonical mass distribution.

**Remark.** It seems that Theorem 8 can be improved to the following: $U(z)$ is representable by a mass distribution on $R - R_0 + B^*_1 : B^*_1 = B - B_0 \subset B - B_0 = B_1$. But in Theorem 9 it is proved that $B^*_0 = B_0$. Hence the above two are equal.

**Proof.** Suppose $V(z) = \int N(z, p) d\mu(p)$. Then by Theorem 7, e) $n_0 V(z) = 0$ and by $c)$ of Theorem 7 $r_m V(z) = 0$. This implies $\lim_{n} r_m, n [V(z)]^* = 0$, where $\Gamma_{m, n} = E \left[ z \in \overline{R} : \delta(z, \Gamma_m) \leq \frac{1}{n} \right]$ Let $z_0$ be a point in $R - R_0$. Then for any given positive number $\epsilon$, there exists a number $n_0(m)$ such that

$$r_m, n [V(z_0)]^* \leq r_m, n V(z_0) \leq \frac{\epsilon}{2^m} \quad \text{for } n \geq n_0(m).$$

For each $m$ select $\Gamma'_m(= \Gamma_{m, n})$ in this fashion. Put $C_m = \sum_{i=1}^{m} \Gamma'_i$. Then $C_m$
is closed and increases as \( m \to \infty \). Denote by \( \bar{A}_m \) and \( A_m \) the closure of the of the complement of \( C_m \) in \( B \) and \( \overline{R} - R_0 \) respectively. Then the distance between \( \bar{A}_m \) and \( \Gamma_m \) \((\Gamma_m \text{ is contained in } B_0 \text{ by the definition of } \Gamma_m) \) is at least \( \frac{1}{n(m)} \). Thus \( \{\bar{A}_n\} \), which forms a decreasing sequence, has an intersection \( \bar{A} \) which is closed and, having no points in common with any \( \Gamma_m \), is a subset of \( B_1 \).

Now

\[
c_m[V(z)]^* \leq c_m V(z) \leq \sum_{i}^{m} \Gamma_i V(z) \leq \sum_{i=1}^{m} 2^{-i} \epsilon < \epsilon
\]

for \( z=z_0 \).

Observing \( \bar{A}_m + \bar{C}_m = B \), we obtain for a subsequence \( n' \) of \( n \) such that

\[
\text{as } n' \to \infty, \tag{7}
\]

\[
\bar{C}_m = C_m \cap B \text{ and } A_m \cap B = \bar{A}_m \text{ and } A_m \text{ is a closed domain in } \overline{R} - R_0,
\]

\[
\bar{A}_m[V(z)]^* \leq z_m[V(z)]^* = V(z) \leq \bar{A}_m[V(z)]^* + c_m[V(z)]^*,
\]

whence \( V(z) \geq \bar{A}_m[V(z)]^* \geq V(z) - \epsilon \) for \( z=z_0 \).

Let \( \mu_1^\prime \) be the restriction of \( \mu_1^\prime\prime \) on \( B_1 \) and put

\[
V_1(z) = \int_{B_1} N(z, p) d(\mu_1^\prime + \mu_1^\prime\prime)(p).
\]

Then

\[
0 \leq V_1(z) \leq V(z) - \bar{A}_m[V(z)]^* < \epsilon
\]

for \( z=z_0 \) and \( V(z) - V_1(z) = \int_{B_1} N(z, p) d(\mu_1^\prime + \mu_1^\prime\prime)(p) \). Put \( \mu_1^\prime = \mu_1^\prime + \mu_1^\prime\prime \) and \( \mu_1^\prime\prime = \mu_1^\prime - \mu_1^\prime\prime \).

Repeat the process (used for \( V(z) \)) for \( V_1(z) \), writing \( V_1(z) = V_2(z) + (V_1(z) - V_2(z)) \), where \( V_2(z) \) and \( V_1(z) - V_2(z) \) are representable by positive mass \( \mu_2^\prime \) and \( \mu_2^\prime\prime \) over \( (C_m \cap B) \) and \( \bar{A}_m \) respectively.

Hence \( V(z_0) \leq \bar{A}_m[V(z)]^* + \epsilon \), where \( \bar{A}_m[V(z)]^* \) and \( \bar{A}_m[V(z)]^* \) are harmonic in \( R - R_0 \), superharmonic in \( \overline{R} - R_0 \) and are representable by positive mass distributions \( \mu_1^\prime \) and \( \mu_1^\prime\prime \) over \( (C_m \cap B) \) and \( \bar{A}_m \) respectively. Let \( \{n''\} \) be a subsequence of \( \{n'\} \) such that

\[
\text{as } n'' \to \infty, \tag{8}
\]

\[
\bar{A}_m[V(z)]^* \leq z_m[\bar{A}_m[V(z)]^*] = V(z) \leq \bar{A}_m[V(z)]^* + c_m[V(z)]^*,
\]

whence \( V(z) \geq \bar{A}_m[V(z)]^* \geq V(z) - \epsilon \) for \( z=z_0 \).

7) \( CR_{n'} \) means the complementary set of \( R_{n'} \).
mass distributions \_\mu^{**} and \_\mu^{*} on \_B_{0} and \_B_{1} respectively such that \_V_{2}(z_{0}) < \frac{\epsilon}{2}.

Proceeding in this way,
\_V_{n}(z) = \_V_{n+1}(z) - (\_V_{n}(z) - \_V_{n+1}(z)),
where \_V_{n+1}(z) and \_V_{n}(z) - \_V_{n+1}(z) are representable by positive mass distributions \_n\mu^{**} and \_n\mu^{*} over \_B_{0} and \_B_{1} respectively such that \_V_{n+1}(z_{0}) < \frac{\epsilon}{2^n}. Then
\_V(z) = \_V(z) - \_V_{1}(z) + \sum_{n=1}^{\infty} (\_V_{n}(z) - \_V_{n+1}(z))
and \_V(z) is represented by a positive mass distribution \mu = \sum_{n=1}^{\infty} n\mu^{*} over \_B_{1}.

Let \_U(z) = \int_{R-R_{0}+B} N(z, p) d\mu(p). Let \_\mu' be the restriction of \mu over \_B_{0}. Then \_\mu' can be replaced by another distribution over \_B_{1} \_\in \_R-R_{0}+B_{1} without any change of \_U(z). Hence we have the theorem.

8. \_N-minimal functions and \_N-minimal points. Let \_U(z) be a positively \_superharmonic function in \_R-R_{0} with \_U(z) = 0 on \_\partial R_{0}. If \_U(z) \geq \_V(z) \geq 0 implies \_V(z) = \_K\_U(z) (0 \leq \_K \leq 1) for every function \_V(z) such that both \_U(z) - \_V(z) and \_V(z) are positively \_superharmonic in \_R-R_{0}, \_U(z) is called \_N-minimal function.

Theorem 9. a). Let \_U(z) be a \_N-minimal function such that \_U(z) = \int_{A} N(z, p) d\mu(p). Then \_U(z) is a multiple of some \_N(z, p): p \_\in \_R-R_{0} + B_{1} \_\cap \_A.

b). \_N(z, p) is \_N-minimal or not according as \_\phi(p) = 1 or = 0, i.e. \_p \_\in \_R-R_{0} + B_{1} or \_p \_\in \_B_{0}.

c). Let \_V_{\_M}(p) = E[z \_\in \_R: \_N(z, p) \_> \_M] and \_\nu_{\_n}(p) = E[z \_\in \_\overline{R}: \_\delta(z, p) < \frac{1}{n}] and suppose \_p \_\in \_R-R_{0} + B_{1}. Then \_N(z, p) = \_V_{\_M}(p) - \_\nu_{\_n}(p) \_N(z, p) = \_\nu_{\_n}(p) \_N(z, p): for \_M < \_\sup_{z \_\in \_R-R_{0}} \_N(z, p), i.e.
\_N(z, p) = \_M\_\omega(\_V_{\_M}(p), z) \_\in \_R-R_{0} - \_V_{\_M}(p).

d). For any given number \_M < \_\sup_{z \_\in \_R-R_{0}} \_N(z, p), there exists a number \_n such that
\_R - \_\nu_{\_n}(p) \_\subset \_V_{\_M}(p) \_\_\text{for} \_p \_\in \_R-R_{0} + B_{1}.
\( B_0^* = B_0 \).

Proof of a). Suppose, \( U(z) \) is \( N \)-minimal and \( U(z) = \int A N(z, p) d\mu(p) \).

Assume, \( \mu \) is not a point mass. Then for any positive mass distribution \( \mu' \) such that \( 0 < \mu' < \mu \), \( \int A N(z, p) d\mu(p) \) and \( \int A N(z, p) d(\mu - \mu')(p) \) are multiples of \( U(z) \) by the \( N \)-minimality of \( U(z) \), because these are superharmonic in \( \overline{R} - R_0 \). Since \( \mu \) is not a point mass, we can find two closed sets \( A_i \) and \( A_z \) such that \( A_i \subset A \) \( (i=1,2) \), \( \text{dist}(A_i, A_z) > 0 \) and the restriction of \( \mu \) on \( A_i \) is positive \( (i=1,2) \). Let \( \{A_{i,n}\} \) be a decreasing sequence of closed subsets of \( A_i \) such that \( A_{i,n} \rightarrow p_i \) as \( n \rightarrow \infty \), \( \mu_{i,n} \) (restriction of \( \mu \) on \( A_{i,n} \)) \( > 0 \) and that the potential of \( \mu_{i,n} \) is a multiple of \( U(z) \).

Put \( \mu_{i,n} = \sim \underline{\mu}_{i,n} \). Then \( \int d\mu_{i,n} \) from \( \{\tilde{\mu}_{i,n}\} \) we can find weak limits \( \tilde{\mu}_i \) \( (i=1,2) \) of unity at \( p_i \in A \) and \( N(z, p_i) = \int A N(z, p) d\tilde{\mu}_i(p) = K \). Assume \( \tilde{\mu}_i \) is not a point mass.

By the minimality of \( U(z) \) \( \mu \) is also a point mass at \( p \in A \) and \( U(z) = \frac{N(z, p)}{K} \). Next we show \( p \in (R - R_0 + B_1) \). Assume \( U(z) = K'N(z, p) \) : \( K' > 0 \) and \( p \in B_o \). Every positive superharmonic function in \( \overline{R} - R_0 \) is representable by a canonical mass distribution \( \mu \) on \( R - R_0 + B_1 \) by Theorem 8 such that \( U(z) = K'N(z, p) = \int B_1 N(z, p) d\mu(p) \). By the minimality of \( U(z) \) \( \mu \) is \( N \)-minimal. Thus by a) \( N(z, p) \) is \( N \)-minimal if and only if \( p \in R - R + B_1 \). Hence we have b).

Proof of c). For \( p \in R - R_0 + B_1 \), \( \nu N(z, p) = N(z, p) \). Hence \( N(z, p) = \)}
We show $v_n(p) \subset V_m(p) N(z, p) = N(z, p)$.

Case 1. $p \in R - R_0 + B_1 - B_S$. In this case we remark $\sup_{z \in R - R_0} N(z, p) = \infty$. In fact, assume $N(z, p) \leq M$ and $p \in R - R_0 + B_1 - B_S$. Then $N(z, p) \leq M \omega(z, p) = 0$. This contradicts $p \in R - R_0 + B_1 - B_S$. Hence $\sup_{z \in R} N(z, p) = \infty$.

Put $\lim_{n \to \infty} CV_m(p) \cap v_n(p) N(z, p) = N'(z, p)$.

Then by $v_n(p) \supset (v_n(p) \cap CV_m(p)) N'(z, p)$ has no mass except at $p$. Hence $N'(z, p) = KN(z, p)$ ($0 \leq K \leq 1$).

But $\sup_{z \in \partial R} N(z, p) = \infty$ and $\sup_{z \in R} N'(z, p) \leq M$ imply $K = 0$ and $N'(z, p) = 0$. Hence $N(z, p) = M \omega(p, z)$.

Therefore $N(z, p) = \lim_{n \to \infty} V_m(p) \subset N(z, p) = \lim_{n \to \infty} V_m(p) N(z, p) \leq N(z, p)$. Now $v_n(p) N(z, p)$ has M.D.I. $\leq 2\pi M$ over $R - R_0 - V_m(z)$ and $N(z, p) = M$ on $\partial V_m(p)$. Hence $N(z, p) = M \omega(V_m(p), z)$ in $R - R_0 - V_m(p)$.

Case 2. $p \in B_S$. In this case by Theorem 7, b) $N(z, p) = K \omega(p, z)$. Hence $\omega(p, z) \geq M \omega(p, z)$.

Put $N_{n}(z, q_{i})$ be a harmonic function in $R_{n} - R_0 - q_{i}$ such that $N_{n}(z, q_{i}) = 0$ on $\partial R_{n}$, $\partial N(z, q_{i}) = 0$ on $\partial R_{n}$ and has a logarithmic singularity at $q_{i}: q_{i} \in R - R_0 - V_m(p)$. Then by the definition of $N(z, q_{i})$, $N_{n}(z, q_{i}) \to N(z, q_{i})$ as $n \to \infty$. Let $N_{n}(z, p)$ be a harmonic function in $R_{n} - R_0 - V_{m'}(p)$ such that $N_{n}(z, p) = 0$ on $\partial R_{n}$, $N_{n}(z, p) = M'$ on $\partial V_{m'}(p) = C_{m'}$ and $\partial N(z, p) = 0$ on $\partial R_{n}$. Then since $N(z, p) = M' \omega(V_{m'}(p), z)$ has M.D.I. over $R - R_0 - V_{m'}(p)$, $N_{n}(z, p) \to N(z, p)$ as $n \to \infty$. By the Green's formula

$$\int_{C_{m'}} \frac{\partial}{\partial n} N(z, q_{i}) ds = 2\pi N_{n}(q_{i}, p)$$

by $q_{i} \in V_{m}(p)$. 

$p N(z, p) \leq N(z, p)$. We show $v_n(p) \cap v_m(p) N(z, p) = N(z, p)$.
Since $C_{M'}$ is regular and $N_n(z, q_i)$ is uniformly bounded on $C_{M'}$ by $q_i \in R - R_0 - V_M(p) = V_M'(p) \subset V_M(p)$, (by Theorem 6) we have by letting $n \to \infty$,

$$M > \frac{1}{2\pi} \int_{C_{M'}} N(z, q_i) \frac{\partial}{\partial n} N(z, p) \, ds = N(q, p) \quad \text{by} \quad q_i \notin V_M(p).$$

(34)

Assume that $d)$ is false. Then there exists a sequence of points $\{q_i\}$ such that $q_i \in CV_M(p) \cap (R - R_0)$ and $\delta(p, q_i) \to 0$. Let $M > M^* < M'$ and put $\epsilon_0 = 2\pi \left(1 - \frac{M}{M^*}\right) > 0$. By the regularity of $C_{M'}$ there exists a number $n_0$ such that

$$\int_{C_{M'} \cap R_n} \frac{\partial}{\partial n} N(z, p) \, ds \geq 2\pi - \epsilon_0 \quad \text{for} \quad n \geq n_0.
$$

If $N(z, q_i) > M^*$ on $R_{n_0} \cap C_M(p)$, then

$$\int_{C_{M'} \cap R_{n_0}} N(z, q_i) \frac{\partial}{\partial n} N(z, p) \, ds \geq M^*(2\pi - \epsilon_0) = M.$$ 

But $N(z, q_i) = M$ for $q_i \notin V_M(p)$.

Hence (34) contradicts (35). Hence $N(z, q_i) \leq M^*$ on $C_{M'} \cap R_{n_0}$ and there exists at least one point $z_i$ on $C_{M'} \cap R_{n_0}$ such that $N(z_i, q_i) \leq M^* < M'$. Since $R_{n_0} \cap C_{M'}$ is compact, there exists a point $\tilde{z}$ which is one of the limiting points $\{z_i\}$. Now $N(\tilde{z}, q) \leq \lim_{i=\infty} N(\tilde{z}, q_i) \leq M^*$, where $q = \lim_{i=\infty} q_i$. On the other hand, $N(\tilde{z}, q) = M' = N(z, p)$ by $\lim_{i=\infty} \delta(p, q_i) = 0$ ($\delta(p, q) = 0$ is equivalent to $N(z, q) = N(z, p)$) and by $\tilde{z} \in C_{M'}$. This is a contradiction. Hence we have $d$.

Proof of e). By g) of Theorem 7, $B_0 \subseteq B_0^*$. We show $B - B_0 = B_1 \subseteq B - B_0^*$. Let $p \in B_1$. Let $N_n^*(z, p)$ be a function in $R - R_0$ such that $N_n^*(z, p) = N(z, p)$ in $R_n - R_0$ and $N_n^*(z, p)$ has M.D.I. over $R - R_n$. Clearly $N_n^*(z, p)$ is superharmonic in $\overline{R} - R_n$. Then since $p \in B_1$, $N_n^*(z, p) = N(z, p)$ is harmonic in $R_n - R_0$ and $N_n^*(z, p)$ is represented by a mass distribution on $\partial R_n$.

Hence $b_m[N_n(z, p)]^* = N_n^*(z, p)$, where $B_m = R - R_m$ and $n > m$.

Now $b_{m \to 1}(p) [N_n^*(z, p)]^* + b_{m \to \infty}(p) [N_n^*(z, p)]^* = b_m N_n^*(z, p) = N_n^*(z, p).$ 8)

Let $\{n'\}$ be a subsequence of $\{n\}$ such that $b_{m \to 1}(p) [N_n^*(z, p)]$ converges uniformly. Then by letting $n' \to \infty$,

$$b_m [N_n^*(z, p)]^* + b_m [N_n(z, p)]^* = N(z, p) \geq b_m [N_n(z, p)]^*.\quad \text{and} \quad b_m [N_n(z, p)]^* \text{ have masses } \mu_n \text{ and } \mu_n$$

8) $C_\nu(p)$ means the complementary set of $\nu(p)$. 


$u_{l}(p) \cap \partial R_{n}$ and $C_{U_{l}}(p) \cap \partial R_{n}$ respectively and $\{_{1}\mu_{n}\}$ and $\{_{2}\mu_{n}\}$ have weak limits $\mu_{\ast}$ and $\mu_{\ast}$ on $B \cap \partial \overline{v_{l}}(p)$ and $C_{U_{l}} \cap \partial R_{n}$ respectively.

$1\mu$ and $2\mu$ on $B_{\ast}$ and $C_{U_{l}} \cap \partial R_{n}$ respectively and $B_{m} \cap v_{l^{(p)}}[N(z, p)]^{*} = \int N(z, p) d_{1}\mu(p)$ and $B_{m} \cap v_{l^{(p)}}[N(z, p)]^{*} = \int N(z, p) d_{2}\mu(p)$.

Since by the assumption $N(z, p)$ is $N$-minimal, whence $b_{m} \cap v_{l^{(p)}}[N(z, p)]^{*}$ and $b_{m} \cap C_{U_{l}}[N(z, p)]^{*}$ are $N$-minimal and $K_{i}N(z, p)$ for every $v_{i}(p)$.

$K_{i}N(z, p)$ is superharmonic by Theorem 7. Hence $p \in (B \cap v_{l}(p))[N(z, p)]^{*}$ and $B_{m} \cap v_{l^{(p)}}[N(z, p)]^{*} = K_{i}N(z, p)$.

Then by a) of Theorem 9 $p \in (B \cap v_{l}(p))[N(z, p)]^{*}$ and $B_{m} \supset B_{0}^{\ast}$.

$B_{m} \cap v_{l}(p)[N(z, p)]^{*} = N(z, p)$ for every $u_{l}(p)$.

Now $\{n^{'}\}$ is any subsequence of $\{n\}$ such that $b_{m} \cap v_{l}(p)[N(z, p)]^{*}$ converges, hence $v_{l^{(p)}}[N_{n^{'}},(z, p)]^{*} \rightarrow N(z, p)$ as $n \rightarrow \infty$ for every $v_{i}(p)$.

Whence $p \in B - B_{0}^{\ast}$.

Hence $B_{0} \supset B_{0}^{\ast}$ and $B_{0} = B_{0}^{\ast}$.

Theorem 10. Let $V_{M}(p) = E[z \in R : N(z, p) > M]$ for $p \in R - R_{0} + B_{1}$.

Then $V_{M}(p)$ may consist of at most an enumerably infinite number of domains $D_{l}$ ($l = 1, 2, \cdots$).

a) $D_{R - R_{0} - V_{M}(p)}(N(z, p)) = 2\pi M$ and $\min(M, N(z, p)) = \mathrm{mod}(V_{M}(p), z)$.

b) Let $D_{i}$ be a component of $V_{M}(p)$. Then $D_{i}$ contains a subset $D$ of $V_{M}(p)$ for $M < M' < \sup_{z \in R} N(z, p)$.

c) Let $U(z)$ be a positive superharmonic function with $U(z) = 0$ on $\partial R_{0}$ and let $C_{M}(p)$ be a regular curve of $N(z, p)$: $p \in R - R_{0} + B_{1}$ such that $\int_{C_{i}} \frac{\partial}{\partial n}N(z, p) ds = 2\pi$. Then for $M_{i} < M_{i+1}$

$$\text{mean}(U(z) \text{ on } C_{M_{i}}) = \frac{1}{2\pi} \int_{C_{M_{i}}} U(z) \frac{\partial}{\partial n}N(z, p) ds = \text{mean}(U(z) \text{ on } C_{M_{i+1}}).$$

d) Let $C_{M_{i}}$ ($i = 1, 2, \cdots$) and $C_{M}$ be a regular curve of $N(z, p)$: $p \in R - R_{0} + B_{1}$ such that $M_{i} \uparrow M$. Then

$$\lim_{i} \text{mean}(U(z) \text{ on } C_{M_{i}}) = \text{mean}(U(z) \text{ on } C_{M}).$$

If $C_{M}$ is not regular, we define $\text{mean}(U(z) \text{ on } C_{M})$ by $\lim_{i \rightarrow \infty} \text{mean}(U(z) \text{ on } C_{M_{i}})$ where $M_{i} \uparrow M$ and $\{C_{M_{i}}\}$ are regular. Then $\text{mean}(U(z) \text{ on } C_{M})$ is defined for every $M < \sup_{z \in R} N(z, p)$ and c) holds for every $M$.

Proof of a). By Theorem 9. c) we have $\mathrm{mod}(V_{M}(p), z) = \min(M, N(z, p))$ and $N(z, p) = \lim N_{n}(z, p)$, where $N_{n}(z, p)$ is a harmonic function in $R_{n} - R_{0} - V_{M}(p)$ such that $N_{n}(z, p) = 0$ on $\partial R_{0}$, $N_{n}(z, p) = M$ on $\partial V_{M}(p)$ and
\[
\frac{\partial}{\partial n}N'(z, p) = 0 \text{ on } \partial R - V_M(p). \quad \text{Clearly } D_{R-R_0-V_M'(p)}(N(z, p)) \leq \lim_{n}D_{R-R_0-V_M'(p)}(N_n'(z, p)) = \lim_{n=\infty}M\int_{\partial V_M(p)} \frac{\partial}{\partial n}N_n'(z, p) ds = M\int_{\partial R_0} \frac{\partial}{\partial n}N(z, p) ds = 2\pi M.
\]

On the other hand, by Fatou's lemma
\[
D(\min(M, N(z, p))) \leq \lim_{n}D(N_n'(z, p)) \leq 2\pi M.
\]

Thus \(D(\min(M, N(z, p))) = 2\pi M\).

Proof of b). Assume that \(D_1 \cap V_M(p) = 0\). Consider \(D(\min(M, N(z, p)))\).

Consider \(D(\min(M, N(z, p)))\).

Let \(N'(z, p) = N(z, p)\) in \(R - R_0 - D_1\), \(N'(z, p) = M\) in \(D_1\).

Then since \(N(z, p)\) is non constant in \(R - R_0\), \(D_{R-R_0-V_M'(p)}(N'(z, p)) < D_{R-R_0-V_M'(p)}(N(z, p))\) and \(N'(z, p) = N(z, p) = M'\) on \(\partial V_M'(p)\).

This contradicts that \(N(z, p)\) has M.D.I. over \(R - R_0 - V_M'(p)\) among all functions with value \(M'\) on \(\partial V_M'(p)\) and 0 on \(\partial R_0\). Hence we have b).

Proof of c). By a) \(N(z, p) = M\omega(V_M(p), z)\) in \(R - R_0 - V_M(p)\) for every \(M < \sup N(z, p)\), whence \(C_M\) is regular for almost all constants \(M' < \sup_{z \in R} N(z, p)\).

\[
\int_{\partial V_{M_{i+1}}(p)} \frac{\partial}{\partial n} N'(z, p) ds = \int \frac{\partial}{\partial n} N(z, p) ds = 2\pi M.
\]

Let \(N_{n}^'(z, p)\) be a harmonic function in \(R_{n} \cap (V_{M_{i}}(p) - V_{M_{i+1}}(p))\) such that \(N_{n}^'(z, p) = M_i\) on \(\partial V_{M_{i}}(p)\), \(N_{n}^'(z, p) = M_{i+1}\) on \(\partial V_{M_{i+1}}(p)\) and \(\underline{\partial}N_{n}^'(z, p) = 0\) on \(\partial R_{n} \cap (V_{M_{i}}(p) - V_{M_{i+1}}(p))\).

Then \(N_{n}^'(z, p) \Rightarrow N(z, p)\).

Hence by the Green's formula
\[
\int_{\partial V_{M_{i+1}}(p) \cap R_{n}} U_{n}^L(z) \frac{\partial}{\partial n} N_{n}^'(z, p) ds = \int_{\partial V_{M_{i+1}}(p) \cap R_{n}} U_{n}^L(z) \frac{\partial}{\partial n} N(z, p) ds = 0.
\]

By the superharmonicity of \(U(z) \leq \overline{U(z)}\) in \(V_{M_{i}}(p)\) and
\[
\lim_{L \to \infty} CV_{M_i}(p)U^L(z) - U(z) \leq 0
\]

Thus we have c).

Proof of d). By c) \[
\int_{\partial V_{M_i}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds = \lim_{L \to \infty} \int_{\partial V_{M_i}(p)} U^L(z) \frac{\partial}{\partial n} N(z, p) ds
\]
is clear. Since \[
\int_{\partial V_{M_i}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds = \lim_{L \to \infty} \int_{\partial V_{M_i}(p)} \min(L, U(z)) \frac{\partial}{\partial n} N(z, p) ds
\]
for any given positive number \(\epsilon\), there exist \(L_0\) and \(n_0\) such that

\[
\int_{\partial V_{M_i}(p)} \min(L, U(z)) \frac{\partial}{\partial n} N(z, p) ds \geq \int_{\partial V_{M_i}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds - \epsilon
\]

for \(n \geq n_0\) and \(L \geq L_0\).

Suppose \(z_i \in \partial V_{M_i}(p), z \in \partial V_{M_i}(p)\) and \(z_i \to z\). Then \(\frac{\partial}{\partial n} N(z_i, p) \to \frac{\partial}{\partial n} N(z, p)\) and since \(\min(L, U(z))\) is continuous in \(\mathbb{R} - R_0\), \(U(z_i) \to U(z)\) in \(\mathbb{R} - R_0\). Hence

\[
\lim_{\partial V_{M_i}(p)} \int U(z) \frac{\partial}{\partial n} N(z, p) ds \geq \lim_{\partial V_{M_i}(p)} \int U(z) \frac{\partial}{\partial n} N(z, p) ds
\]

\[
\geq \int_{\partial V_{M_i}(p)} \min(L, U(z)) \frac{\partial}{\partial n} N(z, p) ds \geq \int_{\partial V_{M_i}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds - \epsilon.
\]

Hence by letting \(\epsilon \to 0\),

\[
\lim_{\partial V_{M_i}(p)} \int U(z) \frac{\partial}{\partial n} N(z, p) ds = \int_{\partial V_{M_i}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds.
\]

9. The value of a superharmonic function on \(B\). Till now the value of a superharmonic function is defined in \(\mathbb{R} - R_0\) only. We shall consider it on the ideal boundary.

Let \(U(z)\) be a positive superharmonic function in \(\mathbb{R} - R_0\) with \(U(z) = 0\) on \(\partial R_0\). Then mean \((U(z) \text{ on } \partial V_M(p))\) (if \(\partial V_M(p)\) is not regular, we use d) of Theorem 9)) \(\uparrow\) as \(M \uparrow \sup N(z, p)\) for \(p \in \mathbb{R} - R_0 + B_1\). We define the value \(U(z)\) at \(p \in \mathbb{R} - R_0 + B_1\) by

\[
\lim_{M \uparrow \sup N(z, p)} \text{mean} (U(z) \text{ on } \partial V_M(p)) = M \uparrow \sup N(z, p).
\]

It is clear, if \(U(z)\) is continuous or \(\infty\) at a point \(z \in \mathbb{R} - R_0\), this coincides
with $U(z)$. Next at $p \in B_0$ we shall define the value of $U(z)$. For $p \in B_0$, $N(z, p) = \int_{B_1} N(z, p_a) d\mu(p_a)$, where $\mu(p_a)$ is a canonical distribution and not necessarily uniquely determined. In this case we define $U(p)$ by

$$\int_{B_1} U(p_a) d\mu(p_a).$$

This definition reduces to the former definition, if $p \in B_1$, because the canonical mass distribution of $N(z, p) : p \in R - R_0 + B_1$ must be a point mass at $p$. Hence our definition is natural. If $U(p) \geq \frac{1}{2\pi} \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds : p \in R - R_0 + B_1$, we say that $U(z)$ is superharmonic locally at a point $p$.

\textbf{Theorem 11. a).} $N(p, q) = N(q, p)$ for $p$ and $q \in R - R_0$.

\textbf{b).} $N(p, p) = \sup_{z \in R} N(z, p) : p \in R - R + B_1$.

\textbf{c).} Let $U(z)$ be a positive superharmonic function in $\bar{R} - R_0$ with $U(z) = 0$ on $\partial R_0$ (of course $N(z, p)$ is a superharmonic function by Theorem 5.a). Then $U(z)$ is lower semicontinuous in $\bar{R} - R_0$ and $U(z)$ is superharmonic locally at every point of $R - R_0 + B_1$. There exists at least one canonical distribution $\mu$ by Theorem 8 such that

$$U(z) = \int_{R - R_0 + B_1} N(z, p) d\mu(p)$$

for $z \in R - R_0$, where the uniqueness of $\mu$ is not proved.

By the definition of the value of $U(z)$ on $B$, $U(z)$ is well defined at any point $p \in R - \bar{R}_0$ and the value of $U(z)$ at a point of $B$ does not depend on a particular distribution and

$$U(z) = \int_{R - R_0 + B_1} N(z, p) d\mu(p)$$

is valid not only in $R - R_0$ but also on $B$.

\textbf{Proof of a).} Case 1. $p$ and $q$ are contained in $R - R_0$. In this case, by the Green’s formula

$$N(p, q) = N(q, p).$$

Case 2. One of $p$ and $q$ is contained in $R - R_0$.

\textbf{Case 2. a).} $p \in R - R_0$ and $q \in B_1$. Then $N(z, q)$ is harmonic in $R - R_0$ and by the maximum principle $V_M(q)$
clusters at $B$ as $M \uparrow \sup_{z \in \mathbb{R}} N(z, q)$. Hence we can find a number $M$ such that $p \not\in V_M(q)$ and $\partial V_M(q)$ is regular. Then by (34)

$$N(p, q) = \frac{1}{2\pi} \int_{\partial V_M(q)} N(z, p) \frac{\partial}{\partial n} N(z, q) \, ds$$

$$= \frac{1}{2\pi} \lim_{M \to M^*} \int_{\partial V_M(q)} N(z, p) \frac{\partial}{\partial n} N(z, q) \, ds = N(q, p),$$

(36)

where $M^* = \sup_{z \in \mathbb{R}} N(z, p)$.

Case 2. b). $p \in R - R_0$, $q \in B_0$. Then $N(z, q) = \int_{B_1} N(z, q_\beta) \mu(q_\beta) : z \in R - R_0$, where $\mu(q_\beta) : q_\beta \in B_1$ is a canonical distribution of $N(z, q)$. Then by case 2, a) $N(q_\beta, p) = N(p, q_\beta)$ and by the definition of the value of $N(z, q)$ at $p \in B_0$, we have

$$N(q, p) = \int_{B_1} N(q_\beta, p) \, d\mu(q_\beta) = \int_{B_1} N(p, q_\beta) \, d\mu(q_\beta) = N(q, p)$$

by $p \in R - R_0$.

Now $N(p, q) : p \in R - R_0$ and $q \in B$ is well defined and $N(q, p) = N(p, q)$, hence $N(q, p)$ does not depend on a particular distribution $\mu(q_\beta)$.

Case 3. $p \in B$ and $q \in B$.

Case 3. a) $p \in B_1$ and $q \in B_1$.

Let $\xi$ and $\eta \in R - R_0$. Then by (36)

$$N(p, \eta) = N(\eta, p) = \frac{1}{2\pi} \int_{\partial V_M(p)} N(z, \eta) \frac{\partial}{\partial n} N(z, p) \, ds \quad \text{for} \quad \eta \not\in V_M(p),$$

(37)

$$N(p, \eta) = N(\eta, p) \geq \frac{1}{2\pi} \int_{\partial V_M(p)} N(z, \eta) \frac{\partial}{\partial n} N(z, p) \, ds \quad \text{for} \quad \eta \in V_M(p),$$

(38)

where $\partial V_M(p)$ is regular.

Since mean $(N(z, q)$ on $\partial V_M(p)) = \frac{1}{2\pi} \int_{\partial V_M(p)} N(\xi, q) \frac{\partial}{\partial n} N(\xi, p) \, ds$ and since $V_M(q)$ clusters at $B$ as $M \uparrow \sup_{z \in \mathbb{R}} N(z, p)$, there exists a number $M'$ for any given positive number $\varepsilon$ such that

$$\text{mean} (N(z, q)$ on $\partial V_M(p)) - \varepsilon \leq \frac{1}{2\pi} \int_{\partial V_M(p)} N(\xi, q) \frac{\partial}{\partial n} N(\xi, p) \, ds,$$

where $\partial V_M(p)$ is the part of $\partial V_M(p)$ outside of $V_{M'}(q)$ and $\partial V_{M'}(q)$ is regular.

Suppose $\xi \in \partial V_M(p)$, then $\xi \not\in V_{M'}(q)$, whence
\[
N(\xi, q) = N(q, \xi) = \frac{1}{2\pi} \int_{\partial V_{M}(q)} N(\eta, \xi) \frac{\partial}{\partial n} N(\eta, q) \, ds.
\]

Accordingly we have
\[
\text{mean } (N(z, q) \text{ on } \partial V_{M}(p)) - \varepsilon \leq \frac{1}{4\pi^{2}} \int_{\partial V_{M}(p)} \left( \int_{\partial V_{M}(q)} N(\eta, \xi) \frac{\partial}{\partial n} N(\eta, q) \, ds \right) \frac{\partial}{\partial n} N(\xi, p) \, ds
= \frac{1}{4\pi^{2}} \int_{\partial V_{M}(q)} \left( \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p) \, ds \right) \frac{\partial}{\partial n} N(\eta, q) \, ds.
\] (39)

By (37) and (38)
\[
\frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p) \, ds \leq \frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, q) \, ds
= N(\eta, p) = N(p, \eta) \quad \text{for } \eta \notin V_{1}(p)
\]
\[
\frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, p) \, ds \leq \frac{1}{2\pi} \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, q) \, ds
= N(\eta, p) = N(p, \eta) \quad \text{for } \eta \in V_{1}(p).
\]

On the other hand,
\[
\text{mean } (N(z, p) \text{ on } \partial V_{M}(q)) = \frac{1}{2\pi} \int_{\partial V_{M}(q)} N(p, \eta) \frac{\partial}{\partial n} N(\eta, q) \, ds.
\]

Hence by (37), (38) and (39)
\[
\text{mean } (N(z, p) \text{ on } \partial V_{M}(q)) - \varepsilon \leq \frac{1}{4\pi^{2}} \int_{\partial V_{M}(q)} \left( \int_{\partial V_{M}(p)} N(\xi, \eta) \frac{\partial}{\partial n} N(\xi, q) \, ds \right) \frac{\partial}{\partial n} N(\eta, q) \, ds
\leq \frac{1}{2\pi} \int_{\partial V_{M}(q)} N(\eta, p) \frac{\partial}{\partial n} N(\eta, q) \, ds = \text{mean } (N(z, p) \text{ on } \partial V_{M}(q)).
\]

Thus by letting \( \varepsilon \rightarrow 0 \)
\[
\text{mean } (N(z, q) \text{ on } \partial V_{M}(p)) \leq \text{mean } (N(z, p) \text{ on } \partial V_{M}(q)).
\]

Since the inverse inequality holds for the other pair of \( V_{M}(p) \) and \( V_{M}(q) \) and since mean \( (N(z, q) \text{ on } \partial V_{M}(p)) \uparrow N(p, q) \) and mean \( (N(z, p) \text{ on } \partial V_{M}(q)) \uparrow N(q, p) \), we have
\[
N(p, q) = N(p, q).
\]

Case 3, b). \( p \in B_{1} \) and \( q \in B_{0} \) or \( p \in B_{0} \) and \( q \in B_{1} \). Without loss of generality we can suppose \( p \in B_{1} \) and \( q \in B_{0} \). In this case \( N(z, q) = \int N(z, q) \, d\mu(q) \) and similarly as in case 2, b) we have \( N(p, q) = N(q, p) \).
Case 4. \( p \in B_0 \): \( N(z, p) = \int_{B_0} N(z, p_\alpha) \, d\mu(p_\alpha) \) and \( q \in B_0 \): \( N(z, q) = \int_{B_0} N(z, q_\beta) \, d\mu(q_\beta) \), \( \mu(p_\alpha) \) and \( \mu(q_\beta) \) are canonical distributions. By Case 3. a) and b)

\[
N(p, q) = \int_{B_1} N(p_\alpha, q) \, d\mu(p_\alpha) = \int_{B_1} \left( \int_{B_1} N(p_\alpha, q_\beta) \, d\mu(p_\alpha) \right) d\mu(q_\beta) = \int_{B_1} N(q_\beta, p_\alpha) \, d\mu(p_\alpha) d\mu(q_\beta) = N(q, p).
\]

By the second and 5-th terms we see that \( N(p, q) \) does not depend on particular distributions \( \mu(p_\alpha) \) and \( \mu(q_\beta) \), whence \( N(q, p) : p \in \overline{R} - R_0 \) and \( q \in \overline{R} - R_0 \) is well defined and \( N(p, q) = N(q, p) \) by Cases 1, 2, 3 and 4.

**Proof of b).** By the definition of the value of \( N(z, p) \) at \( p \in R - R_0 + B_1 \)

\[
N(p, p) = \lim_{M \to M^*} \frac{1}{2\pi} \int_{\partial V_M(p)} N(z, p) \frac{\partial}{\partial n} N(z, p) \, ds = \lim_{M \to M^*} M = \sup_{z \in R} N(z, p),
\]

where \( M^* = \sup N(z, p) \).

**Proof of c).** At first we show that \( U(p) : p \in R - R_0 + B_1 \) is well defined and the representation \( U(z) = \int_{R-R_0+B_1} N(z, p) \, d\mu(p) \) is valid not only in \( R - R_0 \)

but also in \( \overline{R} - R_0 \).

Case 1. \( p \in R - R_0 + B_1 \) and \( U(z) \) is given by

\[
\int_{R-R_0+B_1} N(z, p_\alpha) \, d\mu(p_\alpha) \text{ in } R - R_0 \quad (39)
\]

(\( \mu_\alpha \) is not uniquely determined).

Since \( \int_{\partial V_M(p)} N(z, p) \frac{\partial}{\partial n} N(z, p) \, ds \uparrow \) as \( M \uparrow \sup_{z \in R} N(z, q) \), the order of the integration can be changed. Hence

\[
U(p) = \lim_{M \to M^*} \frac{1}{2\pi} \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} N(z, p) \, ds
\]

\[
= \lim_{M \to M^*} \frac{1}{2\pi} \int_{\partial V_M(p)} \left( \int_{R-R_0+B_1} N(z, p_\alpha) \, d\mu(p_\alpha) \right) \frac{\partial}{\partial n} N(z, p) \, ds
\]

\[
= \frac{1}{2\pi} \int_{R-R_0+B_1} \left( \lim_{M \to M^*} \int_{\partial V_M(p)} N(z, p_\alpha) \frac{\partial}{\partial n} N(z, p) \, ds \right) d\mu(p_\alpha) = \int_{R-R_0+B_1} N(p, p_\alpha) \, d\mu(p_\alpha).
\]

By the second term we see that \( U(p) : p \in B_1 \), depends on the behaviour of
$U(z)$ in $R-R_0$ and does not depend on a particular distribution $\mu(p_a)$. $U(p)$ is uniquely determined and by (40) the representation (39) is also valid on $B_1$.

Case 2. $p \in B_0$: $N(z, p) = \int_{B_1} N(z, p_{\alpha}) d\mu(p_{\alpha})$. In this case by (40) $U(p) = \int_{R-R_0+B_1} N(p, p_{\alpha}) d\mu(p_{\alpha})$ for $p \in B_0 + R-R_0$ and by the definition of $U(p)$ at $p \in B_1$,

$$U(p) = \int_{R-R_0+B_1} U(p_{\alpha}) d\mu(p_{\alpha}) = \int_{R-R_0+B_1} \left( \int_{R-R_0+B_1} N(p_{\beta}, p_{\alpha}) d\mu(p_{\alpha}) \right) d\mu(p_{\alpha}).$$

We see that by the second term $U(p)$ does not depend on $\mu_a$ and by the last term it does not depend on $\mu_b$. Hence $U(p)$ is uniquely determined. Hence (39) is valid also on $B_0$. Thus the representation is valid not only in $R-R_0$ but also on $\overline{R}-R_0$.

Next we show that $U(z)$ is lower semicontinuous in $\overline{R}-R_0$.

1). $N(z, p)$ is lower semicontinuous in $\overline{R}-R_0$ for $p \in R-R+B_1$.

Let $\{z_i\}$ be a sequence in $\overline{R}-R_0$ such that $\delta(z_i, z_0) \rightarrow 0$. Then \( \lim_{i} N(z_i, p) \geq N(z_0, p). \)

Proof of 1). By b) $N(z_i, p) = N(p, z_i)$ and $N(z_0, p) = N(p, z_0)$. Hence it is sufficient to show $\lim_{i} N(p, z_i) \geq N(p, z_0)$. Since $N(p, z_0) = \frac{1}{2\pi} \lim_{M, M^{*} \rightarrow \infty} \int_{\partial V_{M}(p)} N(\zeta, z_0) \frac{\partial}{\partial n} N(\zeta, p) ds$,

$$\delta(z_i, z_0) \rightarrow 0$$

implies that $N(\zeta, z_i) \rightarrow N(\zeta, z_0)$ in $R-R_0$ and $N(\zeta, z_i)$ converges uniformly to $N(\zeta, z_0)$ on $R_m$. Hence

$$N(p, z_0) - \epsilon \leq \frac{1}{2\pi} \lim_{i} \int_{\partial V_{M}(p)} N(\zeta, z_i) \frac{\partial}{\partial n} N(\zeta, p) ds = \frac{1}{2\pi} \lim_{i} \int_{\partial V_{M}(p)} N(\zeta, z_i) \frac{\partial}{\partial n} N(\zeta, p) ds.$$

$$\leq \frac{1}{2\pi} \lim_{i} \left( \lim_{M, M^{*} \rightarrow \infty} \int_{\partial V_{M}(p)} N(\zeta, z_i) \frac{\partial}{\partial n} N(\zeta, p) ds \right) = \lim_{i} N(p, z_i).$$
Let $\varepsilon \to 0$. Then $N(p, z_0) \leq \lim_{i} N(p, z_i)$. Hence $N(z_0, p) \leq \lim_{i} N(z_i, p)$ and $N(z, p)$ is lower semicontinuous in $\overline{R} - R_0$ for $p \in R - R_0 + B_1$.

2). $U(z)$ is lower semicontinuous in $\overline{R} - R_0$ (of course $N(z, p): p \in B_0$ is a superharmonic function, hence $N(z, p)$ is lower semicontinuous).

Proof of 2). By (39), (40) and others the representation $U(z) = \int N(z, p_0) d\mu(p_\alpha): p_\alpha \in R - R_0 + B_1$ is valid in $\overline{R} - R_0$.

Let $\{z_i\}$ be a sequence such that $\delta(z_i, z_0) \to 0: z_i, z_0 \in \overline{R} - R_0$. Then by Fatou's lemma

$$\lim_{i \to \infty} \int_{R - R_0 + B_1} N(z_i, p_\alpha) d\mu(p_\alpha) \geq \int_{R - R_0 + B_1} \lim_{i \to \infty} N(z_i, p_\alpha) d\mu(p_\alpha).$$

$N(z, p_\alpha)$ is lower semicontinuous. Hence

$$\lim_{i \to \infty} U(z_i) = \lim_{i \to \infty} \int_{R - R_0 + B_1} N(z_i, p_\alpha) d\mu(p_\alpha) \geq \lim_{i \to \infty} \int_{R - R_0 + B_1} N(z_i, p_\alpha) d\mu(p_\alpha) = U(z_0).$$

Thus $U(z)$ is lower semicontinuous in $\overline{R} - R_0$.

It is clear that $U(z)$ is superharmonic locally at $p \in R - R + B_1$ by the definition of the value $U(z)$ at $R - R_0 + B_1$.

We have discussed the capacitary potentials of $(G \setminus B)$, of $F$ and of that determined by a sequence of decreasing domain and obtained some properties. Now the method to define the value, on $B$, of superharmonic functions is established. We consider the behaviour of C.P.'s and we shall prove some classical theorems which hold in euclidean space.

10. Capacitary potentials of closed sets, $F_*$ sets and of $F_{\sigma*}$ sets.

Theorem 12. a). Let $p \in R - R_0 + B_1 - B_S$, then $\omega(p, z) = 0$ and $\sup_{z \in \overline{R}} N(z, p) = \infty$. Then

$$\lim_{M \to \infty} V_M(p) - C_n(p) N(z, p) = 0 \quad \text{for every } u_n(p).$$

Let $C_M$ be a regular niveau curve of $N(z, p)$. Then

$$\lim_{M \to \infty} \int_{C_M \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi.$$

b). Let $\omega(F, z)$ be C.P. of a closed set $F$ in $\overline{R} - R_0$ of positive capacity. Then

$$\sup_{z \in F} \omega(F, z) = 1.$$
c). (P.C.7). Let \( \omega(F, z) \) be C.P. of a closed set \( F \) of positive capacity. Then \( \omega(F, z) = 1 \) except at most an \( F' \) set of capacity zero.

d). Let \( G(z, p) \) \( p \in R_0 \) be the Green's function of a Riemann surface \( R = \bigcup R_n \) with positive boundary. Then \( G(z, p) = 0 \) on \( B \) except at most an \( F' \) set of capacity zero.

e). (P.C.8). Let \( F : F \cap \partial R_0 = 0 \) be a closed set of positive capacity: \( \omega(F, z) > 0 \). Let \( \Omega \) be the component of \( R - R_0 - F \) containing \( \partial R_0 \) as its boundary. Then \( E[z \in \Omega \cap CG_n : \omega(F, z) = 1] \) does not contain a closed set of positive capacity: \( G_n = E[z \in \overline{R} : \delta(z, F) < \frac{1}{n}] \).

Proof of a). Assume \( N(z, p) \leq M \). Then \( \frac{N(z, p)}{M} \leq \omega(V_M(p), z) \). Hence \( \frac{N(z, p)}{M} \leq \omega(p, z) = 0 \). This is a contradiction. Hence \( \sup N(z, p) = \infty \). By \( N(z, p) = N(z, p) \) in \( R - R_0 - V'_M(p) \), \( \omega(V_M(p), z) = \frac{\min(M, N(z, p))}{M} \) and \( \lim_{M} \omega(V_M(p), z) = 0 \). Hence by Theorem 6. b) \( N(z, p) - N'(z, p) \) is superharmonic, where \( N'(z, p) = \lim_{M=\infty} V_M(p) \cap C_{U_n}(p) N(z, p) \) and \( N'(z, p) \) is represented by a mass distribution over \( V_M(p) \cap C_{U_n}(p) \) by Theorem 5. b). If \( N'(z, p) \geq 0 \), \( N'(z, p) = KN(z, p) \) by the minimality of \( N(z, p) \), because \( N'(z, p) \) and \( N(z, p) - N'(z, p) \) are superharmonic. Hence \( KN(z, p) \) must be a point mass over \( C_{U_n}(p) \) by Theorem 9. a), whence \( N(z, p) = N(z, q) : q \in C_{U_n}(p) \) by \( \int_{\partial R_0} \frac{\partial}{\partial n} N(z, p) ds = \lim_{m} \int_{\partial R_0} \frac{\partial}{\partial n} N(z, q) ds \). But \( N(z, p) = N(z, q) \) implies \( p = q \). This is a contradiction. Hence \( N'(z, p) = 0 \).

Let \( \omega_m(z) \) be a harmonic function in \( R_m - R_0 - (V_M(p) \cap C_{U_n}(p)) \) such that \( \omega_m(z) = 0 \) on \( \partial R_0 \), \( \omega_m(z) = 1 \) on \( \partial(V_M(p) \cap C_{U_n}(p)) \cap R_m \) and \( \frac{\partial}{\partial n} \omega_m(z) = 0 \) on \( \partial R_m - (V_M(p) \cap C_{U_n}(p)) \). Then \( \omega_m(z) \Rightarrow \omega(V_M(p) \cap C_{U_n}(p), z) \) as \( m \to \infty \). Hence by Fatou's lemma and by the compactness of \( \partial R_0 \) \( \int_{\partial R_0} \frac{\partial}{\partial n} \omega(V_M(p) \cap C_{U_n}(p), z) ds \)

\[
= \lim_{m} \int_{\partial R_0} \frac{\partial}{\partial n} \omega_m(z) ds \leq \lim_{m} \int_{\partial(V_M(p) \cap C_{U_n}(p)) \cap R_m} \frac{\partial}{\partial n} \omega_m(z) ds \geq \int_{\partial(V_M(p) \cap C_{U_n}(p))} \frac{\partial}{\partial n} \omega(V_M(p) \cap C_{U_n}(p), z) ds.
\]

By \( (V_M(p) \cap C_{U_n}(p)) \subset V_M(p) \) and \( N(z, p) \geq M \) on \( C_{U_n}(p) \cap V_M(p) \), by the
maximum principle

\[ M_\omega(V_M(p) \cap C_{U_n}(p), z) \leq V_M(p) \cap C_{U_n}(p) N(z, p) \leq N(z, p) \]

in \( R - R_0 - (V_M(p) \cap C_{U_n}(p)) \). \hfill (42)

On the other hand,

\[ M_\omega(V_M(p) \cap C_{U_n}(p), z) = M_\omega(V_M(p) \cap C_{U_n}(p), N(z, p) = N(z, p) \]

on \( \partial V_M(p) \cap C_{U_n}(p) \). \hfill (43)

Hence by (41) and (43)

\[ \frac{\partial}{\partial n} M_\omega(V_M(p) \cap C_{U_n}(p), z) \geq \frac{\partial}{\partial n} N(z, p) \geq 0 \text{ on } \partial V_M(p) \cap C_{U_n}(p). \] \hfill (44)

Hence by (41) and (43)

\[ \int_{\partial R_0} \frac{\partial}{\partial n} M_\omega(V_M(p) \cap C_{U_n}(p), z) ds \geq \int_{\partial V_M(p) \cap C_{U_n}(p)} \frac{\partial}{\partial n} N(z, p) ds \]

\[ \geq \int_{\partial V_M(p) \cap C_{U_n}(p)} \frac{\partial}{\partial n} N(z, p) ds. \] \hfill (45)

Assume \( \lim_{M \to \infty} \int_{\partial V_M(p) \cap C_{U_n}(p)} \frac{\partial}{\partial n} N(z, p) ds > \delta_0 > 0 \). Then by (45)

\[ \lim_{M \to \infty} \int_{\partial R_0} \frac{\partial}{\partial n} M_\omega(V_M(p) \cap C_{U_n}(p), z) ds \geq \delta_0 \] and \( \lim_{M \to \infty} M_\omega(V_M(p) \cap C_{U_n}(p), z) > 0 \),

whence by (42) \( N'(z, p) = \lim_{M \to \infty} N(z, p) > 0 \). This contradicts \( N'(z, p) = 0 \). Hence

\[ \lim_{M \to \infty} \int_{\partial V_M(p) \cap C_{U_n}(p)} \frac{\partial}{\partial n} N(z, p) ds = 0 \] and \( \lim_{M \to \infty} \int_{\partial V_M(p) \cap C_{U_n}(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \)

by the regularity of \( \partial V_M(p) \). Thus we have a).

Proof of b). Let \( F_m = E \left[ z \in \bar{R} : \delta(z, F) \leq \frac{1}{m} \right] \). Then \( F = \bigcap F_m \) and \( F_m \) can be considered as a non compact domain. Hence \( \sup_{x \in \bar{R}} \omega(F, x) = 1 \) by P.C.2.

But our assertion is not so trivial. If \( F \) has a closed subset \( F' \) of positive capacity in \( R - R_0 \), our assertion is clear. If \( F \) has a point \( p \in B_s \), \( 1 = \omega(p, p) \leq \sup_{x \in F} \omega(z, F') = 1 \) by Theorem 10, b). Hence we can suppose without loss of generality that \( F \subset B \) and \( F \cap B_s = 0 \) and \( \text{Cap}(F) > 0 \). Since \( B_0 \) is a set of capacity zero, \( F \) has at least one point \( p \in B_1 - B_s \). Assume
$\omega(F, z) < K < 1$ on $F$. By the definition \( \omega(F, p) = \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_M(p)} \omega(F, z) \frac{\partial}{\partial n} N(z, p) ds \) for \( p \in B_1 - B_S \). Hence by (a)

$$\omega(F, p) = \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_M(p)} \omega(F, z) \frac{\partial}{\partial n} N(z, p) ds.$$

Let \( G_{K+\delta} = E[z \in R : \omega(F, z) < K + \delta] : \delta > 0 \) and \( 1 > K + \delta > K \). Then by \( \omega(F, z) \leq K \) on \( F \), there exists a positive constant \( \epsilon_0 \) such that

$$\lim_{M=\infty} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \frac{\partial}{\partial n} N(z, p) ds > 2\pi \epsilon_0 \quad \text{and} \quad 0 < \epsilon_0 < \frac{\delta}{K+\delta}.$$ \hspace{1cm} (46)

In fact, if \( \lim_{M=\infty} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \frac{\partial}{\partial n} N(z, p) ds > 2\pi (1 - \epsilon_0) \),

$$\omega(F, p) \geq \lim_{M=\infty} \left( \frac{1}{2\pi} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \omega(F, z) \frac{\partial}{\partial n} N(z, p) ds + \frac{1}{2\pi} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \omega(F, z) \frac{\partial}{\partial n} N(z, p) ds \right) \geq \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \omega(F, z) \frac{\partial}{\partial n} N(z, p) ds \geq \frac{(K+\delta)}{2\pi} 2\pi (1 - \epsilon_0) > K.$$

This contradicts \( \omega(F, p) < K \). Hence (46) holds for every \( \nu_n(p) \). Now by P.C.7, \( \omega(G_{K+\delta} \cap F, z) = 0 \) for \( K + \delta < 1 \), i.e. \( \lim_{m=\infty} \omega(G_{K+\delta} \cap F, m, z) = 0 \), where

$$F_m = E[z \in \overline{R} : \delta(z, F) \leq \frac{1}{m}].$$

Choose a subsequence \( m_1, m_2, \ldots \) of 1, 2, \ldots such that \( \omega(G_{K+\delta} \cap F_{m_i}, z) < \frac{1}{2^i} \) for \( z = z_0 \) (\( i=1, 2, \ldots \)). Then

$$\omega^*(z) = \sum_{i=1}^{\infty} \omega(G_{K+\delta} \cap F_{m_i}, z) < \infty,$$

and \( \omega^*(z) \) is \( \overline{\text{superharmonic}} \) by Theorem 4. (a) and \( \omega^*(z) \geq i_0 \) for \( z \in (\bigcap_{i=1}^{i_0} F_{m_i} \cap G_{K+\delta}) \cap R - R_0 \), hence \( \omega^*(z) \rightarrow \infty \) as \( z \rightarrow p \in F \) inside of \( G_{K+\delta} \).

Let \( p \in R - R_0 + B_1 - B_S \). Then

$$\omega^*(p) = \lim_{M=\infty} \frac{1}{2\pi} \int_{\partial V_M(p)} \omega^*(z) \frac{\partial}{\partial n} N(z, p) ds \geq \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_M(p) \cap G_{K+\delta} \cap \nu_n(p)} \omega^*(z) \frac{\partial}{\partial n} N(z, p) ds \geq i_0 \epsilon_0,$$

for \( \nu_n(p) \subseteq F_{m_{i_0}} \).

This holds for every \( \nu_n(p) \). Hence let \( i_0 \rightarrow \infty \). Then \( \omega^*(p) = \infty \). Now by the lower semicontinuity of \( \omega^*(z) \), \( \omega^*(z) \rightarrow \infty \) as \( z \rightarrow p \in (F \cap (R - R_0 + B_1 - B_0)) \) not only inside of \( G_{K+\delta} \) but also \( \omega^*(z) \rightarrow \infty \) only if \( z \rightarrow p \).
$B_0$ is a sum of closed sets of capacity zero. We can construct as above a superharmonic function $\omega^{**}(z)$ such that $\lim_{z \to p \in B_0} \omega^{**}(z) = 0$. Hence

$$\lim_{z \to p \in B_0} (\omega^*(z) + \omega^{**}(z)) = \infty$$

for any $\varepsilon > 0$. Put $\Delta = E[z \in \overline{R} : \varepsilon(\omega^*(z) + \omega^{**}(z)) \leq 2]$. Then $\Delta$ is closed and $\Delta \cap F = 0$, which implies dist $(\Delta, F) > d_{*} > 0$ and $C\Delta \supset F$. Put $F_{d_{*}} = E[z \in \overline{R} : \delta(z, F) \leq d_{*}]$. Let $d_{*} \omega(z)$ be C.P. of $F_{d_{*}}(\supset F)$. Then $\varepsilon(\omega^*(z) + \omega^{**}(z)) \leq d_{*} \omega(z) \leq \omega(F, z)$. Let $\varepsilon \to 0$. Then $\omega(F, z) = 0$. This is a contradiction. Hence $\sup \omega(F, z) = 1$.

Proof of c). Let $\omega(E_k, z)$ be C.P. of $E_k = E[z \in F : \omega(F, z) \leq 1 - \frac{1}{k}]$ ($k=1, 2, \cdots$). Then $\omega(E_k, z) \leq \omega(F, z)$, whence $\sup_{R - B_0} \omega(E_k, z) \leq 1 - \frac{1}{k} < 1$. Hence by b) $E_k$ is of capacity zero. Then $E = \cup E_k$ is an $F_{\sigma}$ set of capacity zero, because $E_k$ is closed by the semicontinuity of $\omega(F, z)$.

Proof of d). Let $\omega_n(z)$ be a superharmonic function in $\overline{R} - R_0$ such that $\omega_n(z) = 1$ in $R - R_n$, $\omega_n(z) = 0$ on $\partial R_0$, and $\omega_n(z)$ is harmonic in $R_n - R_0$. Then $\lim_{n} \omega_n(z) = \omega(B, z)$. Now $G(z, p) \leq N < \infty$ in $R - R_0$. Hence by the maximum principle $0 < G(z, p) \leq N(1 - \omega(B, z))$. On the other hand, $(1 - \omega(B, z)) = 0$ on $B$ by c) except an $F_*$ set of capacity zero. Whence we have at once d).

Proof of e). Assume $\omega(F, z) = 1$ on a closed set $F^*$ of positive capacity in $\Omega \cap CG_{n_0} : CG_{n_0} = E[z \in \overline{R} : \delta(z, F) \geq \frac{1}{n_0}]$. Clearly $\omega(F, z) < 1$ in $\Omega \cap (R - R_0)$ by the maximum principle and $F^* \subset B$.

$$\omega(F, z) \geq \omega(F^*, z) = \omega(F^*, z) > 0. \quad (47)$$

Let $F^*_{n_0} = E[z \in \overline{R} : \delta(z, F^*) < \frac{1}{n_0}]$. Then $F^*_{n_0} \subset \Omega$ and $\omega(F, z)$ is harmonic and non constant in $F^*_{n_0} \cap (R - R_0)$ and dist $(CF^*_{n_0}, F^*) \geq \frac{1}{n_0}$. By $\omega(F^*, z) = \int_{F^* \cap (R - R_0 + R_1)} N(z, p) d\mu(p)$ by Theorem 13, d) $^9$ and by Theorem 13, b)

$$c_{F^*_{n_0}} \omega(F^*, z) < \omega(F^*, z).$$

Put $V(z) = \omega(F^*, z) - c_{F^*_{n_0}} \omega(F^*, z)$. Then $V(z) > 0$, $V(z) = 0$ on $\delta F^*_{n_0}$ and $D_{F^*_{n_0}}(V(z)) < \infty$. Since $\omega(F, z)$ has M.D.I. over $R - R_0 - F^*_m$ and $c_{F^*_{n_0}} \omega(F^*, z)$ has M.D.I. over $F^*_n$. Hence $V(z)$ has M.D.I. over $F^*_{n_0} - F^*_m$. $F^*_m = E[z \in \overline{R} : \delta(z, F^*) \leq \frac{1}{m}]$ and $m > 2n_0$, whence

$^9$) Theorem 13, d) and a) will be proved independently soon. See p. 60.
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\[ F_m^* + CF_{n_0}^* V(z) = V(z). \]

Put \( G_z = E \{ z \in F_n^* : V(z) > \frac{\delta}{2} \} \); \( \delta = \sup_{z \in F_{n_0}^*} V(z) \). Let \( \omega(G_z, z, F_{n_0}^*) \) be C.P. of \( G_z \) relative \( F_{n_0}^* \) such that \( S(z) = 0 \) on \( \partial F_{n_0}^* \) and \( S(z) = 1 \) on \( \partial G_z \), whence

\[
\omega(G_z \cap F_{m}^*, z, F_{n_0}^*) \leq \omega(G_z, z, F_{n_0}^*) \quad \text{and} \quad D_{F_{n_0}^*} \omega(G_z \leftrightarrow F_{m}^*, z, F_{n_0}^*) \leq \frac{4}{\delta^2} D(V(z)).
\]

Let \( \tilde{V}(z) \) be a harmonic function in \( F_{n_0}^* - F_{m}^* \) such that \( \tilde{V}(z) = \min \left( \frac{\delta}{2}, V(z) \right) \) on \( \partial F_{n_0}^* + \partial F_{m}^* \).

Then also \( D_{F_{n_0}^*} \tilde{V}(z) \leq D_{F_{n_0}^*} (V(z)) \)

\[
\omega(G_z \cap F_{m}^*, z, F_{n_0}^*) \leq \omega(G_z, z, F_{n_0}^*) \quad \text{and} \quad D_{F_{n_0}^*} \omega(G_z \leftrightarrow F_{m}^*, z, F_{n_0}^*) \leq \frac{4}{\delta^2} D(V(z)).
\]

On the other hand, \( \omega(F, z) \) is non constant and harmonic in \( F_{n_0}^* \) by \( F_{n_0}^* \subset \Omega \). Hence \( \omega_n(z) \Rightarrow \omega(F, z) \) as \( n \to \infty \), where \( \omega_n(z) \) is a harmonic function in \( (R_n \cap F_{n_0}^*) \) such that \( \omega_n(z) = \omega(F, z) \) on \( \partial R_n \) and \( \frac{\partial}{\partial n} \omega_n(z) = 0 \) on \( F_{n_0}^* \cap \partial R_n \).

Put \( G_{1,2} = E \{ z \in R : M_1 < \omega(G_z \cap F_{m}^*, z, F_{n_0}^*) < M_2 \} \). Then \( \omega(G_z \cap F_{m}^*, z, F_{n_0}^*) \) has M.D.I. over \( G_{1,2} \). Hence \( \tilde{\omega}_n(z) \Rightarrow \omega(G_z \cap F_{m}^*, z, F_{n_0}^*) \) in \( G_{1,2} \), where \( \tilde{\omega}_n(z) \) is a harmonic function in \( R_n \cap G_{1,2} \) such that \( \tilde{\omega}_n(z) = M_1 \) on \( C_1 \cap R_n \) and \( \frac{\partial}{\partial n} \tilde{\omega}_n(z) = 0 \) on \( G_{1,2} \cap \partial R_n \). We suppose that \( C_1 \) and \( C_2 \) are regular.
Now \( \int_{C_{M_{1}^\cap R_{n}}} \tilde{\omega}_{n}(z) \frac{\partial}{\partial n} \omega_{n}(z) \, ds = M_{i} \int_{C_{M_{1}^\cap R_{n}}} \omega_{n}(z) \frac{\partial}{\partial n} \omega_{n}(z) \, ds = 0 \). Hence by the Green's formula
\[
\int_{C_{M_{1}^\cap R_{n}}} \omega_{n}(z) \frac{\partial}{\partial n} \tilde{\omega}_{n}(z) \, ds = \int_{C_{M_{2}^\cap R_{n}}} \omega_{n}(z) \frac{\partial}{\partial n} \tilde{\omega}_{n}(z) \, ds.
\]
By the regularity of \( C_{M_{i}} \) and by Theorem 3, a). By letting \( n \to \infty \),
\[
\int_{C_{M_{1}}} \omega(F, z) \frac{\partial}{\partial n} \omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*}) \, ds = \int_{C_{M_{2}}} \omega(F, z) \frac{\partial}{\partial n} \omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*}) \, ds.
\]
On \( C_{M_{1}^\cap R} \), \( \omega(F, z) < 1 \) by the non constancy of \( \omega(F, z) \) in \( F_{n_{0}}^{*} \), hence there exists a positive constant \( \delta_{0} \) such that
\[
\int_{C_{M_{1}}} \omega(F, z) \frac{\partial}{\partial n} \omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*}) \, ds \leq \int_{C_{M_{1}}} \frac{\partial}{\partial n} \omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*}) - \delta_{0} = D(\omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*})) - \delta_{0}.
\]
Let \( M_{2} \to 1 \). Then
\[
\lim_{M_{1} \uparrow 1} \int_{C_{M_{2}}} \omega(F, z) \frac{\partial}{\partial n} \omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*}) \, ds < D(\omega(G_{\delta \cap} F^{*}, z, F_{n_{0}}^{*})) - \delta_{0}. \tag{50}
\]
(49) contradicts (50). Hence \( \omega(F^{*}, z) = 0 \). Thus we have e).

Let \( U(z) \) be a positive \( \square \)uperharmonic function in \( \overline{R} - R_{0} \). Then by Theorem 8 there exists a canonical mass distribution of which the uniqueness is not proved. But we shall prove the following

**Theorem 13.** a). Let \( U(z) \) be a positive \( \square \)uperharmonic function in \( \overline{R} - R_{0} \) such that \( U(z) = \int_{F \cap (R - R_{0} + B_{1})} N(z, p) \, d\mu(p) \). Then \( \varphi U(z) = U(z) \).

b). Let \( U(z) \) be a \( \square \)uperharmonic function in a) and let \( F' \) be a closed set such that \( \text{dist} (F, F') > 0 \). Then
\[
\varphi U(z) < \varphi U(z) = U(z).
\]

c). Let \( U(z) \) be a positive \( \square \)uperharmonic function in \( \overline{R} - R_{0} \) and let \( F \) be a closed set such that \( \varphi U(z) = U(z) \). Then \( U(z) \) is represented by a canonical mass distribution on \( F \) such that
\[
U(z) = \int_{F \cap (R - R_{0} + B_{1})} N(z, p) \, d\mu(p)
\]
and any canonical distribution has no mass on \( CF \).

10) If \( \mu = 0 \) on \( B_{0} \), \( \mu \) is called canonical.
d). Let $U(z)$ be a function in a). Let $F$ be the kernel of a canonical mass distribution. Then the kernel of any other canonical mass distribution is also $F$.

e). As a corollary of c) $\omega(F, z) = \int_{F \cap (\mathbb{R} - R_{0} + B_{1})} N(z, p) d\mu(p)$. 

Proof of a). By $\omega_{n}(p)N(z, p) = N(z, p)$ for $p \in R - R_{0} + B_{1}$ and $\omega_{n}(p) \subset F_{m}$: $F_{m} = E\left[z \in \overline{R} : \delta(z, F) \leq \frac{1}{n}\right]$. Whence $F_{n}(z, p) = N(z, p)$. Hence we have at once a).

Proof of b). Assume $\omega_{n}(z) = U(z): F_{n}^{'} = E\left[z \in \overline{R} : \delta(z, F^{'}) \leq \frac{1}{n}\right]$ for every $n$. We cover $F$ by a finite number of closed discs $\mathfrak{F}_{1}, \mathfrak{F}_{2}, \cdots \mathfrak{F}_{i_{0}}$ with diameter $< \frac{1}{2}$. Put $\mu = \mu_{1} + \mu_{2} + \cdots + \mu_{i_{0}}$, where $\mu_{i}$ is the restriction of $\mu$ on $\mathfrak{F}_{i} - \sum_{j=1}^{i-1} \mathfrak{F}_{j}$. Now by the superharmonicity of $\int N(z, p) d\mu(p)$

$$\omega_{n}(U(z)) = \sum_{i=1}^{i_{0}} \left( \int_{\mathfrak{F}_{i}} N(z, p) d\mu_{i}(p) \right) = \sum_{i=1}^{i_{0}} \int N(z, p) d\mu_{i}(p) = U(z)$$

and

$$\omega_{n}(\int N(z, p) d\mu_{i}(p)) \leq \int N(z, p) d\mu_{i}(p)$$ for every $i$. Hence

$$\omega_{n}(\int N(z, p) d\mu_{i}(p)) = \int N(z, p) d\mu_{i}(p) \geq 0$$ for every $i$.

Hence there exists at least one $\mu_{i}$ and $\mathfrak{F}_{i}$ such that $\int N(z, p) d\mu_{i}(p) = \omega_{n}(\int N(z, p) d\mu_{i}(p)) > 0$. We denote them $\mathfrak{F}_{i}$ and $\mu_{i}^{1}$ respectively. As above we choose $\mathfrak{F}_{i}^{2}$ and $\mu_{i}^{2}$ such that $\mathfrak{F}_{i}^{2} \subset \mathfrak{F}_{i}^{1}$, diameter of $\mathfrak{F}_{i}^{2} < \frac{1}{2^{2}}$ and

$$\omega_{n}(\int_{\mathfrak{F}_{i}^{2}} N(z, p) d\mu_{i}^{2}(p)) = \int_{\mathfrak{F}_{i}^{2}} N(z, p) d\mu_{i}^{2}(p) > 0,$$ where $\mu_{i}^{2}(p)$ is the restriction of $\mu$ on $\mathfrak{F}_{i}^{2}$. In this way we can find a sequence $\mathfrak{F}_{i}^{1} \supset \mathfrak{F}_{i}^{2} \cdots$ and $\frac{\mu_{i}}{m_{1}}, \frac{\mu_{i}}{m_{2}}, \cdots$ such that $\bigcap_{i}^{\infty} \mathfrak{F}_{i}^{i} = p \notin B_{0}$, where $m_{i}$ is the total mass of $\mu_{i}^{i}$. Because if every sequence $\mathfrak{F}_{i}^{1} \supset \mathfrak{F}_{i}^{2} \cdots = p \notin B_{0}$, $\mu$ has no mass outside of $B_{0}$. This contradicts that $\mu$ is canonical.

Now $\omega_{n}(\frac{1}{m_{i}} \int N(z, p) d\mu_{i}(p)) = \int \frac{1}{m_{i}} N(z, p) d\mu_{i}(p)$. We can find an weak limit $\mu^{*}$ of $\frac{\mu_{i}}{m_{i}}$ on $\bigcap_{i}^{\infty} \mathfrak{F}_{i}^{i} = p \notin B_{0}$ such that $\int N(z, p) d\mu^{*}(p) = \lim_{i \to \infty} \int \frac{1}{m_{i}} N(z, p) d\mu_{i}(p)$. 


$= N(z, p)$, where $\{i'\}$ is a subsequence such that $\int_{\frac{1}{m_{i'}}} N(z, p) d\mu^{i'}(p) \rightarrow N(z, p)$.

By $p \in (R - R_0 + B_1) \cap F$

$$\lim_{i' \to \infty} \int N(z, p) d\left(\frac{\mu''}{m_{i'}}\right)(p) = N(z, p)$$

is minimal. \hspace{1cm} (51)

On the other hand, by $\int_{F'} \left( \frac{1}{m_i} \int N(z, p) d\mu^{i'}(p) \right) = N(z, p) d\mu^{i'}(p),$

$$\int \frac{1}{m_{i'}} N(z, p) d\mu^{i'}(p)$$

is represented by mass $\mu'$ on $F'$ by Theorem 6.a).

Hence $\lim_{i' \to \infty} \frac{1}{m_{i'}} \int N(z, p) d\mu^{i'}(p)$ is represented by an weak limit $\mu^*$ on $F'$, i.e.

$$\lim_{i' \to \infty} \frac{1}{m_{i'}} \int N(z, p) d\mu'(p) = \int N(z, p) d\mu'(p).$$

By (51) $\int N(z, p) d\mu'(p)$

$= (N(z, p))$ is minimal, whence by Theorem 9, a) $\mu^*$ is a point mass at $q \in F' \cap (R - R_0 + B_1)$. Hence $N(z, p) = N(z, q)$ and $p \in F$ and $q \in F'$. This is a contradiction. Hence $\int_{F'} N(z, p) d\mu'(p) < \int_{F' \cap (R - R_0 + B_1)} N(z, p) d\mu(p)$.

Proof of c). Since $\mu U(z) = U(z)$, by Theorem 6, a) $U(z) = \int N(z, p) d\mu(p)$.

Let $\mu^*$ be a canonical distribution of $\mu$ ($\mu^*$ may be positive over $R - R_0 - F$) and let $\mu^*''$ be the restriction of $\mu^*$ on $F$. Then $\mu^* - \mu^*''$ is also canonical and $\mu^* - \mu^*'' = 0$ on $F$ and $\geq 0$ on $CF$. Assume $\mu^* - \mu^*'' > 0$. Then there exists a closed set $F'$ in $CF$ such that the restriction $\mu^*'''$ of $\mu^*$ on $F' > 0$ and dist $(F, F') > 0$.

$$\mu U(z) = \int_{F'} \left( \int N(z, p) d\mu^*'''(p) + \int N(z, p) d(\mu^* - \mu^*''')(p) \right) = U(z),$$

$$\int_{F'} \left( \int N(z, p) d\mu^*'''(p) \right) \leq \int N(z, p) d\mu^*''''(p)$$

and $\int_{F'} (N(z, p) d(\mu^* - \mu^*''')(p)) \leq \int N(z, p) d(\mu^* - \mu^*''')(p)$.

Whence

$$\int_{F'} \left( \int N(z, p) d\mu^*''''(p) \right) = \int N(z, p) d\mu^*''''(p).$$

(52) contradicts b) by dist $(F, F') > 0$ and $\mu^*'''(p)$ is canonical. Hence $\mu^* - \mu^*'' = 0$ and any canonical distribution has no mass on $CF$. Hence

$$U(z) = \int_{F' \cap (R - R_0 + B_1)} N(z, p) d\mu(p).$$

Proof of d). Let $\mu_i$ ($i=1, 2$) be a canonical mass distribution of $U(z)$ whose kernel is $F_i$. Then by a) $\int_{F_i} U(z) = U(z)$. Hence by c) $\mu_1$ has no mass.
outside of $F_2$, whence $F_2 \supset F_1$. Similarly $F_1 \supset F_2$. Hence $F_1 = F_2$.

Proof of e). $\omega(F, z) = _x \omega(F, z)$ by P.C.1, whence we have e) by c).

Let $A$ be an $F_\varepsilon$ set such that $A = \sim F_n$, $F'_1 \subset F_2 \cdots$ and $F_n$ is closed. We define $\omega(A, z)$ by the limit $\lim_n \omega(F_n, z)$. Then by Theorem 4, h) $\omega(A, z)$ is superharmonic in $\overline{R} - R_0$.

**Theorem 14.** a) P.C.1. $\omega(A, z) = _{A_m} \omega(A, z)$ : $A_m = E \left[ z \in \overline{R} : \delta(z, A) \leq \frac{1}{m} \right]$.

b). P.C.4. $\omega(A, z) = _{G_{n,m}} \omega(A, z) = (1 - \delta) \omega(G_{z}, z)$ in $CG_{z} : G_{z} = E \left[ z \in R : \omega(A, z) > 1 - \frac{1}{m} \right]$.

c). P.C.7. If $\omega(A, z) > 0$, $\omega(A, z) = 1$ on $A$ except at most an $F_\varepsilon$ set of capacity zero.

Let $H$ be an $F_{\sigma}$ set: $H = \cap_{n} A_n$, $A_1 \supset A_2 \supset A_3 \cdots$ and $A_n$ is an $F_{\sigma}$ set. We define $\omega(H, z)$ by the limit $\lim_n \omega(A_n, z)$. Then $\omega(H, z)$ is also superharmonic in $\overline{R} - R_0$ by Theorem 4, h).

d). $\omega(A_n, z) \Rightarrow \omega(H, z)$ as $n \rightarrow \infty$.

e). P.C.4. $\omega(H, z) = _{G_{n,m}} \omega(H, z)$ : $G_{n,m} = E \left[ z \in R : \omega(A_n, z) > 1 - \frac{1}{m} \right]$.

f'). P.C.4. $\omega(H, z) = _{G_{n,m}} \omega(H, z)$ : $G_{n,m} = E \left[ z \in R : \omega(A_n, z) > 1 - \frac{1}{m} \right]$.

g). P.C.7. If $\omega(H, z) > 0$, $\omega(H, z) = (1 - \frac{1}{m}) \omega(G_{m}, z)$ in $CG_{m} : G_{m} = E \left[ z \in R : \omega(H, z) > 1 - \frac{1}{m} \right]$ and $\sup_{z \in H} \omega(H, z) = 1$ and $\sup_{z \in H} \omega(H, z) = 1$.

Proof of a). Put $F_{n,m} = E \left[ z \in \overline{R} : \delta(z, F_n) \leq \frac{1}{m} \right]$. Then $F_{n,m} \subset A_m$. Now by P.C.1. $\omega(F_n, z) \geq _{A_m} \omega(F_n, z) = F_{n,m} \omega(F_n, z) = \omega(F_n, z)$.

Hence $\omega(A, z) \geq _{A_m} \omega(A, z) \geq \lim_n _{A_m} \omega(F_n, z) = \lim_n \omega(F_n, z) = \omega(A, z)$.

Proof of b) Put $G_{n,m} = E \left[ z \in \overline{R} : \omega(F_n, z) > 1 - \frac{1}{m} \right]$. Then $G_{n,m} \subset G_m$.

Hence as above we have b).

Proof of c). If $\omega(A, z) > 0$, there exists a number $n_0$ such that $\omega(F_n, z) > 0$ for $n \geq n_0$. Then by P.C.7 (Theorem 12, c)) $\sup_{z \in A} \omega(A, z) \geq \sup_{z \in F_n} \omega(F_n, z) = 1$.

Put $L_m = E \left[ z \in A : \omega(A, z) \leq 1 - \frac{1}{m} \right]$. Then $L_m \subset \bigcup_{n} E \left[ z \in F_n : \omega(F_n, z) \leq 1 - \frac{1}{m} \right]$.

Now $E \left[ z \in F_n : \omega(F_n, z) \leq 1 - \frac{1}{m} \right]$ is an $F_\varepsilon$ set of capacity zero. Hence $\sum L_m$ is an $F_\varepsilon$ set of capacity zero and we have c).
Proof of d), e) and f). Since \( \omega_{n,m}(A_{n}, z) = \omega(A_{n}, z) \) by b), \( \omega_{n,i}(z) \Rightarrow \omega(A_{n}, z) \), where \( \omega_{n,i}(z) \) is a harmonic function in \( (R_{i} - R_{0} - G_{m,n}) \) such that \( \omega_{n,i}(z) = \omega(A_{n}, z) \) on \( \partial G_{n,m} \cap (R_{i} - R_{0}) \) and \( \partial_{n} \omega_{n,i}(z) = 0 \) on \( \partial R_{i} - G_{n,m} \).

\[
D_{R_{i} - R_{0} - G_{n,m}}(\omega_{n,i}(z), \omega_{n+i,i}(z)) = \int_{\partial G_{n,m} \cap (R_{i} - R_{0})} \omega_{n,i}(z) \frac{\partial}{\partial n} \omega_{n+i,i}(z) \, ds
\]

\[
= \left( 1 - \frac{1}{m} \right) \int_{\partial G_{n,m} \cap (R_{i} - R_{0})} \frac{\partial}{\partial n} \omega_{n+i,i}(z) \, ds
\]

Since \( \omega_{n,i}(z) \Rightarrow \omega(A_{n}, z) \) and \( \omega_{n+i,i}(z) \Rightarrow \omega(A_{n+i}, z) \),

\[
D_{R-R_{0}-G_{n,m}}(\omega(A_{n}, z)) = D_{R-R_{0}-G_{n+i,m}}(\omega(A_{n+i}, z))
\]

Let \( m \to \infty \). Then \( D(\omega(A_{n}, z)) = D(\omega(A_{n+i}, z)) \) and \( \omega(A_{n}, z) \Rightarrow \omega(A_{n+i}, z) \), whence \( D(\omega(A_{n}, z)) = 0 \) as \( n \to \infty \).

Hence \( \omega(A_{n}, z) \Rightarrow \omega(H, z) \). Now \( \omega(A_{n}, z) \) has M.D.I. over \( (R - R_{0} - A_{n,m}) \) by a) and over \( R - R_{0} - G_{n,m} \) by b). Hence by Lemma 1, d) \( \omega(H, z) \) has M.D.I. over \( R - R_{0} - A_{m,n} \) and over \( R - R_{0} - G_{n,m} \). Thus we have e) and f).

Proof of g). By b) \( \omega(A_{n}, z) = (1 - \frac{1}{m}) \omega(G_{n,m}, z) \) in \( R - R_{0} - G_{n,m} : G_{n,m} = E[z \in \overline{R} : \omega(A_{n}, z) > 1 - \frac{1}{m}] \).

Let by \( \omega_{n}(z) = \omega(A_{n}, z) \) and \( \omega_{n+i}(z) = \omega(A_{n+i}, z) \)

\[
\omega(H, z) = \lim_{n} \omega_{n}(z) = \lim_{n} \omega_{n+i}(z)
\]

By \( G_{n} = G_{n} \supset G_{n+1} \supset G_{n+2} \cdots, \omega(G_{n}, z) \Rightarrow a \) function which is equal to \( \omega(H, z) \) by (53). Hence \( \omega(H, z) \) is C.P. \( \omega(G_{n}, z) \) defined by a decreasing sequence of domains \( \{G_{n}\} \). Hence \( \omega(H, z) \) has properties from P.C.1. to P.C.6. and we have (54) and (55).

Hence

\[
\omega(H, z) = \sup_{z \in G_{n}}(H, z) = 1
\]

If \( \omega(H, z) > 0 \), \( \sup_{z \in G_{n}}(H, z) = 1 \) for every \( n \).

Therefore

\[
\omega(H, z) = (1 - \frac{1}{m}) \omega(G_{m}, z) \text{ in } R - R_{0} - G_{m} : G_{m} = E[z \in R : \omega(H, z) > 1 - \frac{1}{m}]
\]

(55)
Next we show $\sup_{z \in H} \omega(H, z) = 1$, if $\omega(H, z) > 0$.

By c) $\omega(A_n, z) = 1$ on $A_n$ except an $F_{\sigma}$ set of capacity zero, hence $\omega(A_n \cap CG_{n}^* = 0$ and $A_n \cap CG_{n}^* \subset F_{\sigma}$. Hence $A_n \subset CG_{n}^* + F_{\sigma}^m$ and

$$H = \bigcap_{n \in \mathbb{N}} A_n \subset (\bigcap_{n \in \mathbb{N}} G_n^*) + \sum_{n \in \mathbb{N}} F_{\sigma}^m,$$

(56)

where $G_n^*$ is an open set by the semicontinuity of $\omega(A_n, z)$.

Put $CG_{n}^* = E \left[ z \in \overline{R} : \omega(H, z) \leq 1 - \frac{1}{m} \right]$. Then by (53)

$$\omega(H \cap CG_{n}^*, z) \leq \omega(G_n^* - CG_{n}^*, z) \leq \omega(G_n^*, z) = \omega(H, z)$$

and

$$\sup_{z \in CG_{n}^*} \omega(G_n^* - CG_{n}^*, z) \leq \sup_{z \in CG_{n}^*} \omega(H, z) < 1 - \frac{1}{m} < 1,$$

where $\omega(G_n^* - CG_{n}^*, z)$ is C.P. defined by sequence $\{G_n^* - CG_{n}^* : n = 1, 2, \cdots \}$. Hence by P.C.2. $\omega(G_n^* - CG_{n}^*, z) = 0$ i.e. $\lim_{n \to \infty} \omega(G_n^* - CG_{n}^*, z) = 0$.

Let $n_1, n_2, \cdots$ be a sequence such that $\int_{\partial R_0} \frac{\partial}{\partial n} \omega(G_n^* - CG_{n}^*, z) ds \leq \frac{1}{2^\ell}$. Then

$$\omega^*(z) = \sum_{i=1}^{\infty} \omega(G_n^* - CG_{n}^*, z)$$

is superharmonic and $\omega^*(z) < \infty$ and

$$\sum_{i=1}^{\infty} \omega(G_n^* - CG_{n}^*, z) \geq \omega_{\infty}(G_n^* - CG_{n}^*, z).$$

If $H \ni p : p \in B_{S}$ then $\omega(H, z) \geq \omega(p, z)$, $\omega(H, p) \geq \omega(p, p) = 1$ by Theorem 10, b). In this case our assertion is trivial. Hence we can suppose that $H \cap B_{S} = 0$. Let $p \in R - R_0 + B_1 - B_S$ be a point in $\bigcap G_n^*$. Then $\sup_{z \in R} N(z, p) = \infty$ and $N(z, p)$ is $N$-minimal. Let $V_M(p)$ be a neighbourhood of $p$ such that $V_M(p) = E \left[ z \in \overline{R} : N(z, p) > M \right]$ and $\partial V_M(p)$ is a regular niveau curve.

Assume $\sup_{z \in H} \omega(H, z) < K < 1 - \frac{2}{m}$. Then $H \subset \overline{CG_{2m}}^*$. Let $\nu_M(p) = E \left[ z \in \overline{R} : \delta(z, p) < \frac{1}{M_{\nu_M(p)}} \right]$ such that $\nu_M(p) \subset G_{n_i}^*$. Such $\nu_M(p)$ can be chosen, because $G_{n_i}^*$ is open by the semicontinuity of $\omega(A_n, z)$. By the definition of the value of a superharmonic functions at a point in $R - R_0 + B_1 - B_S$

$$\omega(H, p) = \frac{1}{2\pi} \lim_{M \to \infty} \int_{\partial V_M(p) \cap \nu_M(p)} \omega(H, p) \frac{\partial}{\partial n} N(z, p) ds \leq (1 - \frac{2}{m})$$

by Theorem 12, a). This implies

$$\lim_{M \to \infty} \int_{\partial V_M(p) \cap \nu_M(p) \cap \overline{CG_{n}}^*} 2\pi \epsilon_0 > 0 \text{ by } \lim_{M \to \infty} \int_{\partial V_M(p) \cap \nu_M(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi.$$
Now by (58) \( \omega^*(z) \geq \omega(z) \), whence\( \omega^*(z) \geq \frac{1}{2\pi} \int N(z, p) \frac{\partial}{\partial n} N(z, p) ds \geq \epsilon_0 \). Let \( \epsilon_0 \to \infty \). Then \( \omega^*(p) = \infty \). On the other hand, \( \sum F_{\sigma}^{n} + B_0 \) is a \( F_{\sigma} \) set of capacity zero, whence we can construct a superharmonic function \( \omega^{**}(z) \) such that \( \omega^{**}(z) \to \infty \) as \( z \to p \in (\sum F_{\sigma}^{n} + B_0) \). Put \( \omega^{***}(z) = \omega^*(z) + \omega^{**}(z) \). Then \( \epsilon \omega^{***}(z) \to \infty \) as \( z \to p \in H \subset (\sum F_{\sigma}^{n} + B_0) \). 

11. Maximum principles.

Theorem 15. a) Let \( U(z) \) be a positive superharmonic function in \( \overline{R} - R_0 \) such that \( U(z) = 0 \) on \( \partial R_0 \) which is represented by a canonical mass distribution \( \mu \) such that \( U(z) = \int N(z, p) d\mu(p) \). Let \( F \) be the kernel of \( \mu \). If \( U(z) \leq M \) at points on which the mass is distributed (this implies \( U(z) \leq M \) on \( F \) by the lower semicontinuity of \( U(z) \)) and if \( \mu = 0 \) on \( B_s \) (set of singular points), then

\[
\int_{C_{\lambda}} U(z) \frac{\partial}{\partial n} U(z) ds \leq 2\pi M \int d\mu,
\]

where \( C_{\lambda} = E[z \in \overline{R} : U(z) = \lambda] \).

b) Let \( U(z) \) be a positive superharmonic function such that \( U(z) = 0 \) on \( \partial R_0 \). Put \( G_{M_i} = E[z \in \overline{R} : U(z) > M_i] : \lim_{i} M_i = \infty \). Put \( U'(z) = \lim_{i} U(z) \). Then \( U'(z) \) is represented by a canonical mass distribution \( \mu \) on \( \bigcap_{i} \overline{G}_{M_i} \). Let \( F \) be the kernel of \( \mu \). Then if \( U'(z) > 0 \), \( \sup_{z \in F} U'(z) = \infty \).

c) Let \( U(z) \) be a positive superharmonic function in \( \overline{R} - R_0 \) with \( U(z) = 0 \) on \( \partial R_0 \) and let \( \mu \) be its canonical mass distribution whose kernel is \( F \). If \( U(z) > 0 \) and \( \operatorname{Cap}(F) = 0 \), \( \sup_{z \in F} U(z) = \infty \).

d) Let \( U(z) \) be a positive superharmonic function in \( \overline{R} - R_0 \) with
$U(z) = \text{on } \partial R_0 \text{ and let } \mu \text{ be its canonical mass distribution whose kernel is } F$. If $U(z) \leq M$ on $F$, then

$$U(z) \leq M \omega(F, z).$$

**Proof of a.** Let $U(z) = \sum_{j=1}^{\epsilon} c_j N(z, p_j)$ such that $p_j \in (F - (R - R_0 + B_1 - B_2))$.

$c_j > 0$ and $\sum c_j = \text{total mass of } \mu$. Put $V_j = E[z \in R : c_j N(z, p_j) > \lambda]$ such that $V_j$ is regular. Such $V_j$ exists, since $\sup_{R} N(z, p) = \infty$ for $p \in R - R_0 + B_1 - B_2$. Then $\sum c_j N(z, p_j)$ has M.D.I. over $R - R_0 - \sum V_j(p_j)$, whence $U(z)$ has M.D.I. over $R - R_0 - \sum V_j(p_j)$. Put $D(z) = E[z \in R : U(z) > \lambda]$. Then $D(z) \supset \sum V_j(p_j)$ and $U(z)$ has M.D.I. over $R - R_0 - D(z)$ i.e. $U(z) = \omega(D(z))$ in $R - R_0 - D(z)$, whence we can find a domain $D(z) = E[z \in R : U(z) > \lambda']$ such that $\lambda > \lambda' > 0$ and $\partial D(z)$ is a regular niveau curve of $U(z)$. Let $CD_{\lambda, i} U(z)$ be a harmonic function in $CD_{\lambda, i}$ with $CD_{\lambda, i} U(z) = U(z)$ on $\partial D_{\lambda, i} + \partial R_0$. Then

$$U(z) \leq U_{\lambda, i} U(z) = \lim_{M \to \infty} U_{\lambda, i} U(z) = \lim_{M \to \infty} U_{\lambda, i} U(z),$$

where $U_{\lambda, i}(z)$ is a harmonic function in $D_{\lambda, i} - (R_0 - R_0)$ such that $U_{\lambda, i}(z) = \min (M, U(z))$ on $\partial D_{\lambda, i} - (R_0 - R_0)$ and $\frac{\partial}{\partial n} U_{\lambda, i}(z) = 0$ on $\partial R_0 - D_{\lambda, i}$.

Let $N(z, p_j)$ be a harmonic function in $(D_{\lambda, i} - V_j(p_j)) \cap (R_0 - R_0)$ such that $N(z, p_j) = N(z, p_j)$ on $\partial (D_{\lambda, i} - V_j(p_j)) \cap (R_0 - R_0)$ and $\frac{\partial}{\partial n} N(z, p_j) = 0$ on $\partial R_0 - (D_{\lambda, i} - V_j(p_j))$. Then $N(z, p_j) \Rightarrow N(z, p_j)$ in $D_{\lambda, i} - V_j(p_j)$ and $U(z) = \sum c_j N(z, p_j) = \lambda'$ on $\partial D_{\lambda, i}$ and $U_{\lambda, i}(z) \Rightarrow U_{\lambda, i}(z)$ as $n \to \infty$ in $D_{\lambda, i}$.

By the Green's formula

$$\int_{\partial D_{\lambda, i} + \partial V_j(p_j) \cap R_0} U_{\lambda, i}(z) \frac{\partial}{\partial n} c_j N(z, p_j) ds = \int_{\partial D_{\lambda, i} + \partial V_j(p_j) \cap R_0} c_j N(z, p_j) \frac{\partial}{\partial n} U_{\lambda, i}(z) ds : j = 1, 2, \ldots, i.$$ 

By

$$\int_{\partial V_j(p_j) \cap R_0} c_j N(z, p_j) \frac{\partial}{\partial n} U_{\lambda, i}(z) ds = \int_{\partial V_j(p_j) \cap R_0} \frac{\partial}{\partial n} U_{\lambda, i}(z) ds = 0$$

we have

$$\int_{\partial D_{\lambda, i} - R_0} U_{\lambda, i}(z) \frac{\partial}{\partial n} c_j N(z, p_j) ds = \int_{\partial D_{\lambda, i} - R_0} U_{\lambda, i}(z) \frac{\partial}{\partial n} c_j N(z, p_j) ds$$

$$+ \int_{\partial D_{\lambda, i} - R_0} c_j N(z, p_j) \frac{\partial}{\partial n} U_{\lambda, i}(z) ds.$$ 

By summing up (59) for $j = 1, 2, \ldots, i$ and by
\[
\sum_{\partial D_{\lambda}^{i}}^{i} \int_{i} c_{j} N_{n}(z, p_{j}) \frac{\partial}{\partial n} U_{n}(z) ds = \lambda' \int \frac{\partial}{\partial n} U_{n}(z) ds = \lambda' \int \frac{\partial}{\partial n} U_{n}(z) ds = 0,
\]
we have
\[
\int_{\partial D_{\lambda}^{i}} U_{n}(z) \frac{\partial}{\partial n} \sum_{j=1}^{i} c_{j} N_{n}(z, p_{j}) ds = \sum_{j=1}^{i} \int_{\partial V_{\lambda}(p_{j})} U_{n}(z) \frac{\partial}{\partial n} c_{j} N_{n}(z, p_{j}) ds.
\]

Put \( U_{i,n}(z) = \sum_{j=1}^{i} c_{j} N_{n}(z, p_{j}) \). Then \( U_{i,n}(z) \leq i \lambda \) in \( R_{n}-R_{0}-\sum V_{\lambda}(p_{j}) \).

Consider \( U_{i,n}(z) \) in \( D_{\lambda_{i}}^{i} - D_{\lambda_{i}}^{i} \). Then \( U_{i,n}(z) = \min_{z \in D_{\lambda_{i}}^{i} - D_{\lambda_{i}}^{i}} U_{n}(z) \leq i \lambda \) on \( \partial D_{\lambda_{i}}^{i} - D_{\lambda_{i}}^{i} \).

By the regularity of \( \partial D_{\lambda_{i}}^{i} \) and by Theorem 3, \( b) \)
\[
\int_{\partial D_{\lambda_{i}}^{i}} U_{n}(z) \frac{\partial}{\partial n} \sum_{j=1}^{i} c_{j} N_{n}(z, p_{j}) ds = \sum_{j=1}^{i} \int_{\partial V_{\lambda}(p_{j})} U_{n}(z) \frac{\partial}{\partial n} c_{j} N_{n}(z, p_{j}) ds.
\]

Assume \( U(z) \leq M \) on \( F \cap (R-R_{0}+B_{1}) \). Then by the local superharmonicity of \( U(z) \) at \( p \)
\[
M \geq \lim_{L \to \infty} \frac{1}{2\pi} \int_{\partial V_{L}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds \geq \frac{1}{2\pi} \int_{\partial V_{\lambda}(p)} U(z) \frac{\partial}{\partial n} N(z, p) ds.
\]
Hence by (63) and by \( c_{j} U_{CD_{\lambda_{i}}^{i}, i} U(z) \to U(z) \) on \( \partial D_{\lambda_{i}}^{i} \), we have
\[
\int_{\partial D_{\lambda_{i}}^{i}} U(z) \frac{\partial}{\partial n} U_{i}(z) ds = \int_{\partial D_{\lambda_{i}}^{i}} U(z) \frac{\partial}{\partial n} U_{i}(z) ds \leq 2\pi \sum_{j=1}^{i} c_{j} M.
\]

By the continuity of \( N(z, p) \) for fixed \( z \) with respect to \( p \), there exists a sequence of linear forms \( \sum_{j=1}^{i} c_{j} N(z, p_{j}) = U_{i}(z) \) such that \( \sum c_{j} = \text{total mass of } U(z) \), \( p_{j} \in F \) and \( U_{i}(z) \to U(z) \) as \( i \to \infty \) in \( R_{m}-R_{0} \) uniformly for any given number \( m \). Now \( U_{i}(z) \to U(z) \) implies \( \frac{\partial}{\partial n} U_{i}(z) \to \frac{\partial}{\partial n} U(z) \) in \( R_{m}-R_{0} \) and \( \partial D_{\lambda_{i}}^{i} = E \{ z \in R : U(z) = \lambda' \} \) tends to \( C = E \{ z \in R : U(z) = \lambda' \} \). Then by Fatou's lemma
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\[ \int_{C_{\lambda^\prime} \cap R_{m}} U(z) \frac{\partial}{\partial n} U(z) ds \leq i \varliminf_{\partial D_{\lambda^\prime \cap R_{m}}} \int U(z) \frac{\partial}{\partial n} U(z) ds \leq 2\pi M \times (\text{total mass of } U(z)). \]

Let \( m \to \infty \). Then

\[ \int_{C_{\lambda^\prime}} U(z) \frac{\partial}{\partial n} U(z) ds \leq 2\pi M \times (\text{total mass of } U(z)). \]

Thus we have \( a \).

**Proof of \( b \).** Put \( G_{M_i} = E[z \in R: U(z) > M_i]: M_1 < M_2, \cdots, \lim_{i} M_i = \infty \).

Then \( G_{M_i} \) is open by the semicontinuity of \( U(z) \) and \( \omega(G_{M_i}, z) \leq \frac{U(z)}{M_i} \).

Let \( M_i \to \infty \). Then \( \lim \omega(G_{M_i}, z) = 0 \). Put \( \lim_{i} \omega(G_{M_i}, z) = U'(z) \). Then by Theorem 6, \( b \)

\[ \lim_{i} \omega(G_{M_i}, z) = U'(z). \]

If \( \sup_{z \in R} U'(z) < M < \infty \), \( U'(z) \leq M \omega(G_{M_i}, z) \to 0 \) as \( M_i \to \infty \). In this case our assertion is trivial. We suppose \( \sup_{z \in R} U'(z) = \infty \). Let \( \mu \) be the canonical mass distribution of \( U'(z) \). We show that \( \mu \) has no mass at any point of \( B_s \). Assume that \( U'(z) \) has a mass \( m \) at \( p \in B_s \). Then \( U'(z) = m \omega(p, z) + U''(z) \) and \( U''(z) \) is also upperharmonic.

Hence

\[ \omega(G_{M_i}, z) = 0. \]

Now \( m \omega(p, z) \leq m \) in \( \overline{R} - R_0 \), whence \( \omega(G_{M_i}, z) \leq m \omega(G_{M_i}, z) \). Let \( M_i \to \infty \). Then \( \lim_{i} \omega(G_{M_i}, z) = 0 \). Hence

\[ \lim_{i} \omega(G_{M_i}, z) = 0 \] and

\[ U'(z) \geq m \omega(p, z) \geq \lim_{i} \omega(G_{M_i}, z) = \lim_{i} U'(z) = U'(z). \]

This is a contradiction. Hence \( U'(z) \) has no mass at any point \( p \in B_s \).

\[ g_{M_i} U'(z) = U'(z) \text{ is clear by definition. We show} \]

\[ \lim_{i} g_{M_i} U'(z) = U'(z) \text{ for any } M_i < \infty, \]

where \( g_{M_i} = E[z \in R: U'(z) > M_i] \).

By \( U(z) \geq U'(z) \), \( \lim_{i} \omega(G_{M_i}, z) = 0 \). Let \( U_{\tilde{M}_i, M_j}(z) \) be a harmonic function in \( R - R_0 - (\tilde{G}_{M_i} + G_{M_j}): M_i < M_j \) such that \( U_{\tilde{M}_i, M_j}(z) = U'(z) \) on \( \partial R_0 + \partial(\tilde{G}_{M_i} + G_{M_j}) \) and \( U_{\tilde{M}_i, M_j}(z) \) has M.D.I. over \( R - R_0 - (\tilde{G}_{M_i} + G_{M_j}) \). Then by \( (\tilde{G}_{M_i} + G_{M_j}) \supset U_{\tilde{M}_i, M_j}(z) \)

\[ U'(z) \geq U_{\tilde{M}_i, M_j}(z) \geq g_{M_j} U'(z) = U'(z). \]

(67)
Let \( U_{M_i, M_j}^*(z) \) be a harmonic function in \( R-R_0-(\tilde{G}_M+G_M): M_i < M_j \) such that \( U_{M_i, M_j}^*(z) = U'(z) \) on \( \partial R_0 + \partial \tilde{G}_M - G_M \) and \( \frac{\partial}{\partial n} U_{M_i, M_j}^*(z) = 0 \) on \( \partial G_M - \tilde{G}_M \) and \( U_{M_i, M_j}^*(z) \) has M.D.I. over \( R-R_0-(\tilde{G}_M+G_M) \). Hence by the maximum principle

\[ |U_{M_i, M_j}^*(z) - U_{M_i}^* M_j(z)| \leq M_0 \omega(G_M, z) \rightarrow 0 \text{ as } j \rightarrow \infty. \]

On the other hand both \( U_{M_i, M_j}(z) \) and \( U_{M_i, M_j}^*(z) \) has M.D.I. over \( R-R_0-(\tilde{G}_M+G_M) \). Hence by the maximum principle

\[ |U_{M_i, M_j}(z) - U_{M_i, M_j}^*(z)| \leq 2\pi M \int d\mu \leq 2\pi M \int d\mu. \]

On the other hand, by (67) \( U'(z) = M_0 \omega(G_M, z) \), in \( R-R_0-\tilde{G}_M \) hence \( \partial \tilde{G}_M \) is a regular niveau curve for almost every number \( M_i \) and

\[ \int_{c_{M_i}} \frac{\partial}{\partial n} U'(z) ds = \int_{c_{M_i}} \frac{\partial}{\partial n} U'(z) ds \quad \text{and} \quad \lim_{M_i \rightarrow \infty} \int_{c_{M_i}} \frac{\partial}{\partial n} U'(z) ds = \infty. \]

This is a contradiction. Hence we have b).

Proof of c). If \( U(z) > 0 \), clearly \( \sup_{z \in K} U(z) = \infty \). Now by the assumption, since \( \text{Cap}(F) = 0 \), \( \mu \) has no mass at any point of \( B_S \). We show \( \frac{\partial}{\partial n} U(z) = 0 \), where \( D_i = E[z \in R: U(z) < \lambda] \). Since by Theorem 13. a) \( U(z) = U(z) \). Now \( U(z) \) has M.D.I. \( \leq 2\pi \lambda \times (\text{total mass of } \mu) \) on \( D_i - F_m, F_m = E[z \in R: \delta(z, F) \leq \frac{1}{m}] \). Hence \( U_{m,n}(z) \rightarrow U_m(z) \) as \( n \rightarrow \infty \) and \( U_m(z) \rightarrow U(z) \)

\[ U_{m,n}(z) = U_{m,n}(z) \text{ on } ((\partial D_i \cap CF_m) \cap (R-R_n)) + \partial R_n \text{ and } \frac{\partial}{\partial n} U_{m,n}(z) = 0 \]

on \( \partial R_n - (D_i - F_m) \). Let \( U_{m,n}(z) \) be a harmonic function in \( (R_n - R_0) \cap (D_i - F_m) \) such that

\[ U_{m,n}(z) = U(z) \text{ on } ((\partial D_i \cap CF_m) \cap (R_n - R_0)) + \partial R_n \text{ and } \frac{\partial}{\partial n} U_{m,n}(z) = 0 \]

on \( \partial F_m - D_i \). Then \( U_{m}(z) \rightarrow U_m(z) \) and \( U'(z) \rightarrow CD_2 U(z) \), because \( CD_2 U(z) \) has M.D.I. over \( D_i \). Now

\[ U_{m,n}(z) = U_{m,n}(z) \text{ on } ((\partial D_i - F_m) \cap (R_n - R_0) + \partial R_n), \]

\[ |U_{m,n}(z) - U_{m,n}(z)| < \lambda \text{ on } \partial E_m - D_i \text{ by } U_{m,n}(z) \text{ and } U_{m,n}(z) < \lambda \text{ in } D_i - F_m \]
and \( \frac{\partial}{\partial n}U_{m,n}(z) = \frac{\partial}{\partial n}U'_{m,n}(z) = 0 \) on \( \partial R_n \cap D' \cap CF_m \).

Hence by the maximum principle

\[ |U_{m,n}(z) - U'_{mn}(z)| < \lambda \omega_n(F_m, z), \]

where \( \omega_n(F_m, z) \) is a harmonic function in \( R_n - R_0 - F_m \) such that \( \omega_n(F_m, z) = 1 \) on \( F_m \), \( \omega_n(F_m, z) = 0 \) on \( \partial R_0 \) and \( \frac{\partial}{\partial n} \omega_n(F_m, z) = 0 \) on \( \partial R_n - F_m \).

Let \( n \to \infty \) and then \( m \to \infty \). Then

\[ |U(z)_{CD_{\lambda}} - U(z)| \leq \lambda \omega(F, z) = 0 \]

by Cap \((F) = 0 \).

Hence \( U(z) \) has M.D.I. over \( D_{\lambda} \) and \( CD_{\lambda}U(z) = U(z) \), whence \( U(z) = \omega(CD_{\lambda}, z) \) in \( D_{\lambda} \).

Thus by a)

\[ \sup_{z \in F} U(z) = \infty. \]

**Proof of d).** Put \( G_{M_i} = E[z \in R: U(z) > M_i], M_1 < M_2, \ldots, \lim_{i} M_i = \infty. \)

Then \( G_{M_i} \) is open and \( \omega(G_{M_i}, z) \leq \frac{U(z)}{M_i} \). Let \( M_i \to \infty \). Then \( \lim_{i} \omega(G_{M_i}, z) = 0 \).

Put \( \lim_{i} G_{M_i}U(z) = U'(z) \). Then by Theorem 6. b) \( U(z) - U'(z) \) is superharmonic in \( \overline{R} - R_{0} \) and \( \lim_{i} G_{M_i}(U'(z)) = U'(z) \). Now the total mass of \( \omega_{M_i}U(z) \leq 1 \).

Hence we can find an weak limit \( \mu' \) of the distribution \( \mu'_i \) of \( \{G_{M_i}U(z)\} \) on \( \bigcap_{i}^\infty G_{M_i} \). Now \( \bigcap_{i}^\infty G_{M_i} \) is of capacity zero, but we don't know that \( \bigcap_{i}^\infty \overline{G}_{M_i} \) is of capacity zero or not. Let \( \mu'^* \) be the canonical distribution of \( \mu' \). Then by \( \overline{G}_{M_i}U'(z) = U'(z) \) by \( \overline{G}_{M_i} \supset G_{M_i} \). \( \mu'^* \) has no mass outside of \( \bigcap_{i}^\infty \overline{G}_{M_i} \) and \( \mu'^* \) is contained in \( \bigcap_{i}^\infty \overline{G}_{M_i} \). Also \( U(z) - U'(z) \) has a canonical mass distribution \( \mu'^{**} \geq 0 \). Then \( \mu'^* + \mu'^{**} \) is a canonical distribution of \( \mu' \) which is the kernel of the distribution \( \mu' \) of \( \mu' \) is contained in \( F \). We show \( U'(z) = 0 \). In fact, by the assumption \( \sup_{z \in F} U'(z) \leq \sup_{z \in F} U(z) \leq M \). Hence by b)

\[ U'(z) = 0. \]

(69)

\( G_{M+a} = E[z \in \overline{R}: U(z) > M+a]: a > 0 \) is open and \( F_n = E[z \in \overline{R}: \delta(z, F') \leq \frac{1}{n}]. \)
is closed and $G_{M+a} \cap F_n$ is an $F_\sigma$ set. Now $U(z) \geq M+a$ in $G_{M+a}$ implies $U(z) \geq (M+a)\omega(G_{M+a} \cap F_n, z)$.

Let $n \to \infty$. Then $U(z) \geq (M+a)\omega(G_{M+a} \cap F, z)$, where $F \cap G_{M+a} = \lim (F_n \cap G_{M+a})$ is an $F_\sigma$ set.

Assume $\omega(G_{M+a} \cap F, z) > 0$. Then by Theorem 14, $\sup_{z \in G_{M+a} \cap F} \omega(G_{M+a} \cap F, z) = 1$ on $G_{M+a} \cap F$.

Hence by $U(z) \geq (M+a)\omega(G_{M+a} \cap F, z)$, $\sup_{z \in F} U(z) \geq M+a$ on $F$. This contradicts $U(z) \leq M$ on $F$.

Hence $\omega(G_{M+a} \cap F, z) = 0$. (70)

By (69) there exists a number $N_0$ such that $a_N U(z_0) < \varepsilon$ for any given point $z_0$ and any given positive number $\varepsilon > 0$ for $N \geq N_0$.

$U(z) = \sup_{z \in G_{M+a} \cap F_n} U(z) \leq a+M$ on $\partial F_n \cap C(F_n \cap G_{M+a})$, $U(z) \leq N$ on $\partial F_n \cap CG_N$ and $U(z) = a_N U(z)$ on $\partial F_n \cap G_N$. Hence by the maximum principle $U(z) = \sup_{z \in F_n} U(z) \leq (a+M)\omega(F_n, z) + N\omega(G_{M+a} \cap F_n, z) + a_N U(z)$.

Let $n \to \infty$. Then by (70)

$U(z) \leq (a+M)\omega(F, z) + \varepsilon$ for $z = z_0$.

Let $\varepsilon \to 0$ and $a \to 0$. Then

$U(z) = \sup_{z \in F} U(z) \leq M\omega(F, z)$.

Thus we have d).

12. Mass distributions. In the sequel we consider Problem of Equilibrium. It is important to summarize the properties of the space and the kernel $N(p, q)$.

1). The space $\overline{R} - R_0$ is composed of a Riemann surface $R-R_0$ and its ideal boundary $B = B_1 + B_0$, where $B_1$ and $B_0$ are the sets of $N$-minimal points and of $N$-non minimal points respectively and $B_0$ is an $F_\sigma$ set of capacity zero. On $B_0$ we cannot distribute any true mass. A distribution $\mu$ on $B_0$ may be called a pseudo distribution in the sense that $\mu$ can be replaced (by Theorem 8) by a canonical distribution on $R-R_0 + B_1$ without any change of $U(z) = \int N(z, p) \alpha \mu(p)$.

2). The kernel $N(p, q)$ satisfies the following conditions:

a) $N(p, q) = N(q, p) : p$ and $q \in \overline{R} - R_0$.

b) $N(p, q)$ is harmonic with respect to $p$ in $R-R_0$ for fixed $q \in \overline{R} - R_0$, whence $N(p, q)$ is continuous in wider sense ($N(p, q)$ may be infinite at $q$) with respect to $p$ for fixed $q \in \overline{R} - R_0$ and $N(p, q)$ is continuous (with
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respect to \( \delta \)-metric) in \( \overline{R} - R_0 \) with respect to \( q \in \overline{R} - R_0 \) for fixed \( p \in R - R_0 \).

c) \( N(p, q) \) is lower semicontinuous in \( \overline{R} - R_0 \) for fixed \( q \in \overline{R} - R_0 \). But it cannot be verified that \( N(p, q) \) is lower semicontinuous in \( \overline{R} - R_0 \) in both arguments \( p \) and \( q \) in \( \overline{R} - R_0 \).

d) The potential \( U(z) = \int N(z, p) d\mu(p) : \mu(p) \geq 0 \) (in the following we call \( U(z) \) the potential of a distribution \( \mu \)) is superharmonic in \( \overline{R} - R_0 \), superharmonic locally at any point of \( R - R_0 + B_1 \) and lower semicontinuous in \( \overline{R} - R_0 \).

3) Maximum principle is valid. Let \( U(z) \) be the potential of a positive canonical mass distribution \( \mu \). If \( U(z) \leq M \) on the kernel \( F \) of \( \mu \), \( U(z) \leq M \omega(F, z) \) in \( \overline{R} - R_0 \), where \( \omega(F, z) \) is C.P. of \( F \).

4) Function theoretic Equilibrium Problem can be solved: let \( F \) be a closed set of positive capacity. Then C.P. \( \omega(F, z) = 1 \) on \( F \) except at most an \( F_{\sigma} \) set of capacity zero and \( \omega(F, z) \) can be represented by a canonical mass distribution \( \mu \) whose kernel is contained in \( F \).

Energy Integral \( I(\mu) \) of a mass distribution \( \mu \) on \( \overline{R} - R_0 \) is defined as

\[
I(\mu) = \int \int N(p, q) d\mu(p) d\mu(q) = \int U(p) d\mu(p).
\]

*Capacity (potential theoretic) of a closed set in \( R - \overline{R}_0 \) is defined by

\[
\frac{1}{\inf_{\mu \in ca} I(\mu)},
\]

where \( \inf I(\mu) \) is the infimum of Energy Integrals of all positive canonical mass distribution on \( F \) of mass unity. If \( F \cap (R - R_0 + B_1) = 0 \), we define \( \text{Cap}(F) = 0 \).

Problem of Equilibrium.

Theorem 16. a) Let \( \mu \) be a positive mass distribution and let \( \mu^* \) be its canonical mass distribution. Then \( I(\mu) = I(\mu^*) \) and \( I(\mu) \) does not depend on a choice of particular distribution.

b) Let \( F \) be a closed set such that \( \text{Cap}(F) > 0 \). Then \( \text{Cap}(F) < 14 \text{Cap}(F) \). If \( F \subseteq B_0 \), \( \text{Cap}(F) = 0 \) and \( \text{Cap}(F) = 0 \) by definition. Hence by Theorem 13. a) \( \text{Cap}(F) > 0 \) if and only if \( \text{Cap}(F) > 0 \).

c) Let \( F \) be a positive capacity (clearly of positive capacity by b)). Let \( \{\mu_n\} \) be a minimizing sequence of positive canonical mass distributions on \( F \) of mass unity such that \( I(\mu_n) \downarrow \inf I(\mu) \). Let \( \mu \) be an weak limit of \( \{\mu_n\} \). Then \( \mu \) is also a positive canonical mass distri-
bution on $F$ of mass unity.

d) Let $V = \inf_{\mu \in \mathcal{C}} I(\mu)$. Then there exists a canonical mass distribution $\mu$ such that $I(\mu) = V$.

e) Let $F$ be a closed set in $\overline{R} - R_{0}$ of positive capacity. Let $\mu$ be a positive canonical mass distribution on $F$ such that $I(\mu) = V$. Then the potential $U(z)$ of $\mu$ satisfies the following conditions:

1) $U(z) \geq V$ on $F$ except at most a set of capacity zero.
2) $U(z) = V\omega(F, z)$ and $I(\mu) = D(\omega(F, z)) = V$.
3) $\text{Cap}(F) = \text{Cap}(F)$.

Proof of $a)$. Suppose $p$ and $q$ are points in $\overline{R} - R_{0}$. Then $N(z, p) = \int N(z, p_{\alpha}) d\mu_{p}(p_{\alpha})$ and $N(z, q) = \int N(z, q_{\beta}) d\mu_{q}(q_{\beta})$, where $\mu_{p}(p_{\alpha})$ and $\mu_{q}(q_{\beta})$ are canonical mass distributions of $N(z, p)$ and $N(z, q)$ respectively. Then

$$
I(\mu) = \int \int N(p, q) d\mu(p) d\mu(q) = \int \int \int \int N(p_{\alpha}, q_{\beta}) d\mu_{p}(p_{\alpha}) d\mu_{q}(q_{\beta}) d\mu(p) d\mu(q)
$$

For other distributions we have the same value, hence $I(\mu)$ does not depend on particular distributions. Thus we have $a$.

Proof of $b)$. Let $V$ be the infimum of all positive canonical mass distributions on $F$ of positive capacity of mass unity. Let $\{\mu_{n}\}$ be a minimizing sequence of canonical distributions of mass unity on $F$ such that $I(\mu_{n}) = V + \varepsilon_{n}: \varepsilon_{n} \downarrow 0$. Put $\varepsilon_{0} = \frac{V}{10}$. Let $n_{0}$ be a number such that $\varepsilon_{n} \leq \frac{\varepsilon_{0}}{10}$ for $n \geq n_{0}$. Let $\mathfrak{M}_{n}$ be the mass of the restriction $\mu_{n}'$ of $\mu_{n}$ on the set $E[z \in F: U_{n}(z) \leq V + \varepsilon_{0}]$, where $U_{n}(z)$ is the potential of $\mu_{0}$. Then since $I(\mu_{n}) = V + \varepsilon_{n}$,

$$
\mathfrak{M}_{n} > \frac{1}{13},
$$

because if the set $E[z \in F: U_{n}(z) > V + \varepsilon_{0}]$ has mass $\frac{V + \varepsilon_{n}}{V + \frac{\varepsilon_{0}}{2}}$,
\[ V + \varepsilon_n = I(\mu_n) \geq \int U_n(z) d\mu_n'(p) > (V + \varepsilon_0) \times \left( \frac{V + \varepsilon_n}{2} \right) > (V + \varepsilon_n). \]

Let \( U_n'(z) \) be the potential of \( \mu_n' \). Then \( U_n'(z) \leq V + \varepsilon_0 \) on the kernel \( F''(\subset F') \) of \( \mu'_n \). Hence by the maximum principle \( U_n'(z) \leq (V + \varepsilon_0) \omega(F', z) \leq (V + \varepsilon_0) \omega(F,z) \), whence

\[ V + \varepsilon_0 \geq \frac{\int \frac{\partial}{\partial n} U_n'(z) ds}{\int \frac{\partial}{\partial n} \omega(F, z) ds} \geq \frac{1}{13} \text{Cap}(F'). \]

Hence by \( \frac{V}{10} = \varepsilon_0 \),

\[ V \geq \frac{10}{143 \text{Cap}(F')} \]

and \( \text{Cap}^*(F) = \frac{1}{V} < 14 \text{Cap}(F') \).

Conversely, if \( \text{Cap}(F') > 0 \), then \( \omega(F, z) = \int \omega(F, z) = \int N(z, p) d\mu^*(p) \) by Theorem 13.c). Now \( \omega(F, z) \leq 1 \) on \( F \) and the total mass of \( \mu^* \) is given by \( \mathfrak{M} = \int \frac{\partial}{\partial n} \omega(F, z) ds \). Hence \( \frac{\mu^*}{\mathfrak{M}} \) is a canonical distribution on \( F \) of mass unity and its Energy Integral \( \leq \frac{1}{\mathfrak{M}} \), whence \( \inf_{\mu \in \text{ca}} I(\mu) \geq V \leq \frac{1}{\mathfrak{M}} \). Hence \( \text{Cap}(F) \geq \mathfrak{M} = \text{Cap}(F) \).

Let \( \{\mu_n\} \) be a minimizing sequence such that \( I(\mu_n) = V + \varepsilon_n, \lim \varepsilon_n = 0 \) and \( \varepsilon_n < \frac{V}{20} \). Let \( \mathfrak{M}_n \) be the mass of the restriction \( \mu'_n \) of \( \mu \) on \( \Gamma_{m,i} \). Then by \( \text{Cap}(\Gamma_{m,i}) < 14 \text{Cap}(\Gamma_{m,i}) \mathfrak{M}_n < \frac{\sqrt{2}}{2^{m+i}} \), because if \( \mathfrak{M}_n \geq \frac{\sqrt{2}}{2^{m+i}}, \frac{21}{20} V \geq V + \varepsilon_n = I(\mu_n) \geq I(\mu'_n) \geq \frac{\mathfrak{M}_n^2}{\text{Cap}(\Gamma_{m,i})} > 2V \). Put

\[ O_{m,2i} = E \left[ z \in \overline{R} : \delta(\Gamma_{m,i}, z) < \frac{2}{2i} \right]. \]

Then \( O_{m,2i} \) is open and \( E \left[ z \in \overline{R} : \delta(\Gamma_{m}, z) \right] \)
\[
\leq \frac{1}{3t} = \Gamma_{m,3t} \subset O_{m,2t} \subset \Gamma_{m,t}.
\]
Hence the mass of \( \mu_n \) on \( O_{m,2t} \) is smaller than \( \frac{\sqrt{2}}{2^{m+l}} \). Let \( \mu \) be an weak limit of \( \{ \mu_n \} \). Then it is known that the mass of \( \mu \) on any open set \( G \) (closed set \( F \)) is smaller (larger) than \( \lim_n (\text{mass of } \mu_n \text{ on } F') \). Hence the mass of \( \mu \) on \( \bigcup_{m=1}^{\infty} O_{m,2t} \) is smaller than \( \sum_{m=1}^{\infty} \frac{\sqrt{2}}{2^{m+l}} = \frac{2+\sqrt{2}}{2^l} \). Let \( A_l = \bigcup_{m=1}^{\infty} \Gamma_{m,3t} \subset (\bigcup_{m=1}^{\infty} O_{m,2t}) \). Then the mass of \( \mu \) on \( A_l \leq \frac{2+\sqrt{2}}{2^l} \). Now \( B_0 \subset (\bigcap_{l=1}^{\infty} A_l) \). Let \( l \to \infty \). Then \( \mu \) has no mass on \( B_0 \), i.e. \( \mu \) is a canonical distribution. Clearly by the closedness of \( F \) \( \mu \) has mass unity on \( \mu \).

Proof of d). Since it cannot be proved that \( N(p,q) \) is lower semi-continuous in \( \overline{R}-R_n \), in both arguments \( p \) and \( q \) in \( \overline{R}-R_n \), it is not so clear that \( I(\mu) \leq \lim_n I(\mu_n) : \mu = \lim_n \mu_n \). Let \( V \) be the infimum of all canonical mass distributions on a closed set \( F \) of positive capacity (\( \text{Cap} (F) > 0 \) by b)) of mass unity. Let \( V > \alpha > 0 \) and let \( \mu \) be a canonical positive mass distribution of mass unity on \( F \). Let \( \mu' \) be the restriction of \( \mu \) on the closed set \( E[z \in F : U(z) \leq V-\alpha] \): \( U(z) \) is the potential of \( \mu \) and let \( 1-\mathfrak{M} \) be the mass of \( \mu' \): \( \mathfrak{M} \geq 0 \). Then \( \frac{\mu'}{1-\mathfrak{M}} \) is a canonical distribution on \( F \) of mass unity and its potential \( \hat{U}(z) = \int N(z,p) \frac{1}{1-\mathfrak{M}} d\mu'(p) \leq \frac{U(z)}{1-\mathfrak{M}} \) on the kernel of \( \mu' \). Assume \( \mathfrak{M} < \frac{\alpha}{V} \). Then

\[
I\left( \frac{\mu'}{1-\mathfrak{M}} \right) = \int U'(z) \frac{1}{1-\mathfrak{M}} d\mu'(p) \leq \int \frac{U(z)}{1-\mathfrak{M}} d\mu'(p) \leq \frac{V-\alpha}{1-\mathfrak{M}} < V.
\]
This contradicts the definition of \( V \). Hence the mass \( \mathfrak{M} \) of any canonical distribution of mass unity on \( E[z \in F : U(z) > V-\alpha] \) satisfies

\[
\mathfrak{M} \geq \frac{\alpha}{V}.
\]

Let \( \{ \mu_n \} \) be a minimizing sequence of canonical mass distributions on \( F \) of mass unity such that \( I(\mu_n) = V + \varepsilon_n : \varepsilon_n \downarrow 0 \). Then for any given positive number \( \alpha < \min (1, V) \) there exists a number \( n \) such that \( \varepsilon_n < \min \left( 1, V \frac{\alpha^4}{V^2 A(\alpha)} \right) : A(\alpha) = 1+2V+\frac{\alpha}{V} \). Suppose \( \varepsilon_n < \frac{\alpha^4}{V^2 A(\alpha)} \). Then the potential \( U_n(z) \) of \( \mu_n \) satisfies the following conditions:
1) The capacity of the closed set $F_{n}^{2\alpha} = E[z \in F : U(z) \leq V - 2\alpha] < \frac{\alpha}{V}$ \hspace{1cm} (72)

2) The mass of $\mu_n$ on $F_{n}^{2\alpha} < \sqrt{28\alpha}$.

We shall prove 1) and 2). Let $\mu_n'$ be the restriction of $\mu_n$ on $E[z \in F : U_n(z) > V - \alpha] : U_n(z) = \int N(z, p) d\mu_n(p)$. Then the mass $M_n$ of $\mu_n' > \frac{\alpha}{V} > 0$ by (71). Put $\delta_n = \frac{M_n}{\frac{\alpha}{V}}$. Then $\delta_n > 1$ and the mass of $\frac{\mu_n'}{\delta_n} = \frac{\alpha}{V}$.

Let $\mu^*$ be the canonical mass distribution of $\frac{\alpha}{VC_n^{2\alpha}} \omega(F_{n}^{2\alpha}, z) : C_{n}^{2\alpha}$ is the capacity of $F_{n}^{2\alpha}$. Then the kernel of $\mu^*$ is contained in $F$ and the mass of $\mu^* = \frac{\alpha}{V}$. Let $\sigma$ be a distribution on $F$ such that $\sigma = \mu^*$ on $F_{n}^{2\alpha}$, $\sigma = 0$ on $E[z \in F : V - \alpha > U_n(z) \geq V - 2]$ and $\sigma = -\frac{\mu_n'}{\delta_n}$ on $E[z \in F : U_n(z) > V - \alpha]$. Put $U_\sigma(z) = \int N(z, p) d\sigma$. Then $|U_\sigma(z)| \leq \int N(z, p)(d\mu_n(p) + d\mu^*(p)) = U_n(z) + \frac{1}{VC_n^{2\alpha}} \omega(F_{n}^{2\alpha}, z)$. Henec $I(\sigma)$

$$\leq \int |U_\sigma(z)||d\sigma| \leq \left(U_n(z) + \frac{d}{VC_n^{2\alpha}}(d\mu_n(p) + d\mu^*(p))\right) \leq I(\mu_n) + \frac{\alpha}{VC_n^{2\alpha}} + \frac{\alpha}{V}(V - 2\alpha) + \frac{1}{C_n^{2\alpha}} \left(\frac{\alpha}{V} + \frac{\alpha^2}{V^2}\right)$

because $\omega(F, z) \leq 1$ on $\overline{R} - R_0$ and $U_n(z) \leq V - 2\alpha$ on the kernel of $\mu^*$.

Assum $C_n^{2\alpha} > \frac{\alpha}{V}$. Then $I(\sigma) \leq 2V + 1 + \frac{\alpha}{V} \leq A(\alpha)$. Now $\mu_n + h\sigma$ is a positive canonical distribution on $F$ of mass unity for $0 \leq h < \delta_n$ ($\delta_n > 1$). Hence

$$I(\mu_n + h\sigma) = V + \eta \geq 0 \text{ and } \eta - \epsilon_n = I(\mu_n + h\sigma) - I(\mu_n)$$

$$= 2h \left(\frac{\alpha}{V}\right)((V - 2\alpha) - (V - \alpha)) + h^2 I(\sigma) = 2h \left(-\frac{\alpha^2}{V} + hI(\sigma)\right),$$

whence

$$\eta_n = h \left(hI(\sigma) - \frac{\alpha^2}{V}\right) + \epsilon_n - \frac{\alpha^2}{V}.$$}

Put $h = \frac{\alpha^2}{VA(\alpha)}$. Then $h < \frac{\alpha^2}{V} < 1$ and $\mu_n + h\sigma$ is a positive canonical distribution on $F$ of mass unity. Now by $I(\sigma) \leq A(\alpha)$ and $\epsilon_n < \frac{\alpha^2}{V^2A(\alpha)}$ we have $\eta_n < 0$. This contradicts that $\eta_n \geq 0$. Hence $C_n^{2\alpha} \leq \frac{\alpha}{V}$.

Next by c) $\text{Cap}(F_{n}^{2\alpha}) < 14 C_n^{2\alpha} \leq \frac{14}{V}$. Let $\mu_n'$ be the restriction of $\mu_n$ on $F_{n}^{2\alpha}$.

Assume mass $M_n$ of $\mu_n' > \sqrt{28\alpha}$. Then by $C_n^{2\alpha} \leq \frac{\alpha}{V}$ we have
\[
I(\mu_n) \geq I(\mu'_n) \geq \frac{\mathfrak{M}_n^2}{\text{Cap}(F_{2^a_n})} > \frac{\mathfrak{M}_n^2}{14 C_{\alpha_n}^2} > 2V.
\]

This contradicts that \( I(\mu_n) = V + \varepsilon_n < 2V \). Hence the mass of \( \mu'_n < \sqrt{28\alpha} \).

Let \( \alpha_1 > \alpha_2 > \cdots \) be a sequence such that \( 2^n \sqrt{\alpha_n} \downarrow 0 \) as \( n \to \infty \). Let \( m(n) \) be the least integer satisfying \( \varepsilon_m < \underline{\alpha_n^4} \).

We make \( \mu_{m(n)} \) correspond to \( \alpha_n \) and denote it by \( \mu_n \) newly.

Then we have subsequence \( \{\mu_n\} \) of former \( \{\mu_n\} \) such that

\[
I(\mu_n) = V + \varepsilon_n \quad \text{and} \quad \varepsilon_n < \frac{\alpha_n^4}{V^2 A(\alpha_n)}.
\]

Hence the mass of \( \mu_n \) on \( E[z \in F: U_n(z) > V + 2^n \sqrt{\alpha_n}] \) and on \( E[z \in F: U_n(z) \leq V - 2\alpha_n] \) respectively, where \( U_n(z) \) is the potential of \( \mu_n \). Then \( \mathfrak{M}_n' \) is the masses of \( \mu_n \) on the set \( E[z \in F: U_n(z) > V + 2^n \sqrt{\alpha_n}] \) and on \( E[z \in F: U_n(z) \leq V - 2\alpha_n] \) respectively, where \( U_n(z) \) is the potential of \( \mu_n \). Then \( \mathfrak{M}_n' < \sqrt{28\alpha_n} \). Consider \( I(\mu_n) \). Then

\[
V + \varepsilon_n = I(\mu_n) \geq \int U_n(z) d\mu_n(p) \geq (V + 2^n \sqrt{\alpha_n}) \mathfrak{M}_n' + (V - 2\alpha)(1 - \mathfrak{M}_n' - \mathfrak{M}_n'').
\]

Whence \( \varepsilon_n > \mathfrak{M}_n' (2^n \alpha_n + 2\alpha_n) - 2\alpha - 2\alpha = \mathfrak{M}_n'' (2^n \alpha_n + 2\alpha_n) - 2\alpha - (2\alpha - V) \sqrt{28\alpha_n} \).

Hence the mass of \( \mu_n \) on \( E[z \in F: V + 2^n \sqrt{\alpha_n} > U_n(z)] > 1 - \frac{6V}{2^n} \). Let \( \mu'_n \) be the restriction of \( \mu_n \) on \( E[z \in F: U_n(z) < V + 2^n \sqrt{\alpha_n}] \) and put \( \mu^*_n = \frac{\mu'_n}{1 - \mathfrak{M}_n'} \).

Then \( \mu^*_n \) is also a canonical distribution on \( F \) of mass unity and \( I(\mu^*_n) \leq \frac{1}{1 - \mathfrak{M}_n'} \int U_n(z) d\mu'_n(p) \leq \frac{V + 2^n \sqrt{\alpha_n}}{1 - \mathfrak{M}_n'} = V + \zeta_n : \zeta_n \downarrow 0 \) as \( n \to \infty \). Hence \( \{\mu^*_n\} \) is also a minimizing sequence of canonical mass distributions on \( F \) of mass unity. On the other hand, \( U^*_n(z) = \int N(z, p) d\mu^*(p) \leq V + \zeta_n \) on the kernel of \( \mu^*_n \). Hence by the maximum principle \( U^*_n(z) \leq V + \zeta_n \) in \( \overline{R} - R_n \). Since the total mass of \( \{\mu^*_n\} \) is unity and \( N(z, p) \) is continuous in \( \overline{R} - R_n \) with respect to \( p \) for \( z \in \overline{R} - R_n \). Hence there exists a subsequence \( \{\mu^*_n\} \) of \( \{\mu^*_n\} \) and an weak limit \( \mu^* \) of \( \{\mu^*_n\} \) such that \( V = \lim_{n \to \infty} (V + \zeta_n) \geq \lim_{n \to \infty} U^*_n(z) = U^*(z) = \int N(z, p) d\mu^*(p) : z \in \overline{R} - R_n \). Further by the semicontinuity of \( U^*(z) \) \( U^*(z) \leq V \) in \( \overline{R} - R_n \), whence \( I(\mu^*) \leq V \). On the other hand, since \( \mu^* \) is also canonical by \( (c) \), \( I(\mu^*) \geq V \). Thus \( \mu^* \) is the required canonical mass
Proof of (e). Suppose \( \mu \) is a canonical distribution on \( F \) such that \( I(\mu) = V \). Put \( I(\mu) = V + \varepsilon \). Then \( \varepsilon = 0 < \frac{\alpha^4}{VA(\alpha)} \) for any given positive number \( \alpha \). Hence by (72) the potential \( U(z) \) of \( \mu \geq V \) on \( F \) except at most a set of capacity zero. Assume \( U(z) > V \) at least one point \( p \) of the kernel of \( \mu \). Then by the lower semicontinuity \( U(z) > V + \varepsilon : \varepsilon > 0 \) in a neighbourhood \( \mathcal{V}(p) \) of \( p \) and the mass \( \mathcal{M} \) of \( \mu \) in \( \mathcal{V}(p) > 0 \). Whence \( I(\mu) \geq \mathcal{M}(V + \varepsilon) + (1 - \mathcal{M})V > V \). This is a contradiction. Hence \( U(z) = V \) on the kernel \( F'(\subset F) \) of \( \mu \). Whence by the maximum principle \( U(z) \leq V \) in \( \overline{R} - R_0 \). Now \( U(z) \) is harmonic in \( R - R_0 - F \), since \( \mu = 0 \) on \( CF \). Hence

\[
U(z) \leq V \omega(F', z) \leq V \omega(F, z).
\]

Inverse inequality is proved as follows: put \( CG_{V - \epsilon} = E\{z \in \overline{R} : U(z) \leq V - \epsilon\} \).

Then \( CG_{V - \epsilon} \cap F \) is closed and of capacity zero (capacity zero), whence we can construct a superharmonic function \( \omega^{**}(z) \) such that \( \omega^{**}(z) \) is continuous in \( R - R_0 \) and \( \omega^{**}(z) \to \infty \) as \( z \to p \in ((CG_{V - \epsilon} \cap F) + B_0) \). Put \( U^*(z) = \alpha \omega^{**}(z) + U(z) : \alpha > 0 \). Then \( U^*(z) \geq V \) on \( F \). Put \( CG_{V - \epsilon}^* = E\{z \in \overline{R} : U^*(z) \leq V - \epsilon\} \).

Then by the lower semicontinuity of \( U^*(z) \) dist \( (CG_{V - \epsilon}^*, F) > \delta_0 > 0 \). Let \( F_m = E\{z \in \overline{R} : \delta(z, F) \leq \frac{1}{m}\} : \frac{1}{m} < \delta_0 \). Let \( G_{V - \epsilon}^* = E\{z \in \overline{R} : U^*(z) > V - \epsilon\} \).

Now \( U^*(z) \) is superharmonic in \( \overline{R} - R_0 \), hence

\[
U^*(z) \geq (V - \epsilon) \omega(G_{V - \epsilon}^*, z) \geq (V - \epsilon) \omega(F_m, z) \geq (V - \epsilon) \omega(F, z),
\]

Let \( \alpha \to 0 \) and then \( \epsilon \to 0 \). Then \( U(z) \geq \omega(F, z) \). Thus \( U(z) = \omega(F, z) \).

Next by \( \int_{\partial R_0} \frac{\partial}{\partial n} U(z) ds = V \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F, z) ds \) we have at once

\[
D(V \omega(F, z)) = V \cdot \frac{1}{\int_{\partial R_2} \frac{\partial}{\partial n} \omega(F, z) ds} = V \quad \text{and}
\]

\[
\text{Cap}(F) = \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F, z) ds = \frac{1}{V} = \ast \text{Cap}(F).
\]

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