<table>
<thead>
<tr>
<th>Title</th>
<th>A REMARK ON MAZUR-ORLICZ'S NORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Koshi, Shôzô</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 16(3-4), 221-224</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1962</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56028">http://hdl.handle.net/2115/56028</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSHIU_16_N3-4_221-224.pdf</td>
</tr>
</tbody>
</table>
A REMARK ON MAZUR-ORLICZ'S NORM

By
Shôzô KOSHI

1. Let Ω be a measure space with finite measure μ, and let $M(u, v)$ be a real valued function which is defined on $[0, \infty) \times \Omega$ such that
   (M. 1) $0 \leq M(u, v) \leq +\infty$ for $(u, v) \in [0, \infty) \times \Omega$ with $M(0, v) = 0$ a.e. $v \in \Omega$;
   (M. 2) $M(u, v)$ is an increasing function and left-hand continuous with $\lim_{u \to 0} M(u, v) < +\infty$ a.e. $v \in \Omega$;
   (M. 3) $M(u, v)$ is a measurable function of $u$ for a fixed $v \in [0, \infty)$;
   (M. 4) $\lim_{u \to \infty} M(u, v) > \lim_{u \to 0} M(u, v)$ a.e. $v \in \Omega$.

Now, we shall consider the function space $L_{M(u,v)}$ whose element $f$ is as follows:

(1) $\rho(\alpha f) = \int_{\Omega} M(\alpha|f(v)|, v) d\mu < +\infty$ for some $\alpha > 0$.

If we identify $f$ and $g$ when $f(v) = g(v)$ except a measure zero set, then we can consider $L_{M(u,v)}$ as a conditionally complete vector lattice with a functional $\rho^1$:

(2) $\rho(f) = \int_{\Omega} M(|f(v)|, v) d\mu$.

In the case that $M(u, v) = M(u)$ for every $v \in \Omega$, and $\lim_{u \to 0} M(u) = 0$, Mazur and Orlicz has considered in his paper [2], the quasi-norm such that

(3) $||f|| = \inf \{ \varepsilon; \rho(f/\varepsilon) < \varepsilon \}$.

$||\cdot||$ has the following properties:

(F. 1) $||f + g|| \leq ||f|| + ||g||$ for $f, g \in L_{M(u)}$;
(F. 2) $\alpha \to 0$, then $||\alpha f|| \to 0$ for each $f \in L_{M(u)}$;
(F. 3) $||f|| \to 0$, then $||\alpha f|| \to 0$ for every real number $\alpha$;
(F. 4) $0 \leq f \leq g$, then $||f|| \leq ||g||$;
(F. 5) $0 \leq f_1 \leq f_2 \leq \cdots$, $\sup_{n} ||f_n|| < +\infty$, then $\bigcup_{n=1}^{\infty} f_n \in L_{M(u)}$ and $||\bigcup_{n=1}^{\infty} f_n|| = \sup_{n} ||f_n||$.

1) This space is an example of quasi-modular spaces. cf. [3].
(F. 6) \( || \cdot || \) is complete.

The fact that \( || \cdot || \) is complete is considered as a generalization of Riesz-Fisher's theorem concerning the completeness of norms. (cf. [1])

In the case of \( L_{M(u,v)} \), we define \( \rho^* \) and \( || \cdot || \) with

\[
\rho^*(f) = \rho(f) - \lim_{\alpha \to 0} \rho(\alpha f) \quad \text{for } f \in L_{M(u,v)}
\]

and

\[
||f|| = \inf \left\{ \varepsilon ; \rho^* \left( \frac{f}{\varepsilon} \right) < \varepsilon \right\} \quad \text{for } f \in L_{M(u,v)}.
\]

Then, \( || \cdot || \) has the properties (F. 1), (F. 2), (F. 3), (F. 4). But, \( || \cdot || \) is not complete in general cases.

We have the following theorem.

**Theorem 1.** \( || \cdot || \) is complete, if and only if \( \lim_{u \to 0} M(u,v) = M(v) \) is an integralable function of \( \Omega \) with respect to \( \mu \).

Moreover, we have

**Theorem 2.** \( L_{M(u,v)} \) has a complete quasi-norm if and only if \( \lim_{u \to 0} M(u,v) = M(v) \) is an integralable function of \( \Omega \) with respect to \( \mu \).

2. From the definition of \( \rho \) and \( \rho^* \), we have

\[
\rho(|f|) = \rho(f), \quad \rho^*(|f|) = \rho^*(f)
\]

and

\[
\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g), \quad \rho^*(\alpha f + \beta g) \leq \rho^*(f) + \rho^*(g)
\]

for \( \alpha + \beta = 1; \alpha, \beta \geq 0 \).

Hence, for \( f_1, f_2, \cdots, f_n \in L_{M(u,v)} \),

\[
\rho^*(|f_1| + |f_2| + \cdots + |f_n|) \leq \rho^*(2f_1) + \cdots + \rho^*(2^n f_n).
\]

**Proof of Theorem 1.** It is easily proved that if \( M(v) \) is an integrable function of \( \Omega \), it follows

\[
\lim_{\alpha \to 0} \rho(\alpha f) < +\infty \quad \text{for each } f \in L_{M(u,v)};
\]

especially

\[
\lim_{\alpha \to 0} \rho(\alpha 1) < +\infty \quad \text{where } 1 \text{ is a characteristic function of } \Omega.
\]

Let \( \{f_n\} \) be a sequence of elements of \( L_{M(u,v)} \) with \( ||f_n|| \leq 1/2^n \). From the definition of \( || \cdot ||, || f_n || \leq 1/2^n \) implies

\[
\rho^*(2^n f_n) \leq \frac{1}{2^n} \quad n = 1, 2, \cdots.
\]
Hence, \( f = \sum_{n=1}^{\infty} |f_n| \) is an element of \( L_{M(u,v)} \) because of

\[
\rho(|f_1| + \cdots + |f_n|) \leq \lim_{\alpha \to 0} \rho(\alpha 1) + \rho^*(|f_1| + \cdots + |f_n|) \\
\leq \lim_{\alpha \to 0} \rho(\alpha 1) + \sum_i \rho^*(2^i f_i) < +\infty;
\]

i.e. \( \rho(f) < +\infty \).

Since \( L_{M(u,v)} \) is a conditionally complete (in order sense), \( \sum_{n=1}^{\infty} f_n \) exists in \( L_{M(u,v)} \) (in order sense).

Let \( \{g_n\} (n=1,2,\cdots) \) be a Cauchy sequence of \( L_{M(u,v)} \). There exists a subsequence \( \{g_{n_i}\} \) of \( \{g_n\} \) with

\[
||g_{n_i} - g_{n_{i+1}}|| \leq \frac{1}{2^i}. 
\]

Hence

\[
h = g_{n_1} + (g_{n_2} - g_{n_1}) + \cdots + (g_{n_{\nu}} - g_{n_{\nu-1}}) + \cdots
\]
is an element of \( L_{M(u,v)} \), and

\[
||h - g_{n_i}|| \leq \frac{1}{2^{i-1}};
\]
i.e. \( g_{n_i} \) is convergent to \( h \) in norm's sense. This shows that \( \{g_n\} \) is a convergent sequence.

Now, we shall prove the converse. We shall assume

\[
\int_\Omega M(v) d\mu = +\infty.
\]

Putting

\[
\inf \{M(v), n\} = M_n(v),
\]
we have

\[
\int_\Omega M_n(v) d\mu < +\infty.
\]

Now, we define the sets as follows:

\[
A_n = \{v \in \Omega \mid M_n(v) \neq 0\}
\]

and

\[
A = \bigcup_{n=1}^{\infty} A_n.
\]

Moreover, if we put

\[
B_n = A_n - A_{n-1} \quad (A_0 = \phi),
\]
then
(20) \[ A = \sum_{n=1}^{\infty} B_n. \]

We can choose the function \( f_n \) with

(21) \[ \rho^*(f_n) \leq \frac{1}{2^n} \]

and

(22) \[ B_n = \{ v; f_n(v) \neq 0 \}. \]

If we put \( f_n^0 = \frac{1}{2^n} f_n \), then we have

(23) \[ ||f_n^0|| \leq \frac{1}{2^n}. \]

Hence, \( g_m = \sum_{n=1}^{m} f_n^0 \) \((m = 1, 2, \cdots)\) is a Cauchy sequence. But \( \sum_{n=1}^{\infty} f_n^0 \in L_{M(u,v)} \).

This shows that \( || \cdot || \) is not complete.

The proof of Theorem 2 is quite similar to that of Theorem 1. Because, if \( \int_{\Omega} M(v) d\mu = +\infty \), then there exists a sequence \( f_n \) \((n = 1, 2, \cdots)\) satisfying (21), (22), (23) for a quasi-norm \( || \cdot ||^* \) defined on \( L_{M(u,v)} \) which is not necessary equal to (5). Similarly to the proof of Theorem 1, \( \{ f_n \} \) is a Cauchy sequence which is not convergent.

Theorem 1 is essentially equal to Theorem 3.2 in [4]. But the proof here is more simpler than that of Theorem 3.2.

References


Department of Mathematics,
Hokkaido University

(Received May 7, 1962)