A NOTE ON THE SIEVE METHOD OF A. SELBERG

By

Saburô UCHIYAMA

The purpose of this note is to obtain a universal upper bound for the remainder term in a useful formula in number theory, known as the sieve of A. Selberg [5, see also 4; Chap. II, Theorem 3.1].

Let $N>1$ and let $a_1\ a_2,\ \cdots,\ a_N$ be natural numbers not necessarily distinct. We wish to evaluate the number S of those a_j $(1\leqq j\leqq N)$ which are not divisible by any prime number $p\leqq z$, where $z\geqq 2$. Let d be a positive integer and let S_d denote the number of a's divisible by d. Suppose that

$$S_d=\frac{\omega(d)}{d}N+R(d),$$

where $R(d)$ is the error term for S_d and where $\omega(d)$ is assumed to be a multiplicative function of d, namely a function such that $(d_1, d_2)=1$ implies

$$\omega(d_1d_2)=\omega(d_1)\omega(d_2):$$

in particular, we have $\omega(1)=1$ if $\omega(d)$ does not vanish identically.

We put

$$f(d)=\frac{d}{\omega(d)}.$$

Then $f(d)$ is a multiplicative function of d. We shall suppose that $1<f(d)\leqq\infty$ for $d>1$; $f(d)=\infty$ only if $\omega(d)=0$ and then $S_d=R(d)$. We now define for positive integers m and d

$$f_1(m)=\sum_{n|m}\mu(n)f\left(\frac{m}{n}\right),$$

$$Z(d)=\sum_{\substack{r\leqq z/d\\(r,d)=1}}\frac{\mu^2(r)}{f_1(r)},\qquad Z=Z(1),$$

$$\lambda(d)=\mu(d)\prod_{p|d}\left(1-\frac{1}{f(p)}\right)^{-1}\frac{Z(d)}{Z},$$

where $\mu(d)$ denotes the Möbius function. It is clear that $f_1(m)$ is a multiplicative function of m and that if $\mu^2(m)=1$ then

$$f_1(m)=f(m)\prod_{p|m}\left(1-\frac{1}{f(p)}\right).$$

The formula of Selberg hereinbefore mentioned is given in the following

THEOREM. *Under the notations and conditions described above we have*

$$S\leqq\frac{N}{Z}+R$$

with

$$R=\sum_{d_1,d_2\leqq z}|\lambda(d_1)\lambda(d_2)R(\{d_1,d_2\})|,$$

where $\{d_1, d_2\}$ *denotes the least common multiple of* d_1 *and* d_2.

We shall suppose in what follows that for all d, d_1, d_2 we have

$$(1)\qquad |R(d)|\leqq\omega(d),\qquad \omega(\{d_1,d_2\})\leqq\omega(d_1)\omega(d_2),$$

the latter inequality being automatically satisfied when $\omega((d_1, d_2))\geqq1$. This condition for $\omega(d)$, as well as the assumption that $\omega(d)$ should be a multiplicative function, is in fact satisfied in many cases of applications of Selberg's sieve method. The remainder term R in the theorem is then not greater than

$$\sum_{d_1,d_2\leqq z}|\lambda(d_1)\lambda(d_2)\omega(d_1)\omega(d_2)|=\left(\sum_{d\leqq z}|\lambda(d)|\omega(d)\right)^2.$$

We show that if the condition (1) is fulfilled, then

$$(2)\qquad R=O(z^2(\log\log z)^2),$$

where, and henceforth, the constants implied in the symbol O are all absolute. Furthermore, if $\omega(p)\leqq1$ for all primes p, then we have, under the condition (1),

$$(3)\qquad R=O\left(\frac{z^2}{Z^2}\right).$$

Indeed, we have by the definition of $\lambda(d)$

$$\begin{aligned}&\sum_{d\leqq z}|\lambda(d)|\omega(d)\\&=\frac{1}{Z}\sum_{d\leqq z}\mu^2(d)\omega(d)\prod_{p|d}\left(1-\frac{1}{f(p)}\right)^{-1}\sum_{\substack{n\leqq z/d\\(n,d)=1}}\mu^2(n)\frac{\omega(n)}{n}\prod_{p|n}\left(1-\frac{1}{f(p)}\right)^{-1}\\&=\frac{1}{Z}\sum_{m\leqq z}\mu^2(m)\frac{\omega(m)}{m}\prod_{p|m}\left(1-\frac{1}{f(p)}\right)^{-1}\cdot\sum_{d|m}d.\end{aligned}$$

Let $\sigma(m)$ be the sum of divisors of m, i.e.

$$\sigma(m) = \sum_{d|m} d.$$

It is known that

$$\sigma(m) = O(m \log\log m)$$

(cf. [3; Theorem 323]). It follows that

$$\begin{aligned}
&\sum_{m\leqq z} \mu^2(m)\frac{\omega(m)}{m} \prod_{p|m}\left(1-\frac{1}{f(p)}\right)^{-1}\sigma(m) \\
&\leqq \left(\sum_{m\leqq z}\frac{\mu^2(m)}{f_1(m)}\right)\cdot \max_{m\leqq z}\sigma(m) \\
&= Z\cdot O(z\log\log z),
\end{aligned}$$

and this proves the assertion (2).

To prove (3) let us suppose that $\omega(p)\leqq 1$ for all primes p. Then we find that

$$\begin{aligned}
&\sum_{m\leqq z} \mu^2(m)\frac{\omega(m)}{m} \prod_{p|m}\left(1-\frac{1}{f(p)}\right)^{-1}\sigma(m) \\
&\leqq \sum_{m\leqq z} \mu^2(m)\frac{1}{m} \prod_{p|m}\left(1-\frac{1}{p}\right)^{-1}\sigma(m) \\
&= \sum_{m\leqq z} \mu^2(m)\frac{\sigma(m)}{\varphi(m)},
\end{aligned}$$

where $\varphi(m)$ is the Euler totient function. It is easily verified that

$$\frac{\sigma(m)}{\varphi(m)} = O\left(\frac{\sigma^2(m)}{m^2}\right),$$

and hence

$$\sum_{m\leqq z} \mu^2(m)\frac{\sigma(m)}{\varphi(m)} = O\left(\sum_{m\leqq z}\frac{\sigma^2(m)}{m^2}\right).$$

By a result due to S. Ramanujan (cf. [2; p. 135]) we see that

$$\sum_{m\leqq n}\sigma^2(m) = O(n^3).$$

Using this relation we obtain by partial summation

$$\begin{aligned}
\sum_{m\leqq z}\frac{\sigma^2(m)}{m^2} &= \sum_{m\leqq z-1}\left(\sum_{r\leqq m}\sigma^2(r)\right)\left(\frac{1}{m^2}-\frac{1}{(m+1)^2}\right)+\frac{\sum_{r\leqq z}\sigma^2(r)}{[z]^2} \\
&= \sum_{m\leqq z-1} O(1)+O(z) = O(z),
\end{aligned}$$

and hence

$$\sum_{m\leqq z} \mu^2(m)\frac{\omega(m)}{m} \prod_{p|m}\left(1-\frac{1}{f(p)}\right)^{-1}\sigma(m) = O(z),$$

completing the proof of (3).

As an easy application of (3) we can prove that the number of positive integers $n \leqq x$ such that $p \nmid n$ for all primes $p \leqq z$ is less than

$$c(a)\frac{x}{\log z},$$

provided that $z \geqq 2$ and $x \geqq z^a$, $a \geqq 2$, where $c(a)$ is a positive constant depending only on a. This result is slightly better than [4; Chap. II, Theorem 4.10].

Also, we may mention the following. Let k and l be integers such that $k \geqq 1$, $0 \leqq l < k$, $(k, l) = 1$. Let $\pi(x, k, l)$ denote, as usual, the number of primes $p \leqq x$ of the form $km+l$. Then, if $k = O(x^a)$, $0 < a < 1$, we have

$$\pi(x, k, l) < \frac{2x}{\varphi(k) \log (x/k)}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

This is a slight improvement of a result due to I.V. Čulanovskiĭ [1; Theorem 1]. (Here the O-constant may possibly depend upon a.)

References

[1] I.V. Čulanovskii: Certain estimates connected with a new method of Selberg in elementary number theory, Doklady Akad. Nauk SSSR (N. S.) vol. 63 (1948), pp. 491-494 (in Russian).

[2] G. H. Hardy, P.V. Seshu Aiyar and B. M. Wilson: Collected papers of Srinivasa Ramanujan, Cambridge 1927.

[3] G. H. Hardy and E. M. Wright: An introduction to the theory of numbers, 3rd edition, Oxford 1954.

[4] K. Prachar: Primzahlverteilung, Springer-Verl., Berlin-Göttingen-Heidelberg 1957.

[5] A. Selberg: On an elementary method in the theory of primes, Norske Vid. Selsk. Forh. Trondhjem, vol. 19, no. 18 (1947), pp. 64-67.

Department of Mathematics,
Hokkaidô University

(Received August 21, 1961)