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A NOTE ON THE SIEVE METHOD OF A. SELBERG

By

Saburō UCHIYAMA

The purpose of this note is to obtain a universal upper bound for the remainder term in a useful formula in number theory, known as the sieve of A. Selberg [5, see also 4; Chap. II, Theorem 3.1].

Let $N>1$ and let $a_1, a_2, \ldots, a_N$ be natural numbers not necessarily distinct. We wish to evaluate the number $S$ of those $a_j$ $(1 \leq j \leq N)$ which are not divisible by any prime number $p \leq z$, where $z \geq 2$. Let $d$ be a positive integer and let $S_d$ denote the number of $a$'s divisible by $d$. Suppose that

$$S_d = \frac{\omega(d)}{d} N + R(d),$$

where $R(d)$ is the error term for $S_d$ and where $\omega(d)$ is assumed to be a multiplicative function of $d$, namely a function such that $(d_1, d_2)=1$ implies

$$\omega(d_1d_2) = \omega(d_1)\omega(d_2);$$

in particular, we have $\omega(1)=1$ if $\omega(d)$ does not vanish identically.

We put

$$f(d) = \frac{d}{\omega(d)},$$

Then $f(d)$ is a multiplicative function of $d$. We shall suppose that $1 < f(d) \leq \infty$ for $d>1$; $f(d)=\infty$ only if $\omega(d)=0$ and then $S_d=R(d)$. We now define for positive integers $m$ and $d$

$$f_1(m) = \sum_{n|m} \mu(n)f\left(\frac{m}{n}\right),$$

$$Z(d) = \sum_{r \leq z/d, (r,d)=1} \frac{\mu^2(r)}{f_1(r)}, \quad Z=Z(1),$$

$$\lambda(d) = \mu(d) \prod_{p|d} \left(1 - \frac{1}{f(p)}\right)^{-1} \frac{Z(d)}{Z},$$

where $\mu(d)$ denotes the Möbius function. It is clear that $f_1(m)$ is a multiplicative function of $m$ and that if $\mu^2(m)=1$ then
$f_1(m) = f(m) \prod_{p|m} \left( 1 - \frac{1}{f(p)} \right)$.

The formula of Selberg hereinbefore mentioned is given in the following

**THEOREM.** Under the notations and conditions described above we have

$$S \leq \frac{N}{Z} + R$$

with

$$R = \sum_{d_1, d_2 \leq z} |\lambda(d_1) \lambda(d_2) R([d_1, d_2])|,$$

where \{d_1, d_2\} denotes the least common multiple of $d_1$ and $d_2$.

We shall suppose in what follows that for all $d, d_1, d_2$ we have

(1) \quad $|R(d)| \leq \omega(d)$, \quad $\omega([d_1, d_2]) \leq \omega(d_1) \omega(d_2)$,

the latter inequality being automatically satisfied when $\omega((d_1, d_2)) \geq 1$. This condition for $\omega(d)$, as well as the assumption that $\omega(d)$ should be a multiplicative function, is in fact satisfied in many cases of applications of Selberg's sieve method. The remainder term $R$ in the theorem is then not greater than

$$\sum_{d_1, d_2 \leq z} |\lambda(d_1) \lambda(d_2) \omega(d_1) \omega(d_2)| = \left( \sum_{d \leq z} |\lambda(d)| \omega(d) \right)^2.$$

We show that if the condition (1) is fulfilled, then

(2) \quad $R = O(z^2 (\log \log z)^2)$,

where, and henceforth, the constants implied in the symbol $O$ are all absolute. Furthermore, if $\omega(p) \leq 1$ for all primes $p$, then we have, under the condition (1),

(3) \quad $R = O \left( \frac{z^2}{Z^2} \right)$.

Indeed, we have by the definition of $\lambda(d)$

$$\sum_{d \leq z} |\lambda(d)| \omega(d)$$

$$= \frac{1}{Z} \sum_{d \leq z} \mu^2(d) \omega(d) \prod_{p|m} \left( 1 - \frac{1}{f(p)} \right)^{-1} \sum_{n \leq z \atop (n,d)=1} \mu^2(n) \frac{\omega(n)}{n} \prod_{p|m} \left( 1 - \frac{1}{f(p)} \right)^{-1}$$

$$= \frac{1}{Z} \sum_{m \leq z} \mu^2(m) \omega(m) \prod_{m \leq d \atop (m,d)=1} \left( 1 - \frac{1}{f(p)} \right)^{-1} \cdot \sum_{d|m} d.$$

Let $\sigma(m)$ be the sum of divisors of $m$, i.e.
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\[ \sigma(m) = \sum_{d|m} d. \]

It is known that
\[ \sigma(m) = O(m \log \log m). \]

(cf. [3; Theorem 323]). It follows that
\[
\sum_{m \leq z} \mu^2(m) \frac{\omega(m)}{m} \prod_{p|m} \left(1 - \frac{1}{f(p)}\right)^{-1} \sigma(m)
\leq \left( \sum_{m \leq z} \frac{\mu^2(m)}{f_1(m)} \right) \cdot \max_{m \leq z} \sigma(m)
= Z \cdot O(z \log \log z),
\]
and this proves the assertion (2).

To prove (3) let us suppose that \( \omega(p) \leq 1 \) for all primes \( p \). Then we find that
\[
\sum_{m \leq z} \mu^2(m) \frac{\omega(m)}{m} \prod_{p|m} \left(1 - \frac{1}{f(p)}\right)^{-1} \sigma(m)
\leq \sum_{m \leq z} \frac{\mu^2(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p}\right)^{-1} \sigma(m)
= \sum_{m \leq z} \frac{\mu^2(m) \sigma(m)}{\varphi(m)},
\]
where \( \varphi(m) \) is the Euler totient function. It is easily verified that
\[ \frac{\sigma(m)}{\varphi(m)} = O\left(\frac{\sigma^2(m)}{m^2}\right), \]
and hence
\[ \sum_{m \leq z} \frac{\mu^2(m) \sigma(m)}{\varphi(m)} = O\left(\sum_{m \leq z} \frac{\sigma^2(m)}{m^2}\right). \]

By a result due to S. Ramanujan (cf. [2; p. 135]) we see that
\[ \sum_{m \leq z} \sigma^2(m) = O(n^3). \]

Using this relation we obtain by partial summation
\[
\sum_{m \leq z} \frac{\sigma^2(m)}{m^2}
= \sum_{m \leq z-1} \left( \sum_{r \leq m} \sigma^2(r) \right) \left( \frac{1}{m^2} - \frac{1}{(m+1)^2} \right)
+ \frac{\sum_{r \leq z} \sigma^2(r)}{[z]^3}
= \sum_{m \leq z-1} O(1) + O(z) = O(z),
\]
and hence
\[ \sum_{m \leq z} \mu^2(m) \frac{\omega(m)}{m} \prod_{p|m} \left(1 - \frac{1}{f'(p)}\right)^{-1} \sigma(m) = O(z), \]
completing the proof of (3).

As an easy application of (3) we can prove that the number of positive integers \( n \leq x \) such that \( p \mid n \) for all primes \( p \leq z \) is less than

\[
c(a) \frac{x}{\log z},
\]

provided that \( z \geq 2 \) and \( x \geq z^a \), \( a \geq 2 \), where \( c(a) \) is a positive constant depending only on \( a \). This result is slightly better than [4; Chap. II, Theorem 4.10].

Also, we may mention the following. Let \( k \) and \( l \) be integers such that \( k \geq 1 \), \( 0 \leq l < k \), \( (k, l) = 1 \). Let \( \pi(x, k, l) \) denote, as usual, the number of primes \( p \leq x \) of the form \( km + l \). Then, if \( k = O(x^a), 0 < a < 1 \), we have

\[
\pi(x, k, l) < \frac{2x}{\varphi(k) \log (x/k)} \left( 1 + O\left( \frac{1}{\log x} \right) \right).
\]

This is a slight improvement of a result due to I.V. Čulanovskii [1; Theorem 1]. (Here the \( O \)-constant may possibly depend upon \( a \).)

References


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