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# ON THE CONTINUITY AND THE MONOTONOUSNESS OF NORMS

By

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§ 1. Let  $R$  be a *universally continuous semi-ordered linear space*<sup>1)</sup> (i.e. *conditionally complete vector lattice* in Birkhoff's sense) and  $\|\cdot\|$  be a norm on  $R$  satisfying the following conditions throughout this paper:

(N. 1)  $|x| \leq |y|$  ( $x, y \in R$ ) implies  $\|x\| \leq \|y\|$ ;

(N. 2)  $0 \leq x_\lambda \uparrow_{\lambda \in A} x$  implies  $\|x\| = \sup_{\lambda \in A} \|x_\lambda\|$ <sup>2)</sup>.

A norm  $\|\cdot\|$  on  $R$  is called *continuous*, if

$$(1.1) \quad \inf_{\nu=1,2,\dots} \|x_\nu\| = 0 \quad \text{for any } x_\nu \downarrow_{\nu=1}^{\infty} 0^{3)}$$

The continuity of norms on  $R$  plays an important rôle in the theory of semi-ordered linear spaces. In fact, it is well known [8, 9; § 31] that *every norm-continuous linear functional  $f$  on  $R$  is (order-) universally continuous*, i.e.

$$(1.2) \quad \inf_{\lambda \in A} |f(x_\lambda)| = 0 \quad \text{for any } x_\lambda \downarrow_{\lambda \in A} 0,$$

and  $R$  becomes *superuniversally continuous*<sup>4)</sup> as a space in this case.

It is clear that if a norm  $\|\cdot\|$  on  $R$  is continuous, the another norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$  is also continuous. As for the conditions under which norms  $\|\cdot\|$  on  $R$  are continuous, there are the detailed investigations by T. Andô [3, 4].

A norm  $\|\cdot\|$  on  $R$  is called *monotone* [8], if

$$(1.3) \quad |x| \leq |y| \quad (x, y \in R) \quad \text{implies} \quad \|x\| \leq \|y\|,$$

and is called *uniformly monotone* [8, 9; § 30], if

1) This terminology is due to H. Nakano [9]. We use mainly notation and terminology of [9] here.

2) A norm satisfying (N. 1) and (N. 2) is called *semi-continuous* in [10]. A norm on  $\|\cdot\|$  satisfying (N. 1) is called *monotone* in [7]. On the other hand, (N. 1) is assumed for any norm of normed lattices in [6].

3) This means  $x_1 \geq x_2 \geq \dots \geq 0$  and  $\bigcap_{\nu=1}^{\infty} x_\nu = 0$ .

4)  $R$  is called *superuniversally continuous*, if for any  $0 \leq x_\lambda, \lambda \in A \leq a$  there exists  $\{x_{\lambda_\nu}\}_{\nu=1}^{\infty} \leq \{x_\lambda\}_{\lambda \in A}$  such that  $\bigcup_{\nu=1}^{\infty} x_{\lambda_\nu} = \bigcup_{\lambda \in A} x_\lambda$ .

(1.4) for any  $\gamma, \varepsilon > 0$  there exists  $\delta > 0$  such that

$$x \wedge y = 0, \|x\| \leq \gamma \text{ and } \|y\| \geq \varepsilon \quad \text{imply} \quad \|x+y\| \geq \|x\| + \delta.$$

When a norm  $\|\cdot\|$  is of  $L_p$ -type<sup>5)</sup> ( $1 \leq p < +\infty$ ), it is both continuous and monotone (uniformly monotone), and when  $\|\cdot\|$  is of  $L_\infty$ -type, it is neither continuous nor monotone. This fact suggests that there may be some correlation between the continuity and the monotonousness of norms on  $R$ , in spite of the existence of a continuous norm which is not monotone.

In § 2 we shall study this relation and show consequently that if a norm  $\|\cdot\|$  on  $R$  is continuous there exists a monotone norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$  (Theorem 3).

In § 3 we shall show a sufficient condition for the continuity of the associated norm of  $\|\cdot\|$  and a necessary and sufficient condition under which the second conjugate norm  $\|\bar{\bar{x}}\|$  ( $\bar{x} \in \bar{R}$ ) is continuous on  $\bar{R}$ .

In the earlier paper [10] the author defined a property of a norm called *finitely monotone*<sup>6)</sup> which is stronger than the continuity. In § 4 we shall prove that a norm  $\|\cdot\|$  on  $R$  is finitely monotone, if and only if there exists an equivalent norm  $\|\cdot\|_1$  which is at the same time a lower semi- $p$ -norm for some  $1 \leq p < +\infty$  (i.e.  $x \wedge y = 0$  implies  $\|x+y\|_1^p \geq \|x\|_1^p + \|y\|_1^p$ ) (Theorem 6). Since a lower semi- $p$ -norm is uniformly monotone, we see that a norm  $\|\cdot\|$  is finitely monotone if and only if we may define an equivalent norm  $\|\cdot\|_1$  which is uniformly monotone. At last some notes on finitely monotone norms shall be made.

In the sequel we denote by  $\tilde{R}$  the *norm associated space* of  $R$  (i.e. the totality of all norm-continuous linear functionals on  $R$ ) and by  $\bar{R}$  the *norm conjugate space* of  $R$  (i.e. the totality of all universally continuous linear functionals on  $R$  which is norm-continuous too)<sup>7)</sup>. The completeness of  $\|\cdot\|$  on  $R$  shall not be assumed, unless otherwise provided.

§ 2. Let  $\|\cdot\|$  be an arbitrary norm on  $R$  satisfying (N. 1) and (N. 2) in the sequel.

**Definition 1.** An element  $a \in R$  is said to be a *continuous element*

5) A norm is called to be of  $L_p$ -type [1], if  $\|x+y\|^p = \|x\|^p + \|y\|^p$  for any  $x, y \in R$  with  $x \wedge y = 0$  in the case  $1 \leq p < +\infty$ , and  $\|x+y\| = \max(\|x\|, \|y\|)$  in the case  $p = +\infty$ .

6) For the definition of the finitely monotone norm see § 4. It was discussed first in [2].

7)  $\tilde{R}$  denotes the totality of all universally continuous linear functionals on  $R$ . When  $\|\cdot\|$  is complete,  $\tilde{R} = \bar{R}$  holds. But  $\tilde{R} \subseteq \bar{R}$  in general.

(with respect to  $\|\cdot\|$ ), if  $|a| \geq a_\nu \downarrow_{\nu=1}^\infty 0$  implies  $\lim_{\nu \rightarrow \infty} \|a_\nu\| = 0$ .

It is easily verified that  $a \in R$  is a continuous element if and only if  $\inf_{\nu=1,2,\dots} \|[p_\nu]a\| = 0$  for any  $[p_\nu] \downarrow_{\nu=1}^\infty 0^8$  (cf. [9; Th. 30.8]). If  $a$  is a continuous element and  $\alpha|a| \geq b$ ,  $b \in R$ , then  $b$  is also a continuous element by the definition. Hence we see that the totality of all continuous elements of  $R$  constitutes a semi-normal manifold<sup>9)</sup> of  $R$  and we denote it by  $R_c$ . A norm  $\|\cdot\|$  on  $R$  is continuous if and only if  $R_c = R$ . We call  $\|\cdot\|$  to be *almost continuous*, if  $R_c$  is a complete semi-normal manifold of  $R$ . Being stated above,  $R$  with a continuous norm is always *semi-regular*<sup>10)</sup>. But this fact remains to be true by replacing the continuity by the almost continuity, that is,  $R$  with an almost continuous norm is semi-regular.

Let  $M$  be a linear manifold of  $R$ . A linear functional  $\bar{a} \in \bar{R}''$  is called *complete* on  $M$  if  $|\bar{a}|(b) = 0$ ,  $b \in M$  implies  $b = 0$ . It is shown [9; § 20] that for any  $0 \neq \bar{a} \in \bar{R}''$  there exists a normal manifold  $N$  on which  $\bar{a}$  is complete. We denote by  $B_a$  ( $a \in R$ ) the semi-normal manifold consisting of all elements  $x \in R$  such that  $|x| \leq \alpha a$  for some real  $\alpha$  (depending on  $x$ ).

Now we have

**Lemma 1.** *If  $0 \leq a \in R$  is a continuous element with respect to  $\|\cdot\|$ , there exists a universally continuous linear functional  $\bar{a} \in \bar{B}_a''$  which is complete on  $B_a$ .*

*Proof.* Since  $B_a$  is a semi-normal manifold,  $B_a$  is a normed semi-ordered linear space with  $\|\cdot\|$  by itself and the norm  $\|\cdot\|$  is continuous on  $B_a$  from the definition. Being stated above,  $B_a$  is semi-regular and hence there exists the system of elements  $\{\bar{a}_\lambda\}_{(\lambda \in A)}$  ( $0 \leq \bar{a}_\lambda \in \bar{B}_a''$ ,  $\lambda \in A$ ) with  $\bigcup_{\lambda \in A} [\bar{a}_\lambda]^{B_a} = 1^{11)}$  and  $\|\bar{a}_\lambda\| = 1$  ( $\lambda \in A$ ). Since  $B_a$  is superuniversally continuous, we can find a subsequence  $\{\bar{a}_{\lambda_\nu}\}_{(\nu=1,2,\dots)}$  of  $\{\bar{a}_\lambda\}_{(\lambda \in A)}$  with  $\bigcup_{\nu=1}^\infty [\bar{a}_{\lambda_\nu}]^{B_a} = 1$ . Now, as  $\bar{B}_a''$  is complete (with respect to the norm),

$$\bar{a}_0 = \sum_{\nu=1}^\infty \frac{1}{2^\nu} \bar{a}_{\lambda_\nu} \in \bar{B}_a''$$

8) For  $p \in R$ ,  $[p]$  denotes the projector by  $p$ . i. e.  $[p]x = \bigcup_{n=1}^\infty (n|p| \wedge x)$  for  $x \geq 0$ .

9) A linear lattice manifold  $M \subseteq R$  is called a *semi-normal manifold*, if  $a \in M$ ,  $|a| \geq |b|$  implies  $b \in M$ . A semi-normal manifold  $M$  is called *complete*, if  $\{M^+\}^+ = \{0\}$ .

10)  $R$  is called *semi-regular*, if  $\bar{a}(a) = 0$  for all  $\bar{a} \in \bar{R}$  implies  $a = 0$ .

11)  $[\bar{a}]^{B_a}$  is a projector on  $B_a$  such that  $\bar{b}([\bar{a}]^{B_a} x) = [\bar{a}]b(x)$  holds for every  $\bar{b} \in \bar{B}_a$  and  $x \in B_a$  [9; § 22].

and  $\bar{a}_0$  is complete on  $B_a$  obviously.

Q. E. D.

**Lemma 2.** Let  $0 \leq a \in R$  be a continuous element. For any  $\varepsilon > 0$ , there exists a positive integer  $n = n(a, \varepsilon)$  such that  $a = \sum_{i=1}^n a_i$ ,  $a_i \geq 0$  and  $\|a_i\| \geq \varepsilon$  ( $i = 1, 2, \dots, n$ ) imply  $n \leq n$ .

*Proof.* There exists a complete linear functional  $0 \leq \bar{a} \in \bar{B}_a$  on  $B_a$  with  $\bar{a}(a) = 1$  by virtue of Lemma 1. If the conclusion of Lemma 2 is not true, we can find a sequence of elements of  $R^+$ :<sup>12)</sup>  $\{a_{\nu, \mu}; \mu = 1, 2, \dots, \kappa_\nu; \nu = 1, 2, \dots\}$  with  $a = \sum_{\mu=1}^{\kappa_\nu} a_{\nu, \mu}$ ,  $2^\nu \leq \kappa_\nu$  and  $\|a_{\nu, \mu}\| \geq \varepsilon$  for each  $1 \leq \nu$  and  $1 \leq \mu \leq \kappa_\nu$ . Since  $\bar{a}$  is linear, there exists  $a_{\nu, \mu}$  with  $\bar{a}(a_{\nu, \mu}) \leq 1/2^\nu$  for each  $1 \leq \nu$ . Putting  $b_i = \bigcup_{\nu \geq i} a_{\nu, \mu} \leq a$  we obtain  $0 \leq b_i \downarrow_{i=1}^\infty$  and  $b_0 = \bigcap_{i=1}^\infty b_i \in B_a$ . Since  $\bar{a}(b_i) \leq 1/2^{i-1}$  and  $\bar{a} \geq 0$ , we have  $\bar{a}(b_0) = 0$  and a fortiori  $b_i \downarrow_{i=1}^\infty b_0 = 0$ , because  $\bar{a}$  is complete on  $B_a$ .

On the other hand, the fact that  $\|b_i\| \geq \|a_{i, \mu_i}\| \geq \varepsilon$  for all  $i \geq 1$  is inconsistent with the assumption, which establishes the proof. Q. E. D.

**Definition 2.**  $a \in R^+$  is called a purely monotone element (with respect to  $\|\cdot\|$ ), if for any  $\varepsilon > 0$  there exists  $\delta = \delta(a, \varepsilon) > 0$  such that  $a \geq b \geq 0$  and  $\|b\| \geq \varepsilon$  imply  $\|a - b\| \leq \|a\| - \delta$ .

Now we obtain

**Theorem 1.** If  $a \in R^+$  is a purely monotone element,  $a$  is a continuous one.

*Proof.* If  $a \geq a_\nu \downarrow_{\nu=1}^\infty 0$  and  $\|a_\nu\| \geq \varepsilon$  ( $\nu = 1, 2, \dots$ ), we have for some  $\delta > 0$

$$\|a_1 - a_\nu\| \leq \|a\| - \delta \quad (\nu = 1, 2, \dots),$$

since  $a \in R^+$  is a purely monotone element. This contradicts (N. 2) in § 1. Therefore  $a$  is a continuous element by the definition. Q. E. D.

**Theorem 2.** For any norm  $\|\cdot\|$  on  $R$ , there exists a norm  $\|\cdot\|_1$  equivalent to  $\|\cdot\|$  such that every continuous element  $a$  with respect to  $\|\cdot\|$  is purely monotone one with respect to  $\|\cdot\|_1$ .

*Proof.* We define  $\|\cdot\|_1$  by the formula:

$$\|x\|_1 = \|x\| + \sup \left\{ \sum_{\nu=1}^\infty \frac{\|y_\nu\|}{2^\nu} \right\} \quad (x \in R).$$

$$|x| = \sum_{\nu=1}^\infty y_\nu, \quad y_\nu \geq 0$$

From the definition of  $\|\cdot\|_1$ , it is clear that  $\|\cdot\|_1$  is a norm on  $R$

12)  $R^+$  denotes the set of all positive elements of  $R$ .

satisfying (N. 1), (N. 2) and

$$\|x\| \leq \|x\|_1 \leq 2\|x\| \quad \text{for every } x \in R.$$

Let  $a \in R^+$  be a continuous element with respect to  $\|\cdot\|$ . By virtue of Lemma 2, for any  $\varepsilon > 0$  there exists an integer  $n_0 = n_0(a, \varepsilon/2)$  such that  $a = \sum_{\nu=1}^n a_\nu$ ,  $a_\nu \in R^+$  and  $\|a_\nu\| \geq \varepsilon/2$  ( $\nu=1, 2, \dots, n$ ) imply  $n \leq n_0$ . Suppose  $0 \leq b \leq a$  and  $\|b\| \geq \varepsilon$ , then for any  $\{b_\nu\}_{\nu=1}^\infty \subset R^+$  with  $a - b = \sum_{\nu=1}^\infty b_\nu$  and  $\|b_1\| \geq \|b_2\| \geq \dots$  we have

$$\|a - b\| + \sum_{\nu=1}^\infty \frac{\|b_\nu\|}{2^\nu} = \|a - b\| + \sum_{\nu=1}^{n_0} \frac{\|b_\nu\|}{2^\nu} + \frac{\|b_{n_0+1}\|}{2^{n_0+1}} + \sum_{\nu=n_0+2}^\infty \frac{\|b_\nu\|}{2^\nu}.$$

Here we have on account of  $\|b_{n_0+1}\| < \varepsilon/2$

$$\frac{\|b_{n_0+1}\|}{2^{n_0+1}} \leq \frac{\varepsilon}{2^{n_0+2}} \leq \frac{\|b\|}{2^{n_0+1}} - \frac{\varepsilon}{2^{n_0+2}},$$

which implies

$$\begin{aligned} \|a - b\| + \sum_{\nu=1}^\infty \frac{\|b_\nu\|}{2^\nu} &\leq \|a - b\| + \sum_{\nu=1}^{n_0} \frac{\|b_\nu\|}{2^\nu} + \frac{\|b\|}{2^{n_0+1}} + \sum_{\nu=n_0+2}^\infty \frac{\|b_\nu\|}{2^\nu} - \frac{\varepsilon}{2^{n_0+2}} \\ &\leq \|a\|_1 - \frac{\varepsilon}{2^{n_0+2}}, \end{aligned}$$

because of  $\sum_{\nu=1}^{n_0} b_\nu + b + \sum_{\nu=n_0+2}^\infty b_\nu \leq a$ . Therefore, we obtain

$$\|a - b\|_1 \leq \|a\|_1 - \frac{\varepsilon}{2^{n_0+2}},$$

which shows that  $a$  is a purely monotone element. Q. E. D.

**Remark 1.** The norm  $\|\cdot\|_1$  constructed in the above theorem has the following property: for any continuous element  $a \in R$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $a \in R^+$ ,  $\|b\|_1 \geq \varepsilon$  implies  $\|a + b\|_1 \geq \|a\|_1 + \delta$ .

From Theorem 2 we obtain immediately

**Corollary 1.** If a norm  $\|\cdot\|$  on  $R$  is almost continuous, then there exists a norm  $\|\cdot\|_1$  equivalent to  $\|\cdot\|$  such that the set of all purely monotone elements coincides with a complete semi-normal manifold  $R_c$ <sup>13)</sup>.

If each  $a \in R^+$  is a purely monotone element with respect to a norm  $\|\cdot\|$ ,  $\|\cdot\|$  is monotone. Therefore, we have

**Theorem 3.** If a norm  $\|\cdot\|$  on  $R$  is continuous, there exists a monotone norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$ <sup>14)</sup>.

13)  $R_c$  is the same for all equivalent norms.

14) Theorems 2 and 3 hold to be true for any norm  $\|\cdot\|$  on  $R$  satisfying (N. 1) only.

**Remark 2.** On the other hand, as an easy example shows, there exists a monotone norm  $\|\cdot\|$  which has no continuous norm equivalent to it.

§ 3. In this section we shall give some notes on the continuity of the norm on  $R$ . First we shall show a simple sufficient condition<sup>15)</sup> for the continuity of the associated norm on  $\tilde{R}$ . We shall consider the following condition (\*) for norms  $\|\cdot\|$  on  $R$ :

$$(*) \quad \|x_\nu\| \leq 1, x_\nu \in R^+ \quad (\nu=1, 2, \dots) \quad \text{implies} \quad \lim_{n \rightarrow \infty} \frac{\|\bigcup_{\nu=1}^n x_\nu\|}{n} = 0.$$

Now we have

**Theorem 4.** Suppose that a norm  $\|\cdot\|$  on  $R$  satisfy (\*). Then the associate norm  $\|\cdot\|$  on  $\tilde{R}$  (or the conjugate norm on  $\bar{R}$ ) is continuous.

*Proof.* If the associated norm  $\|\cdot\|$  on  $\tilde{R}$  is not continuous, we can find a positive number  $\varepsilon > 0$  and a sequence of elements  $0 \leq \tilde{x}_\nu \in \tilde{R}$  ( $\nu=1, 2, \dots$ ) such that  $\|\tilde{x}_\nu\| \geq 2\varepsilon$  and  $\|\sum_{\nu=1}^n \tilde{x}_\nu\| \leq 1$  for  $n \geq 1$ . Now we can find also a sequence of elements  $\{x_\nu\}_{(\nu \geq 1)} \subset R^+$  such that  $\tilde{x}_\nu(x_\nu) \geq \varepsilon$  and  $\|x_\nu\| \leq 1$  for each  $\nu \geq 1$ . Putting  $y_n = \bigcup_{\nu=1}^n x_\nu$ , we obtain for any  $n \geq 1$

$$\|y_n\| \geq \left(\sum_{\nu=1}^n \tilde{x}_\nu\right)y_n \geq \sum_{\nu=1}^n \tilde{x}_\nu(x_\nu) \geq n\varepsilon,$$

hence

$$\lim_{n \rightarrow \infty} \frac{\|y_n\|}{n} = \lim_{n \rightarrow \infty} \frac{\|\bigcup_{\nu=1}^n x_\nu\|}{n} \geq \varepsilon > 0.$$

This contradicts the condition (\*).

Q. E. D.

Since there exists a semi-ordered linear space  $R$  with a continuous norm which has, however, the second conjugate space  $\bar{R}$ <sup>16)</sup> whose norm is continuous no longer, we see that in Theorem 4 we can not exchange (\*) for the continuity of a norm of  $R$  without failing to hold the validity.

Here we give a necessary and sufficient condition for the continuity of the second conjugate norm  $\|X\|$  ( $X \in \bar{R}$ ) of a norm  $\|\cdot\|$  on  $R$ .

15) In [5] T. Andô gave a necessary condition [5; §4 Lemma 4.2].

16) The conjugate space of  $\bar{R}$  is denoted by  $\bar{\bar{R}}$ . If  $R$  is semi-regular,  $R$  can be considered as a complete semi-normal manifold of  $\bar{R}$  [9].

**Theorem 5.** *In order that the second conjugate norm on  $\bar{R}''$  of  $\|\cdot\|$  be continuous, it is necessary and sufficient that the norm  $\|\cdot\|$  on  $R$  satisfies the following condition:*

$$(\#) \quad \sup_{n \geq 1} \left\| \sum_{\nu=1}^n x_\nu \right\| < +\infty, \quad x_\nu \in R^+ \quad (\nu=1, 2, \dots)$$

implies  $\lim_{\nu \rightarrow \infty} \|x_\nu\| = 0$ .

*Proof. Necessity.* Since  $R \subseteq \bar{R}''$ ,  $\|x\| = \|f_x\|^{(17)} = \sup_{\bar{x} \in \bar{R}'', \|\bar{x}\| \leq 1} |\bar{x}(x)|$  for all  $x \in R$  and  $\bar{R}''$  is monotone complete<sup>18)</sup> [9],  $\sup_{n \geq 1} \left\| \sum_{\nu=1}^n f_{x_\nu} \right\| < +\infty$  ( $x_\nu \in R$ ) implies  $X_0 = \sum_{\nu=1}^\infty f_{x_\nu} \in \bar{R}''$ . Then  $Y_n = \sum_{\nu=n}^\infty f_{x_\nu} \in \bar{R}''$ ,  $Y_n \downarrow_{n \rightarrow \infty} 0$ , and hence

$$\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|f_{x_n}\| = \lim_{n \rightarrow \infty} \|Y_n\| = 0,$$

on account of the continuity of the second conjugate norm  $\|X\|$  ( $X \in \bar{R}''$ ).

*Sufficiency.* We shall first show that the norm  $\|\cdot\|$  on  $R$  is continuous in this case. Indeed, let  $R \ni [p_\nu]a \downarrow_{\nu \rightarrow \infty} 0$ . From the assumption  $\{[p_\nu]a\}_{\nu=1}^\infty$  is a Cauchy sequence of  $R$ , whence we have  $\lim_{\nu \rightarrow \infty} \|[p_\nu]a\| = 0$  by virtue of the semi-continuity of  $\|\cdot\|$ .

Now if  $\|X\|$  ( $X \in \bar{R}''$ ) is not continuous, we may find a sequence of elements  $\{X_\nu\}_{\nu=1}^\infty$  of  $\bar{R}''$  such that  $\|X_\nu\| \geq 1$  and  $\left\| \sum_{\nu=1}^n X_\nu \right\| \leq \gamma$  ( $n, \nu=1, 2, \dots$ ) for some  $\gamma \geq 1$ . Since the norm  $\|\cdot\|$  on  $R$  is continuous,  $R$  is semi-regular, and hence  $R$  is a complete semi-normal manifold of  $\bar{R}''$ . Thus there exists a sequence of elements  $\{x_\nu\}_{\nu=1}^\infty$  such that  $X_\nu \geq f_{x_\nu}$ ,  $\|x_\nu\| \geq 1/2^{19)}$  ( $\nu=1, 2, \dots$ ). This contradicts (#), because of  $\sup_{n \geq 1} \left\| \sum_{\nu=1}^n x_\nu \right\| = \sup_{n \geq 1} \left\| \sum_{\nu=1}^n f_{x_\nu} \right\| \leq \gamma$ .

**Corollary 2.** *In order that a norm  $\|\cdot\|$  on  $R$  be monotone complete and continuous, it is necessary and sufficient that  $\|\cdot\|$  is complete<sup>20)</sup> and satisfies the condition (#).*

**§ 4.** A norm  $\|\cdot\|$  on  $R$  is called *finitely monotone* [10], if it satisfies the following:

17)  $f_x$  ( $x \in R$ ) denotes an element of  $\bar{R}''$  for which  $f_x(\bar{x}) = \bar{x}(x)$  holds for each  $\bar{x} \in \bar{R}''$ .

18)  $R$  is called *monotone complete*, if  $0 \leq a_\lambda \uparrow_{\lambda \in A}$  and  $\sup_{\lambda \in A} \|a_\lambda\| < +\infty$  implies  $\bigcup_{\lambda \in A} a_\lambda \in R$ .

19) When  $R$  is semi-regular. A norm satisfying (N. 1) and (N. 2) is reflexive, i.e.  $\|x\| = \sup_{\|\bar{x}\| \leq 1, \bar{x} \in \bar{R}''} |\bar{x}(x)|$  [11].

20) If a norm  $\|\cdot\|$  on  $R$  is monotone complete, it is complete [10; § 30]. The converse of this is not true in general. Cf. Corollary 2 with Theorem 2.1 of [12] in modular spaces.

(4.1) for any  $0 < \varepsilon \leq 1$  there exists a natural number  $N(\varepsilon)$  such that  $\|x_i\| \geq \varepsilon$ ,  $x_i \wedge x_j = 0$ ,  $i \neq j$  ( $i, j = 1, 2, \dots, n$ ) and  $\|\sum_{i=1}^n x_i\| \leq 1$  imply  $n \leq N(\varepsilon)$ .

It is clear that every finitely monotone norm is continuous and any norm  $\|\cdot\|$  which is equivalent to a finitely monotone norm  $\|\cdot\|$  is also such a one. This topologically invariant property of a finitely monotone norm is important and may be utilized. Here we shall characterize a finitely monotone norm by showing the possibility of conversion of it into the another norm of the more familiar and simpler form.

A norm  $\|\cdot\|$  on  $R$  is called a lower semi- $p$ -norm (upper semi- $p$ -norm) if for any  $x \wedge y = 0$ ,  $x, y \in R$ ,  $\|x+y\|^p \geq \|x\|^p + \|y\|^p$  (resp.  $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ ) holds, where  $p$  is a real number with  $1 \leq p \leq +\infty$  [1].

Being well known [8], the lower semi- $p$ -norm and the upper semi- $q$ -norm are of conjugate type<sup>21)</sup>, where  $1/p + 1/q = 1$  and the former is uniformly monotone in the case  $p < +\infty$ , and hence finitely monotone.

At first we shall prove an auxiliary lemma:

**Lemma 3.** Let  $\|\cdot\|$  be a finitely monotone norm and  $p$  be a real number such that  $2^p = N(1/2) + 1$ <sup>22)</sup> holds. Suppose also that  $\varepsilon/2 \leq \|x_\nu\| < \varepsilon$  ( $\nu = 1, 2, \dots, m$ ) and  $x_\nu \wedge x_\mu = 0$  for  $\nu \neq \mu$ . If  $l$  is a natural number such that  $0 \leq m - l2^p < 2^p$  holds, there exist mutually orthogonal elements  $y_\mu$  ( $\mu = 1, 2, \dots, l$ ) such that  $\varepsilon \leq \|y_\mu\|$  ( $\mu = 1, 2, \dots, l$ ) and  $\sum_{\mu=1}^l y_\mu \leq \sum_{\nu=1}^m x_\nu$ .

*Proof.* If  $\{x_{\nu_i}\}_{i=1}^{2^p}$  is arbitrary subsequence of  $\{x_\nu\}_{\nu=1}^m$ , it follows that  $\|(1/\varepsilon)x_{\nu_i}\| \geq 1/2$  ( $i = 1, 2, \dots, 2^p$ ) and  $2^p > N(1/2)$ . This implies  $\|1/\varepsilon \sum_{i=1}^{2^p} x_{\nu_i}\| \geq 1$ , whence we have  $\|\sum_{i=1}^{2^p} x_{\nu_i}\| \geq \varepsilon$ . From this we see that we can find  $\{y_\mu\}_{\mu=1}^l$  which satisfies the above condition.

Now we have

**Theorem 6.** A norm  $\|\cdot\|$  on  $R$  is finitely monotone if and only if there exists a lower semi- $p$ -norm  $\|\cdot\|_1$  equivalent to  $\|\cdot\|$ , where  $1 \leq p < +\infty$ .

*Proof.* Since a lower semi- $p$ -norm is finitely monotone and the finite monotonousness is topological invariant, it suffices to prove the necessity of the theorem.

Let  $\|\cdot\|$  be finitely monotone and  $p$  be a real number satisfying  $2^p = N(1/2) + 1$ . We put now

21) That is, if  $\|\cdot\|$  is a lower semi- $p$ -norm (upper semi- $q$ -norm), the conjugate norm is an upper semi- $q$ -norm (resp. lower semi- $p$ -norm).

22)  $N(1/2)$  is a natural number which appears in (4.1) for  $\varepsilon = 1/2$  with respect to the finitely monotone norm  $\|\cdot\|$ .

$$\|x\|_1 = \sup \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \quad (x \in R).$$

$$x = \bigoplus_{i=1}^n x_i, \quad n = 1, 2, \dots$$

From this definition it is clear that  $\|x+y\|^p \geq \|x\|^p + \|y\|^p$  holds for each  $x, y \in R$  with  $x \wedge y = 0$ . Furthermore it is also evident that  $\|x\| \leq \|x\|_1$  and  $\|\alpha x\|_1 = |\alpha| \|x\|_1$  hold for each  $\alpha$  real and  $x \in R$ . The sub-additivity of  $\|\cdot\|_1$  follows from Minkowski's inequality and  $\|\cdot\|_1$  satisfies (N. 1) and (N. 2), because  $\|\cdot\|$  does. Therefore, it is sufficient to prove that we can find  $\kappa > 0$  for which  $\|x\|_1 \leq \kappa \|x\|$  holds for each  $x \in R$ . Let  $\{x_\nu\}_{\nu=1}^n$  be a mutually orthogonal sequence of positive elements of  $R$  such that  $\|\sum_{\nu=1}^n x_\nu\| = 1$  and  $k_i$  be the number of  $x_\nu$  with  $1/2^i < \|x_\nu\| \leq 1/2^{i-1}$  ( $i = 1, 2, \dots$ ). Then

$$n = k_1 + k_2 + \dots + k_m, \quad k_m \neq 0$$

holds for some  $m \geq 1$ . If  $n = k_1$ ,  $(\sum_{\nu=1}^n \|x_\nu\|^p)^{1/p} \leq (N(1/2))^{1/p} \leq 2$  holds. Thus we shall assume  $n \neq k_1$  in the argument below.

Now we can find an integer  $0 \leq l_m$  such that

$$0 \leq k_m - l_m 2^p < 2^p$$

holds. This implies

$$\sum_{x_\nu \in A_m} \|x_\nu\|^p \leq k_m \frac{1}{2^{p(m-1)p}} \leq \frac{l_m + 1}{2^{p(m-2)}}$$

where  $A_j = \{x_\nu : x_\nu \in \{x_i\}_{i=1}^n, 1/2^j < \|x_\nu\| \leq 1/2^{j-1}\}$  for every  $1 \leq j \leq m$ . We note here that there exists a sequence of mutually orthogonal elements  $\{y_\mu\}_{\mu=1}^{l_m}$  such that  $\sum_{\mu=1}^{l_m} y_\mu \leq \sum_{x_\nu \in A_m} x_\nu$  and  $\|y_\mu\| \geq 1/2^{m-1}$  ( $\mu = 1, 2, \dots, l_m$ ) hold, if  $l_m \neq 0$ , in virtue of Lemma 3.

Next we choose an integer  $0 \leq l_{m-1}$  for which

$$0 \leq k_{m-1} + l_m - l_{m-1} 2^p < 2^p$$

holds. Now this yields

$$\sum_{x_\nu \in A_m \cup A_{m-1}} \|x_\nu\|^p \leq \frac{l_m + 1}{2^{p(m-2)}} + \sum_{x_\nu \in A_{m-1}} \|x_\nu\|^p \leq \frac{l_{m-1}}{2^{p(m-3)}} + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}}$$

and there exists also a mutually orthogonal sequence  $\{z_\mu\}_{\mu=1}^{l_{m-1}}$  such that

$$\sum_{\mu=1}^{l_{m-1}} z_\mu \leq \sum_{x_\nu \in A_m \cup A_{m-1}} x_\nu \quad \text{and} \quad \|z_\mu\| \geq \frac{1}{2^{m-2}}$$

hold for all  $1 \leq \mu \leq l_{m-1}$ . Let  $l_{m-2}$  be similarly defined as before and pro-

23)  $x = \bigoplus_{i=1}^n x_i$  means that  $x = \sum_{i=1}^n x_i$  and  $x_i \wedge x_j = 0$  for  $i \neq j$ .

ceeding this process we obtain finally an integer  $l_2$  such that

$$0 \leq k_2 + l_3 - l_2 2^p < 2^p$$

and

$$\sum_{x_\nu \in A_m \cup A_{m-1} \cup \dots \cup A_2} \|x_\nu\|^p \leq l_2 + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}} + \dots + 1$$

holds. Now, in virtue of Lemma 3 there exists also a mutually orthogonal sequence  $\{\omega_\mu\}_{\mu=1}^{l_2}$  such that  $\sum_{\mu=1}^{l_2} \omega_\mu \leq \sum_{x_\nu \in A_m \cup A_{m-1} \cup \dots \cup A_2} x_\nu$  and  $\|\omega_\mu\| \geq 1/2$ . Therefore we obtain

$$\begin{aligned} \sum_{\nu=1}^n \|x_\nu\|^p &= \sum_{x_\nu \in A_m \cup A_{m-1} \cup \dots \cup A_2} \|x_\nu\|^p + \sum_{x_\nu \in A_1} \|x_\nu\|^p \\ &\leq l_2 + k_1 + \frac{1}{2^{p(m-2)}} + \frac{1}{2^{p(m-3)}} + \dots + 1 \\ &\leq N\left(\frac{1}{2}\right) + 2, \end{aligned}$$

whence  $(\sum_{\nu=1}^n \|x_\nu\|^p)^{1/p} \leq (N(1/2) + 2)^{1/p} \leq 3$ . This shows  $\|x\|_1 = \sup_{x = \bigoplus_{i=1}^n x_i, n=1,2,\dots} (\sum_{i=1}^n \|x_i\|^p)^{1/p}$

$\leq 3$  for each  $x \in R$  with  $\|x\| \leq 1$  and establishes the theorem. Q. E. D.

Since a uniformly monotone norm  $\|\cdot\|$  is finitely monotone [10], it follows from above

**Corollary 3.** *If a norm  $\|\cdot\|$  on  $R$  is uniformly monotone, we can define a lower semi- $p$ -norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$  for some  $p$  with  $1 \leq p < +\infty$ .*

A norm  $\|\cdot\|$  on  $R$  is called *finitely flat* [10], if

(4.2) *for any  $\gamma > 0$  there exists  $\varepsilon > 0$  such that  $\|x_i\| \leq \varepsilon$ ,  $x_i \wedge x_j = 0$ ,  $i \neq j$  ( $i, j = 1, 2, \dots, n$ ) and  $n \leq \gamma/\varepsilon$  imply  $\|\sum_{i=1}^n x_i\| \leq 1$ .*

It is known that the finitely monotonousness and the finite flatness are of conjugate type [10; Th. 1.4, 1.5]. Thus each upper semi- $p$ -norm  $\|\cdot\|$  ( $1 < p$ ) is finitely flat.

Since the norm satisfying (N. 1), (N. 2) is reflexive [11]<sup>24)</sup>, we obtain in virtue of Theorem 6

**Theorem 7.** *Let a norm  $\|\cdot\|$  on  $R$  be finitely flat and  $R$  be semi-regular. Then we can define an upper semi- $p$ -norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$ , where  $1 < p \leq +\infty$ .*

Furthermore we have

24) See the footnote 19) for the definition of *reflexivity*.

**Theorem 8.** *If a norm  $\|\cdot\|$  on  $R$  is both finitely monotone and finitely flat, we can define an upper semi- $p$  and lower semi- $q$ -norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$ , where  $1 < p \leq q < +\infty$ .*

*Proof.* Since the norm  $\|\cdot\|$  is finitely monotone, there exists a lower semi- $q$ -norm  $\|\cdot\|_0$  equivalent to  $\|\cdot\|$  for some  $1 \leq q < +\infty$  in virtue of Theorem 6. Then the conjugate norm  $\|\bar{x}\|_0$  ( $\bar{x} \in \bar{R}$ ) of  $\|\cdot\|_0$  is an upper semi- $q'$ -norm ( $1/q' + 1/q = 1$ ) and it is also finitely monotone, because  $\|\cdot\|$  is finitely flat too. Now we define

$$\|\bar{x}\|_1 = \sup_{\bar{x} = \bigoplus_{i=1}^n \bar{x}_i, n=1,2,\dots} \left( \sum_{i=1}^n \|x_i\|_0^{p'} \right)^{1/p'} \quad (\bar{x} \in \bar{R}),$$

where  $p'$  is a real number such that  $2^{p'} = N(1/2) + 1^{25}$ . It is clear that  $q' \leq p'$  and  $\|\bar{x}\|_1$  is a lower semi- $p'$ -norm and is equivalent to  $\|\bar{x}\|_0$  ( $\bar{x} \in \bar{R}$ ), as is shown in Theorem 6.

On the other hand, from Minkowski's inequality it follows that  $\|\bar{x}\|_1$  ( $\bar{x} \in \bar{R}$ ) remains still to be an upper semi- $q'$ -norm. Thus, the conjugate norm  $\|X\|$  ( $X \in \bar{R}$ ) of  $\|\bar{x}\|_1$  ( $\bar{x} \in \bar{R}$ ) is the upper semi- $p$  and lower semi- $q$ -norm, where  $1/p + 1/p' = 1$  and  $1 < p \leq q < +\infty$ .

Since  $R \subset \bar{R}$  and the norm  $\|x\|$  ( $x \in R$ ) is reflexive, we have obtained an upper semi- $p$ - and lower semi- $q$ -norm  $\|\cdot\|_1$  which is equivalent to  $\|\cdot\|$ .  
Q. E. D.

At last we shall make a note on finitely monotone norms.

**Lemma 4.** *If a norm  $\|\cdot\|$  is finitely monotone, it satisfies the following:*

(4.3) *for any  $1 > \varepsilon > 0$  there exists a natural number  $n(\varepsilon)$  such that  $\varepsilon \leq \|x_i\|$ ,  $x_i \in R^+$  ( $i=1, 2, \dots, n$ ) and  $\|\sum_{i=1}^n x_i\| \leq 1$  implies  $n \leq n(\varepsilon)$ .*

*Proof.* At first suppose that  $\varepsilon_1$  be a real number with  $1/2 < \varepsilon_1 < 1$ ,  $\alpha$  be such that  $1/2 < \alpha < \varepsilon_1$  and  $[p_i] = [(x_i - \alpha x)^+]$ , where  $x = \sum_{i=1}^n x_i$ ,  $x_i \geq 0$  and  $\|x_i\| \geq \varepsilon_1$  ( $i=1, 2, \dots, n$ ). Now we have  $[p_i] \neq 0$  for each  $i$  with  $1 \leq i \leq n$ , because, in the contrary case, we have  $x_i \leq \alpha x$  for some  $i$  ( $1 \leq i \leq n$ ), and hence  $\|x_i\| \leq \alpha \|x\| \leq \alpha < \varepsilon_1$ , which is a contradiction. We have also that  $[p_i][p_j] = 0$  for  $i \neq j$  holds. Because,  $[p_i][p_j] \neq 0$  implies

$$[p_i][p_j](x_i + x_j) \geq \alpha [p_i][p_j]x + \alpha [p_i][p_j]x \geq [p_i][p_j]x$$

which is inconsistent with the fact that  $x = \sum_{i=1}^n x_i$  and  $x_i \geq 0$  ( $i=1, 2, \dots, n$ ).

25) Since  $\|\bar{x}\|_0$  ( $\bar{x} \in \bar{R}$ ) is finitely monotone, there exists an natural number which satisfies (4.1) for  $\varepsilon=1/2$ . We denote it by  $N(1/2)$  here.

Now putting  $y_i = [p_i]x_i$  ( $1 \leq i \leq n$ ), we obtain  $y_i \wedge y_j = 0$  ( $i \neq j$ ),  $\|y_i\| = \|[p_i]x_i\| \geq \|x_i\| - \|(1 - [p_i])x_i\| \geq \|x_i\| - \|\alpha x\| \geq \varepsilon_1 - \alpha$  and  $\|\sum_{i=1}^n y_i\| \leq \|\sum_{i=1}^n x_i\| \leq 1$ . As  $\|\cdot\|$  is finitely monotone, there exists a natural number  $N(\varepsilon_1 - \alpha)$  appeared in (4.1), for which  $n \leq N(\varepsilon_1 - \alpha)^{26)}$  holds.

For any  $0 < \varepsilon < 1$  we choose a natural number  $m$  such that  $0 < \varepsilon_1^m < \varepsilon$ . Putting  $n(\varepsilon) = (N(\varepsilon_1 - \alpha))^m$ , we can verify evidently that  $n(\varepsilon)$  satisfies the condition (4.3). Q. E. D.

Similarly we have

**Lemma 5.** *If a norm  $\|\cdot\|$  is finitely flat, it satisfies the following:*  
 (4.4) *for any  $\gamma > 0$  there exists  $\varepsilon > 0$  such that  $\|a_i\| \leq \varepsilon$ ,  $x_i \in R^+$  ( $i = 1, 2, \dots, n$ ) and  $n \leq \gamma/\varepsilon$  imply  $\|\bigcup_{i=1}^n x_i\| \leq 1$ .*

As the proof is quite similar, we omit it.

From these lemmas we obtain a following theorem which enables us to conclude that "finitely monotone" and "finitely flat" are of dual type (cf. [10; Th. 1.4, 1.5]), that is,

**Theorem 9.** *If a norm  $\|\cdot\|$  on  $R$  is finitely monotone (finitely flat), then the associated norm  $\|\tilde{x}\|$  ( $\tilde{x} \in \tilde{R}^{\text{II}}$ ) is finitely flat (resp. finitely monotone)<sup>27)</sup>.*

The proof is obtained by showing that (4.3) and (4.4) are of dual type in the quite same manner as Theorems 1.4 and 1.5 in [10].

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26)  $N(\varepsilon_1 - \alpha)$  depends only on  $\varepsilon_1 - \alpha$ .

27) This theorem was suggested to the author by Dr. T. Andô.

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