SOME STUDIES ON HOMOLOGICAL ALGEBRA

By

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§ 1. Let A be an algebra over a commutative ring K, and A^e the enveloping algebra of A : $A^e = A \otimes_K A^*$, A^* being the opposite algebra of A. In this paper, we shall mostly assume that A satisfies the condition that A^e is projective as a left A^*-module (or equivalently, as a right A-module), which was first considered by Azumaya [1]. The class of such algebras contains that of algebras which are projective as K-modules. Cartan and Eilenberg proved in [2] that the cohomology groups $H^n(A, M)$ of a K-algebra A with coefficients in a two-sided A-module M coincide with those defined by Hochschild [4] in the case when A is K-projective. Recently Azumaya showed in [1] the validity of the same fact under the weaker condition of the A^*-projectivity of A^e. We shall show in **§2** that the Azumaya theorem can also be proved in the similar way as in Cartan and Eilenberg [2, IX, **§6**]. In **§3** and **§4**, we shall give some results concerning projective dimensions of algebras and concerning supplemented algebras respectively, also under the condition of A^*-projectivity of A^e. Finally, we shall obtain in **§5** a characterization of the Dedekind ring.

Throughout in this paper, we assume that a ring A considered has an identity element and all A-modules are unital, and we use always the notation $\otimes$ instead of $\otimes_K$.

§ 2. Let A be an associative algebra over a commutative ring K, and A^e the enveloping algebra of A: $A^e = A \otimes A^*$, where A^* is the opposite algebra of A.

For each integer $n \geqq -1$, let $S_n(A)$ denote the $(n+2)$-fold tensor product over K of A with itself. Thus $S_{-1}(A) = A$, $S_{n+1}(A) = A \otimes S_n(A)$. We convert $S_n(A)$ into a left A^e-module by setting $(b \otimes c^*)(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = (ba_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes (a_{n+1}c)$.

Lemma 1. *If the enveloping algebra A^e of A is projective as a left A^*-module, then, for $n \geqq 0$, $S_n(A)$ is projective as a left A^e-module.*

Proof. We shall prove this by induction on n. For $n=0$, this is evident since $S_0(A) = A \otimes A$ is isomorphic with $A^e = A \otimes A^*$ as a left A^e-module. Suppose now that we already know that $S_{n-1}(A)$ is A^e-projective. The left A^e-module $S_{n-1}(A)$ may be considered as a left A^*-module by setting

$$a^* s_{n-1} = (1 \otimes a^*) s_{n-1} \qquad \text{for} \quad a^* \in A^*, \ s_{n-1} \in S_{n-1}(A),$$

and from our assumptions that A^e is A^*-projective and $S_{n-1}(A)$ is A^e-projective, $S_{n-1}(A)$ is A^*-projective by [2, II, 6.2]. Now, in the situation $({}_A A, {}_{A^*}S_{n-1}(A))$ we convert $A \otimes S_{n-1}(A)$ into a left A^e-module by setting

$$(b \otimes c^*)(a \otimes s_{n-1}) = ba \otimes c^* s_{n-1} \qquad \text{for} \quad a, b, c \in A, \ s_{n-1} \in S_{n-1}(A).$$

Thus $A \otimes S_{n-1}(A)$ is isomorphic with $S_n(A)$ as a left A^e-module, and so is A^e-projective by [2, IX, 2.5]. Hence $S(A) = \sum_{n \geq 0} S_n(A)$ is, under the same differentiations and an augmentation as in [2, IX, §6], an A^e-projective resolution of A, and, by proceeding in the same way as in [2, IX, §6], we have the following Azumaya theorem:

Theorem 2. *Let A be an algebra over a commutative ring K. If the enveloping algebra A^e of A is projective as a left A^*-module, then the cohomology groups $H^n(A, M)$ of A with coefficients in a two-sided A-module M coincide with those defined by Hochschild* [4].

§ 3. Let A be a K-algebra, L a commutative K-algebra, and $\varphi: A \to L \otimes A$ a ring homomorphism defined by $\varphi a = 1 \otimes a$. Then φ induces a homomorphism φ^e of $A^e = A \otimes A^*$ into $(L \otimes A)^e$ $(\approx L \otimes A^e)$ such that the diagram

$$\begin{tikzcd} A^e \arrow[r, "\rho"] \arrow[d, "\varphi^e"'] & A \arrow[d, "\varphi"] \\ L \otimes A^e \arrow[r, "L \otimes \rho"] & L \otimes A \end{tikzcd}$$

is commutative, where ρ is the augmentation given by $\rho(a \otimes b^*) = ab$.

For each two-sided $L \otimes A$-module M (which using φ may also be regarded as a two-sided A-module) we have the homomorphisms

$$F_n^\varphi : H_n(A, M) \longrightarrow H_n(L \otimes A, M),$$
$$F_\varphi^n : H^n(L \otimes A, M) \longrightarrow H^n(A, M).$$

Suppose now that A^e is A^*-projective. We shall verify conditions (i) and (ii) of the mapping theorem of [2, VIII, 3.1] which in this case become:

(i) $(L \otimes A^e) \otimes_{A^e} A \to L \otimes A$ is an isomorphism:

(ii) $\mathrm{Tor}_n^{A^e}(L \otimes A^e, A) = 0 \qquad \text{for} \quad n > 0.$

Condition (i) follows directly from the associativity of the tensor product. To verify (ii) we consider an A^e-projective resolution X of A. Then

$$\begin{aligned}\mathrm{Tor}_n^{A^e}(L\otimes A^e, A) &= H_n((L\otimes A^e)\otimes_{A^e}X)\\ &\approx H_n(L\otimes X)\\ &\approx H_n(L\otimes A^*\otimes_{A^*}X)\,.\end{aligned}$$

Since A^e is A^*-projective, it follows from [2, II, 6.2] that X is also an A^*-projective resolution of A. Thus

$$H_n(L\otimes A^*\otimes_{A^*}X) = \mathrm{Tor}_n^{A^*}(L\otimes A^*, A)$$

which is equal to 0 for $n>0$ because A is A^*-projective.

Therefore by the mapping theorem of [2, VIII, 3.1] we have the following:

Theorem 3. *Let A be a K-algebra such that its enveloping algebra A^e is A^*-projective, and let L be a commutative K-algebra. Then for each two-sided $L\otimes A$-module M we have the isomorphisms*

$$F_n : H_n(A, M)\approx H_n(L\otimes A, M)\,,$$
$$F^n : H^n(L\otimes A, M)\approx H^n(A, M)\,.$$

Further, if X is an A^e-projective resolution of A, then $L\otimes X$ is an $(L\otimes A)^e$-projective resolution of $L\otimes A$.

Theorem 3 is a generalization of [2, IX, 5.1], and implies the following which is a generalization of [2, IX, 7.1].

Theorem 4. *Let A be a K-algebra such that its enveloping algebra A^e is A^*-projective, and let L be a commutative K-algebra. Then*

$$\dim(L\otimes A)\leqq \dim A\,,$$
$$\text{w. dim}\,(L\otimes A)\leqq \text{w. dim}\,A\,.$$

If further the natural mapping $K\to L$ is a monomorphism of K onto a direct factor of L (as a K-module) then

$$\dim(L\otimes A) = \dim A\,,$$
$$\text{w.dim}\,(L\otimes A) = \text{w. dim}\,A\,.$$

Proof. The first inequalities follow directly from Theorem 3. To prove the second part, consider a K-homomorphism σ: $L\to K$ such that the composition $K\to L\to K$ is the identity. Let M be any two-sided A-module. Then $L\otimes M$ may be regarded as a two-sided $L\otimes A$-module, and by Theorem 3

$$H^n(L\otimes A, L\otimes M)\approx H^n(A, L\otimes M)\,.$$

Since the composition of the homomorphisms

$$H^n(A, M)\longrightarrow H^n(A, L\otimes M)\longrightarrow H^n(A, M)$$

is the identity, it follows that the relation $H^n(L\otimes A,\ L\otimes M)=0$ implies

$H^n(A, M)=0$. Thus $\dim A \leqq \dim(L\otimes A)$.

By the similar method, we have also $\mathrm{w.dim}\, A \leqq \mathrm{w.dim}\,(L\otimes A)$.

We shall give a generalization of [2, IX, 7.4] as follows:

Theorem 5. *Let A be a K-algebra such that the enveloping algebra A^e of A is projective as a left A^*-module, and B a K-algebra. Then*

$$\dim(A\otimes B) \leqq \dim A + \dim B\,.$$

If further K is a field and A and B are finitely K-generated, then

$$\dim(A\otimes B) = \dim A + \dim B\,.$$

Proof. Let X be an A^e-projective resolution of A, of dimension $\leqq p$, and let Y be a B^e-projective resolution of B, of dimension $\leqq q$. Then, by [2, IX, 2.5], $X\otimes Y$ is an $A^e\otimes B^e$-projective left complex over $A\otimes B$. Since the tensor product is right exact, it follows from [2, II, 4.3] that the sequence

$$X_1\otimes Y_0 + X_0\otimes Y_1 \longrightarrow X_0\otimes Y_0 \longrightarrow A\otimes B \longrightarrow 0$$

is exact, and moreover,

$$\begin{aligned} H_n(X\otimes Y) &\approx H_n((X\otimes_A A)\otimes Y) \\ &\approx H_n(X\otimes_A (A\otimes Y))\,. \end{aligned}$$

Since X is a left A^*-projective resolution of A by [2, II, 6.2], that is, X is a right A-projective resolution of A,

$$H_n(X\otimes_A (A\otimes Y)) = \mathrm{Tor}_n^A(A, A\otimes Y) = 0 \qquad \text{for} \quad n>0\,.$$

Thus $(X\otimes Y)$ is acyclic and consequently $X\otimes Y$ is an $A^e\otimes B^e$-projective resolution of $A\otimes B$. Since $A^e\otimes B^e \approx (A\otimes B)^e$ and since $X\otimes Y$ has dimension $\leqq p+q$, the first inequality follows.

The second equality follows by the same method as in [2, IX, 7.4].

Corresponding to [2, IX, 2.8 and 2.8a], we shall give the following two theorems the former of which is due to Azumaya [1, Proposition 1].

Theorem 6. *Let A, B and C be K-algebras. In the situation $({}_AM_B, {}_BN_C, {}_AP_C)$ assume that $A\otimes B^*$ is B^*-projective, and that M is B-projective, and that N is C-projective. Then there is an isomorphism*

$$\mathrm{Ext}^n_{A\otimes B^*}(M, \mathrm{Hom}_C(N, P)) \approx \mathrm{Ext}^n_{A\otimes C^*}(M\otimes_B N, P)\,.$$

Theorem 7. *Let A, B and C be K-algebras. In the situation $({}_BM_A, {}_AN_C, {}_CP_B)$ assume that $A\otimes B^*$ is A-projective, and that M is A-projective, and that N is C-projective. Then there is an isomorphism*

$$\mathrm{Tor}_n^{B^*\otimes C}(M\otimes_A N, P) \approx \mathrm{Tor}_n^{A\otimes B^*}(M, N\otimes_C P)\,.$$

Proof. Let X be an $A\otimes B^*$-projective resolution of M. Since N is C-projective, it follows from [2, IX, 2.3] that $X\otimes_A N$ is $B^*\otimes C$-projective. Since $A\otimes B^*$ is A-projective, X is A-projective by [2, II, 6.2]. Thus X is an A-projective resolution of M, and hence

$$H_n(X\otimes_A N)=\mathrm{Tor}_n^A(M,N)=0 \qquad \text{for} \quad n>0$$

by our assumption that M is A-projective. It follows from this with the right exactness of the tensor product that $X\otimes_A N$ is a $B^*\otimes C$-projective resolution of $M\otimes_A N$. So we have by [2, IX, 2.1]

$$\begin{aligned}\mathrm{Tor}_n^{B^*\otimes C}(M\otimes_A N, P)&=H_n((X\otimes_A N)\otimes_{B^*\otimes C}P)\\ &\approx H_n(X\otimes_{A\otimes B^*}(N\otimes_C P))\\ &=\mathrm{Tor}_n^{A\otimes B^*}(M, N\otimes_C P)\,.\end{aligned}$$

Now we shall give a generalization of [2, IX, 7.5] as follows:

Theorem 8. *For any K-algebra A*

$$\dim A\leqq \mathrm{gl.\,dim}\, A^e\,,$$
$$\mathrm{w.\,dim}\, A\leqq \mathrm{w.\,gl.\,dim}\, A^e\,.$$

If further A is semi-simple, then

$$\dim A=\mathrm{gl.\,dim}\, A^e,$$
$$\mathrm{w.\,dim}\, A=\mathrm{w.\,gl.\,dim}\, A^e\,.$$

Proof. The first part of this theorem follows directly from the definitions of the global dimension and the weak global dimension. To prove the first equality of the second part we use Theorem 6 with $A=B=C$ and $M=A$. By the assumption that A is semi-simple, the opposite algebra A^* is also semi-simple, it follows by [2, I, 4.2] that $A\otimes A^*(=A^e)$ is A^*-projective and that N is A-projective. Thus we obtain an isomorphism

$$H^n(A, \mathrm{Hom}_A(N,P))\approx \mathrm{Ext}_{A^e}^n(N,P)$$

for any two-sided A-modules N and P where $\mathrm{Hom}_A(N,P)$ is the group of right A-homomorphisms $N\to P$. This implies $\mathrm{gl.\,dim}\, A^e\leqq \dim A$, and hence we have $\dim A=\mathrm{gl.\,dim}\, A^e$.

Next, to prove the second equality of the second part we use Theorem 7 with $A=B=C$ and $M=A$. By the assumption that A is semi-simple, the enveloping algebra $A\otimes A^*(=A^e)$ is A-projective and N is also A-projective by [2, I, 4.2]. Thus we obtain an isomorphism

$$\mathrm{Tor}_n^{A^e}(N,P)\approx H_n(A, N\otimes_A P)$$

for any two-sided A-modules N and P. This implies w. gl. dim $A^e \leqq$ w. dim A, and hence w. dim $A =$ w. gl. dim A^e.

§ 4. A K-algebra A together with a K-algebra homomorphism $\varepsilon : A \to K$ is called a *supplemented algebra*. Using this augmentation map $\varepsilon : A \to K$ we may convert any right (or left) A-module M into a two-sided A-module ${}_\varepsilon M$ (or M_ε) by setting $ax = (\varepsilon a)x$ (or $xa = x(\varepsilon a)$) for $a \in A$ and $x \in M$. We consider the diagram

$$\begin{tikzcd} A^e \arrow[r, "\rho"] \arrow[d, "\varphi"'] & A \arrow[d, "\varepsilon"] \\ A \arrow[r, "\varepsilon"'] & K \end{tikzcd}$$

where $\varphi(a \otimes b^*) = a(\varepsilon b)$. Since $\varepsilon\varphi(a \otimes b^*) = \varepsilon(ab) = \varepsilon\rho(a \otimes b^*)$, this diagram is commutative. Thus we find homomorphisms

$$F^\varphi : H_n(A, {}_\varepsilon M) = \mathrm{Tor}_n^{A^e}({}_\varepsilon M, A) \longrightarrow \mathrm{Tor}_n^A(M, K),$$
$$F_\varphi : \mathrm{Ext}_A^n(K, N) \longrightarrow \mathrm{Ext}_{A^e}^n(A, N_\varepsilon) = H^n(A, N_\varepsilon)$$

for right A-module M and a left A-module N.

We shall give a generalization of [2, X, 2. 1] as follows:

Theorem 9. *Let A be a supplemented K-algebra such that the enveloping algebra A^e af A is A^*-projective. Then F^φ and F_φ are isomorphisms, and for each A^e-projective resolution X of A, the complex $X \otimes_A K$ is an A-projective resolution of $K = A \otimes_A K$ as a left A-module.*

Proof. It suffices to verify conditions (i) and (ii) of the mapping theorem of [2, VIII, 3.1]. Applying [2, X, 2.2], we find that the condition (i) holds. To prove the condition (ii), let X be an A^e-projective resolution of A. Then

$$\mathrm{Tor}_n^{A^e}({}_\varepsilon A, A) = H_n({}_\varepsilon A \otimes_{A^e} X) \approx H_n(X \otimes_A K)$$

again by applying [2, X, 2.2]. Since A^e is A^*-projective, it follows from [2, II, 6.2] X is left A^*-projective, that is, X is projective as a right A-module. Therefore

$$H_n(X \otimes_A K) = \mathrm{Tor}_n^A(A, K) = 0 \qquad \text{for} \quad n > 0.$$

This proves the condition (ii) of the mapping theorem and thus completes the proof of Theorem 9.

Let A be a K-algebra and $\eta : K \to A$ the natural map given by $\eta k = k1$. A right A-module M is said to be *weakly projective* if the kernel of the map

$g: M\otimes A\to M$ given by $x\otimes a\to xa$ is a direct summand of $M\otimes A$ regarded as a right A-module using the right operators of A on A.

Similarly a left A-module N is said to be *weakly injective* if the image of the homomorphism $h: N\to \mathrm{Hom}_K(A, N)$ which to each $x\in N$ assigns the homomorphism $a\to ax$ is a direct summand of $\mathrm{Hom}_K(A, N)$ regarded as a left A-module using the right operators of A on A.

Now we shall give a generalization of [2, X, 8.3] as follows:

Theorem 10. *Let A be a supplemented K-algebra such that the enveloping algebra A^e of A is A^*-projective. Then*

$$H_n(A, {}_eM)=0=H^n(A, N_e) \qquad \textit{for} \quad n>0$$

for any weakly projective right A-module M and any weakly injective left A-module N.

Proof. To prove the first equality, we note that following Theorem 9,

$$H_n(A, {}_eM)=\mathrm{Tor}_n^{A^e}({}_eM, A)\approx \mathrm{Tor}_n^A(M, K)\,.$$

Since M is weakly projective, there is an A-homomorphism $M\to M\otimes A$ such that the composition

$$M\longrightarrow M\otimes A\longrightarrow M$$

is the identity, and therefore the composition

$$\mathrm{Tor}_n^A(M, K)\longrightarrow \mathrm{Tor}_n^A(M\otimes A, K)\longrightarrow \mathrm{Tor}_n^A(M, K)$$

is also the identity by [2, II, 1.1]. Thus it suffices to show that $\mathrm{Tor}_n^A(M\otimes A, K)=0$ for $n>0$.

Let X be an A^e-projective resolution of A. Then since $X\otimes_A K$ is an A-projective resolution of K by Theorem 9,

$$\begin{aligned}\mathrm{Tor}_n^A(M\otimes A, K) &= H_n((M\otimes A)\otimes_A(X\otimes_A K))\\ &\approx H_n(M\otimes(X\otimes_A K))\\ &\approx H_n(M\otimes(K\otimes_{A^*}X))\\ &\approx H_n(M\otimes_{A^*}X).\end{aligned}$$

Since A^e is A^*-projective, X is a left A^*-projective resolution of A by [2, II, 6.2]. Consequently

$$H_n(M\otimes_{A^*}X)=\mathrm{Tor}_n^{A^*}(M, A)=0 \qquad \text{for} \quad n>0.$$

To prove the second equality, we note that following Theorem 9,

$$H^n(A, N_e)=\mathrm{Ext}_{A^e}^n(A, N_e)\approx \mathrm{Ext}_A^n(K, N).$$

Since N is weakly injective, there is an A-homomorphism $\mathrm{Hom}_K(A, N) \to N$ such that the composition

$$N \longrightarrow \mathrm{Hom}_K(A, N) \longrightarrow N$$

is the identity, and so the composition

$$\mathrm{Ext}_A^n(K, N) \longrightarrow \mathrm{Ext}_A^n(K, \mathrm{Hom}_K(A, N)) \longrightarrow \mathrm{Ext}_A^n(K, N)$$

is the identity by [2, II, 1.1]. Thus it suffices to show that $\mathrm{Ext}_A^n(K, \mathrm{Hom}_K(A, N)) = 0$ for $n > 0$.

Applying [2, II, 5.2], we have

$$\begin{aligned}\mathrm{Ext}_A^n(K, \mathrm{Hom}_K(A, N)) &= H^n(\mathrm{Hom}_A(X \otimes_A K, \mathrm{Hom}_K(A, N))) \\ &\approx H^n(\mathrm{Hom}_K(A \otimes_A (X \otimes_A K), N)) \\ &\approx H^n(\mathrm{Hom}_K(X \otimes_A K, N)),\end{aligned}$$

and further by [2, II, 5.2′],

$$\begin{aligned}H^n(\mathrm{Hom}_K(X \otimes_A K, N)) &\approx H^n(\mathrm{Hom}_A(X, \mathrm{Hom}_K(K, N))) \\ &\approx H^n(\mathrm{Hom}_A(X, N)) \\ &= \mathrm{Ext}_A^n(A, N)\end{aligned}$$

because X is a right A-projective resolution of A, and this is equal to 0 for $n > 0$.

§ 5. An integral domain A which is hereditary is called *Dedekind ring*.

Lemma 11. *Let A be a Noetherian integral domain and M finitely generated A-module. If $\mathrm{Ext}_A^1(M, N)$ is divisible for all A-module N, then M is projective.*

Proof. For any injective A-module Q, by [2, VI, 5.3], we obtain an isomorphism

$$\mathrm{Tor}_n^A(\mathrm{Hom}_A(N, Q), M) \approx \mathrm{Hom}_A(\mathrm{Ext}_A^n(M, N), Q).$$

Since $\mathrm{Ext}_A^1(M, N)$ is divisible, and since $\mathrm{Hom}_A(\mathrm{Ext}_A^1(M, N), Q)$ is torsion-free by [2, VII, 1.4], $\mathrm{Tor}_1^A(\mathrm{Hom}_A(N, Q), M)$ is torsion-free. On the other hand, by [3, Theorem 1], $\mathrm{Tor}_1^A(\mathrm{Hom}_A(N, Q), M)$ is a torsion module. Thus $\mathrm{Tor}_1^A(\mathrm{Hom}_A(N, Q), M) = 0$, and hence $\mathrm{Hom}_A(\mathrm{Ext}_A^1(M, N), Q) = 0$. As Q is an arbitrary injective module, this implies $\mathrm{Ext}_A^1(M, N) = 0$. As this holds for every A-module N, M is projective.

Finally, we shall give a characterization of the Dedekind ring as follows:

Theorem 12. *A Noetherian integral domain A is a Dedekind ring if and only if for any finitely generated torsion-free A-module M, $\mathrm{Ext}_A^1(M, N)$*

is divisible for all A-module N.

Proof. The necessity of the condition follows from [2, VII, 5.3]. The sufficiency follows from Lemma 11 and [2, VII, 4.1].

Corollary 13. *A Noetherian integral domain A is a Dedekind ring if and only if for any ideal M of A,* $\mathrm{Ext}_A^1(M, N)$ *is divisible for all A-module N.*

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