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SOME THEOREMS ON GALOIS THEORY
OF SIMPLE RINGS

By

Takasi NAGAHARA and Hisao TOMINAGA

Recently, in [7], we have succeeded in constructing Galois theory of simple rings under the assumption that the extension $R/S$ considered is hereditarily Galois (h-Galois) and locally finite. However, we have believed that [7, Theorem 2.1] and [7, Theorem 3.1] should be stated under more desirable assumptions. One of the purposes of the present paper is to give a settlement to this problem. Concerning [7, Theorem 2.1], one will see that the assumption that $R/S$ is locally finite can be excluded from those assumed there (Theorem 1). On the other hand, as was shown in [7, Lemma 3.4], if $R/S$ is h-Galois and locally finite then $\mathfrak{G}R_r$ is dense in $\text{Hom}_{S_l}(R, R)$. In §1, one will see also that if $R/S$ is locally finite and $\mathfrak{G}R_r$ is dense in $\text{Hom}_{S_l}(R, R)$ then the fundamental theorems in Galois theory of finite dimension hold still for regular intermediate rings of $R/S$ left finite over $S$ (Theorems 2 and 3). And, if $R/S$ is locally finite, $\mathfrak{G}R_r$ dense in $\text{Hom}_{S_l}(R, R)$, and $V_R(V_R(S'))$ is simple for each regular intermediate ring $S'$ of $R/S$ left finite over $S$ such that $[S':S]<\infty$, then [7, Theorem 3.1] is still valid even for a regular intermediate ring $R'$ of $R/S$ (Theorem 6). The proof of this improvement will be given in §3. §2 is devoted exclusively to the treaty of algebraic Galois extensions, which is our second purpose. In fact, Theorem 4 may be regarded as a complete extension of [2, Theorem 3] to simple rings as well as an improvement of [7, Lemma 1.9]. §2 contains also a sharpening of [7, Lemma 1.10] (Theorem 5).

Throughout the present paper, $R=\sum_{i=1}^{n}D e_{ij}$ be a simple ring, where $e_{ij}$'s are matrix units and $D=V_R(\{e_{ij}'s\})$ is a division ring. And $S$ be always a simple subring of $R$ (containing 1 of $R$), $\mathfrak{G}$ the group of all the $S$-(ring) automorphisms of $R$. Further, we set $C=V_R(R)$, $Z=V_S(S)$, $V=V_R(S)$, $H=V_R(V)$ and $C_0=V_F(V)$. As to general notations and terminologies used here we follow the previous paper [7].

1. Now, we shall begin our study with the following lemma.

Lemma 1. Let $S$ be a regular subring of $R$, and $\mathfrak{G}$ a subgroup of $\mathfrak{G}$ containing $V$. If $T$ is an intermediate ring of $R/S$ left finite over $S$ such that $R$ is $T$-$R$-irreducible, and $T=J(\mathfrak{G}(T, R))$, then $\infty>[(\mathfrak{G}|T)R_r : R_r]$. 


= [\mathfrak{S}|T'\mathfrak{R}_r : R_\mathfrak{t}_r].

Proof. By [7, Lemma 1.3] and [7, Lemma 1.5], \((\mathfrak{S}|T)\mathfrak{R}_r = \sum_i (\mathfrak{V}\sigma_i|T)\mathfrak{R}_r\) for some \(\sigma_i\)s e \(\mathfrak{S}\). As \(\mathfrak{V}_n(T) = \mathfrak{V}_n(T')\), [7, Lemma 1.4 (iii)] will yield 
\[\infty > [\sum_i (\mathfrak{V}\sigma_i|T)\mathfrak{R}_r : R_\mathfrak{t}_r] = [V : \mathfrak{V}(T)]_r = [V : \mathfrak{V}(T')]_r = [\sum_i (\mathfrak{V}\sigma_i|T')\mathfrak{R}_r : R_\mathfrak{t}_r].\] Further, by [7, Lemma 1.3 (iv)], \(\mathfrak{S}|\mathfrak{V}\sigma_i|T')\mathfrak{R}_r = \sum_i (\mathfrak{V}\sigma_i|T')\mathfrak{R}_r\). Combining these facts mentioned above, it follows that \([\mathfrak{S}|T)\mathfrak{R}_r : R_\mathfrak{t}_r] \leq [\mathfrak{S}|T')\mathfrak{R}_r : R_\mathfrak{t}_r]. \)

Now, if \([\mathfrak{S}|T)\mathfrak{R}_r : R_\mathfrak{t}_r] < [\mathfrak{S}|T')\mathfrak{R}_r : R_\mathfrak{t}_r]\) then there exists some \(\tau \in \mathfrak{S}\) such that \(\tau|T' \notin \sum_i (\mathfrak{V}\sigma_i|T')\mathfrak{R}_r\). While, \(\tau|T \in \sum_i (\mathfrak{V}\sigma_i|T)\mathfrak{R}_r\) implies \(\tau|T = \sigma_j \tilde{v}|T\) with some \(\sigma_j\) and \(\tilde{v} \in \mathfrak{V}\) by [7, Corollary 1.1]. And then, noting that \(\tau' = \tau(\sigma_j \tilde{v})^{-1} \in \mathfrak{S}(T)\), we obtain \(\tau|T' = \tau' \sigma_j \tilde{v}|T' = \sigma_j \tilde{v}|T' \in \sum_i (\mathfrak{V}\sigma_i|T')\mathfrak{R}_r\), which is a contradiction.

Now, by making use of the same method as in the proof of [7, Theorem 2.1], we can prove the following theorem which contains evidently [7, Theorem 2.1].

**Theorem 1.** Let \(S\) be a regular subring of \(\mathfrak{S}\), \(\mathfrak{S}\) a subgroup of \(\mathfrak{S}\) containing \(\mathfrak{V}\), and \(\mathfrak{S}\mathfrak{R}_r\) dense in \(\text{Hom}_{\mathfrak{S}_r}(\mathfrak{R}, \mathfrak{R})\). If \(T\) is an intermediate ring of \(\mathfrak{R}|S\) left finite over \(S\) such that \(T\) is \(T\mathfrak{R}\)-irreducible, then \(J(\mathfrak{S}(T), \mathfrak{R}) = T\).

Proof. Suppose on the contrary \(T' = \mathfrak{S}(\mathfrak{S}(T), \mathfrak{R}) \supset T\). Then, we can find an \(S\)-left submodule \(M\) of \(T'\) such that \([M : S]\_r < [M : S]\_r < \infty\). Since \(\text{Hom}_{\mathfrak{S}_r}(\mathfrak{R}, \mathfrak{R}) = \mathfrak{S}\mathfrak{R}_r\), \(M = ((\mathfrak{S}|T')\mathfrak{R}_r)\mathfrak{M}\) by our assumption, there holds \([M : S]\_r = [\text{Hom}_{\mathfrak{S}_r}(\mathfrak{R}, \mathfrak{R}) : R_\mathfrak{t}_r] = [((\mathfrak{S}|T')\mathfrak{R}_r)\mathfrak{M} : R_\mathfrak{t}_r].\) Hence, we obtain \([((\mathfrak{S}|T')\mathfrak{R}_r : R_\mathfrak{t}_r] = [T : S]\_r < [M : S]\_r \leq [((\mathfrak{S}|T')\mathfrak{R}_r : R_\mathfrak{t}_r]\), which contradicts Lemma 1.

If a division ring \(\mathfrak{R}\) is Galois over \(S\), \(\mathfrak{S}\mathfrak{R}_r\) dense in \(\text{Hom}_{\mathfrak{S}_r}(\mathfrak{R}, \mathfrak{R})\) for any subgroup \(\mathfrak{S}\) of \(\mathfrak{S}\). Consequently, we obtain the following corollary, that is [4, Theorem 1] itself. And, one should remark here that the proof of Theorem 1 is notably easier than that of [4, Theorem 1] given in [4].

**Corollary 1.** Let a division ring \(\mathfrak{R}\) be Galois over \(S\), \(\mathfrak{S}\) a subgroup of \(\mathfrak{S}\) containing \(\mathfrak{V}\) with \(J(\mathfrak{S}(T), \mathfrak{R}) = S\). If \(T\) is an intermediate ring of \(\mathfrak{R}|S\) left finite over \(S\), then \(J(\mathfrak{S}(T), \mathfrak{R}) = T\).

To prove Theorems 2 and 3, the following generalization of [6, Lemma 4] will be needed.

**Lemma 2.** Let \(\mathfrak{R}\) be locally finite over a regular subring \(S\), and \(\mathfrak{S}\mathfrak{R}_r\) dense in \(\text{Hom}_{\mathfrak{S}_r}(\mathfrak{R}, \mathfrak{R})\). If \(\mathfrak{S}'\) is an intermediate simple ring of \(\mathfrak{R}|S\) with \([\mathfrak{S}' : S]\_r < \infty\) then \(\mathfrak{R}\) is \(\mathfrak{S}'\mathfrak{R}\)-completely reducible. In particular, if additionally \(V_\mathfrak{R}(\mathfrak{S}') = 1\) then \(\mathfrak{R}\) is \(\mathfrak{S}'\mathfrak{R}\)-irreducible.

Proof. Let \(M\) be an arbitrary minimal \(\mathfrak{S}'\mathfrak{R}\)-submodule of \(\mathfrak{R}\) such that the composition series of \(M\) as \(\mathfrak{R}\)-right module is of the shortest length among minimal \(\mathfrak{S}'\mathfrak{R}\)-submodules of \(\mathfrak{R}\). Then, \(M = \mathfrak{S}' a \mathfrak{R}\) with non-zero \(a \in M\). Now,
we set $S^* = S'[a, \{e_{ij}'s\}]$, which is a regular subring of $R$ finite over $S$.

And, as $R$ is evidently $S^*$-$R$-irreducible, by [7, Lemma 1.3], \(\text{Hom}_{S^*}(S^*, R) = (\mathfrak{G}|S^*)R_r = \sum_{i=1}^{n}(\sigma_i|S^*)R_r\) with some $\sigma_i \in \mathfrak{G}$, where one may remark that each $(\sigma_i|S^*)R_r$ is $S^*$-$R_r$-irreducible. Accordingly, the $S^*$-$R_r$-submodule $\text{Hom}_{S^*}(S^*, R)$ of $\text{Hom}_{S^*}(S^*, R) = \sum_{i=1}^{n}(\mathfrak{M}_i)$, and each $\mathfrak{M}_i$ is $S^*$-$R_r$-isomorphic to $(\sigma_i|S^*)R_r$ for some $\sigma_i \in \{\sigma_i's\}$. If $\mathfrak{M}_i \ni \epsilon_j \leftrightarrow \sigma_i|S^* \in (\sigma_i|S^*)R_r$, under the above isomorphism, then $x^*\epsilon_j \leftrightarrow x^*(\sigma_i|S^*) = (\sigma_i|S^*)$ $(x^*\sigma_j)_r \leftrightarrow \epsilon_j(x^*\sigma_j)_r$ for each $x^* \in S^*$. And so, we have $x^*\epsilon_j = \epsilon_j(x^*\sigma_j)_r$, whereas it follows $x^*\epsilon_j = (1\epsilon_j) \cdot (x^*\sigma_j)$. Hence, $\mathfrak{M}_i = \epsilon_jR_r = (\sigma_i|S^*)R_r$ with $u_j = 1\epsilon_j$.

Since $\mathfrak{M}_i$ is contained in $\text{Hom}_{S^*}(S^*, R)$, it will be easy to see that $M_j = (S'a)\mathfrak{M}_i$ is an $S'^*$-$R$-submodule of $R$. Moreover, there holds $M_j = (S'a)\alpha_j u_j R = u_j \cdot (S'a \cdot R\alpha_j^-) \alpha_j = u_j \cdot (S'aR) \alpha_j = u_j \cdot M\alpha_j$, whence it follows $[M_j|R]_r = [u_j \cdot (M\alpha_j)|R]_r = [M\alpha_j|R]_r = [M|R]_r$, where $[M|R]_r$ means the length of the composition series of the $R$-right module $M$. Recalling here that $[M|R]_r$ is chosen to be the least, we see that the $S'$-$R$-submodule $M_j$ is 0 or $S'$-$R$-irreducible. Finally, noting that $R$ is $S'$-$\text{Hom}_{S'}(R, R)$-irreducible, there holds $R = (S'a)\text{Hom}_{S'}(R, R) = (S'a)\text{Hom}_{S'}(S^*, R) = (S'a)\sum_{i=1}^{n}(\mathfrak{M}_i)$, which proves evidently the complete reducibility of $R$ as an $S'$-$R$-module. The latter half of our theorem is a direct consequence of the fact $V_{\text{Hom}_{S'}(R, R)}(S'_j \cdot R_j) = (V_{R}(S'))_l$.

Now, we can prove the next theorem:

\textbf{Theorem 2.} Let $R$ be locally finite over a regular subring $S$, $\mathfrak{G}$ a subgroup of $\mathfrak{G}$ containing $\nabla$, and $\mathfrak{S}R_r$ dense in $\text{Hom}_{S^*}(R, R)$. If $T$ is a regular intermediate ring of $R/S$ left finite over $S$ then $J(\mathfrak{G}(T), R) = T$, and in particular $R$ is Galois over $S$.

\textbf{Proof.} Let $V_T(R) = \sum W_{c_{pq}}$, where $c_{pq}$’s are matrix units and $W = V_{W_{pq}}(\mathfrak{G}) (\{c_{pq}'s\})$ is a division ring. Then $T' = T[\{c_{pq}'s\}] = \sum T_{c_{pq}}$ is a simple ring left finite over $S$ and $V_T(T') = W$ is a division ring, whence $R$ is $T'$-$R$-irreducible by Lemma 2. And so, in virtue of Theorem 1, there holds $J(\mathfrak{G}(T')', R) = T'$. Moreover, if we set $V' = \sum Cc_{pq}$ with the center $C' = T'$ then $V'$ is a simple subring of $V'$ and $J(\nabla', T') = V_T(T') = V_{T'}(\{c_{pq}'s\}) = T$. Hence, we obtain $J(\mathfrak{G}(T'), R) = T$.

Further, Lemma 2 enables us to apply the same arguments as in the proofs of [7, Lemma 3.11] and [7, Corollary 3.7] to see the following theorem that contains [7, Corollary 3.7] obviously.

\textbf{Theorem 3.} Let $R$ be locally finite over a regular subring $S$, $\mathfrak{G}$ a subgroup of $\mathfrak{G}$ containing $\nabla$ and $\mathfrak{S}R_r$ dense in $\text{Hom}_{S^*}(R, R)$. If $\sigma$ is an $S'$-(ring) isomorphism of a regular intermediate ring $T$ of $R/S$ left finite over
S onto a regular subring of R, then σ is contained in $\mathfrak{G}|T$.

Here, as careful readers realize at once, in the proofs of Theorems 1–3 no theorems from Galois theory of finite dimension are needed. And, [8, Theorem 5] and [8, Theorem 6] may be regarded as special cases of Theorem 2 and Theorem 3 respectively.

2. If $R$ is left algebraic over a simple subring $S$, and $r$ a regular element of $R$, then $r^{-1}\in S[r]$. In fact, as $R$ is left algebraic over a division subring $E$ of $S$, $[E[r]:E]<\infty$, whence it will be easy to see that $r^{-1}\in E[r]\subseteq S[r]$. As an easy consequence of this fact, we see that if $R$ is left algebraic over $S$ then each intermediate ring of $R/S\{e_{ij}'s\}$ is a regular subring of $R$. In what follows, the above remark will be used often without mention.

The next two lemmas, Lemmas 3 and 4, have been given in [3]. However, for the sake of completeness, we shall present here their proofs those are rather easier than the earlier ones.

**Lemma 3.** Let $R/S$ be outer Galois, and $T$ an intermediate ring of $R/S$ left finite over $S$. If $R$ is $T$-$R$-irreducible then $\#\{x\otimes\}\leq\infty$ for each $x\in T$.

**Proof.** As $\infty>\left[T:S\right]_{r}=\left[\text{Hom}_{S_{r}}(T,R):R_{r}\right]_{r}$, by [7, Lemma 1.3], we have $(\otimes|T)R_{r}=\sum_{\sigma|T}(\sigma|T)R_{r}$ with some $\sigma_i$'s in $\otimes$. And so, if $\sigma$ is an arbitrary element of $\otimes$ then $\sigma|T=\sigma_j|T$ for some $\sigma_j$ by [7, Corollary 1.1]. Hence, it follows that $\#\{x\otimes\}=\#\{x_{\sigma_1},\ldots,x_{\sigma_t}\}$ for $x\in T$.

**Lemma 4.** If $R/S$ is outer Galois and left algebraic then it is locally finite.

**Proof.** If $S\subseteq C$ then $C=V=R$ and our assertion is obviously true. And so, in what follows, we shall assume that $S\lneq C$.

**Case I:** $n=1$. As $R$ is evidently $S$-$R$-irreducible, $S[x]$-$R$-irreducible for each $x\in R$. And so, $\otimes$ is locally finite by Lemma 3. Now, our assertion is clear by [10, (a*)].

**Case II:** $n>1$. Let $a$ be an arbitrary element of $S\setminus C$. By [6, Lemma 7 (i)], there exists such a regular element $r\in R$ that $ar=\sum d_{ij}\epsilon_{ij}$ with $d_{in}=1$ and $d_{in}=0$ ($i\geq2$). $R$ is then outer Galois and left algebraic over $S\tilde{r}$ and $S\tilde{r}\lneq C$. To our end, it suffices to prove that $R/S\tilde{r}$ is locally finite. And so, without loss of generality, we may assume from the beginning that $S$ contains $a=\sum d_{ij}\epsilon_{ij}$ with $d_{in}=1$ and $d_{in}=0$ ($i\geq2$). If $D$ (and so $R$) is finite, there is nothing to prove. And so, we may restrict further our attention to the case where $D$ is infinite. Now, let $Q$ be the set of all $q\in R$ with the property that each intermediate (simple) ring $T$ of $R/S[q,\{e_{ij}'s\}]$ is $S[q]$-$T$-irreducible. By
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[6, Lemma 8], \( t_x = e_{nn-1} + \cdots + xe_{it} + \cdots + e_{it} \in Q \) for each non-zero \( x \in D \). Accordingly, if \( x \in D \) is neither 0 nor 1 then \( e_{it-1} = t_{x-1} - t_x - it \) and \( xe_{it-1} = t_{x-1} - t_{x-1} \) are both contained in \( S[Q] \) (2 \( \leq i \leq n \)), whence we obtain \( D e_{nj} \subseteq S[Q] \) (\( n > j \geq 1 \)) and \( D e_{nj} \subseteq S[Q] \) (\( n > j \geq 1 \)), whence it follows \( e_{nn} = 1 - \sum_i^{n-1} e_{it} e_{it} \in S[Q] \). And then, \( e_{in} = ae_{nn} \in S[Q] \). Now, it is easy to see that \( e_{ij} \)'s and \( D \) are contained in \( S[Q] \), that is, \( S[Q] = R \), whence \( \mathfrak{G} \) is locally finite by Lemma 3. Accordingly, \( R/S \) is locally finite again by [10, (a*)].

**Corollary 2.** If \( R \) is Galois and left algebraic over \( S \) and \([V : C] < \infty \) then \( V/Z \) and \( H[V]/S \) are locally finite.

**Proof.** At first, as \( S[V] = S \times_2 V, V/Z \) is algebraic. We set here \( V = \sum_i C_i v_i \), where \( v_i = 1 \) and \( v_i v_j = \sum c_{ij} v_k \) (\( c_{ij} \in C_i \)). Let \( \{ u_i \} \) be a finite subset of \( V \) and write \( u_k = \sum c_{i} v_k \) (\( c_i \in C_i \)). Evidently, \( C^* = Z[\{ c_{ij} \} \}, \{ c_i \} \) is a subfield of \( C_i \) finite over \( Z \). Hence, \( V^* = \sum_i C_i v_k \) is an intermediate ring of \( V/Z[u_i, \cdots, u_s] \) and finite over \( Z \). This means evidently that \( V/Z \) is locally finite. Next, as the simple ring \( H \) is outer Galois and left algebraic over \( S, H/S \) is locally finite by Lemma 4. If \( F \) is an arbitrary finite subset of \( H[V] \), there exist finite subsets \( F \subseteq H \) and \( F \subseteq V \) such that \( S[F, F] \subseteq S[F] \). We set here \( S[F] = \sum_i Sx_i \), and \( Z[F] = \sum_i Zy_i \). Then, \( S[F] \subseteq S[F, F] = S[F] \). \( Z[F] = \sum_i Sx_i y_j \), whence it follows \( [S[F] : S] < \infty \).

**Corollary 3.** If \( R \) is outer Galois and left algebraic over \( S \), then for each finite subset \( F \) of \( R \) there exists an element \( a \) such that \( S[F] = S[a] \).

**Proof.** As \( R/S \) is locally finite by Lemma 4, [10, (b*)] implies that \( \mathfrak{G} \) is l.f.d. Accordingly, our assertion is a consequence of [6, Theorem 2].

**Lemma 5.** Let \( R/S \) be Galois, \( T \) an intermediate ring of \( R/S \) left finite over \( S \) such that \( R \) is \( T \)-irreducible. If \( H \) is a simple ring left algebraic over \( S \), and \( T' = J(\mathfrak{G}(T), R) \) then \( [T' \cap H : S] < \infty \).

**Proof.** By Lemma 1, \( (\mathfrak{G}) T' r_r = \sum_i \sigma_i T' r_r \) with some \( \sigma_i \)s\( \in \mathfrak{G} \). And so, in virtue of [7, Corollary 1.1], there holds \( \mathfrak{G} T' = \cup_i (\mathfrak{V} \sigma_i T') \), whence \( \mathfrak{G} T' \cap H = \cup_i (\mathfrak{V} \sigma_i T' \cap H) = \{ \sigma_i T' \cap H, \cdots, \sigma_i T' \cap H \} \). Consequently, it follows \( \mathfrak{G} T' \cap H < \infty \). Recalling here that \( H \) is outer Galois and locally finite over \( S \) by Lemma 4 and \( J(\mathfrak{G}, H, H) = S \), [7, Lemma 1.8] yields at once \( [T' \cap H : S] < \infty \).

**Lemma 6.** Let \( R/S \) be Galois and left algebraic over \( S \), \([V : C] < \infty \), \( T \) an intermediate ring of \( R/S \) left finite over \( S \) such that \( R \) is \( T \)-irreducible, and \( T' = J(\mathfrak{G}(T), R) \).

(i) \( H[T'] (= H[T']) \) and \( T' \) are simple rings, and \( H[T] \) is outer Galois and locally finite over \( T' \).
(ii) \([T' : S]_l < \infty\), and so \(H[T]/S\) is locally finite.

**Proof.** (i) Since \(R\) is (inner) Galois and finite over \(H\) and \(H[T]-R\)-irreducible, \(H[T]\) is a simple ring by [6, Lemma 3]. And so, \(H[T]=V_R(V_R(H[T]))=V_R(V_R(T)) \supseteq T'\), whence \(H[T]=H[T']\). We set here \(\mathfrak{T}=\mathfrak{G}(T)\). Then, \(\mathfrak{T}|H[T]\) and \(\mathfrak{T}|H\) are automorphism groups of \(H[T]\) and \(H\) respectively, and \(J(\mathfrak{T}|H[T], H[T]) = T'\), \(J(\mathfrak{T}|H, H) = T' \cap H\). Further, as \(H/S\) is locally finite by Lemma 4, \(\mathfrak{G}(H/S)\) is locally finite by [7, Lemma 1.6], whence \(\mathfrak{T}|H\) is locally finite, that is, \# \{\mathfrak{T} \mathfrak{S}\} \leq \infty\) for each finite subset \(F'\) of \(H\). Let the simple ring \(H[T]\) be represented as a complete matrix ring over a division subring of \(H[T]\) with matrix units \(\{f_{ij}' s\}\). If \(F\) is an arbitrary finite subset of \(H[T]\), there exists a finite subset \(E\) of \(H\) such that \(F \cup \{f_{ij}' s\} \subseteq T[E]\). Since \# \{\mathfrak{T} \mathfrak{S}\} < \infty, T' = T'[\mathfrak{T}\mathfrak{S}]\) is \(\mathfrak{T}\)-normal and \# \{\mathfrak{T} \mathfrak{T}'< \infty\). Moreover, noting that \(H[T] \supseteq T' \supseteq \{f_{ij}' s\}\), we see that \(T'\) is a simple ring. And, as \(H[T]=V_R(V_R(T)) \supseteq T' \supseteq T\), \(\mathfrak{T}|T'\) is a finite outer group. Hence, \(T' = J(\mathfrak{T}|T', T')\) is a simple ring and \(T' : T' = \#(\mathfrak{T}|T') < \infty\), which proves that \(H[T]\) is outer Galois and locally finite over \(T'\).

(ii) We shall use the same notations as in (i), and in particular we set \(F = \{d_k\}'s\) (matrix units of \(H\)). As \(H/S\) is locally finite by Lemma 4, \(\mathfrak{G}(H/S)\) is l.f.d. by [10 (b*)]. Accordingly, every intermediate ring of \(H/S\) is simple by [5, Theorem 1.1]. And, this fact will be used often in the sequel.

Since \(T' \cap H\) is \(\mathfrak{T}\)-normal and \# \{\mathfrak{T} \mathfrak{T}' \cap H\} \leq \# \{\mathfrak{T} \mathfrak{T}' \cap H\} < \infty, \mathfrak{T}|T' \cap H\) is a finite outer group. Hence, \([T' \cap H : T' \cap H] = \#(\mathfrak{T}|T' \cap H) < \infty\). On the other hand, \([T' \cap H : S] < \infty\) by Lemma 5. Consequently, there holds

\[
(T) \quad [T' \cap H : S] = [T' \cap H : T' \cap H] [T' \cap H : S] < \infty.
\]

We set here \(\mathfrak{S}=\mathfrak{G}(T')=\mathfrak{G}(T'), H' = T' \cap H, K' = V_R(\{d_k\}'s)\) and \(K = V_R(\{d_k\}'s)\) (a division ring). Then, as is shown in (i), \([T' : T'] \cap S < \infty\) and \(H[T]\) is outer Galois and locally finite over \(T'\). Since \(H[T]\) is \(T'-H[T]-\)irreducible and \(\mathfrak{T}|H[T]\) is a Galois group of \(H[T]/T'\), [7, Corollary 2.1] enables us to see that \(J(\mathfrak{S}, R) = J(V_R(H[T]), R) = J(\mathfrak{S}, R) = J(\mathfrak{S}, H[T], H[T]) = T'\), whence one will readily see that \(\mathfrak{S}|K\) is a Galois group of \(K/K'\). We shall prove here that \([T' : K'] \leq [R : K]_l\). To this end, suppose \(\{t_1, \ldots, t_m\} \subseteq T'\) is linearly left independent over \(K'\) but linearly dependent over \(K\). Then, without loss of generality, we may assume that \(t_i = \sum_{j=1}^{m}x_{ij}t_i (x_i \in K)\) is a non-trivial relation of the shortest length and \(x_i \in K \setminus K'\). Recalling that \(\mathfrak{S}|K\) is a Galois group of \(K/K'\), there exists some \(\sigma \in \mathfrak{S}\) with \(x_i \sigma \neq x_i\). Accordingly, we obtain \(0=t_i - t_i \sigma = \sum_{j=1}^{m}x_{ij} - x_{ij} \sigma t_i\) and \(x_i - x_i \sigma \in K\). This contradiction proves evidently \([T' : K'] \leq [R : K]_l = [R : H], [H : K] < \infty\). Combining this with (T), we obtain eventually \([T' : S]_l \leq [T' : S]_l = [T' : H']_l, [H' : S] \leq [T' : S]_l\).
$[T^*:K^*], [T^* \cap H:S]<\infty$.

Now, by the light of Lemmas 4, 6 we can prove the following theorem:

**Theorem 4.** If $R$ is Galois and left algebraic over $S$ and $[V:C]<\infty$ then $R/S$ is locally finite.

**Proof.** In case $S \subseteq C$, our assertion is clear by Corollary 2. Thus, in what follows, we shall assume $S \nsubseteq C$.

**Case I:** $n=1$. Since $H \neq C$, $R=H[d]$ with some $d \in R$ by [6, Corollary 2]. Then, $T=S[d]$ is left finite over $S$, $R=H[T]$, and $R$ is $T$-$R$-irreducible. Hence, $R=H[T]$ is locally finite over $S$ by Lemma 6.

**Case II:** $n>1$. By the same reason as in the proof of Lemma 4, without loss of generality, we may assume that $S$ contains $a=\sum_i d_i e_i$ with $d_i \neq 0$ and $d_i=0$ $(i \neq 2)$. We set $R=H[\{e_{ij}\}]=\sum_i D e_i$ with $D=V_R(\{e_{ij}\})$ (a division ring), and distinguish further two cases:

1. $D'=C$. Since $[R:C]=[R':R'] \cdot [R':D']<\infty$, $[S:Z]<\infty$ by [9, Lemma]. The field $Z[C]$ contained in the center of $V$ is locally finite over $Z$ by Corollary 2. If $F$ is an arbitrary finite subset of $C$, $\mathfrak{G}|Z[F\mathfrak{G}]$ and $\mathfrak{G}|(Z \cap C)(F\mathfrak{G})$ are (outer) Galois groups of $Z[F\mathfrak{G}]/Z$ and $(Z \cap C)(F\mathfrak{G})/Z \cap C$ respectively. Further, to be easily seen, they are of same finite order. And so, we have $Z[F\mathfrak{G}]=Z \times \times (Z \cap C)(F\mathfrak{G})$. We have proved therefore that $Z[C]=Z \times (Z \cap C)$, whence it follows particularly $[Z:Z \cap C]=\leq [R:C]<\infty$. Since $V$ is locally finite over $Z$ (Corollary 2) and $D=V_R(R')$ is contained in $V, D$ is evidently locally finite over $Z \cap C$. Accordingly, $R/Z \cap C$ is locally finite. Combining this with $[S:Z \cap C]=[S:Z][Z:Z \cap C]<\infty$, our assertion will be easily seen.

2. $D' \supseteq C$. Since $R$ is Galois and finite over $H$, it will be easy to see that $D$ is Galois and finite over $D$ not contained in $C$. Hence, by [6, Corollary 2], $D=D'[d]$ with some non-zero $d$ in $D$. And then, $R=R'[d]$ of course. We set here $u=d e_n+\sum_i e_{i-1}$, Then, $T=S[u]=S[d, u]$ is left finite over $S$, and $R$ is $T$-$R$-irreducible by [6, Lemma 8]. And so, $H[T]$ is a simple ring by Lemma 6 (i). Since $u^{k-1} au^{n-1}=d^2 e_{k1}$ is a non-zero element of $H[T]\cap e_{kk}R$ $(k=1, \cdots, n)$, $H[T]$ contains $e_{11}, \cdots, e_{nn}$ by [6, Lemma 5]. It follows therefore $e_{1n}=e_{11}a e_{nn}H[T]$ and $d=(u+e_{1n})^n e_{11}$, whence $e_{11} \in H[T]$ $(k=1, \cdots, n)$. Accordingly, $V_R(H[T]) \subseteq V_R(\{e_{11}, e_{21}, \cdots, e_{nn}\})=D$, which implies $H[T]=V_R(H[T]) \supseteq V_R(D) \supseteq [e_{ij}]$. And moreover, $H[T]=H[u, \{e_{ij}\}]=H[d, \{e_{ij}\}]=R[d]=R$. Hence, $R=H[T]$ is locally finite over $S$ by Lemma 6.

Theorem 3 enables us to restate [7, Conclusion 2.1] and [7, Corollary 4.2] under the same assumptions as in [1, VII. §6]:

**Corollary 4.** Let $R$ be Galois and left algebraic over $S$ and $[V:C]<\infty$. 


(i) For each regular intermediate rings $R_1$, $R_2$ of $R/S$, every $S$-ring
isomorphism $\rho$ of $R_1$ onto $R_2$ can be extended to an automorphism of $R$.

(ii) For each regular intermediate ring $R'$ of $R/S$, $R/R'$ is $\mathfrak{G}(R')$-locally
Galois.

(iii) There exists a 1-1 dual correspondence between closed (\*)-regular
subgroups of $\mathfrak{G}$ and regular intermediate rings of $R/S$, in the usual sense
of Galois theory.

Next, we shall improve [7, Lemma 1.10].

Lemma 7. Let $R$ be Galois and left 2-algebraic over $S$, $[V:C_o]<\infty$, and $Q$
the set of all $q\in R$ with the property that each intermediate (simple)
ring $T$ of $R/S[q, \{e_{ij}\}]$ is $S[q]$-T-irreducible. If $S[Q]$ coincides with $R$
and there exists an element $r'\in Q$ such that $R'=V_R(V_R(S[r']))\not\supseteq\{e_{ij}\}$ and
$H\subseteq V_{R'}(R')$ then $H$ is a simple ring and $R/S$ is locally finite.

Proof. If $T$ is an arbitrary intermediate ring of $R/R'$, then $H\subseteq V_T(T)$.
In fact, $H\subseteq V_T(T)$ implies $H\subseteq R'\cap V_T(R')=V_{R'}(R')$, which is a contradiction.
At first, we shall prove that there exists an element $r_i\in R$ such that $R_i=
V_R(V_R(S[r_i]))\not\supseteq\{e_{ij}\}$, $H\subseteq V_{R_i}(R_i)$ and $V_R(S[r_i])\subseteq C_o$.
To this end, we shall distinguish two cases, and set $S'=S[r']$.

Case I: $V_{R'}(R')\subseteq C_o$. As $V$ is (inner) Galois and finite over $C_o$, by [6,
Theorem 5], for arbitrary $x\in V_{R'}(R')(\subseteq V)$ not contained in $C_o$ there exists
some $y\in V$ such that $V=C_o[x, y]$. If we set $S'=S'[y]=S'[r', y]$, then $R''=
V_R(V_R(S''))\not\supseteq S''[V_R(V_R(S'))]=S''[R']\not\supseteq H[x, y, \{e_{ij}\}]$. And so, $R''(\supseteq
\{e_{ij}\})$ is a simple ring containing $V=C_o[x, y]$. Further, recalling that $R''$
is $(S'-R'')$- and so $S''-R''$-irreducible, we see that $\infty >[S'':S]_{1}\geq [V_{R'}(R):V_{R'}(R'')]=
[V:V_{R'}(R'')]$ by [7, Lemma 1.5]. Hence, $R''$ is inner Galois and finite
over (simple) $H=V_{R'}(V)$ and, as is noted at the opening, $H\subseteq V_{R'}(R'')$.
Accordingly, by [6, Corollary 2], $R'''=H[r_i]$ with some $r_i$. Evidently, there
holds $V_R(S[r_i])=V_R(H[r_i])=V_R(R'')=V\cap V_R(R'')\subseteq V\cap V_R(V)=C_o$
and $R_i=V_R(V_R(S[r_i]))\supseteq R'$. Needless to say, as is noted at the opening, $H\subseteq V_{R_i}(R_i)$.

Case II: $V_{R'}(R')\not\subseteq C_o$. Evidently, $V_R(S')=V_R(R')$ is a division subring
of $D$, and its center coincides with the center $C'$ of $R'$. Since $V'=C_o[V_R(S')]$
$=C_o\times C'$, $R'$ and $[V':C_o]<[V:C_o]<\infty$, we see that $[V_R(R'):C']<\infty$
and $V'$ is a simple ring. Consequently, there holds $V_{R'}(V'(V'))=V'$. Now, in
case $V'\subseteq V$, there holds $C_o\subseteq V_{R'}(V')=V_{R'}(V'(S'))\subseteq R'$. And so, by [6, Theorem
5], for an arbitrary $x\in V_{R'}(V'),C_o$ there exists some $y\in V$ such that $V=C_o[x, y]$.
If we set $S'=S'[y]$, it follows $R''=V_R(V_R(S''))\not\supseteq S''[V_R(V_R(S'))]=S''[R']\supseteq
H[x, y, \{e_{ij}\}]\not\supseteq V$. Accordingly the rest of the proof will proceed just as in
the latter half of Case I. While, in case $V=V$, $R''=R':V_{R}(R')=R'\times C V_R(R')$.
is a simple ring containing $V'=V$ and the center $C''$ of $R''$ coincides with $C'$. Noting here that $R$ is $S'$-irreducible and $[V_R(R') : C'] < \infty$, we obtain $\infty > [S' : S] \cdot [V_R(R') : C'] \geq [V : V_R(S')] \cdot [V_R(R') : C'] = [V : C'] = [V : C'']$. Hence, $R''$ is inner Galois and finite over $H = V_{R''}(V)$. Accordingly, the rest of the proof proceeds just as in the last part of Case I.

Thus, we have proved our first step assertion, and at the same time we have seen that $H$ is a simple ring.

Now, let $\{x_1, \ldots, x_m\}$ be an arbitrary finite subset of $R$. As $R=S[Q]$, there exists a finite subset $\{q_1, \ldots, q_t\}$ of $Q$ such that $S[x_1, \ldots, x_m] \subseteq S[q_1, \ldots, q_t]$. We set here $S_r = S[r_1, q_1]$ and $R_r = V_R(V_R(S_r)) (\supseteq R_1 \supseteq V)$. Then, the simple ring $R_r$ being $S_r$-irreducible, $\infty > [S_r : S] \geq [V_R(S_r) : V_{R_r}(S_r)]_r = [V : V_{R_r}(R_r)]_r$ by [7, Lemma 1.5]. $R_r$ is therefore inner Galois and finite over $H$ and $H \subseteq V_{R_r}(R_r)$. And so, by [6, Corollary 2], $R_r = H[r_r]$ with some $r_r$. Noting here that $V_R(S[r_r]) = V_R(H[r_r]) = V_R(R_r) \subseteq V_R(R_r) \subseteq C_0$, one will easily see that $r_r$ possesses the property that $r_r$ enjoyed. Accordingly, we can repeat the same argument for $S[r_r, q_r]$ instead for $S_r = S[r_r, q_r]$ to obtain such an element $r_r$ that $V_R(V_R(S[r_r, q_r])) = H[r_r]$. Continuing the same procedures step by step, we obtain eventually $r_3, \ldots, r_{t+1} \in R$ such that $V_R(V_R(S[r_k, q_k])) = H[r_{k+1}]$ ($k = 1, \ldots, t$). As $q_1, \ldots, q_t \in H[r_{t+1}]$, there exists a finite subset $F$ of $H$ such that $S[q_1, \ldots, q_t] \subseteq S[F, r_{t+1}]$. Recalling here that the simple ring $H$ is outer Galois and left algebraic over $S, S[F] = S[h]$ with some $h \in H$ by Corollary 3. Hence, $S[x_1, \ldots, x_m] (\subseteq S[q_1, \ldots, q_t])$ is contained in $S[h, r_{t+1}]$ that is left finite over $S$.

**Theorem 5.** If $R$ is Galois and left 2-algebraic over $S$ and $[V : C] < \infty$ then $H$ is simple and $R/S$ is locally finite.

**Proof.** If $S \subseteq C$ then $V = R$, whence $[V : C] = [V : C] < \infty$. And so, our theorem for this case is contained in Theorem 4. And next, in case $D$ is finite there is nothing to prove. Thus, in what follows, we shall restrict our attention to the case where $S \not\subseteq C$ and $D$ is infinite. Let $Q$ be the set of all $q \in R$ with the property that each intermediate ring $T$ of $R/S[q, \{e_{ij}\}]$ is $S[q]$-irreducible. We shall distinguish here two cases:

Case I: $n=1$. As each intermediate (division) ring $T$ of $R/S$ is $S$-irreducible, there holds evidently $R = Q$. Now, let $a$ be an arbitrary element of $S \setminus C$. Then, $ar' \neq r'a$ for some $r' \in R$. And, it is evident that $H(\ni a)$ is not contained in the center of $R' = V_R(V_R(S[r']))$. Our assertion is therefore a direct consequence of Lemma 7.

Case II: $n>1$. By the same reason as in the proof of Lemma 4, without loss of generality, we may assume that $S$ contains an element $a = \sum d_{ij}e_{ij}$ with $d_{nn} = 1$ and $d_{ii} = 0 (i \geq 2)$. Then, we have seen in Case II of the proof
of Lemma 4 that $R = S[Q]$. We set here $r' = \sum_{i=1}^{n} e_{ii}^{-1}$, $S' = S[r']$ and $R' = V_{R}(S')$. $r'$ is contained in $Q$ by [6, Lemma 8] and $S'$ contains $e_{ii} = r'^{-1}a_{n-1}$ $(k = 1, \cdots, n)$. And so, $V_{R}(S') \subseteq V_{R}(\{e_{11}, e_{22}, \cdots, e_{nn}\}) = D$, whence $R' \supseteq V_{R}(D) \supseteq \{e_{ij}\}$. And, noting that $ar' \neq r'a$, it will be easy to see that $H$ is not contained in the center of $R'$. Hence, again by Lemma 7, $H$ is simple and $R/S$ is locally finite.

Corresponding to Corollary 4, Theorem 5 together with [7, Conclusion 2.2] and [7, Theorem 4.2] yields the next:

**Corollary 5.** Let $R$ be Galois and left 2-algebraic over $S$, $[V:C\preceq \aleph_{0}]$, and $[R:H]_{l} \leq \aleph_{0}$. And let $R_{1}, R_{2}$ be $f$-regular intermediate rings of $R/S$.

(i) If $\rho$ is an $S$-isomorphism of $R$, onto $R_{i}$ then $\rho$ can be extended to an automorphism of $R$.

(ii) $R/R_{i}$ is $\mathfrak{G}(R_{i})$-locally Galois.

(iii) There exists a 1-1 dual correspondence between closed $\mathfrak{G}$-regular subgroups of $\mathfrak{G}$ and $f$-regular intermediate rings of $R/S$, in the usual sense of Galois theory.

3. In this section, we shall give an improvement of [7, Theorem 3.1]. The realization of our improvement is essentially due to the next lemma.

**Lemma 8.** Let $R$ be locally finite over a regular subring $S$, and $\mathfrak{G}R_{r}$, dense in $\hom_{S}(R, R)$. If $R^{*}$ is a regular intermediate ring of $R/S$, then for each finite subset $F$ of $R^{*}$ there exists a regular subring $S'$ of $R$ such that $R^{*} \supseteq S' \supseteq S[F]$ and $[S' : S]_{l} < \aleph_{0}$.

**Proof.** Let $R^{*} = \sum D^{*}e^{*}_{ij}$ and $V^{*} = V_{R}(R^{*}) = \sum U^{*}g^{*}_{pq}$, where $e^{*}_{ij}$'s, $g^{*}_{pq}$'s are matrix units of $R^{*}$, $V^{*}$ respectively and $D^{*} = V_{R^{*}}(\{e^{*}_{ij}\})$, $U^{*} = V_{R^{*}}(\{g^{*}_{pq}\})$ are division rings. Then, $R^{**} = R^{*}(\{g^{*}_{pq}\}) = \sum R^{*}g^{*}_{pq} = \sum D^{*}(e^{*}_{ij}g^{*}_{pq})$, where $(e^{*}_{ij}g^{*}_{pq})$'s form evidently a system of matrix units and $V^{*}_{R^{*}}(\{e^{*}_{ij}g^{*}_{pq}\}) = D^{*}$. And so, for an arbitrary finite subset $E$ of $R^{**}$, $S_{E} = S(\{e_{ij}\}, \{g_{pq}\})$ is an intermediate simple ring of $R^{**}/S$ left finite over $S$. As $R$ is $S_{E}$-R-completely reducible by Lemma 2, we shall denote by $n(E)$ the length of its composition series, which is evidently bounded with the capacity of $R$ and so finite. Now, we set $n(E_{0}) = \min n(E)$, where $E$ ranges over all the finite subsets of $R^{**}$. Then, $R = N_{0}^{*} \oplus \cdots \oplus N_{n(E_{0})}^{*}$ with $S_{E_{0}}^{*}$-R-irreducible $N_{i}^{*}$'s. Here, we assume that $E$ is an arbitrary finite subset of $R^{**}$ containing $E_{0}$ and $R = N_{1}^{*} \oplus \cdots \oplus N_{n(E_{0})}^{*}$ is a direct decomposition of $R$ into $S_{E_{0}}^{*}$-R-irreducible submodules. Since each $N_{i}$ is yet $S_{E_{0}}^{*}$-R-admissible, we have $n(E) \leq n(E_{0})$, whence it follows $n(E) = n(E_{0})$. Hence, we see that each $N_{i}$ is $S_{E_{0}}^{*}$-R-irreducible as well. And so, if $R = M_{1}^{*} \oplus \cdots \oplus M_{n}^{*}$ is the direct decomposition of the $S_{E_{0}}^{*}$-R-module $R$ into homogeneous components $M_{j}$'s, each $M_{j}$ is $S_{E_{0}}^{*}$-R-admissible. Noting
here that $E$ is an arbitrary finite subset of $R^{**}$ containing $E_{o}$, we see that $M_{j}$ is $R^{**}$-R-admissible. And then, as $V_{R}(R^{**})=U^{*}$ is a division ring, $t$ is to be 1, that is, $R$ is homogeneously completely reducible as an $S_{E_{o}}$-$R$-module. Further, by the same reason, $n(E)=n(E_{o})$ implies that $R$ is homogeneously completely reducible as an $S_{E}$-$R$-module. Accordingly, we see that $V_{R}(S_{E})$ is simpe. Evidently, $S'_{E}=V_{S_{E}}(\{g_{m}'s\})$ is a simple subring of $R^{*}=V_{R^{**}}(\{g_{m}'s\})$ containing $S$ and left finite over $S$. Recalling here that $S_{E}=\sum S'_{E}g_{m}'$, it will be easy to see that $V_{R}(S'_{E})=\sum V_{R}(S'_{E})g_{m}'$, which proves that $S'_{E}$ is a regular subring of $R$. Now, to be easily verified, $S'_{E_{o} \cup F}$ can be adopted as our $S'$ for any finite subset $F$ of $R^{*}$.

**Lemma 9.** Let $R'$ be locally finite over a regular subring $S$, $\mathfrak{G}R_{r}$ dense in $\text{Hom}_{S_{l}}(R, R)$, and let $V_{R}(V_{R}(S'))$ be simple for each regular intermediate ring $S'$ of $R/S$ with $[S':S]<\infty$. If $R'$ is a regular intermediate ring of $R/S$ such that $[H[R'] : H]<\infty$, then $H[R']$ is outer Galois and locally finite over $R'$ and $\mathfrak{G}(H[R']/R') \equiv \mathfrak{G}(H\cap R'/R')$ by the restriction map.

**Proof.** There exists a simple intermediate ring $S'$ of $R'/S$ such that $H[R']=H[S']$ and $[S':S]<\infty$. As $H'=V_{R}(V_{R}(S')) \equiv H[S'] \equiv S'$ and $H'$ is outer Galois and locally finite over $S'$ by Theorem 2, $H[S']$ is simple by [7, Theorem 1.1]. And, noting that $H\mathfrak{G}(S')=H$, Theorem 2 proves that $\mathfrak{G}(S')|H[S']$ is a Galois group of $H[S']/S'$, whence $\mathfrak{G}(S')|H[S']$ is dense in $\mathfrak{G}(H[S']/S')$ by Corollary 4. It follows therefore $H\mathfrak{G}(H[S']/S')=H$. Now, the required isomorphism will be given by the restriction map $\rho : \mathfrak{G}(H[R']/R') \ni \sigma \rightarrow \sigma|H \in \mathfrak{G}(H\cap R')$. Here, one should remark that $\mathfrak{G}(H[R']/R')$ is compact ([7, §1]) and $\mathfrak{G}(H[R']/R')|H$ is dense in $\mathfrak{G}(H\cap R')$ (Corollary 4).

Now, by the validity of Lemmas 8, 9, we can prove the following improvement of [7, Theorem 3.1].

**Theorem 6.** Let $R$ be locally finite over a regular subring $S$, $\mathfrak{G}R_{r}$ dense in $\text{Hom}_{S_{l}}(R, R)$, and let $V_{R}(V_{R}(S'))$ be simple for each regular intermediate ring $S'$ of $R/S$ with $[S':S]<\infty$. If $R'$ is a regular intermediate ring of $R/S$, and $H'$ an intermediate ring of $H/S$ such that $H'|S$ is Galois, then $H[R']$ is outer Galois and locally finite over $R'$, and $\mathfrak{G}(H'[R']/R') \equiv \mathfrak{G}(H'|H\cap R')$ by the restriction map.

**Proof.** As $\mathfrak{G}(H'/S)=\mathfrak{G}(H/S)|H'$ by [7, Corollary 3.9] and Corollary 4, it will suffice to prove our theorem for the case where $H'=H$. Let $R' = \sum D'e_{ij}$, where $D'=V_{R}(\{e_{ij}'s\})$ is a division ring. Then, Lemma 8 enables us to set $H[R'] = \cup R_{s}$, where $R_{s}=H[S_{s}]$ and $S_{s}$ runs over all the regular subrings of $R$ such that $R' \supseteq S_{s} \supseteq \{e_{ij}'s\}$ and $[S_{s}:S]<\infty$. As $[R_{s} : H]<\infty$ by [7, Lemma 3.2 (iii)], $R_{s}$ is simple by Lemma 9, whence so is $H[R'] = \cup R_{s}$.
by [7, Lemma 1.1]. Evidently, \( R' = R' \cap R \) is a simple ring. Moreover, \( V_R(S_\nu) \supseteq V_R(R') = V_R(S_\nu) \) shows that \( R' \) is a regular subring of \( R \). And, one will easily verify that \( R_\nu = H[R'] \), \( R' = \bigcup \nu R'_\nu \) and \( R'_\nu \cap H = R' \cap H \). Now, in virtue of Lemma 9, \( R_\nu \) is outer Galois and locally finite over \( R'_\nu \) and \( \mathfrak G(R_\nu/R'_\nu) \cong \mathfrak G(H[H \cap R'_\nu] = \mathfrak G(H[R' \cap H]) \) by the restriction map. Hence, for each \( \tau \in \mathfrak G(H[R' \cap H]) \) there exists a uniquely determined extension \( \tau \in \mathfrak G(R_\nu/R'_\nu) \) of \( \tau \). Accordingly, one will easily see that if \( R_\nu \subseteq R \), then \( \mathfrak G(R_\nu/R'_\nu)|R_\mu = \mathfrak G(R_\nu/R'_\nu)|H[R'_\mu]\). By the light of this fact, we can define an \( R' \)-(ring) automorphism \( \hat{\tau} \) of \( H[R'] \) by \( \hat{\tau}|R_\nu = \tau_\nu \). We set here \( \hat{\mathfrak G} = \{ \hat{\tau}; \tau \in \mathfrak G(H[R' \cap H]) \} \). Evidently, \( \hat{\mathfrak G} \) forms a group and \( \hat{\mathfrak G}|R_\nu = \mathfrak G(R_\nu/R'_\nu) \). And so, there holds \( J(\hat{\mathfrak G}, H[R']) = \bigcup \nu J(\mathfrak G, R_\nu, R'_\nu) = \bigcup \nu R'_\nu = R' \), which means that \( H[R']/R' \) is outer Galois. Further, to be easily seen \( \hat{\mathfrak G} \) is locally finite (cf. [10, (a*)]). And then, the method used in the proof of [10, (a*]) enables us to see that \( H[R']/R' \) is locally finite. It follows therefore that \( \hat{\mathfrak G} \) is dense in \( \mathfrak G(H[R']/R') \) (Corollary 4), whence it will be easy to see that \( H\mathfrak G(H[R']/R') = H \). Consequently, there holds \( \mathfrak G(H[R']/R')|H = \mathfrak G(H[R' \cap H]) \). Now, the rest of the proof will be easy (cf. the proof of Lemma 9).

Combining Theorem 6 with [5, Corollary 1.4], one will readily obtain

**Corollary 6.** Let \( R \) and \( S \) satisfy the assumptions cited in Theorem 6. If \( R' \) is a regular intermediate ring of \( R/S \), then \( H^* = H^*[R'] \cap H \) for each intermediate ring \( H^* \) of \( H[R' \cap H] \) and \( R^* = (R^* \cap H)[R'] \) for each intermediate ring \( R^* \) of \( H[R']/R' \).

**References**


Some Theorems on Galois Theory of Simple Rings


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