ON HAAR FUNCTIONS IN THE SPACE $L_{M(\xi,t)}$

By

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1. It is well known [6, 8, 12] that Haar functions constitute a (Schauder) basis in Banach spaces $L^{p}[0,1]$ ($1 \leq p < +\infty$) and Orlicz spaces $L_{M}[0,1]$ with the $A_{2}$-condition. Generalizing this fact to an arbitrary separable Banach function space $E$ on a measure space, H. W. Ellis and I. Halperin showed in [3] that Haar system of functions (in an extended sense) composes a basis in $E$, if a norm of $E$ satisfies a condition called levelling length property). Although this condition is sufficiently general, yet it is not always a necessary one.

In this note we shall show a sufficient condition in order that Haar functions be a basis for the Banach function space $L_{M(\xi,t)}[0,1]$ or $L^{p(t)}[0,1]$. In fact, we shall establish, as for the space $L^{p(t)}$, that if $p(t)$ satisfies the Lipschitz $\alpha$-condition ($0 < \alpha \leq 1$) then Haar functions constitute a basis in $L^{p(t)}$ (Theorem 4). As a matter of course, the norms of these spaces do not satisfy the above condition given in [3] except some special cases.

In 2 we shall introduce Haar functions, the function spaces $L_{M(\xi,t)}$ and $L^{p(t)}$ with the notations used here. The main theorems shall be stated in 3, and some remarks shall be presented in 4.

2. A sequence of functions defined on $[0,1]$ : $\{\chi_{\nu}(t)\}_{\nu=1}^{\infty}$ is called a system of Haar functions, if $\chi_{1}(t)=1$ for all $t \in [0,1]$ and for $\nu=2^{n}+k$ ($n=0,1,2,\cdots ; k=1,2,\cdots,2^{n})^{1}$.

$$\chi_{\nu}(t) = \chi_{2^{n}+k}(t) = \begin{cases} \sqrt{2^{n}} & \text{for } t \in \left[ \frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}} \right), \\ \sqrt{-2^{n}} & \text{for } t \in \left( \frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}} \right], \\ 0 & \text{otherwise in } [0,1]. \end{cases}$$

1) A norm $\|\cdot\|$ of $E$ is called to have the levelling length property, if $\|f_{e}\| \leq \|f\|$ holds for any $f \in E$ and measurable set $e$, where $f_{e}$ coincides with $f$ outside the $e$ and on $e$, $f_{e} = \left\{ \frac{1}{d(e)} \int_{e} f(t) dt \right\} C_{e}$ ($C_{e}$ is the characteristic function of $e$). This property was first discussed by them in the earlier paper [4]. At the same time, G. G. Lorentz and D. G. Wertheim also found it independently and named it the average invariant property [9].

2) In the sequel, we eliminate $[0,1]$ and write simply $L_{M(\xi,t)}$ (or $L^{p(t)}$) in place of $L_{M(\xi,t)}[0,1]$ (resp. $L^{p(t)}[0,1]$). $L^{p(t)}$ was first discussed by W. Orlicz in [11], and was investigated precisely by H. Nakano [10].

3) This formulation of Haar functions is due to Z. Ciesielskii [2].
For any $a(t) \in L^1[0,1]$ we denote by $S_n(a) = S_n(a; t) \ (n = 1, 2, \cdots)$ the $n$-th partial sum of Haar Fourier series:

\[(2.2) \quad S_n(a; t) = \sum_{\nu=1}^{n} \alpha_{\nu} \chi_{\nu}(t), \quad \text{where} \quad \alpha_{\nu} = \int_{0}^{1} a(t) \chi_{\nu}(t) \, dt.\]

It is a well-known fact that if $a(t)$ is continuous on $[0,1]$, $S_n(a; t)$ converges uniformly to $a(t)$ and

\[(2.3) \quad S_n(a; t) = \left\{ 2^n \int_{(k-1)/2^n}^{k/2^n} a(s) \, ds \right\} \quad (t \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)) \]

holds for each $n \geq 0$, $k = 1, 2, \cdots, 2^n$.

Now let $M(\xi, t) \ (\xi \geq 0, 0 \leq t \leq 1)$ be a convex function of $\xi \geq 0$ for each $t \in [0,1]$ and a Lebesgue measurable function of $t \in [0,1]$ for each $\xi \geq 0$ with the following properties:

M. 1) $M(0,t) = 0 \quad \text{for a.e.} \quad t \in [0,1]$;
M. 2) $\lim_{\xi \to 0^+} M(\xi, t) = M(\alpha, t) \quad \text{for a.e.} \quad t \in [0,1]$ and each $\alpha > 0$;
M. 3) $\lim_{\xi \to +\infty} M(\xi, t) = +\infty \quad \text{for a.e.} \quad t \in [0,1]$;
M. 4) for any $t \in [0,1]$ there exists $\alpha_t > 0$ such that $M(\alpha_t, t) < +\infty$.

We denote by $L_{M(\xi,t)}$ the set of all measurable functions $x(t)$ satisfying

\[\int_{0}^{1} M(|x(t)|, t) \, dt < +\infty \quad \text{for some} \quad \alpha = \alpha(x) > 0.\]

Then $L_{M(\xi,t)}$ is called a modulared function space and is considered as a modulared semi-ordered linear space [5,10] with a modular $m:

\[(2.4) \quad m(x) = \int_{0}^{1} M(|x(t)|, t) \, dt \quad (x \in L_{M(\xi,t)}),\]

where $0 \leq x, \ x \in L_{M(\xi,t)}$ means that $x(t) \geq 0$ a.e. in $[0,1]$. Furthermore $L_{M(\xi,t)}$ is a Banach space with a norm $\| \cdot \|$ defined by the modular:

\[(2.5) \quad \|x\| = \inf_{m(\xi(x)) \leq 1} \frac{1}{|\xi|} \quad (x \in L_{M(\xi,t)}),\]

and $L_{M(\xi,t)}$ is separable if and only if $m(x) < +\infty$ for every $x \in L_{M(\xi,t)}$, which is also equivalent to the fact that $M(\xi, t)$ satisfies the generalized $\Delta_2$-condition [5,7], i.e.

$(\Delta_2)$ there exist a positive number $\gamma > 0$ and $0 \leq a \in L^1[0,1]$ such that

\[(2.6) \quad M(2\xi, t) \leq \gamma M(\xi, t) + a(t) \quad \text{for all} \quad \xi \geq 0 \text{ and a.e.} \ t \in [0,1].\]

4) This norm is called the modular norm by the modular $m$. In the sequel, we consider $L_{M(\xi,t)}$ with this norm.
This norm $\| \cdot \|$, as is easily seen, does not satisfy the levelling length property in general\(^5\). If there exists a convex function $M(\xi)$ such that $M(\xi, t) = M(\xi)$ holds for every $\xi \geqslant 0$ and a.e. $t \in [0, 1]$, then $L_{M(\xi, t)}$ is nothing but an Orlicz space $L_M$, and if $M(\xi, t) = \xi^p(t)$ holds, where $p(t)$ is a measurable function with $1 \leq p(t) (t \in [0, 1])$, $L_{M(\xi, t)}$ is denoted by $L^p(\xi)$ [10, 11]. $L^p(\xi)$ is separable if and only if $p(t)$ is bounded: $p(t) \leq K$ for a.e. $t \in [0, 1]$ and a constant $K > 0$.

3. For a system of convex $N$-functions $\{ \phi_i(\xi) \}_{\xi \geqslant 0}$, there exist always the join (the least upper bound function) and the meet (the greatest lower bound function) as a convex function in the family $F$ of positive convex functions. In fact, put $\Phi(\xi) = \sup_{\xi \geqslant 0} \phi_i(\xi)$, then $\Phi$ is a convex function which is the join of $\{ \phi_i \}_{\xi \geqslant 0}$ in $F$ (it is possible that $\Phi(\xi) = + \infty$ may hold for each $\xi > 0$). Here we denote this join of $\{ \phi_i \}_{\xi \geqslant 0}$ in $F$ by $\text{conv-} \cup \phi_i$. As for the meet, put $\Psi = \text{conv-} \cap \phi_i$, where $\phi_i(\xi)$ is the complementary function to $\phi_i(\lambda \epsilon \Lambda)$ in the sense of H. W. Young, and further let $\Phi_0$ be the complementary function to $\Psi$ in the above sense too, then $\Phi_0$ comes to be a convex function which is the meet of $\{ \phi_i \}_{\xi \geqslant 0}$ in $F$ (it is possible also that $\Phi_0(\xi) = 0$ may hold for every $\xi \geqslant 0$). We denote the meet of $\{ \phi_i \}_{\xi \geqslant 0}$ in $F$ by $\text{conv-} \cap \phi_i$ as well. From the definitions, it is clear that $\text{conv-} \cap \phi_i(t) \leq \phi_i(t) \leq \text{conv-} \cup \phi_i(t)$ holds for each $t \geqslant 0$ and $\lambda \epsilon \Lambda$.

Now let $L_{M(\xi, t)}$ be a modulated function space. Since $M(\xi, t)$ is convex $N$-functions of $\xi \geqslant 0$ for all $t \in [0, 1]$ by M. 1)—M. 4) in 2, we can define convex functions $\bar{M}_{n,k}(\xi)$ and $\underline{M}_{n,k}(\xi)$ as follows:

\begin{align}
(3.1) \quad & \bar{M}_{n,k}(\xi) = \text{conv-} \cup_{t \in I_{n,k}} M(\xi, t) \leq (\xi \geqslant 0), \\
(3.2) \quad & \underline{M}_{n,k}(\xi) = \text{conv-} \cap_{t \in I_{n,k}} M(\xi, t) \leq (\xi \geqslant 0),
\end{align}

where $I_{n,k} = \left( \frac{k - 1}{2^n}, \frac{k}{2^n} \right)$ ($n = 0, 1, 2, \cdots$; $k = 1, 2, \cdots, 2^n$). We put also

\begin{equation}
(3.3) \quad \omega_n = \max_{k=1,2,\cdots,2^n} \left\{ \frac{\bar{M}_{n,k}(2^n)}{\underline{M}_{n,k}(2^n)} \right\} \quad (n = 1, 2, \cdots),
\end{equation}

if it has a sense.

With these preparations, we have

**Theorem 1.** Haar functions $\{ \lambda_n \}_{n=1}^\infty$ constitute a basis for a modulated function space $L_{M(\xi, t)}$ if $M(\xi, t)$ satisfies the following conditions:

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5) Indeed, we can show that if the norm $\| \cdot \|$ on $L_{M(\xi, t)}$ fulfils the requirement of the levelling length property, $L_{M(\xi, t)}$ reduces to an Orlicz space $L_M$.

6) Since $M(\xi, t)$ satisfies M. 1)—M. 4), $M(\xi, t)$ is considered as convex $N$-functions for a.e. $t \in [0, 1]$. 

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C. 1) the $\Delta_2$-condition in 2 holds true for $M(\xi, t)$;
C. 2) there exists a positive number $\delta$ such that \( \inf_{t \in [0,1]} M(\delta, t) \geq 1 \);
C. 3) there exists a positive number $\kappa$ such that $\lim_{n \to \infty} \omega_n \leq \kappa$.

Before entering into the proof of Theorem 1, we first prove the auxiliary lemmas.

Lemma 1. If $M(\xi, t)$ satisfies C. 2), then $\|x\| \leq 1 \ (x \in L_M(\xi,t))$ implies

$$\int_0^1 |x(t)| dt \leq 2\delta,$$ hence $L_M(\xi,t) \subseteq L^1$.

Proof. If $\|x\| \leq 1 \ (x \in L_M(\xi,t))$, then the formulas (2.4) and (2.5) imply $m(x) \leq 1$. Hence we get

$$1 \geq \int_0^1 M(|x(t)|, t) dt = \int_{e_\delta} M(|x(t)|, t) dt + \int_{e_\delta} M(|x(t)|, t) dt \geq \frac{1}{\delta} \int_{e_\delta} |x(t)| dt,$$

where $e_\delta = \{ t : |x(t)| \geq \delta \}$ and $e_\delta'$ is the complement of $e_\delta$, since $M(\xi, t) \geq \frac{\xi}{\delta} M(\delta, t) \geq \frac{\xi}{\delta}$ holds for every $\xi$ with $\xi \geq \delta > 0$ and a.e. $t \in [0,1]$ by virtue of convexity of $M(\xi, t)$ and C. 2). Therefore we obtain $\int_0^1 |x(t)| dt \leq 2\delta$. Q.E.D.

Lemma 2. If $M(\xi, t)$ satisfies C. 3), then 1(1$=1$ for all $t \in [0,1]$) belongs to $L_M(\xi,t)$.

Proof. As $\Max_{k=1, \ldots, 2^n} \{ \overline{M}_{n, k}(2^n) \} = \omega_n$, we have for some $n$

$$\overline{M}_{1, i}(1) \leq \Max_{k=1, \ldots, 2^n} \{ \overline{M}_{n, k}(2^n) \} < +\infty \quad (i=1, 2),$$

whence

$$m(1) = \int_0^1 M(1, t) dt \leq \int_{1/2}^{1} \overline{M}_{1, i}(1) dt + \int_{1/2}^{1/2} \overline{M}_{1, i}(1) dt < +\infty.$$ 

which implies $1 \in L_M(\xi,t)$. Q.E.D.

Proof of Theorem 1: Putting for $n=1, 2, \ldots; k=1, 2, \ldots, 2^n$

$$T_n x = 2^n \left( \int_{I_{n,k}} x(t) dt \right) c_n^k \quad (x \in L_M(\xi,t)),$$

where $c_n^k$ is the characteristic function of the interval $I_{n,k} = (\frac{k-1}{2^n}, \frac{k}{2^n})$, we

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7) In the definition of $\omega_n$, we can substitute $\conv_{\xi} \cup M(\xi, t)$ (or $\conv_{\xi} \cap M(\xi, t)$) by $\conv_{\xi} \cup M(\xi, t)$ (resp. $\conv_{\xi} \cap M(\xi, t)$) in the formulae (3.1) and (3.2), where $e$ is a set of measure zero, as the proof shows below.
obtain by Lemma 2 a linear operator $T_n^k$ of $L_{M(\sigma,t)}$ into itself for each $n, k$. Then (2.3) can be written as

\begin{equation}
(3.4)
S_{2^n}(x) = \sum_{k=1}^{2^n} T_n^k x.
\end{equation}

Now let $x \in L_{M(\sigma,t)}$ with $\|x\| \leq 1$. From Lemma 1 we have $|(T_n^k x)(t)| \leq \delta 2^{n+1} c_n^k (t \in [0,1])$. According to C.3) we can find a natural number $n_0$ such that $n \geq n_0$ implies $\omega_n \leq 2k$. Then we get for any $n$ with $n > n_0$ and any $k$ $(1 \leq k \leq 2^n)$

\begin{equation}
(3.5)
m\left(\frac{1}{2\delta} T_n^k x \right) \leq \int_{I_{n,k}} \overline{M}_{n,k} \left( \frac{1}{2\delta} |(T_n^k x)(t)| \right) dt
\end{equation}

\begin{equation}
\leq \max \left\{ 2k \int_{I_{n,k}} M_{n,k} \left( \frac{1}{\delta} |(T_n^k x)(t)| \right) dt, \; m(2^n c_n^k) \right\}.
\end{equation}

Because, if $2^n \leq 2^v < \frac{1}{2\delta} |(T_n^k x)(t)| \leq 2^{v+1} \leq 2^n$ $(t \in I_{n,k})$ holds, it follows from the definitions (3.1), (3.2) of $\overline{M}_{n,k}$ and $M_{n,k}$ that

\begin{equation}
\overline{M}_{n,k} \left( \frac{1}{2\delta} |(T_n^k x)(t)| \right) \leq \overline{M}_{v+1,k} \left( 2^{v+1} \right) \leq 2k \overline{M}_{v+1,k} \left( 2^{v+1} \right)
\end{equation}

\begin{equation}
\leq 2k M_{v+1,k'} \left( 2 \cdot \frac{1}{2\delta} |(T_n^k x)(t)| \right) \leq 2k M_{n,k} \left( \frac{1}{\delta} |(T_n^k x)(t)| \right),
\end{equation}

where $k'$ is a suitable natural number such that $I_{v+1,k'} \supset I_{n,k}$.

Now, applying the Jensen's inequality to the last term of (3.5), we get

\begin{equation}
m\left(\frac{1}{2\delta} T_n^k x \right) \leq \max \left\{ 2k \int_{I_{n,k}} M_{n,k} \left( \frac{1}{\delta} |x(t)| \right) dt, \; m(2^n c_n^k) \right\}
\end{equation}

\begin{equation}
\leq 2k \int_{I_{n,k}} M \left( \frac{1}{\delta} |x(t)|, t \right) dt + m(2^n c_n^k) \leq 2km \left( \frac{1}{\delta} x \cdot c_n^k \right) + m(2^n c_n^k).
\end{equation}

Consequently (3.4) gives for $n > n_0$

\begin{equation}
m\left(\frac{1}{2\delta} S_{2^n}(x) \right) = m \left( \frac{1}{2\delta} \sum_{k=1}^{2^n} T_n^k x \right) = \sum_{k=1}^{2^n} m \left( \frac{1}{2\delta} T_n^k x \right)
\end{equation}

\begin{equation}
\leq 2k \sum_{k=1}^{2^n} m \left( \frac{1}{\delta} x \cdot c_n^k \right) + \sum_{k=1}^{2^n} m(2^n c_n^k) = 2km \left( \frac{1}{\delta} x \right) + m(2^n 1).
\end{equation}

Therefore, $\|x\| \leq 1$ implies $m \left(\frac{1}{2\delta} S_{2^n}(x) \right) \leq k' (n > n_0)^8$ for a suitable chosen $k' > 0$, which also shows sup $\|S_{2^n}x\| < +\infty$.

8) Since $M(\xi, t)$ satisfies the $\Delta_2$-condition, we have sup $m \left( \frac{1}{\delta} x \right) < +\infty$. 

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As the result of the above, we can see directly that the operator norms of $S_\nu$ ($\nu=1,2,\cdots$) are uniformly bounded. Since the set of all continuous functions is dense in $L_{(\xi,t)}$, in case $M(\xi,t)$ satisfies the $(\Delta_2)$-condition, and uniform convergence implies the norm convergence in $L_{(\xi,t)}$, our assertion is obtained.

Q. E. D.

Next we shall replace the condition C.3) in Theorem 1 by a somewhat simpler one. For this purpose we define from $M(\xi,t)$

\[(3.6)\quad L(\xi,t) = \log M(\xi,t)/\log \xi ,\]

if $M(\xi,t)$ and $\xi$ are both greater than 1, and

$L(\xi,t) = 0$

otherwise, where $t\in[0,1]$ and $\xi \geq 0$.

Using $L(\xi,t)$ we shall prove

**Theorem 2.** Suppose that C.1) and C.2) hold for $M(\xi,t)$. If $L(\xi,t)$ (defined by (3.6) from $M(\xi,t)$) satisfies the Lipschitz $\alpha$-condition ($0<\alpha \leq 1$) with a constant $\gamma > 0$ for all $\xi \geq \xi_0$, i.e.

C.3') $|L(\xi,t) - L(\xi,t')| \leq \gamma |t-t'|^\alpha$ for all $t,t' \in [0,1]$, $\xi \geq \xi_0$, where $\alpha, \gamma$ and $\xi_0 \geq 0$ are all certain fixed constants.

Then Haar functions $\{\chi_\nu(t)\}_{\nu=1}^{\infty}$ constitute a basis in $L_{(\xi,t)}$.

**Proof.** It follows by C.2) $L(\xi,t) = \log M(\xi,t)/\log \xi$ for all $\xi$ with $\xi \geq \text{Max}(\delta,1)$, which implies also for sufficiently large $\xi_0 > 1$

\[(3.7)\quad M(\xi,t) \leq \xi^{|t-t'|^\alpha} M(\xi,t') \quad (t,t' \in [0,1], \xi \geq \xi_0)\]

by virtue of C.3').

Let $n_0$ be a natural number such that both $n_0 \gamma 2^{n_0} \leq 1$ and $2^{n_0} > \xi_0$ hold. Then for any $n > n_0$, the inequality (3.7) gives $M(\xi,t) \leq \xi^{\frac{|t-t'|^\alpha}{2^{n_0}}} M(\xi,t')$ for all $t,t' \in I_{n,k}$ and $\xi \geq \xi_0$, where $k=1,2,\cdots,2^n$. Recalling the definition of $\overline{M}_{n,k}(\xi)$, we obtain for every $t,t' \in I_{n,k}$ and $\xi \geq \xi_0$

\[(3.8)\quad M(\xi,t') \leq \overline{M}_{n,k}(\xi) \leq \xi^{\frac{|t-t'|^\alpha}{2^{n_0}}} M(\xi,t) .\]

Now we put

9) In view of the proof of Theorem 2, we can see that Theorem 2 remains to be true if we replace C.3') by a somewhat general condition: C.3'') $|L(\xi,t) - L(\xi,t')| \leq \omega(|t-t'|)$ ($\xi \geq \xi_0$, $t,t' \in [0,1]$), where $\omega(\delta)$ is a function defined on $[0,\infty)$ satisfying $\varlimsup_{\delta \to 0} (\frac{1}{\delta})^{w(\delta)} < +\infty$. 

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(3.9) \[ \beta_{n}^{k} = \text{ess. inf}_{t \in I_{n,k}} \varphi(2^{n}, t), \]

where \[ \varphi(2^{n}, t) = \lim_{\epsilon \to 0} \frac{M(2^{n}, t) - M(2^{n} - \epsilon, t)}{\epsilon} \quad \text{for} \quad t \in I_{n,k}, \quad k = 1, 2, \ldots, 2^{n}. \]

From the condition C.2 and the fact that $M(\xi, t)$ is a convex function for each $t \in [0,1]$, we have $\beta_{n}^{k} > 0$ for every $n > n_{0}$ and $1 \leq k \leq 2^{n}$. Thus, for each $n, k$ ($n > n_{0}, 1 \leq k \leq 2^{n}$) there exists $t_{0} \in I_{n,k}$ such that

\[ \frac{1}{2} \varphi(2^{n}, t_{0}) \leq \beta_{n}^{k}. \]

For this $t_{0}$, we put

(3.10) \[ \phi_{n,k}(\xi, t) = \begin{cases} \frac{1}{2} M(\xi, t_{0}) & \text{for } 0 \leq \xi \leq 2^{n}; \\ \varphi(2^{n}, t_{0}) (\xi - 2^{n}) + \frac{M(2^{n}, t_{0})}{2}, & \text{for } 2^{n} \leq \xi. \end{cases} \]

Then, according to (3.8), $\phi_{n,k}(\xi, t)$ is a convex N-function satisfying for each $\xi \geq 2^{n}$

(3.11) \[ \phi_{n,k}(\xi) \leq M(\xi, t) \quad \text{for a.e. } t \in I_{n,k} \]

and

(3.12) \[ \overline{M}_{n,k}(\xi) \leq 4 \phi_{n,k}(\xi) \quad \text{for all } 2^{n} \leq \xi \geq 2^{n_{0}}, \]

because, $2^{n_{0}} \leq 2^{\frac{m_{n}}{2^{n}}} \leq 2$ holds for $\xi \leq 2^{n}$.

Then, substituting $M_{n,k}$ in the proof of Theorem 1 by $\phi_{n,k}$, we can prove by (3.11) and (3.12) that

\[ m\left(\frac{1}{2\alpha}S_{2n}x\right) \leq 4m(2^{n}1) + 4m\left(\frac{1}{2\delta}x\right) \]

holds for all $n \geq n_{0}$, hence sup $\|S_{2n}x\| < +\infty$ ($n > n_{0}$), in the quite same way. From this the proof is immediately established.

Q. E. D.

For the $L^{p(t)}$ spaces ($1 \leq p(t)$ for a.e. $t \in [0,1]$), the matters in question come to be quite simple. In this case, $M(\xi, t) = \xi^{p(t)}$ satisfies C.2 always, and C.1 is also implied from the condition corresponding to C.3) or C.3'). In fact, we obtain

**Theorem 3.** Haar functions $\{\chi_{\nu}(t)\}_{\nu=1}^{\infty}$ constitute a basis in $L^{p(t)}$, if

(3.4) \[ \lim_{\delta \to 0} \omega(\delta) \log \frac{1}{\delta} < +\infty \]

holds, where $\omega(\delta) = \text{ess. sup}_{t, t' \in [0,1], |t - t'| \leq \delta} |p(t) - p(t')|$. Furthermore from Theorem 2 we have
Theorem 4. If \( p(t) \) satisfies the Lipschitz \( \alpha \)-condition \( (0 < \alpha \leq 1) \), then Haar functions \( \{ \chi_{\nu}(t) \}_{\nu=1}^{\infty} \), constitute a basis in \( L^{p(t)} \).

In these cases, the conditions C.3) and C.3') hold respectively, whence C.1) is also necessarily fulfilled, as easily verified. Therefore the assertion is obtained from Theorem 1 and 2.

4. In this section some remarks concerning the theorems in 3 shall be presented.

Remark 1. The condition C.2) in Theorems 1 and 2 can not be erased. Indeed, in case \( M(\xi, t) \) satisfies only C.1) and C.3), it may occur, as an easy example shows, that \( \alpha_{\nu} = \int_{0}^{1} a(t) \chi_{\nu}(t) dt = +\infty \) holds for some \( \nu \) (hence necessarily for an infinite number of \( \nu \)), where \( a(t) \) is an element of \( L_{M(\xi, t)} \).

Remark 2. Theorems 3 and 4 have the direct extensions, without adding any assumption, to the following special modular function spaces: \( L_{M^{p}} \) or \( L_{p(M)} \), where \( M \) and \( p^{M} \) are defined such that

\[
M^{p}(\xi, t) = M(\xi)^{p(t)} \quad \text{and} \quad p^{M}(\xi, t) = M(\xi^{p(t)})
\]

hold respectively for all \( \xi \geq 0 \) and \( t \in [0,1] \) with a convex N-function \( M(\xi) \) satisfying the \( \Delta_{2} \)-condition and \( p(\cdot) \geq 1 \).

For the proof based on Theorems 1 and 2, we only note here that if a convex N-function \( M(\xi) \) satisfy the \( \Delta_{2} \)-condition \( M(\xi) \leq r \xi^{p} \) holds \( (\xi \geq \xi_{0}) \) for some \( r > 0 \), \( p \geq 1 \) and \( \xi_{0} \geq 0 \).

Remark 3. In Theorems 3 and 4 we can not weaken the assumption by the continuity of \( p(t) \) without failing to hold the validity, as the following example shows.

Example: Let \( \{ \nu_{i} \}_{i=1}^{\infty} \) be a sequence of natural numbers such that and \( (2^{\nu_{i}+1})^{\frac{1}{2^{i+1}}} > i \) and \( \nu_{1} < \nu_{2} \cdots \) for all \( i \geq 1 \), and \( I_{i} = \left( \frac{1}{2^{\nu_{i}+1}}, \frac{1}{2^{\nu_{i}}} \right) \).

Now we put

\[
(4.2) \quad p_{0}(t) = \begin{cases} 
1 + \frac{1}{2^{n}} & \text{for } t \in \left[ \frac{1}{2^{n}}, \frac{1}{3 \cdot 2^{n+1}}, \frac{1}{2^{n}} \right]; \\
1 + \frac{1}{2^{n+1}} & \text{for } t \in \left( \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{3 \cdot 2^{n+1}} \right]; \\
\text{linear} & \text{otherwise.}
\end{cases}
\]

for all \( n \geq 1 \), and also

\[
(4.3) \quad f_{n}(t) = \begin{cases} 
\beta_{n} & \text{for } t \in \left( \frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{3 \cdot 2^{n+1}} \right]; \\
0 & \text{otherwise},
\end{cases}
\]
On Haar functions in the space $L_{M(\xi,t)}$

where $\beta_n = \frac{3 \cdot 2^{n+1}}{2^{n+1} + 1}$ for all $n \geq 1$. Then $p_0(t)$ is continuous, $1 \leq p_0(t) \leq 2$ for all $t \in [0,1]$, $f_n \in L^{p_0(t)}$ and $\|f_n\| = 1$ for all $n \geq 1$.

Now, suppose that the set of Haar functions be a basis for the space $L^{p_0(t)}$, and by virtue of the Banach’s Theorem [1], the norms $\|S_n\|$ ($n \geq 1$) must be bounded from above. On the other hand, we can deduce without difficulty that for a sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ suitable chosen we have $m(S_{k_n}f_n) \geq \sqrt{n}/9$ for all $n \geq 1$, whence we have $\sup_{n \geq 1} \|S_nf_n\| = \sup_{n \geq 1} \|S_n\| = +\infty$. Therefore we obtain a contradiction. Consequently, we conclude that Haar functions $\{\chi_n(t)\}_{n=1}^{\infty}$ do not compose a basis in the space $L^{p_0(t)}$ thus constructed.

References


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