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<th>CORRESPONDENCE OF BOUNDARIES OF RIEMANN SURFACFS</th>
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<tr>
<td>Author(s)</td>
<td>Kuramochi, Zenjiro</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要 17(3-4): 096-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1963</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56039">http://hdl.handle.net/2115/56039</a></td>
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<td>Type</td>
<td>bulletin (article)</td>
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<td>File Information</td>
<td>JFSHIU_17_N3-4_096-122.pdf</td>
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CORRESPONDENCE OF BOUNDARIES OF RIEMANN SURFACES

By

Zenjiro KURAMOCHI

Let $R$ be a Riemann surface with positive boundary and let $R_n$ ($n=0, 1, 2, \cdots$) be its exhaustion with compact relative boundary $\partial R_n$. Let $N(z, p)$ be a positive harmonic function in $R-R_0-\ p : p \in R-R_0$ such that $N(z, p)=0$ on $\partial R_0$, $N(z, p)$ has a logarithmic singularity at $p$ and $N(z, p)$ has minimal $D^\ast$irichlet integral over $R-R_0$, where $D^\ast$irichlet integral is taken with respect to $N(z, p)+\log|z-p|$ in a neighbourhood of $p$. We call such $N(z, p)$ the $N$-Green's function of $R-R_0$ with pole at $p$.

Consider now a sequence of points $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$. Since the functions $N(z, p_i)$ ($i=1, 2, \cdots$) forms, from some $i$ on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of $R-R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R-R_0$ having no points of accumulation in $R-R_0+\partial R_0$, for which the corresponding $N(z, p_i)$'s have the property just mentioned, that is, $\{N(z, p_i)\}$ converges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z, p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of $R$. The set of all the ideal boundary points of $R$ will be denoted by $B$ and the set $R-R_0+B$ by $\overline{R}-R_0$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p)=\lim_{i}N(z, p_i)$ ($z \in R-R_0, p \in B$), where $\{p_i\}$ is any fundamental sequence determining $p$. The function $N(z, p)$ is characteristic of the point $p$ of their corresponding $N(z, p)$ as a function of $z$. The function $\delta(p_1, p_2)$ of two points $p_1$ and $p_2$ in $\overline{R}-R_0$ is defined as

$$\delta(p_1, p_2) = \sup_{z \in R_1-R_0} \left| \frac{N(z, p_1)}{1+N(z, p_1)} - \frac{N(z, p_2)}{1+N(z, p_2)} \right| .$$

Evidently, $\delta(p_1, p_2)=0$ is equivalent to $N(z, p_1)=N(z, p_2)$ for all points $z$ in $R_1-R_0$. Therefore we have $N(z, p_1)=N(z, p_2)$ in $\overline{R}-R_0$, i.e. $\delta(p_1, p_2)=0$ implies $p_1=p_2$ and it is clear that $\delta(p_1, p_2)$ satisfies the axioms of distance. Therefore $\delta(p_1, p_2)$ can be considered as the distance between $p_1$ and $p_2$ of $\overline{R}-R_0$.
The topology (we call N-Martin’s topology) induced by this metric is homeomorphic to the original topology, when it is restricted in $R - R_0$ and we continue this topology into $R_0$. Since $\{N(z, p_i) : p_i \in \bar{R} - R_0\}$ is also a normal family, both $\bar{R} - R_0 + \partial R + B$ and $B$ are closed and compact. For a fixed point $z$, $N(z, p)$ is continuous with respect to this metric as a function of $p$ in $\bar{R} - R_0$ except at $p$. Other topologies can be introduced on any Riemann surfaces (with null or positive boundary). For example K-Martin’s topology, Green’s topology, and Stoilow’s topology. We proved that N-Martin’s, and Stoilow’s and Green’s topologies are Dirichlet-seperative. We define the ideal boundary $B$ of $R$ by the completion of $R$. Then $\bar{R} + B$ is compact with respect to N-Martin’s and Stoilow’s topology.

We use in the present paper the notions of capacity, superharmonic functions, $N$-minimal points and others which are referred to the papers\(^{1}\). Suppose N-Martin’s topology is introduced on $\bar{R} = R + B$. Let $G$ be a domain and $N(z, p)$ be the least positive superharmonic function in $R - R_0$ larger than $N(z, p)$ on $CG$ (complementary set of $G$), where $p \in B^n$ ($B^n$ is the set of $N$-minimal boundary points). If $N(z, p) > \alpha_0 N(z, p)$, we say that $G$ contains $p$ $N$-approximately and denote it by $G \ni p$.

Let $w = f(z) : z \in R$, $w \in R$ be an analytic function whose values fall on a basic surface $R$, where $R$ is a Riemann surface with positive or null-boundary and a topology is defined on $R + B$. Put

$$M(p) = \bigcap G(G), \text{ for } p \in B^n,$$

where $\{G\}$ means the all sets $G$, such that $G \ni p$.

1). For any point $p \in R$, there exists an open disc $C(p)$ in $R$ such that the area of $R$ over $C(p)$ is finite.

2). There exists a number $n_0$ such that $n(w) \leq M < \infty$ in $R - R_{n}$, where $n(w)$ is the number of points of $R$ over $p$ and $\{R_{n}\}$ is an exhaustion of $R$ with compact relative boundary $\partial R_{n}$.

If the above two conditions are satisfied, we say that $w = f(z)$ is of almost finitely sheeted. Then we proved

**Theorem 1\(^{2}\).** Let $R$ be a covering surface with positive boundary and

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with N-Martin's topology over $\mathbb{R}$ with D. S. (Dirichlet separative\(^ {9} \)) topology and suppose $\mathbb{R}+B$ is compact with respect to the topology on $\mathbb{R}$. If $\mathcal{R}$ is a covering surface of almost finitely sheeted: $w=f(z), z \in \mathcal{R}$ and $w \in \mathbb{R}$. Then $M(p)$ is defined in $\mathcal{B}$ except an $F_\alpha$ set of capacity zero and $\mathcal{S}=\mathcal{E}[p \in B_1^N: \text{diameter of } M(p)>0]$ is a $G_\delta$ set of inner capacity zero, where $B_1^N$ is the set of $N$-minimal boundary points of $\mathcal{R}$.

This is an extension of Beurling’s theorem\(^ {9} \).

Let $w=f(z)$ be an analytic function in $|z|<1$ and let $\mathcal{R}$ be the Riemann surface, generated by $w=f(z)$ over the $\omega$-sphere $K$. Let $a$ be any point of $\mathcal{K}$ and $\mathcal{R}_\alpha$ be the part of $\mathcal{R}$, which lies over a disc $C_r(a)$ of radius $r$ with $a$ as its centre and $A(r)$ be the spherical area of $\mathcal{R}_\alpha$. If

$$
\lim_{r \to 0} \frac{A(r)}{r^2} < \infty,
$$

then $a$ is called an ordinary value of $f(z)$ in Beurling's sense. Then

**Theorem 2\(^ {9} \) (A. Beurling).** Let $w=f(z)$ be an analytic function in $|z|<1: w \epsilon \omega$-sphere. Suppose the spherical area of the covering surface generated by $w=f(z)$ is finite. Let $a$ be an ordinary value of $f(z)$ in Beurling's sense. Then the set $\mathcal{E}$ of $e^{\alpha}$ such that $\lim_{r \to 1} f(re^{i\theta})=a$ is a set of capacity zero.

M. Tsuji\(^ {10} \) proved the above theorem under the condition that $f(z)$ takes three values $\alpha, \beta, \gamma$ only a finite number of times in $|z|<1$ instead of the condition that the spherical area of the image is finite.

The purpose of the present paper is to extend Theorem 2 to Riemann surfaces and to study the correspondence of boundaries of Riemann surfaces.

1). **Properties of domains\(^ {7} \) containing $p \in B_1^N$ $N$-approximately.**

We shall study the properties of a domain $G \supset p: p \in B_1^N$, some of which are useful in the following. We proved the following

**Lemma\(^ {9} \).** a). For $p \in B_1^N$ we have $1^\circ$. If $G_{i} \ni p, \cap_{i=1}^N G_{i} \ni p$. $2^\circ$. If $G_{i} \ni p, \cap_{i=1}^N G_{i} \ni p$. $3^\circ$. $E[z \in \mathcal{R}: \text{dist}(z, p)<\frac{1}{n}]=v_n(p) \ni p$. Since $\mathcal{R}+B$ is closed.

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4) See 1).

5) A. BEURLING : Ensembles exceptionnelles, Act Math., 72 (1940)


7) In the present paper we suppose for any domain $G$ in Riemann surfaces $\mathcal{R}$ that $\partial G$ consists of at most enumerably infinite number of analytic curves clustering nowhere in $\mathcal{R}$.

8) See 1) of 3) and 2).
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by 1) we have $M(p) = \cap_i N(G_i) \neq 0$, and since $R + B$ is separable with respect to the given topology on $R + B$, we can find a sequence $\{G_i\}_{i=1, 2, 3, \cdots}$ from $\{G_r\}$ such that $M(p) = \cap_i N(G_i)$. We see by 2) that there exists only one component $G'$ of $G$ such that $G' \supset p$. Let $G$ and $G'$ be domains such that $G \supset G'$ and $D(\omega(G' \cap B, z, G)) > 0$. Then there exists at least one point $p \in B_N^r$ such that $G' \ni p$, where $\omega(G' \cap B, z, G) = \lim_{n} \omega_{n,n+i} (z)$ and $\omega_{n,n+i} (z)$ is a harmonic function in $(G \cap R_{n+i}) - (G' \cap (R_{n+i} - R_n)) - R_n$ such that $\omega_{n,n+i} (z) = 0$ on $\partial G$, $\omega_{n,n+i} (z) = 1$ on $\partial (G' \cap (R_{n+i} - R_n))$, $\frac{\partial}{\partial n} \omega_{n,n+i} (z) = 0$ on $\partial R_{n+i} - G'$ and $D(\omega_{n,n+i} (z)) \leq M < \infty$ for a certain $n_0$ and for every $i$.

b). Let $G$ be a domain in $R$. Then the set of points $\in B_N^r$ not contained in $G$ N-approximately is a $G_\delta$ set, i.e. the set of points $\in B_N^r$ contained in $G$ is an $F_\sigma$ set.

It is known$^9$ that for any $V_M(p) = E(z \in R : N(z, p) > M) : M < M^* = \sup N(z, p)$, $p \in B_N^r$, there exists a $v_n(p)$ such that $(R \cap v_n(p)) \subset V_M(p)$. Hence we have by 1) of a)

c). For any $V_M(p) : M < M^*$ and for any domain $G' \ni p$ we can find a domain $G' (= G \cap v_n(p))$ such that $p \in G' \subset v_n(p) \subset V_M(p)$.

d). If $\omega(p, z) = \lim_{n} \omega(v_n(p), z) > 0 : p \in B_N^r$ (this is equivalent to sup $N(z, p) < \infty$), we call $p$ a singular point. Let $B_N^s$ be the set of singular points. Then $B_N^s$ is an $F_\sigma$ set and $B_N^s \subset B_N^r$. Then

for any $v_n(p)$ $\lim_{M=\infty} N(v_n(p)) = 0$ and $\lim_{M=\infty} \int_{\partial V_M(p) \cap C(v_n(p))} \frac{\partial}{\partial n} N(z, p) ds = 0$ for $p \in B_N^r - B_N^s$.

Since $\min (M, N(z, p)) = M \omega(V_M(p), z)$ and $\int_{\partial V_M(p) \cap C(v_n(p))} \frac{\partial}{\partial n} N(z, p) ds = 2\pi$ for almost $M < M^*$, d) means that the system $\{V_M(p)\} (M < M_2 < M_3 \cdots)$ is almost equivalent to the neighbourhood system $\{v_n(p)\}$. By c) we know $\{G_i\} : G_i \ni p$ is finer than $\{v_n(p)\}$. We shall prove the following theorem to show that $G$ has the similar properties as $\{v_n(p)\}$.

Theorem 3. Suppose $G \ni p : p \in B_N^r$. Then

1). $\omega^\infty (\omega(\alpha N(z, p)) = \lim_{n=\infty} v_n(p)(\omega N(z, p)) = 0$ for $p \in B_N^r - B_N^s$.

9) See p. 42 of P. and see 2).
2). \( p \cap \alpha N(z, p) = \lim_{n} v_{n}(p) \cap \alpha N(z, p) = 0 \) and \( \alpha \cap N(z, p) = \alpha N(z, p) = N(z, p) \) for \( p \in B_{1}^{N} \).

3). \( \lim_{M \rightarrow M^{*}} v_{M}(p) \cap \alpha N(z, p) = N(z, p) \) and \( \lim_{M \rightarrow M^{*}} \int_{\partial V_{M}(P) \setminus CG} \frac{\partial}{\partial n} N(z, p) ds = 0 \) for \( p \in B_{1}^{N} \), where \( M^{*} = \sup N(z, p) \).

By \( v_{n}(p) \ni p \) implies d) of the Lemma.

Proof of 1). Assume \( p(\alpha N(z, p)) > 0 \). Since \( p(\alpha N(z, p)) \) has mass only at \( p \), \( p(\alpha N(z, p)) = a \), \( N(z, p) \), where \( 1 \geq a > 0 \), by \( N(z, p) \geq \alpha N(z, p) \). Now \( p \) is a set of capacity zero and \( 0 \leq U(z) = \alpha N(z, p) - p(\alpha N(z, p)) \) is also superharmonic. Hence similarly as before \( U_{1}(z) = a_{1} N(z, p) \) and \( U_{i}(z) = a_{i} N(z, p) + U_{i+1}(z) \), \( i = 1, 2, 3, \ldots \).

Then \( a_{i} \geq a_{i+1} \) and \( \Sigma a_{n} \leq 1 \), \( \lim a_{n} = 0 \) and \( U_{n}(z) \downarrow \). Put \( U^{*}(z) = \lim U_{n}(z) \).

Then \( U^{*}(z) \) is also superharmonic and \( U^{*}(z) \leq a_{1} N(z, p) \) for any \( n \). Thus \( U^{*}(z) = 0 \) and \( N(z, p) = (\Sigma a_{n}) N(z, p) \).

Proof of 2). Case 1. \( p \in B_{1}^{N} - B_{S}^{N} \). In this case \( v_{n}(p) \cap \alpha N(z, p) = N(z, p) \) on \( v_{n}(p) \cap CG \). Now since \( A \supseteq B \), \( \alpha N(z, p) = v_{n}(p) \cap \alpha N(z, p) \) by \( v_{n}(p) \cap CG \).

Let \( n \rightarrow \infty \). Then by 1)
\[
\alpha N(z, p) = \lim_{n} v_{n}(p) \cap \alpha N(z, p) = 0 .
\] (1)

Case 2. \( p \in B_{S}^{N} \) in this case it was proved that \( \omega(p \cap CG, z) = 0 \).

Whence \( \omega(p \cap CG, z) = a \omega(p, z) \), where \( a = 2 \pi / \int_{R_{1}} \frac{\partial}{\partial n} \omega(p, z) ds \). We shall

---

10) p. 32 of P (see 2).

11) We can prove \( D(\min(N, U^{*}(z)) \leq D(\min(M, N(z, p)) \leq 2\pi M \).

prove the latter part of 2). \( \nu_{\cap CG} N(z, p) + \nu_{\cap \nu MG} N(z, p) \geq \nu_{\cap \nu MG} N(z, p) \). Hence by (1) and (1') we have \( \nu_{\cap CG} N(z, p) = 0 \) and \( \nu_{\cap \nu MG} N(z, p) \geq \nu_{\cap \nu MG} N(z, p) = N(z, p) \).

**Proof of 3.** Case 1. \( \nu \in B_{1}^{\nu} - B_{S}^{\nu} \). \( S(z) = \lim_{M=\infty} CG \cap V_{M}(p) N(z, p) \).

Assume \( S(z) > 0 \). Then \( S(z) \) is superharmonic and since \( \lim \text{Cap}(CG \cap V_{M}(p)) \leq \lim \text{Cap}(V_{M}(p)) = 0 \), \( N(z, p) - S(z) \) is also superharmonic. On the other hand, \( N(z, p) \) is \( N \)-minimal, whence \( S(z) \) must be a multiple of \( N(z, p) \), i.e. \( S(z) = a N(z, p) : a > 0 \). Clearly \( \nu_{\cap \nu MG} N(z, p) \geq \nu_{\cap \nu MG} N(z, p) \geq S(z) = a N(z, p) \). Whence \( \nu_{\cap \nu MG} N(z, p) = \nu(a N(z, p)) = a \). This contradicts 1). Hence \( \lim_{M=\infty} \nu_{\cap \nu MG} N(z, p) = 0 \).

Case 2. \( \nu \in B_{S}^{\nu} \). In this case it was proved\(^{13}\) that \( \lim \omega(V_{M}(p) \cap CG, z) = 0 \) to show \( \lim \omega(C_{\nu MG}(p) \cap V_{M}(p), z) = 0 \). But \( \nu_{\cap \nu MG} N(z, p) = a \omega(V_{M}(p) \cap CG, z) \).

Hence \( \lim_{M=\infty} \nu_{\cap \nu MG} N(z, p) = 0 \).

We show \( \lim_{M=\infty} \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} N(z, p) ds = 0 \) for \( \nu \in B_{S}^{\nu} \). \( M \omega(V_{M}(p) \cap CG, z) \) is \( M \) on \( V_{M}(p) \cap CG \) and superharmonic in \( \overline{R} \). Hence \( M \omega(V_{M}(p) \cap CG, z) \leq \nu_{\cap \nu MG} N(z, p) \leq N(z, p) \). Since \( N(z, p) = M \) on \( \partial V_{M}(p) \),

\[
\int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} M \omega(V_{M}(p) \cap CG, z) ds \geq \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} M \omega(V_{M}(p) \cap CG, z) ds \geq \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} N(z, p) ds.
\]

Assume \( \lim_{M=\infty} \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} N(z, p) ds \geq \delta > 0 \). Then there exists a sequence \( M_{i} < M_{i+1} \cdots M_{i} \) such that \( 0 \leq \lim M \omega(V_{M_{i}}(p) \cap CG, z) \) by \( \int_{\partial V_{M_{i}}} \frac{\partial}{\partial n} M \omega(V_{M_{i}}(p) \cap CG, z) ds > \frac{\delta}{2} \). On the other hand, \( \lim M \omega(V_{M}(p) \cap CG, z) = \lim_{M=\infty} \nu_{\cap \nu MG} N(z, p) \).

This contradicts \( \lim_{M=\infty} \nu_{\cap \nu MG} N(z, p) = 0 \). Hence \( \lim_{M=\infty} \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} N(z, p) ds = 0 \) for \( \nu \in B_{S}^{\nu} \). We consider the domain \( \overline{G} \ni p \) and we shall show that \( \nu_{\cap \nu MG} N(z, p) (= N(z, p) - CG N(z, p) > 0) \) in \( G \) has the similar properties as those of \( N(z, p) \) in \( R - R_{0} \).

**Theorem 4.** Let \( G \) be a domain in \( R - R_{0} \) such that \( \overline{G} \ni p (\nu \in B_{S}^{\nu}) \).

\(^{13}\) See p. 97 of 12) and p. 21) of P.
Then

1. a). \( p \cap V_{M}(p) = N(z, p) \) and \( \lim_{M \to \infty} \) \( p \cap V_{M}(p) = N(z, p) \) for \( p \in B_{1}^{N} - B_{S}^{N} \), where \( V_{M}(p) = E[z \in G : N(z, p) > M] \).

1. b). \( \omega(p, z) = p \cap V_{M}(p) \) \( \omega(p, z) : M < 1 - M_{0} = M^{**} \) for \( p \in B_{s}^{N} \), where \( M_{0} \) is the infimum of numbers \( M_{i} \) such that \( \omega(p \cap V_{M_{i}}^{c}, z) = 0, M_{0} < 1^{12} \).

1. c). \( D_{G}(\min(M, N'(z, p)) \leq 2\pi M \) for \( p \in B_{1}^{N} - B_{S}^{N} \).

2. a). Let \( \omega^{*} N'(z, p) \) be a function in \( G \) such that \( \omega^{*} N'(z, p) = \min(L, N'(z, p)) \) on \( G' + \delta G \) and \( \omega^{*} N'(z, p) \) has M. D. I. \( (\leq 2\pi L \) by 1. c)) over \( G - G' \). Put \( \omega^{*} N'(z, p) = \lim_{L \to \infty} \omega^{*} N'(z, p) \). Then \( \omega^{*} N'(z, p) \subseteq N'(z, p) \) for any domain \( G' \) in \( G \). Thus \( N'(z, p) \) is superharmonic in \( G \).

2. b). As a special case we have

\[
\min(M, N'(z, p)) = M \omega(V_{M}(p), z, G) \text{ for } M < \infty \text{ and } p \in B_{1}^{N} - B_{S}^{N},
\]

\[
\min(M, \omega'(p, z)) = M \omega(V_{M}(p), z, G) \text{ for } M < M^{**} \text{ for } p \in B_{s}^{N},
\]

where \( \omega'(p, z) = \omega(p, z) - \omega(p, z) \) and \( \omega(V_{M}(p), z, G) \) is C. P. of \( V_{M}(p) \) relative to \( G \).

And

\[
V_{M}(p) \ni p.
\]

By 2. a) and 2. b) it is seen that \( V_{M}(p) \) has similar properties as \( V_{M}(p) \) in \( R - R_{0} \),

where \( V_{M}(p) = E[z \in G : N(z, p) > M] \) for \( p \in B_{1}^{N} - B_{S}^{N} \) and \( V_{M}(p) = E[z \in G : \omega(p, z) > M] \) for \( p \in B_{s}^{N} \).

3. Let \( G' \) be a domain in \( G \) such that \( \omega N'(z, p) = N(z, p) \). Then \( \omega N'(z, p) = N'(z, p) \). As a special case by \( v_{n}(p) \ni p \) we have

\[
N'(z, p) = \omega N'(z, p) = \lim_{n \to \infty} v_{n}(p) \ni N'(z, p) = \omega N'(z, p) \text{ for } p \in B_{1}^{N}.
\]

Now \( v_{n}(p) \ni p \), hence by considering \( v_{n}(p) \) a domain \( G \) we have by 2. b) the following fact: for any \( v_{n}(p) \) there exists a domain \( G' = V_{M}(p) \) such that \( v_{n}(p) \ni G' \ni p \). Hence for any \( V_{M}(p) \) there exist \( v_{n}(p) \) and \( V_{M}(p) \) such that \( V_{M}(p) \ni v_{n}(p) \ni V_{M}(p) \ni p \). This means \( \{V_{M}(p)\}, \{v_{n}(p)\} \) and \( \{V_{M}(p)\} \) are equivalent.

Proof of 1). a). Now \( p \in B_{1}^{N} - B_{S}^{N} \). In this case \( N(z, p) \leq c_{0} N(z, p) + M \).
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on $CV_{M}^{r}(p)$. Hence

$$cr_{M^{(p)\cap p}}^{N(z,p)}\leq\omega_{CV_{1\wedge f}^{r}(p)\cap p}(CGN(z, p)) + M\omega(p\cap CV_{M}^{r}(p), z)\leq\omega_{CV_{1\wedge f}^{r}(p)\cap p}(CGN(z, p)) = 0$$

by $\omega(p, z) = 0$ and by 2) of Theorem 3. On the other hand,

$$cr_{M^{(p)\cap p}}^{N(z,p)} + V_{M^{(p)}}^{r}(p)N(z,p) \geq N(z,p).$$

Hence

$$\lim_{M^{f} = \infty}V_{M^{(p)}}^{r}(p)N(z,p) = N(z,p).$$

Proof of 1). b). Since $p \in B_{b}^{N}$, $N(z,p) = a \omega(p, z) : a > 0$. Let $1 > M' > M_{o} > 0$. Then $\omega(p \cap V_{M}^{r}, z) = 0$, where $V_{M}^{r} = E[z \in R : \omega(p, z) > M']$. Further we have

$$\omega(\tilde{V}_{1-}, p, z) = \omega(p, z)^{12),}$$

where $\tilde{V}_{1-} = CV_{M}^{r} \cap V_{1-}^{r}(p) = E[z \in G : \omega(p, z) - \omega(p, z) > 1 - \varepsilon - M']$. Since $\omega(p, z) > 1 - \varepsilon$ on $\tilde{V}_{1-}$, $\omega(p, z) \geq \omega(p, z)(1 - \varepsilon) = (1 - \varepsilon)\omega(p \cap \tilde{V}_{1-}, z)$. We have by letting $\varepsilon \rightarrow 0 \lim_{\varepsilon \rightarrow 0} \omega(p, z) = \omega(p, z)$.

Because $\lim_{\varepsilon \rightarrow 0} \omega(p \cap \tilde{V}_{1-}, z) = 1$. Clearly $V_{M}(p) \supset \tilde{V}_{1-}$, for $M < 1 - \varepsilon - M'$ and

$$\omega(p, z) \geq \omega(p \cap V_{M}^{r}, z) \geq \lim_{\varepsilon \rightarrow 0} \omega(p, z) = \omega(p, z).$$

Thus

$$\omega(p, z) = \omega(p \cap V_{M}^{r}, z).$$

Proof of 1). c). Case 1. $p \in B_{b}^{N} - B_{c}^{N}$. By 1). a) $r_{M^{(p)\cap p}}^{N(z,p)}N(z,p) = N(z,p)$ and since $r_{M^{(p)\cap p}}^{N(z,p)} - r_{M^{(p)\cap p}}^{N(z,p)} \uparrow r_{M^{(p)\cap p}}^{N(z,p)} = N(z,p)$ as $m \rightarrow \infty$, there exists a number $m_{o}$ such that

$$\omega(p, z) = \omega(p \cap V_{M}^{r}, z).$$

Fig. 1.
\[ 0 \leq N(z, p) - v_M \cap R_m N(z, p) > \varepsilon \text{ on } R_l \text{ for } m \geq m_0, \quad (2) \]

for any given \( R_l \) (\( R_l \) is an exhaustion of \( R \)) and \( \varepsilon > 0 \). Let \( N_{m,n}(z, p) \) be a harmonic function in \( R_n - R_l \) such that \( N_{m,n}(z, p) = 0 \) on \( \partial R_l \), \( N_{m,n}(z, p) = N(z, p) \) on \( R_m \cap V_M(p) \) and \( \frac{\partial}{\partial n} N_{m,n}(z, p) = 0 \) on \( \partial R_n \). Then

\[ N_{m,n}(z, p) \Rightarrow v_M \cap R_m N(z, p) \quad (\leq v_M N(z, p)) \text{ as } n \to \infty, \]

where \( \Rightarrow \) means convergence in mean. Hence there exists a number \( n_1(m) \) such that

\[ |N_{m,n}(z, p) - v_M N(z, p)| < \varepsilon \text{ on } R_l \text{ for } n > n_1(m) \text{ and } n > m. \quad (3) \]

Let \( U_{l,m,n}(z) \) be a harmonic function in \( R_n - (R_l \cap CG) \) such that \( U_{l,m,n}(z) = N_{m,n}(z, p) \) on \( \partial (R_l \cap CG) \), \( \frac{\partial}{\partial n} U_{l,m,n}(z) = 0 \) on \( \partial R_n \). Then by (2) and (3)

\[ |U_{l,m,n}(z) - N(z, p)| < 2\varepsilon \text{ on } \partial (R_l \cap CG) \text{ for } n > n_1(m) \text{ and } n > m. \quad (4) \]

On the other hand, let \( U'_{l,m,n}(z) \) be a harmonic function in \( R_n - (R_l \cap CG) \) such that \( U'_{l,m,n}(z) = N(z, p) \) on \( \partial (R_l \cap CG) \), \( \frac{\partial}{\partial n} U'_{l,m,n}(z) = 0 \) on \( \partial R_n \). Then by the maximum principle

\[ |U_{l,m,n}(z) - U'_{l,m,n}(z)| < 2\varepsilon \text{ in } R_n - (R_l \cap CG). \quad (5) \]

Let \( n \to \infty \). Then by \( \lim^* U_{l,m,n}(z) \to \alpha \cap R, N(z, p) \)

\[ \lim^* U_{l,m,n}(z) - \alpha \cap R, N(z, p) | < 2\varepsilon \text{ in } R_n - (R_l \cap CG), \quad (6) \]

where \( \lim^* U_{l,m,n}(z) \) means a limit by a converging subsequence of \( \{U_{l,m,n}(z)\} \). Hence there exists a number \( n_2 > n_1 \) such that

\[ |U_{l,m,n}(z) - \alpha \cap R, N(z, p)| < 3\varepsilon \text{ for } n > n_2 \text{ in } (R_m \cap G) \supset (R_m \cap V_M(p)). \quad (7) \]

Now by \( \alpha \cap R, N(z, p) \uparrow \alpha \cap N(z, p) \) as \( l \to \infty \) we have \( U_{l,m,n}(z) < c_\alpha N(z, p) + 3\varepsilon \) on \( R_m \cap V_M(p)\). On the other hand, \( M \leq N(z, p) - c_\alpha N(z, p) \leq N(z, p) - U_{l,m,n}(z) + 3\varepsilon \) on \( R_m \cap V_M(p) \). Since \( N_{m,n}(z) = U_{l,m,n}(z) \) on \( \partial (R_l \cap CG) \) and \( c_\alpha N(z, p) = N(z, p) \) on \( R_m \cap V_M(p) \),

\[ V^* = E[z \in R_n - (R_l \cap CG) : N_{m,n}(z, p) - U_{l,m,n}(z) > M - 3\varepsilon] \supset E[z \in R_m : N(z, p) - c_\alpha N(z, p) > M] = V^*_M(p) \cap R_m. \]

Whence \( N_{m,n}(z, p) - U_{l,m,n}(z) \) is harmonic in \( R_n - (R_l \cap CG) - V^* \). It follows that
On the other hand, \( \int_{\partial\psi\cap R_{n}}\frac{\partial}{\partial n}U_{l,m,n}(z)ds=\int_{\partial R_{n}\cap V^{2}}\frac{\partial}{\partial n}U_{l,m,n}(z)ds=0 \).

Hence \( \int_{\partial^{n}\cap R_{2}}\frac{\partial}{\partial n}(N_{m,n}(z,P)-U_{l,m,n}(z))ds\leqq 2\pi+\epsilon \) for \( n>n_{3}(m, l) \), whence

\[ R_{n}-(CG\cap R_{l})-V^{*}(N_{m,n}(z,p)-U_{l,m,n}(z))\leqq(2\pi+\epsilon)(M-3\epsilon) \]  

(8)

Let \( \mathfrak{G} \) be a compact domain completely contained in \( G-V^{*}_{M}(p) \). Then since

\( (N_{m,n}(z,p)-U_{l,m,n}(z))\rightarrow(r_{M}(p)\cap R_{n}N(z,p)-\partial R_{n}R_{i}N(z,p)+\delta(z)) \) as \( n\rightarrow\infty \), (where \( |\delta(z)|<2\epsilon \)) by (6) and since \( CG\cap R_{i} \) is compact, \( (r_{M}(p)\cap R_{n}N(z,p)-\partial R_{n}R_{i}N(z,p)+\delta(z)\rightarrow(r_{M}(p)N(z,p)-\partial R_{n}R_{i}N(z,p)+\delta(z)) \) as \( m\rightarrow\infty \). Further \( r_{M}(p)N(z,p)-\partial R_{n}R_{i}N(z,p)+\delta(z) \rightarrow N(z,p)-\alpha_{n}N(z,p) \) as \( l\rightarrow\infty \) by \( r_{M}(p)N(z,p)=N(z,p) \). Hence there exist numbers \( l, m, n \) such that \( \mathfrak{G}\subset(R_{n}-(CG\cap R_{i})-V^{*}) \) for any \( \epsilon>0 \) by (8) and

\[ D\mathfrak{G}(N(z,p)-CGN(z,p))=D(\min(M, N(z,p)))\leqq 2\pi M \]  

Case 2. \( p\in B^{N}_{s} \). In this case it is evident that \( D(\omega(p, z))<\infty \) and \( D(\omega(p, z))<\infty \) and we have at once \( D(\omega(p, z))<\infty \).

**Proof of 2). a.** Let \( pN^{T}(z,p) \) be a function such that \( pN^{T}(z,p)=\min(T, N(z,p)) \) on \( D \) and \( pN^{T}(z,p) \) has M. D. I. over \( R-R_{0}-D \) and let \( p^{*}N^{T}(z,p) \): \( G^{*}\subset G \) be a function in \( G-G^{*} \) such that \( p^{*}N^{T}(z,p)=0 \) on \( \partial G \), \( p^{*}N^{T}(z,p)=\min(T, N^{*}(z,p)) \) on \( G^{*} \) and \( p^{*}N^{T}(z,p) \) has M. D. I. over \( G-G^{*} \) (this can be defined by 1. c)). Then by \( N(z,p)=N^{*}(z,p)+\alpha_{n}N(z,p) \) we have \( \alpha_{n}N^{*}(z,p)\geqq \delta_{n}(\alpha_{n}N(z,p)) \) on \( \partial G+\partial G^{*} \) and by the maximum principle*) \( \alpha_{n}N^{*}(z,p)\geqq \alpha_{n}(\alpha_{n}N^{*}(z,p)+N^{*}(z,p)) \). Let \( L\rightarrow\infty \). Then

\[ \alpha_{n}N^{*}(z,p)\geqq \alpha_{n}(\alpha_{n}N(z,p)+N^{*}(z,p)) \]
Next similarly we have \( G + a \cdot N^r(z, p) \leq \lambda_{G + a} (G + N^r(z, p)) \) and \( G + a \cdot N(z, p) \leq G + a \cdot (G + N(z, p)) \). On the other hand, by \( CG + G' \supset CG \) we have \( G + a \cdot N^r(z, p) \leq G + a \cdot N(z, p) \).

Thus \( N^r(z, p) \) is superharmonic in \( G \).

**Proof of 2. b.** By 1. a) and 1. b) \( v_{M}(p)N(z, p) = N(z, p) \) whence

\[
N(z, p) \leq c v_{M}(p)N(z, p) = v_{M}(p)N(z, p) = N(z, p)
\]

for \( M < \infty \) for \( p \in B_{1}^{N} - B_{S}^{N} \),

\[
G \supset v_{M}(p)N(z, p) = v_{M}(p)N(z, p) = N(z, p)
\]

Hence

\[
N(z, p) = \omega(p, z) \quad \text{for} \quad p \in B_{1}^{N} - B_{S}^{N}.
\]

This implies \( N(z, p) = 0 \). This is a contradiction. Hence \( v_{M}(p) \not\in p \).

**Case 2.** \( p \in B_{1}^{N} \). It is sufficient to prove for \( \omega(p, z) \). Now \( \omega(p, z) = c v_{M}(p)(c v_{M}(p) \omega(p, z)) = c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) \).

Assume \( V_{M}(p) \not\in p \). Then \( c v_{M}(p) \omega(p, z) = c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) = \omega(p, z) \). On the other hand, \( v_{M}(p) \omega(p, z) = M \omega(V_{M}(p), z, G) \) for \( M < M^{**} \). Hence \( c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) \leq c v_{M}(p) \omega(p, z) \).

This is a contradiction. Hence \( V_{M}(p) \not\in p \) for \( M < M^{**} \).
Proof of 3). By \( q \cdot N(z, p) = N(z, p) \) we have \( c_{G_0} N(z, p) = N(z, p) \). This implies \( c_{G_0} N(z, p) + N'(z, p) = N(z, p) \leq c_{G_0} N(z, p) + s N'(z, p) \).

This implies \( c_{G_0} N(z, p) + N'(z, p) \leq N(z, p) \).

Hence \( s N'(z, p) = N'(z, p) \).

2. On capacitary potentials. Let \( \{G_n\} \) \( (n=1, 2, 3, \cdots) \) be a decreasing sequence of domains in \( R - R_0 \). Let \( G \) be a domain in \( R - R_0 \) such that \( G \supset G_1 \) and let \( \Gamma \) be a compact arc on \( \partial G \). Let \( \omega_{n, n+i}^G(z) \) be a harmonic function in \( (R_{n+i} \cap G) - G_n \) such that \( \omega_{n, n+i}^G(z) = 0 \) on \( \partial G, \omega_{n, n+i}^G(z) = 1 \) on \( G_n \) and \( \frac{\partial}{\partial n} \omega_{n, n+i}^G(z) = 0 \) on \( \partial R_{n+i} - G_n \).

Then if \( D(\omega_{n, n+i}^G(z)) < M < \infty \) for a number \( n_0 \) and for every \( i \), then \( \omega_{n, n+i}^G(z) \Rightarrow \omega_n^G(z) \) as \( i \rightarrow \infty \) and \( \omega_n^G(z) \Rightarrow \omega(\{G_n\}, z, G) \). Let \( \omega_{n, n+i}^\mathfrak{R}(z) \) be a harmonic function in \( (R_{n+i} \cap G) - G_n \) such that \( \omega_{n, n+i}^\mathfrak{R}(z) = 1 \) on \( G_n \), \( \omega_{n, n+i}^\mathfrak{R}(z) = 0 \) on \( \partial \Gamma, \frac{\partial}{\partial n} \omega_{n, n+i}^\mathfrak{R}(z) = 0 \) on \( \partial R_{n+i} - G_n \). Then since \( \Gamma \) is compact and by \( \text{dist} (\partial R_0, G_n) > 0 \), \( D(\omega_{n, n+i}^\mathfrak{R}(z)) < M < \infty \) for every \( n \geq n_0 \) and every \( i \), then \( \omega_{n, n+i}^\mathfrak{R}(z) \Rightarrow \omega_n^\mathfrak{R}(z) \) and \( \omega_n^\mathfrak{R}(z) \Rightarrow \omega(\{G_n\}, z, G, \Gamma) \). It can be proved easily if \( \omega(\{G_n\}, z, R - R_0) > 0, \omega(\{G_n\}, z, R - R_0') > 0 \) for any compact set \( R_0' \) such that \( R_0' \cap G_n = 0 \) \( (n_0 \) is a certain number).

Theorem 5. If the condition \( D(\omega_{n, n+i}^G(z)) < M < \infty \) for a number \( n_0 \) and for every number \( i \) is satisfied, then

1. \( \omega(\{G_n\}, z) > 0 \) if and only if \( \omega(\{G_n\}, z, G) > 0 \),
2. \( \omega(\{G_n\}, z, G) > 0 \) if and only if \( \omega(\{G_n\}, z, G, \Gamma) > 0 \),
3. If \( \omega(\{G_n\}, z, G, \Gamma) > 0, \omega(\{G_n\}, z) > 0 \) without the above condition.

Proof. Suppose \( \omega(\{G_n\}, z) > 0 \). Then by the Dirichlet principle \( D(\omega_{n, n+i}^\mathfrak{R}(z)) \leq D(\omega_{n, n+i}^G(z)) \). By the mean covergency of \( \omega_{n, n+i}^\mathfrak{R}(z) \) and \( \omega_{n, n+i}^G(z) \) we have

\[ D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z)) > 0. \]

Hence \( \omega(\{G_n\}, z, G) > 0. \)

Next suppose \( \omega(\{G_n\}, z, G) > 0. \) Then by the maximum principle \( \omega_{n, n+i}^\mathfrak{R}(z) \) \( \geq \omega_{n, n+i}^G(z) \). Hence

\[ \omega(\{G_n\}, z) > 0. \]

14) See p. 21 of P.
By \( \omega(\{G_n\}, z, G) \leq \omega(\{G_n\}, z, G, I) \) and by \( D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z, G, I)) \) we have 2. Also by the Dirichlet principle \( D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z, G, \Gamma)) \) and we have 3).

It is known that \( \omega(\{G_n\}, z, G) \) (or \( \omega(\{G_n\}, z, G, \Gamma) \)) attains 1 almost on \( \{G_n\} \) in the following sense: \( \omega(\{G_n \cap V_{1-} \cdot \}, z, G) = 0 \) (or \( \omega(\{G_n \cap V_{1-} \cdot \}, z, G, \Gamma) \)) for any \( \epsilon > 0 \). Conversely we shall prove. Let \( G_1 \) and \( G_2 \) be two domains (or closed domains), if there exists at least one \( C_1 \)-function \( U(z) \) such that \( U(z) = 0 \) on \( G_1 \), \( U(z) = 1 \) on \( G_2 \) and \( D(U(z)) < \infty \), we say that \( G_1 \) and \( G_2 \) are Dirichlet-disjoint.

**Lemma 1.1** If there exist numbers \( i_0 \) and \( n_0 \) such that \( CG_{n_0} \) and \( G_{n_0 + i_0} \) are Dirichlet-disjoint,

\[
\omega(CG_{n_0} \cap V_{1-} \cdot, z, G-G_{n_0+i_0}) \Rightarrow 0 \text{ as } \epsilon \to 0 , \quad \text{and} \\
\omega(CG_{n_0} \cap V_{1-} \cdot, z, G) \Rightarrow 0 \text{ as } \epsilon \to 0 ,
\]

where \( V_{1-} \cdot = E[z \in G : \omega(\{G_n\}, z, G) > 1 - \epsilon] \).

This fact means that \( \omega(\{G_n\}, z, G) < 1 \) almost everywhere outside of \( G_n \).

1'). If there exist numbers \( i_0 \) and \( n_0 \) such that \( CG_{n_0} \) and \( G_{n_0 + i_0} \) are Dirichlet-disjoint,

\[
\omega(CG_{n_0} \cap V_{1-} \cdot, z, G, \Gamma) \Rightarrow 0 \text{ as } \epsilon \to 0 ,
\]

where \( V_{1-} \cdot = E[z \in G : \omega(\{G_n\}, z, G, \Gamma) > 1 - \epsilon] \).

2'). Under the condition of 1')

\[
\int_{\partial V_{1-} \cdot \cap CG_{n_0}} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G) \downarrow 0 \text{ as } \epsilon \to 0 .
\]

Proof. Since \( \omega(V_{1-} \cdot, z, G) \) has M. D. I. over \( G-V_{1-} \cdot, \) \( D(\omega(V_{1-} \cdot, z, G)) \leq D \left( \frac{1}{1-\epsilon} \omega(\{G_n\}, z, G) \right) = M' < \infty \), because \( \omega(\{G_n\}, z, G) \geq (1-\epsilon) \) on \( V_{1-} \cdot \), and \( =0 \) on \( \partial G \). Now \( CG_{n_0} \) and \( G_{n_0 + i_0} \) are Dirichlet-disjoint, whence there exists a \( C_1 \)-function \( U(z) \) such that \( U(z) = 0 \) on \( CG_{n_0 + i_0} \), \( U(z) = 1 \) on \( G_{n_0} \) and \( D(U(z)) < M'' < \infty \). Put \( S(z) = \min(\omega(V_{1-} \cdot, z, G), U(z)) \). Then \( S(z) = 0 \) on \( \partial G + \partial G_{n_0 + i_0} \), \( S(z) \geq 1 \) on \( V_{1-} \cdot \cap CG_{n_0} \) and \( D(S(z)) \leq M + M'' < \infty \). Hence \( D(\omega(V_{1-} \cdot \cap CG_{n_0}, z, G-G_{n_0+i_0})) \leq D(S(z)) < \infty \) and lim \( \omega(V_{1-} \cdot \cap CG_{n_0}, z, G-G_{n_0+i_0}) \) exists. By

15) If \( Uz \) has partial derivatives almost everywhere, we call \( U(z) \) a \( C_1 \)-function.
Correspondence of Boundaries of Riemann Surfaces

\[ D(\omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G)) \leq D(\omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G - G_{n_{0}+i_{0}}) \lim_{\epsilon \to 0} \omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G - G_{n_{0}+i_{0}}) \lim_{\epsilon \to 0} \]\n
such \( C_{r} \) is called a regular niveau curve.\(^{16}\)

Now \( \omega(\{G_{n}\}, z, G) \) is a harmonic function in \( G - G_{n_{0}+i_{0}} \), having M.D.I. over \( G - G_{n_{0}+i_{0}} \) among all functions with the same value on \( \partial G + \partial G_{n_{0}+i_{0}} \), whence

\[ A_{m}(z) \Rightarrow \omega(\{G_{n}\}, z, G) \text{ as } m \to \infty, \]

where \( A_{m}(z) \) is a harmonic function in \( R_{m} \cap (G - G_{n_{0}+i_{0}}) \) such that \( A(z) = \omega(\{G_{n}\}, z, G) \) on \( \partial G + \partial G_{n_{0}+i_{0}} \cap R_{m} \) and \( \frac{\partial}{\partial n} A_{m}(z) = 0 \) on \( (G - G_{n_{0}+i_{0}}) \cap R_{m} \). Also \( \omega(z) \) has M.D.I. over \( G - G_{n_{0}+i_{0}} - V_{1-\epsilon} \), whence \( \omega_{m}(z) \Rightarrow \omega(z) \), where \( \omega_{m}(z) \) is a harmonic function in \( R_{m} \cap (G - G_{n_{0}+i_{0}} - \tilde{V}_{1-\epsilon}) \) such that \( \omega_{m}(z) = \omega(z) \) on \( \partial \tilde{V}_{1-\epsilon} \), \( \frac{\partial \omega_{m}}{\partial n}(z) = 0 \) on \( \partial R_{m} \cap (G - G_{n_{0}+i_{0}} - \tilde{V}_{1-\epsilon}) \), where \( \tilde{V}_{1-\epsilon} = E[z \in G - G_{n_{0}+i_{0}} : \omega(z) > 1 - \epsilon] \). Then by Green's formula

\[ \int_{c_{1-\epsilon} \cap R_{m}} A_{m}(z) \frac{\partial}{\partial n} \omega_{m}(z) ds = \int_{c_{\delta} \cap R_{m}} A_{m}(z) \frac{\partial \omega_{m}}{\partial n}(z) ds, \]

where \( C_{1-\epsilon} \) and \( C_{\delta} \) are regular niveau curves of \( \omega(z) \). Then by letting \( m \to \infty \)

\[ \int_{c_{1-\epsilon}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) ds = \int_{c_{\delta}} \omega(\{G_{n}\}, z, G) \frac{\partial \omega}{\partial n}(z) ds. \]

By \( \omega(z) \leq \omega(\{G_{n}\}, z, G) \) we have

\[ (1 - \epsilon) D(\omega(z)) = \int_{c_{1-\epsilon}} \omega(z) \frac{\partial \omega}{\partial n}(z) ds \leq \int_{c_{1-\epsilon}} \omega(\{G_{n}\}, z, G) \frac{\partial \omega}{\partial n}(z) ds = \int_{c_{\delta}} \omega(\{G_{n}\}, z, G) \frac{\partial \omega}{\partial n}(z) ds. \]

On the other hand, \( C_{\epsilon} \subset G, \omega(\{G_{n}\}, z, G) < 1 \text{ in } G \) and there exists a number \( \epsilon_{0} > 0 \) such that

---

16) See theorem 2 of P.
17) See See 15)
\[
\int_{c_{\varepsilon}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds < (1-\varepsilon_{0}) \int_{c_{1-\varepsilon}} \frac{\partial}{\partial n} \omega(z) \, ds = (1-\varepsilon_{0}) D(\omega(z)).
\]

Take \( \varepsilon < \frac{\varepsilon_{0}}{2} \). Then (11) will be a contradiction. Hence
\[
\omega(z) = \lim_{i \to 0} \omega(V_{i-}, \cap CG_{n_{i}}, z, G-G_{n_{i}+i_{0}}) = 0.
\]

Also by the Dirichlet principle we have \( \lim \omega(V_{i-}, \cap CG, z, G)=0 \).

**Proof of 2).** Assume \( \limsup_{i \to \infty} \int_{c_{n_{i}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds > \delta > 0 \). Then by
\[
r \omega(V_{r} \cap CG_{n_{i}}, z, G-G_{n_{i}+i_{0}}) \leq \omega(\{G_{n}\}, z, G) \quad \text{and} \quad r \omega(V_{r} \cap CG_{n_{i}}, z, G-G_{n_{i}+i_{0}}) = \omega(\{G_{n}\}, z, G) = r \quad \text{on } \partial V_{r}
\]
and we have
\[
r \delta \leq r \int_{c_{n_{i}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds \leq r \int_{c_{n_{i}}} \frac{\partial}{\partial n} \omega(V_{r} \cap CG_{n_{i}}, z, G-G_{n_{i}+i_{0}}) \, ds \leq \int_{\partial(V_{r} \cap CG_{n_{i}})} r \frac{\partial}{\partial n} \omega(V_{r} \cap CG_{n_{i}}, z, G) \, ds \leq r D(\omega(V_{r} \cap CG_{n_{i}})).
\]

Let \( r \to 1 \). Then by \( r \omega(V_{r} \cap CG_{n_{i}}, z, G-G_{n_{i}+i_{0}}) \Rightarrow \omega(z) \) we have \( D(\omega(z)) > \delta \) and \( \omega(z) > 0 \). This is a contradiction. Hence
\[
\lim_{i \to \infty} \int_{c_{n_{i}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds = 0.
\]

The other part can be proved similarly.

**Angular domain** \( \Delta(p) \) of \( p \in B^{N} \). Let \( \Delta(p) \) be a domain. If
\[
\lim_{n \to \infty} \Delta(p) \cap c_{n} \ni N(z, p) > 0,
\]
we call \( \Delta(p) \) an angular domain with vertex at \( p \). Then

**Theorem 6.** 1). Let \( G \) be a domain. If \( \lim_{M \to M^{*}} \int_{c_{M}} \frac{\partial}{\partial n} N(z, p) \, ds > 0 \), \( G \) is an angular domain, where \( C_{M} = E[z \in R : N(z, p) = M] \): \( M < M^{*} = \sup_{z \in R} N(z, p) \).

2). If \( G^{N} \ni p \), \( G \) is an angular domain.

3). Let \( \Delta(p) \) be an angular domain. If \( G^{N} \ni p \), then \( G \cap \Delta(p) \) is also an angular domain.

**Proof of 1.** Suppose \( \lim_{M \to M^{*}} \int_{c_{M}} \frac{\partial}{\partial n} N(z, p) \, ds > \delta > 0 \). Then there exists a sequence \( M_{1}, M_{2}, \ldots \) and \( M_{i} \uparrow M^{*} \) such that \( \int_{c_{M \cap \theta}} \frac{\partial}{\partial n} N(z, p) \, ds > \frac{2\delta}{3} \) for \( i \geq 1 \).
Now for any given \( v_n(p) \) \( \lim_{M \to M^*} \int_{\partial V_M(p) \cap v_n(p) \cap G} \frac{\partial}{\partial n} N(z, p) \, ds = 0 \) by Theorem 3. 3).

Hence by \((\partial V_M(p) \cap v_n(p) \cap G) + (\partial V_M(p) \cap C v_n(p) \cap G) = \partial V_M(p) \cap G\), there exists a number \( i_0(n) \) such that 
\[
\int_{\partial V_M(p) \cap v_n(p) \cap G} \frac{\partial}{\partial n} N(z, p) \, ds \geq \frac{\delta}{2}
\]
for \( i > i_0(n) \). Now \( M_0 \omega(V_M(p) \cap v_n(p) \cap G, z) = M_0 = N(z, p) \) on \( \partial V_M(p) \), whence
\[
\int_{\partial V_M(p) \cap v_n(p) \cap G} \frac{\partial}{\partial n} N(z, p) \, ds \geq \frac{\delta}{2}.
\]

On the other hand, by \( M_0 \omega(V_M(p) \cap v_n(p) \cap G, z) \leq v_M(p) \cap v_n(p) \cap G \), we have
\[
\int_{\partial V_M(p) \cap v_n(p) \cap G} \frac{\partial}{\partial n} N(z, p) \, ds \geq \frac{\delta}{2}.
\]

Let \( n \to \infty \). Then \( V_M(p) \cap v_n(p) \cap G \to p \cap G \) for any given \( M_0 < M^* \) by \( V_M(p) \supset v_n(p) : n = n(M) \) and \( \lim_{n=8} v_n(p) \cap G \cap V_M(p) \).

Hence its mass only at \( p \) and \( \lim_{n=\infty} v_n(p) \cap G \cap V_M(p) \geq \frac{2\pi}{2} N(z, p) > 0 \). Thus \( G \)

is an angular domain.

**Proof of 2**. If \( G \ni p \), 
\[
\int_{\partial V_M(p) \cap \partial G} \frac{\partial}{\partial n} N(z, p) \, ds \downarrow 0 \text{ as } M \uparrow M^* \text{ by Theorem 3). 3).}
\]

But \( \int_{\partial V_M(p) \cap \partial G} \frac{\partial}{\partial n} N(z, p) \, ds = 2\pi \) for almost \( M \). Hence \( \lim_{M \to M^*} \int_{\partial V_M(p) \cap \partial G} \frac{\partial}{\partial n} N(z, p) \, ds = 2\pi \) and by 1) \( G \) is an angular domain.

**Proof of 3**. By \( v_n(p) \cap \Delta(p) = (v_n(p) \cap \Delta(p) \cap CG) + (v_n(p) \cap \Delta(p) \cap G) \) and by \( \lim_{n} v_n(p) \cap \Delta(p) \cap G = 0 \) by Theorem 3). 2) we have
\[
\lim_{n} v_n(p) \cap \Delta(p) \cap G N(z, p) = 0 \quad \text{and} \quad \lim_{n} v_n(p) \cap \Delta(p) \cap G N(z, p) \geq \frac{2\pi}{2} N(z, p) > 0.
\]

Hence \( G \cap \Delta(p) \) is also an angular domain.

Let \( R \) be a disc: \(|z| < 1\) and \( R_0 \) be \(|z| < \frac{1}{2}\). Then \( N(z, p) = U(z) + \log \frac{1}{|z-p|} : p = e^{i\theta} \), where \( U(z) \) is a bounded harmonic function in \( E\left[z : \frac{1}{2} < |z| < 2\right] \). Hence it is easily verified that \( E\left[z : \arg \frac{z-e^{i\theta}}{e^{i\theta}} \right] < \frac{\pi}{2} \] = \( A(e^{i\theta}) \)
satisfies the condition of Theorem 6). 1) and \( A(e^{i\theta}) \) is an angular domain in our sense.

3. Capacitary potentials of the boundary determined by a domain \( G \) containing an end part \( \Delta(p) \) or containing \( p \) \( N \) approximately.

**Theorem 7**. 1). Let \( F \) be a closed set (with respect to \( N \)-Martin’s topology) on \( B \) of positive capacity. Let \( G \) be a domain such that \( G \supset (\Delta(p) \cap v_{n}(p)) \) for \( p \in (B_{1}^{N} \cap F) \), where \( n \) depends on \( p \) and \( \Delta(p) \) is an angular domain of \( p \). Then

\[
\lim_{n} \omega(G \cap F_{n}, z) > 0 ,
\]

where \( F_{n} = E[z \in \overline{R} : \text{dist} (z, F) \leq \frac{1}{n}] \).

2). Let \( G \) be a domain such that \( G \supset G_{n}(n=1, 2, \ldots) \), where \( \{G_{n}\} \) is a decreasing sequence of domains and \( G_{n} \supset p \) for any \( p \in (B_{1}^{N} \cap F) \). Let \( \Gamma \) be a compact arc on \( G \). Then

\[
\omega(\{G_{n}\}, z, G, \Gamma) > 0 .
\]

Further if \( CG \) and \( G_{n_{0}} \) (\( n_{0} \) is a certain number) are Dirichlet-disjoint, then

\[
\omega(\{G_{n}\}, z) > 0 \quad \text{and} \quad \omega(\{G_{n}\}, z, G) > 0 .
\]

**Proof of 1).** Since \( \text{Cap}(B_{0}^{N})=0 \), we can suppose that \( \text{Cap}(F \cap B_{1}^{N}) > 0 \). By the condition \( G \supset (\Delta(p) \cap v_{n}(p)) \) for \( n > n(p) \) and \( F_{n} \supset v_{n}(p) : p \in F \cap B_{1}^{N} \), we have

\[
a_{\cap F}N(z, p) = \lim_{n} a_{\cap F_{n}}N(z, p) \geq \lim_{n} a_{\cap \Delta(p) \cap v_{n}(p)}N(z, p) > 0 . \quad (11)
\]

Now \( N(z, p) \) is continuous for \( z \in R \) with respect to \( p \) and \( \int_{R} \frac{\partial}{\partial n} a_{\cap F_{n} \cap F_{m}}N(z, p) \, ds \) is also continuous with respect to \( p \) and \( a_{\cap F_{m} \cap F_{n}}N(z, p) \uparrow a_{\cap F_{m}}N(z, p) \) as \( m \rightarrow \infty \), \( a_{\cap F_{n}}N(z, p) \downarrow a_{\cap F_{m}}N(z, p) \) as \( n \rightarrow \infty \). Hence the set \( A_{t} = E[z \in F \cap B_{t}^{N} : \int_{R_{n}} a_{\cap F}N(z, p) \, ds > \frac{1}{t}] \) is a Borel set and

\[
\sum_{t=1}^{\infty} A_{t} = F \cap B_{1}^{N} .
\]

Capacity of Borel sets in \( B \) is the supremum of total mass of positive canonical distributions \( \{\mu\} \) such that the potential \( U(z) \) of \( \mu \) satisfies the condition that \( U(z) \leq 1 \) (this is equivalent to that the infimum of energy integrals \( I(\mu) \)}
of canonical distributions of unit masses $= 1/\text{Cap}$. Hence $\text{Cap}(A_i) \uparrow \text{Cap}(F)$ as $l \to \infty$, because $\mu$ is Borel measurable. Therefore we can find a set $A_i \subset F$ such that $\text{Cap}(A_i) > \frac{1}{2} \text{Cap}(F)$. Since $\omega(F_n, z) \downarrow \omega(F, z)$, for any given number $\left(\frac{1}{2l}\right) > \epsilon > 0$ there exists a number $n(\epsilon)$ such that

$$\int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F, z) ds \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F_n, z) ds - \epsilon \left(\text{clearly} \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F \cap R_{m}, z) ds - \epsilon\right).$$

(12)

Since $\xi_n \cap R_{m}N(z, p) \uparrow \xi_n N(z, p) \geq \xi N(z, p)$ as $m \to \infty$,

$$\sum_{m} A_{n,m} = A_i,$$

(13)

where $A_{n,m} = E[p \in A_i \cap B_{1}^{N} : \int_{\partial R_{0}} \frac{\partial}{\partial n} N(z, p) ds > \frac{1}{2l}]$.

Hence there exists a number $m(n(\epsilon))$ for any number $n(\epsilon)$ such that

$$\text{Cap}(A_{n,m}) \geq \frac{1}{2} \text{Cap}(A_i) \geq \frac{1}{4} \text{Cap}(F) \text{ for } m > m(n(\epsilon)).$$

By the definitions of $\text{Cap}(A_{n,m})$, there exists a positive canonical mass distribution $\mu$ on $A_{n,m}$ such that $\int d\mu = \mathfrak{M} > 0$ and its potential $U(z)$ satisfies $U(z) \leq 1$ in $R$. Since $N(z, p)$ is continuous with respect to $p$, we can find a sequence $\{U_i(z)\}$ of the form $U_i(z) = \sum_{j=1}^{i} c_j N(z, p_j) : p_j \in A_{n,m} : c_j > 0$, $\sum c_j = \mathfrak{M} < \frac{1}{4} \text{Cap}(F)$ such that $U_i(z) \to U(z)$ for any compact set $R_m$ of $R$. Hence we can suppose $U_i(z) < 1 + \epsilon$ on $R_{m(n(\epsilon))}$, whence $(1 + \epsilon)\omega(F_n \cap R_{m(n(\epsilon))}, z) \geq U_i(z)$ on $R_{m(n(\epsilon))}$. Hence by (12) and (13) we have by $\omega(F_n \cap R, z) \geq \omega(F_n \cap R_m, z)$

$$\int_{\partial R_{0}} (1 + \epsilon) \frac{\partial}{\partial n} \omega(F, z) ds + \epsilon \geq \int_{\partial R_{0}} (1 + \epsilon) \frac{\partial}{\partial n} \omega(F_n \cap R_{m(n(\epsilon))}, z) ds \geq \int_{\partial R_{0}} (1 + \epsilon) U(z) \frac{\partial}{\partial n} N(z, p_j) ds \geq \mathfrak{M} \geq \frac{\mathfrak{M}}{2l} > 0.$$

Thus by letting $\epsilon \to 0$ $\omega(F, z) > 0$.

Proof of 2). By $G \supseteq G_{n} \ni p (p \in F \cap B_{1}^{N})$ we have by Theorem 3). 2) $\alpha_{n} N(z, p) = N(z, p)$ and by Theorem 4). 3) $\alpha_{n}^{*} N'(z, p) = N'(z, p) > 0$ for any $n$ and
\[
\int_{r} \frac{\partial}{\partial n} N'(z, p) ds > 0,
\]
where \( N'(z, p) = N(z, p) - c_q N(z, p) \). Let \( A_{t, n} = E \left[ p \in F \cap B_r^n : \int_{r} \frac{\partial}{\partial n} N'(z, p) ds > \frac{1}{l} \right] \). Then \( A_{t, n} \) and \( A_t \) are measurable, \( A_{t, n} = A_t \) and \( \sum A_{t, n} = F \cap B_r^n \). Hence there exists a set \( A_{t, n} \) such that \( \text{Cap}(A_{t, n}) \geq \frac{1}{2} \text{Cap}(F) \) for any \( n \). Now \( N'(z, p) = \lim_{L=\infty} N'(z, p) = \lim_{m=\infty} D \cap R_m \), for any domain \( D \) in \( G \) can be proved as super-harmonic functions in \( R - R_m \), where \( D \cap R_m \) is a function in \( G \) such that \( N'(z, p) = N(z, p) \) on \( (D \cap R_m) \cap G + \partial G \) and \( D \cap R_m \) has M. D. I. over \( G - (D \cap R_m) \). Hence \( \sum_{m} A_{t, n, m} = A_{t, n} = A_t \), where

\[
A_{t, n, m} = E \left[ p \in A_t : \int_{r} \frac{\partial}{\partial n} N'(z, p) ds > \frac{1}{2l} \right].
\]

There exists a number \( m(n) \) such that \( \text{Cap}(A_{t, n, m}) > \frac{1}{4} \text{Cap}(F) = \mathfrak{W} \) for \( m > m(n) \). By the definition of \( \text{Cap}(A_{t, n, m}) \), there exists a positive canonical mass distribution \( \mu \) on \( A_{t, n, m} \) such that \( \frac{\mathfrak{W}}{2} = \int d\mu \) and the potential \( U(z) \) of \( \mu < 1 \) in \( R - R_m \). Since \( G \cap R_m \) is compact, \( U(z) \) can be approximated on \( G \cap R_m \) by a sequence \( U_i(z) = \sum_{j=1}^{i} c_j N(z, p_j) : p_j \in A_{i, m}, \sum c_j = \frac{\mathfrak{W}}{2} \). Hence for any given number \( \epsilon > 0 \) \( U_i(z) < 1 + \epsilon \) on \( G \cap R_m \) for \( i > i(\epsilon) \). Whence

\[
(1 + \epsilon) \omega(G_n \cap R_m, z, G, \Gamma) \geq U_i(z) = \sum_{j=1}^{i} c_j N(z, p_j) \text{ on } G \cap R_m.
\]

On the other hand, clearly \( N(z, p) \geq N'(z, p) \) in \( G \) and \( \sum c_j N'(z, p_j) = 0 \leq \omega(G_n \cap R_m, z, G, \Gamma) \) on \( \partial G \). Whence by the maximum principle

\[
(1 + \epsilon) \omega(G_n \cap R_m, z, G, \Gamma) \geq \sum_{j=1}^{i} c_j N'(z, p_j) \quad \text{and}
\]

\[
\int (1 + \epsilon) \frac{\partial}{\partial n} \omega(G_n \cap R_m, z, G, \Gamma) ds \geq \int \frac{\partial}{\partial n} (\sum c_j N'(z, p_j)) ds \geq \frac{\text{Cap}(F)}{8}.
\]

Let \( i \to \infty \), then \( m \to \infty \), then \( n \to \infty \) and then \( \epsilon \to 0 \). Then

18) See theorem 3 of P.
\[
\int \frac{\partial}{\partial n} \omega(\{G_n\}, z, \Gamma) \, ds \geq \frac{\text{Cap}(F)}{8}. \quad \text{Thus } \omega(\{G_n\}, z, \Gamma) > 0.
\]

By Theorem 5). 2) we have at once \(\omega(\{G_n\}, z, G) > 0\) from \(\omega(\{G_n\}, z, G, \Gamma) > 0\). Theorem 7) is an extension of the following

**Theorem**. Let \(F\) be a closed set of positive logarithmic capacity in the \(z\)-plane. Let \(S(p)\) be a sector with vertex at \(p\) and with aperture \(\theta: 2\pi > \theta > 0\). Let \(G\) be a domain (its connectivity may be \(\infty\)). Suppose \(G\) contain an endpart of \(S(p)\) for any \(p \in F\). Then \(\omega(F, z, G, \Gamma) > 0\), where \(\Gamma\) is a compact arc on \(\partial G\).

4. Correspondence of sets on boundaries of Riemann surfaces by analytic functions.

Let \(R\) be a covering surface over \(\mathcal{R}\). We suppose \(R\) is a surface with positive boundary and \(N\)-Martin's topology is defined on \(R + B = \mathcal{R} (N(z, p)\) is defined in \(R - R_0\) and suppose that \(R + B\) (\(B\) is the boundary of \(R\) and \(R\) is of with null or positive boundary) is compact with respect to the topology defined by a metric in \(R\). Let \(\mathcal{F}\) be a closed set in \(R + B\) and let \(D\) be a compact disc in \(R\) such that \(\text{dist} (\mathcal{F}, D) > 0\). We define the capacity

\[
\int \frac{\partial}{\partial n} \omega(\mathcal{F}, w, R - D) \, ds
\]

of \(\mathcal{F}\), where \(\omega(\mathcal{F}, w, R - D)\) is C. P. of \(\mathcal{F}\). Clearly if \(\mathcal{F}\) is contained in a local parameter disc \(C(p)\) of \(p \in \mathcal{R}\), \(\mathcal{F}\) is of capacity zero if and only if \(\mathcal{F}\) is of logarithmic capacity zero.

Put \(\mathcal{F}_n = \bigcap \{z \in \mathcal{R} : \text{dist}(w, \mathcal{F}) \leq \frac{1}{n}\}\). If for any given \(n\), there exists a number \(i(n)\) such that \(\mathcal{F}_n\) and \(\mathcal{F}_{n+i}\) are Dirichlet-disjoint, we call \(\mathcal{F}\) a \(D\)-regularly closed set. Of course, if \(\mathcal{F}\) is a compact set in \(\mathcal{R}\), we can construct a domains \(D_1\) and \(D_2\) such that \(D_1 \subset D_2, D_2 \supset \mathcal{F}_{n+i}, D \subset \mathcal{F}_n, \text{dist} (\partial D_1, \partial D_2) > 0\), \(\partial D_i (i = 1, 2)\) is composed of a finite number of analytic curves and there exists a harmonic function in \(D_1 - D_2\) such that \(U(w) = 1\) on \(\partial D_1, U(w) = 0\) on \(\partial D_2\) and \(D(U(w)) < \infty\). Put \(U(w) = 1\) in \(R - D_1, U(w) = 0\) in \(D_2\). Thus \(\mathcal{F}\) is \(D\)-regularly closed.

**Cluster sets at \(p \in B_n\).** Let \(w = f(z)\) be an analytic function: \(z \in \mathcal{R}, w \in \mathcal{R}\). Put \(M(f(p)) = \bigcap f(G_i)\), where \(G_i\) runs over all domains \(G_i, a\) and

\[
f(p) = \bigcap \{f(D(p) \cap \mathcal{V}_n(p))\}.
\]

Then we have the following

**Lemma.** Let $\mathfrak{F}$ be a closed set on $R+B$. Then

1. Suppose $f(p) \in \mathfrak{F}$. Then $f^{-1}(\mathfrak{F}_n)$ contains an endpart of $A(p)$.

2. Suppose $M(f(p)) \in \mathfrak{F}$. Then $f^{-1}(\mathfrak{F}_n) \ni p$ for any $n$.

In fact, clearly $f(p)$ is closed and non void. For any given number $n$ there exists a number $m$ such that $f(\Delta(p) \cap v_m(p)) \subset \mathfrak{F}_n$. Hence $f^{-1}(\mathfrak{F}_n) \supset (\Delta(p) \cap v_n(p))$. Next $M(f(p))$ is closed. Since $R+B$ is a metric space and separable, we can find a sequence $G_1, G_2, \cdots$ from $\{G_i : \exists \rho \text{ such that } \prod_{i=1}^{\infty} f(G_i) \cap \mathfrak{F}_{n^\prime} \supset \mathfrak{F}_n \}$. Hence for any given $n$ there exists a number $m$ such that $\prod_{i=1}^{\infty} f(G_i) \cap \mathfrak{F}_{n^\prime} \supset \mathfrak{F}_n$. Put $G_m' = G_1 \cap G_2 \cdots \cap G_m$. Then $G_m' \ni p$. Whence by $f^{-1}(\mathfrak{F}_m) \supset G_m'$ we have $f^{-1}(\mathfrak{F}_n) \ni p$ for any $n$.

Let $n(w)$ be the number of points lying over $w \in R$. Put $\bar{n}(w) = \lim_{n \to \infty} n(w)$: $w \in R+B$. If $\bar{n}(w) < \infty$ for any point $p$ of $\mathfrak{F}$, we can find a number $M$ such that $n(w) < M < \infty$ for $w \in (v_n(w) \cap R)$. We shall prove the following

**Theorem 8.** Let $w = f(z)$ be an analytic function from $R$ into $R$. Then

1. Let $\mathfrak{F} \subset (R+B)$ be a $D$-regularly closed set of capacity zero and suppose $\bar{n}(w) < \infty$ for $w \in \mathfrak{F}$. Then the set $A = E[p \in B_n^1 : f(p) \subset \mathfrak{F}]$ is of capacity zero.

2. Let $\mathfrak{F} \subset (R+B)$ be a (not necessarily $D$-regularly) closed set of capacity zero and suppose $\bar{n}(w) < \infty$ for $w \in \mathfrak{F}$. Then the set $A = E[p \in B_n^1 : M(f(p)) \subset \mathfrak{F}]$ is of capacity zero.

**Proof of 1.** By the condition of 1) we can find a number $n'$ such that $n(w) \leq M < \infty$ in $\mathfrak{F}_{n'}$ and a number $n(n')$ such that $R - f^{-1}(\mathfrak{F}_{n'})$ has a compact disc $D$. In fact, let $G$ be a sufficiently small disc. Then $f(\text{interior of } G)$ is an open set and $f(G) \cap \mathfrak{F}_{n'} = 0$ for $n \geq n'$. Hence there exists at least one point $w_0$ such that $w_0 - f(G)$ and $w_0 \in \mathfrak{F}_{n'}$. $f^{-1}(\mathfrak{F}_{n'} \cap R)$ is closed and $f^{-1}(\mathfrak{F}_{n'}) \ni p_0 \in f^{-1}(w_0)$. Hence there exists a neighbourhood $v(p_0)$ such that $v(p_0) \cap f^{-1}(\mathfrak{F}_{n'}) = 0$ ($n > n'$). Take $v(p_0)$ as $D$. Then $D$ is the required domain.

Assume $A$ is of positive capacity. Then there exists a positive canonical mass distribution $\mu$ whose $I(\mu) < \infty$. In this case we can find also a closed set $F$ in $A$ of positive capacity. Now a closed set $F$ is of positive capacity if and only if $F$ is of function theoretic capacity positive$^{20}$, whence $\omega(F, z) (= \omega(F, z, R-R_0)) > 0$. Let $D$ be a compact disc in $R$ such that $f(D) \subset D$.  

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20) See p. 73 of P.
Then $\omega(F, z) > 0$ if and only if $\omega(F, z, R - D) > 0$. Similarly by $\text{Cap}(\mathfrak{F}) > 0$ we have $\omega(\mathfrak{F}, w, R - D) > 0$. Let $n''$ be a number such that $D \subset R - \mathfrak{F}_{n''}$ ($n'' > n'$) and fix $n''$ at present.

Since $f^{-1}(p) \subset \mathfrak{F}$, by lemma $f^{-1}(\mathfrak{F}_{m})$ contains an endpart of $J(p)$ for any $p$ and $m$. Then by Theorem 7). 1) by putting $G = f^{-1}(\mathfrak{F}_{m})$

$$\lim_{m \to \infty} \omega(f^{-1}(\mathfrak{F}_{m}), z, R - D) \geq \lim_{m \to \infty} \omega(F \cap f^{-1}(\mathfrak{F}_{m}), z, R - D) > 0. \quad (14)$$

Now $\mathfrak{F}$ is $D$-regularly closed and there exists a $C$-function $U(w)$ in $\mathbb{R}$ such that $U(w) = 0$ on $\partial \mathfrak{F}_{n''}$, $U(w) = 1$ on $\mathfrak{F}_{n''} + i_{0}$ and $D(U(w)) \leq L < \infty$. Hence $D(\omega(\mathfrak{F}_{m}, w, \mathfrak{F}_{n''})) \leq L$ for $m > n'' + i_{0}$ and since $\mathfrak{F}$ is of capacity zero

$$D(\omega(\mathfrak{F}_{m}, w, \mathfrak{F}_{n''})) \downarrow 0 \quad \text{as} \quad m \to \infty. \quad (15)$$

Consider $U(z) = U(f^{-1}(w))$ in $f^{-1}(\mathfrak{F}_{n''}) - f^{-1}(\mathfrak{F}_{m})$. Then by $n(w) \leq M$ in $\mathfrak{F}_{n''}$

$$D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n''})) \leq MD(U(w)) \leq ML < \infty$$

$$D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n''})) \downarrow 0 \quad \text{as} \quad m \to \infty. \quad (16)$$

Let $m \to \infty$. Then by (15) and (16)

$$D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n''})) \downarrow 0 \quad \text{as} \quad m \to \infty. \quad (17)$$

On the other hand, by the Dirichlet principle and by (14) $D(\omega(f^{-1}(\mathfrak{F}_{m}) \cap F, f^{-1}(\mathfrak{F}_{n''})) \geq D(\omega(f^{-1}(\mathfrak{F}_{m}) \cap F, z, R - D)) > 0$. This contradicts (17). Hence $A$ is a set of capacity zero.

Proof of 2). As (1) we can find a number $n'$ such that $n'(w) \leq M < \infty$ in $\mathfrak{F}_{n'}$ and $R - f^{-1}(\mathfrak{F}_{n'})$ contains a compact disc $D$. Fix $n'$ at present. Put $G = R \cap f^{-1}(\mathfrak{F}_{n'})$. Then $G$ may consist of an enumerably infinite number of domains (components) $G^{1}, G^{2}, \cdots$. Since $M(f(p)) \subset \mathfrak{F}, G = f^{-1}(\mathfrak{F}_{n})$ contains any point of $A \cap B^{n}$ by Lemma 4). Let $A_{t}$ be the set of points contained $N$-approximately in $G^{t}$. Then $A_{t}$ is a Borel set by Lemma b) and $\sum A_{t} = A$.

Hence there exists at least one component $G'$ of $G$ such that $G' \ni \rho, \rho \in A'$ and $\text{Cap}(A') > 0$, where $A'$ is the set of points contained in $G'$. Hence we can find a closed set $F$ of positive capacity in $A'$. Since $\mathfrak{F}$ is a set of capacity zero, we have by Theorem 5). 3) $0 = \omega(\mathfrak{F}, w, R - D)$ implies

$$D(\omega(\mathfrak{F}_{n}, w, \mathfrak{F}_{n}, \Gamma_{w})) \to 0 \quad \text{as} \quad n \to \infty, \quad (18)$$

where $\Gamma_{w}$ is a compact arc on $\partial \mathfrak{F}_{n}$.

Let $\Gamma_{z}$ be a compact arc on $\partial G'$ such that $f(\Gamma_{z}) \subset \Gamma_{w}$. Since $G_{n} \ni \rho : \rho \in (B_{n} \cap F)$, we have by Theorem 7). 2)

$$\omega(\{F \cap G_{n}\}, z, G', \Gamma_{z}) = \lim \omega(F \cap G_{n}, z, G', \Gamma_{n}) > 0. \quad (19)$$
Now \( U(z) = f^{-1}(\omega(\mathfrak{F}_n, \omega, \mathfrak{F}_n, \Gamma_w)) \) is a harmonic function in \( G' - (f^{-1}(\mathfrak{F}_n) \cap G') \) such that \( U(z) = 1 \) on \( f^{-1}(\mathfrak{F}_n) = G_n \cup (G_n \cap F) \) and \( U(z) = 0 \) on \( \Gamma \subset f^{-1}(\Gamma_w) \) and since \( \omega(G_n, z, \mathfrak{F}_n, \Gamma_z) \) has M. D. I.,

\[
D(\omega(G_n, z, G', \Gamma_z)) \leq MD(\omega(\mathfrak{F}_n, w, \mathfrak{F}_n, \Gamma_w)) \downarrow 0 \quad \text{as} \quad n \to \infty. \quad (20)
\]

But \( \omega(G_n, z, G', \Gamma_z) \geq \omega(G_n \cap F, z, G', \Gamma_z) \).

This contradicts (20). Thus \( A \) is a set of capacity zero.

Similarly as (1) we can prove the following

Theorem 7.3. Suppose \( n(w) \leq M < \infty \) outside of a compact set \( D \) of \( R \). Then (1) is valid even if \( \mathfrak{F} \) is not \( D \)-regularly closed.

In fact, in the proof of 1) take \( R - \Omega' \) instead of \( \mathfrak{F}'_n \), where \( \Omega' \) is a compact domain such that \( \Omega' \supset \Omega, \Omega' \cap \mathfrak{F}'_n = 0 \) (\( n_0 \) is a certain number such that \( n_0 > 1 \)) and \( \text{dist} (\partial \Omega, \partial \Omega') > 0 \). Then \( D(\omega(f^{-1}(\mathfrak{F}_n), z, \Omega - f^{-1}(\Omega'))) \leq MD(\omega(\mathfrak{F}_n, w, \Omega - \Omega')) \leq L < \infty \) for \( n > n_n \) and \( \omega(\{G_n\}, z, \Omega - f^{-1}(\Omega')) \) can be defined, where \( G_n = f^{-1}(\mathfrak{F}_n) \). We used the \( D \)-regularity of \( \mathfrak{F}'_n \) only to define \( \omega(\{G_n\}, z, f^{-1}(\mathfrak{F}'_n)) \) in the proof of 1). Hence the above theorem can be proved similarly as 1).

**Remark.** The assumption of 1) on \( \mathfrak{F} \) is stronger than that of 2) but the assumption of 1) on the cluster set at \( p \) is weaker than that of 2). The assumption of 3) on the cluster set is that of 1) but the assumption of 3) on \( n(w) \) is strongest.

We shall consider an extension of Beurling's theorem. Let \( \mathfrak{F} \) be a compact set in \( R \) and suppose \( \mathfrak{F} \) is contained in a local parameter disc \( \Omega : \Omega \) can be thought equivalent to \( |z| < 1 \) and suppose \( \mathfrak{F} \) is contained in \( E(z : |z| < \frac{1}{2}) \).

In this case we suppose that the distance between two points in \( \Omega \) means the euclidean distance. Let \( A(r) \) be the area of \( R \) over \( \mathfrak{F} = E(z \in \Omega : \text{dist} (z, \mathfrak{F}) \leq r: r < \frac{1}{2} \) \). Then we have the following

**Theorem 9.** Let \( w = f(z) \) be an analytic function from \( R \) (with positive boundary and with N-Martin's topology) into \( R \). Let \( \mathfrak{F} \) be a closed set contained in a local parameter disc \( \Omega = E(|z| < 1) \) and \( \mathfrak{F} \) in \( E(|z| < \frac{1}{2}) \).

Suppose

\[
\lim_{r \to 0} \frac{A(r)}{r} < \infty.
\]

Then \( A = E(z \in B_1^N : M(f(p)) \subset \mathfrak{F} \) is a set of capacity zero.
Proof. We can find a sequence $r_i$, $r_z$, \ldots such that \( \frac{A(r_i)}{r_i^2} < K < \infty \). Put $G_i = f^{-1}(\mathfrak{F}_i)$, where $\mathfrak{F}_i = E[\omega \in \Omega : \text{dist}(w, \mathfrak{F}_i) \leq r_i]$. Then we see for sufficiently large number $i$ \( R - f^{-1}(\mathfrak{F}_i) \neq 0 \). This implies that $R - f^{-1}(\mathfrak{F}_i)$ contains a compact disc. Hence we can suppose without loss of generality that $R - f^{-1}(\mathfrak{F}_i)$

$\supset R_d \{ R_n \} (n = 0, 1, 2, \ldots)$ is an exhaustion of $R$ for $i \geq 1$ and $\frac{A(r_i)}{r_i^2} < K$ for $i \geq 1$. Now $G_i = f^{-1}(\mathfrak{F}_i)$ consists of enumerable infinite number of components $G', G', \ldots \colon \sum_j G_j = G_i$. By Lemma b) \( A^j = E[p \in A : p \in G'] \) and $E[p \in A : p \in G]$ are measurable. Since $G_i \supset p$ implies that there exists only one component $G_j$ of $G_i$ containing $p$ $N$-approximately. Hence

$$\sum_j A^j = A.$$  

Assume $A$ is a set of positive capacity. Then there exists at least one $G_i$ such that $\text{Cap}(A^i) > 0$. We fix $G_i$ at present and denote it by $G$. We can find a closed set $F$ of positive capacity in $A^i$. Put $G_i = G \cap f^{-1}(\mathfrak{F}_i)$. Then also $G_i \supset p$ for any $p$ of $A^i$ and $G_i$ is clearly non compact. Hence by Theorem 7.2) $\omega(F \cap \{ F_i \}, z, G, \Gamma) > 0$ and by Theorem 5). 3) $\omega(\{ F_i \}, z, R - R_0) > 0$, where $\Gamma$ is a compact arc on $\partial G$. Put $\omega(\{ F_i \}, z, R - R_0) = U(z) > 0$. Then $U(z) = 0$ on $\partial R_0$, \( \int_{\partial R_0} \frac{\partial}{\partial n} U(z) ds = D(U(z)) = \int_{\partial V_M} \frac{\partial}{\partial n} U(z) ds \) for almost $M : 0 < M < 1$, where $V_M = E[z \in R : U(z) > M]$ and such $\partial V_M$ is called a regular niveau curve.

We shall show that $G_i$ and $CG_i$, $G_i, CG_i = f^{-1}(\mathfrak{F}_{i+j}) \cap G$ are Dirichlet-disjoint. In fact, $\text{dist}(\partial \mathfrak{F}_i, \partial \mathfrak{F}_{i+j}) > 0$ and $\partial \mathfrak{F}_i$ and $\partial \mathfrak{F}_{i+j}$ are compact. We can construct compact domains $\Omega_1$ and $\Omega_2$ such that $\mathfrak{F}_i \supset \Omega_1 \supset \Omega_2 \supset \mathfrak{F}_{i+j}$, $\text{dist}(\partial \Omega_1, \partial \Omega_2) > 0$ and $\partial \Omega_i (i = 1, 2)$ is composed of a finite number of components of which each one is consists of only one analytic curve. Hence we can define a $C_{\Gamma}$-function $V(w)$ in $R$ such that $V(w)$ is harmonic in $\Omega_1 - \Omega_2$, $V(w) = 1$ in $\Omega_1$, $V(w) = 0$ on $\partial \Omega_1$ and $|\frac{\partial}{\partial u} V(w)| < L$, $|\frac{\partial}{\partial v} V(w)| < L : L < \infty$, where $w = u + iv$. Put $V(z) = V(f^{-1}(w))$. Then $D(V(z)) \leq KL^2 r_i^2$. Thus $G_i$ and $G_{i+j}$ are Dirichlet-disjoint. Hence by Lemma 1). 2)

\[
(21) \quad \int_{\partial V_M} \frac{\partial}{\partial n} U(z) ds \downarrow 0 \quad \text{as } M \uparrow 1 \quad \text{for any given } G_i.
\]

Put $\alpha = D(U(z))$. Then $\int_{\partial V_M} \frac{\partial}{\partial n} U(z) ds = \alpha$ for a regular niveau curve $\partial V_M$. Let
$U_{n}^{M}(z)$ be a harmonic function in $R_{n}-R_{0}-V_{M}$ such that $U_{n}^{M}(z)=0$ on $\partial R_{0}$, $U_{n}^{M}(z)=M$ on $\partial V_{M}-R_{n}$, $\frac{\partial}{\partial n}U_{n}^{M}(z)=0$ on $\partial R_{n}-V_{M}$ and put $\alpha_{n}^{M}=D(U_{n}^{M}(z))$

$$= \int_{\partial R_{0}} \frac{\partial}{\partial n}U_{n}^{M}(z)ds \quad \text{since } R_{n}-R_{0}-V_{M} \text{ is compact}.$$ Then

$$U_{n}^{M}(z) \Rightarrow U(z) \quad \text{and} \quad \alpha_{n}^{M} \rightarrow \alpha, \text{ as } M \rightarrow 1.$$ Suppose $\partial V_{M}$ is regular. Then by $\alpha_{n}^{M} \rightarrow \alpha$ and by $\lim_{n} \int_{\partial V_{n} \cap \partial V} \frac{\partial}{\partial n}U_{n}^{M}(z)ds \geq \int_{\partial V_{M} \cap \partial V} \frac{\partial}{\partial n}U(z)ds$, we can find a numbers $M$ and $n_{1}(\epsilon, i, M)$ for any given numbers $\epsilon$ and $i$ such that

$$\int_{(\partial V_{M} \cap R_{n}) \cap G} \frac{\partial}{\partial n}U_{n}^{M}(z)ds \geq \alpha-2\epsilon \quad \text{for } n>n_{1}(\epsilon, i, M). \quad (22)$$

Put $D(\mathfrak{F}_{k})=\iint_{f^{-1}(\mathfrak{F}_{k})} \left| f^\prime(z) \right| \left\{ \left( \frac{\partial}{\partial x}U(z) \right)^{2} + \left( \frac{\partial}{\partial y}U(z) \right)^{2} \right\}^{\frac{1}{2}} dxdy$ and

$$D(\mathfrak{F}_{k}, CV_{M} \cap R_{n})=\iint_{f^{-1}(\mathfrak{F}_{k}) \cap CV_{M} \cap R_{n}} \left| f^\prime(z) \right| \left\{ \left( \frac{\partial}{\partial x}U_{n}^{M}(z) \right)^{2} + \left( \frac{\partial}{\partial y}U_{n}^{M}(z) \right)^{2} \right\}^{\frac{1}{2}} dxdy.$$ Clearly $\lim_{r \to 0} \frac{A(r)}{r^{2}} < \infty$ implies that area of $f^{-1}(\mathfrak{F}_{k}) \to 0$ as $k \to \infty$. Since $f(z)$ is analytic in $R$ and for any $R_{n}$ area of $f^{-1}(\mathfrak{F}_{k})$ in $R_{n} \to 0$ as $k \to \infty$, $D(U(z))$ over $R_{n} \cap f^{-1}(\mathfrak{F}_{k}) \to 0$ as $k \to \infty$. Further $D(U(z))<\infty$ and $D(U(z)) \to 0$ as $n \to \infty$ and

$$D(U(z)) \leq D(U(z)) + D(U(z)) \leq A(r_{k})<\epsilon \quad \text{for } k>k_{1}(\epsilon). \quad (23)$$

Put $D(\mathfrak{F}_{k}, CV_{M})=\iint_{CV_{M} \cap f^{-1}(\mathfrak{F}_{k})} \left| f^\prime(z) \right| \left\{ \left( \frac{\partial}{\partial x}U(z) \right)^{2} + \left( \frac{\partial}{\partial y}U(z) \right)^{2} \right\}^{\frac{1}{2}} dxdy$. Then
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by \( U^M_n(z) \Rightarrow U(z) \) \( D(\mathfrak{F}_k, CV_M \cap R_n) \rightarrow D(\mathfrak{F}_k, CV_M) \) as \( n \rightarrow \infty \). Hence for any \( \varepsilon \) we can find a number \( n_2(\varepsilon, k, M) \) such that

\[
D(\mathfrak{F}_k, CV_M \cap R_n) \leq D(\mathfrak{F}_k, CV_M) + \varepsilon \leq D(\mathfrak{F}_k) + \varepsilon
\]

for \( n \geq n_2 \).

Next by (22) for any given \( \mathfrak{F}_k \) and \( \mathfrak{F}_i \) \((i > k)\) and \( \varepsilon \) we can find \( M' \) and \( n_3 \) such that

\[
\int \frac{1}{r} \left| f'(re^{i\theta}) \right| \left| \frac{\partial}{\partial r} U^M_n(re^{i\theta}) \right| r \, dr \, d\theta = \alpha - 2\varepsilon
\]

and

\[
D(\mathfrak{F}_k, CV_{M'} \cap R_n) \leq \sqrt{K\varepsilon} r_k + \sqrt{\varepsilon}
\]

(24)

for \( n > n_3 > \max(n_1, n_2) \).

We consider \( D(\mathfrak{F}_k, CV_{M'} \cap R_n) \) in the following. Let \( \zeta = \exp(U^M_n(z) + iV^M_n(z)) = re^{i\theta} \), where \( V^M_n(z) \) is the conjugate function of \( U^M_n(z) \). Then \( |\zeta| = e^{M'} \) on \( \partial V_{M'} \cap R_n \) and by (22)

\[
\int_{\partial V_{M'} \cap R_n \cap G_i} d\theta = \int_{\partial V_{M'} \cap R \cap G_i} \frac{\partial}{\partial n} U^M_n(z) \, ds \geq \alpha - 2\varepsilon
\]

(25)

Now

\[
D(\mathfrak{F}_k, CV_{M'} \cap R_n) = \int \int_{\partial V_{M'} \cap R_n \cap \partial \mathfrak{F}} \left| f'(z) \right| \left\{ \left( \frac{\partial}{\partial r} U^M_n(z) \right)^2 + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} U^M_n(z) \right)^2 \right\}^\frac{1}{2} \, r \, dr \, d\theta
\]

(26)

is harmonic in a compact domain \( R_n - R_0 - V_{M'} \). Hence \( U^M_n(z) \) has a finite number of branch points and we can trace the trajectories \( T_{\theta} \) along which \( \theta = \text{const} \) from \( r = 1 \) \((U^M_n(z) = 0 \text{ on } \partial R_0)\) to \( r = e^{M'} \) \((U^M_n(z) = M' \text{ on } \partial V_{M'})\). Let \( \Theta \) be the set of \( \theta \) such that \( T_{\theta} \) intersects \( G_i \) when \( z \) goes from \( R_0 \) to \( \partial V_{M'} \) along \( T_{\theta} \). Then by (25) \( \Theta \geq \alpha - 2\varepsilon \). If \( \theta \in \Theta \), image \( f(T_{\theta}) \) of \( T_{\theta} \) must intersect \( G_k \) and \( G_i \) whence

\[
D(\mathfrak{F}_k) + \varepsilon \geq D(\mathfrak{F}_k, CV_{M'} \cap R_n) \geq (\alpha - 2\varepsilon) (r_k - r_i)
\]

(26)

Let \( r_i \rightarrow 0 \) and then \( \varepsilon \rightarrow 0 \). Then (26) contradicts (23). Hence \( A \) is a set of capacity zero.

As an application of Theorem 9 we shall prove

**Theorem 10.** Suppose \( R \) is a covering surface almost finitely sheeted over \( \mathbb{R} \) and \( \mathfrak{k} = R + B \) is compact. Then the set \( A \) of point \( p \in B_{1}^{N} \) such
that \( \mathcal{F} \) is a set of capacity zero, where \( \mathcal{F} \) satisfies the condition
\[
\lim_{r \to 0} \frac{A(r)}{r^2} < \infty,
\]
and \( A(r) \) is the area of \( R \) over \( \mathcal{F}_r = E[w: \text{euclidean dist}(w, \mathcal{F}) \leq r] \).

**Proof.** By Theorem 1. \( M(f(p)) = \) one point \( q \) for any point \( p \) of \( B_i^N \) except at most a set of capacity zero. Suppose \( M(f(p)) = q \). We can find a sequence \( G_i(i=1,2, \cdots) \) such that \( G_i \ni p \) and \( \bigcap_i \overline{f(G_i)} = q \) from \( \{G_i\} : G_i \ni p \).

Put \( G'_i = G_i \cap v_i(p) \). Then also \( G'_i \ni p \) and by the Theorem 6). 3) \( \Delta(p) \cap G'_i \) is an angular domain, whence \( f(\Delta(p) \cap G'_i) \neq 0 \) for every \( i \). On the other hand, \( \bigcap_i \overline{f(\Delta(p) \cap G'_i)} \subset \bigcap_i \overline{f(\Delta(p) \cap v_i(p))} = f(\Delta(p) \cap G'_i) = q \), hence \( f(\Delta(p)) = M(f(p)) = q \in \mathcal{F} \). Thus by Theorem 9) we have Theorem 10.

In the previous paper\(^{21}\) we constructed a covering surface \( R \) over the \( w \)-plane: \( w = f(z) : z \in R \) with finite area such that \( R \) has a singular point \( p \in B_i^N \) with the following properties: 1) \( \omega(p, z) > 0 \) and 2) \( \bigcap_n \overline{f(v_n(p))} = \) one point.

This example shows that even when \( \mathcal{F} = \) one point, some condition (for instance \( \lim_{r \to 0} \frac{A(r)}{r^2} < \infty \)) is necessary for the validity of Theorems 9) or 10.)

*(Remark to the condition in Theorem 9).* The condition \( \lim_{r \to 0} \frac{A(r)}{r^2} < \infty \) in Theorem 9 imposed on a closed set \( \mathcal{F} \) in \( R \) does not necessarily mean that \( \mathcal{F} \) is so small as \( \text{Cap}(\mathcal{F}) = 0 \) but means the part of \( R \) over \( \mathcal{F}_r \) is so small as \( \omega(\{\mathcal{F}_r\}, z) \to 0 \) as \( r \to 0 \). In reality the part of \( R \) over \( \mathcal{F}_r \) may be so small even when \( \mathcal{F} \) is a segment.

Department of Mathematics
Hokkaido University

(Received June 24, 1963)

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