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CORRESPONDENCE OF BOUNDARIES
OF RIEMANN SURFACES

By

Zenjiro KURAMOCHI

Let $R$ be a Riemann surface with positive boundary and let $R_n \ (n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundary $\partial R_n$. Let $N(z, p)$ be a positive harmonic function in $R - R_0 - p : p \in R - R_0$ such that $N(z, p) = 0$ on $\partial R_0$, $N(z, p)$ has a logarithmic singularity at $p$ and $N(z, p)$ has minimal $D*irichlet$ integral over $R - R_0$, where $D*irichlet$ integral is taken with respect to $N(z, p) + \log|z-p|$ in a neighbourhood of $p$. We call such $N(z, p)$ the $N$-Green's function of $R - R_0$ with pole at $p$.

Consider now a sequence of points $\{p_i\}$ of $R - R_0$ having no points of accumulation in $R - R_0 + \partial R_0$. Since the functions $N(z, p_i) \ (i=1, 2, \cdots)$ forms, from some $i$ on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore is convergent in every compact part of $R - R_0$ to a positive harmonic function. A sequence $\{p_i\}$ of $R - R_0$ having no point of accumulation in $R - R_0$, for which the corresponding $N(z, p_i)$'s have the property just mentioned, that is, $\{N(z, p_i)\}$ coverges to a harmonic function—will be called fundamental. If two fundamental sequences determine the same limit function $N(z, p)$, we say that they are equivalent. Two fundamental sequences equivalent to a given one determine an ideal boundary point of $R$. The set of all the ideal boundary points of $R$ will be denoted by $B$ and the set $R - R_0 + B$ by $\overline{R} - R_0$. The domain of definition of $N(z, p)$ may now be extended by writing $N(z, p) = \lim_{i} N(z, p_i) \ (z \in R - R_0, p \in B)$, where $\{p_i\}$ is any fundamental sequence determining $p$. The function $N(z, p)$ is characteristic of the point $p$ of their corresponding $N(z, p)$ as a function of $z$. The function $\delta(p_i, p_2)$ of two points $p_i$ and $p_2$ in $\overline{R} - R_0$ is defined as

$$\delta(p_i, p_2) = \sup_{z \in \overline{R} - R_0} \left| \frac{N(z, p_i)}{1 + N(z, p_i)} - \frac{N(z, p_2)}{1 + N(z, p_2)} \right|.$$ 

Evidently, $\delta(p_i, p_2)=0$ is equivalent to $N(z, p_i)=N(z, p_2)$ for all points $z$ in $\overline{R} - R_0$. Therefore we have $N(z, p_i) = N(z, p_2)$ in $\overline{R} - R_0$, i.e. $\delta(p_i, p_2)=0$ implies $p_i=p_2$ and it is clear that $\delta(p_i, p_2)$ satisfies the axioms of distance. Therefore $\delta(p_i, p_2)$ can be considered as the distance between $p_i$ and $p_2$ of $\overline{R} - R_0$. 


The topology (we call $N$-Martin's topology) induced by this metric is homeomorphic to the original topology, when it is restricted in $R-R_0$ and we continue this topology into $R_0$. Since $\{N(z, p_i)\} : p_i \in \overline{R}-R_0$ is also a normal family, both $\overline{R}-R_0 + \partial R_0 + B$ and $B$ are closed and compact. For a fixed point $z$, $N(z, p)$ is continuous with respect to this metric as a function of $p$ in $\overline{R}-R_0$ except at $p$. Other topologies can be introduced on any Riemann surfaces (with null or positive boundary). For example $K$-Martin's topology, Green's topology, and Stoilow's topology. We proved that $N$-Martin's, and Stoilow's and Green's topologies are Dirichlet-separating. We define the ideal boundary $B$ of $R$ by the completion of $R$. Then $\overline{R}+B$ is compact with respect to $N$-Martin’s and Stoilow’s topology.

We use in the present paper the notions of capacity, superharmonic functions, $N$-minimal points and others which are referred to the papers\(^{9}\). Suppose $N$-Martin's topology is introduced on $\overline{R}=R+B$. Let $G$ be a domain and $caN(z, p)$ be the least positive superharmonic function in $R-R_0$ larger than $N(z, p)$ on $CG$ (complementary set of $G$), where $p \in B^N (B^N$ is the set of $N$-minimal boundary points). If $N(z, p)>caN(z, p)$, we say that $G$ contains $p$ $N$-approximately and denote it by $G \ni p$.

Let $w=f(z) : z \in R$, $w \in R$ be an analytic function whose values fall on a basic surface $R$, where $R$ is a Riemann surface with positive or null-boundary and a topology is defined on $R+B$. Put

$$M(p) = \cap_{t} f(G_t) \text{ for } p \in B^N,$$

where $\{G_t\}$ means the all sets $G_t$ such that $G_t \ni p$.

1. For any point $p \in R$, there exists an open disc $C(p)$ in $R$ such that the area of $R$ over $C(p)$ is finite.

2. There exists a number $n_0$ such that $n(w) \leq M < \infty$ in $R-R_{n}$, where $n(w)$ is the number of points of $R$ over $p$ and $\{R_n\}$ is an exhaustion of $R$ with compact relative boundary $\partial R_n$.

If the above two conditions are satisfied, we say that $w=f(z)$ is of almost finitely sheeted. Then we proved

**Theorem 1**. Let $R$ be a covering surface with positive boundary and

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with N-Martin's topology over $\mathbb{R}$ with D.S. (Dirichlet separative\textsuperscript{4}) topology and suppose $\mathbb{R} + \mathbb{B}$ is compact with respect to the topology on $\mathbb{R}$. If $\mathbb{R}$ is a covering surface of almost finitely sheeted: $w = f(z)$, $z \in \mathbb{R}$ and $w \in \mathbb{R}$. Then $M(p)$ is defined in $\mathbb{B}$ except an $F_\varepsilon$ set of capacity zero and $S = E[p \in B_i^N \colon \text{diameter of } M(p) > 0]$ is a $G_\delta$ set of inner capacity zero, where $B_i^N$ is the set of N-minimal boundary points of $\mathbb{R}$.

This is an extension of Beurling's theorem\textsuperscript{5}.

Let $w = f(z)$ be an analytic function in $|z| < 1$ and let $\mathbb{R}$ be the Riemann surface, generated by $w = f(z)$ over the w-sphere $K$. Let a be any point of $K$ and $R_r$ be the part of $\mathbb{R}$, which lies over a disc $C_r(a)$ of radius $r$ with $a$ as its centre and $A(r)$ be the spherical area of $R_r$. If

$$\lim_{r \to 0} \frac{A(r)}{r^2} < \infty,$$

then a is called an ordinary value of $f(z)$ in Beurling's sense. Then

**Theorem 2**\textsuperscript{5} (A. Beurling). Let $w = f(z)$ be an analytic function in $|z| < 1 : w \in w$-sphere. Suppose the spherical area of the covering surface generated by $w = f(z)$ is finite. Let a be an ordinary value of $f(z)$ in Beurling's sense. Then the set $E$ of $e^i$ such that $\lim_{r \to 1} f(re^i) = a$ is a set of capacity zero.

M. Tsuji\textsuperscript{6} proved the above theorem under the condition that $f(z)$ takes three values $\alpha, \beta, \gamma$ only a finite number of times in $|z| < 1$ instead of the condition that the spherical area of the image is finite.

The purpose of the present paper is to extend Theorem 2 to Riemann surfaces and to study the correspondence of boundaries of Riemann surfaces.

1. **Properties of domains**\textsuperscript{7} containing $p \in B_i^N$. $N$-approximately.

We shall study the properties of a domain $G \ni p : p \in B_i^N$some of which are useful in the following. We proved the following

**Lemma**\textsuperscript{5}. a). For $p \in B_i^N$ we have $1^\circ$. If $G_i \ni p, \bigcap G_i \ni p$. $2^\circ$. If $G \ni p$, $(\text{int } CG) \notin p. 3^\circ$. $E[z \in \mathbb{R} : \text{dist}(z, p) < \frac{1}{n}] = v_n(p) \ni p$. Since $\mathbb{R} + \mathbb{B}$ is closed,

\textsuperscript{4}. See 1).

\textsuperscript{5} A. BEURLING: Ensembles exceptionnelles. Act Math., 72 (1940)

\textsuperscript{6} See 5) also M. TSUJI: Beurling's theorem on exceptional sets. Tohoku Math. Journ., 2 (1950),

\textsuperscript{7} In the present paper we suppose for any domain $G$ in Riemann surfaces $\mathbb{R}$ that $\emptyset G$ consists of at most enumerably infinite number of analytic curves clustering nowhere in $\mathbb{R}$.

\textsuperscript{8} See 1) of 3) and 2).
Correspondence of Boundaries of Riemann Surfaces

by 1°) we have $M(p) = \bigcap_{\tau} \overline{f(G_{\tau})} \neq 0$, and since $R + B$ is separable with respect to the given topology on $R + B$, we can find a sequence $\{G_{\tau}\} : i = 1, 2, 3, \cdots$ from $\{G_{\tau}\}$ such that $M(p) = \bigcap_{\tau} \overline{f(G_{\tau})}$. We see by 2°) that there exists only one component $G'$ of $G$ such that $G' \ni p$. Let $G$ and $G'$ be domains such that $G \supset G'$ and $D(\omega(G' \cap B, z, G)) > 0$. Then there exists at least one point $p \in B^{\infty}_{N}$ such that $G \ni p$, where $\omega(G' \cap B, z, G) = \lim_{n} \omega_{n,n+i}(z)$ and $\omega_{n,n+i}(z)$ is a harmonic function in $(G \cap R_{n+i}) - (G' \cap (R_{n+i} - R_{n})) - R_{0}$ such that $\omega_{n,n+i}(z) = 0$ on $\partial G$, $\omega_{n,n+i}(z) = 1$ on $\delta(G' \cap (R_{n+i} - R_{n}))$, $\frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G'$ and $D(\omega_{n,n+i}(z)) \leq M < \infty$ for a certain $n_{0}$ and for every $i$.

b). Let $G$ be a domain in $R$. Then the set of points $\in B^{\infty}_{N}$ not contained in $G$ $N$-approximately is a $G_{\delta}$ set, i.e. the set of points $\in B^{\infty}_{N}$ contained in $G$ is an $F_{\sigma}$ set. It is known9) that for any $V_{M}(p) = E[z \in R : N(z, p) > M] : M < M^{*} = \sup N(z, p)$, $p \in B^{\infty}_{N}$, there exists a $v_{n}(p)$ such that $(R \cap v_{n}(p)) \subset V_{M}(p)$. Hence we have by 1°) of a)

c). For any $V_{M}(p) : M < M^{*}$ and for any domain $G \supset p$ we can find a domain $G' (= G \cap v_{n}(p))$ such that $p \in G' \subset v_{n}(p) \subset V_{M}(p)$.

d). If $\omega(p, z) = \lim_{n} \omega(v_{n}(p), z) > 0 : p \in B^{\infty}_{N}$ (this is equivalent to sup $N(z, p) < \infty)$, we call $p$ a singular point. Let $B^{\infty}_{N}$ be the set of singular points. Then $B^{\infty}_{N}$ is an $F_{\sigma}$ set and $B^{\infty}_{N} \subset B^{\infty}_{N}$. Then

definition{for any $v_{n}(p)$}$ lim_{M \to \infty} N_{V_{M}(p) \cap C_{v_{n}(p)}}(z, p) = 0$ and $\lim_{M \to \infty} \int_{\delta V_{M}(p) \cap C_{v_{n}(p)}} \frac{\partial}{\partial n} N(z, p)ds = 0$ for $p \in B^{\infty}_{N} - B^{\infty}_{N}$.

Since $\min_{M} (M, N(z, p)) = M \omega(V_{M}(p), z)$ and $\int_{\delta V_{M}(p) \cap C_{v_{n}(p)}} \frac{\partial}{\partial n} N(z, p)ds = 2\pi$ for almost $M < M^{*}$, d) means that the system $\{V_{M_{n}}(p)\}$ $(M_{1} < M_{2} < M_{3} \cdots)$ is almost equivalent to the neighbourhood system $\{v_{n}(p)\}$. By c) we know $\{G_{\tau}\} : G \ni p$ is finer than $\{v_{n}(p)\}$. We shall prove the following theorem to show that $G$ has the similar properties as $\{v_{n}(p)\}$.

**Theorem 3.** Suppose $G \ni p : p \in B^{\infty}_{N}$. Then

1). $p_{(\infty, N(z, p))} = \lim_{n \to \infty} v_{n}(p, (\infty, N(z, p))) = 0$ for $p \in B^{\infty}_{N} - B^{\infty}_{N}$.

9) See p. 42 of P. and see 2).
2.  \( p \cap c_{0}N(z, p) = \lim_{n} v_{n}(p) \cap c_{0}N(z, p) = 0 \) and \( a \cap pN(z, p) = aN(z, p) = N(z, p) \) for \( p \in B_{1}^{N} \).

3.  \( \lim_{M \to M*} v_{M}(p) \cap c_{0}N(z, p) = 0 \) and \( \int_{\partial V_{M}(p) \cap c_{0}N(z, p)} \frac{\partial}{\partial n} N(z, p) \hspace{1cm} ds = 0 \) for \( p \in B_{1}^{N} \).

where \( M^{*} = \sup N(z, p) \).

By \( v_{n}(p) \ni p \) 3) implies d) of the Lemma.

Proof of 1). Assume \( p(c_{0}N(z, p)) > 0 \). Since \( p(c_{0}N(z, p)) \) has mass only at \( p \), \( p(c_{0}N(z, p)) = a \cdot N(z, p) \), where \( 1 \geq a_{1} > 0 \), by \( N(z, p) \subseteq c_{0}N(z, p) \). Now \( p \) is a set of capacity zero and \( 0 \leq U_{1}(z) = c_{0}N(z, p) - p(c_{0}N(z, p)) \) is also superharmonic\(^{10} \). Hence similarly as before \( \frac{\partial}{\partial n} N(z, p) \) and \( U_{1}(z) = a_{2} N(z, p) + U_{2}(z) \) and \( U_{2}(z) \) is superharmonic.

Thus \( a_{1} \geq a_{2} \cdots \) and \( \Sigma a_{n} \leq 1 \), \( \lim_{n} a_{n} = 0 \) and \( U_{n}(z) \). Put \( U^{*}(z) = \lim U_{n}(z) \).

Then \( a_{1} \geq a_{2} \cdots \) and \( \Sigma a_{n} \leq 1 \), \( \lim_{n} a_{n} = 0 \) and \( U_{n}(z) \). Put \( U^{*}(z) = \lim U_{n}(z) \).

Then \( U^{*}(z) \) is also superharmonic\(^{11} \) and \( U^{*}(z) \leq a_{n} N(z, p) \) for any \( n \). Thus \( U^{*}(z) = 0 \) and \( N(z, p) = \Sigma a_{n} N(z, p) \) on \( CG \). Whence \( \Sigma a_{n} \leq 1 \) and \( c_{0}N(z, p) = N(z, p) \). This contradicts \( G^{N} \in p \). Hence \( p(c_{0}N(z, p)) = 0 \).

Proof of 2). Case 1. \( p \in B_{1}^{N} - B_{2}^{N} \). In this case \( v_{n}(p) \cap c_{0}N(z, p) = N(z, p) = c_{0}N(z, p) = v_{n}(p) \cap c_{0}N(z, p) \) on \( v_{n}(p) \cap CG \). Now since \( aU(z) \leq bU(z) \) for a superharmonic function \( U(z) \) for \( A \supseteq B \),

\[
\chi_{v_{n}(p) \cap c_{0}N(z, p)} = \chi_{v_{n}(p) \cap c_{0}N(z, p)} \leq \chi_{v_{n}(p) \cap c_{0}N(z, p)} \leq \chi_{v_{n}(p) \cap c_{0}N(z, p)} \leq \chi_{v_{n}(p) \cap c_{0}N(z, p)} \leq \chi_{v_{n}(p) \cap c_{0}N(z, p)}
\]

Let \( n \to \infty \). Then by 1)

\[
\chi_{v_{n}(p) \cap c_{0}N(z, p)} = \chi_{v_{n}(p) \cap c_{0}N(z, p)} = 0 .
\]

Case 2. \( p \in B_{1}^{N} \) in this case it was proved that \( \omega(p \cap CG, z) = 0 \). Whence

\[
\chi_{v_{n}(p) \cap c_{0}N(z, p)} = \omega(p \cap CG, z) = 0 ,
\]

by \( \omega(p, z) \leq 1 \) and \( N(z, p) = a \cdot \omega(p, z) \), where \( a = 2\pi \int_{B_{1}} \frac{\partial}{\partial n} \omega(p, z) \hspace{1cm} ds \). We shall

\( \text{p. 32 of P (see 2))} \).

\( \text{11) We can prove } D(\min(N, U^{*}(z)) \leq D(\min(M, N(z, p)) \leq 2\pi M \). Then by p. 23 of P } U(z) \text{ is superharmonic.} \)

prove the latter part of 2). \( \nu \cap CG N(z, p) + \nu \cap \Omega N(z, p) \geq \nu N(z, p) \). Hence by (1) and (1') we have \( \nu \cap CG N(z, p) = 0 \) and \( \nu N(z, p) \geq \nu \cap \Omega N(z, p) \geq \nu N(z, p) = N(z, p) \).

**Proof of 3.** Case 1. \( p \in B_{1}^{N} - B_{s}^{N} \). Put \( S(z) = \lim_{M=\infty} CG \cap V_{M}(p) N(z, p) \). Assume \( S(z) > 0 \). Hence by (1) and (1) we have \( p \cap CG N(z, p) = 0 \) and \( GN(z, p) \geq p\cap G \Omega N(z, p) \geq p N(z, p) = N(z, p) \).

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Then

1. a). \( \nu \cap v_M(p)N(z, p) = v_M(p)N(z, p) = N(z, p) \) and \( \lim_{M \to \infty} \nu \cap v_M(p)N(z, p) = N(z, p) \) for \( p \in B^N \), where \( V_M(p) = E[z \in G : N(z, p) > M] \).

1. b). \( \omega(p, z) = \nu \cap v_M(p)\omega(p, z) \) for \( M < 1 - M_0 = M^{**} \) and \( M_0 < 1^{12} \) for \( p \in B^N \), where \( M_0 \) is the infimum of numbers \( M_i \) such that \( \omega(p \cap V_{M_i}^\prime, z) = 0 \), \( M_0 \). \( \end{equation} \)

2. a). Let \( \om(N, z, p) \) be a function in \( G \) such that \( \om(N, z, p) = \om(V_M(p), z, G) \) and \( \om \) is C.P. of \( V_M(p) \) relative to \( G \).

Then \( \om(N, z, p) \) is superharmonic in \( G \).

2. b). As a special case we have

\[ \begin{align*}
\min(M, N(z, p)) & = M \omega(V_M(p), z, G) \text{ for } M < \infty \text{ and } p \in B^N \ni p \neq p \text{ for } p \in B^N, \\
\min(M, \omega(p, z)) & = M \omega(V_M(p), z, G) \text{ for } M < M^{**} \text{ for } p \in B^N, \\
\end{align*} \]

where \( \omega(p, z) = \omega(p, z) - C_2 \omega(p, z) \) and \( \omega(V_M(p), z, G) \) is C.P. of \( V_M(p) \) relative to \( G \).

As a special case by \( v_n(p) \ni p \) we have

\[ N(z, p) = v_n(p) \ni p \text{ for } p \in B^N. \]

Now \( v_n(p) \ni p \), hence by considering \( v_n(p) \) a domain \( G \) we have by 2. b) the following fact: for any \( v_n(p) \) there exists a domain \( G = V_M(p) \) such that \( v_n(p) \supset G \ni p \text{ for } p \in B^N. \]

Hence for any \( V_M(p) \) there exist \( v_n(p) \) and \( V_M(p) \) such that \( V_M(p) \supset v_n(p) \supset V_M(p) \ni p \). This means \( \{V_M(p)\}, \{v_n(p)\} \) and \( \{V_M(p)\} \) are equivalent.

Proof of 1. a). Now \( p \in B^N \). In this case \( N(z, p) \leq C_2 N(z, p) + M \)
on $CV_M'(p)$. Hence
\[ cr_{M(p)\cap p}^{N(z,p)} + cr_{M(p)\cap p}^{N(z,p)} = cr_{M(p)\cap p}^{N(z,p)} + M\omega(p \cap CV_M'(p), z) \leq M(\omega(N(z,p)) = 0 \text{ by } \omega(p, z) = 0 \text{ and by 2) of Theorem 3.}

On the other hand,
\[ cr_{M(p)\cap p}^{N(z,p)} + v_{M(p)}^{N(z,p)} \geq v_{M(p)}^{N(z,p)} = N(z,p) \text{.}
\]

Whence
\[ N(z,p) \geq v_{M(p)}^{N(z,p)} \geq v_{M(p)}^{N(z,p)} \geq v_{M(p)}^{N(z,p)} \geq N(z,p) \text{.}
\]

Hence
\[ \lim_{M^{f} \to \infty} v_{M(p)}^{N(z,p)} = N(z,p) \text{.}
\]

Proof of 1). b). Since $p \in B_N^S$, $N(z,p) = a \omega(p, z) : a > 0$. Let $1 > M' > M_0 > 0$. Then $\omega(p \cap V_M'(z) = 0$, where $V_M'^r = E[z \in R : \omega(p, z) > M']$. Further we have
\[ \omega(\tilde{V}_{1-}, p, z) = \omega(p, z)^{12},
\]
where $\tilde{V}_{1-} = CV_M' \cap V_{1-}(p) = E[z \in G : \omega(p, z) - \omega(p, z) > 1 - \varepsilon - M']$. Since $\omega(p, z) > 1 - \varepsilon$ on $\tilde{V}_{1-} \cap \tilde{V}_{1-}, \omega(p, z) \geq \omega(p, z) - (1 - \varepsilon) = 1 - \varepsilon \omega(p \cap \tilde{V}_{1-}, z)$. We have by letting $\varepsilon \to 0$ $\lim_{\varepsilon \to 0} \omega(p \cap \tilde{V}_{1-}, z) = \omega(p, z)$.

Because $\lim_{M \to \infty} \omega(p \cap \tilde{V}_{1-}, z)$ has mass only at $p$ and $\sup \omega(p \cap \tilde{V}_{1-}, z) = 1$.

Clearly $V_M'(p) \supset \tilde{V}_{1-}$, for $M < 1 - \varepsilon - M'$ and $\omega(p, z) \geq \omega(p, z) \geq \lim_{M \to \infty} \omega(p, z) = \omega(p, z)$. Thus
\[ \omega(p, z) = \omega(p, z).
\]

Proof of 1). c). Case 1. $p \in B_N^S - B_N^S$. By 1). a) $v_{M(p)}^{N(z,p)} = N(z,p)$ and since $v_{M(p)\cap R_m}^{N(z,p)} \uparrow v_{M(p)}^{N(z,p)}$ as $m \to \infty$, there exists a number $m_0$ such that

\[ \omega(p, z) = \omega(p, z).
\]

Fig. 1.
$0 \leq N(z, p) - v_M(p) \cap \Omega M N(z, p) > \epsilon$ on $R_l$ for $m \geq m_0$, \quad (2)

for any given $R_l$ ($R_l$ is an exhaustion of $R$) and $\epsilon > 0$. Let $N_{m,n}(z, p)$ be a harmonic function in $R_n - \Omega$ such that $N_{m,n}(z, p) = 0$ on $\partial R_n$, $N_{m,n}(z, p) = N(z, p)$ on $R_m \cap V_M \cap R_n$ and $\frac{\partial}{\partial n} N_{m,n}(z, p) = 0$ on $\partial R_n$. Then

$N_{m,n}(z, p) \Rightarrow v_M(p) \cap \Omega M N(z, p) \leq \epsilon$ as $n \to \infty$, where $\Rightarrow$ means convergence in mean. Hence there exists a number $n_1(m)$ such that

$|N_{m,n}(z, p) - v_M(p)\cap \Omega M N(z, p)| < \epsilon$ on $R_l$ for $n \geq n_1(m)$ and $n > m$. \quad (3)

Let $U_{l,m,n}(z)$ be a harmonic function in $R_n - (R_l \cap CG)$ such that $U_{l,m,n}(z) = N_{m,n}(z, p)$ on $\partial (R_l \cap CG)$, $\frac{\partial}{\partial n} U_{l,m,n}(z) = 0$ on $\partial R_n$. Then by (2) and (3)

$|U_{l,m,n}(z) - N(z, p)| < 2\epsilon$ on $\partial (R_l \cap CG)$ for $n > n_1(m)$ and $n > m$. \quad (4)

On the other hand, let $U_{l,m,n}^*(z)$ be a harmonic function in $R_n - (R_l \cap CG)$ such that $U_{l,m,n}^*(z) = N(z, p)$ on $\partial (R_l \cap CG)$, $\frac{\partial}{\partial n} U_{l,m,n}^*(z) = 0$ on $\partial R_n$. Then by the maximum principle

$|U_{l,m,n}(z) - U_{l,m,n}^*(z)| < 2\epsilon$ in $R_n - (R_l \cap CG)$. \quad (5)

Let $n \to \infty$. Then by $U_{l,m,n}(z) \Rightarrow \Omega \cap R M N(z, p)$

$\lim^* U_{l,m,n}(z) - \Omega \cap R M N(z, p) \leq 2\epsilon$ in $R_n - (R_l \cap CG)$, \quad (6)

where $\lim^* U_{l,m,n}(z)$ means a limit by a converging subsequence of $\{U_{l,m,n}(z)\}$. Hence there exists a number $n_2 > n_1$ such that

$|U_{l,m,n}(z) - \Omega \cap R M N(z, p)| < 3\epsilon$ for $n > n_2$ in $(R_m \cap G) \supset (R_m \cap V_M \cap R_n)$. \quad (7)

Now by $\Omega \cap R M N(z, p) \uparrow \Omega \cap N(z, p)$ as $l \to \infty$ we have $U_{l,m,n}(z) < \Omega \cap N(z, p) + 3\epsilon$ on $R_m \cap V_M \cap R_n$. On the other hand, $M \leq N(z, p) - \Omega \cap N(z, p) \leq N(z, p) - U_{l,m,n}(z) + 3\epsilon$ on $R_m \cap V_M \cap R_n$. Since $N_{m,n}(z) = U_{l,m,n}(z)$ on $\partial (R_l \cap CG)$ and $\Omega \cap N(z, p) = N(z, p)$ on $R_m \cap V_M \cap R_n$,

$V^* = E[z \in R_n - (R_l \cap CG) : N_{m,n}(z, p) - U_{l,m,n}(z) > M - 3\epsilon] \supset E[z \in R_m : N(z, p) - \Omega \cap N(z, p) > M] = V_M \cap R_m \cap R_m$.

Whence $N_{m,n}(z, p) - U_{l,m,n}(z)$ is harmonic in $R_n - (R_l \cap CG) - V^*$. It follows that
\[0 \leq \int_{\partial V^* \cap R_{n}} \frac{\partial}{\partial n} N_{m,n}(z,p) \, ds = \int_{\partial R_{n}} \frac{\partial}{\partial n} N_{m,n}(z,p) \, ds \uparrow \int_{\partial R_{0}} \frac{\partial}{\partial n} N_{m,n}(z,p) \, ds \leq 2\pi\] as \( n \to \infty \).

On the other hand, \[
\int_{\partial \psi \cap R_{n}} \frac{\partial}{\partial n} U_{l,m,n}(z) \, ds = \int_{\partial R_{n} \cap V^2} \frac{\partial}{\partial n} U_{l,m,n}(z) \, ds = 0.
\]

Hence \[
\int_{\partial^{*} \cap R_{2}} \frac{\partial}{\partial n} (N_{m,n}(z,P)-U_{l,m,n}(z)) \, ds \leq 2\pi + \epsilon
\]
for \( n > n_{3}(m, l) \), whence

\[
R_{n}-(CG \cap R_{l})-V^* (N_{m,n}(z,p)-U_{l,m,n}(z)) \leq (2\pi + \epsilon)(M-3\epsilon)
\]
for any \( \epsilon > 0 \) by (8)

Let \( \mathfrak{G} \) be a compact domain completely contained in \( G-V'_{\alpha}(p) \). Then since \( (N_{m,n}(z,p)-U_{l,m,n}(z)) \to (r_{\alpha}(p)+r_{m,n}N(z,p)+d(z)) \) as \( n \to \infty \), (where \( |d(z)| < 2\varepsilon \)) by (6) and since \( C \cap R_{l} \) is compact, \( (r_{\alpha}(p)+r_{m,n}N(z,p)+d(z) \to (r_{\alpha}(p))N(z,p)+c_{\alpha}N(z,p)+\delta(z) \) as \( m \to \infty \). Further \( r_{\alpha}(p)N(z,p)-c_{\alpha}N(z,p)+\delta(z) \to N(z,p)-c_{\alpha}N(z,p) \) as \( l \to \infty \) by \( r_{\alpha}(p)N(z,p)=N(z,p) \). Hence there exist numbers \( l, m, n \) such that \( \mathfrak{G} \subset (R_{n}-(CG \cap R_{l})-V^* \) for any \( \epsilon > 0 \) by (8) and

\[
D_{\omega(P, z)}(N_{m,n}(z,p)-U_{l,m,n}(z)) \leq (2\pi + \epsilon)(M-3\epsilon).
\]

Let \( n \to \infty \). Then \( D_{\omega(P, z)}(N_{m,n}(z,p)-U_{l,m,n}(z)) \leq \lim_{n} D_{\omega(P, z)}(N_{m,n}(z,p)-U_{l,m,n}(z)) \). Similarly let \( m \to \infty \) and then \( l \to \infty \) and then \( \epsilon \to 0 \) and then \( \mathfrak{G} \to (G-V'_{\alpha}(p)) \). Then

\[
D_{\omega(P, z)}(N_{m,n}(z,p)-c_{\alpha}N(z,p)) = D_{\omega(P, z)}(\min(M, N(z,p))) \leq 2\pi M.
\]

Case 2. \( p \in B_{s}^{N} \). In this case it is evident that \( D(\omega(p, z)) < \infty \) and \( D(c_{\alpha}\omega(p, z)) < \infty \) and we have at once \( D(\omega(p, z)-c_{\alpha}\omega(p, z)) < \infty \).

Proof of 2. a). Let \( d N^T(z,p) \) be a function such that \( d N^T(z,p) = \min(T, N(z,p)) \) on \( D \) and \( d N^T(z,p) \) has M. D. I. over \( R-R_{0}-D \) and let \( \kappa_{\gamma} N^T(z,p) : G' \subset G \) be a function in \( G-G' \) such that \( \kappa_{\gamma} N^T(z,p) = 0 \) on \( \partial G \), \( \kappa_{\gamma} N^T(z,p) = \min(T, N(z,p)) \) on \( G' \) and \( \kappa_{\gamma} N^T(z,p) \) has M. D. I. over \( G-G' \) (this can be defined by 1). c)). Then by \( N(z,p) = N(z,p)+c_{\alpha}N(z,p) \) we have \( \kappa_{\gamma}+\gamma N^{T}(z,p) \geq \kappa_{\gamma}+\gamma (c_{\alpha}N(z,p)+\gamma N^{T}(z,p)) \) on \( \partial G+\partial G' \) and by the maximum principle*13 \( \kappa_{\gamma}+\gamma N^{T}(z,p) \geq \kappa_{\gamma}+\gamma (c_{\alpha}N^{*}(z,p)+\gamma N^{T}(z,p)) \). Let \( L \to \infty \). Then

\[
\kappa_{\gamma}+\gamma N(z,p) \geq \kappa_{\gamma}+\gamma (c_{\alpha}N(z,p)+N^{*}(z,p)).
\]
Next similarly we have $c_{a+\theta}N^L(z,p)\leqq \frac{1}{c_{a+\theta}(c_\theta N(z,p)+N(z,p))}$ and $c_{a+\theta}N(z,p)\leqq c_{a+\theta}(c_\theta N(z,p)+N(z,p))$. On the other hand, by $CG+G'\supset CG$ we have $c_{a+\theta}N(z,p) = c_{\theta}N(z,p)$. Hence by $c_{a+\theta}N(z,p) = \frac{1}{c_{a+\theta}}N(z,p)$ we have

$$c_{\theta}N(z,p) + N(z,p) = N(z,p) \leqq c_{a+\theta}N(z,p) = c_{\theta}N(z,p) + \frac{1}{c_{a+\theta}}N(z,p) \quad (9)$$

and $N(z,p) \leqq \frac{1}{c_{a+\theta}}N(z,p)$. Thus $N(z,p)$ is superharmonic in $G$.

**Proof of 2). b).** By 1). a) and 1. b) $\nu_{\triangledown}(p)N(z,p) = N(z,p)$ whence

$$N(z,p) \leqq c_{a+\nu_{\triangledown}(p)}N(z,p) = \nu_{\triangledown}(p)N(z,p) = N(z,p) \quad \text{for } M<\infty \text{ for } p\in B_{1}^{N} - B_{1}^{N},$$

$$c_{a+\nu_{\triangledown}(p)}N(z,p) = \nu_{\triangledown}(p)N(z,p) = N(z,p) \quad \text{for } M<M^{*} :$$

$$N(z,p) = a_{\omega}(p,z) \quad \text{for } p\in B_{1}^{N}.$$

Hence $c_{\theta}N(z,p) + N(z,p) = N(z,p) = c_{a+\nu_{\triangledown}(p)}N(z,p) = c_{a+\nu_{\triangledown}(p)}(c_{\theta}N(z,p) + N(z,p)) = c_{\theta}N(z,p) + \nu_{\triangledown}(p)N(z,p),$ because $c_{a+\nu_{\triangledown}(p)}(c_{\theta}N(z,p)) = c_{\theta}N(z,p)$ by $(CG + V'_{M}(p)) \supset CG$ and $c_{a+\nu_{\triangledown}(p)}N(z,p)$ $= \nu_{\triangledown}(p)N(z,p)$. Hence $N(z,p) = \nu_{\triangledown}(p)N(z,p)$. Clearly by definition $\nu_{\triangledown}(p)N(z,p)$ $= M\omega(V_{\triangledown}(p), z, G)$. Hence

$$N(z,p) = M\omega(V_{\triangledown}(p), z, G) \quad \text{in } G - V'_{M}(p).$$

We show $V_{\triangledown}(p) \not\supset p$. Case 1. $p\not\supset B_{1}^{N} - B_{1}^{N}$. Assume $V_{\triangledown}(p) \not\supset p$. Then $c_{\nu_{\triangledown}(p)}N(z,p) = N(z,p)$ and $c_{\theta}N(z,p) + N(z,p) = N(z,p) = c_{\nu_{\triangledown}(p)}N(z,p) = c_{\nu_{\triangledown}(p)}(c_{\theta}N(z,p)) + c_{\nu_{\triangledown}(p)}N^{*}(z,p) \leqq c_{\theta}N(z,p) + M$. Whence $N(z,p) = \nu_{\triangledown}(p)N(z,p) \leqq \frac{1}{c_{\theta}}N(z,p) + M\omega(p, z)$. But by a) of Theorem 3 $\rho(c_{\theta}N(z,p)) = 0$ and $\omega(p, z) = 0$ by $p\in B_{1}^{N} - B_{1}^{N}$. This implies $N(z,p) = 0$. This is a contradiction. Hence $V_{\triangledown}(p) \not\supset p$.

Case 2. $p\in B_{1}^{N}$. It is sufficient to prove for $\omega(p, z)$. Now $\omega(p, z) = \omega(p, z) - c_{\theta}\omega(p, z)$ and

$$c_{\theta} + c_{\nu_{\triangledown}(p)}\omega(p, z) = c_{\theta} + c_{\nu_{\triangledown}(p)}(c_{\theta}\omega(p, z)) + c_{\nu_{\triangledown}(p)}\omega'(p, z) \quad (10).$$

Assume $V_{\triangledown}(p) \not\supset p$. Then $c_{\nu_{\triangledown}(p)}\omega(p, z) = \omega(p, z)$ and $\omega(p, z) \leqq c_{\theta} + c_{\nu_{\triangledown}(p)}\omega(p, z) \geqq c_{\nu_{\triangledown}(p)}\omega(p, z) = \omega(p, z)$. On the other hand, $\nu_{\triangledown}(p)\omega(p, z) = M\omega(V_{\triangledown}(p), z, G)$ for $M<\infty$. Hence $\sup \omega(p, z) \geqq M^{*}$ and $\omega(p, z) = \nu_{\triangledown}(p)\omega(p, z) \leqq M\omega(p, z)$. Hence by (10) and by $c_{\theta} + c_{\nu_{\triangledown}(p)}(c_{\theta}\omega(p, z)) = c_{\theta}\omega(p, z)$ we have

$$\omega(p, z) = c_{\theta} + c_{\nu_{\triangledown}(p)}\omega(p, z) = c_{\theta}\omega(p, z) + c_{\nu_{\triangledown}(p)}\omega(p, z) < c_{\theta}\omega(p, z) + \omega(p, z) = \omega(p, z).$$

This is a contradiction. Hence $V_{\triangledown}(p) \not\supset p$ for $M<\infty$. 

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Proof of 3). By \( e \cdot N(z, p) = N(z, p) \) we have \( e \cdot e \cdot N(z, p) = N(z, p) \). This implies \( e \cdot e \cdot N(z, p) + N'(z, p) = N(z, p) \leq e \cdot e \cdot e \cdot N(z, p) + e \cdot e \cdot N'(z, p) \).

2. On capacitary potentials. Let \( \{ G_n \} \) \( (n = 1, 2, 3, \cdots) \) be a decreasing sequence of domains in \( R - R_0 \). Let \( G \) be a domain in \( R - R_0 \) such that \( G \supset G_1 \) and let \( \Gamma \) be a compact arc on \( \partial G \). Let \( \omega_{n,n+i}^G(z) \) be a harmonic function in \( (R_{n+i} \cap G) - G_n \) such that \( \omega_{n,n+i}^G(z) = 0 \) on \( \partial G \), \( \omega_{n,n+i}^G(z) = 1 \) on \( G_n \) and \( \frac{\partial}{\partial n} \omega_{n,n+i}^G(z) = 0 \) on \( \partial R_{n+i} - G_n \). Then if \( D(\omega_{n,n+i}^G(z)) < M < \infty \) for a number \( n_0 \) and for every \( i \), then \( \omega_{n,n+i}^G(z) \Rightarrow \omega_n^G(z) \) as \( i \to \infty \) and \( \omega_n^G(z) \Rightarrow \omega(\{G_n\}, z, G) \) called C.P. (capacitary potential) of the set determined by \( \{G_n\} \) as \( n \to \infty \). If \( G = R - R_0 \), we denote \( \omega_{n,n+i}^G(z) \) by \( \omega_{n,n+i}(z) \). Then \( D(\omega_{n,n+i}(z)) < M < \infty \) for any and every \( i \) by the fact that \( \text{dist} (\partial R_0, G_i) > 0 \). We denote \( \omega(\{G_n\}, z, R - R_0) \) simply by \( \omega(\{G_n\}, z) \). Let \( \omega_n^R(z) \) be a harmonic function in \( (R_{n+i} \cap G) - G_n \) such that \( \omega_n^R(z) = 1 \) on \( G_n \), \( \omega_n^R(z) = 0 \) on \( \partial \Gamma \), \( \frac{\partial}{\partial n} \omega_n^R(z) = 0 \) on \( \partial G - \Gamma \) and \( \partial R_{n+i} - G_n \). Then since \( \Gamma \) is compact and by \( \text{dist} (\Gamma, G_i) > 0 \), \( D(\omega_{n,n+i}(z)) < M < \infty \) for \( n \geq n_0 \) and every \( i \). Then \( \omega_{n,n+i}(z) \Rightarrow \omega_n^R(z) \) and \( \omega_n^R(z) \Rightarrow \omega(\{G_n\}, z, G, \Gamma) \). It can be proved easily if \( \omega(\{G_n\}, z, R - R_0) > 0 \), \( \omega(\{G_n\}, z, R - R_0) > 0 \) for any compact set \( R_0' \) such that \( R_0' \cap G_n = 0 \) \( (n_0 \) is a certain number).

Theorem 5. If the condition \( D(\omega_{n,n+i}(z)) < M < \infty \) for a number \( n_0 \) and for any number \( i \) is satisfied, then

1. \( \omega(\{G_n\}, z) > 0 \) if and only if \( \omega(\{G_n\}, z, G) > 0 \),
2. \( \omega(\{G_n\}, z, G) > 0 \) if and only if \( \omega(\{G_n\}, z, G, \Gamma) > 0 \),
3. If \( \omega(\{G_n\}, z, G, \Gamma) > 0 \), \( \omega(\{G_n\}, z) > 0 \) without the above condition.

Proof. Suppose \( \omega(\{G_n\}, z) > 0 \). Then by the Dirichlet principle \( D(\omega_{n,n+i}(z)) \leq D(\omega_{n,n+i}(z)) \). By the mean covergency of \( \omega_{n,n+i}(z) \) and \( \omega_{n,n+i}(z) \) we have

\[
D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z)) > 0.
\]

Hence \( \omega(\{G_n\}, z) > 0 \).

Next suppose \( \omega(\{G_n\}, z, G) > 0 \). Then by the maximum principle \( \omega_{n,n+i}(z) \geq \omega_{n,n+i}(z) \). Hence

\[
\omega(\{G_n\}, z) > 0.
\]

14) See p. 21 of P.
By \(\omega(\{G_n\}, z, G)\leq \omega(\{G_n\}, z, G, \Gamma)\) and by \(D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z, G, \Gamma))\) we have 2. Also by the Dirichlet principle \(D(\omega(\{G_n\}, z, G)) \geq D(\omega(\{G_n\}, z, G, \Gamma))\) and we have 3).

It is known that \(\omega(\{G_n\}, z, G)\) (or \(\omega(\{G_n\}, z, G, \Gamma)\)) attains 1 almost on \(\{G_n\}\) in the following sense: \(\omega(\{G_n\} \cap V_{1-}^{c}, z, G) > 0\) (or \(\omega(\{G_n\} \cap V_{1-}^{c}, z, G, \Gamma) > 0\)) for any \(\epsilon > 0\). Conversely we shall prove. Let \(G_1\) and \(G_2\) be two domains (or closed domains), if there exists at least one \(C_1\)-function \(U(z)\) such that \(U(z) = 0\) on \(G_1\), \(U(z) = 1\) on \(G_2\) and \(D(U(z)) < \infty\), we say that \(G_1\) and \(G_2\) are Dirichlet-disjoint.

**Lemma 1.1** If there exist numbers \(i_0\) and \(n_0\) such that \(CG_{n_0}\) and \(G_{n_0+i_0}\) are Dirichlet-disjoint,

\[
\omega(CG_{n_0} \cap V_{1-}^{c}, z, G-G_{n_0+i_0}) \Rightarrow 0 \text{ as } \epsilon \to 0,
\]

where \(V_{1-}^{c} = E[z \in G : \omega(\{G_n\}, z, G) > 1-\epsilon]\) (or \(V_{1-}^{c} = E[z \in G : \omega(\{G_n\}, z, G, \Gamma) > 1-\epsilon]\)) for any \(\epsilon > 0\).

This fact means that \(\omega(\{G_n\}, z, G) < 1\) almost everywhere outside of \(G_n\).

1'). If there exist numbers \(i_0\) and \(n_0\) such that \(CG_{n_0}\) and \(G_{n_0+i_0}\) are Dirichlet-disjoint,

\[
\omega(CG_{n_0} \cap V_{1-}^{c}, z, G, \Gamma) \Rightarrow 0 \text{ as } \epsilon \to 0,
\]

where \(V_{1-}^{c} = E[z \in G : \omega(\{G_n\}, z, G, \Gamma) > 1-\epsilon]\).

2). Under the condition of 1)

\[
\int_{\partial V_{1-}^{c} \cap CG_{n_0}} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G) \downarrow 0 \text{ as } \epsilon \to 0.
\]

2'). Under the condition of 1')

\[
\int_{\partial V_{1-}^{c} \cap CG_{n_0}} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G, \Gamma) \downarrow 0 \text{ as } \epsilon \to 0.
\]

**Proof.** Since \(\omega(V_{1-}^{c}, z, G)\) has M. D. I. over \(G-V_{1-}^{c}\), \(D(\omega(V_{1-}^{c}, z, G)) \leq D\left(\frac{1}{1-\epsilon} \omega(\{G_n\}, z, G)\right) = M' < \infty\), because \(\omega(\{G_n\}, z, G) \geq (1-\epsilon)\) on \(V_{1-}^{c}\) and \(= 0\) on \(\partial G\). Now \(CG_{n_0}\) and \(G_{n_0+i_0}\) are Dirichlet-disjoint, whence there exists a \(C_1\)-function \(U(z)\) such that \(U(z) = 0\) on \(CG_{n_0+i_0}\), \(U(z) = 1\) on \(G_{n_0}\) and \(D(U(z)) < M'' < \infty\). Put \(S(z) = \min(\omega(V_{1-}^{c}, z, G), U(z))\). Then \(S(z) = 0\) on \(\partial G + \partial G_{n_0+i_0}\), \(S(z) \geq 1\) on \(V_{1-}^{c} \cap CG_{n_0}\) and \(D(S(z)) \leq M + M'' < \infty\). Hence \(D(\omega(V_{1-}^{c} \cap CG_{n_0}, z, G-G_{n_0+i_0})) \leq D(S(z)) < \infty\) and lim \(\omega(V_{1-}^{c} \cap CG_{n_0}, z, G-G_{n_0+i_0})\) exists. By

15) If \(Uz\) has partial derivatives almost everywhere, we call \(U(z)\) a \(C_1\)-function.
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$D(\omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G)) \leq D(\omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G-G_{n_{\hat{\cup}}+i_{0}}) \lim_{\epsilon \rightarrow 0}$

exists. Assume $\omega(z) = \lim_{\epsilon \rightarrow 0} \omega(V_{1-\epsilon} \cap CG_{n_{0}}, z, G-G_{n_{\hat{\cup}}+i_{0}}) > 0$. Let $C_{r} = E[z : \omega(z) = r]$. Then

$$\int_{c_{r}} \frac{\partial}{\partial n} \omega(z) \, ds = D(\omega(z))$$

for almost $r$,

such $C_{r}$ is called a regular niveau curve

Now $\omega(\{G_{n}\}, z, G)$ is a harmonic function in $G-G_{n_{0}+i_{0}}$ having M. D. I. over $G-G_{n_{0}+i_{0}}$ among all functions with the same value on $\partial G + \partial G_{n_{0}+i_{0}}$.

$A_{m}(z) \Rightarrow \omega(\{G_{n}\}, z, G)$ as $m \rightarrow \infty$,

where $A_{m}(z)$ is a harmonic function in $R_{m} \cap (G-G_{n_{0}+i_{0}})$ such that $A(z) = \omega(\{G_{n}\}, z, G)$ on $(\partial G + \partial G_{n_{0}+i_{0}}) \cap R_{m}$ and $\frac{\partial}{\partial n} A_{m}(z) = 0$ on $(G-G_{n_{0}+i_{0}}) \cap R_{m}$. Also $\omega(z)$ has M. D. I. over $G-G_{n_{0}+i_{0}}-V_{1-\epsilon}$, whence $\omega_{m}(z) \Rightarrow \omega(z)$, where $\omega_{m}(z)$ is a harmonic function in $R_{m} \cap (G-G_{n_{0}+i_{0}}-\tilde{V}_{1-\epsilon})$ such that $\omega_{m}(z) = \omega(z)$ on $\partial \tilde{V}_{1-\epsilon}$, $\frac{\partial \omega_{m}}{\partial n}(z) = 0$ on $\partial R_{m} \cap (G-G_{n_{0}+i_{0}}-\tilde{V}_{1-\epsilon})$, where $\tilde{V}_{1-\epsilon} = E[z \in G-G_{n_{0}+i_{0}} : \omega(z) > 1 - \epsilon]$. Then by Green’s formula

$$\int_{C_{1-\epsilon} \cap R_{m}} A_{m}(z) \frac{\partial}{\partial n} \omega_{m}(z) \, ds = \int_{C_{\delta} \cap R_{m}} A_{m}(z) \frac{\partial \omega_{m}}{\partial n} (z) \, ds ,$$

where $C_{1-\epsilon}$ and $C_{\delta}$ are regular niveau curves of $\omega(z)$. Then by letting $m \rightarrow \infty$

$$\int_{C_{1-\epsilon}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds = \int_{C_{\delta}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds .$$

By $\omega(z) \leq \omega(\{G_{n}\}, z, G)$ we have

$$(1-\epsilon)D(\omega(z)) = \int_{C_{1-\epsilon}} \omega(z) \frac{\partial \omega}{\partial n} (z) \, ds \leq \int_{C_{1-\epsilon}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds$$

$$= \int_{C_{\delta}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds . \quad (11)$$

On the other hand, $C_{\delta} \subset G$, $\omega(\{G_{n}\}, z, G) < 1$ in $G$ and there exists a number $\epsilon_{0} > 0$ such that

16) See theorem 2 of P.
17) See See 15)
\[
\int_{c_{\delta}} \omega(\{G_{n}\}, z, G) \frac{\partial}{\partial n} \omega(z) \, ds < (1 - \varepsilon_{0}) \int_{c_{1-\varepsilon}} \frac{\partial}{\partial n} \omega(z) \, ds = (1 - \varepsilon_{0}) D(\omega(z)).
\]

Take \( \varepsilon < \frac{\varepsilon_{0}}{2} \). Then (11) will be a contradiction. Hence

\[
\omega(z) = \lim_{i \to 0} \omega(V_{1-\varepsilon} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) = 0.
\]

Also by the Dirichlet principle we have \( \lim_{i \to 0} \omega(V_{1-\varepsilon} \cap CG, z, G) = 0 \).

**Proof of 2**). Assume \( \liminf_{r \to 1} \int_{C_{0} \cap CG_{n_{0}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds > \delta > 0 \). Then by

\[
r \omega(V_{r} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) \leq \omega(\{G_{n}\}, z, G) \quad \text{and} \quad r \omega(V_{r} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) = \omega(\{G_{n}\}, z, G) = r \quad \text{on} \quad \partial V_{r},
\]

we have

\[
r \delta \leq r \int_{C_{0} \cap CG_{n_{0}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds \leq r \int_{C_{0} \cap CG_{n_{0}}} \frac{\partial}{\partial n} \omega(V_{r} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) \, ds \leq \int_{\partial V_{r} \cap CG_{n_{0}}} r \frac{\partial}{\partial n} \omega(V_{r} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) \, ds.
\]

Let \( r \to 1 \). Then by \( r \omega(V_{r} \cap CG_{n_{0}}, z, G - G_{n_{0}+\varepsilon_{0}}) \to \omega(z) \) we have \( D(\omega(z)) > \delta \) and \( \omega(z) > 0 \). This is a contradiction. Hence

\[
\lim_{r \to 1} \int_{C_{0} \cap CG_{n_{0}}} \frac{\partial}{\partial n} \omega(\{G_{n}\}, z, G) \, ds = 0.
\]

The other part can be proved similarly.

**Angular domain** \( \Delta(p) \) of \( p \in B_{1}^{N} \). Let \( \Delta(p) \) be a domain. If

\[
\lim_{n \to \infty} \Delta(p \cap v_{n}(p), z, p) > 0,
\]

we call \( \Delta(p) \) an angular domain with vertex at \( p \). Then

**Theorem 6.** 1). Let \( G \) be a domain. If \( \lim_{M \to M_{*}} \int_{C_{M} \cap G} \frac{\partial}{\partial n} N(z, p) \, ds > 0 \), \( G \) is an angular domain, where \( C_{M} = E[z \in R : N(z, p) = M] : M < M_{*} = \sup_{z \in R} N(z, p) \).

2). If \( G^{N} \ni p \), \( G \) is an angular domain.

3). Let \( \Delta(p) \) be an angular domain. If \( G^{N} \ni p \), then \( G \cap \Delta(p) \) is also an angular domain.

**Proof of 1.** Suppose \( \lim_{M \to M_{*}} \int_{C_{M} \cap G} \frac{\partial}{\partial n} N(z, p) \, ds > \delta > 0 \). Then there exists a sequence \( M_{1}, M_{2}, \ldots \) and \( M_{i} \uparrow M_{*} \) such that

\[
\int_{C_{M_{i}} \cap G} \frac{\partial}{\partial n} N(z, p) \, ds > \frac{2\delta}{3} \quad \text{for} \quad i \geq 1.
\]
Now for any given $v_{n}(p)$ \[ \lim_{M \to M^{*}} \int_{\partial V_{M}(p) \cap v_{n}(p)} \frac{\partial}{\partial n} N(z, p) ds = 0 \] by Theorem 3. 3).

Hence by $(\partial V_{M}(p) \cap v_{n}(p) \cap G) + (\partial V_{M}(p) \cap C v_{n}(p) \cap G) = \partial V_{M}(p) \cap G$, there exists a number $i_{0}(n)$ such that
\[ \int_{\partial R_{0}} \frac{\partial}{\partial n} M_{i}(p) \omega(V_{M_{i}}(p) \cap v_{n}(p) \cap G) ds \geq \frac{\delta}{2} \]
for $i > i_{0}(n)$. Now $M_{i} \omega(V_{M}(p) \cap v_{n}(p) \cap G) \leq M_{i} \omega(V_{M}(p) \cap v_{n}(p))$ and
\[ M_{i} \omega(V_{M}(p) \cap v_{n}(p) \cap G, z) \leq M_{i} = N(z, p) \]
on $\partial V_{M}(p)$, whence
\[ \int_{\partial R_{0}} \frac{\partial}{\partial n} M_{i} \omega(V_{M}(p) \cap v_{n}(p) \cap G, z) ds \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} M_{i} \omega(V_{M}(p) \cap v_{n}(p) \cap G) ds \geq \frac{\delta}{2}. \]
On the other hand, by $M_{i} \omega(V_{M}(p) \cap v_{n}(p) \cap G, z) \leq M_{i} \omega(V_{M}(p) \cap v_{n}(p))$ we have
\[ \int_{\partial R_{0}} \frac{\partial}{\partial n} v_{n}(p) \cap \Delta(p) \cap G N(z, p) ds \geq \frac{\delta}{2}. \]
Let $n \to \infty$. Then $V_{M}(p) \cap v_{n}(p) \cap G \to p \cap G$ for any given $M_{i} < M^{*}$ by $V_{M}(p) \supset v_{n}(p) : n = n(M)$ and
\[ \lim_{n=\infty} v_{n}(p) \cap \Delta(p) \cap G \leq \lim_{n=\infty} v_{n}(p) \cap \Delta(p) \cap G \cap V_{M_{i}}(p) N(z, p) \geq \frac{2}{2\pi} N(z, p) > 0. \]
Thus $G$ is an angular domain.

**Proof of 2.** If $G \ni p$, \[ \int_{\partial V_{M}(p) \cap CG} \frac{\partial}{\partial n} N(z, p) ds \downarrow 0 \] as $M \uparrow M^{*}$ by Theorem 3. 3). But \[ \int_{\partial V_{M}(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \] for almost $M$. Hence \[ \lim_{M \to M^{*}} \int_{\partial V_{M}(p) \cap G} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \] and by 1) $G$ is an angular domain.

**Proof of 3.** By $v_{n}(p) \cap \Delta(p) = (v_{n}(p) \cap \Delta(p) \cap CG) + (v_{n}(p) \cap \Delta(p) \cap G)$ and by \[ \lim_{n} v_{n}(p) \cap \Delta(p) N(z, p) = \lim_{n} v_{n}(p) \cap \Delta(p) \cap G N(z, p) = 0 \] by Theorem 3. 2) we have
\[ \lim_{n} v_{n}(p) \cap \Delta(p) N(z, p) \leq \lim_{n} v_{n}(p) \cap \Delta(p) \cap G N(z, p) + \lim_{n} v_{n}(p) \cap \Delta(p) \cap G N(z, p) \] and
\[ \lim_{n} v_{n}(p) \cap \Delta(p) \cap G N(z, p) > 0. \] Hence $G \cap \Delta(p)$ is also an angular domain.

Let $R$ be a disc: $|z| < 1$ and $R_{0}$ be $|z| < \frac{1}{2}$. Then $N(z, p) = U(z) + \log \frac{1}{|z-p|} : p = e^{i\theta}$, where $U(z)$ is a bounded harmonic function in $E \left[ z : \frac{1}{2} < |z| < 2 \right]$. Hence it is easily verified that $E \left[ z : \arg \frac{z-e^{i\theta}}{e^{i\theta}} < \frac{\pi}{2} \right] = A(e^{i\theta})$. 

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satisfies the condition of Theorem 6). 1) and $A(e^{i\theta})$ is an angular domain in our sense.

3. Capacitary potentials of the boundary determined by a domain $G$ containing an end part $\Delta(p)$ or containing $p$ $N$-approximately.

**Theorem 7.** 1). Let $F$ be a closed set (with respect to $N$-Martin's topology) on $B$ of positive capacity. Let $G$ be a domain such that $G \supset (\Delta(p) \cap v_n(p))$ for $p \in (B_1^N \cap F)$, where $n$ depends on $p$ and $\Delta(p)$ is an angular domain of $p$. Then

$$\lim_{n} \omega(G \cap F_n, z) > 0,$$

where $F_n = E[z \in \bar{R} : \text{dist}(z, F) \leq \frac{1}{n}]$.

2). Let $G$ be a domain such that $G \supset G_n (n = 1, 2, \cdots)$, where $\{G_n\}$ is a decreasing sequence of domains and $G_n \ni p$ for any $p \in (B_1^N \cap F)$. Let $\Gamma$ be a compact arc on $G$. Then

$$\omega(\{G_n\}, z, G, \Gamma) > 0.$$

Further if $CG$ and $G_{n_0}$ ($n_0$ is a certain number) are Dirichlet-disjoint, then

$$\omega(\{G_n\}, z) > 0 \quad \text{and} \quad \omega(\{G_n\}, z, G) > 0.$$

**Proof of 1).** Since $\text{Cap}(B_0^N) = 0$, we can suppose that $\text{Cap}(F \cap B_1^N) > 0$. By the condition $G \supset (\Delta(p) \cap v_n(p))$ for $n > n(p)$ and $F_n \supset v_n(p): p \in F \cap B_1^N$, we have

$$a_{\cap F_n}N(z, p) = \lim_{n} a_{\cap F_n}N(z, p) \geq \lim_{n} a_{\cap (\Delta(p) \cap v_n(p))}N(z, p) > 0.$$ (11)

Now $N(z, p)$ is continuous for $z \in R$ with respect to $p$ and $\int_{\partial R} \frac{\partial}{\partial n} a_{\cap F_n}N(z, p) ds$ is also continuous with respect to $p$ and $a_{\cap F_n}N(z, p)$ as $m \to \infty$, $a_{\cap F_n}N(z, p) \downarrow a_{\cap F_n}N(z, p)$ as $n \to \infty$. Hence the set $A_t = E[z \in F \cap B_1^N : \int_{\partial R} \frac{\partial}{\partial n} a_{\cap F_n}N(z, p) ds > \frac{1}{l}]$ is a Borel set and

$$\sum_{t=1}^{\infty} A_t = F \cap B_1^N.$$ 

Capacity of Borel sets in $B$ is the supremum of total mass of positive canonical distributions $\{\mu\}$ such that the potential $U(z)$ of $\mu$ satisfies the condition that $U(z) \leq 1$ (this is equivalent to that the infimum of energy integrals $I(\mu)$
of canonical distributions of unit masses $=1/\text{Cap}$. Hence $\text{Cap}(A_{l}) \uparrow \text{Cap}(F)$ as $l \to \infty$, because $\mu$ is Borel measurable. Therefore we can find a set $A_{l} \subset F$ such that $\text{Cap}(A_{l}) > \frac{1}{2} \text{Cap}(F)$. Since $\omega(F_{n}, z) \downarrow \omega(F, z)$, for any given number $(\frac{1}{2l} > \varepsilon > 0$ there exists a number $n(\varepsilon)$ such that

$$\int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F, z) ds \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F_{n}, z) ds - \varepsilon \left(\text{clearly} \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(F_{n} \cap R_{m}, z) ds - \varepsilon\right).$$

(12)

Since $\xi_{n \cap R_{m}} N(z, p) \uparrow \xi_{n} N(z, p) \geq \xi_{n} N(z, p)$ as $m \to \infty$,

$$\sum_{m} A_{n, m} = A_{l},$$

where $A_{n, m} = E[p \in A_{l} \cap B_{1}^{m} : \int_{\partial R_{0}} \frac{\partial}{\partial n} F_{l} \cap R_{m} N(z, p) ds > \frac{1}{2l}]$. Hence there exists a number $m(\varepsilon)$ for any number $n(\varepsilon)$ such that

$$\text{Cap}(A_{n, m}) \geq \frac{1}{2} \text{Cap}(A_{l}) \geq \frac{1}{4} \text{Cap}(F).$$

By the definitions of $\text{Cap}(A_{n, m})$, there exists a positive canonical mass distribution $\mu$ on $A_{n, m}$ such that $\int d\mu = \Re > 0$ and its potential $U(z)$ satisfies $U(z) \leq 1$ in $R$. Since $N(z, p)$ is continuous with respect to $p$, we can find a sequence $\{U_{i}(z)\}$ of the form $U_{i}(z) = \sum_{j=1}^{i} c_{j} N(z, p_{j}) : p_{j} \in A_{n, m}: c_{j} > 0$, $\sum c_{j} = \Re < \frac{1}{4} \text{Cap}(F)$ such that $U_{i}(z) \to U(z)$ for any compact set $R_{m}$ of $R$. Hence we can suppose $U_{i}(z) < 1 + \varepsilon$ on $R_{m(\varepsilon)}$, whence $(1 + \varepsilon)\omega(F_{n} \cap R_{m(\varepsilon)}), z) \geq U_{i}(z)$ on $R_{m(\varepsilon)}$. Hence by (12) and (13) we have by $\omega(F_{n} \cap R, z) \geq \omega(F_{n} \cap R_{m}, z)$

$$\int_{\partial R_{0}} (1 + \varepsilon) \frac{\partial}{\partial n} \omega(F, z) ds + \varepsilon \geq \int_{\partial R_{0}} (1 + \varepsilon) \frac{\partial}{\partial n} \omega(F_{n} \cap R_{m(\varepsilon)}, z) ds \geq$$

$$\int_{\partial R_{0}} (1 + \varepsilon) U(z) \frac{\partial}{\partial n} x_{n \cap R_{m(\varepsilon)}} U_{i}(z) ds = \int_{\partial R_{0}} \frac{\partial}{\partial n} \sum c_{j} N(z, p_{j}) ds \geq \frac{\Re}{2l} > 0.$$ 

Thus by letting $\varepsilon \to 0$, $\omega(F, z) > 0$.

**Proof of 2.** By $G \supset G_{n} \ni p (p \in F \cap B_{1}^{m})$ we have by Theorem 3), 2) $a_{n} N(z, p) = N(z, p)$ and by Theorem 4), 3) $a_{n}^{*} N'(z, p) = N'(z, p) > 0$ for any $n$ and
\[
\int r \frac{\partial}{\partial n} N'(z, p) \, ds > 0, \text{ where } N'(z, p) = N(z, p) - c_{0} N(z, p). \] 
Let \( A_{t, n} = E \left[ p \in F \cap B_{1}^{n} : \int r \frac{\partial}{\partial n} a_{n} N'(z, p) \, ds > \frac{1}{l} \right] \) and \( A_{t} = E \left[ p \in F \cap B_{1}^{n} : \int r \frac{\partial}{\partial n} N'(z, p) \, ds > \frac{1}{l} \right] \). Then \( A_{t, n} \) and \( A_{t} \) are measurable, \( A_{t, n} = A_{t} \) and \( \sum_{t} A_{t, n} = F \cap B_{1}^{n} \). Hence there exists a set \( A_{t, n} \) such that \( \text{Cap}(A_{t, n}) \geq \frac{1}{2} \text{Cap}(F) \) for any \( n \). Now \( z N'(z, p) = \lim_{L}^{\infty} \{ D \cap R_{m}^{*} N'(z, p) \} \) for any domain \( D \) in \( G \) can be proved as super-harmonic functions in \( R - R_{0} \), where \( D \cap R_{m}^{*} N'(z, p) \) is a function in \( G \) such that \( z N'(z, p) = N(z, p) \) on \( (D \cap R_{m}) \cap G + \partial G \) and \( D \cap R_{m}^{*} N'(z, p) \) has M.D.I. over \( G - (D \cap R_{m}) \). Hence \( \sum_{m} A_{t, m, n} = A_{t, n} = A_{t} \), where

\[
A_{t, m, n} = E \left[ p \in A_{t} : \int r \frac{\partial}{\partial n} a_{n} \cap R_{m}^{*} N'(z, p) \, ds > \frac{1}{2l} \right].
\]

There exists a number \( m(n) \) such that \( \text{Cap}(A_{t, m, n}) > \frac{1}{4} \text{Cap}(F) = \mathfrak{M} \) for \( m > m(n) \). By the definition of \( \text{Cap}(A_{t, m, n}) \), there exists a positive canonical mass distribution \( \mu \) on \( A_{t, m, n} \) such that \( \frac{\mathfrak{M}}{2} = \int d\mu \) and the potential \( U(z) \) of \( \mu < 1 \) in \( R - R_{0} \). Since \( G \cap R_{m} \) is compact, \( U(z) \) can be approximated on \( G \cap R_{m} \) by a sequence \( U_{i}(z) = \sum_{j} c_{j} N(z, p_{j}) : p_{j} \in A_{t, m, n}, \sum c_{j} = \frac{\mathfrak{M}}{2} \). Hence for any given number \( \varepsilon > 0 \), \( U_{i}(z) < 1 + \varepsilon \) on \( G \cap R_{m} \) for \( i > i(\varepsilon) \). Whence

\[
(1 + \varepsilon) \omega(G_{n} \cap R_{m}, z, G, \Gamma) \geq U_{i}(z) = \sum_{j=1}^{i} c_{j} N(z, p_{j}) \text{ on } G \cap R_{m}.
\]

On the other hand, clearly \( N(z, p) \geq N'(z, p) \) in \( G \) and \( \sum_{j} c_{j} N'(z, p_{j}) \leq \omega(G_{n} \cap R_{m}, z, G, \Gamma) \) on \( \partial G \). Whence by the maximum principle

\[
(1 + \varepsilon) \omega(G_{n} \cap R_{m}, z, G, \Gamma) \geq \sum_{j}^{i} c_{j} N'(z, p_{j}) \quad \text{and}
\]

\[
\int r (1 + \varepsilon) \frac{\partial}{\partial n} \omega(G_{n} \cap R_{m}, z, G, \Gamma) \, ds \geq \int r \frac{\partial}{\partial n} (\sum_{j} c_{j} N'(z, p_{j})) \, ds \geq \frac{\text{Cap}(F)}{8}.
\]

Let \( i \to \infty \), then \( m \to \infty \), then \( n \to \infty \) and then \( \varepsilon \to 0 \). Then

18) See theorem 3 of P.
\[ \int_{\Gamma} \frac{\partial}{\partial n} \omega(\{G_n\}, z, G, \Gamma) \, ds \geq \frac{\text{Cap}(F)}{8}. \] Thus, \( \omega(\{G_n\}, z, G, \Gamma) > 0. \)

By Theorem 5. 2) we have at once \( \omega(\{G_n\}, z, G) > 0 \) from \( \omega(\{G_n\}, z, G, \Gamma) > 0. \)

Theorem. Let \( F \) be a closed set of positive logarithmic capacity in the \( z \)-plane. Let \( S(p) \) be a sector with vertex at \( p \) and with aperture \( \theta : 2\pi > \theta > 0. \) Let \( G \) be a domain (its connectivity may be \( \infty \)). Suppose \( G \) contain an endpart of \( S(p) \) for any \( p \in F. \) Then \( \omega(F, z, G, \Gamma) > 0, \) where \( \Gamma \) is a compact arc on \( \partial G. \)

4. Correspondence of sets on boundaries of Riemann surfaces by analytic functions.

Let \( R \) be a covering surface over \( \mathbb{R}. \) We suppose \( R \) is a surface with positive boundary and \( N \)-Martin's topology is defined on \( R + B = \mathbb{R} (N(z, p) \) is defined in \( R - R_c \) and suppose that \( R + B \) (\( B \) is the boundary of \( R \) and \( R \) is of with null or positive boundary) is compact with respect to the topology defined by a metric in \( R. \) Let \( \mathcal{F} \) be a closed set in \( R + B \) and let \( D \) be a compact disc in \( R \) such that dist \((\mathcal{F}, D) > 0. \) We define the capacity

\[ \int_{\partial D} \omega(\mathcal{F}, w, R - D) \, ds \] of \( \mathcal{F}, \) where \( \omega(\mathcal{F}, w, R - D) \) is C. P. of \( \mathcal{F}. \) Clearly if \( \mathcal{F} \) is contained in a local parameter disc \( C(p) \) of \( p \in R, \) \( \mathcal{F} \) is of capacity zero if and only if \( \mathcal{F} \) is of logarithmic capacity zero.

Put \( \mathcal{F}_n = E \left\{ z \in \mathbb{R} : \text{dist}(w, \mathcal{F}) \leq \frac{1}{n} \right\}. \) If for any given \( n, \) there exists a number \( i(n) \) such that \( \mathcal{F}_n \) and \( C_{\mathcal{F}_{n + i}} \) are Dirichlet-disjoint, we call \( \mathcal{F} \) a \( D \)-regularly closed set. Of course, if \( \mathcal{F} \) is a compact set in \( R, \) we can construct a domains \( D_1 \) and \( D_2 \) such that \( D_1 \subset D_2, \) \( \partial D_1 \supset \mathcal{F}_{n + i}, \) \( D \subset \mathcal{F}_n, \) dist \((\partial D_1, \partial D_i) > 0, \) \( \partial D_1 (i = 1, 2) \) is composed of a finite number of analytic curves and there exists a harmonic function in \( D_1 - D_2 \) such that \( U(w) = 1 \) on \( \partial D_1, U(w) = 0 \) on \( \partial D_2 \) and \( D(U(w)) < \infty. \) Put \( U(w) = 1 \) in \( R - D_1, U(w) = 0 \) in \( D_2. \) Thus \( \mathcal{F} \) is \( D \)-regularly closed.

Cluster sets at \( p \in \mathbb{R}. \) Let \( w = f(z) \) be an analytic function : \( z \in R, \) \( w \in \mathbb{R}. \) Put \( M(f(p)) = \bigcap_{n} f(G_{\tau}), \) where \( G_{\tau} \) runs over all domains \( G_{\tau} \) of \( p \) and

\[ f_N(p) = \bigcap_{n} f(D(p) \cap v_n(p)). \]

Then we have the following

**Lemma.** Let \( \mathcal{F} \) be a closed set on \( \mathbb{R} + \mathcal{B} \). Then

1. Suppose \( s_f(p) \in \mathcal{F} \). Then \( f^{-1}(\mathcal{F}_n) \) contains an endpart of \( \Delta(p) \).

2. Suppose \( M(f(p)) \in \mathcal{F} \). Then \( f^{-1}(\mathcal{F}_n) \supset \mathcal{F} \) for any \( n \).

In fact, clearly \( s_f(p) \) is closed and non void. For any given number \( n \) there exists a number \( m \) such that \( f(\Delta(p) \cap v_m(p)) \subset \mathcal{F}_n \). Hence \( f^{-1}(\mathcal{F}_n) \supset (\Delta(p) \cap v_m(p)) \). Next \( M(f(p)) \) is closed. Since \( \mathbb{R} + \mathcal{B} \) is a metric space and separable, we can find a sequence \( G_1, G_2, \ldots \) from \( \{ G_i \} : G_i \supset p \) such that \( \bigcap_{n=1}^{\infty} \mathcal{F}_n \supset \mathcal{F}_n \). Put \( G'_m = G_1 \cap G_2 \cap \cdots \cap G_m \). Then \( G'_m \supset p \). Whence by \( f^{-1}(\mathcal{F}_n) \supset G'_m \) we have \( f^{-1}(\mathcal{F}_n) \supset \mathcal{F} \) for any \( n \).

Let \( n(w) \) be the number of points lying over \( w \in \mathbb{R} \). Put \( \bar{n}(w) = \lim_{n \to \infty} \sup_{w \in \mathbb{R} + \mathcal{B}} n(w) : w \in \mathbb{R} + \mathcal{B} \). If \( \bar{n}(w) < \infty \) for any point \( p \) of \( \mathcal{F} \), we can find a number \( M \) such that \( n(w) < M < \infty \) for \( w \in (v_n(w) \cap R) \). We shall prove the following

**Theorem 8.** Let \( w = f(z) \) be an analytic function from \( \mathbb{R} \) into \( \mathbb{R} \). Then

1. Let \( \mathcal{F} \subset (\mathbb{R} + \mathcal{B}) \) be a \( \mathcal{D} \)-regularly closed set of capacity zero and suppose \( \bar{n}(w) < \infty \) for \( w \in \mathcal{F} \). Then the set \( A = E[p \in B_\mathfrak{F} : s_f(p) \subset \mathcal{F}] \) is of capacity zero.

2. Let \( \mathcal{F} \subset (\mathbb{R} + \mathcal{B}) \) be a (not necessarily \( \mathcal{D} \)-regularly) closed set of capacity zero and suppose \( \bar{n}(w) < \infty \) for \( w \in \mathcal{F} \). Then the set \( A = E[p \in B_\mathfrak{F} : M(f(p)) \in \mathcal{F}] \) is of capacity zero.

**Proof of 1.** By the condition of 1) we can find a number \( n' \) such that \( n(\omega) \leq M < \infty \) in \( \mathcal{F}_n \). and a number \( n(\omega) \leq n' \) such that \( \mathbb{R} - f^{-1}(\mathcal{F}_n) \) has a compact disc \( \mathcal{D} \). In fact, let \( G \) be a sufficiently small disc. Then \( s_f(G) \) is an open set and \( s_f(G) \cap \mathcal{F}_n = 0 \) for \( n \geq n' \) ( \( n' \) is a certain number). Hence there exists at least one point \( \omega_0 \) such that \( \omega_0 \in \mathcal{F}_n \). \( \mathcal{F}_n \cap R \) is closed and \( f^{-1}(\mathcal{F}_n) \) is a closed set of points \( p \) such that \( v(p) \cap f^{-1}(\mathcal{F}_n) = 0 \). Take \( v(p_0) \) as \( D \). Then \( D \) is the required domain.

Assume \( A \) is of positive capacity. Then there exists a positive canonical mass distribution \( \mu \) whose \( I(\mu) < \infty \). In this case we can find also a closed set \( F \) of positive capacity. Now a closed set \( F \) is of positive capacity if and only if \( F \) is of function theoretic capacity positive\(^{20} \), whence \( \omega(F, z) = \omega(F, z, R - R_0) > 0 \). Let \( D \) be a compact disc in \( R \) such that \( f(D) \subset D \).

---

\(^{20}\) See p. 73 of P.
Then \(\omega(F, z) > 0\) if and only if \(\omega(F, z, R-D) > 0\). Similarly by \(\text{Cap}(\mathfrak{F}) > 0\) we have \(\omega(\mathfrak{F}, w, R-D) > 0\). Let \(n''\) be a number such that \(D \subset R-\mathfrak{F}_{n''}, (n'' > n')\) and fix \(n''\) at present.

Since \(f(p) \subset \mathfrak{F}'\), by lemma \(f^{-1}(\mathfrak{F}_{m})\) contains an endpart of \(D(p)\) for any \(p\) and \(m\). Then by Theorem 7). 1) by putting \(G_{m} = f^{-1}(\mathfrak{F}_{m})\)

\[
\lim_{m \to \infty} \omega(f^{-1}(\mathfrak{F}_{m}), z, R-D) \geq \lim_{m \to \infty} \omega(F \cap f^{-1}(\mathfrak{F}_{m}), z, R-D) > 0 . \tag{14}
\]

Now \(\mathfrak{F}\) is \(D\)-regularly closed and there exists a \(C\)-function \(U(w)\) in \(R\) such that \(U(w) = 0\) on \(\partial \mathfrak{F}_{n''}, U(w) = 1\) on \(\mathfrak{F}_{n''} \cap A\), and \(D(U(w)) \leq L < \infty\). Hence \(D(\omega(\mathfrak{F}_{m}, w, \mathfrak{F}_{n'\prime})) \leq L\) for \(m > n'' + i\), and since \(\mathfrak{F}\) is of capacity zero

\[
D(\omega(\mathfrak{F}_{m}, w, \mathfrak{F}_{n'\prime})) \downarrow 0 \quad \text{as} \quad m \to \infty . \tag{15}
\]

Consider \(U(z) = U(f^{-1}(w))\) in \(f^{-1}(\mathfrak{F}_{n'\prime}) - f^{-1}(\mathfrak{F}_{m})\). Then by \(n(w) \leq M\) in \(\mathfrak{F}_{n'\prime}\)

\[
D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n'\prime}))) \leq MD(U(w)) \leq ML < \infty
\]

and

\[
D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n'\prime}))) \leq MD(\omega(\mathfrak{F}_{m}, w, \mathfrak{F}_{n'\prime})) . \tag{16}
\]

Let \(m \to \infty\). Then by (15) and (16)

\[
D(\omega(f^{-1}(\mathfrak{F}_{m}), z, f^{-1}(\mathfrak{F}_{n'\prime}))) \downarrow 0 \quad \text{as} \quad m \to \infty . \tag{17}
\]

On the other hand, by the Dirichlet principle and by (14) \(D(\omega(f^{-1}(\mathfrak{F}_{m}) \cap F, f^{-1}(\mathfrak{F}_{n'\prime}) \cap F, z, R-D)) > 0\). This contradicts (17). Hence \(A\) is a set of capacity zero.

**Proof of 2).** As (1) we can find a number \(n'\) such that \(n'(w) \leq M < \infty\) in \(\mathfrak{F}_{n'}\) and \(R-f^{-1}(\mathfrak{F}_{n'})\) contains a compact disc \(D\). Fix \(n'\) at present. Put \(G = R \cap f^{-1}(\mathfrak{F}_{n'})\). Then \(G\) may consist of an enumerably infinite number of domains (components) \(G_{i}, G_{2}, \ldots\). Since \(M(f(p)) \subset \mathfrak{F}_{n}, G_{n}' = f^{-1}(\mathfrak{F}_{n})\) contains any point of \(A \cap B_{n}\) by Lemma 4). Let \(A_{i}\) be the set of points contained \(N\)-approximately in \(G_{i}\). Then \(A_{i}\) is a Borel set by Lemma b) and \(\sum A_{i} = A\).

Hence there exists at least one component \(G'\) of \(G\) such that \(G' \ni p, p \in A'\) and \(\text{Cap}(A') > 0\), where \(A'\) is the set of points contained in \(G'\). Hence we can find a closed set \(F\) of positive capacity in \(A'\). Since \(\mathfrak{F}\) is a set of capacity zero, we have by Theorem 5). 3) \(0 = \omega(\mathfrak{F}, w, R-D)\) implies

\[
D(\omega(\mathfrak{F}_{n}, w, \mathfrak{F}_{n'}, \Gamma_{\mathfrak{F}_{n}})) \to 0 \quad \text{as} \quad n \to \infty , \tag{18}
\]

where \(\Gamma_{w}\) is a compact arc on \(\partial \mathfrak{F}_{w}\).

Let \(\Gamma_{s}\) be a compact arc on \(\partial G'\) such that \(f(\Gamma_{s}) \subset \Gamma_{w}\). Since \(G_{n} \ni p: p \in (B_{n} \cap F)\), we have by Theorem 7). 2)

\[
\omega\{F \cap G_{n}\}, z, G', \Gamma_{s}\} = \lim \omega(F \cap G_{n}, z, G', \Gamma_{s}) > 0 . \tag{19}
\]
Now \( U(z) = f^{-1}(\omega(\mathfrak{F}_n, w, \mathfrak{F}_n, \Gamma_w)) \) is a harmonic function in \( G' - (f^{-1}(\mathfrak{F}_n) \cap G') \) such that \( U(z) = 1 \) on \( f^{-1}(\mathfrak{F}_n) = G_n \supset (G_n \cap F) \) and \( U(z) = 0 \) on \( \Gamma_z \subset f^{-1}(\Gamma_w) \) and since \( \omega(G_n, z, G', \Gamma_z) \) has M. D. I.,
\[
D(\omega(G_n, z, G', \Gamma_z)) \leq MD(\omega(\mathfrak{F}_n, w, \mathfrak{F}_n, \Gamma_w)) \downarrow 0 \quad \text{as } n \to \infty.
\]
But \( \omega(G_n, z, G', \Gamma_z) \geq \omega(G_n \cap F, z, G', \Gamma_z) \). This contradicts (20). Thus \( A \) is a set of capacity zero.

Similarly as (1) we can prove the following

Theorem 7). 3). Suppose \( n(w) \leq M < \infty \) outside of a compact set \( D \) of \( R \). Then (1) is valid even if \( \mathfrak{F} \) is not \( D \)-regularly closed.

In fact, in the proof of (1) take \( R - D' \) instead of \( \mathfrak{F}_n \), where \( D' \) is a compact domain such that \( D' \supset D, \, D' \cap \mathfrak{F}_n = 0 \) (\( n_0 \) is a certain number such that \( n_0 > 1 \)) and dist \( (\partial D, \partial D') > 0 \). Then \( D(\omega(f^{-1}(\mathfrak{F}_n), z, \Omega - f^{-1}(D'))) \leq MD(\omega(\mathfrak{F}_n, w, \Omega - f^{-1}(D')) \leq L < \infty \) for \( n > n_0 \), and \( \omega(\{G_n\}, z, \Omega - f^{-1}(D')) \) can be defined, where \( G_n = f^{-1}(\mathfrak{F}_n) \). We used the \( D \)-regularity of \( \mathfrak{F}_n \) only to define \( \omega(\{G_n\}, z, f^{-1} (\mathfrak{F}_n')) \) in the proof of (1). Hence the above theorem can be proved similarly as (1).

Remark. The assumption of (1) on \( \mathfrak{F} \) is stronger than that of (2) but the assumption of (1) on the cluster set at \( p \) is weaker than that of (2). The assumption of (3) on the cluster set is that of (1) but the assumption of (3) on \( n(w) \) is strongest.

We shall consider an extension of Beurling’s theorem. Let \( \mathfrak{F} \) be a compact set in \( R \) and suppose \( \mathfrak{F} \) is contained in a local parameter disc \( \Omega : \Omega \) can be thought equivalent to \( |z| < 1 \) and suppose \( \mathfrak{F} \) is contained in \( E \{ z : |z| < \frac{1}{2} \} \). In this case we suppose that the distance between two points in \( \Omega \) means the euclidean distance. Let \( A(r) \) be the area of \( R \) over \( \mathfrak{F}_r = E \{ z \in \Omega : \text{dist}(z, \mathfrak{F}) \leq r : r < \frac{1}{2} \} \). Then we have the following

**Theorem 9.** Let \( w = f(z) \) be an analytic function from \( R \) (with positive boundary and with N-Martin’s topology) into \( R \). Let \( \mathfrak{F} \) be a closed set contained in a local parameter disc \( \Omega = E \{ |z| < 1 \} \) and \( \mathfrak{F} \) in \( E \{ |z| < \frac{1}{2} \} \).

Suppose
\[
\lim_{r \to 0} \frac{A(r)}{r^2} < \infty.
\]

Then \( A = E \{ z \in B^N : M(f(p)) \subset \mathfrak{F} \} \) is a set of capacity zero.
Proof. We can find a sequence \( r_i, r_n, \ldots \) such that \( \frac{A(r_i)}{r_i^2} < K < \infty \). Put \( G_i = f^{-1}(\mathcal{F}_i) \), where \( \mathcal{F}_i = E[w \in \Omega : \text{dist}(w, \mathcal{F}) \leq r_i] \). Then we see for sufficiently large number \( i \) at \( R - f^{-1}(\mathcal{F}_i) \neq 0 \). This implies that \( R - f^{-1}(\mathcal{F}_i) \) contains a compact disc. Hence we can suppose without loss of generality that \( R - f^{-1}(\mathcal{F}_i) \)

\[ \supset R \{ R_n \} (n = 0, 1, 2, \ldots) \] is an exhaustion of \( R \) for \( i \geq 1 \) and \( \frac{A(r_i)}{r_i^2} < K \) for \( i \geq 1 \). Now \( G_i = f^{-1}(\mathcal{F}_i) \) consists of enumerable infinite number of components \( G', G', \ldots : \sum G' = G_i \). By Lemma b) \( A^I = E[p \in A : \check{p} \in G'] \) and \( E[p \in A : \check{p} \in G] \) are measurable. Since \( G \uparrow \check{p} \) implies that there exists only one component \( G' \) of \( G_i \) containing \( p \) \( N \)-approximately. Hence

\[ \sum \frac{A_j}{j} = A. \]

Assume \( A \) is a set of positive capacity. Then there exists at least one \( G' \) such that \( \text{Cap}(A') > 0 \). We fix \( G' \) at present and denote it by \( G \). We can find a closed set \( F \) of positive capacity in \( A' \). Put \( G_i = G \cap f^{-1}(\mathcal{F}_i) \). Then also \( G_i \uparrow \check{p} \) for any \( \check{p} \) of \( A' \) and \( G_i \) is clearly non compact. Hence by Theorem 7.2) \( \omega(F \cap \{ G_i \}, z, G, G') > 0 \) and by Theorem 5) \( \omega(\{ G_i \}, z, R - R_0) > 0 \), where \( G' \) is a compact arc on \( \partial G \). Put \( \omega(\{ G_i \}, z, R - R_0) = U(z) > 0 \). Then

\[ U(z) = 0 \text{ on } \partial R, \int_{\partial R} \frac{\partial U(z)}{\partial n}ds = D(U(z)) = \int_{\partial V_M} \frac{\partial U(z)}{\partial n}ds \]

for almost \( M : 0 < M < 1 \), where \( V_M = E[z \in R : U(z) > M] \) and such \( \partial V_M \) is called a regular niveau curve.

We shall show that \( G_i \) and \( CG_{i+j} : G_{i+j} = f^{-1}(\mathcal{F}_{i+j}) \) are Dirichlet-disjoint. In fact, \( \text{dist}(\partial \mathcal{F}_i, \partial \mathcal{F}_{i+j}) > 0 \) and \( \partial \mathcal{F}_i \) and \( \partial \mathcal{F}_{i+j} \) are compact. We can construct compact domains \( \Omega_1 \) and \( \Omega_2 \) such that \( \mathcal{F}_i \supset \Omega_1 \supset \Omega_2 \supset \mathcal{F}_{i+j} \), \( \text{dist}(\partial \Omega_1, \partial \Omega_2) > 0 \) and \( \partial \Omega_i \) \( (i = 1, 2) \) is composed of a finite number of components of which each one is consists of only one analytic curve. Hence we can define a \( C_1 \)-function \( V(w) \) in \( R \) such that \( V(w) \) is harmonic in \( \Omega_1 - \Omega_2 \), \( V(w) = 1 \) in \( \Omega_1 \), \( V(w) = 0 \) on \( C \Omega_2 \) and \( \left| \frac{\partial}{\partial u} V(w) \right| < L, \left| \frac{\partial}{\partial v} V(w) \right| < L : L < \infty \), where \( w = u + iv \). Put \( V(z) = V(f^{-1}(w)) \). Then \( D(V(z)) \leq KL^2 r_i^2 \). Thus \( G_i \) and \( G_{i+j} \) are Dirichlet-disjoint. Hence by Lemma 1) 2)

\[ (21) \sum \frac{\partial}{\partial n} U(z)ds \downarrow 0 \text{ as } M \uparrow 1 \text{ for any given } G_i. \]

Put \( \alpha = D(U(z)) \). Then \( \int_{\partial V_M} \frac{\partial}{\partial n} U(z)ds = \alpha \) for a regular niveau curve \( \partial V_M \). Let
$U_n^M(z)$ be a harmonic function in $R_n-R_0-V_M$ such that $U_n^M(z)=0$ on $\partial R_0$, $U_n^M(z)=M$ on $\partial V_M-R_n$, $\frac{\partial}{\partial n} U_n^M(z)=0$ on $\partial R_n-V_M$ and put $\alpha_n^M=D(U_n^M(z))$ 

$$ = \int_{\partial R_0} \frac{\partial}{\partial n} U_n^M(z) ds \left( = \int_{\partial V_M\cap R_n} \frac{\partial}{\partial n} U_n^M(z) ds, \text{ since } R_n-R_0-V_M \text{ is compact} \right).$$

Then $U_n^M(z) \Rightarrow U(z)$ and $\alpha_n^M \rightarrow \alpha$, as $M \rightarrow 1$.

Suppose $\partial V_M$ is regular. Then by $\alpha_n^M \rightarrow \alpha$ and by $\lim_{n} \int_{\partial V_M\cap R_n} \frac{\partial}{\partial n} U_n^M(z) ds \geq \int_{\partial V_M\cap G_{l}} \frac{\partial}{\partial n} U(z) ds$ we can find a numbers $M$ and $n_{1}(\epsilon, i, M)$ for any given numbers $\epsilon$ and $i$ such that

$$ \int_{(\partial V_M\cap R_n)\cap G} \frac{\partial}{\partial n} U_n^M(z) ds \geq \alpha - 2\epsilon \text{ for } n > n_{1}(\epsilon, i, M). \quad (22)$$

Put $D(\mathfrak{F}_{k}) = \iint_{f^{-1}(\mathfrak{F}_{k})\cap CV_M\cap R_n} |f'(z)| \left\{ \left( \frac{\partial}{\partial x} U(z) \right)^2 + \left( \frac{\partial}{\partial y} U(z) \right)^2 \right\}^{\frac{1}{2}} dxdy$ and

$$ D(\mathfrak{F}_{k}, CV_M\cap R_n) = \iint_{f^{-1}(\mathfrak{F}_{k})\cap CV_M\cap R_n} |f'(z)| \left\{ \left( \frac{\partial}{\partial x} U_n^M(z) \right)^2 + \left( \frac{\partial}{\partial y} U_n^M(z) \right)^2 \right\}^{\frac{1}{2}} dxdy. $$

Clearly $\lim_{r \rightarrow 0} \frac{A(r)}{r^2} < \infty$ implies that area of $f^{-1}(\mathfrak{F}_{k}) \rightarrow 0$ as $k \rightarrow \infty$. Since $f(z)$ is analytic in $R$ and for any $R_n$ area of $f^{-1}(\mathfrak{F}_{k})$ in $R_n \rightarrow 0$ as $k \rightarrow \infty$, $D(U(z))$ over $R_n \cap f^{-1}(\mathfrak{F}_{k}) \rightarrow 0$ as $k \rightarrow \infty$. Further $D(U(z)) < \infty$ and $D(U(z)) \rightarrow 0$ as $n \rightarrow \infty$ and $D(U(z)) \leq D(U(z)) + D(U(z))$. Hence for any given number $\epsilon$ we can find a number $k(\epsilon)$ such that

$$ D(U(z)) < \epsilon \text{ for } k > k(\epsilon). $$

Hence $D(\mathfrak{F}_{k}) \leq \iint_{f^{-1}(\mathfrak{F}_{k})\cap CV_M\cap R_n} |f'(z)|^2 dxdy$ for $k > k(\epsilon)$ and

$$ D(\mathfrak{F}_{k}) \leq \sqrt{K_\epsilon} r_k \text{ for } k > k_1. \quad (23)$$

Put $D(\mathfrak{F}_{k}, CV_M) = \iint_{CV_M\cap f^{-1}(\mathfrak{F}_{k})} |f'(z)| \left\{ \left( \frac{\partial}{\partial x} U(z) \right)^2 + \left( \frac{\partial}{\partial y} U(z) \right)^2 \right\}^{\frac{1}{2}} dxdy$. Then
by $U_n^M(z) \Rightarrow U(z)$ \ $D(\mathfrak{F}_k, CV_M \cap R_n) \rightarrow D(\mathfrak{F}_k, CV_M)$ as $n \to \infty$. Hence for any $\varepsilon$ we can find a number $n_2(\varepsilon, k, M)$ such that

\[ D(\mathfrak{F}_k, CV_M \cap R_n) \leq D(\mathfrak{F}_k, CV_M) + \varepsilon \leq D(\mathfrak{F}_k) + \varepsilon \quad \text{for} \quad n \geq n_2. \]

Next by (22) for any given $\mathfrak{F}_k$ and $\mathfrak{F}_i$ ($i > k$) and $\varepsilon$ we can find $M'$ and $n_3$ such that

\[ \int_{\partial V_{M^\prime} \cap R_n} \left| \frac{\partial}{\partial n} U_n^{M^\prime}(z) \right| ds > \alpha - 2\varepsilon \quad \text{and} \quad D(\mathfrak{F}_k, CV_{M^\prime} \cap R_n) \leq \sqrt{K\varepsilon} r_k + \sqrt{\varepsilon} \quad (24) \]

for $n > n_3 > \max(n_1, n_2)$.

We consider $D(\mathfrak{F}_k, CV_{M^\prime} \cap R_n)$ in the following. Put $\mathfrak{F}_k = \exp(U_n^M(z) + iV_n^{M^\prime}(z)) = re^{i\theta}$, where $V_n^{M^\prime}(z)$ is the conjugate function of $U_n^{M^\prime}(z)$. Then $|\mathfrak{F}_k| = e^{M^\prime}$ on $\partial V_{M^\prime} \cap R_n$ and by (22)

\[ \Theta > \alpha - 2\varepsilon \quad (25) \]

Now

\[ D(\mathfrak{F}_k, CV_{M^\prime} \cap R_n) = \int \int_{\partial V_{M^\prime} \cap R_n \cap \partial \mathfrak{F}_k} \left| f'(z) \left\{ \left( \frac{\partial}{\partial r} U_n^{M^\prime}(z) \right)^2 + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} U_n^{M^\prime}(z) \right)^2 \right\} \right| r \, dr \, d\theta = \int \int \frac{1}{r} \left| f'(re^{i\theta}) \right| \left| \frac{\partial}{\partial r} U_n^{M^\prime}(re^{i\theta}) \right| \, r \, dr \, d\theta \quad \text{and} \quad U_n^{M^\prime}(z) \]

is harmonic in a compact domain $R_n - R_0 - V_{M^\prime}$. Hence $U_n^{M^\prime}(z)$ has a finite number of branch points and we can trace the trajectories $T_\theta$ along which $\theta = \text{const}$ from $r = 1$ ($U_n^{M^\prime}(z) = 0$ on $\partial R_0$) to $r = e^{M^\prime}$ ($U_n^{M^\prime}(z) = M^\prime$ on $\partial V_{M^\prime}$). Let $\Theta$ be the set of $\theta$ such that $T_\theta$ intersects $G_\epsilon$ when $z$ goes from $R_0$ to $\partial V_{M^\prime}$ along $T_\theta$. Then by (25) $\Theta > \alpha - 2\varepsilon$. If $\theta \in \Theta$, image $f(T_\theta)$ of $T_\theta$ must intersect $G_k$ and $G_\epsilon$, whence $\int \left| f'(re^{i\theta}) \right| \, dr \geq r_k - r_\epsilon$ and

\[ D(\mathfrak{F}_k) + \varepsilon \geq D(\mathfrak{F}_k, CV_M \cap R_n) \geq (\alpha - 2\varepsilon)(r_k - r_\epsilon) \quad (26) \]

Let $r_\epsilon \to 0$ and then $\varepsilon \to 0$. Then (26) contradicts (23). Hence $A$ is a set of capacity zero.

As an application of Theorem 9 we shall prove

**Theorem 10.** Suppose $R$ is a covering surface almost finitely sheeted over $\mathfrak{R}$ and $\mathfrak{K} = \mathfrak{R} + B$ is compact. Then the set $A$ of point $p \in B_1^\mathfrak{K}$ such
that $\cup f(p) \subset \mathfrak{F}$ is a set of capacity zero, where $\mathfrak{F}$ satisfies the condition

$$\lim_{r \to 0} \frac{A(r)}{r^2} < \infty,$$

and $A(r)$ is the area of $R$ over $\mathfrak{F}_r = E[w: \text{euclidean dist } (w, \mathfrak{F}) \leq r]$.

**Proof.** By Theorem 1. $M(f(p)) = \text{one point } q$ for any point $p$ of $B_i^N$ except at most a set of capacity zero. Suppose $M(f(p)) = q$. We can find a sequence $G_i(i=1, 2, \cdots)$ such that $G_i \ni p$ and $\bigcap_i f(G_i) = q$. Thus by Theorem 9) we have Theorem 10.

In the previous paper¹¹ we constructed a covering surface $R$ over the $w$-plane: $w = f(z): z \in R$ with finite area such that $R$ has a singular point $p \in B_i^N$ with the following properties: 1) $\omega(p, z) > 0$ and 2) $\bigcap_n f(v_n(p)) = \text{one point}$. This example shows that even when $\mathfrak{F} = \text{one point}$, some condition (for instance $\lim_{r \to 0} \frac{A(r)}{r^2} < \infty$) is necessary for the validity of Theorems 9) or 10.)

( remark to the condition in Theorem 9). The condition $\lim_{r \to 0} \frac{A(r)}{r^2} < \infty$ in Theorem 9 imposed on a closed set $\mathfrak{F}$ in $R$ does not necessarily mean that $\mathfrak{F}$ is so small as $\text{Cap}(\mathfrak{F}) = 0$ but means the part of $R$ over $\mathfrak{F}_r$ is so small as $\omega(\{\mathfrak{F}_r\}, z) \to 0$ as $r \to 0$. In reality the part of $R$ over $\mathfrak{F}_r$ may be so small even when $\mathfrak{F}$ is a segment.

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¹¹ The last example of the paper "Examples of Singular points: Journ. Fac. Sci. Hokkaido Univ., XVI (1962)."