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AN INVERSE THEOREM OF GROSS'S
STAR THEOREM

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

Zenjiro KURAMOCHI

Let \( w = w(z) \) be an analytic function of \( z \) in a Riemann surface \( R \) whose values fall on the \( w \)-sphere. Let \( z = z^{-1}(w) \) be its inverse. Let \( e(w, w_0) \) be an arbitrary regular element of \( z^{-1}(w) \). We continue analytically \( e(w, w_0) \), using only regular element (without any algebraic element) along every ray: \( \arg(w - w_0) = \theta \ (0 \leq \theta < 2\pi) \) toward infinity. Then, there arise two cases whether the continuation defines a singularity \( \omega_{\theta} \) in a finite distance or not, in the former case, we call the ray a singular ray. For each singular ray: \( \arg(w - w_0) = \theta \), we exclude the segment between the singularity \( \omega_{\theta} \) and \( w = \infty \) from the \( w \)-plane. The remaining domain \( \Omega \) is clearly a (single valued) regular branch of \( z = z^{-1}(w) \). Let \( \rho = \rho(\theta) \) the polar coordinate of the singularity \( \omega_{\theta} \) or \( \infty \) according as the singular ray exists or not. Then \( \rho(\theta) \) is clearly lower semicontinuous and \( S_{n} = E[\theta; \rho(\theta) \leq n] \) is closed. We call the set \( E[\theta; \rho(\theta) < \infty] \) the singular set \( S \) of \( \Omega \). Then by \( S = \sum_{n=1}^{\infty} S_{n} \) \( S \) is an \( F_{c} \) set. Then the famous Gross's Star Theorem is as follows:

**Theorem.** Let \( R \) be a domain such that \( R = E[z; |z| < \infty] \) in the \( z \)-plane and let \( f(z) \) be an analytic function of \( z \in R \) whose values fall on the \( w \)-plane. Let \( \Omega \) be a star domain. Then \( S \) is a set of linear measure zero.

This theorem was extended by M. Tsuji\(^1\) to the case: \( R \) is a domain in the \( z \)-plane such that the boundary of \( R \) is a set of capacity zero and also extended by Z. Yûjôbô\(^2\) to the case: \( R \) is a Riemann surface with null-boundary. The method used by them is essentially the same as used by W. Gross. On the other hand, T. Yoshida\(^3\) showed that the Gross's theorem holds for not only conformal mappings but also for quasiconformal mappings.

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and M. Ohtsuka$^9$ extended the class of conformal mappings to a little wider class than quasiconformal mappings in which the Gross's theorem holds. Further we proved that there exists a Riemann surface $R \in O_{AB}^6$ such that the covering surface over the $w$-plane (mapped by an analytic function $w = w(z)$: $z \in R$) has not Gross's property (singular set of $\Omega$ is $|w| = 1$) and also there exists a domain $D \in O_{AB}^6$ in the $z$-plane such that $\partial D$ is a set of linear measure zero on a straight and its covering surface (mapped by an analytic function in $D$) has not Gross's property. Above two examples show that the validity of the Gross's theorem depends on the size of the boundary (boundary of $R$ must be so small that $R$ has null-boundary) but on the complexity of the boundary. In the present paper we consider an inverse of Gross's theorem i.e. to consider "how to construct a covering surface for given singular set?".

Let $F_i (i = 1, 2, \ldots)$ be a closed set on $|w| = 1$. If $\text{dist}(F_i, \sum_{j \neq i} F_j) > 0$, we call $\sum F_i$ a discrete $F_*$ set. We shall prove

**Theorem.** Let $S$ be an arbitrary discrete $F_*$ set of linear measure zero on $|w| = 1$. Then we can construct a covering surface $\mathfrak{R}$ which is conformally equivalent to a planer domain with null-boundary such that $\mathfrak{R}$ has a star domain $\Omega$ whose singular set is $S$.

At present we cannot prove the above theorem under the condition that the connectivity of $\mathfrak{R}$ is one. We suppose that the above theorem is valid for arbitrary $F_*$ but it is complicated too much to construct a covering surface for any $F_*$. Now by this theorem we know that Gross's theorem cannot be improved for Riemann surface of connectivity $\infty$ but it remains the problem: Is the singular set $S$ of a star domain of a covering surface (which is conformally equivalent to $|z| < \infty$) smaller than sets of measure zero?

I. Extension of $\mathcal{L}$ through $D$. Let $\mathcal{L}$ be a leaf identical to the whole $w$-plane. Let $D$ be a circular echelon

$$D: \quad R e^{-\alpha} < |w| < R, \quad -\frac{\theta}{2} < \arg w < \frac{\theta}{2}, \quad \theta < \pi, \quad \alpha > 0.$$ 

Let $U(w)$ be a $C_1$-function in $D$ such that $U(w) = 2(\alpha + \log R - \log |w|)/\alpha$ in

7) If a continuous function has partial derivatives almost everywhere, we call it a $C_1$ function.
the part $D': \text{Re}^{-\alpha}(|w| < \text{Re}^{-\frac{\alpha}{2}}$ of $D$ and $U(w)=1$ in the part $D'': \text{Re}^{|w|} \geq \text{Re}^{-\frac{\alpha}{2}}$ of $D$. Let $V(w)$ be a harmonic function in the complementary set of $D+C:C=E[w:|w|\leq 1]$ such that $V(w)=0$ on $\partial C$ and $V(w)=U(w)$ on $\partial D$. Then $D(V(w))$ depends on $U(w)$ and $D$. Let $\bar{U}(w)$ be a $C_1$-function in the complementary set of $C$ such that $\bar{U}(w)=U(w)$ in $D$ and $U(w)=0$ on $\partial C$. Then by the Dirichlet principle

$$D(\bar{U}(w)) \geq D(U(w)) + D(V(w)).$$

As $\theta \to 0$, i.e. $D$ becomes narrow, $D(U(w)) \to 0$ but $D(V(w))$ does not tend to zero. Now we shall prove the following

**Lemma.** Let $C$ and $D$ and $U(w)$ be as above. We can construct a closed Riemann surface $\Re_D$, covering surface of a finite number of sheets $\mathcal{L}, \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}$ over the $w$-plane of genus 0 satisfying the following conditions:

1). Every branch point lies on $J$: $\arg w = -\frac{\theta}{2}$, $\text{Re}^{-\alpha} < |w| < \text{Re}^{|w|}$ and $J'$: $\arg w = \frac{\theta}{2}$, $\text{Re}^{-\alpha} < |w| < \text{Re}^{|w|}$, where $\text{Re}^{-\alpha} > 1$.

2). $D$ connects $\mathcal{L}, \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}$ so that every $\mathcal{L}_i$ ($i=1, 2, \cdots, n$) contains a part $D_i$ of $D$, $\mathcal{L}$ and $\mathcal{L}_{n+1}$ do not contain any part of $D$.

3). There exists a $C_1$-function $\bar{U}(w)$ in $\Re_D$ such that $\bar{U}(w)=U(w)$ in $D$, $\bar{U}(w)=0$ in $\mathcal{L}$, $\bar{U}(w)=1$ in $\mathcal{L}_{n+1}$ and

$$D(\bar{U}(w)) \leq 3D(U(w)) = \frac{6\theta}{\alpha}.$$

Such operation (to construct $\Re_D$ through $D$) will be called extension of $\mathcal{L}$ though $D$.

**Proof.** Let $D_i$ and $I_i$ and $I_i'$ ($i=1, 2, \cdots, n$) be an echelon and a segment as follows:

- $D_i$: $\text{Re}^{-\alpha+\frac{\theta}{2}} < |w| < \text{Re}^{-\frac{\theta}{2}}$, $-\frac{\theta}{2} < \arg w < \frac{\theta}{2}$,
- $I_i$: $\text{Re}^{-\alpha+\frac{\theta}{2}} < |w| < \text{Re}^{-\frac{\theta}{2}}$, $\arg w = -\frac{\theta}{2}$,
- $I_i'$: $\text{Re}^{-\alpha+\frac{\theta}{2}} < |w| < \text{Re}^{-\frac{\theta}{2}}$, $\arg w = \frac{\theta}{2}$,

$$\sum_{i=1}^{n} D_i = D', \text{ where } \gamma = \frac{\alpha}{2n}.$$
Then $D_i$'s are conformally equivalent. Map $D_i$ by $\xi = \log w$ onto a rectangle $K_i$ such that $K_i: \log R - \alpha + (i-1)\tau < \eta < \log R - \alpha + i\tau$, $-\frac{\theta}{2} < \zeta < \frac{\theta}{2}$, where $\xi = \gamma + i\zeta$. Then $U(w) \rightarrow U_i(\xi) = 2(\alpha - \log R + \gamma)/\alpha$, $U_i(\xi) = (i-1)/n$ on $\eta = \log R - \alpha + (i-1)\tau$ and $U_i(\xi) = i/n$ on $\eta = \log R - \alpha + i\tau$. We shall define a function $\tilde{U}_i(\xi)$ corresponding to $K_i$ from $U_i(\xi)$. Let $K'_i$ be the symmetric image of $K_i$ with respect to $\eta = \log R + i\gamma - \alpha$:

$$K'_i: \quad \log R - \alpha + i\tau < \eta < \log R - \alpha + (i+1)\tau, \quad -\frac{\theta}{2} < \zeta < \frac{\theta}{2}.$$  

Let $\Gamma_U$ and $\Gamma_L$ be semicircles as follows:

$\Gamma_U: \quad |\xi - p_U| < \tau, \quad \pi \geq \arg(\xi - p_U) \geq 0$,

$\Gamma_L: \quad |\xi - p_L| < \tau, \quad 2\pi \geq \arg(\xi - p_L) \geq \pi$.

Consider the Dirichlet integral of $\tilde{U}_i(\xi)$. Then $D_{K_i + K'_i}(\tilde{U}_i(\xi)) = \frac{2}{n}$, $D(U(w)) = \frac{2}{n} \times \frac{2\theta}{\alpha} = \frac{4\theta}{n\alpha}$ and $D_{\Gamma_U}(\tilde{U}_i(\xi)) = D_{\Gamma_L}(\tilde{U}_i(\xi)) = \int_{\rho\Rightarrow 0}^{\frac{\alpha}{2n}} \int_{0}^{\pi} \left[ \left( \frac{\partial}{\partial\rho} \tilde{U}_i(\xi) \right)^2 + \frac{1}{\rho^2} \left( \frac{\partial}{\partial\varphi} \tilde{U}_i(\xi) \right)^2 \right] \rho d\rho d\varphi = \left( \frac{\alpha}{2n} \right)^2 \frac{4\pi}{\alpha^2} = \frac{\pi}{n^2}$, where $(\xi - p_U) = \rho e^{i\varphi}$ in $\Gamma_U$ and
$(\xi - p_{L}) = \rho \ e^{\varphi}$ in $\Gamma_{L}$

Hence

$$D_{\xi_{g}}(\tilde{U}_{\xi}(\xi)) = \frac{4\theta}{n\alpha} + \frac{2\pi}{n^{2}}.$$ 

Map $K_{L} + K_{L} + \Gamma_{U} + \Gamma_{L}$ in the $\xi$-plane to $D_{\xi} + D_{\xi} + \Gamma_{U} + \Gamma_{L}$ in the $w$-plane by $w = e^{\xi}$ and consider the function $\tilde{U}_{\xi}(w)$ such that $\tilde{U}_{\xi}(w) = \tilde{U}_{\xi}(\log w) = \tilde{U}_{\xi}(\xi)$ in $D_{\xi} + D_{\xi} + \Gamma_{U} + \Gamma_{L}$ and $\tilde{U}_{\xi}(w) = \frac{i - 1}{n}$ outside of $D_{\xi} + D_{\xi} + \Gamma_{U} + \Gamma_{L}$. This $w$-plane is denoted by $\mathcal{L}_{\xi}$ in which $\tilde{U}_{\xi}(w)$ is defined. Clearly

$$D_{\xi_{g}}(\tilde{U}_{\xi}(w)) = \frac{4\theta}{n\alpha} + \frac{2\pi}{n^{2}}.$$ (1)

**Structure of $\mathcal{R}_{D}$.** Let $S_{i}^{+}$ and $S_{i}^{-}$ be slits as follows:

$S_{0}^{-}: |w| = Re^{-\alpha}, -\theta < \arg w < \theta$ in $\mathcal{L}$.

$S_{i}^{+}: |w| = Re^{-\alpha + (i-1)\gamma}, -\theta < \arg w < \theta$ in $\mathcal{L}_{i}$, $i = 1, 2, \ldots, n$.

$S_{i}^{-}: |w| = Re^{-\alpha + i\gamma}, -\theta < \arg w < \theta$ in $\mathcal{X}_{i}$.

$S_{n+1}^{-}: |w| = Re^{\alpha - \frac{\theta}{2}}$, $-\frac{\theta}{2} < \arg w < \frac{\theta}{2}$ in $\mathcal{L}_{n+1}$.

where $\mathcal{L}_{n+1}$ is a leaf identical to the $w$-plane.

Connect $\mathcal{L}$ with $\mathcal{L}_{1}$ crosswise on $S_{0}^{-} (= S_{1}^{+})$, connect $\mathcal{L}_{i}$ and $\mathcal{L}_{i+1}$ crosswise on $S_{i}^{-} (= S_{i+1}^{+})(i = 1, 2, \ldots, n)$. Then we have an $n + 2$ sheeted covering surface over the $w$-plane. Clearly $\mathcal{R}_{D}$ is closed and of genus zero. We define a new $C_{1}$-function $\hat{U}(w)$ in $\mathcal{R}_{D}$ as follows: Put $\hat{U}(\xi) = \hat{U}_{\xi}(w) = 0$ in $\mathcal{L}$, $\hat{U}(w) = \tilde{U}_{\xi}(w)$ in $\mathcal{L}_{i}$ ($i = 1, 2, \ldots, n$) and $\hat{U}(w) = 1$ in $\mathcal{L}_{n+1}$. Then since $\hat{U}(w) = \tilde{U}_{\xi}(w) = U(w)$ on $S_{i}^{-} (= S_{i+1}^{+})(i = 0, 1, 2, \ldots, n)$ through which $\mathcal{L}_{i}$ and $\mathcal{L}_{i+1}$ are connected and since $\tilde{U}_{\xi}(w)$ is a $C_{1}$-function, $\hat{U}(w)$ is a $C_{1}$-function in $\mathcal{R}_{D}$, where $\mathcal{L}_{0}$ means $\mathcal{L}$. Then the Dirichlet integral of $\hat{U}(w)$ is given as $D(\hat{U}(w)) = n\left(\frac{4\theta}{n\alpha} \right) + n\left(\frac{4\pi}{n^{2}}\right)$. Choose a number $n$ such that $\frac{4\pi}{n} < \frac{2\theta}{\alpha}$. Then $D(\hat{U}(w)) < 3D(U(w)) = \frac{6\theta}{\alpha}$, hence we have the lemma.

**II. Extension of $\mathcal{L}$ through $\sum D_{m}$ ($m = 1, 2, \ldots, m_{0}$).** Let

$$D_{m}: \ Re^{-\alpha} < |w| < R, \ \theta_{m} < \arg w < \theta_{m}', \ \theta_{m}' < \theta_{m+1}.$$ 

In every $D_{m}$ let $U(w) = 2(\alpha - \log R + \log w)/\alpha$ for $Re^{-\alpha} < |w| < Re^{-\frac{\theta}{2}}$ and $U(w)$
An Inverse Theorem of Gross's Star Theorem

$=1$ for $Re^{-\frac{\alpha}{a}} \leq |w| < R$. We define $L_{m,1}, L_{m,2}, \cdots, L_{m,n(m)+1}$ and connect them on slits contained in $D_m$ as mentioned in $I$ such that there exists a $C_1$-function $\hat{U}_m(w)$ in $L_{m,1} + L_{m,2} + \cdots + L_{m,n(m)+1}$, $\hat{U}_m(w) = 0$ in $L$, $\hat{U}_m(w) = 1$ in $L_{m,n(m)+1}$ and $D(\hat{U}_m(w)) \leq 3D_{p_m}(U(w))$. Then we have a covering surface $\mathcal{R}_{\Sigma D}$ of $1 + (n(1)+1) + (n(2)+1) + \cdots + (n(m)+1)$ number of sheets and of genus zero. Put $\hat{U}(w) = \hat{U}_m(w)$ in $L + \sum_{m=1}^{m_0} \sum_{n=1}^{n(m)+1} L_{m,n}$. Then since $\hat{U}_m(w) = 0$ in $L$, $\hat{U}(w)$ is a $C_1$-function in $\mathcal{R}_{\Sigma D}$ and

$$D(\hat{U}(w)) = \sum_{m=1}^{m_0} D(\hat{U}_m(w)) \leq 3 \sum_{m=1}^{m_0} D_{p_m}(U(w)) = \frac{6}{\alpha} \sum_{m=1}^{m_0} (\theta'_m - \theta_m).$$

Now the projection of every branchpoint of $\mathcal{R}_{\Sigma D}$ lies on arg$w = \theta_m$ and arg$w = \theta'_m (m = 1, 2, \cdots, m_0)$. We consider the star domain $\Omega$ of $\mathcal{R}_{\Sigma D}$ with centre at $w = 0$ of $L$. Then $\partial \Omega$ consists of segments $|w| > Re^{-\alpha}$, arg$w = \theta_m$ and $|w| > Re^{-\alpha}$, arg$w = \theta'_m (m = 1, 2, \cdots, m_0)$ and $\Omega$ is composed of the following parts:

$$\Omega = \hat{\mathcal{L}} + \sum_{m=1}^{m_0} \sum_{n=1}^{n(m)+1} \hat{L}_{m,n}$$

and $D_m \subset \sum_{n=1}^{n(m)} \approx \hat{\mathcal{L}}_{m,n}$ where

$$\hat{\mathcal{L}}: E[w: |w| < Re^{-\alpha}] + \sum_{m=1}^{m_0} E[w: |w| > Re^{-\alpha}, \theta'_m < \arg w < \theta_{m+1}]$$

$$+ E[w: |w| > Re^{-\alpha}, \theta'_m < \arg w < \theta_m] \text{ of } L.$$

$$\hat{\mathcal{L}}_{m,1}: E[w: Re^{-\alpha + m}\theta_m > |w| > Re^{-\alpha}, \theta_m < \arg w < \theta'_m] \text{ of } L_{m,1}$$

$$\cdots$$

$$\hat{\mathcal{L}}_{m,n}: E[w: Re^{-\alpha + nt_m}\theta_m > |w| > Re^{-\alpha + (n-1)\tau_m}, \theta_m < \arg w < \theta'_m] \text{ of } L_{m,n}$$

$(n = 2, 3, \cdots, n(m))$$$

$$\cdots$$

$$\hat{\mathcal{L}}_{m,n+1}: E[w: |w| > Re^{-\frac{\alpha}{2}}, \theta_m < \arg w < \theta'_m], \text{ where } \tau_m = \alpha/2n(m)$

and $m = 1, 2, \cdots, m_0$ and $n = 1, 2, \cdots, n(m)+1$.

The function $\hat{U}(w)$ in $\Omega$ is as follows: $\hat{U}(w) = 0$ in $\hat{\mathcal{L}}$, $\hat{U}(w) = 2(\alpha - \log R + \log |w|)/\alpha$ in $\sum_{m=1}^{m_0} \sum_{n=1}^{n(m)} \hat{\mathcal{L}}_{m,n}$ and $\hat{U}(w) = 1$ in $\sum_{m=1}^{m_0} \mathcal{L}_{m,n(m)+1}$.

III. Extension of $\mathcal{L}$ through a closed set $F$ of linear measure zero on $|w| = 1$. The complementary set of $F = CF = \sum_{l=1}^{\infty} I_l$, where $I_l$ is an open interval. Put $F_i = \Gamma \sum_{l=1}^{I_i} I_i (\Gamma: |w| = 1)$. Then $F_i = J_{i,1} + J_{i,2} + \cdots + J_{i,m(l)+1} + p_{i,1}$.
$+p_{t,2}+\cdots+p_{t,m^{t}(l)}$, where $J_{t,i}$ is a closed interval and $p_{t,i}$ is an isolated point of $F_t$. Clearly $F=\bigcap_{t=1}^{\infty} F_t$. Let $l'(l)$ be the smallest number such that $\text{mes } F_{l'(l)}<\frac{1}{4^l}$. For simplicity we denote $F_{l'(l)}$ by $F_l$. Then $F=\bigcap_{l=1}^{\infty} F_l$ and $\text{mes } F_{l}<\frac{1}{4^l}$. Put $A_l=J_{l,1}+J_{l,2}+\cdots+J_{l,m(l)}$ and $B_l=p_{l,1}+\cdots+p_{l,m(l)}$. Then $F_l=A_l+B_l$ and $F_{l+1}=A_{l+1}+B_{l+1}$. Since every $p_{l,m}$ is isolated in $F_l$ and is contained in $F$, $B_l\subset B_{l+1}$. Hence by $F_{l}\supset F_{l+1}$

\[ B_{l+1}-B_l \subset A_l \]

And

\[ F_l=A_l+\sum_{t=1}^{l} (B_t-B_{t-1}), \quad F=\bigcap_{t=1}^{\infty} A_t+\sum_{t=1}^{\infty} (B_t-B_{t-1}), \]

where $B_0=0$.

Since $\text{mes } F=0$, $F$ does not contain any closed interval and

\[ F=\sum_{t=1}^{\infty} \sum_{m=1}^{m(l)} (q_{t,m}+q_{t,m}') + \lim_{l=\infty} B_l, \]

where $q_{t,m}$ and $q_{t,m}'$ are endpoints of $J_{t,m}$.

We define echelons $D_{t,m}$, $\hat{D}_{t,m}$, $\tilde{D}_{t,m}$ ($D_{t,m} = \hat{D}_{t,m} + \tilde{D}_{t,m}$) and a slit $t_{l,m}$ from $F_l$ as follows:

\[ D_{t,m} : \quad \Re^{-\frac{\alpha}{2^{l-1}}}<|\omega|<R, \quad \theta_{t,m}<\arg \omega<\theta_{t,m}', \]

\[ \hat{D}_{t,m} : \quad \Re^{-\frac{\alpha}{2^{l-1}}}<|\omega|<\Re^{-\frac{\alpha}{2^{l}}}, \quad \theta_{t,m}<\arg \omega<\theta_{t,m}', \]

\[ \tilde{D}_{t,m} : \quad \Re^{-\frac{\alpha}{z^{l}}}<|\omega|<R, \quad \theta_{lm}<\arg \omega<\theta_{t,m}'-\frac{\alpha}{l}, \quad m=1,2,\cdots,m(l) \]

\[ t_{l,m} : \quad \Re^{-\frac{\alpha}{2^{l-1}}}<|\omega|<\Re^{-\frac{\alpha}{2^{l}}}, \quad \arg \omega = \arg p_{t,m}\in(B_l-B_{l-1}), \]

\[ \overline{t_{l,1}} \]

Fig. 3.
An Inverse Theorem of Gross's Star Theorem

Let \( \theta_{l,m} = \min_{W \subseteq -J_l,m} \arg w \) and \( \theta_{l,m}' = \max_{W_{\overline{c}}J_l,m} \arg w \).

\[ \sum_{m=1}^{m(l)} D_{l,m} \supset \sum_{m=1}^{m(l+1)} D_{l+1,m}, \] \[ \sum_{m=1}^{m(l)} \hat{D}_{l,m} \cap \sum_{m=1}^{m(l+1)} \hat{D}_{l+1,m} = 0 \]

Then \[ \sum_{m=1}^{m(l)} D_{l,m} \supset \sum_{m=1}^{m'(l+1)} t_{l+1,m}, \sum_{m=1}^{m(l)} \hat{D}_{l,m} \cap \sum_{m=1}^{m'(l)} t_{l,m} = 0 \].

IV. Extension of 1st step of \( L \) through \( F \). Let \( L \) be a leaf. Let \( U_i(w) = 2(\alpha - \log R + \log |w|)/\alpha \) in \( \sum_{m=1}^{m(1)} \hat{D}_{1,m} \) and \( U_i(w) = 1 \) in \( \sum_{m=1}^{m(1)} \hat{D}_{1,m} \).

We extend \( L \) through \( \sum_{m=1}^{m(1)} D_{1,m} \) (see II) to \( \mathcal{R}' = \mathcal{L} + \sum_{m=1}^{m(1)} \sum_{n=1}^{n(1,m)+1} L_{1,m,n} \). Then since \( \mathcal{L} \) exists a \( C_1 \)-function \( U_i(w) \) in \( \mathcal{R}' \) such that \( U_i(w) = 0 \) in \( L \), \( U_i(w) = 1 \) in \( \sum_{m=1}^{m(1)} X_{1,m,n(m)+1} \) and \( D(U_i(w)) = \frac{6\theta_1}{\alpha} \), where \( \theta_1 = \sum_{m=1}^{m(1)} (\theta_{1,m}' - \theta_{1,m}) \leq \frac{1}{4} \).

Next we connect a leaf \( L_{1,m}' \) with \( L \) crosswise on \( t_{1,m} (m=1,2, \cdots, m'(1)) \). Put \( \mathcal{R}'(F, 1) = \mathcal{R}' + \sum_{m=1}^{m(1)} \approx^\Gamma_{1,m} \) and put \( U(w, F, 1) = 0 \) in \( \sum_{m=1}^{m(1)} \hat{D}_{1,m} \) and \( U(w, F, 1) = U_1(w) \) in \( \mathcal{R}' \). Then since \( U_i(w) = 0 \) in \( L \), \( U(w, F, 1) \) is also a \( C_1 \)-function in \( \mathcal{R}'(F, 1) \) and \( D(U(w, F, 1)) = D(U_i(w)) \). Put

\[ \mathcal{R}(F, 1) = \mathcal{R}'(F, 1) - \sum_{m=1}^{m(1)} L_{1,m,n(m)+1} \]

Then \( \partial \mathcal{R}(F, 1) \) is composed of \( m(1) \) number of compact relative boundary components \( B(F, 1) \) such that each component lies on the slits on which \( L_{1,m,n(m)+1} \) is connected. Such operation is called the extension of first step of \( L \) through \( F \).

Extension of 2nd step of \( L \) through \( F \). We extend every \( L_{1,m,n(m)+1} (m = 1, 2, \cdots, m'(1)) \) through \( \sum_{n'} D_{z,m} \) (\( \sum' \) means the sum over \( D_{z,m} \) contained in \( D_{1,m} \)) by defining \( L_{z,m,n} (n = 1, 2, \cdots, n(2,m)+1) \) and connect \( L_{z,m} \) on \( t_{z,m} (m = 1, 2, \cdots, m'(2)) \) crosswise to obtain \( \mathcal{R}'(F, 2) = \mathcal{R}'(F, 1) + \sum_{m=1}^{m(2)} \sum_{n=1}^{n(2,m)+1} L_{z,m,n} \). Put \( \mathcal{R}(F, 2) = \mathcal{R}'(F, 2) - \sum_{m=1}^{m(2)} L_{z,m,n(2,m)+1} \). Then there exists a \( C_1 \)-function \( U(F, w, 2) \) in \( \mathcal{R}(F, 2) \) such that \( U(F, w, 2) = 0 \) in \( \mathcal{R}'(F, 1) \), \( U(F, w, 2) = 1 \) on \( B(F, 2) = \partial \mathcal{R}(F, 2) \) and

\[ D(U(F, w, 2)) = 3D(U_2(w)) = \frac{6\theta_2}{\alpha} \],

where \( U_2(w) = 2(\alpha - \log R + \log |w|)/\alpha \) in \( \sum_{m=1}^{m(2)} D_{z,m} \) and \( U_2(w) = 1 \) in \( \sum_{m=1}^{m(2)} D_{z,m} \) and \( \theta_2 = \sum_{m=1}^{m(2)} (\theta_{2,m}' - \theta_{2,m}) \leq \frac{1}{4^2} \).
Suppose $\mathcal{R}(F, l)$ is defined, we define $(l+1)$-th step and $\mathcal{R}(F, l+1)$ as follows: we extend $L_{l,m,n(l,m)+1}$ (m = 1, 2, ..., m(l)) through $\sum' D_{l+1,m}$ ($\sum'$ means over $D_{l+1,m}$ contained in $D_{l,m}$) by defining $\mathcal{L}_{l+1,m,n(t.m)+1}(m=1,2,\cdots,m(l))$ through $\sum D_{l+1,m}$ (\sum means over $D_{l+1,m}$). We extend $X_{l+1,m}^f$ and connecting $\sum_{m=1}^{m(l+1)}L_{l+1,m}$ on $\sum_{m=1}^{m(l+1)}t_{l+1m}$ ($\sum \subset \sum_{m=1}^{m(l)}D_{l,m}$). Put

$$\mathcal{R}'(F, l+1) = \mathcal{R}'(F, l) + \sum_{m=1}^{m(l+1)}L_{l+1,m} + \sum_{m=1}^{m(l+1)}X_{l+1,m}^f$$

and

$$\mathcal{R}(F, l+1) = \mathcal{R}'(F, l+1) - \sum_{m=1}^{m(l+1)}X_{l+1,m,n(l+1,m)+1}$$

There exists a $C_1$ function $U(F, w, l+1)$ in $\mathcal{R}(F, l+1)$ such that $U(F, w, l+1) = 0$ in $\mathcal{R}'(F, l)$, $U(F, w, l+1) = 1$ on $B(F, l+1)$ and $D(U(F, w, l+1)) \leq \frac{6\theta_{l+1}}{2^{l+1}}$. Such extension is called the extension of $l+1$-th step of $\mathcal{L}$ through $F$. Put $\mathcal{R}_F = \lim_{l} \mathcal{R}(F, l)$. Then $\mathcal{R}_F$ has the following properties:

1). $\mathcal{R}_F$ is a Riemann surface of planer character of connectivity $\leq \infty$ and has null-boundary.

2). Let $\Omega_F$ be a star domain of $\mathcal{R}_F$ with centre at $w = 0$ of $\mathcal{L}$. Then $\Omega_F$ contains the part of $\mathcal{L}$ outside of $\sum_{m=1}^{m(1)}K_{1,m} + \sum_{m=1}^{m(1)}K_{1,m}$, where

$K_{1,m}$: $Re^{-a} < |w| < \infty$, $\theta_{1,m} < \arg w < \theta_{1,m}'$, $m = 1, 2, \cdots, m(1)$

$K_{1,m}'$: $Re^{-a} < |w| < \infty$, $\arg w = \theta_{1,m}' = \arg t_{1,m}$, $m = 1, 2, \cdots, m'(1)$

3). Let $\Omega_F$ be as above. Then the singular set of $\Omega_F$ is $F$.

1). Clearly every $\mathcal{R}(F, l)$ is of planer character and $B(F, l)$ consists of $n(l)$ number of components. Hence $\mathcal{R}_F$ is of planer character and its connectivity $\leq \infty$. Now $\mathcal{R}(F, l)$ (l = 1, 2, ..., ) is an exhaustion of $\mathcal{R}_F$, let $\omega_l(w)$ be a harmonic function in $\mathcal{R}(F, l) - C$ such that $\omega_l(w) = 0$ on $\partial C$ and $\omega_l(w) = 1$ on $B(F, l)$, where $C = E[w:|w| < 1]$ of $\mathcal{L}$. Then by the Dirichlet principle $D(\omega_l(w)) \leq D(U(F, w, l)) \leq \frac{6}{4^l \times 2^{l+1}}$. Whence $\lim_{l=\infty} \omega_l(w) = 0$ and $\mathcal{R}_F$ has null-boundary.

2). is clear from the structure of $\mathcal{R}(F, 1)$.

3). If $p$ is an accumulating point of $\lim B_l$, $p \in A_l$ for any $l$ and $p \in \bigcap_{l=1}^{\infty} A_l$, by $F_i = A_i + (B_{l-1} - B_{l-2}) \supset F$. Now $\mes F = 0$ and $F$ does not contain any arc. Whence if $p \in \bigcap_{l=1}^{\infty} A_l$, $p \in \sum_{l=1}^{\infty} \sum_{m=1}^{m(l)} (q_{l,m} + q_{l,m}^f)$. Corresponding fact occurs for
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In fact for \( p \in \sum_{l=1}^{\infty} \sum_{m=1}^{m(l)} (q_{l,m} + q'_{l,m}) - \lim_{l=\infty} B_l \), there exists a ray: \( r(p) \) such that \( r(p): Re^{-\frac{a}{2^l}} \leq |w| < \infty, \arg w = \arg p \). Suppose \( p \in \sum_{l=1}^{\infty} \sum_{m=1}^{m(l)} (q_{l,m} + q'_{l,m}) - \lim_{l=\infty} B_l \). Then since there exist only a finite number of points \( \{q_{l,m}\} \) and \( \{q'_{l,m}\} \) for given \( l \), there exists a sequence \( q_{l_1,i_1}, q_{l_2,i_2}, \ldots \rightarrow p \), \( l_1 < l_2 \ldots \). Hence there exists a sequence of rays \( r_{l,i}: Re^{-\frac{a}{2^i}} \leq |w| < \infty, \arg w = \arg q_{l,i} \) tending to the ray \( r(p) \). Thus to every \( p \in F \) a singular ray corresponds and the singular set \( S \) of \( \Omega_F \) is \( F \).

V. Extension of \( L \) through a discrete \( F \) set of measure zero.

Let \( F_0 = \sum_{i=1}^{\infty} F_i \), \( \delta_i = \text{dist}(F_i, \sum_{j \neq i} F_j) > 0 \) and \( F_{i,0} = \{ p : \text{dist}(F_i, p) \leq \frac{\delta_i}{2} \} \).

Then \( \overline{F}_{i,0} \supset F_i \) and \( \overline{F}_{i,0} \cap \overline{F}_{j,0} = 0 \) for \( i \neq j \). Every \( F_i \) is expressed by

\[
F_i = \cap_{l=1}^{\infty} A_{i,l} + \sum_{l=1}^{\infty} (B_{i,l} - B_{i,l-1}),
\]

where \( B_{i,0} = 0 \) and

\[
A_{i,l} = J_{i,l,1} + J_{i,l,2} + \cdots + J_{i,l,m(i,l)} \quad \text{and} \quad B_{i,l} = p_{i,l,1} + p_{i,l,2} + \cdots + p_{i,l,m'(i,l)}.
\]

Since \( \text{mes} F_i = 0 \), there exists a number \( l(i) \) such that \( \text{mes} A_{i,l} < \frac{\delta_i}{4} \) and \( A_{i,l} + B_{i,l} \subset \overline{F}_{i,0} \) for \( l > l(i) \). On the other hand, by (3) we suppose without loss of generality that

\[
A_{i,l} + B_{i,l} \subset \overline{F}_{i,0}, \quad i = 1, 2, \ldots.
\]

Also we can suppose

\[
\text{mes} A_{i,l} \leq 1/2^l, \quad l = 1, 2, \ldots.
\]

We define \( D_{i,l,m}(m = 1, 2, \ldots, m(i,l)) \) and \( t_{i,l,m}(m = 1, 2, \ldots, m'(i,l)) \) from \( F_i = \bigcap_{l=1}^{\infty} A_{i,l} + \lim_{l=\infty} B_{i,l} \) for every \( i \). Let

\[
R_1 = e^{\beta l} \quad \text{and} \quad \alpha = \alpha^l, \quad \beta > \alpha > 2.
\]

By (6) we have

\[
\log(R_1 e^{-\frac{\alpha l}{2}}/R_i) = \beta^{l+1} - \beta^l - \frac{\alpha^{l+1}}{2} > \beta^l \left( \frac{\beta}{2} - 1 \right) \rightarrow \infty \quad \text{as} \quad i \rightarrow \infty.
\]
At first we extend $\mathcal{L}$ through $F_1$ by $\{D_{1,l,m}\} + \{t_{1,l,m}\}$ and we obtain $\Re_{F_1}$. Then

1°. By (2) of III $\Omega_{F_1}$ (star domain of $\Re_{F_1}$ with centre at $w=0$ of $\mathcal{L}$ contains the part of $\mathcal{L}$ outsider of $K_1$: $R e^{-\frac{a_i}{2}} \leq |w| \leq \infty$, $\arg w = \theta \in \overline{F}_1$.

2°. There exists a sequence of bordered Riemann surface $\{\Re(F_l, l)\}$ $(l=1, 2, \ldots)$, where $\Re(F_1, l) \rightarrow \Re(F_1)$ as $l \rightarrow \infty$ and $\Re(F_1, l)$ has compact relative boundary $B(F_1, l)$. This $\Re(F_1, l)$ is extended through $(\sum_{m=1}^{m(1,1)}D_{1,1,m} + \sum_{m=1}^{m(1,1)}t_{1,1,m}) + (\sum_{m=1}^{m(1,2)}D_{1,2,m} + \sum_{m=1}^{m(1,2)}t_{1,2,m}) + \cdots + (\sum_{m=1}^{m(1,l)}D_{1,l,m} + \sum_{m=1}^{m(1,l)}t_{1,l,m})$. We call $\Re(F_1l)$, a surface of $l$-th step through $F_1$.

3°. There exists a $C_1$-function $U(F_1, w, l)$ in $\Re(F_1, l)$ such that $U(F_1, w, l) = 0$ in $\mathcal{L}$, $U(F_1, w, l) = 1$ on $B(F_1, l)$ and $D(U(F_1, w, l)) \leq \frac{6}{\alpha_1} \text{mes}(A_1,l) \leq \frac{6}{2^{l}\alpha_1}$.

By (1) $\Omega_{F_1}$ contains the part of $\mathcal{L}$ not lying on $(K_1+K_2+\cdots+K_t)$ where $K_t: R e^{-\frac{a_{i+1}}{2}} \leq |w| \leq \infty \arg w = \theta \in \overline{F}_t$. Hence

\[ \sum_{m=1}^{m(t+1,1)}D_{t+1,1,m} + \sum_{m=1}^{m(t+1,1)}t_{t+1,1,m} \]

is contained in $\Omega_{\Sigma F_1}$. Whence the extension of $\Omega_{\Sigma F_1}$ through $F_{t+1}$ can be performed. This is the extension of $\mathcal{L}$ through $F_{t+1}$. Thus we can define the extension of $\Omega_{\Sigma F_1}$ through $F_{t+1}$ to obtain $\Re_{\Sigma F_1}$. Also there exists a $C_1$-function $U(F_{t+1}, w, l)$ in $\Re(F_{t+1}, l)$ such that $U(F_{t+1}, w, l) = 0$ in $\mathcal{L}$, $U(F_{t+1}, w, l) = 1$ on $B(F_{t+1}, l)$ and

\[ D(U(F_{t+1}, w, l)) \leq \frac{6}{2^{t}\alpha_{t+1}}. \]

Put

\[ \Re_{F_0} = \lim_{t \to \infty} \Re_{\Sigma F_t}. \quad \text{Then } \Re_{F_0} \]

has the following properties:

1°. $\Re_{F_0}$ is the surface of planer character and $\Re_{F_0}$ has null-boundary.

2°. The singular set $S$ of the star domain $\Omega$ of $\Re_{F_0}$ with centre at $w=0$ on $\mathcal{L}$ is $F_0$. 

\[ \text{Fig. 4.} \]
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Proof of 1°). We must define an exhaustion \( M_n (n=1, 2, \cdots) \) of \( M_{F_s} \) with compact relative boundary \( \partial M_n \). Let \( C \) be the circle \(|w|<1\) in \( C \). We extend \( \mathcal{L} \) through \( F_i \) till \( n \)-th step, \( \mathcal{L} \) through \( F_i \) till \( (n-1) \)-th step... and through \( F_n \) till first step. Then we have the surface composed of \( \mathcal{L} + (\mathcal{R}(F_1, n)-\mathcal{L}) + (\mathcal{R}(F_2, n-1)-\mathcal{L}) + \cdots + B(F_n, 1) \). This surface has compact relative boundary consisting of \( B(F_1, n)+B(F_2, n-1)+\cdots+B(F_n, 1) \). We extract the part \( \Gamma_n : R_{n+1}e^{-a_{n+1}}<|w|<\infty \) (in which \( \mathcal{L} \) will be extended through \( F_{n+1} \), \( F_{n+2} \) ...). Then we have the surface composed of \( \mathcal{L} + (\mathcal{R}(F_1, n)-\mathcal{L}) + (\mathcal{R}(F_2, n-1)-\mathcal{L}) + \cdots + (\mathcal{R}(F_n, 1)-\mathcal{L}) \). This surface has compact relative boundary consisting of \( B(F_1, n)+B(F_2, n-1)+\cdots+B(F_n, 1) \).

We extract the part \( \Gamma_n \):

\[ \begin{align*}
&\sum_{j=1}^{n} B(F_1, n+1-j) + \partial \Gamma_n \\text{from} \\ \mathcal{L}
&\text{of this surface.} \\
&\text{The remaining surface} \ M_n \ \text{has relative boundary} \\
&\sum_{j=1}^{n} B(F_1, n+1-j) + \partial \Gamma_n \\
\end{align*} \]

Let \( G_n \) be a ring in \( C \) such that \( R_n<|w|<R_{n+1}e^{-a_{n+1}} \) and let \( V_n(w) \) be a \( C_1 \)-function in \( M_n \) such that \( V_n(w) \) is harmonic in \( G_n \), \( V_n(w)=0 \) in \( M_n-G_n \), \( V_n(w)=1 \) on \( \partial G_n \) lying on \( |w|=R_{n+1}e^{-a_{n+1}} \). Then by (6)

\[ D(V_n(w))=2\pi/\log\frac{R_{n+1}e^{-a_{n+1}}}{R_n} \leq \frac{2\pi}{\beta^{n} \left( \frac{\beta}{2} - 1 \right)} \]

Let \( \mathcal{U}_n(w)=V_n(w) \) in \( G_n \) and \( \mathcal{U}_n(w)=U(F_1, w, n)(=U(F_2, w, n-1)=\cdots=U(F_n, w, 1)) \) in \( X-G_n \), \( \mathcal{U}_n(w)=U(F_1, w, n) \) in \( \mathcal{R}(F_1, n)-X \), \( \mathcal{U}_n(w)=U(F_2, w, n-1) \) in \( \mathcal{R}(F_2, n-1)-X \) ... and \( \mathcal{U}_n(w)=V(w) \) in \( G_n \).

Then \( \mathcal{U}_n(w) \) is a \( C_1 \)-function in \( M_n \) such that \( \mathcal{U}_n(w)=0 \) on \( \partial C \) and \( \mathcal{U}_n(w) = 1 \) on \( \partial M_n \). Then by the Dirichlet principle and by (6) and (7)

\[ D(\omega_n(w)) \leq D(\mathcal{U}_n(w)) \leq \sum_{i=1}^{n} D(U(F_i, w, n+1-i)) + D(V_n(w)) \leq \]

\[ \frac{6}{2^{n+1} \alpha_i} + \frac{2\pi}{\beta^n \left( \frac{\beta}{2} - 1 \right)} + \frac{6n}{2^{n+1}} + \frac{2\pi}{\alpha^n \left( \frac{\beta}{2} - 1 \right)} \]

where \( \omega_n(w) \) is a harmonic function in \( M_n-C \) such that \( \omega_n(w)=1 \) on \( \partial M_n \) and \( \omega_n(w)=0 \) on \( \partial C \). It is evident that \( \lim_{n} \omega_n(w)=0 \) and \( \{M_n\} \) is an exhaustion of \( M_{F_s} \). Hence \( M_{F_s} \) has null-boundary. Next since every \( M_n \) is a surface of planer character, \( M_{F_s} \) is of planer character.

2°). It can be proved that the singular set \( S \) of \( \Omega \) satisfies \( S=\sum F_i=F_s \) as III. Hence \( M_{F_s} \) is the Riemann surface required and the Theorem is proved.

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