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<tr>
<td>Author(s)</td>
<td>Koizumi, Sumiyuki</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 17(3-4): 065-072</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1963</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56041">http://hdl.handle.net/2115/56041</a></td>
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<td>Type</td>
<td>bulletin (article)</td>
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<tr>
<td>File Information</td>
<td>JFSHIU_17_N3-4_065-072.pdf</td>
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<td>Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP</td>
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ON THE SPECTRUM OF FUNCTION IN THE WIELY SPACE

By

Sumiyuki KOIZUMI

1. Introduction. Let $f$ be a bounded measurable function defined on the real line. By $\Lambda(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

\[(1.1)\quad K*f = \int_{-\infty}^{\infty} K(x-y)f(y)dy = 0 \quad (-\infty < x < \infty)\]

implies

\[(1.2)\quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0.\]

(c.f. A. Beurling [1].)

In the previous paper we introduced the analogous definitions. By $\Lambda_*(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

\[(1.3)\quad K*f = \int_{-\infty}^{\infty} K(x-y)f(y)dy \sim 0\]

implies

\[(1.4)\quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0\]

where the notation $K*f \sim 0$ means

\[(1.5)\quad \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{-\infty}^{\infty} |K*f(x)|^2 dx = 0.\]

(c.f. S. Koizumi [4]. It is clear that

\[(1.6)\quad \Lambda_*(f) \subseteq \Lambda(f).\]

The purpose of this paper is to investigate properties of the above defined set as for functions represented by the Fourier-Stieltjes transform and almost periodic functions in the sense of Weyl.

2. General property of $\Lambda_*(f)$.

Theorem 1. The set $\Lambda_*(f)$ is closed.

This is a trivial result.
\textbf{Theorem 2.} Let $f$ be a bounded measurable function and $f \sim 0$. Then we have $\Lambda_*(f) = 0$.

\textit{Proof of Theorem 2.} Using the same notation in our previous paper (S. Koizumi [4]) we get

\begin{equation}
\Lambda_*(f) = \Lambda_{Wy}(f)
\end{equation}

From the assumption $f \sim 0$ we get

\begin{equation}
\lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{-\infty}^{\infty} |s(u+\varepsilon, x) - s(u-\varepsilon, x)|^2 du = 0
\end{equation}

where $s(u)$ is the Fourier-Wiener transform of $f(x)$. Therefore we get $\Lambda_{Wy}(f) = 0$ and thus we get $\Lambda_*(f) = 0$.

3. Fourier-Stieltjes transform.

\textbf{Theorem 3.} Let $f$ be represented by the Fourier-Stieltjes transform. That is

\begin{equation}
f(x) = \frac{1}{\sqrt{2\pi l}} \int_{-\infty}^{\infty} e^{itx} d\sigma(u)
\end{equation}

where $\sigma(u)$ is a function of bounded variation on the real line. Then we have

\begin{equation}
\lim_{l \to \infty} \sup_{-\infty < z < \infty} \frac{1}{l} \int_{x}^{x+l} |f(t)|^z dt = \frac{1}{2\pi} \sum |\sigma(\lambda+0) - \sigma(\lambda-0)|^z.
\end{equation}

\textit{proof of Theorem 3.} Let $\sigma_1(u) + \sigma_2(u)$ be the Lebesgue decomposition of $\sigma(u)$. The $\sigma_1(u)$ is its continuous part and the $\sigma_2(u)$ is its saltus part. Let $f_i(x)$ be the Fourier-Stieltjes transform of $\sigma_i(u)$ ($i=1, 2$) respectively. Then we have

\begin{equation}
\frac{1}{l} \int_{x}^{x+l} |f(t)|^z dt = \frac{1}{l} \int_{x}^{x+l} |f_i(t)|^z dt + 2\Re \left( \frac{1}{l} \int_{x}^{x+l} f_i(t) \overline{f_i(t)} dt \right) + \frac{1}{l} \int_{x}^{x+l} |f_2(t)|^z dt.
\end{equation}

Firstly we shaly estimate the part of $f_i(t)$. We have

\begin{equation}
\frac{1}{l} \int_{x}^{x+l} |f_i(t)|^z dt = \frac{1}{2\pi l} \int_{x}^{x+l} dt \int_{-\infty}^{\infty} e^{itx} d\sigma_i(u) \int_{-\infty}^{\infty} e^{-ivx} \overline{d\sigma_i(v)}
\end{equation}

\begin{equation}
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma_i(u) \int_{-\infty}^{\infty} \overline{d\sigma_i(v)} e^{i(\lambda-\lambda')x} \frac{1}{i(\lambda-\lambda')l} e^{i(\lambda-\lambda')x}
\end{equation}

Here if we put $\sigma_i^*(u) = \int_{-\infty}^{u} |d\sigma_i(v)|$, then the $\sigma_i^*(u)$ is a continuous, nondecreas-
ing and bounded function and we get
\[
\sup_{-\infty<x<\infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2}dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{-\infty}^{\infty} \sigma_{1}^{*}(v) \left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^{2}d\sigma_{1}^{*}(u)d\sigma_{1}^{*}(v)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{|u-v|\leq\delta} \sigma_{1}^{*}(v) \left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^{2}d\sigma_{1}^{*}(u)d\sigma_{1}^{*}(v)
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{|u-v|>\delta} \sigma_{1}^{*}(v) \left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^{2} = I_{1} + I_{2}, \text{ say.}
\]

The number \(\delta\) will be determined in later. As for the \(I_{2}\): for any given positive number \(\eta\) there exist a number \(N\) such that
\[
|\sigma_{1}^{*}(\pm\infty) - \sigma_{1}^{*}(\pm N)| \leq \eta/4V_{1},
\]
where \(V_{1} = \sigma_{1}^{*}(\infty) = \int_{-\infty}^{\infty} |d\sigma(v)|\). Then if we fixe the number \(N\), there exist a number \(\delta\) such that
\[
\sup_{-N \leq u \leq N} \sigma_{1}^{*}(u + \delta) - \sigma_{1}^{*}(u - \delta) < \eta/4V_{1}.
\]

Hence we get
\[
I_{2} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{-\infty}^{\infty} \sigma_{1}^{*}(v)
\]
\[
= \frac{1}{2\pi} \left(\int_{-\infty}^{-N} + \int_{-N}^{N} \right) \left(\sigma_{1}^{*}(u + \delta) - \sigma_{1}^{*}(u - \delta)\right)d\sigma_{1}^{*}(u)
\]
\[
= \frac{1}{2\pi} \left(\int_{-\infty}^{-N} + \int_{-N}^{N} \right) \left(\sigma_{1}^{*}(u + \delta) - \sigma_{1}^{*}(u - \delta)\right)d\sigma_{1}^{*}(u) + \frac{1}{2\pi} \int_{-N}^{N} \left(\sigma_{1}^{*}(u + \delta) - \sigma_{1}^{*}(u - \delta)\right)d\sigma_{1}^{*}(u)
\]
\[
\leq (1/2\pi)(\eta/2 + \eta/2) + (1/2\pi)(\eta/4) = 5\eta/8\pi.
\]

As for \(I_{1}\): the condition \(|u-v| \geq \delta\) imply
\[
\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^{2} \to 0 \quad \text{(as } l \to \infty)\).
\]

Therefore we get
\[
\lim_{l \to \infty} \sup_{-\infty<x<\infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2}dt \leq 5\eta/8\pi.
\]

and the number \(\eta\) is an arbitrarily small. Thus we get
\[
\lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} dt = 0.
\]

Secondly we shall estimate the part of \( f_{2}(t) \). The \( \sigma_{2}(u) \) has enumerable number of jump point \( (\lambda_{n}) \) \((n=1, 2\cdots)\). Here if we put
\[
a_{n} = \sigma(\lambda_{n} + 0) - \sigma(\lambda_{n} - 0)
\]
and
\[
\sum |a_{n}| = \int_{-\infty}^{\infty} d\sigma_{2}(u) = V_{2}.
\]
Then we have
\[
f_{2}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma_{2}(u) = \frac{1}{\sqrt{2\pi}} \sum a_{n} e^{i\lambda_{n}x}.
\]
Therefore we get
\[
\frac{1}{l} \int_{x}^{x+l} |f_{2}(t)|^{2} dt = \lim_{N \to \infty} \frac{1}{2\pi l} \sum_{n=1}^{N} a_{n} e^{i\lambda_{n}x} \frac{1}{2\pi} \sum_{n=1}^{N} a_{n} e^{i\lambda_{n}x} = J_{1} + J_{2}, \text{ say.}
\]
As for \( J_{2} \) : From \( \sum_{m \neq n} a_{m} a_{n} \leq \sum |a_{m}| \sum |a_{n}| = V_{2}^{2} \), we get
\[
\lim_{l \to \infty} \sup_{-\infty < x < \infty} |J_{2}| \leq \sum |a_{m}| \sum |a_{n}| \lim_{l \to \infty} \frac{|a_{m} a_{n}|}{i(\lambda_{m} - \lambda_{n})l} = 0.
\]
Therefore we get
\[
(3.4) \quad \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{2}(t)|^{2} dt = \frac{1}{2\pi} \sum |a_{n}|^{2}.
\]
The estimations (3.3) and (3.4) read
\[
(3.5) \quad \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} f_{1}(t) \overline{f_{2}(t)} dt \leq \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} \overline{f_{2}(t)} dt \leq \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} \overline{f_{2}(t)} dt \leq \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} \overline{f_{2}(t)} dt \leq 0.
\]
Thus we obtain the (3.2).

The following results is well known: if \( f(x) \) can be represented by the Fourier-Stieltjes transform (3.1), then we have
\[
(3.6) \quad \Lambda(f) = \{ \lambda \mid \sigma(\lambda + \epsilon) - \sigma(\lambda - \epsilon) > 0 \text{ for any } \epsilon > 0 \}
\]
(c.f. H. Pollard [5]). Now we shall prove the following theorem.

**Theorem 4.** If \( f(x) \) can be represented by the formula (3.1), then we get
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(3.7) \[ \Lambda_{*}(f) = \{ \lambda | \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \]

where the notation \( \{ \} \) means the closure of the set \{ \}. Proof of Theorem 4. The proof can be done by decomposing into three parts.

(i) Firstly we shall prove

(3.8) \[ \Lambda^{*}(f) \supseteq \{ \lambda | \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \]

We have

\[
g(x) = K * f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x-y) dy \int_{-\infty}^{\infty} e^{iuy} d\sigma(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\sigma(u) \int_{-\infty}^{\infty} K(x-y) e^{iuy} dy = \frac{1}{2\pi} \sum |k(\lambda)|^{2} |\sigma(\lambda+0) - \sigma(\lambda-0)|^{2}.
\]

Hence if \( g(x) = K * f \sim 0 \), we obtain

\[
\frac{1}{2\pi} \sum |k(\lambda)|^{2} |\sigma(\lambda+0) - \sigma(\lambda-0)|^{2} = 0
\]

and we can conclude that the condition \( \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \) implies \( k(\lambda) = 0 \). Thus we get the formula (i).

(ii) Secondly we shall prove

(3.9) \[ \Lambda_{*}(f) \supseteq \overline{\{ \lambda | \sigma(\lambda+0) - \sigma(\lambda-0) > 0 \}} \]

This is immediate by the fact \( \Lambda_{*}(f) \) to be closed and the formula (i).

(iii) Lastly we shall prove

(3.10) \[ \Lambda^{*}(f) = \{ \lambda | \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \]

If there is an \( \lambda_{0} \in \Lambda_{*}(f) - \{ \lambda | \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \), there exist a neighborhood of \( \lambda_{0}, (\lambda_{0} - \delta, \lambda_{0} + \delta) \) where the \( \sigma(u) \) is continuous. Let \( k_{\lambda_{0}}(u) \) be an ordinary triangular function on this interval. Let \( K_{\lambda_{0}}(x) \) be its Fourier transform. Let us put

\[
g_{\lambda_{0}}(x) = \int_{-\infty}^{\infty} K_{\lambda_{0}}(x-y) f(y) dy.
\]

Then we get
\[
\lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+t} |g_{\lambda}(t)|^2 dt = \sum |k_{\lambda}(\lambda)|^2 |\sigma(\lambda+0)-\sigma(\lambda-0)|^2 = 0.
\]

On the other hand we have \( k_{\lambda}(u) \neq 0 \) at \( u = \lambda \) by the definition of \( k_{\lambda}(u) \). Therefore we get \( \lambda_0 \in \Lambda_{\star}(f) \). This reads a contradiction. Thus we get the formula (iii).

**Theorem 5.** Let \( f(x) \) be represented by the formula (3.1). Let us assume that \( K(x) \) belong to the class \( L_{1} \) and its Fourier transform vanish on the set of \( \Lambda_{\star}(f) \). Then we get

(3.12) \[ K \ast f \sim 0. \]

Proof of Theorem 5. This is a corollary to the formula (3.9.)

4. **The almost periodic function in the sense of Weyl.** For almost periodic function there always exist

(4.1) \[ a(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t} dt. \]

Then we have

**Theorem 6.** Let \( f(x) \) be an almost periodic function in the sense of Weyl. Let us put

(4.2) \[ \Lambda_0(f) = \{ \lambda | a(\lambda) \neq 0 \}. \]

Then we have

(4.3) \[ \Lambda_0(f) = \Lambda_{Wy}(f). \]

Proof of Theorem 6. (i) Proof of \( \Lambda_0(f) \subseteq \Lambda_{Wy}(f) \). We have by the Wiener formula

\[
\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\lambda-\epsilon}^{\lambda+\epsilon} \{ s(u+\epsilon)-s(u-\epsilon) \} du = \sqrt{2\pi} a(\lambda).
\]

Therefore for any positive number \( \delta \) we get

\[ |a(\lambda)| \leq \lim_{\epsilon \to 0} \sup_{-\infty < x < \infty} \left( \frac{1}{2\epsilon} \int_{\lambda-\epsilon}^{\lambda+\epsilon} |s(u+\epsilon,x)-s(u-\epsilon,x)|^2 du \right)^{\frac{1}{2}}. \]

Thus we attain \( \Lambda_0(f) \subseteq \Lambda_{Wy}(f) \). Since \( \Lambda_{Wy}(f) \) is a closed set we conclude \( \Lambda_0(f) \subseteq \Lambda_{Wy}(f) \).

(ii) Proof of \( \Lambda_{Wy}(f) \subseteq \Lambda_0(f) \). We shall prove that if \( \lambda \notin \Lambda_0(f) \) then \( \lambda \notin \Lambda_{Wy}(f) \) too. From \( \lambda \in \Lambda_0(f) \) there exists an neighbourhood of \( \lambda \), \( (\lambda-\delta,\lambda+\delta) \) which does not contain any element of \( \Lambda_0(f) \). On the other hand since \( f \) is an almost periodic, there exist an trigonometric polynomial
such that

\[ p_N(x) = \sum_{1}^{N} a_n e^{i\lambda_n x} \quad \lambda_n \in \Lambda_0(f) \quad (n = 1, 2, \ldots, N) \]

Therefore by the one-sided Wiener formula we get

\[ \lim_{\epsilon \to 0} \sup_{-\infty < x < \infty} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} \left| s(u + \epsilon, x) - s(u - \epsilon, x) \right|^z du \to 0 \quad (N \to \infty) \]

where we mean by \( s_N(u) \) the Fourier-Wiener transform of \( p_N(x) \). Since \( s_N(u) \) do not contain only elements of \( \Lambda_0(f) \) as a spectre we get

\[ s_N(u + \epsilon, x) - s_N(u - \epsilon, x) = \sum_{1}^{N} a_n e^{i\lambda_n x} \sqrt{2\pi} \chi_n(u), \]

where \( \chi_n(u) \) is an characteristic function of an interval \([\lambda_n - \epsilon, \lambda_n + \epsilon]\). Thus we can conclude

\[ s_N(u + \epsilon, x) - s_N(u - \epsilon, x) = 0 \quad \text{on} \quad (\lambda - \delta, \lambda + \delta). \]

Hence we have \( \lambda \notin \Lambda_{Wy}(f) \). Thus we have completed the Proof of Theorem 6.

**Theorem 7.** Let \( f(x) \) be a bounded almost periodic function in the sense of Weyl. Let \( K(x) \) be an integrable function and its Fourier transform vanish on the set of \( \Lambda_*(f) \). Then we have

\[ K*f \sim 0. \]

**Proof of Theorem 7.** By the hypotheses we get

\[ \int_{-\infty}^{\infty} K(y) p_N(x-y) dy = \sum_{1}^{N} a_n \int_{-\infty}^{\infty} K(y) e^{i\lambda_n (x-y)} dy = 0. \]

Thus we get

\[ g(x) = \int_{-\infty}^{\infty} K(y) f(x-y) dy = \int_{-\infty}^{\infty} K(y) \{ f(x-y) - p_N(x-y) \} dy \]

and

\[ \frac{1}{l} \int_{u}^{u+l} |g(x)|^2 dx \leq \left| \int_{-\infty}^{\infty} K(y) dy \right|^2 \sup_{-\infty < u < \infty} \frac{1}{l} \int_{u}^{u+l} |f(y) - p_N(y)|^2 dy. \]
Therefore we get by (4.5)

\[ \lim_{l \to \infty} \sup_{-\infty < u < \infty} \frac{1}{l} \int_{u}^{u+l} |g(x)|^2 dx = 0. \]

References.


Department of Mathematics,
Hokkaido University

(Received March 20, 1963)