ON THE SPECTRUM OF FUNCTION IN THE WEYL SPACE

By

Sumiyuki KOIZUMI

1. Introduction. Let f be a bounded measurable function defined on the real line. By $\Lambda(f)$ we mean the set of real number λ which satisfies the following property. For any integrable function $K(x)$ the condition

$$K*f=\int_{-\infty}^{\infty}K(x-y)f(y)dy=0 \qquad (-\infty<x<\infty) \tag{1.1}$$

implies

$$\int_{-\infty}^{\infty}e^{i\lambda y}K(y)dy=0. \tag{1.2}$$

(c. f. A. Beurling [1].

In the previous paper we introduced the analogous definitions. By $\Lambda_*(f)$ we mean the set of real number λ which satisfies the following property. For any integrable function $K(x)$ the condition

$$K*f=\int_{-\infty}^{\infty}K(x-y)f(y)dy\sim 0 \tag{1.3}$$

implies

$$\int_{-\infty}^{\infty}e^{i\lambda y}K(y)dy=0 \tag{1.4}$$

where the notation $K*f\sim 0$ means

$$\overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\frac{1}{l}\int_{-\infty}^{\infty}|K*f(x)|^2dx=0. \tag{1.5}$$

(c. f. S. Koizumi [4]. It is clear that

$$\Lambda_*(f)\subseteqq\Lambda(f). \tag{1.6}$$

The purpose of this paper is to investigate properties of the above defined set as for functions represented by the Fourier-Stieltjes transform and almost periodic functions in the sense of Weyl.

2. General property of $\Lambda_*(f)$.

Theorem 1. *The set $\Lambda_*(f)$ is closed.*

This is a trivial result.

Theorem 2. *Let f be a bounded measurable function and $f\sim 0$. Then we have $\Lambda_*(f)=\emptyset$.*

Proof of Theorem 2. Using the same notation in our previous paper (S. Koizumi [4]) we get

$$\Lambda_*(f)=\Lambda_{Wy}(f) \tag{2.1}$$

From the assumption $f\sim 0$ we get

$$\overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon,x)-s(u-\varepsilon,x)|^2du=0 \tag{2.2}$$

where $s(u)$ is the Fourier-Wiener transform of $f(x)$. Therefore we get $\Lambda_{Wy}(f)=\emptyset$ and thus we get $\Lambda_*(f)=\emptyset$.

3. Fourier-Stieltjes transform.

Theorem 3. *Let f be represented by the Fourier-Stieltjes transform. That is*

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}d\sigma(u) \tag{3.1}$$

where $\sigma(u)$ is a function of bounded variation on the real line. Then we have

$$\overline{\lim_{l\to\infty}}\ \sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f(t)|^2dt=\frac{1}{2\pi}\sum|\sigma(\lambda+0)-\sigma(\lambda-0)|^2. \tag{3.2}$$

proof of Theorem 3. Let $\sigma_1(u)+\sigma_2(u)$ be the Lebesgue decomposition of $\sigma(u)$. The $\sigma_1(u)$ is its continuous part and the $\sigma_2(u)$ is its saltus part. Let $f_1(x)$ be the Fourier-Stieltjes transform of $\sigma_i(u)$ $(i=1,2)$ respectively. Then we have

$$\frac{1}{l}\int_{x}^{x+l}|f(t)|^2dt=\frac{1}{l}\int_{x}^{x+l}|f_1(t)|^2dt+2\Re\left(\frac{1}{l}\int_{x}^{x+l}f_1(t)\overline{f_2(t)}dt\right)$$
$$+\frac{1}{l}\int_{x}^{x+l}|f_2(t)|^2dt.$$

Firstly we shaly estimate the part of $f_1(t)$. We have

$$\frac{1}{l}\int_{x}^{x+l}|f_1(t)|^2dt=\frac{1}{2\pi l}\int_{x}^{x+l}dt\int_{-\infty}^{\infty}e^{iux}d\sigma_1(u)\int_{-\infty}^{\infty}e^{-ivx}\overline{d\sigma_1(v)}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1(u)\int_{-\infty}^{\infty}\overline{d\sigma_1(v)}\frac{e^{i(u-v)l}-1}{i(u-v)l}e^{i(u-v)x}$$

Here if we put $\sigma_*^1(u)=\int_{-\infty}^{u}|d\sigma_1(v)|$, then the $\sigma_1^*(u)$ is a continuous, nondecreasing and bounded function and we get

$$\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{-\infty}^{\infty}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{|u-v|\geqq\delta}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{|u-v|\geqq\delta}d\sigma_1^*(v)\left(\frac{\sin(u-v)l/2}{(u-v)l/2}\right)^2=I_1+I_2,\text{ say.}$$

The number δ will be determined in later. As for the I_2: for any given positive number η there exist a number N such that

$$|\sigma_1^*(\pm\infty)-\sigma_1^*(\pm N)|\leqq\eta/4V_1,$$

where $V_1=\sigma_1^*(\infty)=\int_{-\infty}^{\infty}|d\sigma(v)|$. Then if we fixe the number N, there exist a number δ such that

$$\sup_{-N\leqq u\leqq N}\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta)<\eta/4V_1.$$

Hence we get

$$I_2\leqq\frac{1}{2\pi}\int_{-\infty}^{\infty}d\sigma_1^*(u)\int_{u-\delta}^{u+\delta}d\sigma_1^*(v)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta))d\sigma_1^*(u)$$

$$=\frac{1}{2\pi}\left(\int_{-\infty}^{-N}+\int_N^{\infty}\right)(\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta))d\sigma_1^*(u)+\frac{1}{2\pi}\int_{-N}^{N}(\sigma_1^*(u+\delta)$$

$$-\sigma_1^*(u-\delta)d\sigma_1^*(u)\leqq\frac{1}{2\pi}2V_1[\{\sigma_1^*(-N)-\sigma_1^*(-\infty)\}+\{\sigma_1^*(\infty)$$

$$-\sigma_1^*(N)\}]+\frac{1}{2\pi}\int_{-N}^{N}\sup_{-N\leqq u\leqq N}\{\sigma_1^*(u+\delta)-\sigma_1^*(u-\delta)\}\,d\sigma_1^*(u)$$

$$\leqq(1/2\pi)(\eta/2+\eta/2)+(1/2\pi)(\eta/4)=5\eta/8\pi.$$

As for I_1: the condition $|u-v|\geqq\delta$ imply

$$\left(\frac{\sin(u-v)l/2}{(u-v)/2}\right)^2\to0\qquad(\text{as }l\to\infty).$$

Therefore we get

$$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f_1(t)|^2dt\leqq5\eta/8\pi.$$

and the number η is an arbitrarily small. Thus we get

$$\text{(3. 3)} \qquad \overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f_1(t)|^2 dt = 0.$$

Secondly we shall estimate the part of $f_2(t)$. The $\sigma_2(u)$ has enumerable number of jump point (λ_n) $(n=1, 2\cdots)$. Here if we put

$$a_n = \sigma(\lambda_n+0) - \sigma(\lambda_n-0)$$

and

$$\sum |a_n| = \int_{-\infty}^{\infty} d\sigma_2(u) = V_2.$$

Then we have

$$f_2(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iux} d\sigma_2(u) = \frac{1}{\sqrt{2\pi}} \sum a_n e^{i\lambda_n x}.$$

Therefore we get

$$\frac{1}{l}\int_x^{x+l} |f_2(t)|^2 dt = \lim_{N\to\infty} \frac{1}{2\pi l}\int_x^{x+l} \sum_{-N}^{N} a_n e^{i\lambda_n t} \overline{\sum_{-N}^{N} a_n e^{i\lambda_n t}}\, dt$$

$$= \frac{1}{2\pi}\sum_{-\infty}^{\infty} |a_n|^2 + \frac{1}{2\pi}\sum_{m\neq n} a_m \bar{a}_n e^{i(\lambda_m-\lambda_n)x} \frac{e^{i(\lambda_m-\lambda_n)l}-1}{i(\lambda_m-\lambda_n)l} = J_1 + J_2, \text{ say.}$$

As for J_2: From $|\sum_{m\neq n} a_m \bar{a}_n| \leqq \sum |a_m| \sum |a_n| = V_2^2$, we get

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} |J_2| \leqq \sum_{m,n} |a_m||\bar{a}_n| \overline{\lim_{l\to\infty}} \frac{e^{i(\lambda_m-\lambda_n)l}-1}{i(\lambda_m-\lambda_n)l} = 0.$$

Therefore we get

$$\text{(3. 4)} \qquad \overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f_2(t)|^2 dt = \frac{1}{2\pi}\sum |a_n|^2.$$

The estimations (3. 3) and (3. 4) read

$$\text{(3. 5)} \qquad \overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \left| \frac{1}{l}\int_x^{x+l} f_1(t)\overline{f_2(t)}\, dt \right|$$

$$\leqq \overline{\lim_{l\to\infty}} \sup_{-\infty<x<\infty} \left(\frac{1}{l}\int_x^{x+l} |f_1(t)|^2 dt\right)^{\frac{1}{2}} \left(\frac{1}{l}\int_x^{x+l} |f_2(t)|^2 dt\right)^{\frac{1}{2}} = 0.$$

Thus we obtain the (3. 2).

The following results is well known: if $f(x)$ can be represented by the Fourier-Stieltjes transform (3. 1), then we have

$$\text{(3. 6)} \qquad \Lambda(f) = \{\lambda \mid \sigma(\lambda+\varepsilon) - \sigma(\lambda-\varepsilon) > 0 \text{ for any } \varepsilon > 0\}$$

(c. f. H. Pollard [5]). Now we shall prove the following theorem.

Theorem 4. *If $f(x)$ can be represented by the formula* (3. 1), *then we get*

(3. 7) $$\Lambda_*(f)=\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$$

where the notation $\overline{\{\ \}}$ *means the closure of the set* $\{\ \}$.

Proof of Theorem 4. The proof can be done by decomposing into three parts.

(i) Firstly we shall prove

(3. 8) $$\Lambda^*(f)\supseteqq\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}.$$

We have

$$g(x)=K*f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K(x-y)\,dy\int_{-\infty}^{\infty}e^{iuy}d\sigma(u)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\sigma(u)\int_{-\infty}^{\infty}K(x-y)e^{iuy}dy=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{iux}k(u)\,d\sigma(u),$$

where the $k(u)$ is the Fourier transform of $K(x)$ and a bounded continuous function on the real line. Repeating the same arguments as the proof of Theorem 3 we get

(3. 9) $$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|g(t)|^2dt=\frac{1}{2\pi}\sum|k(\lambda)|^2\,|\sigma(\lambda+0)-\sigma(\lambda-0)|^2.$$

Hence if $g(x)=K*f\sim 0$, we obtain

$$\frac{1}{2\pi}\sum|k(\lambda)|^2\,|\sigma(\lambda+0)-\sigma(\lambda-0)|^2=0$$

and we can conclude that the condition $\sigma(\lambda+0)-\sigma(\lambda-0)>0$ implies $k(\lambda)=0$. Thus we get the formular (i).

(ii) Secondly we shall prove

(3. 10) $$\Lambda_*(f)\supseteqq\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0)\}}$$

This is immediate by the fact $\Lambda_*(f)$ to be closed and the formula (i).

(iii) Lastly we shall prove

(3. 11) $$\Lambda_*(f)=\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$$

If there is an $\lambda_0\in\Lambda_*(f)-\overline{\{\lambda\,|\,\sigma(\lambda+0)-\sigma(\lambda-0)>0\}}$, there exist an nighbourhood of λ_0, $(\lambda_0-\delta,\lambda_0+\delta)$ where the $\sigma(u)$ is continuous. Let $k_{\lambda_0}(u)$ be an ordinary triangular function on this interval. Let $K_{\lambda_0}(x)$ be its Fourier transform. Let us put

$$g_{\lambda_0}(x)=\int_{-\infty}^{\infty}K_{\lambda_0}(x-y)f(y)\,dy.$$

Then we get

$$\overline{\lim_{l\to\infty}}\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|g_{\lambda_0}(t)|^2dt=\sum|k_{\lambda_0}(\lambda)|^2|\sigma(\lambda+0)-\sigma(\lambda-0)|^2=0.$$

On the other hand we have $k_{\lambda_0}(u)\neq 0$ at $u=\lambda_0$ by the definition of $k_{\lambda_0}(u)$. Therefore we get $\lambda_0 \overline{\in} \Lambda_*(f)$. This reads a contradiction. Thus we get the formula (iii).

Theorem 5. *Let $f(x)$ be represented by the formula* (3.1). *Let us assume that $K(x)$ belong to the class L_1 and its Fourier transform vanish on the set of $\Lambda_*(f)$. Then we get*

$$K*f\sim 0. \tag{3.12}$$

Proof of Theorem 5. This is a corollary to the formula (3.9.)

4. The almost periodic function in the sense of Weyl. For almost perioic function there always exist

$$a(\lambda)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)e^{-i\lambda t}dt. \tag{4.1}$$

Then we have

Theorem 6. *Let $f(x)$ be an almost periodic function in the sense of Weyl. Let us put*

$$\Lambda_0(f)=\{\lambda\,|\,a(\lambda)\neq 0\}. \tag{4.2}$$

Then we have

$$\overline{\Lambda_0(f)}=\Lambda_{Wy}(f). \tag{4.3}$$

Proof of Theorem 6. (i) Proof of $\overline{\Lambda_0(f)}\subseteqq\Lambda_{Wy}(f)$. We have by the Wiener formula

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon)-s(u-\varepsilon)\}\,du=\sqrt{2\pi}\,a(\lambda).$$

Therefore for any positive number δ we get

$$|a(\lambda)|\leqq\overline{\lim_{\varepsilon\to 0}}\sup_{-\infty<x<\infty}\left(\frac{1}{2\varepsilon}\int_{\lambda-\delta}^{\lambda+\delta}|s(u+\varepsilon,x)-s(u-\varepsilon,x)|^2du\right)^{\frac{1}{2}}.$$

Thus we attain $\Lambda_0(f)\subseteqq\Lambda_{Wy}(f)$. Since $\Lambda_{Wy}(f)$ is a closed set we conclude $\overline{\Lambda_0(f)}\subseteqq\Lambda_{Wy}(f)$.

(ii) Proof of $\Lambda_{Wy}(f)\subseteqq\overline{\Lambda_0(f)}$. We shall prove that if $\lambda\overline{\in}\overline{\Lambda_0(f)}$ then $\lambda\notin\Lambda_{Wy}(f)$ too. From $\lambda\overline{\in}\overline{\Lambda_0(f)}$ there exists an neighbourhood of λ, $(\lambda-\delta,\lambda+\delta)$ which doen not contain any element of $\Lambda_0(f)$. On the other hand since f is an almost periodic, there exist an trigonometric polynomial

(4. 4) $$p_N(x)=\sum_1^N a_n e^{i\lambda_n x} \qquad \lambda_n \in \Lambda_0(f)\ (n=1,2,\cdots,N)$$

such that

(4. 5) $$\overline{\lim_{T\to\infty}} \sup_{-\infty<x<\infty} \frac{1}{2T}\int_{-T}^{T} |f(t+x)-p_N(t+x)|^2 dt \to 0 \qquad (N\to\infty.)$$

Therefore by the one-sided Wiener formula we get

$$\overline{\lim_{\varepsilon\to 0}} \sup_{-\infty<x<\infty} \frac{1}{2\varepsilon}\int_{-\infty}^{\infty} |\{s(u+\varepsilon,x)-s(u-\varepsilon,x)\} - \{s_N(u+\varepsilon,x) - s_N(u-\varepsilon,x)\}|^2 du \to 0\ (N\to\infty)$$

where we mean by $s_N(u)$ the Fourier-Wiener transform of $p_N(x)$. Since $s_N(u$ doen contain only elements of $\Lambda_0(f)$ as an spectre we get

$$s_N(u+\varepsilon,x)-s_N(u-\varepsilon,x)=\sum_1^N a_n e^{i\lambda_n x}\sqrt{2\pi}\,\chi_n(u),$$

where $\chi_n(u)$ is an characteristic function of an interval $[\lambda_n-\varepsilon, \lambda_n+\varepsilon]$. Thus we can conclude

$$s_N(u+\varepsilon,x)-s_N(u-\varepsilon,x)=0 \qquad \text{on } (\lambda-\delta, \lambda+\delta).$$

Therefore we get

$$\overline{\lim_{\varepsilon\to 0}} \sup_{-\infty<x<\infty} \frac{1}{2\pi}\int_{\lambda-\delta}^{\lambda+\delta} |s(u+\varepsilon,x)-s(u-\varepsilon,x|^2 du=0.$$

Hence we have $\lambda \notin \Lambda_{Wy}(f)$. Thus we have completed the Proof of Theorem 6.

Theorem 7. *Let $f(x)$ be a bounded almost periodic function in the sense of Weyl. Let $K(x)$ be an integrable function and its Fourier transform vanish on the set of $\Lambda_*(f)$. Then we have*

(4. 6) $$K*f\sim 0.$$

Proof of Theorem 7. By the hypotheses we get

$$\int_{-\infty}^{\infty} K(y)p_N(x-y)dy=\sum_1^N a_n\int_{-\infty}^{\infty} K(y)e^{i\lambda_n(x-y)}dy=0.$$

Thus we get

$$g(x)=\int_{-\infty}^{\infty}K(y)f(x-y)dy=\int_{-\infty}^{\infty}K(y)\{f(x-y)-p_N(x-y)\}dy$$

and

$$\frac{1}{l}\int_u^{u+l}|g(x)|^2dx \leqq \left|\int_{-\infty}^{\infty}K(y)dy\right|^2 \sup_{-\infty<u<\infty}\frac{1}{l}\int_u^{u+l}|f(y)-p_N(y)|^2dy.$$

Therefore we get by (4. 5)

$$\overline{\lim_{l\to\infty}} \sup_{-\infty<u<\infty} \frac{1}{l}\int_u^{u+l} |g(x)|^2 dx = 0.$$

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Department of Mathematics,
Hokkaido University

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