<table>
<thead>
<tr>
<th>Title</th>
<th>ON THE SPECTRUM OF FUNCTION IN THE WEYL SPACE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Koizumi, Sumiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 17(3-4): 065-072</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1963</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56041">http://hdl.handle.net/2115/56041</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSHIU_17_N3-4_065-072.pdf</td>
</tr>
</tbody>
</table>

**Summary:**

The document discusses the spectrum of function in the Weyl space, which is a significant topic in the field of functional analysis. It is published in the Journal of the Faculty of Science, Hokkaido University, and the citation includes the volume and page numbers. The file associated with the document is available for download.
ON THE SPECTRUM OF FUNCTION IN THE WEYL SPACE

By

Sumiyuki KOIZUMI

1. Introduction. Let $f$ be a bounded measurable function defined on the real line. By $\Lambda(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

$$(1.1) \quad K*f = \int_{-\infty}^{\infty} K(x-y)f(y)dy = 0 \quad (-\infty < x < \infty)$$

implies

$$(1.2) \quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0.$$

(c.f. A. Beurling [1].)

In the previous paper we introduced the analogous definitions. By $\Lambda_{*}(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

$$(1.3) \quad K*f = \int_{-\infty}^{\infty} K(x-y)f(y)dy \sim 0$$

implies

$$(1.4) \quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0$$

where the notation $K*f \sim 0$ means

$$(1.5) \quad \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{-\infty}^{\infty} |K*f(x)|^2 dx = 0.$$

(c.f. S. Koizumi [4]. It is clear that

$$(1.6) \quad \Lambda_{*}(f) \subseteq \Lambda(f).$$

The purpose of this paper is to investigate properties of the above defined set as for functions represented by the Fourier-Stieltjes transform and almost periodic functions in the sense of Weyl.

2. General property of $\Lambda_{*}(f)$.

Theorem 1. The set $\Lambda_{*}(f)$ is closed.

This is a trivial result.
Theorem 2. Let $f$ be a bounded measurable function and $f \sim 0$. Then we have $A_*(f) = 0$.

Proof of Theorem 2. Using the same notation in our previous paper (S. Koizumi [4]) we get

\[(2.1) \quad A_*(f) = A_{Wy}(f)\]

From the assumption $f \sim 0$ we get

\[(2.2) \quad \lim_{l \to \infty} \sup_{-\infty < \varepsilon < \infty} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |s(u + \varepsilon, x) - s(u - \varepsilon, x)|^2 du = 0\]

where $s(u)$ is the Fourier-Wiener transform of $f(x)$. Therefore we get $A_{Wy}(f) = 0$ and thus we get $A_*(f) = 0$.

3. Fourier-Stieltjes transform.

Theorem 3. Let $f$ be represented by the Fourier-Stieltjes transform. That is

\[(3.1) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma(u)\]

where $\sigma(u)$ is a function of bounded variation on the real line. Then we have

\[(3.2) \quad \lim_{l \to \infty} \sup_{-\infty < \varepsilon < \infty} \frac{1}{l} \int_{x}^{x+l} |f(t)|^2 dt = \frac{1}{2\pi} \sum |\sigma(\lambda+0) - \sigma(\lambda-0)|^2.\]

Proof of Theorem 3. Let $\sigma_1(u) + \sigma_2(u)$ be the Lebesgue decomposition of $\sigma(u)$. The $\sigma_1(u)$ is its continuous part and the $\sigma_2(u)$ is its saltus part. Let $f_i(x)$ be the Fourier-Stieltjes transform of $\sigma_i(u) (i=1, 2)$ respectively. Then we have

\[
\frac{1}{l} \int_{x}^{x+l} |f_i(t)|^2 dt = \frac{1}{2\pi l} \int_{x}^{x+l} dt \int_{-\infty}^{\infty} e^{iux} d\sigma_1(u) \int_{-\infty}^{\infty} e^{-ivx} d\overline{\sigma_1(v)}
\]

Here if we put $\sigma^*_i(u) = \int_{-\infty}^{u} |d\sigma_i(v)|$, then the $\sigma^*_i(u)$ is a continuous, nondecreas-
ing and bounded function and we get

$$\sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{-\infty}^{\infty} \sigma_{1}^{*}(v) \left( \frac{\sin(u-v)l/2}{(u-v)l/2} \right)^{2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{|u-v| \geq \delta} \sigma_{1}^{*}(v) \left( \frac{\sin(u-v)l/2}{(u-v)l/2} \right)^{2} = I_{1} + I_{2}, \text{ say.}$$

The number $\delta$ will be determined in later. As for the $I_{2}$: for any given positive number $\eta$ there exist a number $N$ such that

$$|\sigma_{1}^{*}(\pm\infty) - \sigma_{1}^{*}(\pm N)| \leq \frac{\eta}{4V_{1}},$$

where $V_{1} = \sigma_{1}^{*}(\infty) = \int_{-\infty}^{\infty} |d\sigma(v)|$. Then if we fixe the number $N$, there exist a number $\delta$ such that

$$\sup_{-N \leq u \leq N} \sigma_{1}^{*}(u) - \sigma_{1}^{*}(u-\delta) < \frac{\eta}{4V_{1}}.$$

Hence we get

$$I_{2} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_{1}^{*}(u) \int_{u-\delta}^{u+\delta} \sigma_{1}^{*}(v)$$

$$= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} (\sigma_{1}^{*}(u+\delta) - \sigma_{1}^{*}(u-\delta))d\sigma_{1}^{*}(u) \right)$$

$$= \frac{1}{2\pi} \left( \int_{-\infty}^{-N} + \int_{N}^{\infty} \right) (\sigma_{1}^{*}(u+\delta) - \sigma_{1}^{*}(u-\delta))d\sigma_{1}^{*}(u) + \frac{1}{2\pi} \int_{-N}^{N} (\sigma_{1}^{*}(u+\delta) - \sigma_{1}^{*}(u-\delta))d\sigma_{1}^{*}(u)$$

$$\leq \frac{1}{2\pi} 2V_{1} \left[ \{\sigma_{1}^{*}(-N) - \sigma_{1}^{*}(-\infty)\} + \{\sigma_{1}^{*}(\infty) - \sigma_{1}^{*}(N)\} \right] + \frac{1}{2\pi} \int_{-N}^{N} \sup_{-N \leq u \leq N} (\sigma_{1}^{*}(u+\delta) - \sigma_{1}^{*}(u-\delta))d\sigma_{1}^{*}(u)$$

$$\leq (1/2\pi)(\eta/2 + \eta/2) + (1/2\pi)(\eta/4) = 5\eta/8\pi.$$

As for $I_{1}$: the condition $|u-v| \geq \delta$ imply

$$\left( \frac{\sin(u-v)l/2}{(u-v)l/2} \right)^{2} \rightarrow 0 \quad \text{as} \; l \rightarrow \infty.$$

Therefore we get

$$\lim_{l \rightarrow \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} dt \leq 5\eta/8\pi.$$

and the number $\eta$ is an arbitrarily small. Thus we get
(3.3) \[ \lim_{l \to \infty} \sup_{-\infty < z < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} \, dt = 0. \]

Secondly we shall estimate the part of \( f_{2}(t) \). The \( \sigma_{2}(u) \) has enumerable number of jump point \( (\lambda_{n}) \) \( (n=1, 2\cdots) \). Here if we put
\[ a_{n} = \sigma(\lambda_{n}+0) - \sigma(\lambda_{n}-0) \]
and
\[ \sum |a_{n}| = \int_{-\infty}^{\infty} d\sigma_{2}(u) = V_{2}. \]
Then we have
\[ f_{2}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma_{2}(u) = \frac{1}{\sqrt{2\pi}} \sum_{n} a_{n} e^{i\lambda_{n}x}. \]
Therefore we get
\[ \frac{1}{l} \int_{x}^{x+l} |f_{2}(t)|^{2} \, dt = \lim_{N \to \infty} \frac{1}{N} \int_{x}^{x+N} \sum_{n} a_{n} e^{i\lambda_{n}t} \, dt \]
\[ = \frac{1}{2\pi} \sum_{n} |a_{n}|^{2} + \frac{1}{2\pi} \sum_{m \neq n} \frac{a_{m}a_{n}e^{i(\lambda_{m}-\lambda_{n})x}}{i(\lambda_{m}-\lambda_{n})} \frac{e^{i(\lambda_{m}-\lambda_{n})l} - 1}{l} = J_{1} + J_{2}, \text{ say.} \]

As for \( J_{2} \): From \( \sum_{m \neq n} |a_{m}a_{n}| \leq \sum |a_{m}| \sum |a_{n}| = V_{2}^{2} \), we get
\[ \lim_{l \to \infty} \sup_{-\infty < z < \infty} |J_{2}| \leq \sum |a_{m}| \sum |a_{n}| \lim_{l \to \infty} \frac{e^{i(\lambda_{m}-\lambda_{n})l} - 1}{i(\lambda_{m}-\lambda_{n})l} = 0. \]
Therefore we get
(3.4) \[ \lim_{l \to \infty} \sup_{-\infty < z < \infty} \frac{1}{l} \int_{x}^{x+l} |f_{2}(t)|^{2} \, dt = \frac{1}{2\pi} \sum |a_{n}|^{2}. \]
The estimations (3.3) and (3.4) read
(3.5) \[ \lim_{l \to \infty} \sup_{-\infty < z < \infty} \left| \frac{1}{l} \int_{x}^{x+l} f_{1}(t) \overline{f_{2}(t)} \, dt \right| \]
\[ \leq \lim_{l \to \infty} \sup_{-\infty < z < \infty} \left( \frac{1}{l} \int_{x}^{x+l} |f_{1}(t)|^{2} \, dt \right)^{\frac{1}{2}} \left( \frac{1}{l} \int_{x}^{x+l} |f_{2}(t)|^{2} \, dt \right)^{\frac{1}{2}} = 0. \]
Thus we obtain the (3.2).

The following results is well known: if \( f(x) \) can be represented by the Fourier-Stieltjes transform (3.1), then we have
(3.6) \[ \Lambda(f) = \{ \lambda | \sigma(\lambda + \epsilon) - \sigma(\lambda - \epsilon) > 0 \text{ for any } \epsilon > 0 \} \]
(c.f. H. Pollard [5]). Now we shall prove the following theorem.

Theorem 4. If \( f(x) \) can be represented by the formula (3.1), then we get
(3. 7) \[ A_*(f) = \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \]

where the notation \( \{ \} \) means the closure of the set \( \{ \} \).

Proof of Theorem 4. The proof can be done by decomposing into three parts.

(i) Firstly we shall prove

\[ A^*(f) \supseteq \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \}. \]

We have

\[
g(x) = K * f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x-y) dy \int_{-\infty}^{\infty} e^{iuy} d\sigma(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\sigma(u) \int_{-\infty}^{\infty} K(x-y) e^{iuy} dy =_{2\pi} \frac{1}{\Gamma} \int_{-\infty}^{\infty} e^{iux} k(u) d\sigma(u),
\]

where the \( k(u) \) is the Fourier transform of \( K(x) \) and a bounded continuous function on the real line. Repeating the same arguments as the proof of Theorem 3 we get

\[ \limsup_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |g(t)|^z dt = \frac{1}{2\pi} \sum |k(\lambda)|^z |\sigma(\lambda + 0) - \sigma(\lambda - 0)|^2. \]

Hence if \( g(x) = K * f \to 0 \), we obtain

\[
\frac{1}{2\pi} \sum |k(\lambda)|^z |\sigma(\lambda + 0) - \sigma(\lambda - 0)|^2 = 0
\]

and we can conclude that the condition \( \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \) implies \( k(\lambda) = 0 \). Thus we get the formula (i).

(ii) Secondly we shall prove

\[ A_*(f) \equiv \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \}. \]

This is immediate by the fact \( A_*(f) \) to be closed and the formula (i).

(iii) Lastly we shall prove

\[ A_*(f) = \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \}\]

If there is an \( \lambda_0 \in A_*(f) \to \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \), there exist a nighbourhood of \( \lambda_0, (\lambda_0 - \delta, \lambda_0 + \delta) \) where the \( \sigma(u) \) is continuous. Let \( k_{\lambda_0}(u) \) be an ordinary triangular function on this interval. Let \( K_{\lambda_0}(x) \) be its Fourier transform. Let us put

\[
g_{\lambda_0}(x) = \int_{-\infty}^{\infty} K_{\lambda_0}(x-y)f(y) dy.
\]

Then we get
\[
\lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |g_{\lambda}(t)|^{2} dt = \sum |k_{\lambda}(\lambda)|^{2} |\sigma(\lambda+0) - \sigma(\lambda-0)|^{2} = 0.
\]

On the other hand we have \( k_{\lambda}(u) \neq 0 \) at \( u = \lambda \) by the definition of \( k_{\lambda}(u) \). Therefore we get \( \lambda \in \Lambda_{\ast}(f) \). This reads a contradiction. Thus we get the formula (iii).

**Theorem 5.** Let \( f(x) \) be represented by the formula (3.1). Let us assume that \( K(x) \) belong to the class \( L_{1} \) and its Fourier transform vanish on the set of \( \Lambda_{\ast}(f) \). Then we get

\[
(3.12)
K*f \sim 0.
\]

**Proof of Theorem 5.** This is a corollary to the formula (3.9.)

4. The almost periodic function in the sense of Weyl. For almost perioic function there always exist

\[
(4.1)
\quad a(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t} dt.
\]

Then we have

**Theorem 6.** Let \( f(x) \) be an almost periodic function in the sense of Weyl. Let us put

\[
(4.2)
\Lambda_{0}(f) = \{ \lambda | a(\lambda) \neq 0 \}.
\]

Then we have

\[
(4.3)
\overline{\Lambda_{0}(f)} \subseteqq \Lambda_{Wy}(f).
\]

**Proof of Theorem 6.** (i) Proof of \( \overline{\Lambda_{0}(f)} \subseteqq \Lambda_{Wy}(f) \). We have by the Wiener formula

\[
\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \{ s(u+\epsilon) - s(u-\epsilon) \} du = \sqrt{2\pi} a(\lambda).
\]

Therefore for any positive number \( \delta \) we get

\[
|a(\lambda)| \leq \lim_{\epsilon \to 0} \sup_{-\infty < \epsilon < \infty} \left( \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |s(u+\epsilon,x) - s(u-\epsilon,x)|^{2} du \right)^{\frac{1}{2}}.
\]

Thus we attain \( \Lambda_{0}(f) \subseteqq \Lambda_{Wy}(f) \). Since \( \Lambda_{Wy}(f) \) is a closed set we conclude \( \Lambda_{0}(f) \subseteqq \Lambda_{Wy}(f) \).

(ii) Proof of \( \Lambda_{Wy}(f) \subseteqq \overline{\Lambda_{0}(f)} \). We shall prove that if \( \lambda \in \overline{\Lambda_{0}(f)} \) then \( \lambda \notin \Lambda_{Wy}(f) \). For if \( \lambda \notin \overline{\Lambda_{0}(f)} \) there exists an neighbouroughd of \( \lambda, (\lambda-\delta, \lambda+\delta) \) which does not contain any element of \( \Lambda_{0}(f) \). On the other hand since \( f \) is an almost periodic, there exist an trigonometric polynomial
On the Spectrum of Function in the Weyl Space

\[ \psi_N(x) = \sum_{i=1}^{N} a_ne^{i\lambda_n x} \quad \lambda_n \in \Lambda_0(f) (n = 1, 2, \ldots, N) \]
such that

\[ \lim_{T \to \infty} \sup_{-\infty < x < \infty} \frac{1}{2T} \int_{-T}^{T} |f(t + x) - \psi_N(t + x)|^2 \, dt \to 0 \quad (N \to \infty). \]

Therefore by the one-sided Wiener formula we get

\[ \lim_{\epsilon \to 0} \sup_{-\infty < x < \infty} \frac{1}{2\epsilon} \int_{-\infty}^{\infty} |s(u + \epsilon, x) - s(u - \epsilon, x) - s_N(u + \epsilon, x) - s_N(u - \epsilon, x)|^2 \, du \to 0 \quad (N \to \infty) \]

where we mean by \( s_N(u) \) the Fourier-Wiener transform of \( \psi_N(x) \). Since \( s_N(u) \) do not contain only elements of \( \Lambda_0(f) \) as a spectre we get

\[ s_N(u + \epsilon, x) - s_N(u - \epsilon, x) = \sum_{i=1}^{N} a_ne^{i\lambda_n x} \sqrt{2\pi} \chi_n(u), \]

where \( \chi_n(u) \) is a characteristic function of an interval \([\lambda_n - \epsilon, \lambda_n + \epsilon]\). Thus we can conclude

\[ s_N(u + \epsilon, x) - s_N(u - \epsilon, x) = 0 \quad \text{on} \quad (\lambda - \delta, \lambda + \delta). \]

Therefore we get

\[ \lim_{\epsilon \to 0} \sup_{-\infty < x < \infty} \frac{1}{2\pi} \int_{\lambda - \delta}^{\lambda + \delta} |s(u + \epsilon, x) - s(u - \epsilon, x)|^2 \, du = 0. \]

Hence we have \( \lambda \notin \Lambda_{\text{Wy}}(f) \). Thus we have completed the Proof of Theorem 6.

**Theorem 7.** Let \( f(x) \) be a bounded almost periodic function in the sense of Weyl. Let \( K(x) \) be an integrable function and its Fourier transform vanish on the set of \( \Lambda_*(f) \). Then we have

\[ K * f \sim 0. \]

**Proof of Theorem 7.** By the hypotheses we get

\[ \int_{-\infty}^{\infty} K(y) \psi_N(x - y) \, dy = \sum_{i=1}^{N} a_n \int_{-\infty}^{\infty} K(y) e^{i\lambda_n (x - y)} \, dy = 0. \]

Thus we get

\[ g(x) = \int_{-\infty}^{\infty} K(y) f(x - y) \, dy = \int_{-\infty}^{\infty} K(y) \{ f(x - y) - \psi_N(x - y) \} \, dy \]

and

\[ \frac{1}{l} \int_{-l}^{l} |g(x)|^2 \, dx \leq \left| \int_{-\infty}^{\infty} K(y) \, dy \right|^2 \sup_{-\infty < y < \infty} \frac{1}{l} \int_{-l}^{l} |f(y) - \psi_N(y)|^2 \, dy. \]
Therefore we get by (4.5)

$$\lim_{l \to \infty} \sup_{-\infty < u < \infty} \frac{1}{l} \int_{u}^{u+l} |g(x)|^2 dx = 0.$$ 

References.


Department of Mathematics, Hokkaido University

(Received March 20, 1963)