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ON THE SPECTRUM OF FUNCTION IN
THE WEYL SPACE

By

Sumiyuki KOIZUMI

1. Introduction. Let $f$ be a bounded measurable function defined on
the real line. By $\Lambda(f)$ we mean the set of real number $\lambda$ which satisfies the
following property. For any integrable function $K(x)$ the condition

\[(1.1) \quad K \ast f = \int_{-\infty}^{\infty} K(x-y)f(y)dy = 0 \quad (-\infty < x < \infty)\]

implies

\[(1.2) \quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0.\]

(c.f. A. Beurling [1].)

In the previous paper we introduced the analogous definitions. By $\Lambda_*(f)$ we mean the set of real number $\lambda$ which satisfies the following property. For any integrable function $K(x)$ the condition

\[(1.3) \quad K \ast f = \int_{-\infty}^{\infty} K(x-y)f(y)dy \sim 0\]

implies

\[(1.4) \quad \int_{-\infty}^{\infty} e^{i\lambda y}K(y)dy = 0\]

where the notation $K \ast f \sim 0$ means

\[(1.5) \quad \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{-\infty}^{\infty} |K \ast f(x)|^2 dx = 0.\]

(c.f. S. Koizumi [4].) It is clear that

\[(1.6) \quad \Lambda_*(f) \subseteq \Lambda(f).\]

The purpose of this paper is to investigate properties of the above defined
set as for functions represented by the Fourier-Stieltjes transform and almost
periodic functions in the sense of Weyl.

2. General property of $\Lambda_*(f)$.

Theorem 1. The set $\Lambda_*(f)$ is closed.

This is a trivial result.
Theorem 2. Let $f$ be a bounded measurable function and $f \sim 0$. Then we have $\Lambda_*(f) = 0$.

Proof of Theorem 2. Using the same notation in our previous paper (S. Koizumi [4]) we get

\[(2.1) \quad \Lambda_*(f) = \Lambda_{Wy}(f)\]

From the assumption $f \sim 0$ we get

\[(2.2) \quad \lim_{l \to \infty} \sup_{-\infty < \varepsilon < \infty} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |s(u + \varepsilon, x) - s(u - \varepsilon, x)|^2 du = 0\]

where $s(u)$ is the Fourier-Wiener transform of $f(x)$. Therefore we get $\Lambda_{Wy}(f) = 0$ and thus we get $\Lambda_*(f) = 0$.

3. Fourier-Stieltjes transform.

Theorem 3. Let $f$ be represented by the Fourier-Stieltjes transform. That is

\[(3.1) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma(u)\]

where $\sigma(u)$ is a function of bounded variation on the real line. Then we have

\[(3.2) \quad \lim_{l \to \infty} \sup_{-\infty < \varepsilon < \infty} \frac{1}{l} \int_{x}^{x+l} |f(t)|^2 dt = \frac{1}{2\pi} \sum |\sigma(\lambda + 0) - \sigma(\lambda - 0)|^2.\]

proof of Theorem 3. Let $\sigma_1(u) + \sigma_2(u)$ be the Lebesgue decomposition of $\sigma(u)$. The $\sigma_1(u)$ is its continuous part and the $\sigma_2(u)$ is its saltus part. Let $f_i(x)$ be the Fourier-Stieltjes transform of $\sigma_i(u)$ ($i=1,2$) respectively. Then we have

\[\frac{1}{l} \int_{x}^{x+l} |f(t)|^2 dt = \frac{1}{2\pi l} \int_{x}^{x+l} dt \int_{-\infty}^{\infty} e^{iux} d\sigma_1(u) \int_{-\infty}^{\infty} e^{-ivx} d\overline{\sigma_1(v)}\]

Here if we put $\sigma_1^*(u) = \int_{-\infty}^{u} |d\sigma_1(v)|$, then the $\sigma_1^*(u)$ is a continuous, nondecreas-
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get

\[
\sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f(t)|^2 dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma_1^*(u) \int_{-\infty}^{\infty} d\sigma_1^*(v) \left( \frac{\sin(u-v)}{(u-v)} \right)^2 
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma_1^*(u) \int_{|u-v| \geq \delta} d\sigma_1^*(v) \left( \frac{\sin(u-v)}{(u-v)} \right)^2 = I_1 + I_2, \text{ say.}
\]

The number \( \delta \) will be determined in later. As for the \( I_z \): for any given positive number \( \eta \) there exist a number \( N \) such that

\[
|\sigma_1^*(\pm \infty) - \sigma_1^*(\pm N)| \leq \eta/4V_1,
\]

where \( V_1 = \sigma_1^*(\infty) = \int_{-\infty}^{\infty} |d\sigma(v)| \). Then if we fixe the number \( N \), there exist a number \( \delta \) such that

\[
\sup_{-N \leq u \leq N} \sigma_1^*(u+\delta) - \sigma_1^*(u-\delta) < \eta/4V_1.
\]

Hence we get

\[
I_z \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma_1^*(u) \int_{u+\delta}^{u+\delta} d\sigma_1^*(v) 
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sigma_1^*(u+\delta) - \sigma_1^*(u-\delta)) d\sigma_1^*(u) 
\]

\[
= \frac{1}{2\pi} \left( \int_{-N}^{N} (\sigma_1^*(u+\delta) - \sigma_1^*(u-\delta)) d\sigma_1^*(u) + \frac{1}{2\pi} \int_{N}^{N} (\sigma_1^*(u+\delta) 
\]

\[
- \sigma_1^*(u-\delta) d\sigma_1^*(u) \leq \frac{1}{2\pi} 2V_1 \left( \{ \sigma_1^*(-N) - \sigma_1^*(-\infty) \} + \{ \sigma_1^*(N) - \sigma_1^*(\infty) \} \right) \right) 
\]

\[
\leq (1/2\pi)(\eta/2 + \eta/2) + (1/2\pi)(\eta/4) = 5\eta/8\pi.
\]

As for \( I_1 \) : the condition \( |u-v| \geq \delta \) imply

\[
\left( \frac{\sin(u-v)}{(u-v)} \right)^2 \rightarrow 0 \quad \text{(as } l \rightarrow \infty). \]

Therefore we get

\[
\lim_{l \rightarrow \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_1(t)|^2 dt \leq 5\eta/8\pi.
\]

and the number \( \eta \) is an arbitrarily small. Thus we get
$$\varlimsup_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_1(t)|^2 dt = 0.$$ (3.3)

Secondly we shall estimate the part of $f_2(t)$. The \( \sigma_2(u) \) has enumerable number of jump point \((\lambda_n) (n=1, 2\cdots)\). Here if we put

$$a_n = \sigma(\lambda_n + 0) - \sigma(\lambda_n - 0)$$

and

$$\sum |a_n| = \int_{-\infty}^{\infty} d\sigma_2(u) = V_2.$$

Then we have

$$f_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\sigma_2(u) = \frac{1}{\sqrt{2\pi}} \sum a_n e^{i\lambda_n x}.$$

Therefore we get

$$\frac{1}{l} \int_{x}^{x+l} |f_2(t)|^2 dt = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{-N}^{N} a_n e^{i\lambda_n t} \bar{a_n} e^{i\lambda_n t} dt$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} |a_n|^2 + \frac{1}{2\pi} \sum_{m \neq n} a_m a_n e^{i(\lambda_m - \lambda_n)l} \frac{e^{i(\lambda_m - \lambda_n)l} - 1}{i(\lambda_m - \lambda_n)l} = J_1 + J_2,$$

say.

As for \( J_2 \): From \( \sum_{m \neq n} a_m a_n \leq \sum |a_m| \sum |a_n| = V_2^2 \), we get

$$\limsup_{l \to \infty} \sup_{-\infty < x < \infty} |J_2| \leq \sum |a_m| |a_n| \limsup_{l \to \infty} \frac{e^{i(\lambda_m - \lambda_n)l} - 1}{i(\lambda_m - \lambda_n)l} = 0.$$

Therefore we get

$$(3.4) \quad \limsup_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |f_2(t)|^2 dt = \frac{1}{2\pi} \sum |a_n|^2.$$

The estimations (3.3) and (3.4) read

$$(3.5) \quad \limsup_{l \to \infty} \sup_{-\infty < x < \infty} \left| \frac{1}{l} \int_{x}^{x+l} f_2(t)\overline{f_2(t)}dt \right|$$

$$\leq \limsup_{l \to \infty} \sup_{-\infty < x < \infty} \left( \frac{1}{l} \int_{x}^{x+l} |f_1(t)|^2 dt \right)^{\frac{1}{2}} \left( \frac{1}{l} \int_{x}^{x+l} |f_2(t)|^2 dt \right)^{\frac{1}{2}} = 0.$$}

Thus we obtain the (3.2).

The following results is well known: if \( f(x) \) can be represented by the Fourier-Stieltjes transform (3.1), then we have

$$(3.6) \quad \Lambda(f) = \{ \lambda | \sigma(\lambda + \epsilon) - \sigma(\lambda - \epsilon) > 0 \text{ for any } \epsilon > 0 \}$$

(c.f. H. Pollard [5]). Now we shall prove the following theorem.

**Theorem 4.** If \( f(x) \) can be represented by the formula (3.1), then we get
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(3.7) \[ \Lambda_{*}(f) = \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \} \]

where the notation \( \{ \} \) means the closure of the set \( \{ \} \).

Proof of Theorem 4. The proof can be done by decomposing into three parts.

(i) Firstly we shall prove

(3.8) \[ \Lambda_{*}(f) \supseteq \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \}. \]

We have

\[
g(x) = K * f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x - y) dy \int_{-\infty}^{\infty} e^{iyu} d\sigma(u)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\sigma(u) \int_{-\infty}^{\infty} K(x - y) e^{iyu} dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyu} k(u) d\sigma(u),
\]

where the \( k(u) \) is the Fourier transform of \( K(x) \) and a bounded continuous function on the real line. Repeating the same arguments as the proof of Theorem 3 we get

(3.9) \[ \lim_{l \to \infty} \sup_{-\infty < x < \infty} \frac{1}{l} \int_{x}^{x+l} |g(t)|^2 dt = \frac{1}{2\pi} \sum |k(\lambda)|^2 |\sigma(\lambda + 0) - \sigma(\lambda - 0)|^2. \]

Hence if \( g(x) = K * f \sim 0 \), we obtain

\[
\frac{1}{2\pi} \sum |k(\lambda)|^2 |\sigma(\lambda + 0) - \sigma(\lambda - 0)|^2 = 0
\]

and we can conclude that the condition \( \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \) implies \( k(\lambda) = 0 \). Thus we get the formalular (i).

(ii) Secondly we shall prove

(3.10) \[ \Lambda_{*}(f) \supseteq \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) < 0 \}. \]

This is immediate by the fact \( \Lambda_{*}(f) \) to be closed and the formula (i).

(iii) Lastly we shall prove

(3.11) \[ \Lambda_{*}(f) = \{ \lambda \mid \sigma(\lambda + 0) - \sigma(\lambda - 0) > 0 \}. \]

If there is an \( \lambda_0 \in \Lambda_{*}(f) \), there exist an neighbourhood of \( \lambda_0 \), \( (\lambda_0 - \delta, \lambda_0 + \delta) \) where the \( \sigma(u) \) is continuous. Let \( k_{\lambda_0}(u) \) be an ordinary triangular function on this interval. Let \( K_{\lambda_0}(x) \) be its Fourier transform. Let us put

\[
g_{\lambda_0}(x) = \int_{-\infty}^{\infty} K_{\lambda_0}(x - y) f(y) dy.
\]

Then we get
\[
\lim_{l \to \infty} \sup_{-\infty < \lambda < \infty} \frac{1}{l} \int_{x}^{x+t} |g_{\lambda}(t)|^{2} dt = \sum |k_{\lambda_{0}}(\lambda)|^{2} |\sigma(\lambda+0) - \sigma(\lambda-0)|^{z} = 0.
\]
On the other hand we have \(k_{\lambda_{0}}(u) \neq 0\) at \(u = \lambda_{0}\) by the definition of \(k_{\lambda_{0}}(u)\). Therefore we get \(\lambda_{0} \in \Lambda_{w}(f)\). This reads a contradiction. Thus we get the formula (iii).

**Theorem 5.** Let \(f(x)\) be represented by the formula (3.1). Let us assume that \(K(x)\) belong to the class \(L_{1}\) and its Fourier transform vanish on the set of \(\Lambda_{w}(f)\). Then we get

\[(3.12) \quad K*f \sim 0.\]

**Proof of Theorem 5.** This is a corollary to the formula (3.9.)

4. **The almost periodic function in the sense of Weyl.** For almost periodic function there always exist

\[(4.1) \quad a(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-it\lambda} dt.
\]

Then we have

**Theorem 6.** Let \(f(x)\) be an almost periodic function in the sense of Weyl. Let us put

\[(4.2) \quad \Lambda_{0}(f) = \{ \lambda | a(\lambda) \neq 0 \}.\]

Then we have

\[(4.3) \quad \Lambda_{0}(f) \subseteq \Lambda_{Wy}(f).\]

**Proof of Theorem 6.** (i) Proof of \(\Lambda_{0}(f) \subseteq \Lambda_{Wy}(f)\). We have by the Wiener formula

\[
\lim_{t \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \{s(u+\epsilon) - s(u-\epsilon)\} du = \sqrt{2\pi} a(\lambda).
\]

Therefore for any positive number \(\delta\) we get

\[
|a(\lambda)| \leq \lim_{\epsilon \to 0} \sup_{-\infty < \epsilon < \infty} \left( \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |s(u+\epsilon, x) - s(u-\epsilon, x)|^{2} du \right)^{\frac{1}{2}}.
\]

Thus we attain \(\Lambda_{0}(f) \subseteq \Lambda_{Wy}(f)\). Since \(\Lambda_{Wy}(f)\) is a closed set we conclude \(\Lambda_{0}(f) \subseteq \Lambda_{Wy}(f)\).

(ii) Proof of \(\Lambda_{Wy}(f) \subseteq \Lambda_{0}(f)\). We shall prove that if \(\lambda \in \Lambda_{0}(f)\) then \(\lambda \notin \Lambda_{Wy}(f)\). From \(\lambda \in \Lambda_{0}(f)\) there exists an neighbourhood of \(\lambda, (\lambda - \delta, \lambda + \delta)\) which does not contain any element of \(\Lambda_{0}(f)\). On the other hand since \(f\) is an almost periodic, there exist an trigonometric polynomial
(4.4) \[ p_N(x) = \sum_{1}^{N} a_ne^{i\lambda_n x}, \quad \lambda_n \in \Lambda_0(f) \quad (n = 1, 2, \ldots, N) \]
such that

\[ \lim_{T \to \infty} \sup_{-\infty < t < \infty} \frac{1}{2T} \int_{-T}^{T} |f(t+x) - p_N(t+x)|^2 dx \to 0 \quad (N \to \infty). \]

Therefore by the one-sided Wiener formula we get

\[ \lim_{\epsilon \to 0} \sup_{-\infty < u < \infty} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |s(u+\epsilon, x) - s(u-\epsilon, x) - s_N(u+\epsilon, x) - s_N(u-\epsilon, x)|^2 du \to 0 \quad (N \to \infty) \]

where we mean by \( s_N(u) \) the Fourier-Wiener transform of \( p_N(x) \). Since \( s_N(u) \) doen contain only elements of \( \Lambda_0(f) \) as an spectre we get

\[ s_N(u+\epsilon, x) - s_N(u-\epsilon, x) = \sum_{1}^{N} a_ne^{i\lambda_n x} \sqrt{2\pi} \chi_n(u), \]

where \( \chi_n(u) \) is an characteristic function of an interval \([\lambda_n-\epsilon, \lambda_n+\epsilon]\). Thus we can conclude

\[ s_N(u+\epsilon, x) - s_N(u-\epsilon, x) = 0 \quad \text{on} \quad (\lambda-\delta, \lambda+\delta). \]

Therefore we get

\[ \lim_{\epsilon \to 0} \sup_{-\infty < u < \infty} \frac{1}{2\epsilon} \int_{\lambda-\delta}^{\lambda+\delta} |s(u+\epsilon, x) - s(u-\epsilon, x)|^2 du = 0. \]

Hence we have \( \lambda \notin \Lambda_{Wy}(f) \). Thus we have completed the Proof of Theorem 6.

**Theorem 7.** Let \( f(x) \) be a bounded almost periodic function in the sense of Weyl. Let \( K(x) \) be an integrable function and its Fourier transform vanish on the set of \( \Lambda_*^*(f) \). Then we have

(4.6) \[ K* f \sim 0. \]

**Proof of Theorem 7.** By the hypotheses we get

\[ \int_{-\infty}^{\infty} K(y)p_N(x-y)dy = \sum_{1}^{N} a_n \int_{-\infty}^{\infty} K(y)e^{i\lambda_n(x-y)}dy = 0. \]

Thus we get

\[ g(x) = \int_{-\infty}^{\infty} K(y)f(x-y)dy = \int_{-\infty}^{\infty} K(y)\{f(x-y) - p_N(x-y)\}dy \]

and

\[ \frac{1}{l} \int_{u}^{u+l} |g(x)|^2 dx \leq \int_{-\infty}^{\infty} K(y)dy \sup_{-\infty < u < \infty} \frac{1}{l} \int_{u}^{u+l} |f(y) - p_N(y)|^2 dy. \]
Therefore we get by (4.5)

$$\lim_{l \to \infty} \sup_{-\infty < u < \infty} \frac{1}{l} \int_{u}^{u+l} |g(x)|^2 dx = 0.$$ 

References.


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