ON QUASI-GALOIS EXTENSIONS OF
DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

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Throughout the present paper, $R$ be always a division ring, and $S$ a division subring of $R$. And, we use the following conventions: $C=V_R(R)$, $V=V_R(S)$, $H=V_R^2(S)=V_R(V_R(S))$, and further for any subrings $R_1 \supseteq R_2$ of $R$ the set of all $R_2$-(ring) isomorphisms of $R_1$ into $R$ will be denoted as $\Gamma(R_1/R_2)$. As to other notations and terminologies used in this paper, we follow the previous one [3]. We consider here the following conditions:

(I) If $S'$ is a subring of $R$ properly containing $S$ with $[S':S]<\infty$ then $\Gamma(S'/S)\neq 1$.

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(II) If $S_1 \supseteq S_2$ are intermediate rings of $R/S$ with $[S_1:S]<\infty$ then $\Gamma(S_1/S)|S_2=\Gamma(S_2/S)$.

(II') If $T_1 \supseteq T_2$ are intermediate rings of $R/H$ with $[T_1:H]<\infty$ then $\Gamma(T_1/S)|T_2=\Gamma(T_2/S)$.

Following [5], $R/S$ is said to be (left-)quasi-Galois when (I) and (II) are fulfilled. Symmetrically, if (I') and (II') are done, we shall say $R/S$ is right-quasi-Galois. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

Lemma 1. If $S'$ is an intermediate ring of $R/S$ then $[V:V_R(S')]$. 

1) In [5], the condition that if $T$ is an intermediate ring of $R/H$ with $[T:H]<\infty$ then $\Gamma(T/S)|H=\Gamma(H/S)$ was cited as (II'). However, it will be rather natural to alter it like above.
[S':S]_l, and particularly in case V^2_R(S)=S the equality holds (provided we
do not distinguish between two infinite dimensions). If [S':S]_l<\infty then
V^2_R(S')=H[S'], and if R|S is (left-) locally finite then so is R/H.

Lemma 2. Let R/S be locally finite. In order that R/S is quasi-Galois it is necessary and sufficient that (I') and (II) are satisfied, and if (I') and (II') are satisfied then R/S is quasi-Galois.

Lemma 3. Let R be locally finite and quasi-Galois over S. If T is
an intermediate ring of H|S then T\Gamma(T|S)\subseteq H, whence it follows \Gamma(H|S)
=\mathfrak{G}(H|S).

Lemma 4. Let R be locally finite and quasi-Galois over S. If S' is
an intermediate ring of R/S with [S':S]_l<\infty then R/S' is quasi-Galois,
V^2_R(S')|S' is outer Galois and \mathfrak{G}(V^2_R(S')|S')\approx \mathfrak{G}(H[H\cap S']) by contraction, and
\Gamma(V^2_R(S')|S') |S'=\Gamma(S'/S).

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we
can prove that if R is locally finite and quasi-Galois over S and R' an in-
termediate ring of R/S with [H[R']:H]_l<\infty then H[R'] is locally finite and outer Galois over R' and \mathfrak{G}(H[R']|R')\approx \mathfrak{G}(H[H\cap R']) by contraction.
Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

Theorem 1. Let R be locally finite and quasi-Galois over S. If R'
is an intermediate ring of R/S, and H' an intermediate ring of H/S that
is Galois over S, then H'[R'] is locally finite and outer Galois over R' and
\mathfrak{G}(H'[R']|R')\approx \mathfrak{G}(H[H\cap R']) (algebraically and topologically) by contraction.

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

Lemme 5. Let T be an intermediate division ring of R/S, and \mathfrak{G} an
automorphism group of H[T]. If J(\mathfrak{G},H[T])=T and H\mathfrak{G}=H then [H^*:T:H^*]_l^\mathfrak{G}=[T:H\cap T]_l and [T\cdot H^*:H^*]_l=[T:H\cap T]_l, for each intermediate
division ring H* of H/H\cap T.

Now, Lemmas 4 and 5 enable us to apply the argument used in the
proof of [3, Lemma 3.2] to obtain the next lemma.

Lemma 6. Let R be locally finite and quasi-Galois over S. If S' is
an intermediate ring of R/S with [S':S]_l<\infty then [H^*[S']:H^*]_l=[R^*:H
\cap R^*]_l=[S':H\cap S']_l, for each intermediate rings H* of H[H\cap S'] and R*
of H[S']/S'.

By the validity of Lemma 6, the proof of the next theorem proceeds
evidently just like that of [3, Theorem 3.2] did.

2) $H^*\cdot T$ means the module product of $H^*$ and $T$. 
Theorem 2. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R|S$ then $[T:H\cap T]=[V:V_R(T)],<\infty$.

Lemma 7. If $R/H$ is locally finite and $R'$ is an intermediate ring of $R/H$ with $[R':H],<\infty$ then $R/H$ is right-locally finite and $[R':H]_r=[R':H]_r$.

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let $X$ be an arbitrary finite subset of $V$ that is linearly left-independent over $V'=V_R(R')$, and let $R_i=R'[X]$, that is evidently left-finite over $H$. We set here $V_i=V_{R_i}(H)$, $V'_i=V_{R_i}(R')$, and $C_i=V_{R_i}(R_i)$. Then, $[V_i:C_i]_{\infty}[R_i:H],<\infty$ by Lemma 1, whence it follows $[V_i:V'_i]=(V_i:V'_i)<\infty$. On the other hand, Lemma 1 yields also $[V_i:V'_i]<[R':H]_r$, whence we obtain $[V_i:V'_i]<[R':H]_r$. Recalling here that $X\subseteq V_i$ and $V'_i\subseteq V'$, we obtain $\bigoplus X<\bigoplus V_i<\bigoplus V', that is, $[V:V']_r<\bigoplus V'_i$. Lemma 1 yields therefore $[R':H]_r=[V:V']_r<\bigoplus V'_i$, because $V'_iV'=igoplus V'$. Now, the right-local finiteness of $R/H$ is evident, and so it follows symmetrically $[R':H]_r<[R':H]_r$. We have proved therefore that $[R':H]_r=[R':H]_r$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. Let $R$ be Galois over $S$ and locally finite over $H$. If $S'$ is an intermediate ring of $R|S$ with $[S':S],<\infty$ then $[S':S]_r=[S':S]_r$.

Proof. At first, if $R/S$ is outer Galois, [3, Lemma 1.3] yields at once $\infty>[S':S]_r=[(\mathfrak{u}|S')R:R],\infty=[(\mathfrak{u}|S')C:R].\infty=[(\mathfrak{u}|S')R:R],\infty=[S':S]_r$. Next, for general case, $R/S'$ is Galois by [2, Theorem 1] and there holds $\infty>[H[S']:H]_r=[H[S']:H], by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty>[H[S']:H]_r>[S':H]_r>[V:V_{R}(S')_r]=[H[S']:H]_r, and \infty>[H[S']:H]_r>[H\cdot S':H]_r=[S':H]_r>[V:V_{R}(S')_r]=[H[S']:H]_r$. Accordingly, it follows $[S':H\cap S']_r=[H[S']:H],=[H[S']:H]_r=[S':H\cap S']_r<\infty$. Recalling here that $H/S$ is outer Galois, as is noted above, there holds $[H\cap S':S]_r=[H\cap S':S],<\infty$. Now, combining these equalities, our assertion $[S':S]=S:S]_r$ will be evident.

Now, we shall prove the next theorem.

Theorem 3. The following conditions are equivalent to each other:

3) Since the division ring $R$ is $\mathfrak{G}r_{i}$-irreducible and $V_{hom(R,R)}(\mathfrak{G}r_{i})=S_{i}$, $\mathfrak{G}r_{i}$ is dense in $Hom_{S_{i}}(R,R)$ by Jacobson’s density theorem [1, p. 28].
(1) $R/S$ is locally finite and quasi-Galois, (1a) $R/S$ is right-locally finite and right-quasi-Galois, (2) $R/S$ is locally finite and (I)', (II) are fulfilled, (2a) $R/S$ is right-locally finite and (I)', (IIa) are fulfilled, (3) $R/S$ is locally finite and (I)', (II') are fulfilled, and (3a) $R/S$ is right-locally finite and (I)', (II') are fulfilled.

Proof. In virtue of Lemma 2, one will readily see that only the implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (1a) are left to be shown.

(1) $\Rightarrow$ (3). Let $T_1 \supseteq T_2$ be intermediate rings of $R/H$ with $[T_1:H]_r < \infty$. Choose an intermediate ring $S'_1$ of $T_1/S$ such that $[S'_1:S]_r < \infty$ and $T_1 = H[S'_1]$ and an intermediate ring $S_i$ of $T_i/S'_i$ such that $[S_i:S]_r < \infty$ and $T_i = H[S_i] = V_H(S_i)$. If we set $S_2 = T_2 \cap S_i (\supseteq S'_2)$, then $[S_2:S]_r < \infty$ and $T_2 = H[S_2] = V_H(S_2)$ evidently. As $R/S_i$ is quasi-Galois, $\mathfrak{G}(T_2/S_2) = \mathfrak{G}(T_i/S_i) | T_2$ by Lemma 4. Noting that $\Gamma(T_i/S_i)|S_2 = \Gamma(T_2/S_2)$, by Lemma 4, for each $\sigma \in \Gamma(T_i/S_i)$ we can find some $\rho \in \Gamma(T_i/S_i)$ such that $\sigma | S_2 = \rho | S_2$. By Lemma 3, $T_2 \sigma = H[S_2] \subseteq H[S_2] \subseteq T_2$. Let $T_2 = H[S_2]$ and $\sigma^{-1}\rho^{-1} \in \Gamma(T_i/S_2) = \mathfrak{G}(T_2/S_2) = \mathfrak{G}(T_i/S_i) | T_2$. Accordingly, $\rho$ is contained in $\Gamma(T_i/S_i)$ $\Rightarrow$ (3) obviously.

(1) $\Rightarrow$ (1a). Let $S'$ be an intermediate ring of $R/S$ with $[S':S]_r < \infty$. Since $\mathfrak{G}(H[S']/S') \approx \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), Lemmas 1, 5 and 7 yield $[S':H \cap S']_r = [S':H:H]_r \leq [H[S']:H]_r < \infty$. On the other hand, recalling that $H/S$ is outer Galois by Lemma 2, we obtain $[H \cap S':S]_r = [H \cap S':S]_r < \infty$. (See the proof of Corollary 1.) Combining those, we obtain $[S':S]_r < \infty$, which proves evidently the right-local finiteness of $R/S$. Now, our assertion will be obvious.

Corollary 2. Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R/S$ finitely generated over $S$ then $[S':S]_r = [S':S]_r$.

Proof. As $R/H$ is locally finite by Lemma 1 and $R$ is right-locally finite and right-quasi-Galois over $S$ by Theorem 3, Lemmas 6 and 7 together with their symmetries yield $[S':H \cap S']_r = [H[S']:H]_r = [H[S']:H]_r = [S':H \cap S']_r$. Hence, we readily obtain $[S':S]_r = [S':S]_r$. (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of (II'). This fact requested is now secured by Theorem 3.

Corollary 3. If $R$ is locally finite and quasi-Galois over $S$ and $[R:H]_r \leq \aleph_0$, then $R/S$ is Galois.

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].
Theorem 4. If $R/S$ is locally finite and quasi-Galois then so is $R/T$ for each $f$-regular intermediate ring $T$ of $R/S$.

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that $T\subseteq H$. Let $F$ be an arbitrary finite subset of $R$, and set $S' = S[F]$, $H' = T[H \cap S']$, $R' = H'[S'] = T[F]$. Then, $[R':H]' = [S':H \cap S'] < \infty$ by Lemma 6. On the other hand, noting that $H$ is locally finite and outer Galois over $S$, there holds $[H':T] < \infty$ by [3, Conclusion 2.1]. Hence, we have $[T[F]:T], [R':H]', [H':T] < \infty$, which means evidently the local finiteness of $R/T$. Moreover, as $V^\tau_k(T) = H$ and the condition (II') holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R/S$ then $\Gamma(V^\tau_k(T)/S)|T = \Gamma(T'/S)$.

Proof. Take an intermediate ring $S'$ of $T/S$ such that $[S':S], < \infty$ and $V^\tau_k(S') = V^\tau_k(T)$. Then, $T' = V^\tau_k(T) = V^\tau_k(S') = H[S']$ and $[T':H], < \infty$ by Lemma 1. As $\mathfrak{B}(T'/S) \approx \mathfrak{B}(H/H \cap S')$ by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once $T = (H \cap T)[S']$. Now, let $\sigma$ be an arbitrary element of $F(T/S)$. Then, by Lemma 4 $\sigma | S' = \tau | S'$ for some $\tau \in \Gamma(T'/S)$, and by Lemma 3 we see that $T_\sigma = (H \cap T)\sigma[S'\sigma] \subseteq H[S'\sigma] = H(S'\sigma) = T'\sigma$. And so, $\sigma^{-1} \in \Gamma(T'/S') = \mathfrak{B}(T'/S')|T$ by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have $\sigma = (\sigma^{-1})\tau \in \Gamma(T'/S)|T$.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. Let $R$ be locally finite and quasi-Galois over $S$, and $T_i \supseteq T_i$ intermediate rings of $R/S$. If $T_i$ is $f$-regular then $\Gamma(T_i/S) = \Gamma(T_i/S)|T_i$.

Proof. Setting here $T_i = V^\tau_k(T_i)(i = 1, 2)$, we have $T_i \supseteq T_i \supseteq H$, $T_i \supseteq T_i \supseteq T_i$, $[T_i':H], < \infty$ by Lemma 1, and so $\Gamma(T_i/S) = \Gamma(T_i'/S)|T_i(i = 1, 2)$ and $\Gamma(T_i'/S) = \Gamma(T_i'/S)|T_i'$ by Lemma 8 and Theorem 3 respectively. It follows therefore $\Gamma(T_i'/S) = \Gamma(T_i'/S)|T_2$, $T_2 = \Gamma(T_i'/S) | T_2 = (\Gamma(T_i'/S)|T_2) | T_2 = \Gamma(T_i/S)|T_2$.

References


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