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ON QUASI-GALOIS EXTENSIONS OF
DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

Takasi NAGAHARA and Hisao TOMINAGA

Throughout the present paper, $R$ be always a division ring, and $S$ a
division subring of $R$. And, we use the following conventions: $C=V_{R}(R), V = V_{R}(S), H=V_{R}^{2}(S)=V_{R}(V_{R}(S))$, and further for any subrings $R_{1}\supseteq R_{2}$ of $R$ the set of all $R_{2^{-}}$-ring isomorphisms of $R_{1}$ into $R$ will be denoted as $\Gamma(R_{1}/R_{2})$. As to other notations and terminologies used in this paper, we follow the previous one [3]. We consider here the following conditions:

(I) If $S'$ is a subring of $R$ properly containing $S$ with $[S':S]<\infty$ then $\Gamma(S'/S)\neq 1$.

(I$_{0}$) If $S'$ is a subring of $R$ properly containing $S$ with $[S':S]<\infty$ then $\Gamma(S'/S)\neq 1$.

(I') $H/S$ is Galois.

(II) If $S_{1}\supseteq S_{2}$ are intermediate rings of $R/S$ with $[S_{1}:S]_{r}<\infty$ then $\Gamma(S_{1}/S_{2})=\Gamma(S_{1}/S)$.

(II$_{0}$) If $S_{1}\supseteq S_{2}$ are intermediate rings of $R/S$ with $[S_{1}:S]_{r}<\infty$ then $\Gamma(S_{1}/S)|S_{2}=\Gamma(S_{1}/S)$. 

(II') If $T_{1}\supseteq T_{2}$ are intermediate rings of $R/H$ with $[T_{1}:H]_{l}<\infty$ then $\Gamma(T_{1}/T_{2})=\Gamma(T_{1}/S)$. 

(II'$_{0}$) If $T_{1}\supseteq T_{2}$ are intermediate rings of $R/H$ with $[T_{1}:H]_{r}<\infty$ then $\Gamma(T_{1}/T_{2})=\Gamma(T_{1}/S)$.

Following [5], $R/S$ is said to be (left-)quasi-Galois when (I) and (II) are fulfilled. Symmetrically, if (I$_{0}$) and (II$_{0}$) are done, we shall say $R/S$ is right-quasi-Galois. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

**Lemma 1.** If $S'$ is an intermediate ring of $R/S$ then $[V:V_{R}(S')]<\infty$.

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1) In [5], the condition that if $T$ is an intermediate ring of $R/H$ with $[T:H]_{r}<\infty$ then $\Gamma(T/S)|H=\Gamma(H/S)$ was cited as (II'). However, it will be rather natural to alter it like above.
if $S':S_l<\infty$ then $V^*_k(S')=H[S']$, and if $R|S$ is (left-) locally finite then so is $R/H$.

**Lemma 2.** Let $R|S$ be locally finite. In order that $R|S$ is quasi-Galois it is necessary and sufficient that (I) and (II) are satisfied, and if (I') and (II') are satisfied then $R|S$ is quasi-Galois.

**Lemma 3.** Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an intermediate ring of $H|S$ then $T\Gamma(T|S)\subseteq H$, whence it follows $\Gamma(H|S)=\mathfrak{G}(H|S)$.

**Lemma 4.** Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R|S$ with $[S':S_l]<\infty$ then $R|S'$ is quasi-Galois, $V^*_k(S')/S'$ is outer Galois and $\mathfrak{G}(V^*_k(S')/S')\approx \mathfrak{G}(H|H\cap S')$ by contraction, and $\Gamma(V^*_k(S')/S')|S'=\Gamma(S'/S)$.

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we can prove that if $R$ is locally finite and quasi-Galois over $S$ and $R'$ an intermediate ring of $R|S$ with $[H[R']:H]<\infty$ then $H[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H[R']/R')\approx \mathfrak{G}(H/H\cap R')$ by contraction. Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

**Theorem 1.** Let $R$ be locally finite and quasi-Galois over $S$. If $R'$ is an intermediate ring of $R|S$, and $H'$ an intermediate ring of $H|S$ that is Galois over $S$, then $H'[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H'[R']/R')\approx \mathfrak{G}(H/H\cap R')$ (algebraically and topologically) by contraction.

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

**Lemme 5.** Let $T$ be an intermediate division ring of $R|S$, and $\mathfrak{G}$ an automorphism group of $H[T]$. If $J(\mathfrak{G},H[T])=T$ and $H\mathfrak{G}=H$ then $[H^*\cdot T,H^*]_p=[T,H\cap T]_l$ and $[T\cdot H^*,H^*]=[T,H\cap T]$ for each intermediate division ring $H^*$ of $H/H\cap T$.

Now, Lemmas 4 and 5 enable us to apply the argument used in the proof of [3, Lemma 3.2] to obtain the next lemma.

**Lemma 6.** Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R|S$ with $[S':S]<\infty$ then $[H^*[S']:H^*]=[R^*:H\cap R^*]_{l}=[S':H\cap S']_l$ for each intermediate rings $H^*$ of $H/H\cap S'$ and $R^*$ of $H[S']/S'$.

By the validity of Lemma 6, the proof of the next theorem proceeds evidently just like that of [3, Theorem 3.2] did.

2) $H^*\cdot T$ means the module product of $H^*$ and $T$. 
Theorem 2. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R/S$ then \([T:H \cap T] = [V : V_R(T)]_l < \infty\).

Lemma 7. If $R/H$ is locally finite and $R'$ is an intermediate ring of $R/H$ with \([R' : H] < \infty\) then $R/H$ is right-locally finite and $[R' : H]_r = [R' : H]_l$.

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let $X$ be an arbitrary finite subset of $V$ that is linearly left-independent over $V' = V_R(R')$, and let $R_l = R'[X]$, that is evidently left-finite over $H$. We set here $V_l = V_R(H)$, $V_l' = V_R(R')$, and $C_l = V_R(R')$. Then, \([V_l : C_l] \leq [R_l : H]_l < \infty\) by Lemma 1, whence it follows $[V_l' : V_l]' = [V_l' : V_l]_r < \infty$. On the other hand, Lemma 1 yields also $[V_l : V_l]' \leq [R' : H]_l$, whence we obtain $[V_l' : V_l]' \leq [R' : H]_l$. Recalling here that $X \subseteq V_l$ and $V_l' \subseteq V'$, we obtain $\infty \geq [V_l : V_l]' \leq [R' : H]_l$, that is, $[V : V_l]' \leq [R' : H]_l$. Lemma 1 yields therefore $[R' : H]_l = [V : V_l]' \leq [R' : H]_l$, because $V_l^H(V') = V'$. Now, the right-local finiteness of $R/H$ is evident, and so it follows symmetrically $[R' : H]_l \leq [R' : H]_r$. We have proved therefore that $[R' : H]_l = [R' : H]_r$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. Let $R$ be Galois over $S$ and locally finite over $H$. If $S'$ is an intermediate ring of $R/S$ with \([S' : S]_l < \infty\) then \([S' : S]_r = [S' : S]_l\).

Proof. At first, if $R/S$ is outer Galois, [3, Lemma 1.3] yields at once $\infty > [S' : S]_l = (\mathfrak{G}S')R_l : R_l]_r = (\mathfrak{G}S')C_l : C_l]_r = (\mathfrak{G}S')R_l : R_l]_r = [S' : S]_r$. Next, for general case, $R/S'$ is Galois by [2, Theorem 1] and there holds $\infty > [H[S'] : H]_l = [H[S'] : H]_r$ by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty > [H[S'] : H]_l \geq [S' : H \cap S']_r \geq [V : V_R(S')_l = [H[S'] : H]_r$, and $\infty > [H[S'] : H]_l \geq [H : S' : H]_l = [S' : H \cap S']_r \geq [V : V_R(S')_l = [H[S'] : H]_r$. Accordingly, it follows $[S' : H \cap S']_r = [H[S'] : H]_l = [S' : H \cap S']_l < \infty$. Recalling here that $H/S$ is outer Galois, as is noted above, there holds $[H \cap S' : S]_r = [H \cap S' : S]_r < \infty$. Now, combining these equalities, our assertion \([S' : S]_l = [S' : S]_l\) will be evident.

Now, we shall prove the next theorem.

Theorem 3. The following conditions are equivalent to each other:

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3) Since the division ring $R$ is $\mathfrak{G}R_l$-irreducible and $V_{\text{Hom}(R,K)}(\mathfrak{G}R_l) = S_l$, $\mathfrak{G}R_l$ is dense in $H_{\text{Hom}}(R,R)$ by JACOBSON's density theorem [1, p. 28].
(1) \(R/S\) is locally finite and quasi-Galois, \((1)\) \(R/S\) is right-locally finite and right-quasi-Galois, (2) \(R/S\) is locally finite and \((I')\), \((II)\) are fulfilled, (2) \(R/S\) is right-locally finite and \((I')\), \((II)\) are fulfilled, (3) \(R/S\) is locally finite and \((I')\), \((II)\) are fulfilled, and (3) \(R/S\) is right-locally finite and \((I')\), \((II)\) are fulfilled.

Proof. In virtue of Lemma 2, one will readily see that only the implications \((1)\Rightarrow(3)\) and \((1)\Rightarrow(1_0)\) are left to be shown.

\((1)\Rightarrow(3)\): Let \(T_1 \supseteq T_2\) be intermediate rings of \(R/H\) with \([T_1:H], < \infty\). Choose an intermediate ring \(S'_1\) of \(T_1/S\) such that \([S'_1:S], < \infty\) and \(T_1 = H[S'_1]\) and an intermediate ring \(S_1\) of \(T_1/S_1\) such that \([S_1:S], < \infty\) and \(T_1 = H[S_1] = V^2(S'_1)\). If we set \(S_2 = T_1 \cap S_1(\supseteq S'_2)\), then \([S_2:S], < \infty\) and \(T_1 = H[S_2] = V^2(S'_2)\) evidently. As \(R/S_1\) is quasi-Galois, \(\mathfrak{H}(T_2/S_2) = \mathfrak{H}(T_1/S_1)\) by Lemma 4. Noting that \(\Gamma(T_2/S_2)\) is a quasi-Galois ring, we can find some \(\rho \in \Gamma(T_2/S_2)\) such that \(\rho | S_2 = \rho | S_1\). By Lemma 3, \(T_2 \sigma = H[S_2\sigma] \subseteq H[S_2\rho] = T_1\). Accordingly, \(\sigma\) is contained in \(\Gamma(T_2/S_2)\) obviously.

\((1)\Rightarrow(1_0)\): Let \(S'\) be an intermediate ring of \(R/S\) with \([S':S], < \infty\). Since \(\mathfrak{H}(H[S']/S') \approx \mathfrak{H}(H/H \cap S')\) by contraction (Lemma 4), Lemmas 1, 5 and 7 yield \([S':H \cap S'], = [S':H:H], \leq [H[S']:H], < \infty\). On the other hand, recalling that \(H/S\) is outer Galois by Lemma 2, we obtain \([H \cap S':S], = [H \cap S':S], < \infty\). (See the proof of Corollary 1.) Combining those, we obtain \([S':S], < \infty\), which proves evidently the right-local finiteness of \(R/S\). Now, our assertion will be obvious.

Corollary 2. Let \(R\) be locally finite and quasi-Galois over \(S\). If \(S'\) is an intermediate ring of \(R/S\) finitely generated over \(S\) then \([S':S], = [S':S],\).

Proof. As \(R/H\) is locally finite by Lemma 1 and \(R\) is right-locally finite and right-quasi-Galois over \(S\) by Theorem 3, Lemmas 6 and 7 together with their symmetries yield \([S':H \cap S'], = [H[S']:H], = [H[S']:H], < \infty\). Hence, we readily obtain \([S':S], = [S':S],\). (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of \((II')\). This fact requested is now secured by Theorem 3.

Corollary 3. If \(R\) is locally finite and quasi-Galois over \(S\) and \([R:H], \leq \aleph_0\), then \(R/S\) is Galois.

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].
Theorem 4. If R|S is locally finite and quasi-Galois then so is R/T for each f-regular intermediate ring T of R/S.

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that T\subseteq H. Let F be an arbitrary finite subset of R, and set S' = S[F], H' = T[H \cap S'], R' = H'[S'] = T[F]. Then, [R' : H']_r = [S' : H \cap S'], < \infty by Lemma 6. On the other hand, noting that H is locally finite and outer Galois over S, there holds [H' : T] < \infty by [3, Conclusion 2.1]. Hence, we have [T[F] : T]_r = [R' : H' \cdot H' : T] < \infty, which means evidently the local finiteness of R/T. Moreover, as V^a_T(H) = H and the condition (II') holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. Let R be locally finite and quasi-Galois over S. If T is an f-regular intermediate ring of R|S then \Gamma(V^a_T(S)/T) = \Gamma(T/S).

Proof. Take an intermediate ring S' of T/S such that [S' : S]_< \infty and V^a_R(S') = V^a_R(T). Then, T' = V^a_T(S') = V^a_R(S') = H[S'] and [T' : H]_< \infty by Lemma 1. As \mathcal{O}(T' / S') \approx \mathcal{O}(H[H \cap S']) by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once T = (H \cap T)[S']. Now, let \sigma be an arbitrary element of F(T/S). Then, by Lemma 4 \sigma | S' = \tau | S' for some \tau \in \Gamma(T'/S), and by Lemma 3 we see that T'\sigma = (H \cap T)\sigma[S] = H[S'] = H[S'\tau] = T'\tau. And so, \sigma^{-1} \in \Gamma(T'/S') = \mathcal{O}(T'/S') | T by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have \sigma = (\sigma^{-1}) \tau \in \Gamma(T'/S) | T.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. Let R be locally finite and quasi-Galois over S, and T_i \supseteq T_i intermediate rings of R|S. If T_i is f-regular then \Gamma(T_i/S) = \Gamma(T_i/S) | T_i.

Proof. Setting here T_i = V^a_T(S_i) (i = 1, 2), we have T_i \supseteq T_i \supseteq H, T_i \supseteq T_i \supseteq T_i, [T'_i : H]_< \infty by Lemma 1, and so \Gamma(T_i/S) = \Gamma(T_i/S) | T_i (i = 1, 2) and \Gamma(T'_i/S) = \Gamma(T'_i/S) | T'_i by Lemma 8 and Theorem 3 respectively. It follows therefore \Gamma(T_i/S) = \Gamma(T'_i/S) | T_i | T_i = \Gamma(T'_i/S) | T_i = (\Gamma(T'_i/S) | T_i) | T_i = \Gamma(T_i/S) | T_i.

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