ON QUASI-GALOIS EXTENSIONS OF
DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

Takasi NAGAHARA and Hisao TOMINAGA

Throughout the present paper, $R$ be always a division ring, and $S$ a division subring of $R$. And, we use the following conventions: $C=V_R(R)$, $V=V_R(S)$, $H=V^2_R(S)=V_R(V_R(S))$, and further for any subrings $R_1 \supseteq R_2$ of $R$ the set of all $R_2$-(ring) isomorphisms of $R_1$ into $R$ will be denoted as $\Gamma(R_1/R_2)$. As to other notations and terminologies used in this paper, we follow the previous one [3].

We consider here the following conditions:

(I) If $S'$ is a subring of $R$ properly containing $S$ with $[S':S]_l<\infty$ then $\Gamma(S'/S)\neq 1$.

(II') If $S_1 \supseteq S_2$ are intermediate rings of $R/S$ with $[S_1:S]_l<\infty$ then $\Gamma(S_1/S)|S_2=\Gamma(S_2/S)$.

Following [5], $R/S$ is said to be (left-)quasi-Galois when (I) and (II) are fulfilled. Symmetrically, if (I') and (II') are done, we shall say $R/S$ is right-quasi-Galois. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

**Lemma 1.** If $S'$ is an intermediate ring of $R/S$ then $[V:V_R(S')]<\infty$.  

---

1) In [5], the condition that if $T$ is an intermediate ring of $R/H$ with $[T:H]_l<\infty$ then $\Gamma(T/S)|H=\Gamma(H/S)$ was cited as (II'). However, it will be rather natural to alter it like above.
[S':S], and particularly in case $V^2_R(S)=S$ the equality holds (provided we do not distinguish between two infinite dimensions). If $[S':S]<\infty$ then $V^2_R(S)=H[S']$, and if $R'S$ is (left)-locally finite then so is $R/H$.

**Lemma 2.** Let $R/S$ be locally finite. In order that $R/S$ is quasi-Galois it is necessary and sufficient that $\langle I \rangle$ and $\langle II \rangle$ are satisfied, and if $\langle I' \rangle$ and $\langle II' \rangle$ are satisfied then $R/S$ is quasi-Galois.

**Lemma 3.** Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an intermediate ring of $H/S$ then $\Gamma(T/S)\subseteq H$, whence it follows $\Gamma(H/S) = \mathfrak{G}(H/S)$.

**Lemma 4.** Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R/S$ with $[S':S]<\infty$ then $R/S'$ is quasi-Galois, $V^2_R(S')/S'$ is outer Galois and $\mathfrak{G}(V^2_R(S')/S') \approx \mathfrak{G}(H[H \cap S'])$ by contraction, and $\Gamma(V^2_R(S')/S') = \Gamma(S'/S).

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we can prove that if $R$ is locally finite and quasi-Galois over $S$ and $R'$ an intermediate ring of $R/S$ with $[H[R']*H]<\infty$ then $H[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H[R']/R') \approx \mathfrak{G}(H[H \cap R'])$ by contraction. Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

**Theorem 1.** Let $R$ be locally finite and quasi-Galois over $S$. If $R'$ is an intermediate ring of $R/S$, and $H'$ an intermediate ring of $H/S$ that is Galois over $S$, then $H'[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H[R']/R') \approx \mathfrak{G}(H[H \cap R'])$ (algebraically and topologically) by contraction.

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

**Lemme 5.** Let $T$ be an intermediate division ring of $R/S$, and $\mathfrak{G}$ an automorphism group of $H[T]$. If $J(\mathfrak{G}, H[T]) = T$ and $H\mathfrak{G} = H$ then $[H*: T : H*] = [T : H \cap T]$, and $[T : H : H*] = [T : H \cap T]$, for each intermediate division ring $H*$ of $H/H \cap T$.

Now, Lemmas 4 and 5 enable us to apply the argument used in the proof of [3, Lemma 3.2] to obtain the next lemma.

**Lemma 6.** Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R/S$ with $[S':S]<\infty$ then $[H[S'] : H*] = [R* : H \cap R*] = [S' : H \cap S']$, for each intermediate rings $H*$ of $H/H \cap S'$ and $R*$ of $H[S']/S'$.

By the validity of Lemma 6, the proof of the next theorem proceeds evidently just like that of [3, Theorem 3.2] did.

---

2) $H* \cdot T$ means the module product of $H*$ and $T$. 

Theorem 2. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R/S$ then $[T:H\cap T]=[V:V_{R}(T)]_{l}<\infty$.

Lemma 7. If $R/H$ is locally finite and $R'$ is an intermediate ring of $R/H$ with $[R':H]<\infty$ then $R/H$ is right-locally finite and $[R':H]_{r}=[R':H]$.

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let $X$ be an arbitrary finite subset of $V$ that is linearly left-independent over $V'=V_{R}(R')$, and let $R_{1}=R'[X]$, that is evidently left-finite over $H$. We set here $V_{i}=V_{R_{i}}(H)$, $V'_{i}=V_{R_{i}}(R')$, and $C_{i}=V_{R_{i}}(R_{i})$. Then, $[V_{i}:C_{i}]<[R_{i}:H]_{r}<\infty$ by Lemma 1, whence it follows $[V_{i}:V'_{i}]_{l}=[V_{i}:V'_{i}]_{r}<\infty$. On the other hand, Lemma 1 yields also $[V_{i}:V'_{i}]_{l}<[R':H]_{r}$, whence we obtain $[V_{i}:V'_{i}]_{l}<[R':H]_{r}$. Recalling here that $X\subseteq V_{i}$ and $V'_{1}\subseteq V'$, we obtain $X\subseteq[V_{i}:V'_{i}]_{l}<[R':H]_{r}$, that is, $[V:V']_{l}<[R':H]_{r}$. Lemma 1 yields therefore $[R':H]_{r}=[V:V']_{l}<[R':H]_{r}$, because $V'_{l}(V')=V'$. Now, the right-local finiteness of $R/H$ is evident, and so it follows symmetrically $[R':H]_{r}<[R':H]_{r}$. We have proved therefore that $[R':H]_{r}=[R':H]_{r}$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. Let $R$ be Galois over $S$ and locally finite over $H$. If $S'$ is an intermediate ring of $R/S$ with $[S':S]_{l}<\infty$ then $[S':S]_{r}=[S':S]_{r}$.

Proof. At first, if $R/S$ is outer Galois, [3, Lemma 1.3] yields at once $[S':S]_{r}=[(S|S)'R_{r}:R_{r}]_{r}^{p}=[(S'|S)'C_{r}:C_{r}]_{r}=[(S'|S)'C_{r}:C_{r}']_{r}=[(S'|S)'R_{r}:R_{r}]_{r}=[S':S]_{r}$. Next, for general case, $R/S'$ is Galois by [2, Theorem 1] and there holds $[S':S]_{r}=[H(S')':H]_{r}$, by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty>[H(S')':H]_{r}=[S':H\cap S']_{l}=[V:V_{R}(S')]_{l}=[H(S')':H]_{r}$, and $\infty>[S':S']_{r}=[H(S':H)_{r}]=[S':H\cap S']_{l}=[V:V_{R}(S')]_{l}=[H(S)'H]_{r}$. Accordingly, it follows $[S':S']_{r}=[S':H\cap S']_{l}=[V:V_{R}(S')]_{l}=[H(S')':H]_{r}=[S':H\cap S']_{l}<\infty$. Recalling here that $H/S$ is outer Galois, as is noted above, there holds $[H\cap S':S']_{r}=[H\cap S':S']_{r}<\infty$. Now, combining these equalities, our assertion $[S':S]_{r}=[S':S]_{r}$, will be evident.

Now, we shall prove the next theorem.

Theorem 3. The following conditions are equivalent to each other:

---

3) Since the division ring $R$ is $\emptyset R_{r}$-irreducible and $V_{Hom(R,R_{r})}(\emptyset R_{r})=S_{l}$, $\emptyset R_{r}$ is dense in $Hom_{S_{l}}(R,R)$ by JACOBSON's density theorem [1, p. 28].
(1) \( R/S \) is locally finite and quasi-Galois, (1\(_o\)) \( R/S \) is right-locally finite and right-quasi-Galois, (2) \( R/S \) is locally finite and (I'), (II) are fulfilled, (2\(_o\)) \( R/S \) is right-locally finite and (I'), (II\(_o\)) are fulfilled, (3) \( R/S \) is locally finite and (I'), (II') are fulfilled, and (3\(_o\)) \( R/S \) is right-locally finite and (I'), (II\(_o\)) are fulfilled.

**Proof.** In virtue of Lemma 2, one will readily see that only the implications (1) \( \Rightarrow \) (3) and (1) \( \Rightarrow \) (1\(_o\)) are left to be shown.

(1) \( \Rightarrow \) (3). Let \( T_1 \supseteq T_2 \) be intermediate rings of \( R/H \) with \( [T_1:H]_r < \infty \). Choose an intermediate ring \( S'_1 \) of \( T_1/S \) such that \( [S'_1:S]_r < \infty \) and \( T_1 = H[S'_1] \) and an intermediate ring \( S_1 \) of \( T_1/S'_1 \) such that \( [S_1:S]_r < \infty \) and \( T_1 = H[S_1] = V_H^r(S_1) \). If we set \( S_2 = T_2 \cap S_1(\supseteq S'_2) \), then \( [S_2:S]_r < \infty \) and \( T_2 = H[S_2] = V_H^r(S_2) \) evidently. As \( R/S_1 \) is quasi-Galois, \( \mathfrak{G}(T_2/S_2) = \mathfrak{G}(T_1/S_1)|T_2 \) by Lemma 4. Noting that \( \Gamma(T_1/S_1)|S_1 \subseteq \Gamma(T_1/S_1)|S_1 = \Gamma(T_1/S)|S_1 \) by Lemma 4, for each \( \sigma \in \Gamma(T_1/S) \) we can find some \( \rho \in \Gamma(T_1/S) \) such that \( \sigma|S_1 = \rho|S_1 \). By Lemma 3, \( T_2 \sigma = H[S_2 \sigma] = H[S_1 \sigma] = T_1 \sigma \). Consequently, \( \sigma \) is contained in \( \Gamma(T_1/S)|T_2 \) obviously.

(1) \( \Rightarrow \) (1\(_o\)). Let \( S' \) be an intermediate ring of \( R/S \) with \( [S':S]_r < \infty \). Since \( \mathfrak{G}(H[S']/S') : \mathfrak{G}(H/H \cap S') \) by contraction (Lemma 4), Lemmas 1, 5 and 7 yield \( [S':H \cap S']_r = [S':H:H]_r \leq [H[S']:H]_r < \infty \). On the other hand, recalling that \( H/S \) is outer Galois by Lemma 2, we obtain \( [H \cap S':S]_r = [H \cap S':S]_r < \infty \) (See the proof of Corollary 1). Combining those, we obtain \( [S':S]_r < \infty \), which proves evidently the right-local finiteness of \( R/S \). Now, our assertion will be obvious.

**Corollary 2.** Let \( R \) be locally finite and quasi-Galois over \( S \). If \( S' \) is an intermediate ring of \( R/S \) finitely generated over \( S \) then \( [S':S]_r = [S':S]_r \).

**Proof.** As \( R/H \) is locally finite by Lemma 1 and \( R \) is right-locally finite and right-quasi-Galois over \( S \) by Theorem 3, Lemmas 6 and 7 together with their symmetries yield \( [S':H \cap S']_r = [H[S']:H]_r = [H[S']:H]_r = [S':H \cap S']_r \). Hence, we readily obtain \( [S':S]_r = [S':S]_r \). (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of (II'). This fact requested is now secured by Theorem 3.

**Corollary 3.** If \( R \) is locally finite and quasi-Galois over \( S \) and \( [R:H]_r \leq \aleph_0 \), then \( R/S \) is Galois.

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].
Theorem 4. If $R/S$ is locally finite and quasi-Galois then so is $R/T$ for each $f$-regular intermediate ring $T$ of $R/S$.

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that $T \subseteq H$. Let $F$ be an arbitrary finite subset of $R$, and set $S' = S[F]$, $H' = T[H \cap S']$, $R' = H'[S'] = T[F]$. Then, $[R' : H'] = [S' : H \cap S'] < \infty$. By Lemma 5, we have $[T[F] : T] = [H' : T] < \infty$, which means evidently the local finiteness of $R/T$. Moreover, as $V_{k}(T) = H$ and the condition (II') holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R/S$ then $\Gamma(V_{k}(T)/S)|T = \Gamma(T/S)$.

Proof. Take an intermediate ring $S'$ of $T/S$ such that $[S' : S] < \infty$ and $V_{k}(S') = V_{k}(T)$. Then, $T' = V_{k}(T) = V_{k}(S') = H[S']$ and $[T' : H] < \infty$ by Lemma 1. As $\mathfrak{G}(T'/S') \subseteq \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once $\Gamma(T'/S')|T_{T} = \Gamma(T/S)$. Now, let $\sigma$ be an arbitrary element of $\Gamma(T'/S')$. Then, by Lemma 4 $\sigma|S' = \tau|S'$ for some $\tau \in \Gamma(T'/S')$, and by Lemma 3 we see that $T' \subseteq T$ for all $\tau \in \Gamma(T'/S')$, and so, $\sigma^{-1} \in \Gamma(T'/S') = \mathfrak{G}(T'/S')|T$ by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have $\sigma = (\sigma^{-1}) \tau \in \Gamma(T'/S')|T$.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. Let $R$ be locally finite and quasi-Galois over $S$, and $T_i \supseteq T_{i+1}$ intermediate rings of $R/S$. If $T_i$ is $f$-regular then $\Gamma(T_i/S) = \Gamma(T_{i+1}/S)$.

Proof. Setting here $T_i = V_{k}(T_i)$ ($i = 1, 2$), we have $T_i \supseteq T_i \supseteq H$, $T_i \supseteq T_i \supseteq T_i \supseteq T_i$, $[T_i : H] < \infty$ by Lemma 1, and so $\Gamma(T_i/S) = \Gamma(T_{i+1}/S)|T_i$ ($i = 1, 2$) and $\Gamma(T_i/S) = \Gamma(T_{i+1}/S)|T_i$ by Lemma 8 and Theorem 3 respectively. It follows therefore $\Gamma(T_i/S) = (\Gamma(T_{i+1}/S)|T_i) | T_i = \Gamma(T_{i+1}/S)|T_i = (\Gamma(T_i/S)|T_i) | T_i = \Gamma(T_i/S)|T_i$.

References


Department of Mathematics, Okayama University
and
Department of Mathematics, Hokkaido University

(Received May 31, 1963)