ON QUASI-GALOIS EXTENSIONS OF DIVISION RINGS

Dedicated to Prof. Kinjiro Kunugi on his 60th birthday

By

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Throughout the present paper, \( R \) be always a division ring, and \( S \) a division subring of \( R \). And, we use the following conventions: \( C=V_{R}(R) \), \( V=V_{R}(S) \), \( H=V_{R}^{2}(S)=V_{R}(V_{R}(S)) \), and further for any subrings \( R_{1}\supseteq R_{2} \) of \( R \) the set of all \( R_{2} \)-(ring) isomorphisms of \( R_{1} \) into \( R \) will be denoted as \( \Gamma(R_{1}/R_{2}) \). As to other notations and terminologies used in this paper, we follow the previous one [3]. We consider here the following conditions:

(I) If \( S' \) is a subring of \( R \) properly containing \( S \) with \( [S':S]_{l}<\infty \) then \( \Gamma(S'/S)\neq 1 \).

(I') If \( S' \) is a subring of \( R \) properly containing \( S \) with \( [S':S]_{r}<\infty \) then \( \Gamma(S'/S)\neq 1 \).

(II) If \( T_{1}\supseteq T_{2} \) are intermediate rings of \( R/H \) with \( [T_{1}:H]_{l}<\infty \) then \( \Gamma(T_{1}/S)|H=\Gamma(H/S) \).

(II') If \( T_{1}\supseteq T_{2} \) are intermediate rings of \( R/H \) with \( [T_{1}:H]_{r}<\infty \) then \( \Gamma(T_{1}/S)|H=\Gamma(H/S) \).

Following [5], \( R/S \) is said to be (left-)quasi-Galois when (I) and (II) are fulfilled. Symmetrically, if (I') and (II') are done, we shall say \( R/S \) is right-quasi-Galois. In [5], we can find some fundamental theorems of quasi-Galois extensions. The purpose of the present paper is to expose several additional theorems concerning such extensions. At first, we shall recall the following lemmas which have been obtained in [4] and [5].

**Lemma 1.** If \( S' \) is an intermediate ring of \( R/S \) then \( [V:V_{R}(S')]_{r}\leq \)

\[1)\] In [5], the condition that if \( T \) is an intermediate ring of \( R/H \) with \( [T:H]_{r}<\infty \) then \( \Gamma(T/S)|H=\Gamma(H/S) \) was cited as (II'). However, it will be rather natural to alter it like above.
[S':S]_l, and particularly in case $V^*_S(S) = S$ the equality holds (provided we do not distinguish between two infinite dimensions). If $[S':S]_l < \infty$ then $V^*_S(S) = H[S']$, and if $R|S$ is (left-) locally finite then so is $R|H$.

Lemma 2. Let $R|S$ be locally finite. In order that $R|S$ is quasi-Galois it is necessary and sufficient that (V) and (II) are satisfied, and if (V) and (II) are satisfied then $R|S$ is quasi-Galois.

Lemma 3. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an intermediate ring of $H|S$ then $T \cap (T|S) \subseteq H$, whence it follows $\Gamma(H|S) = \mathfrak{G}(H|S)$.

Lemma 4. Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R|S$ with $[S':S]_l < \infty$ then $R|S'$ is quasi-Galois, $V^*_S(S')/S'$ is outer Galois and $\mathfrak{G}(V^*_S(S')/S') \approx \mathfrak{G}(H|H \cap S')$ by contraction, and $\Gamma(V^*_S(S')/S') \mid S' = \Gamma(S'/S)$.

By Lemma 4, in the same way as in the proof of [3, Lemma 3.5], we can prove that if $R$ is locally finite and quasi-Galois over $S$ and $R'$ is an intermediate ring of $R|S$ with $[H[R']:H]_l < \infty$ then $H[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H[R']/R') \approx \mathfrak{G}(H|H \cap R')$ by contraction. Accordingly, we can apply the same argument as in the proof of [4, Lemma 4] to obtain the next

Theorem 1. Let $R$ be locally finite and quasi-Galois over $S$. If $R'$ is an intermediate ring of $R|S$, and $H'$ an intermediate ring of $H|S$ that is Galois over $S$, then $H'[R']$ is locally finite and outer Galois over $R'$ and $\mathfrak{G}(H[R']/R') \approx \mathfrak{G}(H[H \cap R'])$ (algebraically and topologically) by contraction.

The proof of the next lemma will be easy from that of [3, Lemma 3.2].

Lemma 5. Let $T$ be an intermediate division ring of $R|S$, and $\mathfrak{G}$ an automorphism group of $H[T]$. If $J(\mathfrak{G}, H[T]) = T$ and $H^\mathfrak{G} = H$ then $[H^*: T:H^*]_l = [T: H \cap T]_l$ and $[T \cdot H^*: H^*]_l = [T: H \cap T]_l$ for each intermediate division ring $H^*$ of $H|H \cap T$.

Now, Lemmas 4 and 5 enable us to apply the argument used in the proof of [3, Lemma 3.2] to obtain the next lemma.

Lemma 6. Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R|S$ with $[S':S]_l < \infty$ then $[H^*: S': H^*]_l = [R^*: H \cap R^*]_l = [S': H \cap S']_l$ for each intermediate rings $H^*$ of $H|H \cap S'$ and $R^*$ of $H[S']/S'$.

By the validity of Lemma 6, the proof of the next theorem proceeds evidently just like that of [3, Theorem 3.2] did.

2) $H^* \cdot T$ means the module product of $H^*$ and $T$. 

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Theorem 2. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R/S$ then $[T:H \cap T] = [V:V_R(T)], < \infty$.

Lemma 7. If $R/H$ is locally finite and $R'$ is an intermediate ring of $R/H$ with $[R':H], < \infty$ then $R/H$ is right-locally finite and $[R':H]_r = [R':H]$.

Proof. Although the first assertion is [2, Lemma 4] itself, we shall prove here both. Let $X$ be an arbitrary finite subset of $V$ that is linearly left-independent over $V' = V_R(R')$, and let $R_i = R'[X]$, that is evidently left-finite over $H$. We set here $V_r = V_r(H)$, $V'_i = V_r(R')$, and $C_i = V_r(R_i)$. Then, $[V_i:C_i] \leq [R_i:H], < \infty$ by Lemma 1, whence it follows $[V_i:V'_i] = [V_i:V'_i]_r < \infty$. On the other hand, Lemma 1 yields also $[V_i:V'_i] \leq [R':H]_r$, whence we obtain $[V_i:V'_i] \leq [R':H]$. Recalling here that $X \subseteq V_i$ and $V'_i \subseteq V'$, we obtain $X \subseteq [V_i:V'_i] \leq [R':H]$, that is, $[V':V'_i] \leq [R':H]$. Lemma 1 yields therefore $[R':H] = [V:V'_i] \leq [R':H]$, because $V_r^+(V') = V'$. Now, the right-local finiteness of $R/H$ is evident, and so it follows symmetrically $[R':H]_r \leq [R':H]$.

We have proved therefore that $[R':H] = [R':H]$.

The next corollary has been stated in [2, Theorem 2], whose proof was essentially due to [1, Theorem 7.9.2]. However, we have recently found that the proof of [1, Theorem 7.9.2] would be open to doubt—we are afraid that the proof of [1, Theorem 7.8.1] was no longer efficient in that of [1, Theorem 7.9.2]. Because of this reason, we should like to present a new proof without making use of [1, Theorem 7.9.2] to our corollary.

Corollary 1. Let $R$ be Galois over $S$ and locally finite over $H$. If $S'$ is an intermediate ring of $R/S$ with $[S':S], < \infty$ then $[S':S]_r = [S':S]$.

Proof. At first, if $R/S$ is outer Galois, [3, Lemma 1.3] yields at once $\infty > [S':S] = [(\mathfrak Z|S''):R_1:S_1] = [(\mathfrak Z|S')C_1:C_1] = [(\mathfrak Z|S')R_1:R_1]_r = [S':S]$. Next, for general case, $R/S'$ is Galois by [2, Theorem 1] and there holds $\infty > [H[S']:H]_r = [H[S']:H]$. by Lemma 7. And so, by Lemmas 1 and 5, we obtain $\infty > [H[S']:H]_r = [S':H]_r = [S':H]_r = [V:V_R(S')]_r = [H[S']:H]_r$ and $\infty > [H[S']:H]_r = [H[S']:H]_r = [V:V_R(S')]_r = [H[S']:H]_r$. Accordingly, it follows $[S':H]_r = [H[S']:H]_r = [H[S']:H]_r = [S':H]_r = [S':H]_r < \infty$. Recalling here that $H/S$ is outer Galois, as is noted above, there holds $[H\cap S':S]_r = [H\cap S':S]_r < \infty$. Now, combining these equalities, our assertion $[S':S] = [S':S]$, will be evident.

Now, we shall prove the next theorem.

Theorem 3. The following conditions are equivalent to each other:

\[ \text{3) Since the division ring } R \text{ is } \mathfrak{G}R_r \text{-irreducible and } V_{\text{Hom}(R,R)}(\mathfrak{G}R_r) = S_t, \mathfrak{G}R_r \text{ is dense in } \text{Hom}_{S_t}(R,R) \text{ by Jacobson's density theorem [1, p. 28].} \]
(1) $R/S$ is locally finite and quasi-Galois, (1) $R/S$ is right-locally finite and right-quasi-Galois, (2) $R/S$ is locally finite and (I'), (II) are fulfilled, (2) $R/S$ is right-locally finite and (I'), (II') are fulfilled, (3) $R/S$ is locally finite and (I'), (II') are fulfilled, and (3) $R/S$ is right-locally finite and (I'), (II') are fulfilled.

Proof. In virtue of Lemma 2, one will readily see that only the implications (1)⇒(3) and (1)⇒(1) are left to be shown.

(1)⇒(3). Let $T_1 \supseteq T_2$ be intermediate rings of $R/H$ with $[T_1:H], < \infty$. Choose an intermediate ring $S'_i$ of $T_i/S_i$, such that $[S'_i:S_i]< \infty$ and $T_i = H[S'_i]$ and an intermediate ring $S_i$ of $T_i/S_i$ such that $[S_i:S_i]< \infty$ and $T_i = H[S_i] = V^p(S_i)$. If we set $S_i = T_i \cap S_i (\supseteq S'_i)$, then $[S_i:S_i]< \infty$ and $T_i = H[S_i] = V^p(S_i)$ evidently. As $R/S_i$ is quasi-Galois, $\mathfrak{G}(T_i/S_i) = \mathfrak{G}(T_i/S_i) | T_i$ by Lemma 4. Noting that $\Gamma(T_i/S)|S_i \subseteq \Gamma(T_i/S)|S_i = \Gamma(T_i/S)|S_i$ by Lemma 4, for each $\sigma \in \Gamma(T_i/S)$ we can find some $\rho \in \Gamma(T_i/S)$ such that $\sigma | S_i = \rho | S_i$. By Lemma 3, $T_2 \rho = H[S_i] \rho = T_2 \rho$ and $\sigma \rho^{-1} \in \Gamma(T_i/S)|S_i \subseteq \mathfrak{G}(T_i/S)|S_i | T_i$. Accordingly, $\sigma$ is contained in $\Gamma(T_i/S)|T_i$ obviously.

(1)⇒(1). Let $S'$ be an intermediate ring of $R/S$ with $[S':S_i]< \infty$. Since $\mathfrak{G}(H[S']) \approx \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), Lemmas 1, 5 and 7 yield $[S':H \cap S']_r = [S':H:H]_r < \infty$. On the other hand, recalling that $H/S$ is outer Galois by Lemma 2, we obtain $[H \cap S':S]_r = [H \cap S':S]_r < \infty$. (See the proof of Corollary 1.) Combining those, we obtain $[S':S]_r < \infty$, which proves evidently the right-local finiteness of $R/S$. Now, our assertion will be obvious.

Corollary 2. Let $R$ be locally finite and quasi-Galois over $S$. If $S'$ is an intermediate ring of $R/S$ finitely generated over $S$ then $[S':S]_r = [S':S]_r$.

Proof. As $R/H$ is locally finite by Lemma 1 and $R$ is right-locally finite and right-quasi-Galois over $S$ by Theorem 3, Lemmas 6 and 7 together with their symmetries yield $[S':H \cap S']_r = [H[S']:H]_r = [H[S']:H]_r = [S':H \cap S']_r$. Hence, we readily obtain $[S':S]_r = [S':S]_r$. (Cf. the proof of Corollary 1.)

The following corollary is [3, Corollary 3.5] itself. However, its proof contained a gap. In fact, in order to be able to apply the argument used in the proof of [3, Lemma 3.9], we had to prove the validity of (II'). This fact requested is now secured by Theorem 3.

Corollary 3. If $R$ is locally finite and quasi-Galois over $S$ and $[R:H]_r < \infty$, then $R/S$ is Galois.

Further, for the sake of completeness, we shall give here the proof of the following theorem [5, Theorem 2].
Theorem 4. If $R|S$ is locally finite and quasi-Galois then so is $R/T$ for each $f$-regular intermediate ring $T$ of $R|S$.

Proof. Obviously, by Lemma 4, we may restrict our proof to the case that $T \subseteq H$. Let $F$ be an arbitrary finite subset of $R$, and set $S' = S[F]$, $H' = T[H \cap S']$, $R' = H'[S'] = T[F]$. Then, $[R':H']_r = [S':H \cap S']_r < \infty$ by Lemma 6. On the other hand, noting that $H$ is locally finite and outer Galois over $S$, there holds $[H':T]<\infty$ by [3, Conclusion 2.1]. Hence, we have $[T[F]:T]_r = [R':H']_r \cdot [H':T]<\infty$, which means evidently the local finiteness of $R/T$. Moreover, as $V^{\frac{1}{2}}(T) = H$ and the condition (II') holds by Theorem 3, our assertion is a consequence of Theorem 3.

Lemma 8. Let $R$ be locally finite and quasi-Galois over $S$. If $T$ is an $f$-regular intermediate ring of $R|S$ then $\Gamma(V^{\frac{1}{2}}(T)/S)|T = \Gamma(T|S)$.

Proof. Take an intermediate ring $S'$ of $T|S$ such that $[S':S]_r < \infty$ and $V_R(S') = V_R(T)$. Then, $T' = V^{\frac{1}{2}}(T) = V^{\frac{1}{2}}(S') = H[S']$ and $[T':H]<\infty$ by Lemma 1. As $\mathfrak{G}(T'/S') \cong \mathfrak{G}(H/H \cap S')$ by contraction (Lemma 4), [3, Conclusion 2.1] will yield at once $T = (H \cap T)[S']$. Now, let $\sigma$ be an arbitrary element of $\Gamma(T|S)$. Then, by Lemma 4 $\sigma|S' = \tau|S'$ for some $\tau \in \Gamma(T'/S')$, and by Lemma 3 we see that $T\sigma = ((H \cap T)\sigma)[S'\sigma] = H[S'\sigma] = H[S^2\tau] = T^\tau$. And so, $\sigma^\tau \in \Gamma(T'/S') = \mathfrak{G}(T'/S')|T$ by Lemmas 3, 4 and [3, Conclusion 2.1], whence we have $\sigma = (\sigma^{-1})^\tau \in \Gamma(T'/S')|T$.

By the light of Lemma 8, we can prove the following extension theorem of isomorphisms that corresponds to [3, Theorem 3.5].

Theorem 5. Let $R$ be locally finite and quasi-Galois over $S$, and $T_i \supseteq T_i$ intermediate rings of $R|S$. If $T_i$ is $f$-regular then $\Gamma(T_i/S) = \Gamma(T_i/S)|T_i$.

Proof. Setting here $T_i = V^{\frac{1}{2}}(T_i)$ $(i = 1, 2)$, we have $T_i \supseteq T_i' \subseteq H$, $T_i' \supseteq T_i \supseteq T_i$, $[T_i':H]_r < \infty$ by Lemma 1, and so $\Gamma(T_i/S) = \Gamma(T_i'/S)|T_i$ $(i = 1, 2)$ and $\Gamma(T_i'/S) = \Gamma(T_i'/S)|T_i'$ by Lemma 8 and Theorem 3 respectively. It follows therefore $\Gamma(T_i/S) = (\Gamma(T_i'/S)|T_i)$ $T_i = \Gamma(T_i'/S)|T_i = (\Gamma(T_i'/S)|T_i)|T_i = \Gamma(T_i'/S)|T_i$.

References


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