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ON A THEOREM CONCERNING THE DISTRIBUTION OF ALMOST PRIMES

By

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By an *almost prime* is meant a positive rational integer the number of prime factors of which is bounded by a certain constant. Let us denote by $\Omega(n)$ the total number of prime factors of a positive integer n . In 1920 Viggo Brun [2] elaborated an elementary method of the sieve of Eratosthenes to prove that for all sufficiently large x there exists at least one integer n with $\Omega(n) \leq 11$ in the interval $x \leq n \leq x + x^{\frac{1}{2}}$. Quite recently W. E. Mientka [4] improved this result of Brun, showing that for all large x there exists at least one integer n with $\Omega(n) \leq 9$ in the interval $x \leq n \leq x + x^{\frac{1}{2}}$. To establish this Mientka makes use of the sieve method due to A. Selberg instead of Brun's method (cf. [3] and [4]). By refining the argument of Mientka [4] we can further improve his result. Indeed, we shall prove in this paper the following

Theorem. *Let $k \geq 2$ be a fixed integer. Then, for all sufficiently large x , there exists at least one integer n with $\Omega(n) \leq 2k$ in the interval $x < n \leq x + x^{1/k}$.*

Thus, in particular, if $k=2$ then for all large x the interval $x < n \leq x + x^{\frac{1}{2}}$ always contains an integer n such that $\Omega(n) \leq 4$. Of course, the restriction in the theorem that k be integral may be relaxed without essential changes in the result.

Let us mention that the existence of a prime number p in the interval $x < p \leq x + x^{1/k}$ for all large x could not be deduced, as is well known, even from the Riemann hypothesis if only $k=2$.

Note. It is possible to generalize our theorem presented above so as to concern with the distribution of almost primes in an arithmetic progression. Thus, if a and b are integers such that $a \geq 1$, $0 \leq b \leq a-1$, $(a, b)=1$, then we can prove the existence of an integer n satisfying

$$x < n \leq x + x^{1/k}, \quad n \equiv b \pmod{a}, \\ \Omega(n) \leq 2k,$$

provided that x be sufficiently large, $k \geq 2$ being a fixed integer. Here, in particular, in the case of $k=2$, the inequality $\Omega(n) \leq 4$ may be replaced by $\Omega(n) \leq 3$: this result is apparently stronger than the above theorem for the

corresponding case. Proof is similar to that of our theorem but somewhat more complicated arguments are needed.

1. Let $M > 0$ and $N > 1$ be integers and let $z \geq 2$ and $w > 0$ be any real numbers such that $w^2 \geq z$. We denote by S the number of those integers n in the interval $M < n \leq M + N$ which are not divisible by any prime number $p \leq z$. Then, by making use of the 'lower' sieve of A. Selberg (cf. [3] and [6]) we can show that

$$S \geq (1 - Q)N - R_1,$$

where

$$Q = \sum_{p \leq z} \frac{1}{pZ_p} \text{ with } Z_p = \sum_{\substack{1 \leq m \leq w/\sqrt{p} \\ g(m) < p}} \frac{\mu^2(m)}{\phi(m)}$$

and

$$R_1 = O\left(w^2 \sum_{p \leq z} \frac{1}{pZ_p^2}\right).$$

Here $g(1) = 1$ and for $m > 1$ $g(m)$ denotes the greatest prime divisor of m , and the O -constant for R_1 is absolute. It will be shown later that $Z_p > c \log p$ for all $p \leq z$, where $c > 0$ is a constant, so that we have $R_1 = O(w^2)$.

Now we take

$$z = (2N)^{\frac{1}{4}}, \quad w = (2N)^{\frac{1}{2} - \varepsilon},$$

where $0 < \varepsilon < \frac{1}{4}$. If we fix ε sufficiently small then there holds the following

Lemma 1. *For all sufficiently large N we have*

$$S > 1.6054 \frac{N}{\log N}.$$

Our proof of Lemma 1 runs essentially on the same lines as in [4]; we shall give an outline of the proof of this lemma in § 3.

Throughout in the following the constants implied in the symbol O are all absolute (apart from the possible dependence on the parameter ε), and c represents positive constants not necessarily the same in each occurrence.

2. In order to prove Lemma 1 we require some auxiliary results due to N. G. de Bruijn [1] on the number $\Psi(x, y)$ of integers $n \leq x$ and free of prime factors $> y$.

It is proved by de Bruijn [1] that we have

$$(1) \quad \Psi(x, y) = O(xe^{-cu})$$

and more precisely

$$(2) \quad \Psi(x, y) = x\rho(u) + O(1) \\ + O(xu^2e^{-c\sqrt{\log y}}) + O\left(\frac{x\rho(u)\log(2+u)}{\log y}\right),$$

where $x > 1$, $y \geq 2$, $u = (\log x)/\log y$, and the function $\rho(u)$ is defined by the following conditions:

$$(3) \quad \rho(u) = 0 \quad (u < 0); \quad \rho(u) = 1 \quad (0 \leq u \leq 1); \\ u\rho'(u) = -\rho(u-1) \quad (u > 1); \quad \rho(u) \text{ continuous for } u > 0.$$

Lemma 2. We have for $t \geq t_0 \geq 1$

$$\rho(t) \leq \rho(t_0)e^{-(t-t_0)},$$

so that

$$\int_{t_0}^{\infty} \rho(u) du \leq \rho(t_0) \quad (t_0 \geq 1).$$

This is stated and employed without proof in [4] as a lemma of N. C. Ankeny. By integrating by parts we deduce from (3) that for $t \geq 1$

$$t\rho(t) = \int_0^t \rho(u) du - \int_0^t \rho(u-1) du = \int_{t-1}^t \rho(u) du \leq \rho(t-1),$$

since $\rho(u)$ decreases monotonously for $u \geq 0$. Hence

$$\frac{\rho'(u)}{\rho(u)} = -\frac{\rho(u-1)}{u\rho(u)} \leq -1 \quad (u \geq 1)$$

and the result follows at once.

3. Following Mientka [4] let us put

$$H_p = \prod_{q < p} \left(1 - \frac{1}{q}\right)^{-1} \quad (p \leq z),$$

where in the product on the right-hand side q runs through the prime numbers less than p , and

$$T_p = \sum_{\substack{m > w/\sqrt{p} \\ g(m) \leq p}} \frac{1}{m} \quad (p \leq z).$$

Then we have $|H_p - Z_p| \leq T_p$ and

$$S \geq N \prod_{p \leq z} \left(1 - \frac{1}{p}\right) - N \sum_{p \leq z} \frac{T_p}{pH_p(H_p - T_p)} - R_1$$

(cf. [3] and [4]). Since it is well known that

$$\prod_{p \leq z} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log z} + O\left(\frac{1}{\log^2 z}\right),$$

C being the Euler constant, it remains only to evaluate the middle term on the right-hand side of the above inequality for S .

By partial summation we have

$$T_p = \sum_{m > w/\sqrt{p}} \frac{\Psi(m, p)}{m^2} + O(N^{-\frac{3}{8} + \epsilon}).$$

We find easily that

$$T_p = O\left(\frac{1}{\log^2 N}\right)$$

for every $p \leq \exp(\log N)^{\frac{2}{3}}$, on taking account of (1). For $\exp(\log N)^{\frac{2}{3}} < p \leq z$ we have

$$T_p = \sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{\Psi(m, p)}{m^2} + O\left(\frac{1}{\log^2 N}\right),$$

where, by (2),

$$\begin{aligned} & \sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{\Psi(m, p)}{m^2} \\ &= \sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{1}{m} \rho\left(\frac{\log m}{\log p}\right) \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) \\ & \quad + O\left(\frac{1}{\log^2 N}\right), \end{aligned}$$

and this is equal to

$$\left(\int_{w/\sqrt{p}}^{\infty} \frac{1}{x} \rho\left(\frac{\log x}{\log p}\right) dx + O(N^{-\frac{3}{8} + \epsilon})\right) \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right).$$

Hence

$$T_p = \log p \int_{\log(w/\sqrt{p})/\log p}^{\infty} \rho(u) du \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right)$$

for $\exp(\log N)^{\frac{2}{3}} < p \leq z$.

Put

$$I_p = \int_{\log(w/\sqrt{p})/\log p}^{\infty} \rho(u) du \quad (p \leq z).$$

Then it follows immediately from the above results that

$$\begin{aligned} & \sum_{p \leq z} \frac{T_p}{pH_p(H_p - T_p)} \\ &= e^{-C} \sum_{\exp(\log N)^{\frac{2}{3}} < p \leq z} \frac{1}{p \log p} \frac{I_p}{e^C - I_p} + O\left(\frac{\log \log N^2}{(\log N)^{4/3}}\right). \end{aligned}$$

For p in the interval $(2N)^{\frac{1}{\nu+1}} < p \leq (2N)^{\frac{1}{\nu}}$ ($\nu \geq 4$) we have, by Lemma 2,

$$I_p \leq \rho \left(\frac{\log(\omega/\sqrt{p})}{\log p} \right) \leq \rho(t_\nu),$$

where we have put

$$t_\nu = \left(\frac{1}{2} - \varepsilon \right) \nu - \frac{1}{2}.$$

Therefore

$$\begin{aligned} & \sum_{\exp(\log N)^{\frac{2}{3}} < p \leq z} \frac{1}{p \log p} \frac{I_p}{e^c - I_p} \\ & \leq \sum_{4 \leq \nu < c(\log N)^{\frac{1}{3}}} \left(\sum_{(2N)^{\frac{1}{\nu+1}} < p \leq (2N)^{\frac{1}{\nu}}} \frac{1}{p \log p} \right) \frac{\rho(t_\nu)}{e^c - \rho(t_\nu)} \\ & = \frac{1}{\log N} \sum_{\nu=4}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} \frac{\rho(t_\nu)}{e^c - \rho(t_\nu)} + O\left(\frac{1}{(\log N)^{4/3}}\right). \end{aligned}$$

Here we used the relation

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c_1 + O\left(\frac{1}{\log x}\right),$$

c_1 being a constant. Hence

$$\begin{aligned} & \sum_{p \leq z} \frac{T_p}{p H_p (H_p - T_p)} \\ & \leq \frac{e^{-c}}{\log N} \sum_{\nu=4}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} \frac{\rho(t_\nu)}{e^c - \rho(t_\nu)} + O\left(\frac{(\log \log N)^2}{(\log N)^{4/3}}\right). \end{aligned}$$

Now, by the definition of $\rho(u)$, we have

$$\rho(u) = 1 - \log u \quad (1 \leq u \leq 2).$$

If we take $\varepsilon = 10^{-4}$, then we find that

$$\begin{aligned} \rho(t_4) &= \rho(1.4996) < 0.5949, \\ \rho(t_5) &= \rho(1.9995) < 0.3072, \end{aligned}$$

so that

$$5 \log \frac{5}{4} \frac{\rho(t_4)}{e^c - \rho(t_4)} < 0.5597,$$

and

$$\begin{aligned} & \sum_{\nu=5}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} \frac{\rho(t_\nu)}{e^c - \rho(t_\nu)} \\ & \leq 6 \log \frac{6}{5} \frac{\rho(t_5)}{e^c - \rho(t_5)} \frac{1}{1 - e^{-0.4999}} < 0.5807, \end{aligned}$$

on appealing to Lemma 2. We thus have proved that

$$\sum_{p \leq z} \frac{T_p}{pH_p(H_p - T_p)} < 1.1404e^{-c} \frac{1}{\log N} + O\left(\frac{(\log \log N)^2}{(\log N)^{4/3}}\right)$$

and this completes the proof of Lemma 1 since

$$(4 - 1.1404)e^{-c} > 1.6055.$$

4. Let q be any prime number in the interval $z < q \leq z^2$, where, as before, $z = (2N)^{\frac{1}{4}}$. We next estimate the number $S(q)$ of those integers n in $M < n \leq M + N$ which are multiples of q and are not divisible by any prime number $p \leq z$. We have by the ‘upper’ sieve of A. Selberg (cf. [5])

$$S(q) \leq \frac{N}{qZ} + R_2,$$

where

$$Z = \sum_{1 \leq m \leq z} \frac{\mu^2(m)}{\phi(m)}$$

and

$$R_2 = O\left(\frac{z^2}{Z^2}\right).$$

It is easily verified that

$$Z \geq \sum_{1 \leq m \leq z} \frac{1}{m} = \log z + O(1),$$

and therefore

$$(4) \quad S(q) \leq \frac{4N}{q \log N} + O\left(\frac{N}{q \log^2 N}\right).$$

Lemma 3. *Let U denote the number of those integers n in $M < n \leq M + N$ which are divisible by no primes $p \leq z$, by at most two primes q with $z < q \leq z^2$, and by no integers of the form q^2 , q being a prime in $z < q \leq z^2$. Then, for all sufficiently large N , we have*

$$U > 0.6811 \frac{N}{\log N}.$$

Let N be a sufficiently large positive number. The number of those integers n in $M < n \leq M + N$ which are not divisible by any prime $p \leq z$ and are divisible by some q^2 , where q is a prime in $z < q \leq z^2$, does not exceed

$$\sum_{z < q \leq z^2} \left(\left[\frac{M+N}{q^2} \right] - \left[\frac{M}{q^2} \right] \right) = O(N^{\frac{3}{4}}).$$

Now, the number of those integers n with $M < n \leq M + N$ which are not divisible by any prime $p \leq z$ and are divisible by at least three (distinct) primes q in $z < q \leq z^2$ is, by (4), not greater than

$$\frac{1}{3} \sum_{z < q \leq z^2} S(q) \leq \frac{4 \log 2}{3} \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

It thus follows from Lemma 1 that

$$U > \left(1.6054 - \frac{4 \log 2}{3}\right) \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right),$$

which proves our lemma since $(4/3) \log 2 < 0.9242$.

5. We can now conclude the proof of our theorem. Let x be a sufficiently large positive real number and put

$$M = [x], \quad N = [x^{1/k}].$$

Then, by Lemma 3, there exists at least one integer n in the interval $M < n \leq M + N$, i. e. in the interval

$$x < n \leq x + x^{1/k},$$

such that it is not divisible by any prime $p \leq (2N)^{\frac{1}{4}}$ and is divisible by at most two primes q in $(2N)^{\frac{1}{4}} < q \leq (2N)^{\frac{1}{2}}$ but not divisible by the squares of these q , where

$$(2N)^k > (2(x^{1/k} - 1))^k > x + x^{1/k}$$

since $k \geq 2$. Therefore, according as n has no, one or two prime factors q in $(2N)^{\frac{1}{4}} < q \leq (2N)^{\frac{1}{2}}$ it has at most $2k-1$, $2k-1$ or $2k-2$ additional prime factors. Hence the total number of prime factors of n is at most $2k$, i. e. $\Omega(n) \leq 2k$. This completes the proof of the theorem.

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