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<td>Author(s)</td>
<td>Uchiyama, Saburô</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 17(3-4): 152-159</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1963</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56044">http://hdl.handle.net/2115/56044</a></td>
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<td>Type</td>
<td>bulletin (article)</td>
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<tr>
<td>File Information</td>
<td>JFSHIU_17_N3-4_152-159.pdf</td>
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ON A THEOREM CONCERNING
THE DISTRIBUTION OF ALMOST PRIMES

By

Saburō UCHIYAMA

By an almost prime is meant a positive rational integer the number of prime factors of which is bounded by a certain constant. Let us denote by \( \Omega(n) \) the total number of prime factors of a positive integer \( n \). In 1920 Viggo Brun [2] elaborated an elementary method of the sieve of Eratosthenes to prove that for all sufficiently large \( x \) there exists at least one integer \( n \) with \( \Omega(n) \leq 11 \) in the interval \( x \leq n \leq x + x^{3/4} \). Quite recently W. E. Mientka [4] improved this result of Brun, showing that for all large \( x \) there exists at least one integer \( n \) with \( \Omega(n) \leq 9 \) in the interval \( x \leq n \leq x + x^{3/4} \). To establish this Mientka makes use of the sieve method due to A. Selberg instead of Brun's method (cf. [3] and [4]). By refining the argument of Mientka [4] we can further improve his result. Indeed, we shall prove in this paper the following

**Theorem.** Let \( k \geq 2 \) be a fixed integer. Then, for all sufficiently large \( x \), there exists at least one integer \( n \) with \( \Omega(n) \leq 2k \) in the interval \( x < n \leq x + x^{1/k} \).

Thus, in particular, if \( k = 2 \) then for all large \( x \) the interval \( x < n \leq x + x^{3/4} \) always contains an integer \( n \) such that \( \Omega(n) \leq 4 \). Of course, the restriction in the theorem that \( k \) be integral may be relaxed without essential changes in the result.

Let us mention that the existence of a prime number \( p \) in the interval \( x < p \leq x + x^{1/k} \) for all large \( x \) could not be deduced, as is well known, even from the Riemann hypothesis if only \( k = 2 \).

**Note.** It is possible to generalize our theorem presented above so as to concern with the distribution of almost primes in an arithmetic progression. Thus, if \( a \) and \( b \) are integers such that \( a \geq 1 \), \( 0 \leq b \leq a - 1 \), \( (a, b) = 1 \), then we can prove the existence of an integer \( n \) satisfying

\[
x < n \leq x + x^{1/k}, \quad n \equiv b \pmod{a},
\]

\[
\Omega(n) \leq 2k,
\]

provided that \( x \) be sufficiently large, \( k \geq 2 \) being a fixed integer. Here, in particular, in the case of \( k = 2 \), the inequality \( \Omega(n) \leq 4 \) may be replaced by \( \Omega(n) \leq 3 \): this result is apparently stronger than the above theorem for the
corresponding case. Proof is similar to that of our theorem but somewhat more complicated arguments are needed.

1. Let \( M > 0 \) and \( N > 1 \) be integers and let \( z \geq 2 \) and \( w > 0 \) be any real numbers such that \( w^2 \geq z \). We denote by \( S \) the number of those integers \( n \) in the interval \( M < n \leq M + N \) which are not divisible by any prime number \( p \leq z \). Then, by making use of the 'lower' sieve of A. Selberg (cf. [3] and [6]) we can show that

\[
S \geq (1 - Q) N - R_1,
\]

where

\[
Q = \sum_{p \leq z} \frac{1}{pZ_p} \quad \text{with} \quad Z_p = \sum_{1 \leq m \leq w/p} \frac{\mu^2(m)}{\phi(m)}
\]

and

\[
R_1 = O \left( w^2 \sum_{p \leq z} \frac{1}{pZ_p^2} \right).
\]

Here \( q(1) = 1 \) and for \( m > 1 \) \( g(m) \) denotes the greatest prime divisor of \( m \), and the \( O \)-constant for \( R_1 \) is absolute. It will be shown later that \( Z_p > c \log p \) for all \( p \leq z \), where \( c > 0 \) is a constant, so that we have \( R_1 = O(w^2) \).

Now we take

\[
z = (2N)^{\frac{1}{4}}, \quad w = (2N)^{\frac{1}{2} - \varepsilon},
\]

where \( 0 < \varepsilon < \frac{1}{4} \). If we fix \( \varepsilon \) sufficiently small then there holds the following

**Lemma 1.** For all sufficiently large \( N \) we have

\[
S > 1.6054 \frac{N}{\log N}.
\]

Our proof of Lemma 1 runs essentially on the same lines as in [4]; we shall give an outline of the proof of this lemma in § 3.

Throughout in the following the constants implied in the symbol \( O \) are all absolute (apart from the possible dependence on the parameter \( \varepsilon \)), and \( c \) represents positive constants not necessarily the same in each occurrence.

2. In order to prove Lemma 1 we require some auxiliary results due to N. G. de Bruijn [1] on the number \( \Psi(x, y) \) of integers \( n \leq x \) and free of prime factors \( > y \).

It is proved by de Bruijn [1] that we have

\[
\Psi(x, y) = O(xe^{-cy})
\]

and more precisely
\( (2) \quad \Psi(x, y) = x \rho(u) + O(1) \\
+ O(xu^{2}e^{-c\sqrt{\log y}}) + O\left(\frac{x \rho(u) \log(2+u)}{\log y}\right) \),

where \( x > 1, \ y \geq 2, \ u = (\log x)/\log y, \) and the function \( \rho(u) \) is defined by the following conditions:

\( (3) \quad \rho(u) = 0 \ (u < 0); \ \rho(u) = 1 \ (0 \leq u \leq 1); \)

\[ u \rho'(u) = -\rho(u-1) \ (u > 1); \ \rho(u) \text{ continuous for } u > 0. \]

**Lemma 2.** We have for \( t \geq t_{0} \geq 1 \)

\[ \rho(t) \leq \rho(t_{0})e^{-(t-t_{0})}, \]

so that

\[ \int_{t_{0}}^{\infty} \rho(u)du \leq \rho(t_{0}) \quad (t_{0} \geq 1). \]

This is stated and employed without proof in [4] as a lemma of N. C. Ankeny. By integrating by parts we deduce from (3) that for \( t \geq 1 \)

\[ t \rho(t) = \int_{0}^{t} \rho(u)du - \int_{0}^{t} \rho(u-1)du = \int_{t-1}^{t} \rho(u)du \leq \rho(t-1), \]

since \( \rho(u) \) decreases monotonously for \( u \geq 0. \) Hence

\[ \frac{\rho'(u)}{\rho(u)} = -\frac{\rho(u-1)}{u \rho(u)} \leq -1 \quad (u \geq 1) \]

and the result follows at once.


\[ H_{p} - \prod_{q < p} \left(1 - \frac{1}{q}\right)^{-1} \quad (p \leq z), \]

where in the product on the right-hand side \( q \) runs through the prime numbers less than \( p, \) and

\[ T_{p} = \sum_{m > w/\sqrt{p} \atop \rho(m) \leq p} \frac{1}{m} \quad (p \leq z). \]

Then we have \( |H_{p} - Z_{p}| \leq T_{p} \) and

\[ S \geq N \prod_{p \leq z} \left(1 - \frac{1}{p}\right) - N \sum_{p \leq z} \frac{T_{p}}{pH_{p}(H_{p} - T_{p})} - R, \]

(cf. [3] and [4]). Since it is well known that

\[ \prod_{p \leq z} \left(1 - \frac{1}{p}\right) = \frac{e^{-c}}{\log z} + O\left(\frac{1}{\log^{2} z}\right), \]
$C$ being the Euler constant, it remains only to evaluate the middle term on the right-hand side of the above inequality for $S$.

By partial summation we have

$$T_p = \sum_{m > \sqrt{p}} \frac{\Psi(m, p)}{m^2} + O(N^{-3/4}).$$

We find easily that

$$T_p = O\left(\frac{1}{\log^2 N}\right)$$

for every $p \leq \exp(\log N)^{3/2}$, on taking account of (1). For $\exp(\log N)^{3/2} < p \leq z$ we have

$$T_p = \sum_{\sqrt{p} < m \leq \exp(\log N)^{2}} \frac{\Psi(m, p)}{m^2} + O\left(\frac{1}{\log^2 N}\right),$$

where, by (2),

$$\sum_{\sqrt{p} < m \leq \exp(\log N)^{2}} \frac{\Psi(m, p)}{m^2} = \sum_{\sqrt{p} < m \leq \exp(\log N)^{2}} \frac{1}{m} \rho\left(\frac{\log m}{\log p}\right)\left(1 + O\left(\frac{\log\log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right),$$

and this is equal to

$$\left(\int_{\sqrt{p}}^{\infty} \frac{1}{x} \rho\left(\frac{\log x}{\log p}\right) dx + O(N^{-3/4})\right)\left(1 + O\left(\frac{\log\log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right).$$

Hence

$$T_p = \log p \int_{\sqrt{p}}^{\infty} \rho(u) du \left(1 + O\left(\frac{\log\log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right)$$

for $\exp(\log N)^{3/2} < p \leq z$.

Put

$$I_p = \int_{\sqrt{p}}^{\infty} \rho(u) du \quad (p \leq z).$$

Then it follows immediately from the above results that

\[
\frac{T_p}{pH_p(H_p - T_p)} = e^{-c} \sum_{\exp(\log N)^{3/2} < p \leq z} \frac{1}{p \log p} \frac{I_p}{e^c - I_p} + O\left(\frac{\log\log N}{(\log N)^{4/3}}\right).
\]
For $p$ in the interval $(2N)^{\frac{1}{\nu-1}} < p \leq (2N)^{\frac{1}{\nu}} (\nu \geq 4)$ we have, by Lemma 2,

$$I_p \leq \rho \left( \frac{\log (\omega/\sqrt{p})}{\log p} \right) \leq \rho(t_\nu),$$

where we have put

$$t_\nu = \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2}.$$

Therefore

$$\sum_{\exp(\log N)^{\frac{3}{4}} < p \leq z} \frac{1}{\rho \log p} \frac{I_p}{e^\epsilon - I_p} \leq \sum_{\nu=4}^{\infty} \frac{1}{\nu} \sum_{\nu=4}^{\infty} \frac{1}{\rho \log p} \frac{\rho(t_\nu)}{e^\epsilon - \rho(t_\nu)} = \frac{1}{\log N} \sum_{\nu=4}^{\infty} (\nu + 1) \log \frac{\nu}{\nu-1} \frac{\rho(t_\nu)}{e^\epsilon - \rho(t_\nu)} + O\left( \frac{1}{\log N} \right).$$

Here we used the relation

$$\sum_{p \leq x} \frac{1}{p \log p} = \log \log x + c_1 + O\left( \frac{1}{\log x} \right),$$

$c_1$ being a constant. Hence

$$\sum_{p \leq x} \frac{T_p}{p H_p (H_p - T_p)} \leq \frac{e^{-c}}{\log N} \sum_{\nu=4}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} \frac{\rho(t_\nu)}{e^\epsilon - \rho(t_\nu)} + O\left( \frac{\log \log N}{\log N} \right).$$

Now, by the definition of $\rho(u)$, we have

$$\rho(u) = 1 - \log u \quad (1 \leq u \leq 2).$$

If we take $\epsilon = 10^{-4}$, then we find that

$$\rho(t_4) = \rho(1.4996) < 0.5949,$$

$$\rho(t_5) = \rho(1.9995) < 0.3072,$$

so that

$$5 \log \frac{5}{4} \frac{\rho(t_4)}{e^\epsilon - \rho(t_4)} < 0.5597,$$

and
Theorem Concerning the Distribution of Almost Primes

\[ \sum_{\nu \geq 5}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} \frac{\rho(t)}{e^C - \rho(t)} \leq 6 \log \frac{6}{5} \frac{\rho(t_5)}{e^C - \rho(t_5)} \frac{1}{1 - e^{-0.4999}} < 0.5807 , \]

on appealing to Lemma 2. We thus have proved that

\[ \sum_{p \leq z} \frac{T_p}{p H_p (H_p - T_p)} < 1.1404 e^{-C} \frac{1}{\log N} + O(\frac{(\log \log N)^2}{(\log N)^{4/3}}) \]

and this completes the proof of Lemma 1 since

\[ (4 - 1.1404) e^{-C} > 1.6055 . \]

4. Let \( q \) be any prime number in the interval \( z < q \leq z^2 \), where, as before, \( z = (2N)^{1/3} \). We next estimate the number \( S(q) \) of those integers \( n \) in \( M < n \leq M + N \) which are multiples of \( q \) and are not divisible by any prime number \( p \leq z \). We have by the 'upper' sieve of A. Selberg (cf. [5])

\[ S(q) \leq \frac{N}{qZ} + R_z , \]

where

\[ \sum_{1 \leq m \leq z} \frac{\mu^2(m)}{\phi(m)} \]

and

\[ R_z = O \left( \frac{z^2}{Z^2} \right) . \]

It is easily verified that

\[ Z \geq \sum_{1 \leq m \leq z} \frac{1}{m} = \log z + O(1) , \]

and therefore

\[ S(q) \leq \frac{4N}{q \log N} + O(\frac{N}{q \log^2 N}) . \]

Lemma 3. Let \( U \) denote the number of those integers \( n \) in \( M < n \leq M + N \) which are divisible by no primes \( p \leq z \), by at most two primes \( q \) with \( z < q \leq z^2 \), and by no integers of the form \( q^2 \), \( q \) being a prime in \( z < q \leq z^2 \). Then, for all sufficiently large \( N \), we have

\[ U > 0.6811 \frac{N}{\log N} . \]
Let $N$ be a sufficiently large positive number. The number of those integers $n$ in $M<n\leqq M+N$ which are not divisible by any prime $p\leqq z$ and are divisible by some $q^2$; where $q$ is a prime in $z<q\leqq z^2$, does not exceed

$$\sum_{z<q<z^2} \left( \left\lfloor \frac{M+N}{q^2} \right\rfloor - \left\lfloor \frac{M}{q^2} \right\rfloor \right) - O(N^{\frac{3}{2}}).$$

Now, the number of those integers $n$ with $M<n\leqq M+N$ which are not divisible by any prime $p\leqq z$ and are divisible by at least three (distinct) primes $q$ in $z<q\leqq z^2$ is, by (4), not greater than

$$\frac{1}{3} \sum_{z<q<z^2} S(q) \leqq \frac{4\log 2}{3} \frac{N}{\log N} + O \left( \frac{N}{\log^2 N} \right).$$

It thus follows from Lemma 1 that

$$U > \left( 1.6054 - \frac{4\log 2}{3} \right) \frac{N}{\log N} + O \left( \frac{N}{\log^2 N} \right),$$

which proves our lemma since $(4/3)\log 2 < 0.9242$.

5. We can now conclude the proof of our theorem. Let $x$ be a sufficiently large positive real number and put

$$M = \lfloor x \rfloor, \quad N = \lfloor x^{1/k} \rfloor.$$

Then, by Lemma 3, there exists at least one integer $n$ in the interval $M<n\leqq M+N$, i.e. in the interval

$$x < n \leqq x + x^{1/k},$$

such that it is not divisible by any prime $p\leqq (2N)^{\frac{1}{2}}$ and is divisible by at most two primes $q$ in $(2N)^{\frac{1}{2}} < q \leqq (2N)^{\frac{1}{2}}$ but not divisible by the squares of these $q$, where

$$(2N)^{\frac{1}{2}} > (2(x^{1/k}-1))^k > x + x^{1/k}$$

since $k \geqq 2$. Therefore, according as $n$ has no, one or two prime factors $q$ in $(2N)^{\frac{1}{2}} < q \leqq (2N)^{\frac{1}{2}}$ it has at most $2k-1$, $2k-1$ or $2k-2$ additional prime factors. Hence the total number of prime factors of $n$ is at most $2k$, i.e. $\Omega(n) \leqq 2k$. This completes the proof of the theorem.

References

On a Theorem Concerning the Distribution of Almost Primes


Department of Mathematics,
Hokkaido University

(Received July 19, 1963)