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ON A THEOREM CONCERNING
THE DISTRIBUTION OF ALMOST PRIMES

By

Saburō UCHIYAMA

By an *almost prime* is meant a positive rational integer the number of prime factors of which is bounded by a certain constant. Let us denote by \( \Omega(n) \) the total number of prime factors of a positive integer \( n \). In 1920 Viggo Brun [2] elaborated an elementary method of the sieve of Eratosthenes to prove that for all sufficiently large \( x \) there exists at least one integer \( n \) with \( \Omega(n) \leq 11 \) in the interval \( x \leq n \leq x + x^{\frac{1}{2}} \). Quite recently W. E. Mientka [4] improved this result of Brun, showing that for all large \( x \) there exists at least one integer \( n \) with \( \Omega(n) \leq 9 \) in the interval \( x \leq n \leq x + x^{\frac{1}{2}} \). To establish this Mientka makes use of the sieve method due to A. Selberg instead of Brun's method (cf. [3] and [4]). By refining the argument of Mientka [4] we can further improve his result. Indeed, we shall prove in this paper the following theorem.

**Theorem.** Let \( k \geq 2 \) be a fixed integer. Then, for all sufficiently large \( x \), there exists at least one integer \( n \) with \( \Omega(n) \leq 2k \) in the interval \( x < n \leq x + x^{1/k} \).

Thus, in particular, if \( k = 2 \) then for all large \( x \) the interval \( x < n \leq x + x^{\frac{1}{2}} \) always contains an integer \( n \) such that \( \Omega(n) \leq 4 \). Of course, the restriction in the theorem that \( k \) be integral may be relaxed without essential changes in the result.

Let us mention that the existence of a prime number \( p \) in the interval \( x < p \leq x + x^{1/k} \) for all large \( x \) could not be deduced, as is well known, even from the Riemann hypothesis if only \( k = 2 \).

**Note.** It is possible to generalize our theorem presented above so as to concern with the distribution of almost primes in an arithmetic progression. Thus, if \( a \) and \( b \) are integers such that \( a \geq 1 \), \( 0 \leq b \leq a - 1 \), \((a, b) = 1\), then we can prove the existence of an integer \( n \) satisfying

\[
x < n \leq x + x^{1/k}, \quad n \equiv b \pmod{a}, \quad \Omega(n) \leq 2k,
\]

provided that \( x \) be sufficiently large, \( k \geq 2 \) being a fixed integer. Here, in particular, in the case of \( k = 2 \), the inequality \( \Omega(n) \leq 4 \) may be replaced by \( \Omega(n) \leq 3 \): this result is apparently stronger than the above theorem for the
corresponding case. Proof is similar to that of our theorem but somewhat more complicated arguments are needed.

1. Let $M>0$ and $N>1$ be integers and let $z \geq 2$ and $w > 0$ be any real numbers such that $w^2 \geq z$. We denote by $S$ the number of those integers $n$ in the interval $M < n \leq M + N$ which are not divisible by any prime number $p \leq z$. Then, by making use of the ‘lower’ sieve of A. Selberg (cf. [3] and [6]) we can show that

$$S \geq (1 - Q)N - R_1,$$

where

$$Q = \sum_{p \leq z} \frac{1}{pZ_p} \text{ with } Z_p = \sum_{1 \leq m < w/\sqrt{p}} \frac{\mu^2(m)}{\phi(m)}$$

and

$$R_1 = O \left( w^2 \sum_{p \leq z} \frac{1}{pZ_p^z} \right).$$

Here $g(1)=1$ and for $m > 1$ $g(m)$ denotes the greatest prime divisor of $m$, and the $O$-constant for $R_1$ is absolute. It will be shown later that $Z_p > c \log p$ for all $p \leq z$, where $c > 0$ is a constant, so that we have $R_1 = O(w^2)$.

Now we take

$$z = (2N)^{\frac{1}{4}}, \quad w = (2N)^{\frac{1}{2} - \epsilon},$$

where $0 < \epsilon < \frac{1}{4}$. If we fix $\epsilon$ sufficiently small then there holds the following

**Lemma 1.** For all sufficiently large $N$ we have

$$S > 1.6054 \frac{N}{\log N}.$$

Our proof of Lemma 1 runs essentially on the same lines as in [4]; we shall give an outline of the proof of this lemma in § 3.

Throughout in the following the constants implied in the symbol $O$ are all absolute (apart from the possible dependence on the parameter $\epsilon$), and $c$ represents positive constants not necessarily the same in each occurrence.

2. In order to prove Lemma 1 we require some auxiliary results due to N. G. de Bruijn [1] on the number $\Psi(x, y)$ of integers $n \leq x$ and free of prime factors $> y$.

It is proved by de Bruijn [1] that we have

$$\Psi(x, y) = O(xe^{-cy})$$

and more precisely
\( \Psi(x, y) = x\rho(u) + O(1) \)
\[ + O(xue^{-e\sqrt{\log y}}) + O\left(\frac{xp(u)\log(2+u)}{\log y}\right), \]
where \( x > 1, y \geq 2, \ u = (\log x)/\log y, \) and the function \( \rho(u) \) is defined by the following conditions:
\[
\rho(u) = 0 \quad (u < 0); \quad \rho(u) = 1 \quad (0 \leq u \leq 1); \\
u\rho'(u) = -\rho(u-1) \quad (u > 1); \quad \rho(u) \text{ continuous for } u > 0.
\]

**Lemma 2.** We have for \( t \geq t_0 \geq 1 \)
\[
\rho(t) \leq \rho(t_0) e^{-\left(t - t_0\right)},
\]
so that
\[
\int_{t_0}^{\infty} \rho(u) du \leq \rho(t_0) \quad (t_0 \geq 1).
\]

This is stated and employed without proof in [4] as a lemma of N. C. Ankeny. By integrating by parts we deduce from (3) that for \( t \geq 1 \)
\[
t\rho(t) = \int_{0}^{t} \rho(u) du - \int_{0}^{t} \rho(u-1) du = \int_{t-1}^{t} \rho(u) du \leq \rho(t-1),
\]
since \( \rho(u) \) decreases monotonously for \( u \geq 0 \). Hence
\[
\frac{\rho'(u)}{\rho(u)} = -\frac{\rho(u-1)}{u\rho(u)} \leq -1 \quad (u \geq 1)
\]
and the result follows at once.

\[
H_p = \prod_{q < p} \left(1 - \frac{1}{q}\right)^{-1} \quad (p \leq z),
\]
where in the product on the right-hand side \( q \) runs through the prime numbers less than \( p \), and
\[
T_p = \sum_{\substack{m > w/\sqrt{p} \\ \nu(m) \leq p}} \frac{1}{m} \quad (p \leq z).
\]
Then we have \( |H_p - Z_p| \leq T_p \) and
\[
S \geq N \prod_{p \leq z} \left(1 - \frac{1}{p}\right) - N \sum_{p \leq z} \frac{T_p}{pH_p(H_p - T_p)} - R,
\]
(cf. [3] and [4]). Since it is well known that
\[
\prod_{p \leq z} \left(1 - \frac{1}{p}\right) = \frac{e^{-c}}{\log z} + O\left(\frac{1}{\log^2 z}\right),
\]
C being the Euler constant, it remains only to evaluate the middle term on the right-hand side of the above inequality for $S$.

By partial summation we have

$$T_p = \sum_{m \geq w/\sqrt{p}} \frac{\Psi(m, p)}{m^2} + O(N^{-\frac{3}{2}}).$$

We find easily that

$$T_p = O\left(\frac{1}{\log^2 N}\right)$$

for every $p \leq \exp(\log N)^\frac{3}{2}$, on taking account of (1). For $\exp(\log N)^\frac{3}{2} < p \leq z$ we have

$$T_p = \sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{\Psi(m, p)}{m^2} + O\left(\frac{1}{\log^2 N}\right),$$

where, by (2),

$$\sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{\Psi(m, p)}{m^2} = \sum_{w/\sqrt{p} < m \leq \exp(\log N)^2} \frac{1}{m} \rho\left(\frac{\log m}{\log p}\right) \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right),$$

and this is equal to

$$\left(\int_{w/\sqrt{p}}^\infty \frac{1}{x} \rho\left(\frac{\log x}{\log p}\right) dx + O(N^{-\frac{3}{2}})\right) \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right).$$

Hence

$$T_p = \log p \int_{\log(\log p)/\log p}^\infty \rho(u) du \left(1 + O\left(\frac{\log \log N}{\log p}\right)\right) + O\left(\frac{1}{\log^2 N}\right)$$

for $\exp(\log N)^\frac{3}{2} < p \leq z$.

Put

$$I_p = \int_{\log(\log p)/\log p}^\infty \rho(u) du \quad (p \leq z).$$

Then it follows immediately from the above results that

$$\sum_{p \leq z} \frac{T_p}{pH_p(H_p - T_p)} = e^{-C} \sum_{\exp(\log N)^\frac{3}{2} < p \leq z} \frac{1}{p \log p} \frac{I_p}{e^C - I_p} + O\left(\frac{\log \log N)^3}{(\log N)^{4/3}}\right).$$
For $p$ in the interval $(2^N)^{\frac{1}{\nu+1}} < p \leq (2^N)^{\frac{1}{\nu}} (\nu \geq 4)$ we have, by Lemma 2,

$$I_p \leq \rho \left( \frac{\log (w/\sqrt{p})}{\log p} \right) \leq \rho(t_v),$$

where we have put

$$t_v = \left( \frac{1}{2} - \varepsilon \right) \nu - \frac{1}{2}.$$

Therefore

$$\sum_{\exp(\log N)^{\frac{3}{v}} < p \leq z} \frac{1}{\rho \log p} \frac{I_p}{e^c - I_p} \leq \frac{\rho(t_v)}{e^c - \rho(t_v)}$$

$$= \frac{1}{\log N} \sum_{v=4}^{\infty} (v+1) \log \frac{v+1}{v} \frac{\rho(t_v)}{e^c - \rho(t_v)} + O \left( \frac{1}{(\log N)^{1/3}} \right).$$

Here we used the relation

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c_i + O \left( \frac{1}{\log x} \right),$$

c_i being a constant. Hence

$$\sum_{v=4}^{\infty} \frac{T_p}{pH_p (H_p - T_p)} \leq \frac{e^{-c}}{\log N} \sum_{v=4}^{\infty} (v+1) \log \frac{v+1}{v} \frac{\rho(t_v)}{e^c - \rho(t_v)} + O \left( \frac{(\log \log N)^2}{(\log N)^{1/3}} \right).$$

Now, by the definition of $\rho(u)$, we have

$$\rho(u) = 1 - \log u \quad (1 \leq u \leq 2).$$

If we take $\varepsilon = 10^{-4}$, then we find that

$$\rho(t_4) = \rho(1.4996) < 0.5949,$$

$$\rho(t_5) = \rho(1.9996) < 0.3072,$$

so that

$$5 \log \frac{5}{4} \frac{\rho(t_4)}{e^c - \rho(t_4)} < 0.5597,$$

and
\[ \sum_{\nu=5}^{\infty} \frac{\nu+1}{\nu} \log \frac{\nu+1}{\nu} \frac{\rho(t_{\nu})}{e^{\nu} - \rho(t_{\nu})} \leq 6 \log \frac{6}{5} \frac{\rho(t_{5})}{e^{C} - \rho(t_{5})} \frac{1}{1 - e^{-0.4999}} < 0.5807, \]

on appealing to Lemma 2. We thus have proved that

\[ \sum_{p \leq z} \frac{T_{p}}{pH_{p}(H_{p} - T_{p})} < 1.1404e^{-c} \frac{1}{\log N} + O \left( \frac{(\log \log N)^{2}}{(\log N)^{4/3}} \right) \]

and this completes the proof of Lemma 1 since

\[ (4 - 1.1404)e^{-c} > 1.6055. \]

4. Let \( q \) be any prime number in the interval \( z < q \leq z^{2} \), where, as before, \( z = (2N)^{1/3} \). We next estimate the number \( S(q) \) of those integers \( n \) in \( M < n \leq M + N \) which are multiples of \( q \) and are not divisible by any prime number \( p \leq z \). We have by the 'upper' sieve of A. Selberg (cf. [5])

\[ S(q) \leq \frac{N}{qZ} + R_{s}, \]

where

\[ Z = \sum_{1 \leq m \leq z} \frac{\mu^{2}(m)}{\phi(m)} \]

and

\[ R_{s} = O \left( \frac{z^{2}}{Z^{2}} \right). \]

It is easily verified that

\[ Z \geq \sum_{1 \leq m \leq z} \frac{1}{m} = \log z + O(1), \]

and therefore

\[ S(q) \leq \frac{4N}{q \log N} + O \left( \frac{N}{q \log^{2} N} \right). \]

**Lemma 3.** Let \( U \) denote the number of those integers \( n \) in \( M < n \leq M + N \) which are divisible by no primes \( p \leq z \), by at most two primes \( q \) with \( z < q \leq z^{2} \), and by no integers of the form \( q^2 \), \( q \) being a prime in \( z < q \leq z^{2} \). Then, for all sufficiently large \( N \), we have

\[ U > 0.6811 \frac{N}{\log N}. \]
Let $N$ be a sufficiently large positive number. The number of integers $n$ in $M<n\leqq M+N$ which are not divisible by any prime $p\leqq z$ and are divisible by some $q^2$, where $q$ is a prime in $z<q\leqq z^2$, does not exceed

$$\sum_{z<q\leqq z^2} \left( \left\lfloor \frac{M+N}{q^2} \right\rfloor - \left\lfloor \frac{M}{q^2} \right\rfloor \right) - O(N^{\frac{3}{2}}).$$

Now, the number of those integers $n$ with $M<n\leqq M+N$ which are not divisible by any prime $p\leqq z$ and are divisible by at least three (distinct) primes $q$ in $z<q\leqq z^2$ is, by (4), not greater than

$$\frac{1}{3} \sum_{z<q\leqq z^2} S(q) \leqq \frac{4\log 2}{3} \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right).$$

It thus follows from Lemma 1 that

$$U > \left(1.6054 - \frac{4\log 2}{3}\right) \frac{N}{\log N} + O\left(\frac{N}{\log^2 N}\right),$$

which proves our lemma since $(4/3)\log 2 < 0.9242$.

5. We can now conclude the proof of our theorem. Let $x$ be a sufficiently large positive real number and put

$$M = \lfloor x \rfloor, \quad N = \lfloor x^{1/k} \rfloor.$$

Then, by Lemma 3, there exists at least one integer $n$ in the interval $M<n\leqq M+N$, i.e. in the interval

$$x<n\leqq x+x^{1/k},$$

such that it is not divisible by any prime $p\leqq (2N)^{\frac{1}{2}}$ and is divisible by at most two primes $q$ in $(2N)^{\frac{1}{2}}<q\leqq (2N)^{\frac{3}{2}}$ but not divisible by the squares of these $q$, where

$$(2N)^k > (2(x^{1/k}-1))^k > x+x^{1/k}$$

since $k\geqq 2$. Therefore, according as $n$ has no, one or two prime factors $q$ in $(2N)^{\frac{1}{2}}<q\leqq (2N)^{\frac{3}{2}}$ it has at most $2k-1$, $2k-1$ or $2k-2$ additional prime factors. Hence the total number of prime factors of $n$ is at most $2k$, i.e. $\Omega(n)\leqq 2k$. This completes the proof of the theorem.

References


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