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REMARKS ON COMPLETENESS OF CONTINUOUS FUNCTION LATTICE

By

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Let E be an arbitrary topological space and $C(E)$ be a vector lattice of all real valued continuous functions on E . In general the lattice $C(E)$ is neither conditionally complete¹⁾ nor conditionally σ -complete²⁾. H. Nakano shows in [1] that a sufficient condition for $C(E)$ to be conditionally σ -complete (conditionally complete) is that E is σ -universal (universal), that is, every open F_σ -set has an open closure (every open set has an open closure) (cf. [2] Chap. VII, Theorem 41.1, Theorem 41.4). Under the assumption that E is normal (completely regular) σ -universality (universality) of E is a necessary condition for $C(E)$ to be conditionally σ -complete (conditionally complete). L. Gillman and M. Jerison in their book [3] show that for a completely regular space E the necessary and sufficient condition for $C(E)$ to be conditionally σ -complete is that E is *basically disconnected*, that is, every cozero-set³⁾ has an open closure ([3] p. 51, 3N). In this note we shall remark the necessary and sufficient topological condition for $C(E)$ on an arbitrary topological space E to be conditionally σ -complete or conditionally complete.

In the sequel a cozero-set P of $f \in C(E)$ will be denoted by $P(f)$; $P(f) = \{x \mid f(x) \neq 0\} = \{x \mid |f|(x) > 0\}$.

Theorem 1. *$C(E)$ is a conditionally σ -complete lattice if and only if the following two conditions are satisfied*

a) *there exists the smallest open-closed set $U(P)$ containing P for any cozero-set P .*

b) *if $P_1 \cap P_2 = \emptyset$ for two cozero-sets P_1 and P_2 , then $U(P_1) \cap U(P_2) = \emptyset$.*

Proof. Suppose $C(E)$ is conditionally σ -complete and P is a cozero-set of some $f \in C(E)$, $P = P(f)$, then by the conditional σ -completeness of $C(E)$ f gives the orthogonal decomposition of the constant function $\mathbf{1}$ as follows

$$\mathbf{1} = [f]\mathbf{1} + [f]^\perp\mathbf{1}$$

- 1) every family with an upper bound in $C(E)$ has a supremum in $C(E)$.
- 2) every countable family with an upper bound in $C(E)$ has a supremum in $C(E)$.
- 3) $\{x \mid f(x) = 0\}$ is a zero-set of $f \in C(E)$, cozero-set is a complement of a zero-set.

where $[f]1 = \bigcup_{n=1}^{\infty} (1 \wedge n|f|)$ and $[f]^{\perp}1 = 1 - [f]1$. $[f]1 \wedge [f]^{\perp}1 = 0$ implies that $[f]1$ is a characteristic function χ_U for some open-closed set U , and $|f| \wedge [f]^{\perp}1 = 0$ implies $U \supset P(f)$. The fact that $[f]1 \wedge |g| = 0$ for all $g \in C(E)$ such that $|f| \wedge |g| = 0$ shows that U is the smallest open-closed set containing $P(f)$. If $P_1 \wedge P_2 = \phi$ for cozero-sets $P_1 = P(f_1)$ and $P_2 = P(f_2)$, then by $|f_1| \wedge |f_2| = 0$ we have $[f_1]1 \wedge [f_2]1 = 0$, namely $U(P_1)$ and $U(P_2)$ are disjoint from the above argument $\chi_{U(P_1)} = [f_1]1$ and $\chi_{U(P_2)} = [f_2]1$.

Conversely, let a) and b) satisfied. To prove the conditional σ -completeness of $C(E)$ it is sufficient to show the existence of an infimum $\bigcap_{n=1}^{\infty} f_n$ for any sequence $\{f_n\}$ of non-negative continuous functions. If we put $E_{\alpha}^{(n)} = \{x \mid f_n(x) < \alpha\}$ and $E_{\alpha} = \bigcup_{n=1}^{\infty} E_{\alpha}^{(n)}$ for all $\alpha > 0$, then obviously E_{α} is a cozero-set of a continuous function $g_{\alpha} = \sum_{n=1}^{\infty} \frac{1}{2^n} (\alpha 1 - f_n)^+$ ⁴⁾. Hence from a) we can find the smallest open-closed set U_{α} containing E_{α} ($\alpha > 0$). We have then

$$(1) \quad U_{\alpha} \supset U_{\beta} \quad (\alpha > \beta > 0), \quad (2) \quad \bigcup_{\alpha > 0} U_{\alpha} = E.$$

If we put $f_0(x) = \inf_{x \in U_{\alpha}} \alpha$ ($x \in E$), then by (2) f_0 is a non-negative real valued function on E , and by (1) we see

$$\{x \mid f_0(x) < \alpha\} = \bigcup_{\alpha > \beta > 0} U_{\beta}, \quad \{x \mid f_0(x) \leq \alpha\} = \bigcap_{\beta > \alpha} U_{\beta} \quad (\alpha > 0).$$

This implies the continuity of f_0 . Since $E_{\alpha}^{(n)} \subset E_{\alpha} \subset U_{\alpha} \subset \{x \mid f_0(x) \leq \alpha\}$ ($n = 1, 2, \dots; \alpha > 0$), we have $f_n \geq f_0$ ($n = 1, 2, \dots$), that is, f_0 is a lower bound of $\{f_n\}$. And if $f_n \geq g \geq 0$ ($n = 1, 2, \dots$), for some $g \in C(E)$, then we have $E_{\alpha} \wedge \{x \mid g(x) > \alpha\} = \phi$ ($\alpha > 0$). Hence from the assumption b) we see $U_{\alpha} \wedge \{x \mid g(x) > \alpha\} = \phi$ ($\alpha > 0$), and so $\{x \mid f_0(x) < \alpha\} \subset U_{\alpha} \subset \{x \mid g(x) \leq \alpha\}$ ($\alpha > 0$). hence $g \geq f_0$. Therefore f_0 is an infimum of $\{f_n\}$.

Similarly it is easy to give a necessary and sufficient condition for $C(E)$ to be a conditionally complete lattice.

Let $C(E)$ be conditionally complete, then E satisfies the following condition c) in addition to a) and b) in Theorem 1;

c) *all of open-closed sets of E constitutes a complete lattice.*

In fact, let U_{λ} ($\lambda \in A$) be any system of open-closed sets in E , then since $\chi_{U_{\lambda}}$ ($\lambda \in A$) has an infimum $f = \bigcap_{\lambda \in A} \chi_{U_{\lambda}}$ and a supremum $g = \bigcup_{\lambda \in A} \chi_{U_{\lambda}}$ in $C(E)$, easily it is shown that $U(P(f))$ and $U(P(g))$ are respectively an infimum and a supremum of U_{λ} ($\lambda \in A$) in all of open-closed sets of E . Conversely, suppose E

4) the positive part of $\alpha 1 - f_n$; $(\alpha 1 - f_n) \cup 0$.

satisfies a), b) and c), to show the conditional completeness of $C(E)$ only a slight modification of the definition of U_α in the proof of Theorem 1 is necessary. Namely, for any system $f_\lambda \geq 0$ ($\lambda \in A$) of non-negative continuous functions, putting $E_\alpha^{(\lambda)} = \{x | f_\lambda(x) < \alpha\}$ and $E_\alpha = \bigcup_{\lambda \in A} E_\alpha^{(\lambda)}$ ($\alpha > 0$), by the condition c) we can define U_α as the smallest open-closed set containing E_α ; U_α is the supremum of $U_\alpha^{(\lambda)}$ ($\lambda \in A$) in all of open-closed sets, where $U_\alpha^{(\lambda)}$ is the smallest open-closed set containing $E_\alpha^{(\lambda)}$.

Theorem 2. $C(E)$ is a conditionally complete lattice if and only if E satisfies the conditions a), b) and c).

Finally we shall remark an extension theorem. If we replace cozero-sets in a) and b) by open F_σ -sets;

a') there exists the smallest open-closed set $U(F)$ containing F for any open F_σ -set F ,

b') if $F_1 \cap F_2 = \emptyset$ for two open F_σ -sets F_1 and F_2 , then $U(F_1) \cap U(F_2) = \emptyset$, then we have a purely topological sufficient condition for $C(E)$ to be conditionally σ -complete. Obviously it is weaker than σ -universality in [1].

Under the assumptions a') and b') we have a following extension theorem which is a slight generalization of Theorem 41.2 of [2].

Suppose E satisfies a') and b'), then a continuous function φ defined on an open F_σ -set F has a continuous extension ψ over E , provided ψ may take values $+\infty$ and $-\infty$.

To prove this it is sufficient to show that φ has a continuous extension over $U(F)$. Putting $F_\alpha = \{x | \varphi(x) < \alpha\}$ ($+\infty > \alpha > -\infty$) since F is an open F_σ -set in E , F_α is also an open F_σ -set in E . Hence by a') we can find the smallest open-closed set U_α containing F_α for all α , and $\{U_\alpha\}$ has the properties $U_\alpha \supset U_\beta$ ($\alpha > \beta$) and $U(F) \supset \bigcup_{+\infty > \alpha > -\infty} U_\alpha \supset F$. Similarly to the latter part of the proof of Theorem 1 we define $\psi(x) = \inf_{x \in U_\alpha} \alpha$ ($x \in \bigcup_{+\infty > \alpha > -\infty} U_\alpha$), $\psi(x) = +\infty$ ($x \in U(F) - \bigcup_{+\infty > \alpha > -\infty} U_\alpha$), then we see easily ψ is a continuous function on $U(F)$ and $\varphi \geq \psi$ on F . Since by b') $\{x | \varphi(x) > \alpha\} \cap F_\alpha = \emptyset$ implies $\{x | \varphi(x) > \alpha\} \cap U_\alpha = \emptyset$, we have $\psi \geq \varphi$ on F .

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