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<thead>
<tr>
<th>Item</th>
<th>Details</th>
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</thead>
<tbody>
<tr>
<td>Title</td>
<td>REMARKS ON COMPLETENESS OF CONTINUOUS FUNCTION LATTICE</td>
</tr>
<tr>
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REMARKS ON COMPLETENESS OF CONTINUOUS FUNCTION LATTICE

By

Takashi ITÔ

Let $E$ be an arbitrary topological space and $C(E)$ be a vector lattice of all real valued continuous functions on $E$. In general the lattice $C(E)$ is neither conditionally complete\(^1\) nor conditionally $\sigma$-complete\(^2\). H. Nakano shows in [1] that a sufficient condition for $C(E)$ to be conditionally $\sigma$-complete (conditionally complete) is that $E$ is $\sigma$-universal (universal), that is, every open $F_\omega$-set has an open closure (every open set has an open closure) (cf. [2] Chap. VII, Theorem 41.1, Theorem 41.4). Under the assumption that $E$ is normal (completely regular) $\sigma$-universality (universality) of $E$ is a necessary condition for $C(E)$ to be conditionally $\sigma$-complete (conditionally complete). L. Gillman and M. Jerison in their book [3] show that for a completely regular space $E$ the necessary and sufficient condition for $C(E)$ to be conditionally $\sigma$-complete is that $E$ is basically disconnected, that is, every cozero-set\(^3\) has an open closure ([3] p. 51, 3N). In this note we shall remark the necessary and sufficient topological condition for $C(E)$ on an arbitrary topological space $E$ to be conditionally $\sigma$-complete or conditionally complete.

In the sequel a cozero-set $P$ of $f\in C(E)$ will be denoted by $P(f)$; $P(f) = \{x|f(x)\neq 0\} = \{x||f|(x)>0\}$.

**Theorem 1.** $C(E)$ is a conditionally $\sigma$-complete lattice if and only if the following two conditions are satisfied

a) there exists the smallest open-closed set $U(P)$ containing $P$ for any cozero-set $P$.

b) if $P_1 \cap P_2 = \phi$ for two cozero-sets $P_1$ and $P_2$, then $U(P_1) \cap U(P_2) = \phi$.

**Proof.** Suppose $C(E)$ is conditionally $\sigma$-complete and $P$ is a cozero-set of some $f\in C(E)$, $P = P(f)$, then by the conditional $\sigma$-completeness of $C(E)$ $f$ gives the orthogonal decomposition of the constant function 1 as follows

\[
1 = [f]1 + [f]^\perp 1
\]

1) every family with an upper bound in $C(E)$ has a supremum in $C(E)$.
2) every countable family with an upper bound in $C(E)$ has a supremum in $C(E)$.
3) $\{x|f(x)=0\}$ is a zero-set of $f\in C(E)$, cozero-set is a complement of a zero-set.
where $[f]1=\bigcup_{n=1}^{\infty}(1_{\cap}n|f|)$ and $[f]1=1-[f]1$. $[f]1\cup[f]1=0$ implies that $[f]1$ is a characteristic function $\chi_U$ for some open-closed set $U$, and $|f|1\cup[f]1=0$ implies $U\supset P(f)$. The fact that $[f]1\cup|g|=0$ for all $g\in C(E)$ such that $|f|1\cup|g|=0$ shows that $U$ is the smallest open-closed set containing $P(f)$. If $P_1\cap P_2=\emptyset$ for cozero-sets $P_i=P(f_i)$ and $P_2=P(f_2)$, then by $|f_i|1\cup|f_i|=0$ we have $[f_i]1\cup[f_i]1=0$, namely $U(P_i)$ and $U(P_2)$ are disjoint from the above argument $\chi_{U(P_i)}=[f_i]1$ and $\chi_{U(P_2)}=[f_2]1$.

Conversely, let a) and b) satisfied. To prove the conditional $\sigma$-completeness of $C(E)$ it is sufficient to show the existence of an infimum $\bigcap_{n=1}^{\infty}f_n$ for any sequence $\{f_n\}$ of non-negative continuous functions. If we put

$$E_{\alpha}^{(n)}=\{x|f_n(x)<\alpha\} \quad \text{and} \quad E_{\alpha}^{(\alpha)}=\bigcup_{n=1}^{\infty}E_{\alpha}^{(n)}$$

for all $\alpha>0$, then obviously $E_{\alpha}$ is a cozero-set of a continuous function $g_{\alpha}=\sum_{n=1}^{\infty}\frac{1}{2^{n}}(\alpha-1-f_n)^{+}$. Hence from a) we can find the smallest open-closed set $U_{\alpha}$ containing $E_{\alpha}$ $(\alpha>0)$. We have then

$$1) \quad U_{\alpha}\supset U_\beta \quad (\alpha>\beta>0), \quad 2) \quad \bigcup_{\alpha>0}E_{\alpha}=E.$$

If we put $f_0(x)=\inf_{\alpha>0}x$ $(x\in E)$, then by (2) $f_0$ is a non-negative real valued function on $E$, and by (1) we see

$$\{x|f_0(x)<\alpha\}=\bigcup_{\alpha>\beta>0}U_{\beta}, \quad \{x|f_0(x)\leq\alpha\}=\bigcap_{\alpha>\beta>0}U_{\beta} \quad (\alpha>0).$$

This implies the continuity of $f_0$. Since $E_{\alpha}^{(n)}\subset E_{\alpha}\subset E_{\alpha}^{(\alpha)} \quad (n=1, 2, \cdots; \alpha>0)$, we have $f_n\geq f_0 \quad (n=1, 2, \cdots)$, that is, $f_0$ is a lower bound of $\{f_n\}$. And if $f_n\geq g\geq 0 \quad (n=1, 2, \cdots)$, for some $g\in C(E)$, then we have $E_{\alpha}^{(n)}\cap\{x|g(x)>\alpha\}=\phi \quad (\alpha>0)$. Hence from the assumption b) we see $U_{\alpha}^{(n)}\cap\{x|g(x)>\alpha\}=\phi \quad (\alpha>0)$, and so $\{x|f_0(x)<\alpha\}\subset U_{\alpha}\subset \{x|g(x)\leq\alpha\} \quad (\alpha>0)$, hence $g\geq f_0$. Therefore $f_0$ is an infimum of $\{f_n\}$.

Similarly it is easy to give a necessary and sufficient condition for $C(E)$ to be a conditionally complete lattice.

Let $C(E)$ be conditionally complete, then $E$ satisfies the following condition c) in addition to a) and b) in Theorem 1;

c) all of open-closed sets of $E$ constitutes a complete lattice.

In fact, let $U_1(\lambda\in\Lambda)$ be any system of open-closed sets in $E$, then since $\chi_{U_1}(\lambda\in\Lambda)$ has an infimum $f=\bigcap_{\lambda\in\Lambda}\chi_{U_1}$ and a supremum $g=\bigcup_{\lambda\in\Lambda}\chi_{U_1}$ in $C(E)$, easily it is shown that $U(P(f))$ and $U(P(g))$ are respectively an infimum and a supremum of $U_1(\lambda\in\Lambda)$ in all of open-closed sets of $E$. Conversely, suppose $E$
satisfies a), b) and c), to show the conditional completeness of $C(E)$ only a slight modification of the definition of $U_a$ in the proof of Theorem 1 is necessary. Namely, for any system $f_i \geq 0 \ (\lambda \in \Lambda)$ of non-negative continuous functions, putting $E_{\alpha}^{(\psi)} = \{ x | f_i(x) < \alpha \}$ and $E_a = \bigcup_{\alpha>0} E_{\alpha}^{(\psi)}$, by the condition c) we can define $U_a$ as the smallest open-closed set containing $E_a$; $U_a$ is the supremum of $U_{\alpha}^{(\psi)} \ (\lambda \in \Lambda)$ in all of open-closed sets, where $U_{\alpha}^{(\psi)}$ is the smallest open-closed set containing $E_{\alpha}^{(\psi)}$.

**Theorem 2.** $C(E)$ is a conditionally complete lattice if and only if $E$ satisfies the conditions a), b) and c).

Finally we shall remark an extension theorem. If we replace cozero-sets in a) and b) by open $F_{\gamma}$-sets:

a') there exists the smallest open-closed set $U(F)$ containing $F$ for any open $F_{\gamma}$-set $F$;

b') if $F_1 \cap F_2 = \phi$ for two open $F_{\gamma}$-sets $F_1$ and $F_2$, then $U(F_1) \cap U(F_2) = \phi$, then we have a purely topological sufficient condition for $C(E)$ to be conditionally $\sigma$-complete. Obviously it is weaker than $\sigma$-universality in [1].

Under the assumptions a') and b') we have a following extension theorem which is a slight generalization of Theorem 41.2 of [2].

**Suppose $E$ satisfies a') and b'), then a continuous function $\varphi$ defined on an open $F_{\gamma}$-set $F$ has a continuous extension $\bar{\varphi}$ over $E$, provided $\bar{\varphi}$ may take values $+\infty$ and $-\infty$.

To prove this it is sufficient to show that $\varphi$ has a continuous extension over $U(F)$. Putting $F_{\alpha} = \{ x | \varphi(x) < \alpha \} \ ( + \infty > \alpha > - \infty )$ since $F$ is an open $F_{\gamma}$-set in $E$, $F_{\alpha}$ is also an open $F_{\gamma}$-set in $E$. Hence by a') we can find the smallest open-closed set $U_a$ containing $F_{\alpha}$ for all $\alpha$, and $\{ U_a \}$ has the properties $U_a \supset U_B ( \alpha > \beta )$ and $U(F) \supset \bigcup_{\alpha>0} U_a \supset F$. Similarly to the latter part of the proof of Theorem 1 we define $\bar{\varphi}(x) = \inf_{x \in U_a} \bigcup_{\alpha>0} U_a$, $\bar{\varphi}(x) = + \infty \ ( x \in U(F) \bigcup_{\alpha>0} U_a )$, then we see easily $\bar{\varphi}$ is a continuous function on $U(F)$ and $\varphi \geq \bar{\varphi}$ on $F$. Since by b') $\{ x | \varphi(x) > \alpha \} \cap F_a = \phi$ implies $\{ x | \varphi(x) > \alpha \} \cap U_a = \phi$, we have $\phi \geq \bar{\varphi}$ on $F$.

**References**


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