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ON THE DISTRIBUTION OF ALMOST PRIMES IN AN ARITHMETIC PROGRESSION

By

Saburô UCHIYAMA

1. Introduction. An almost prime is a positive integer the number of whose prime divisors is bounded by a certain constant. The purpose of this paper is to deal with an existence problem of almost primes in a short arithmetic progression of integers. We shall prove the following

Theorem. Let $k$ and $l$ be two integers with $k \geq 1$, $0 \leq l \leq k-1$, $(k, l)=1$. There exists a numerical constant $c_{1}>0$ such that for every real number $x \geq c_{1}k^{3.5}$ there is at least one integer $n$ satisfying

$$ x < n \leq 2x, \quad n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

where $V(n)$ denotes the total number of prime divisors of $n$. In particular, if we write $a(k, l)$ for the least positive integer $n (>1)$ satisfying

$$ n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

then we have

$$ a(k, l) < c_{2}k^{3.5}$$

with some absolute constant $c_{2}>0$.

It is of some interest to compare our results presented above, though they are not the best possible, with a recent result of T. Tatuzawa [5] on the existence of a prime number $p$ satisfying $x < p \leq 2x$, $p \equiv l \pmod{k}$ and a celebrated theorem of Yu. V. Linnik concerning the upper bound for the least prime $p \equiv l \pmod{k}$ (cf. [3: X]).

Our proof of the theorem is based essentially upon the general sieve methods due to A. Selberg. The deepest result which we shall refer to is:

$$ \pi(x) = li x + O \left( x \exp(-c_{3}(\log x)^{1/2}) \right)$$

with a positive constant $c_{3}$, where $\pi(x)$ denotes, as usual, the number of primes not exceeding $x$ (in fact, a slightly weaker result will suffice for our purpose). Apart from this, the proof is entirely elementary.

Notations. Throughout in the following, $k$ represents a fixed positive
integer, \( l \) an integer with \( 0 \leq l \leq k - 1 \), \( (k, l) = 1 \). The letters \( p, q \) are used to denote prime numbers and, \( d, m, n, r \) to denote positive integers. The functions \( \mu(n) \) and \( \varphi(n) \) are Möbius’ and Euler’s functions, respectively. The function \( g(n) \) is defined as follows: \( g(1) = 1 \) and for \( n > 1 \) \( g(n) \) is the greatest prime divisor of \( n \).

\( s, t, u, v, w, x, y, z \) will be used to denote real numbers, constant or variable. \( c \) represents positive constants, not depending on \( k \) and \( l \), which are not necessarily the same in each occurrence; the constants implied in the symbol \( O \) are either absolute or else uniform in \( k \) and \( l \).

2. Preliminaries. There needs the following lemma for later calculations:

**Lemma 1.** We have

\[
\sum_{p \leq t} \frac{1}{p} = \log \log t + c_4 + O\left(\frac{1}{\log t}\right),
\]

where \( c_4 \) is a constant;

\[
\sum_{p \leq t} \frac{\log p}{p} = \log t + O(1);
\]

\[
\prod_{p \leq t} \left(1 - \frac{1}{p}\right)^{-1} = e^C \log t + O(1),
\]

where \( C \) is Euler’s constant; and

\[
\varphi(m) > c \frac{m}{\log \log 3m}.
\]

These results are well known. For a proof see [3: I, Theorems 3.1, 4.1 and 5.1].

Let \( M \geq 0, N \geq 2 \) be arbitrary but fixed integers and put

\[
y = 2k(N + 1), \quad w = y^{\frac{1}{2} - \varepsilon},
\]

where \( 0 < \varepsilon < \frac{1}{4} \): we shall fix \( \varepsilon = \frac{1}{7} \) later on. Further we put

\[
z = y^{\frac{1}{\alpha}}, \quad z_1 = y^{\frac{1}{\beta}}, \quad z_2 = y^{\frac{1}{\gamma}},
\]

where \( \alpha, \beta, \gamma \) are fixed real numbers satisfying

\[
10 \geq \gamma \geq 4 \geq \alpha > 2 \geq \beta > 1.
\]
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First we wish to evaluate from below the number $S_1$ of those integers of the form $kn+l$ $(M<n\leqq M+N)$ which are not divisible by any prime $p\leqq z$. Applying the ‘lower’ sieve of A. Selberg (see [2] and [7]), we find that

$$S_1 \geqq (1-Q)N-R_1,$$

where

$$Q = \sum_{p \leqq z, (p,k)=1} \frac{1}{pZ_p}$$

with

$$Z_p = \sum_{n \leqq w/p} \frac{\mu^2(n)}{\varphi(n)},$$

and

$$R_1 = O\left(\frac{w^3}{\sqrt{z}}\right).$$

It will be shown later that

$$Z_p > c \frac{\varphi(k)}{k} \log p$$

for all $p \leqq z$,

and so we have, by Lemma 1,

$$R_1 = O\left(\frac{w^3(\log \log 3k)^2}{z \log z}\right).$$

We put

$$H_p = \prod_{q \leqq p, (q,k)=1} \left(1 - \frac{1}{q}\right)$$

for $p \leqq z$.

Then it is easily verified that

$$1-Q = \prod_{p \leqq z, (p,k)=1} \left(1 - \frac{1}{p}\right) - \sum_{p \leqq z, (p,k)=1} \frac{H_p - Z_p}{pH_pZ_p}.$$

Lemma 2. We have

$$S_1 \geqq \frac{KN}{\varphi(k)} \prod_{p \leqq z} \left(1 - \frac{1}{p}\right) - N \sum_{p \leqq z, (p,k)=1} \frac{H_p - Z_p}{pH_pZ_p} + O\left(\frac{N(\log \log 3k)^3}{z \log z}\right) + O\left(\frac{w^3(\log \log 3k)^2}{z \log z}\right).$$

Proof. We have only to prove that

$$\prod_{p \leqq z, (p,k)=1} \left(1 - \frac{1}{p}\right) = \frac{k}{\varphi(k)} \prod_{p \leqq z} \left(1 - \frac{1}{p}\right) + O\left(\frac{(\log \log 3k)^3}{z \log z}\right)$$
\[
(1) \quad \prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\varphi(k)} + O\left(\frac{(\log \log 3k)^3}{z}\right).
\]

Now we have
\[
0 \leq \prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right) - \frac{\varphi(k)}{k} = \prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right) - \prod_{p \mid k} \left(1 - \frac{1}{p}\right) = \sum_{d > z, d \mid k \delta(p)} \sum_{a \mid k, \equiv 0(p)} \frac{\mu^2(d)}{d} = \sum_{p > z, p \mid k} \frac{1}{p} \sum_{d \mid k, (d, p) = 1} \frac{\mu^2(d)}{d} = o\left(\frac{1}{z} \log z \log \log 3k\right) = o\left(\frac{\log \log 3k}{z}\right),
\]
from which follows (1) at once.

Let \( q \) be a prime number in the interval \( z < q \leq z \), with \( (q, k) = 1 \). We next evaluate from above the number \( S(q) \) of those integers \( kn + l \ (M < n \leq M + N) \) which are not divisible by any prime \( p \leq z \) and are divisible by the prime \( q \). Applying the ‘upper’ sieve of A. Selberg (see the Appendix below), we find that
\[
S(q) \leq \frac{N}{qW_q} + R(q),
\]
where
\[
W_q = \sum_{n \leq z, (n, k) = 1} \frac{\mu^2(n)}{\varphi(n)}
\]
with
\[
a = \frac{\alpha}{2} \left(1 - 2\varepsilon - \frac{\log q}{\log y}\right)
\]
and
\[
R(q) = O\left(\frac{z^a}{W_q^2}\right) = O\left(\frac{w^a}{qW_q^2}\right).
\]

Now, let \( r \geq 1 \) be a fixed integer and let \( S_r \) denote the number of those integers of the form \( kn + l \ (M < n \leq M + N) \) which are not divisible by any prime \( p \leq z \) and are divisible by at least \( r + 1 \) distinct primes \( q \) in the interval \( z < q \leq z \), with \( (q, k) = 1 \). Clearly \( S_r \) is not greater than
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\[ \sum_{z < q \leq z_1} \frac{1}{q W_q} \leq \frac{N}{r+1} \sum_{z < q \leq z_1} \frac{1}{q W_q} + O \left( \frac{w^r (\log \log 3k)^2}{\log^2 y} \right). \]

Hence:

**Lemma 3.** We have

\[ S_2 \leq \frac{N}{r+1} \sum_{z < q \leq z_1} \frac{1}{q W_q} + o \left( \frac{w^r (\log \log 3k)^2}{\log^2 y} \right). \]

**Proof.** It will later be shown that

\[ W_q > c \frac{\varphi(k)}{k} \log y \quad \text{for } z < q \leq z_1. \]

It follows that

\[ \sum_{z < q \leq z_1} \frac{1}{q W_q} = O \left( \frac{w^r (\log \log 3k)^2}{\log^2 y} \right), \]

since

\[ \sum_{z < q \leq z_1} \frac{1}{q} = \log \frac{\alpha}{\beta} + O(1) = O(1). \]

3. **Some lemmas.** Here we collect some auxiliary results which will be needed in the next two sections.

**Lemma 4.** We have

\[ \sum_{d|m} \frac{\mu^2(d) \log d}{d} = O \left( (\log \log 3m)^2 \right). \]

**Proof.** The left-hand side is equal to

\[ \sum_{d|m} \frac{\mu^2(d)}{d} \sum_{p|d} \log p = \sum_{p|m} \log p \sum_{d|m} \frac{\mu^2(d)}{d} \]

\[ = \sum_{p|m} \log p \sum_{d|m/p} \frac{\mu^2(d)}{d}, \]

where we have

\[ \sum_{d|m/p \equiv 0 (p)} \frac{\mu^2(d)}{d} \leq \sum_{d|m} \frac{1}{d} = O(\log \log 3m) \]
and

\[ \sum_{p|m} \frac{\log p}{p} = \sum_{p\leq \log m} \frac{\log p}{p} + O(1) \]

\[ = O(\log \log 3m). \]

This proves Lemma 4.

**Lemma 5.** We have

\[ \sum_{n \leq t} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(m)}{m} \log t + O(\log \log 3m). \]

**Proof.** H. N. Shapiro and J. Warga [4: Appendix I] have proved that

\[ \sum_{n \leq t} \frac{\mu^2(n)}{n} = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \log t + O(\log \log 3m). \]

Using this inequality we obtain

\[ \sum_{n \leq t} \frac{\mu^2(n)}{\varphi(n)} = \sum_{n \leq t} \frac{\mu^2(n)}{n} \prod_{p|n} \left(1 + \frac{1}{p-1}\right) \]

\[ = \sum_{n \leq t} \frac{\mu^2(n)}{n} \sum_{d|n} \frac{1}{\varphi(d)} \]

\[ = \sum_{d \leq t} \frac{\mu^2(d)}{d \varphi(d)} \sum_{n \leq \frac{t}{d}} \frac{\mu^2(n)}{n} \prod_{p|n} \left(1 - \frac{1}{p^2}\right) \log \frac{t}{d} + O(\log \log 3dm) \]

\[ = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log \frac{t}{d} \]

\[ + O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d \varphi(d)} \log \log 3dm \right) \]

\[ = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log t \]

\[ + O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log d \right) \]

\[ + O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d \varphi(d)} \log \log 3dm \right). \]
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\[ \frac{\varphi(m)}{m} \log t + O(\log \log 3m), \]

since

\[ \sum_{d|m} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left( 1 - \frac{1}{p^s} \right)^{-1} = \prod_{p|m} \left( 1 + \frac{1}{p^s} \right)^{-1} = \prod_{p|m} \left( 1 - \frac{1}{p^s} \right)^{-1}. \]

Now, for \( u > 0, \ v \geq 2 \), let \( G(u, v) \) denote the number of positive integers \( n \leq u \) with \( g(n) \leq v \).

Define the function \( \rho(s) \) by the following properties:

\[ \rho(s) = \begin{cases} 0 & (s < 0); \\ 1 & (0 \leq s \leq 1); \\ \rho'(s) = -\rho(s-1) & (s > 1); \\ \rho(s) \text{ continuous for } s > 0. \end{cases} \]

Then the following result has been proved by N. G. de Bruijn [1]:

**Lemma 6.** Let \( u > 0, \ v \geq 2 \), and put \( t = (\log u)/\log v \). Then we have

\[ G(u, v) = O(u e^{-ct}) \]

and, more precisely,

\[ G(u, v) = u \rho(t) \left( 1 + O\left( \frac{\log(2 + t)}{\log v} \right) \right) + O(1) + O\left( u t^2 P(v) \right), \]

where \( P(v) \) is a function satisfying

\[ \begin{align*} P(v) & \downarrow 0 \quad (v \to \infty), \\ P(v) & > (\log v)/v \quad (v \geq 2), \\ |\pi(v) - \text{li} v| & < v P(v)/\log v \quad (v \geq 2). \end{align*} \]

As to the function \( \rho(s) \) itself, it is not difficult to prove the following result, which is known as a lemma of N. C. Ankeny:

**Lemma 7.** For \( s_1 \geq s_2 \geq 1 \) we have

\[ \rho(s_1) \leq \rho(s_2) e^{- (s_1 - s_2)}, \]

so that

\[ \int_s^\infty \rho(t) dt \leq \rho(s) \quad (s \geq 1). \]

For a proof of this result see [8].

4. **Evaluation of \( S_1 \).** We are now going to find an explicit lower bound for \( S_1 \) on the basis of Lemma 2.
First we have to evaluate $Z_p$ and $H_p - Z_p$ for $p \leq z$. To accomplish this we distinguish three cases on the magnitude of the prime $p$.

It is clear that

$$T_p \overset{\text{def}}{=} \sum_{\substack{n > w/\sqrt{p} \atop g(n) \leq p \atop (n, k) = 1}} \frac{1}{n} \geq H_p - Z_p \geq 0.$$  

Case 1: $2 \leq p \leq \exp(\log y)^{3}$. By partial summation we get

$$T_p \leq \sum_{\substack{n > w/\sqrt{p} \atop g(n) \leq p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O(y^{-c}),$$

where $c_z = \frac{1}{2} - \varepsilon - \frac{1}{2\alpha}$. By Lemma 6 we have

$$\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = O\left( \sum_{n > w/\sqrt{p}} n^{-\left(1 + c/\log p\right)} \right) = O\left((\log y)^{\frac{2}{3}} \exp\left(-c(\log y)^{\frac{1}{3}}\right)\right).$$

It follows that

$$H_p - Z_p = O\left(\frac{1}{\log^{2} y}\right), \quad Z_p > c \frac{\varphi(k)}{k} \log p,$$

since, by Lemma 1,

$$H_p = \prod_{q \leq \exp(\log y)^{3}} \left(1 - \frac{1}{q}\right)^{-1} \geq \frac{\varphi(k)}{k} e^{c \log p} + O(1).$$

Case 2: $\exp(\log y)^{3} < p \leq z$. We have

$$T_p = \sum_{\substack{n > w/\sqrt{p} \atop g(n) \leq p \atop (n, k) = 1}} \frac{1}{n} \sum_{d | (n, k)} \mu(d)$$

$$= \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{\substack{n > w/\sqrt{p} \atop g(n) \leq p}} \frac{1}{n}$$

$$= \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{\substack{n > w/\sqrt{p} \atop g(n) \leq p}} \frac{1}{n}$$

$$+ \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{\substack{w/\sqrt{p} \leq n > w/\sqrt{p} \atop g(n) \leq p}} \frac{1}{n}$$
\[
\prod_{q|k, q \leqq p} \left( 1 - \frac{1}{q} \right) \sum_{g(n) \leqq p} \frac{1}{n} + O \left( (\log \log 3k)^2 \right),
\]

since we have, by Lemma 4,

\[
\sum_{d|k, g(d) \leqq p} \frac{\mu(d)}{d} \sum_{\simeq, q(n) \leqq p} \frac{1}{n} = O \left( \frac{\log \log 3k}{\sqrt{p}} \right) = O \left( (\log \log 3k)^2 \right).
\]

Now, by partial summation, we have

\[
\sum_{n > w/\sqrt{p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O \left( y^{-c}\right),
\]

Here, by Lemma 6, we find that

\[
\sum_{n > \exp(\log y)^2} \frac{G(n, p)}{n^2} = O \left( \sum_{n > \exp(\log y)^2} n^{-\frac{1+c}{\log p}} \right)
\]

\[
= O \left( \exp(-c \log y) \right)
\]

so that

\[
\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = \sum_{\exp(\log y)^2 < n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O \left( \frac{1}{\log^2 y} \right).
\]

Let us write \( I \) for the interval \( w/\sqrt{p} < n \leqq \exp(\log y)^2 \). Then, by making use of the result in Lemma 6, we obtain

\[
\sum_{n \in I} \frac{G(n, p)}{n^2} = \sum_{n \in I} \frac{1}{n} \rho \left( \frac{\log n}{\log p} \right) \left( 1 + O \left( \frac{\log \log y}{\log p} \right) \right)
\]

\[
+ O \left( \sum_{n \in I} \frac{1}{n} \right) + O \left( \sum_{n \in I} \left( \frac{\log n}{\log p} \right)^2 P(p) \right).
\]

It is easily verified that

\[
\sum_{n \in I} \frac{1}{n} \rho \left( \frac{\log n}{\log p} \right) = \log p \int_{t_p}^{\infty} \rho(t) dt + O \left( y^{-c} \right),
\]

where \( t_p = (\log w/\sqrt{p}) / \log p \);
\[
\sum_{n \in \mathcal{I}} \frac{1}{n} = O(y^{-\epsilon_5}); \quad \sum_{n \in \mathcal{I}} \left( \frac{\log n}{\log p} \right)^2 P(p) = O\left( \frac{1}{\log^2 y} \right),
\]
where we have taken \( P(v) = c \exp(-c(\log v)^{\frac{1}{2}}) \).

We thus have
\[
T_p = \log p \int_{t_p}^{\infty} \rho(t) dt \left( 1 + O\left( \frac{\log \log y}{\log p} \right) \right) + O\left( \frac{1}{\log^2 y} \right).
\]

Hence
\[
H_p - Z_p \leq \prod_{q \uparrow k, q \leq p} \left( 1 - \frac{1}{q} \right) \log p \int_{t_p}^{\infty} \rho(t) dt \left( 1 + O\left( \frac{\log \log y}{\log p} \right) \right)
\]
\[
+ O\left( (\log \log 3k)^2 \right),
\]
\[
Z_p \geq \prod_{q \mid k, q \leq p} \left( 1 - \frac{1}{q} \right) \left( e^c - \int_{t_p}^{\infty} \rho(t) dt \log p + O(\log \log y) \right)
\]
\[
+ O\left( (\log \log 3k)^2 \right).
\]

**Case 3:** \( z_1 < p \leq z \).

Put \( t_p = (\log w/\sqrt{p})/\log p \), as before. If \( 0 < t_p \leq 1 \) then we have
\[
Z_p = \sum_{\substack{n \leq w/\sqrt{p} \\gcd(n,k)=1 \\gcd(n,q)=1}} \frac{\mu^2(n)}{\varphi(n)}
\]
\[
= \frac{\varphi(k)}{k} \log \frac{w}{\sqrt{p}} + O(\log \log 3k)
\]
\[
= \frac{\varphi(k)}{k} t_p \log p + O(\log \log 3k),
\]
\[
H_p - Z_p = \frac{\varphi(k)}{k} (e^c - t_p) \log p + O(\log \log 3k),
\]
by Lemma 5. If \( t_p > 1 \) then
\[
Z_p \geq \sum_{n \leq w/\sqrt{p} \\gcd(n,k)=1} \frac{\mu^2(n)}{\varphi(n)} - \sum_{p \mid q, \gcd(n,k)=1} \sum_{n \leq w/\sqrt{p}} \frac{\mu^2(n)}{\varphi(n)}
\]
\[
= \sum_{n \leq w/\sqrt{p} \\gcd(n,k)=1} \frac{\mu^2(n)}{\varphi(n)} - \sum_{p \mid q, \gcd(n,k)=1} \frac{1}{\varphi(q)} \sum_{n \leq w/\sqrt{p} \\gcd(n,q)=1} \frac{\mu^2(n)}{\varphi(n)},
\]
where, again by Lemma 5,
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\[
\sum_{p \leq q \leq w^{1/2}} \frac{1}{\varphi(q)} \sum_{n \leq w^{1/2}} \frac{\mu^{2}(n)}{\varphi(n)} n \leq w/q \overline{p} = \sum_{(q,k) = 1} \frac{1}{\varphi(q)} \sum_{(n,qk) = 1} \frac{\mu^{2}(n)}{\varphi(n)} n \leq w/q \overline{p} (\frac{\varphi(qk)}{qk} \log \frac{w}{q\sqrt{p}} + O(\log \log 3qk))
\]

\[
= \frac{\varphi(k)}{k} \sum_{p \leq q \leq w^{1/2}} \frac{1}{q} \log \frac{w}{q\sqrt{p}} + O\left( \sum_{p \leq q \leq w^{1/2}} \frac{\log \log 3qk}{\varphi(q)} \right)
\]

\[
= \frac{\varphi(k)}{k} \left( \log \frac{w}{\sqrt{p}} \log \frac{\log \frac{w}{\sqrt{p}}}{p} - \log \frac{w}{\sqrt{p}} + \log p \right) + O(\log \log y),
\]

and hence

\[
Z_p \geq \frac{\varphi(k)}{k} (2t_p - 1 - t_p \log t_p) \log p + O(\log \log y).
\]

Therefore

\[
H_p - Z_p \leq \frac{\varphi(k)}{k} \left( e^C - (2t_p - 1 - t_p \log t_p) \right) \log p + O(\log \log y).
\]

Here we have, as in the proof of Lemma 2,

\[
H_p = \prod_{q < p, q \mid k} \left( 1 - \frac{1}{q} \right) \prod_{q < p} \left( 1 - \frac{1}{q} \right)^{-1}
\]

\[
= \left( \frac{\varphi(k)}{k} + O\left( \frac{\log \log 3k}{p} \right) \right) (e^C \log p + O(1))
\]

\[
= \frac{\varphi(k)}{k} e^C \log p + O(\log \log 3k).
\]

We are now in position to be able to evaluate the sum

\[
\sum_{p \leq z} \frac{H_p - Z_p}{pH_p Z_p}.
\]

Define:

\[
A(t) = \begin{cases} 
   \frac{e^C - t}{t} & (0 < t \leq 1), \\
   \frac{e^C - (2t - 1 - t \log t)}{2t - 1 - t \log t} & (1 < t < e^C), 
\end{cases}
\]
where \( t = e^\epsilon \) is the unique solution of
\[
2t - 1 - t \log t = 0, \quad t > 1,
\]
so that \( 1.8 < c_6 < 1.9 \); and
\[
B(t) = \frac{\int_t^\infty \rho(s) \, ds}{e^\epsilon - \int_t^\infty \rho(s) \, ds} \quad (t > \frac{1}{4}).
\]

Let us put, for the sake of brevity,
\[
z_3 = \exp(\log y)^{2/3}.
\]

Then we have
\[
\sum_{\substack{z_3 < p \leq z_1 \atop (p, k) = 1}} \frac{H_p - Z_p}{pH_pZ_p} = O \left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} \right)
= O \left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} \right),
\]
and, noticing that
\[
\prod_{1 < q \leq p} \left( 1 - \frac{1}{q} \right)^{-1} \leq \frac{k}{\varphi(k)}
\]
for every \( p \),
\[
\sum_{z_3 < p \leq z_1 \atop (p, k) = 1} \frac{H_p - Z_p}{pH_pZ_p} \leq \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} + O \left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} \right),
\]
where we have used the inequality
\[
\sum_{z_3 < p \leq z_1} \frac{1}{p \log^2 p} = O \left( \sum_{z_3 < n \leq z_1} \frac{1}{n \log^2 n} \right) = O \left( \frac{1}{\log^2 z_3} \right).
\]

We now assume that \( r, 4 \leq r \leq 10 \), be integral. Write \( J_r \) for the interval \( y^{1/r+1} < p \leq y^{1/r} (\nu \geq r) \). Then, since the function \( B(t) \) is monotone decreasing,
\[
\sum_{z_3 < p \leq z_1} \frac{B(t_p)}{p \log p} = \sum_{\gamma \leq \nu < \varphi(y)} \sum_{p \in J_r} \frac{B(t_p)}{p \log p}
\]
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\[
\begin{align*}
\leq & \sum_{\gamma \leq \nu < \log y} \left( \sum_{\nu \in J_{\nu}} \frac{1}{\nu} \right) \max_{\nu \leq \nu \leq \nu} B(t_{\nu}) \\
\leq & \sum_{\gamma \leq \nu < \log y} \log \frac{\nu+1}{\nu} B \left( \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2} \right) \\
& + O \left( \sum_{\gamma \leq \nu < \log y} \frac{1}{\nu} \right) \\
\leq & \frac{1}{\log y} \sum_{\nu = 0}^{\infty} (\nu + 1) \log \frac{\nu+1}{\nu} B \left( \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2} \right) + O \left( \frac{1}{\log^{\gamma/3} y} \right).
\end{align*}
\]

Thus we obtain

\[
\sum_{\gamma \leq \nu < \log y} \frac{H_{p} - Z_{p}}{p \log Z_{p}} \leq \frac{k}{\varphi(k)} e^{-c} \sum_{\nu = 0}^{\infty} (\nu + 1) \log \frac{\nu+1}{\nu} B \left( \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2} \right) \\
+ O \left( \frac{k}{\varphi(k)} \frac{(\log \log 3k)^3}{\log y} \right) + O \left( \frac{k}{\varphi(k)} \frac{\log \log y}{\log y} \right).
\]

We have similarly

\[
\sum_{\gamma \leq \nu < \log y} \frac{H_{p} - Z_{p}}{p \log Z_{p}} \leq \frac{k}{\varphi(k)} e^{-c} \sum_{\nu = 0}^{\infty} A(t_{\nu}) \\
+ O \left( \frac{k}{\varphi(k)} \frac{(\log \log 3k) \log \log y}{\log y} \right).
\]

Put

\[
n = [\log^{1/2} y], \quad u_{j} = \alpha + \frac{\gamma - \alpha}{n} j \quad (j \geq 0),
\]

and write \( K_{j} \) for the interval

\[
y^{1/u_{j+1}} < p \leq y^{1/u_{j}} \quad (0 \leq j \leq n-1).
\]

Now, the function \( A(t) \) is continuous, monotone decreasing in the interval \( 0 < t \leq e \) and monotone increasing in the interval \( e < t < e^{n} \). Thus, if we denote by \( m \) the integer for which

\[
\left( \frac{1}{2} - \epsilon \right) u_{m} - \frac{1}{2} \leq e < \left( \frac{1}{2} - \epsilon \right) u_{m+1} - \frac{1}{2},
\]

then
\[
\sum_{z, p | z, (p, k) = 1} \frac{A(t_p)}{p \log p} = \sum_{j=0}^{n-1} \sum_{p \in K_j} \frac{A(t_p)}{p \log p}
\]

\[
\leq \sum_{j=0}^{n-1} \left( \sum_{p \in K_j} \frac{1}{p} \right) \max_{\not\in x_j} \frac{A(t_p)}{\log p}
\]

\[
\leq \sum_{j=0}^{n-1} \log \frac{u_{j+1}}{u_j} \frac{u_{j+1}}{u_j} A \left( \left( \frac{1}{2} - \varepsilon \right) u_j - \frac{1}{2} \right)
\]

\[
+ \log \frac{u_{m+1}}{u_m} \frac{u_{m+1}}{u_m} \max \left( A \left( \left( \frac{1}{2} - \varepsilon \right) u_m - \frac{1}{2} \right), A \left( \left( \frac{1}{2} - \varepsilon \right) u_{m+1} - \frac{1}{2} \right) \right)
\]

\[
+ \sum_{j=m+1}^{\iota-1} \log \frac{u_{j+1}}{u_j} \frac{u_{j+1}}{\log y} A \left( \left( \frac{1}{2} - \varepsilon \right) u_j - \frac{1}{2} \right)
\]

\[
+ O \left( \sum_{j=0}^{n-1} \frac{u_j}{\log^2 y} \right)
\]

\[
= \frac{1}{\log y} \int_{\alpha}^{\gamma} A \left( \left( \frac{1}{2} - \varepsilon \right) u - \frac{1}{2} \right) du + O \left( \frac{1}{\log^{3/2} y} \right),
\]

where it should be noticed that we have uniformly

\[
u_{j+1}, \log \frac{u_{j+1}}{u_j} = \frac{r - \alpha}{n} + O \left( \frac{1}{n^2} \right)
\]

Hence

\[
\sum_{z, p | z, (p, k) = 1} \frac{H_p - Z_p}{p H_p Z_p} \leq \frac{k}{\varphi(k)} \frac{e^{-C}}{\log y} \int_{\alpha}^{\gamma} A \left( \left( \frac{1}{2} - \varepsilon \right) u - \frac{1}{2} \right) du
\]

\[
+ O \left( \frac{k}{\varphi(k)} \frac{\log \log y \log \log 3k}{\log^2 y} \right) + O \left( \frac{k}{\varphi(k)} \frac{1}{\log^{3/2} y} \right).
\]

Collecting these results, we thus obtain, via Lemma 2, the following

**Lemma 8.** We have

\[
S_i \geq \frac{kN}{\varphi(k)} \frac{e^{-c}}{\log y} \left( \alpha - \int_{\alpha}^{\gamma} A \left( \left( \frac{1}{2} - \varepsilon \right) u - \frac{1}{2} \right) du \right.
\]

\[
- \sum_{\nu=0}^{m} (\nu+1) \log \frac{\nu+1}{\nu} B \left( \left( \frac{1}{2} - \varepsilon \right) \nu - \frac{1}{2} \right)
\]

\[
+ O \left( \frac{kN}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^{1/3} y} \right) + O \left( \frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{1/3} y} \right)
\]

\[
+ O \left( \frac{N(\log \log 3k)^3}{y^{1/4} \log y} \right) + O \left( y^{1-2\epsilon} (\log \log 3k)^2 \right).
\]
5. Evaluation of $S_t$. By virtue of Lemma 3, our present task is only to estimate the quantity

$$\sum_{z<q\leq z_1} \frac{1}{qW_q}.$$ 

We set

$$C(t) = \begin{cases} \frac{\alpha}{a} & (0<a\leq 1), \\ \frac{\alpha}{2a - 1 - a \log a} & (1<a\leq 2), \end{cases}$$

where

$$a = \frac{\alpha}{2} \left(1 - 2\varepsilon - \frac{1}{t}\right).$$

Then, it is not difficult to verify, by Lemma 5, that, with $t = t_q = (\log y)/\log q$

$$W_q = \sum_{n \leq q, \varphi(n) \leq \varphi(k) \varphi(n)} \frac{\mu^2(n)}{\varphi(n)} (n,k) = 1 \geq \frac{\varphi(k)}{k} \frac{\log y}{C(t_q)} + O(\log \log 3k) \quad (z < q \leq z_1),$$

and consequently

$$\sum_{z<q\leq z_1} \frac{C(t_q)}{q} \leq \frac{k}{\varphi(k)} \frac{1}{\log y} \sum_{z<q\leq z_1} \frac{C(t_q)}{q} + O\left(\frac{k}{\varphi(k)} (\log \log 3k)^2 \frac{\log^2 y}{\log y}\right).$$

Put

$$n = \lfloor \log^{1/2} y \rfloor, \quad u_j = \beta + \frac{\alpha - \beta}{n} j \quad (0 \leq j \leq n),$$

and write $L_j$ for the interval

$$y^{1/u_{j+1}} < q \leq y^{1/u_j} \quad (0 \leq j \leq n-1).$$

Then we have

$$\sum_{z<q\leq z_1} \frac{C(t_q)}{q} = \sum_{j=0}^{n-1} \sum_{q \in L_j} \frac{C(t_q)}{q} \leq \sum_{j=0}^{n-1} \left(\sum_{q \in L_j} \frac{1}{q}\right) \max_{q \in L_j} C(t_q) \leq \sum_{j=0}^{n-1} \left(\log \frac{u_{j+1}}{u_j} + O\left(\frac{1}{\log y}\right)\right) C(u_j) = \int_{\beta}^{\alpha} C(u) \frac{du}{u} + O\left(\frac{1}{\log^{1/2} y}\right).$$
since the function $C(t)$ is continuous and decreases monotonously for $t>(1-2\varepsilon)^{-1}$ and since we have uniformly
\[
\log \frac{u_{j+1}}{u_j} = \frac{\alpha-\beta}{n} \frac{1}{u_j} + O\left(\frac{1}{n^2}\right) \quad (0 \leq j \leq n-1).
\]

We thus have proved the following

**Lemma 9.** We have
\[
S_2 \leq \frac{1}{r+1} \frac{kN}{\varphi(k)} \frac{1}{\log y} \int_{\beta}^{\alpha} \frac{C(u)}{u} du + O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^2}{\log^2 y}\right) + O\left(\frac{1}{\log^{3/2} y}\right).
\]

6. **Numerical computations.** We need the following easy lemma, a part of which has already been used in the proof of Lemmas 8 and 9.

**Lemma 10.** The function
\[
f(s) = \frac{1}{2s-1-s \log s} \quad (1<s<e^{c_6})
\]
is positive, convex, and monotone decreasing for $1<s \leq e$ and monotone increasing for $e<s<e^{c_6}$.

Putting $f_1(s) = (f(s))^{-1}$, we see that $f_1(s)>0$, $f_1'(s) = 1 - \log s$ and $f_1''(s) = -1/s$, and the result follows at once.

We now choose $\varepsilon = \frac{1}{7}$ and take
\[
\alpha = 4, \quad \beta = 2, \quad \text{and} \quad r = 10.
\]

Our aim in this section is to compute numerically two integrals and a sum appearing in Lemmas 8 and 9.

(i) Computation of
\[
\int_{\alpha}^{\gamma} A\left(\frac{1}{2} - \varepsilon\right) du = \int_{4}^{10} A\left(\frac{5}{14} u - \frac{1}{2}\right) du.
\]

The integral is equal to
\[
\int_{4}^{4.2} A\left(\frac{5}{14} u - \frac{1}{2}\right) du + \int_{4.2}^{10} A\left(\frac{5}{14} u - \frac{1}{2}\right) du,
\]
where the first integral is found to be
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\[ \int_{\frac{5}{14} u - \frac{1}{2}}^{\frac{5}{14} u - \frac{1}{2}} \frac{1}{14} \log \frac{14}{13} du - 0.2 < 0.1696, \]

while the second is

\[ \int_{\frac{5}{14} u - \frac{1}{2}}^{\frac{5}{14} u - \frac{1}{2}} F(u) du - \int_{\frac{5}{14} u - \frac{1}{2}}^{\frac{5}{14} u - \frac{1}{2}} F(u) du \]

with \( F(u) = f(s(u)) \), where \( f(s) \) is the function defined in Lemma 10 and \( s(u) = \frac{5}{14} u - \frac{1}{2} \). To estimate the integral of \( F(u) \) over \((4.2, 10)\) we proceed as follows.

We find:

\[ F(4.2) = 1.0000; \quad F(4.5) < 0.9080; \]
\[ F(5) < 0.8011; \quad F(6) < 0.6803; \]
\[ F(7) < 0.6197; \quad F(8) < 0.5907; \]
\[ F(9) < 0.5820; \quad F(10) < 0.5896. \]

By Lemma 10, the function \( F(u) \) is convex for \( 4.2 \leq u \leq 10 \). Hence

\[ \int_{\frac{5}{14} u - \frac{1}{2}}^{\frac{5}{14} u - \frac{1}{2}} F(u) du \leq \frac{3}{20} \left( F(4.2) + F(4.5) \right) + \frac{1}{4} \left( F(4.5) + F(5) \right) \]
\[ + \frac{1}{4} \left( F(5) + F(10) \right) + \left( F(6) + F(7) + F(8) + F(9) \right) \]
\[ < 3.8817, \]

and the second integral in (3) is less than

\[ 3.8817 e^c - 5.8 < 1.1137. \]

Thus we have

\[ \int_{\frac{5}{14} u - \frac{1}{2}}^{\frac{5}{14} u - \frac{1}{2}} \frac{1}{14} \log \frac{14}{13} du < 0.1696 + 1.1137 = 1.2833. \]

(ii) Computation of

\[ \sum_{\nu=7}^{\infty} \frac{1}{\nu+1} \log \frac{\nu+1}{\nu} B\left( \frac{1}{2} - \epsilon, \nu - \frac{1}{2} \right) \]
\[ = \sum_{\nu=10}^{\infty} \frac{1}{\nu+1} \log \frac{\nu+1}{\nu} B\left( \frac{5}{14}, \nu - \frac{1}{2} \right). \]
By the definition (2), the function $\rho(s)$ is positive and monotone decreasing for $s > 0$, and moreover

$$\rho(s) = 1 - \log s \quad \text{for} \quad 1 \leq s \leq 2.$$  

Put $s(\nu) = \frac{5}{14}\nu - \frac{1}{2}$. Then we have $s(10) = \frac{43}{14} > 3$ and

$$\rho\left(s(10)\right) \leq \rho(3) \leq \rho(2)e^{-1} = (1 - \log 2)e^{-1} < 0.1129,$$

by Lemma 7. Now, using Lemma 7 again, we find that for $\nu \geq 10$

$$B\left(s(\nu)\right) \leq \frac{\rho(s(\nu))}{e^{\nu} - \rho(s(\nu))} \leq \frac{\rho(s(10))}{e^{\nu} - \rho(s(10))} e^{-\frac{s}{14}(\nu - 10)}.$$

Since $(\nu + 1)\log((\nu + 1)/\nu)$ decreases monotonously as $\nu \to \infty$, we thus obtain

$$\sum_{\nu=10}^{\infty} (\nu + 1) \log\frac{\nu + 1}{\nu} B\left(s(\nu)\right) \leq 11 \log \frac{11}{10} \frac{\rho(s(10))}{e^{\nu} - \rho(s(10))} \frac{1}{1 - e^{-s/14}} < 0.2366.$$

(iii) Computation of

$$\int_{\beta}^{\alpha} \frac{C(u)}{u} \, du = \int_{2}^{4} \frac{C(u)}{u} \, du.$$

For $2 \leq u \leq 4$ we have

$$\frac{3}{7} \leq a = 2\left(\frac{5}{7} - \frac{1}{u}\right) \leq \frac{13}{14}.$$

Hence

$$\int_{2}^{4} \frac{C(u)}{u} \, du = 2 \int_{2}^{4} \left(\frac{5}{7} u - 1\right)^{-1} \, du$$

$$= \frac{14}{5} \log \frac{13}{3} < 4.1058.$$

7. Proof of the theorem. Let $1 \leq k < x$, $0 \leq l \leq k - 1$, $(k, l) = 1$. Take

$$M = \left[\frac{x - l}{k}\right], \quad N = \left[\frac{x}{k}\right],$$

and put
Let \( y = 2k(N+1), \quad z = y^{1/4}, \quad z_1 = y^{1/2}, \quad w = y^{5/14} \).

Then it is clear that \( y > 2x \) and that \( M < n \leq M + N \) implies \( x < kn + l \leq 2x \).

By \( D(x; k, l) \) we denote the number of those integers of the form \( kn + l \) \((M < n \leq M + N)\) which are divisible by no primes \( p \leq z \), by at most two primes \( q \) in \( z < q \leq z_1 \), and by no integers of the form \( q^2 \), \( q \) being a prime in \( z < q \leq z_1 \): clearly such an integer \( kn + l \) \((M < n \leq M + N)\), if it exists, has at most two prime factors, i.e. \( V(kn + l) \leq 2 \).

In order to estimate \( D(x; k, l) \) from below, we apply Lemma 8 and Lemma 9 with \( r = 2 \). Let us note that we have from the data in §6

\[
e^{-c} \left( 4 - \int_{4}^{10} A(s(u)) du - \sum_{\nu=10}^{\infty} (\nu+1) \log \frac{\nu+1}{\nu} B(s(\nu)) \right)
> e^{-c} (4 - 1.2833 - 0.2366) > 1.3923
\]

and

\[
\frac{1}{3} \int_{2}^{4} \frac{C(u)}{u} du < \frac{4.1058}{3} = 1.3686.
\]

Now, the number \( R_2 \) of those integers \( kn + l \) \((M < n \leq M + N)\) which are not divisible by any prime \( p \leq z \) and are divisible by some integer \( q^2 \) with \( q \) in \( z < q \leq z_1 \) does not exceed

\[
\sum_{z < q \leq z_1} \left( \frac{N}{q^2} + 1 \right) = O \left( \frac{N}{z} \right) + O(z_1).
\]

We find, therefore, that

\[
D(x; k, l) \geq S_1 - S_2 - R_2 \geq (1.3923 - 1.3686) \frac{kN}{\varphi(k)} \frac{1}{\log y} 
+ O \left( \frac{kN}{\varphi(k)} \frac{\log \log 3k^3}{\log^{1/3} y} \right) + O \left( \frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{1/3} y} \right) 
+ O \left( \frac{N}{\log \log 3k^3} \frac{y^{1/4}}{\log y} \right) + O \left( y^{5/14} \log \log 3k^3 \right) + O \left( \frac{N}{y^{1/4}} \right).
\]

Since \( N = \frac{x}{k} + O(1), \ \ 2x < y \leq 4x \), it follows that

\[
D(x; k, l) \geq 0.0237 \frac{1}{\varphi(k)} \frac{x}{\log x} 
+ O \left( \frac{1}{\varphi(k)} \frac{x(\log \log 3k^3)}{\log^{1/3} x} \right) + O \left( \frac{1}{\varphi(k)} \frac{x \log \log x}{\log^{1/3} x} \right)
\]
$+O\left(\frac{1}{k} \frac{x^{3/4} (\log \log 3k)^3}{\log x}\right) + O\left(x^{5/7} (\log \log 3k)^2\right) + O\left(\frac{1}{k} x^{3/4}\right)$.

Let $c_7 > 3.5$ be a fixed number. If $x \geqq k^{c_7}$ and $k$ is sufficiently large, then all the error terms on the right-hand side of the above inequality for $D(x; k, l)$ are of negligible order of magnitude, with respect to the leading term. Thus, for all large enough $k$, $x \geqq k^{c_7}$ implies that

$$D(x; k, l) > 0.0236 \frac{1}{\varphi(k)} \frac{x}{\log x} > 1.$$ 

Hence, by continuity argument, we conclude that there is a (finite) natural number $k_0$ such that, if $k \geqq k_0$ and $x \geqq k^{c_7}$ then we have $D(x; k, l) > 0$. Therefore there exists an absolute constant $c_1 > 0$ such that

$$D(x; k, l) > 0 \quad \text{for all} \quad x \geqq c_1 k^{c_7}, \quad k \geqq 1.$$ 

* This completes the proof of our theorem.

**Appendix**

ON THE 'UPPER' SIEVE OF A. SELBERG

Here we aim at generalizing the results obtained in [6].

Let $N > 1$ and let $a_1, a_2, \cdots, a_N$ be rational integers not necessarily different from each other. Let $S$ denote the number of those integers $a_j (1 \leqq j \leqq N)$ which are not divisible by any prime number $p \leqq z$, where $z \geqq 2$. Suppose that for every positive integer $d$

$$S_d \overset{\text{def}}{=} \sum_{\substack{n \leqq N \atop a_n \equiv 0 (d)}} 1 = \frac{\omega(d)}{d} N + R(d),$$

where $R(d)$ is the error term for $S_d$ and $\omega(d)$ is a multiplicative function of $d$. We put

$$f(d) = \frac{d}{\omega(d)}$$

and suppose that $f(d) > 1$ for all $d > 1$.

Let $w$ be an arbitrary but fixed real number such that $w \geqq 2$. We define for positive integers $m$ and $d$

$$f_i(m) = \sum_{n|m} \mu(n) f\left(\frac{m}{n}\right),$$
\[ W(d) = \sum_{r \leq w/d, (r, d) = 1} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)}, \quad W = W(1), \]
\[ \lambda(d) = \epsilon_z(d) \mu(d) \prod_{p \mid d} \left(1 - \frac{1}{f(p)}\right)^{-1} \cdot \frac{W(d)}{W}, \]
where \( \epsilon_z(n) = 0 \) or 1 according as \( n \) has or has not a prime factor \( \geq z \). Then we have, since \( \lambda(1) = 1 \),
\[ S \leq \sum_{n \leq N} \left( \sum_{d \mid n} \lambda(d) \right)^2 = \sum_{d \leq w} \left( \frac{\lambda(d_1) \lambda(d_2)}{f(d)} \right) \frac{N}{f(d)} + \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})|, \]
where \( \{d_1, d_2\} \) denotes the least common multiple of \( d_1 \) and \( d_2 \).

Now
\[ \sum_{d \leq w} \left( \sum_{d_1, d_2 \leq w} \lambda(d_1) \lambda(d_2) \right) \frac{1}{f(d)} = \sum_{r \leq w} f_1(r) \left( \sum_{d \leq w} \lambda(d) \frac{1}{f(d)} \right)^2 = \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \sum_{d \leq w} \epsilon_z(d) \mu(d) \frac{\mu^2(d)}{f_1(r)} \sum_{m \leq w/d, (m, d) = 1} \epsilon_z(m) \frac{\mu^2(m)}{f_1(m)} \right)^2 = \frac{1}{W^2} \sum_{r \leq w} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)} = \frac{1}{W}. \]

We thus have proved the following

**Theorem.** Under the notations and conditions described above we have
\[ S \leq \frac{N}{W} + R \]
with
\[ R = \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})| \]

This is a generalization of [3: II, Theorem 3.1].

To evaluate the remainder term \( R \) explicitly, let us suppose that for all positive integers \( d, d_1, d_2 \).
\[ |R(d)| \leq B \omega(d), \quad \omega(\{d_1, d_2\}) \leq \omega(d_1) \omega(d_2), \]

where \( B > 0 \) is a constant independent of \( d \). These conditions imply

\[ R \leq B \left( \sum_{d \leq w} \lambda(d) \omega(d) \right)^2. \]

Then, it is not difficult to show that we have, in general,

\[ R = O \left( w^2 (\log \log w)^2 \right), \]

and, in the special case where \( \omega(p) \leq 1 \) for all primes \( p \leq x \),

\[ R = O \left( \frac{w^2}{W^2} \right), \]

where the constants implied in the symbol \( O \) depend only on the constant \( B \).

The proof of these estimates of the remainder term \( R \) can easily be carried out just in the same way as in [6], and we shall omit the details (cf. also [7]).

References


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