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ON THE DISTRIBUTION OF ALMOST PRIMES IN AN ARITHMETIC PROGRESSION

By

Saburô UCHIYAMA

1. Introduction. An almost prime is a positive integer the number of whose prime divisors is bounded by a certain constant. The purpose of this paper is to deal with an existence problem of almost primes in a short arithmetic progression of integers. We shall prove the following

Theorem. Let \( k \) and \( l \) be two integers with \( k \geq 1 \), \( 0 \leq l \leq k-1 \), \( (k, l) = 1 \). There exists a numerical constant \( c_1 > 0 \) such that for every real number \( x \geq c_1 k^{3.5} \) there is at least one integer \( n \) satisfying

\[
x < n \leq 2x, \quad n \equiv l \pmod{k}, \quad V(n) \leq 2,
\]

where \( V(n) \) denotes the total number of prime divisors of \( n \). In particular, if we write \( a(k, l) \) for the least positive integer \( n (>1) \) satisfying

\[
n \equiv l \pmod{k}, \quad V(n) \leq 2,
\]

then we have

\[
a(k, l) < c_2 k^{3.5}
\]

with some absolute constant \( c_2 > 0 \).

It is of some interest to compare our results presented above, though they are not the best possible, with a recent result of T. Tatuzawa [5] on the existence of a prime number \( p \) satisfying \( x < p \leq 2x \), \( p \equiv l \pmod{k} \) and a celebrated theorem of Yu. V. Linnik concerning the upper bound for the least prime \( p \equiv l \pmod{k} \) (cf. [3: X]).

Our proof of the theorem is based essentially upon the general sieve methods due to A. Selberg. The deepest result which we shall refer to is:

\[
\pi(x) = \text{li} \ x + O \left( x \exp(-c_3 (\log x)^{1/2}) \right)
\]

with a positive constant \( c_3 \), where \( \pi(x) \) denotes, as usual, the number of primes not exceeding \( x \) (in fact, a slightly weaker result will suffice for our purpose). Apart from this, the proof is entirely elementary.

Notations. Throughout in the following, \( k \) represents a fixed positive
integer, \( l \) an integer with \( 0 \leq l \leq k - 1 \), \((k, l) = 1\). The letters \( p, q \) are used to denote prime numbers and, \( d, m, n, r \) to denote positive integers. The functions \( \mu(n) \) and \( \varphi(n) \) are Möbius’ and Euler’s functions, respectively. The function \( g(n) \) is defined as follows: \( g(1) = 1 \) and for \( n > 1 \) \( g(n) = \) the greatest prime divisor of \( n \).

\( s, t, u, v, w, x, y, z \) will be used to denote real numbers, constant or variable. \( c \) represents positive constants, not depending on \( k \) and \( l \), which are not necessarily the same in each occurrence; the constants implied in the symbol \( O \) are either absolute or else uniform in \( k \) and \( l \).

2. Preliminaries. There needs the following lemma for later calculations:

**Lemma 1.** We have

\[
\sum_{p \leq t} \frac{1}{p} = \log \log t + c_4 + O\left(\frac{1}{\log t}\right),
\]

where \( c_4 \) is a constant;

\[
\sum_{p \leq t} \frac{\log p}{p} = \log t + O(1);
\]

\[
\prod_{p \leq t} \left(1 - \frac{1}{p}\right)^{-1} = e^C \log t + O(1),
\]

where \( C \) is Euler’s constant; and

\[
\varphi(m) > c \frac{m}{\log \log 3m}.
\]

These results are well known. For a proof see [3: I, Theorems 3.1, 4.1 and 5.1].

Let \( M \geq 0, N \geq 2 \) be arbitrary but fixed integers and put

\[
y = 2k(N+1), \quad w = y^{\frac{1}{2}},
\]

where \( 0 < \varepsilon < \frac{1}{4} \): we shall fix \( \varepsilon = \frac{1}{7} \) later on. Further we put

\[
x = y^{\frac{1}{a}}, \quad x_1 = y^{\frac{1}{b}}, \quad x_2 = y^{\frac{1}{c}},
\]

where \( \alpha, \beta, \gamma \) are fixed real numbers satisfying

\[
10 \geq \gamma \geq 4 \geq \alpha > 2 \geq \beta > 1.
\]
First we wish to evaluate from below the number $S_1$ of those integers of the form $kn+l$ ($M<n\leqq M+N$) which are not divisible by any prime $p\leqq z$. Applying the ‘lower’ sieve of A. Selberg (see [2] and [7]), we find that

$$S_1 \geqq (1-Q)N-R_1,$$

where

$$Q = \sum_{p \leqq z} \frac{1}{pZ_p} \quad \text{with} \quad Z_p = \sum_{n \leqq z/p} \frac{\mu^2(n)}{\varphi(n)},$$

and

$$R_1 = O\left(\sum_{p \leqq z} \frac{1}{pZ_p^2}\right).$$

It will be shown later that

$$Z_p > c \frac{\varphi(k)}{k} \log p \quad \text{for all} \quad p \leqq z,$$

and so we have, by Lemma 1,

$$R_1 = O\left(\frac{w^3}{(\log \log 3k)^2}\right).$$

We put

$$H_p = \prod_{q<p, (q,k)=1} \left(1-\frac{1}{q}\right)^{-1} \quad (p \leqq z).$$

Then it is easily verified that

$$1-Q = \prod_{p \leqq z} \left(1-\frac{1}{p}\right) - \sum_{p \leqq z} \frac{H_p - Z_p}{pH_p Z_p}.$$

**Lemma 2.** We have

$$S_1 \geqq \frac{kN}{\varphi(k)} \prod_{p \leqq z} \left(1-\frac{1}{p}\right) - N \sum_{p \leqq z} \frac{H_p - Z_p}{pH_p Z_p} + O\left(\frac{N(\log \log 3k)^3}{z \log z}\right) + O\left(\frac{w^3}{(\log \log 3k)^2}\right).$$

**Proof.** We have only to prove that

$$\prod_{p \leqq z} \left(1-\frac{1}{p}\right) = \frac{k}{\varphi(k)} \prod_{p \leqq z} \left(1-\frac{1}{p}\right) + O\left(\frac{(\log \log 3k)^3}{z \log z}\right).$$
or

$$(1) \quad \prod_{p \leq z \atop p \parallel k} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\varphi(k)} + O \left(\frac{(\log \log 3k)^3}{z}\right).$$

Now we have

$$0 \leq \prod_{p \leq z \atop p \parallel k} \left(1 - \frac{1}{p}\right) - \frac{\varphi(k)}{k} = \prod_{p \leq z \atop p \parallel k} \left(1 - \frac{1}{p}\right) - \prod_{p \parallel k} \left(1 - \frac{1}{p}\right) \leq \sum_{p > z, p \parallel k} \sum_{a \parallel k, \equiv 0(p)} \frac{\mu^2(d)}{d} = \sum_{p > z, p \parallel k} \frac{1}{p} \sum_{d \parallel k, (d, p) = 1} \frac{\mu^2(d)}{d} = o\left(\frac{1}{z} \log z \log k_{\log \log 3k} = o\left(\frac{\log \log 3k}{z}\right)\right),$$

from which follows (1) at once.

Let $q$ be a prime number in the interval $z < q \leq z$, with $(q, k) = 1$. We next evaluate from above the number $S(q)$ of those integers $kn + l$ ($M < n \leq M + N$) which are not divisible by any prime $p \leq z$ and are divisible by the prime $q$. Applying the ‘upper’ sieve of A. Selberg (see the Appendix below), we find that

$$S(q) \leq \frac{N}{q W_q} + R(q),$$

where

$$W_q = \sum_{n \leq z \atop (n, k) = 1} \frac{\mu^2(n)}{\varphi(n)} \left(\frac{\log n}{\log y}\right).$$

with

$$a = \frac{\alpha}{2} \left(1 - \frac{1}{z} - \frac{\log q}{\log y}\right)$$

and

$$R(q) = O \left(\frac{z^{2a}}{W_q^2}\right) = O \left(\frac{w^2}{q W_q^2}\right).$$

Now, let $r \geq 1$ be a fixed integer and let $S_r$ denote the number of those integers of the form $kn + l$ ($M < n \leq M + N$) which are not divisible by any prime $p \leq z$ and are divisible by at least $r + 1$ distinct primes $q$ in the interval $z < q \leq z$, with $(q, k) = 1$. Clearly $S_r$ is not greater than
\[
\frac{1}{r+1} \sum_{z < q \leq q_{1}} S(q).
\]

Hence:

**Lemma 3.** We have

\[
S_{z} \leq \frac{N}{r+1} \sum_{z < q \leq q_{1}} \frac{1}{q W_{q}} + O \left( \frac{w^{2} (\log \log 3k)^{3}}{\log^{2} y} \right).
\]

**Proof.** It will later be shown that

\[
W_{q} > c \frac{\varphi(k)}{k} \log y \quad \text{for } z < q \leq q_{1}.
\]

It follows that

\[
\frac{1}{r+1} \sum_{z < q \leq q_{1}} R(q) = O \left( \frac{w^{2} (\log \log 3k)^{3}}{\log^{2} y} \sum_{z < q \leq q_{1}} \frac{1}{q} \right)
\]

\[= O \left( \frac{w^{2} (\log \log 3k)^{3}}{\log^{2} y} \right), \]

since

\[
\sum_{z < q \leq q_{1}} \frac{1}{q} = \log \frac{\alpha}{\beta} + O(1) = O(1).
\]

**3. Some lemmas.** Here we collect some auxiliary results which will be needed in the next two sections.

**Lemma 4.** We have

\[
\sum_{d \mid m} \frac{\mu^{2}(d) \log d}{d} = O \left( (\log \log 3m)^{2} \right).
\]

**Proof.** The left-hand side is equal to

\[
\sum_{d \mid m} \frac{\mu^{2}(d)}{d} \sum_{p \mid d} \log p = \sum_{p \mid m} \log p \sum_{d \mid m} \frac{\mu^{2}(d)}{d}
\]

\[= \sum_{p \mid m} \frac{\log p}{p} \sum_{d \mid m, (d, p) = 1} \frac{\mu^{2}(d)}{d}, \]

where we have

\[
\sum_{d \mid m, (d, p) = 1} \frac{\mu^{2}(d)}{d} \leq \sum_{d \mid m} \frac{1}{d} = O(\log \log 3m).
\]
and
\[
\sum_{p|m} \frac{\log p}{p} = \sum_{p\leq \log m} \frac{\log p}{p} + O(1)
= O(\log \log 3m).
\]

This proves Lemma 4.

**Lemma 5.** We have
\[
\sum_{\substack{n \leq t \\ (n, m) = 1}} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(m)}{m} \log t + O(\log \log 3m).
\]

**Proof.** H. N. Shapiro and J. Warga [4: Appendix I] have proved that
\[
\sum_{\substack{n \leq t \\ (n, m) = 1}} \frac{\mu^2(n)}{n} = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \log t + O(\log \log 3m).
\]

Using this inequality we obtain
\[
\sum_{\substack{n \leq t \\ (n, m) = 1}} \frac{\mu^2(n)}{\varphi(n)} = \sum_{\substack{n \leq t \\ (n, m) = 1}} \frac{\mu^2(n)}{n} \prod_{p \mid n} \left(1 + \frac{1}{p-1}\right)
= \sum_{\substack{n \leq t \\ (n, m) = 1}} \frac{\mu^2(n)}{n} \sum_{d \mid n} \frac{1}{\varphi(d)}
= \sum_{\substack{d \leq t \\ (d, m) = 1}} \frac{\mu^2(d)}{d \varphi(d)} \sum_{\substack{n \leq t/d \mid dm \\ (n, dm) = 1}} \frac{\mu^2(n)}{n}
= \frac{\varphi(m)}{m} \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right) \log \frac{t}{d} + O(\log \log 3dm)
+ O\left(\sum_{a \leq t} \frac{\mu^2(d)}{d \varphi(d)} \log \log 3dm\right)
= \frac{\varphi(m)}{m} \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right) \log t
+ O\left(\sum_{a \leq t} \frac{\mu^2(d)}{d^2} \prod_{p \mid d} \left(1 - \frac{1}{p^2}\right)^{-1} \log d\right)
+ O\left(\sum_{a \leq t} \frac{\mu^2(d)}{d \varphi(d)} \log \log 3dm\right).
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\[ \frac{\varphi(m)}{m} \log t + O(\log \log 3m), \]

since

\[ \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{d \mid a} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{p \mid m} \left(1 + \frac{1}{p^2 - 1}\right) = \prod_{p \mid m} \left(1 - \frac{1}{p^2}\right)^{-1}. \]

Now, for \( u > 0, \ v \geq 2 \), let \( G(u, v) \) denote the number of positive integers \( n \leq u \) with \( g(n) \leq v \).

Define the function \( \rho(s) \) by the following properties:

\[
\begin{align*}
\rho(s) &= 0 \quad (s < 0); \\
\rho(s) &= 1 \quad (0 \leq s \leq 1); \\
\rho'(s) &= -\rho(s-1) \quad (s > 1); \\
\rho(s) & \text{ continuous for } s > 0.
\end{align*}
\]

Then the following result has been proved by N. G. de Bruijn [1]:

Lemma 6. Let \( u > 0, \ v \geq 2 \), and put \( t = (\log u)/\log v \). Then we have

\[ G(u, v) = O(ue^{-ct}) \]

and, more precisely,

\[ G(u, v) = u\rho(t) \left(1 + O\left(\frac{\log(2+t)}{\log v}\right)\right) + O(1) + O(ut^2P(v)), \]

where \( P(v) \) is a function satisfying

\[ P(v) \downarrow 0 \quad (v \to \infty), \quad P(v) > (\log v)/v \quad (v \geq 2), \]

\[ |\pi(v) - \text{li} v| < vP(v)/\log v \quad (v \geq 2). \]

As to the function \( \rho(s) \) itself, it is not difficult to prove the following result, which is known as a lemma of N. C. Ankeny:

Lemma 7. For \( s_1 \geq s_2 \geq 1 \) we have

\[ \rho(s_1) \leq \rho(s_2)e^{-(s_1-s_2)}, \]

so that

\[ \int_{s}^{\infty} \rho(t) \, dt \leq \rho(s) \quad (s \geq 1). \]

For a proof of this result see [8].

4. Evaluation of \( S_1 \). We are now going to find an explicit lower bound for \( S_1 \) on the basis of Lemma 2.
First we have to evaluate $Z_p$ and $H_p - Z_p$ for $p \leq z$. To accomplish this we distinguish three cases on the magnitude of the prime $p$.

It is clear that

$$T_p \overset{\text{def}}{=} \sum_{n > w/\sqrt{p}, \frac{w}{\sqrt{p}} \leq n \leq p, (n, k) = 1} \frac{1}{n} \geq H_p - Z_p \geq 0.$$ 

**Case 1**: $2 \leq p \leq \exp(\log y)^{3}$. By partial summation we get

$$T_p \leq \sum_{n > w/\sqrt{p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O(y^{-c_5}),$$

where $c_5 = \frac{1}{2} - \epsilon - \frac{1}{2\alpha}$. By Lemma 6 we have

$$\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = O\left(\sum_{n > w/\sqrt{p}} n^{-(1+\epsilon)/\log p}\right) = O\left((\log y)^{\frac{5}{3}} \exp\left(-c(\log y)^{\frac{1}{3}}\right)\right).$$

It follows that

$$H_p - Z_p = O\left(\frac{1}{\log^2 y}\right), \quad Z_p > c \frac{\varphi(k)}{k} \log p,$$

since, by Lemma 1,

$$H_p = \prod_{\frac{w}{\sqrt{p}} \leq n \leq p, (n, k) = 1} \left(1 - \frac{1}{q}\right)^{-1} \geq \frac{\varphi(k)}{k} e^{C \log p} + O(1).$$

**Case 2**: $\exp(\log y)^{\frac{5}{3}} < p \leq z$. We have

$$T_p = \sum_{n > w/\sqrt{p}, \frac{w}{\sqrt{p}} \leq n \leq p, (n, k) = 1} \frac{1}{n} \sum_{d | (n, k)} \mu(d)$$

$$= \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{n > w/\sqrt{p}} \frac{1}{n}$$

$$= \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{n > w/\sqrt{p}} \sum_{w < n/d < w/\sqrt{p}} \frac{1}{n}$$

$$+ \sum_{d | (n, k)} \frac{\mu(d)}{d} \sum_{w < n/d < w/\sqrt{p}} \sum_{n > w/\sqrt{p}} \frac{1}{n}$$
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\[
\prod_{q \mid k} \left( 1 - \frac{1}{q} \right) \sum_{g(n) \leq p} \frac{1}{n} = O \left( (\log \log 3k)^2 \right),
\]

since we have, by Lemma 4,

\[
\sum_{d \mid k} \frac{\mu(d)}{d} \sum_{g(d) \leq p} \frac{1}{n} = O \left( \sum_{d \mid k} \frac{\mu^2(d) \log d}{d} \right)
\]

\[
= O \left( (\log \log 3k)^2 \right).
\]

Now, by partial summation, we have

\[
\sum_{n > w/\sqrt{p}} \frac{1}{n} = \sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O \left( y^{-c} \right).
\]

Here, by Lemma 6, we find that

\[
\sum_{n > \exp(\log y)^2} \frac{G(n, p)}{n^2} = O \left( \sum_{n > \exp(\log y)^2} n^{-\left( 1 + \frac{c}{\log p} \right)} \right)
\]

\[
= O \left( \log y \exp(-c \log y) \right)
\]

\[
= O \left( (\log y) y^{-\alpha} \right),
\]

so that

\[
\sum_{n > w/\sqrt{p}} \frac{G(n, p)}{n^2} = \sum_{\exp(\log y)^2 < n > w/\sqrt{p}} \frac{G(n, p)}{n^2} + O \left( \frac{1}{\log^2 y} \right).
\]

Let us write \( I \) for the interval \( w/\sqrt{p} < n \leq \exp(\log y)^2 \). Then, by making use of the result in Lemma 6, we obtain

\[
\sum_{n \in I} \frac{G(n, p)}{n^2} = \sum_{n \in I} \frac{1}{n} \rho \left( \frac{\log n}{\log p} \right) \left( 1 + O \left( \frac{\log \log y}{\log p} \right) \right)
\]

\[
+ O \left( \sum_{n \in I} \frac{1}{n} \right) + O \left( \sum_{n \in I} \frac{\left( \frac{\log n}{\log p} \right)^2 P(p)}{n} \right).
\]

It is easily verified that

\[
\sum_{n \in I} \frac{1}{n} \rho \left( \frac{\log n}{\log p} \right) = \log p \int_{t_p}^{\infty} \rho(t) dt + O(1)
\]

where \( t_p = (\log w/\sqrt{p})/\log p \).
\[
\sum_{n \in I} \frac{1}{n} = O(y^{-c_5}); \quad \sum_{n \in I} \frac{1}{n} \left(\frac{\log n}{\log p}\right)^2 P(p) = O\left(\frac{1}{\log^2 y}\right),
\]
where we have taken \(P(v) = c \exp(-c(\log v)^{1/2})\).

We thus have

\[
T_p = \log p \int_{t_p}^{\infty} \rho(t) dt \left(1 + O\left(\frac{\log \log y}{\log p}\right)\right) + O\left(\frac{1}{\log^2 y}\right).
\]

Hence

\[
H_p - Z_p \leq \prod_{q \downarrow k} \left(1 - \frac{1}{q}\right) \log p \int_{t_p}^{\infty} \rho(t) dt \left(1 + O\left(\frac{\log \log y}{\log p}\right)\right)
+ O\left(\left(\log \log 3k\right)^2\right),
\]

\[
Z_p \geq \prod_{q \downarrow k} \left(1 - \frac{1}{q}\right) \left(e^c - \int_{t_p}^{\infty} \rho(t) dt\right) \log p + O\left(\log \log y\right)
+ O\left(\left(\log \log 3k\right)^2\right).
\]

**Case 3:** \(x < p \leq z\). Put \(t_p = (\log w/\sqrt{p})/\log p\), as before. If \(0 < t_p \leq 1\) then we have

\[
Z_p = \sum_{n \leq w/\sqrt{p}} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(k)}{k} \log \frac{w}{\sqrt{p}} + O(\log \log 3k)
+ \frac{\varphi(k)}{k} t_p \log p + O(\log \log 3k),
\]

\[
H_p - Z_p = \frac{\varphi(k)}{k} (e^c - t_p) \log p + O(\log \log 3k),
\]

by Lemma 5. If \(t_p > 1\) then

\[
Z_p \geq \sum_{n \leq w/\sqrt{p}} \frac{\mu^2(n)}{\varphi(n)} = \sum_{n \leq w/\sqrt{p}} \frac{\mu^2(n)}{\varphi(n)} - \sum_{n \leq w/\sqrt{p}} \frac{1}{\varphi(n)} \sum_{n \leq w/\sqrt{p}} \frac{\mu^2(n)}{\varphi(n)}
\]

where, again by Lemma 5,
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\[
\sum_{p \leq q \leq w/\sqrt{p}} \frac{1}{\varphi(q)} \sum_{n \leq q \sqrt{p}} \frac{\mu^2(n)}{\varphi(n)} = \sum_{p \leq q \leq w/'\sqrt{p}} \frac{1}{\varphi(q)} \left( \frac{\varphi(qk)}{qk} \log \frac{w}{q\sqrt{p}} + O(\log \log 3qk) \right)
\]

\[
= \frac{\varphi(k)}{k} \sum_{p \leq q \leq w/\sqrt{p}} \frac{1}{q} \log \frac{w}{q \sqrt{p}} + O\left( \sum_{p \leq q \leq w/\sqrt{p}} \frac{\log \log 3qk}{\varphi(q)} \right)
\]

\[
= \frac{\varphi(k)}{k} \left( \log \frac{w}{\sqrt{p}} \log \frac{\log \frac{w}{\sqrt{p}}}{\log} - \log \frac{w}{\sqrt{p}} + \log p \right) + O(\log \log y),
\]

and hence

\[
Z_p \geq \frac{\varphi(k)}{k} \left( 2t_p - 1 - t_p \log t_p \right) \log p + O(\log \log y).
\]

Therefore

\[
H_p - Z_p \leq \frac{\varphi(k)}{k} \left( e^C - \left( 2t_p - 1 - t_p \log t_p \right) \right) \log p + O(\log \log y).
\]

Here we have, as in the proof of Lemma 2,

\[
H_p = \prod_{q < p} \left( 1 - \frac{1}{q} \right) \prod_{q < p} \left( 1 - \frac{1}{q} \right)^{-1}
\]

\[
= \left( \frac{\varphi(k)}{k} + O\left( \frac{\log \log 3k}{p} \right) \right) \left( e^C \log p + O(1) \right)
\]

\[
= \frac{\varphi(k)}{k} e^C \log p + O(\log \log 3k).
\]

We are now in position to be able to evaluate the sum

\[
\sum_{p \leq \infty} \frac{H_p - Z_p}{pH_pZ_p}.
\]

Define:

\[
A(t) = \begin{cases} 
\frac{e^t - t}{t} & (0 < t \leq 1), \\
\frac{e^t - (2t - 1 - t \log t)}{2t - 1 - t \log t} & (1 < t < e^t), 
\end{cases}
\]
where $t = e^\epsilon$ is the unique solution of

$$2t - 1 - t \log t = 0, \quad t > 1,$$

so that $1.8 < c_\epsilon < 1.9$; and

$$B(t) = \frac{\int_t^\infty \rho(s) ds}{e^\epsilon - \int_t^\infty \rho(s) ds} \quad (t > \frac{1}{4}).$$

Let us put, for the sake of brevity,

$$z_\epsilon = \exp(\log y)^{2/3}.$$

Then we have

$$\sum_{\substack{2 \leq p \leq z_\epsilon \\ (p, k) = 1}} \frac{H_p - Z_p}{pH_p Z_p} = O\left(\frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^2 y} \sum_p \frac{1}{p \log^2 p}\right).$$

and, noticing that

$$\prod_{q \leq p} \left(1 - \frac{1}{q}\right)^{-1} \leq \frac{k}{\varphi(k)},$$

for every $p$,

$$\sum_{\gamma \leq \nu < c(\log y)} g \sum_{p \in J_\nu} \frac{B(t_p)}{p \log p} = \sum_{\gamma \leq \nu < c(\log y)} g \sum_{p \in J_\nu} \frac{B(t_p)}{p \log p},$$

where we have used the inequality

$$\sum_{\zeta \leq p \leq z_\epsilon} \frac{1}{p \log^2 p} = O\left(\frac{1}{\log^2 z_\epsilon}\right).$$

We now assume that $\gamma, 4 \leq \gamma \leq 10$, be integral. Write $J_\gamma$ for the interval $y^{1/\nu+1} < p \leq y^{1/\nu}(\nu \geq \gamma)$. Then, since the function $B(t)$ is monotone decreasing,

$$\sum_{\gamma \leq \nu < c(\log y)} g \sum_{\gamma \leq \nu < c(\log y)} g \sum_{p \in J_\nu} \frac{B(t_p)}{p \log p} = \sum_{\gamma \leq \nu < c(\log y)} g \sum_{p \in J_\nu} \frac{B(t_p)}{p \log p}.$$
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\[
\sum_{\nu=\nu_0}^{\nu_1} B(t_\nu) \leqq \sum_{\nu=\nu_0}^{\nu_1} \frac{1}{\log p} \max_{\nu_0 \leqq \nu < e(\log y)^{\frac{1}{3}}} \sum_{p \in J_{\nu}} \frac{1}{p}
\]

\[
= \sum_{\nu=\nu_0}^{\nu_1} \log \frac{\nu+1}{\nu} B \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2}
\]

\[
+ O \left( \sum_{\nu=\nu_0}^{\nu_1} \frac{\nu}{\log^2 y} \right)
\]

\[
\leqq \frac{1}{\log y} \sum_{\nu=\nu_0}^{\nu_1} (\nu+1) \log \frac{\nu+1}{\nu} B \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2}
\]

Thus we obtain

\[
\sum_{z_2 < p \leqq z_2} \frac{H_p - Z_p}{pH_pZ_p} \leqq \frac{k}{\varphi(k)} \frac{e^{-C}}{\log y} \sum_{\nu=\nu_0}^{\nu_1} (\nu+1) \log \frac{\nu+1}{\nu} B \left( \frac{1}{2} - \epsilon \right) \nu - \frac{1}{2}
\]

\[
+ O \left( \frac{k}{\varphi(k)} \frac{(\log \log 3k)^3}{\log^3 y} \right) + O \left( \frac{k}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y} \right)
\]

We have similarly

\[
\sum_{z_2 < p \leqq z_2} \frac{H_p - Z_p}{pH_pZ_p} \leqq \frac{k}{\varphi(k)} e^{-C} \sum_{\nu=\nu_0}^{\nu_1} A(t_\nu) \frac{\nu}{p \log p}
\]

\[
+ O \left( \frac{k}{\varphi(k)} \frac{(\log \log 3k) \log \log y}{\log^3 y} \right)
\]

Put

\[ n = [\log^{1/2} y], \quad u_j = \alpha + \frac{\gamma - \alpha}{n} j \quad (j \geqq 0), \]

and write \( K_j \) for the interval

\[ y^{1/u_{j+1}} < p \leqq y^{1/u_j} \quad (0 \leqq j \leqq n-1). \]

Now, the function \( A(t) \) is continuous, monotone decreasing in the interval \( 0 < t \leqq e \) and monotone increasing in the interval \( e < t < e^\alpha \). Thus, if we denote by \( m \) the integer for which

\[ \left( \frac{1}{2} - \epsilon \right) u_m - \frac{1}{2} \leqq \epsilon < \left( \frac{1}{2} - \epsilon \right) u_{m+1} - \frac{1}{2}, \]

then
\[
\sum_{\epsilon_{2} < p \leq z} \frac{A(t_{p})}{p \log p} = \sum_{j=0}^{n-1} \sum_{p \in \mathcal{K}_{j}} \frac{A(t_{p})}{p \log p}
\]

\[
\leq \sum_{j=0}^{n-1} \left( \sum_{p \in \mathcal{K}_{j}} \frac{1}{p} \right) \max_{\not\in x_{j}} \frac{A(t_{p})}{\log p}
\]

\[
\leq \sum_{j=0}^{n-1} \log \frac{u_{j+1}}{u_{j}} \log y \ A\left(\frac{1}{2} - \varepsilon u_{j} - \frac{1}{2}\right)
\]

\[
+ \log \frac{u_{m+1}}{u_{m}} \frac{u_{m+1}}{\log y} \max\left(A\left(\frac{1}{2} - \varepsilon u_{m} - \frac{1}{2}\right), A\left(\frac{1}{2} - \varepsilon u_{m+1} - \frac{1}{2}\right)\right)
\]

\[
\leq \int_{\alpha}^{\gamma} A\left(\frac{1}{2} - \varepsilon u - \frac{1}{2}\right) du + O\left(\frac{1}{\log^{3/2} y}\right).
\]

where it should be noticed that we have uniformly

\[
u_{j+1}, \log \frac{u_{j+1}}{u_{j}} = \frac{r-\alpha}{n} + O\left(\frac{1}{n^{2}}\right) \quad (0 \leq j \leq n-1).
\]

Hence

\[
\sum_{\epsilon_{2} < p \leq z} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} \leq \frac{k}{\varphi(k)} \frac{e^{-C}}{\log y} \int_{\alpha}^{\gamma} A\left(\frac{1}{2} - \varepsilon u - \frac{1}{2}\right) du
\]

\[
+ O\left(\frac{k}{\varphi(k)} \frac{(\log \log y) \log \log 3k}{\log^{2} y}\right) + O\left(\frac{k}{\varphi(k)} \frac{1}{\log^{3/2} y}\right).
\]

Collecting these results, we thus obtain, via Lemma 2, the following

**Lemma 8.** We have

\[
\sum_{\epsilon_{2} < p \leq z} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} \geq \frac{kN}{\varphi(k)} \frac{e^{-C}}{\log y} \left(\alpha - \int_{\alpha}^{\gamma} A\left(\frac{1}{2} - \varepsilon u - \frac{1}{2}\right) du\right)
\]

\[
- \sum_{\nu=1}^{\nu} (\nu+1) \log \frac{\nu+1}{\nu} B\left(\frac{1}{2} - \varepsilon \nu - \frac{1}{2}\right)
\]

\[
+ O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^{3}}{\log^{3/2} y}\right) + O\left(\frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{2} y}\right)
\]

\[
+ O\left(\frac{N(\log \log 3k)^{3}}{y^{1/4} \log y}\right) + O\left(y^{1-2} (\log \log 3k)^{2}\right).\]
5. Evaluation of $S_{2}$. By virtue of Lemma 3, our present task is only to estimate the quantity

$$\sum_{z<q\leq z_{1}} \frac{1}{qW_{q}}.$$  

We set

$$C(t) = \begin{cases} \frac{\alpha}{a} & (0 < a \leq 1), \\ \frac{\alpha}{2a - 1 - a \log a} & (1 < a \leq 2), \end{cases}$$

where

$$a = \frac{\alpha}{2} \left( 1 - 2\epsilon - \frac{1}{t} \right).$$

Then, it is not difficult to verify, by Lemma 5, that, with $t = t_{q} = (\log y)/\log q$

$$W_{q} = \sum_{n \leq q, \varphi(n) \leq z} \frac{\mu^{2}(n)}{\varphi(n)} (n, k) = 1 \geq \frac{\varphi(k)}{k} \frac{\log y}{C(t_{q})} + O(\log \log 3k) (z < q \leq z_{1}),$$

and consequently

$$\sum_{z<q\leq z_{1}} \frac{1}{qW_{q}} \leq \frac{1}{\varphi(k)} \frac{1}{\log y} \sum_{z<q\leq z_{1}} \frac{C(t_{q})}{q} + O\left( \frac{1}{\log^{\frac{1}{2}} y} \right).$$

Put

$$n = \lfloor \log^{\frac{1}{2}} y \rfloor, \quad u_{j} = \beta + \frac{\alpha - \beta}{n} j \quad (0 \leq j \leq n),$$

and write $L_{j}$ for the interval

$$y^{1/u_{j+1}} < q \leq y^{1/u_{j}} \quad (0 \leq j \leq n-1).$$

Then we have

$$\sum_{z<q\leq z_{1}} \frac{C(t_{q})}{q} \leq \sum_{j=0}^{n-1} \sum_{q \in L_{j}} \frac{C(t_{q})}{q} + \frac{1}{\varphi(k)} \frac{(\log \log 3k)^{3}}{\log^{2} y}.$$
since the function $C(t)$ is continuous and decreases monotonously for $t > (1 - 2\epsilon)^{-1}$ and since we have uniformly

$$\log \frac{u_{j+1}}{u_j} = \frac{\alpha - \beta}{n} \frac{1}{u_j} + O\left(\frac{1}{n^2}\right) \quad (0 \leq j \leq n - 1).$$

We thus have proved the following

**Lemma 9.** We have

$$S_2 \leq \frac{1}{r+1} \frac{kN}{\varphi(k)} \frac{1}{\log y} \int_{\beta}^{\alpha} \frac{C(u)}{u} \, du$$

$$+ O\left(\frac{kN}{\varphi(k)} \frac{1}{\log^2 y}\right) + O\left(\frac{\log \log 3k}{\log^3 y}\right) + O\left(\frac{\gamma^{1-2\epsilon}}{\log^2 y}\right).$$

**6. Numerical computations.** We need the following easy lemma, a part of which has already been used in the proof of Lemmas 8 and 9.

**Lemma 10.** The function

$$f(s) = \frac{1}{2s - 1 - s \log s} \quad (1 < s < e^c)$$

is positive, convex, and monotone decreasing for $1 < s \leq e$ and monotone increasing for $e < s < e^c$.

Putting $f_1(s) = (f(s))^{-1}$, we see that $f_1(s) > 0$, $f'_1(s) = 1 - \log s$ and $f''_1(s) = -1/s$, and the result follows at once.

We now choose $\epsilon = \frac{1}{7}$ and take

$$\alpha = 4, \quad \beta = 2, \quad \text{and} \quad \gamma = 10.$$

Our aim in this section is to compute numerically two integrals and a sum appearing in Lemmas 8 and 9.

(i) Computation of

$$\int_{\alpha}^{\gamma} A(\left(\frac{1}{2} - \epsilon\right)u - \frac{1}{2}) \, du = \int_{4}^{10} A(\frac{5}{14}u - \frac{1}{2}) \, du.$$

The integral is equal to

$$\int_{4}^{1.2} A(\frac{5}{14}u - \frac{1}{2}) \, du + \int_{1.2}^{10} A(\frac{5}{14}u - \frac{1}{2}) \, du,$$

where the first integral is found to be
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\[ e^C \int_{4}^{4.2} \left( \frac{5}{14}u - \frac{1}{2} \right)^{-1} du - \int_{4}^{4.2} du = e^C \frac{14}{5} \log \frac{14}{13} - 0.2 < 0.1696, \]

while the second is

\[ e^C \int_{4}^{10} F(u) du - \int_{4}^{10} du \]

with \( F(u) = f(s(u)) \), where \( f(s) \) is the function defined in Lemma 10 and \( s(u) = \frac{5}{14}u - \frac{1}{2} \). To estimate the integral of \( F(u) \) over \((4.2, 10)\) we proceed as follows.

We find:

\[
\begin{align*}
F(4.2) &= 1.0000; & F(4.5) &< 0.9080; \\
F(5) &= 0.8011; & F(6) &< 0.6803; \\
F(7) &= 0.6197; & F(8) &< 0.5907; \\
F(9) &= 0.5820; & F(10) &< 0.5896.
\end{align*}
\]

By Lemma 10, the function \( F(u) \) is convex for \( 4.2 \leq u \leq 10 \). Hence

\[
\int_{4.2}^{10} F(u) du \leq \frac{3}{20} (F(4.2) + F(4.5)) + \frac{1}{4} (F(4.5) + F(5)) + \frac{1}{2} (F(5) + F(10)) + (F(6) + F(7) + F(8) + F(9)) < 3.8817,
\]

and the second integral in (3) is less than

\[ 3.8817 e^C - 5.8 < 1.1137. \]

Thus we have

\[
\int_{4}^{10} A \left( \frac{5}{14}u - \frac{1}{2} \right) du < 0.1696 + 1.1137 = 1.2833.
\]

(ii) Computation of

\[
\sum_{\nu=10}^{\infty} \frac{(\nu + 1) \log \frac{\nu + 1}{\nu} B\left(\frac{5}{14} \nu - \frac{1}{2}\right)}{\nu} = \sum_{\nu=10}^{\infty} \frac{(\nu + 1) \log \frac{\nu + 1}{\nu} B\left(\frac{5}{14} \nu - \frac{1}{2}\right)}{\nu}.
\]
By the definition (2), the function $\rho(s)$ is positive and monotone decreasing for $s > 0$, and moreover

$$\rho(s) = 1 - \log s \quad \text{for} \quad 1 \leq s \leq 2.$$ 

Put $s(\nu) = \frac{5}{14} \nu - \frac{1}{2}$. Then we have $s(10) = \frac{43}{14} > 3$ and

$$\rho(s(10)) \leq \rho(3) \leq \rho(2)e^{-1} = (1 - \log 2)e^{-1} < 0.1129,$$

by Lemma 7. Now, using Lemma 7 again, we find that for $\nu \geq 10$

$$B(s(\nu)) \leq \frac{\rho(s(\nu))}{e^{\nu} - \rho(s(\nu))} \leq \frac{\rho(s(10))}{e^{\nu} - \rho(s(10))} e^{-\frac{5}{14}(\nu - 10)}.$$ 

Since $(\nu + 1)\log((\nu + 1)/\nu)$ decreases monotonously as $\nu \to \infty$, we thus obtain

$$\sum_{\nu=10}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} B(s(\nu)) \leq 11 \log \frac{11}{10} \frac{\rho(s(10))}{e^{C} - \rho(s(10))} \frac{1}{1 - e^{-5/14}} < 0.2366.$$

(iii) Computation of

$$\int_{\beta}^{\alpha} \frac{C(u)}{u} \, du = \int_{\beta}^{\alpha} \frac{C(u)}{u} \, du.$$

For $2 \leq u \leq 4$ we have

$$\frac{3}{7} \leq a = 2 \left( \frac{5}{7} - \frac{1}{u} \right) \leq \frac{13}{14}.$$

Hence

$$\int_{\beta}^{4} \frac{C(u)}{u} \, du = 2 \int_{\beta}^{\frac{5}{7} \left( u - 1 \right)^{-1}} \, du$$

$$= \frac{14}{5} \log \frac{13}{3} < 4.1058.$$

7. **Proof of the theorem.** Let $1 \leq k < x$, $0 \leq l \leq k - 1$, $(k, l) = 1$. Take

$$M = \left[ \frac{x - l}{k} \right], \quad N = \left[ \frac{x}{k} \right],$$

and put
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Then it is clear that $y > 2x$ and that $M < n \leq M + N$ implies $x < kn + l \leq 2x$.

By $D(x; k, l)$ we denote the number of those integers of the form $kn + l$ ($M < n \leq M + N$) which are divisible by no primes $p \leq z$, by at most two primes $q$ in $z < q \leq z_1$, and by no integers of the form $q^2$, $q$ being a prime in $z < q \leq z_1$: clearly such an integer $kn + l$ ($M < n \leq M + N$), if it exists, has at most two prime factors, i.e. $V(kn + l) \leq 2$.

In order to estimate $D(x; k, l)$ from below, we apply Lemma 8 and Lemma 9 with $r = 2$. Let us note that we have from the data in §6

$$e^{-c} \left(4 - \int_{4}^{10} A(s(u)) du - \sum_{
u=10}^{\infty} \left(\nu + 1\right) \log \frac{\nu + 1}{\nu} B(s(\nu))\right)$$

$$> e^{-c}(4 - 1.2833 - 0.2366) > 1.3923$$

and

$$\frac{1}{3} \int_{2}^{4} \frac{C(u)}{u} du < \frac{4.1058}{3} = 1.3686.$$ 

Now, the number $R_2$ of those integers $kn + l$ ($M < n \leq M + N$) which are not divisible by any prime $p \leq z$ and are divisible by some integer $q^2$ with $q$ in $z < q \leq z_1$ does not exceed

$$\sum_{z < q \leq z_1} \left(\frac{N}{q^2} + 1\right) = O\left(\frac{N}{z}\right) + O(z_1).$$

We find, therefore, that

$$D(x; k, l) \geq S_1 - S_2 - R_2 \geq (1.3923 - 1.3686) \frac{kN}{\varphi(k)} \frac{1}{\log y}$$

$$+ O\left(\frac{kN \log \log 3k^3}{\varphi(k) \log^{3/2} y}\right) + O\left(\frac{kN \log \log y}{\varphi(k) \log^{3/2} y}\right)$$

$$+ O\left(\frac{N \log \log 3k^3}{y^{1/4} \log y}\right) + O\left(y^{5/7} \log \log 3k^3\right) + O\left(\frac{N}{y^{1/4}}\right).$$

Since $N = \frac{x}{k} + O(1)$, $2x < y \leq 4x$, it follows that

$$D(x; k, l) \geq 0.0237 \frac{1}{\varphi(k)} \frac{x}{\log x}$$

$$+ O\left(\frac{1}{\varphi(k)} \frac{x \log \log 3k^3}{\log^{3/2} x}\right) + O\left(\frac{1}{\varphi(k)} \frac{x \log \log x}{\log^{3/2} x}\right).$$
\[ + O \left( \frac{1}{k} \frac{x^{3/4} (\log \log 3k)^{3}}{\log x} \right) + O \left( x^{5/7} (\log \log 3k)^{2} \right) + O \left( \frac{1}{k} x^{3/4} \right). \]

Let \( c_{1} > 3.5 \) be a fixed number. If \( x \geq k^{c_{1}} \) and \( k \) is sufficiently large, then all the error terms on the right-hand side of the above inequality for \( D(x; k, l) \) are of negligible order of magnitude, with respect to the leading term. Thus, for all large enough \( k \), \( x \geq k^{c_{1}} \) implies that

\[ D(x; k, l) > 0.0236 \frac{1}{\varphi(k)} \frac{x}{\log x} > 1. \]

Hence, by continuity argument, we conclude that there is a (finite) natural number \( k_{0} \) such that, if \( k \geq k_{0} \) and \( x \geq k^{3.5} \) then we have \( D(x; k, l) > 0 \). Therefore there exists an absolute constant \( c_{1} > 0 \) such that

\[ D(x; k, l) > 0 \quad \text{for all} \quad x \geq c_{1} k^{3.5}, \quad k \geq 1. \]

This completes the proof of our theorem.

**Appendix**

**ON THE 'UPPER' SIEVE OF A. SELBERG**

Here we aim at generalizing the results obtained in [6]. Let \( N > 1 \) and let \( a_{1}, a_{2}, \ldots, a_{N} \) be rational integers not necessarily different from each other. Let \( S \) denote the number of those integers \( a_{j} (1 \leq j \leq N) \) which are not divisible by any prime number \( p \leq x \), where \( x \geq 2 \). Suppose that for every positive integer \( d \)

\[ S_{d} \overset{\text{def}}{=} \sum_{\substack{n \leq N \\mid a_{n} \not\equiv 0(d)}} 1 = \frac{\omega(d)}{d} N + R(d), \]

where \( R(d) \) is the error term for \( S_{d} \) and \( \omega(d) \) is a multiplicative function of \( d \). We put

\[ f(d) = \frac{d}{\omega(d)} \]

and suppose that \( f(d) > 1 \) for all \( d > 1 \).

Let \( w \) be an arbitrary but fixed real number such that \( w \geq 2 \). We define for positive integers \( m \) and \( d \)

\[ f_{1}(m) = \sum_{n \mid m} \mu(n) f \left( \frac{m}{n} \right), \]
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\[ W(d) = \sum_{r \leq w/d \atop (r, d) = 1} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)} , \quad W = W(1) , \]

\[ \lambda(d) = \epsilon_z(d) \mu(d) \prod_{p | d} \left( 1 - \frac{1}{f(p)} \right)^{-1} . \frac{W(d)}{W} , \]

where \( \epsilon_z(n) = 0 \) or 1 according as \( n \) has or has not a prime factor \( > z \). Then we have, since \( \lambda(1) = 1 \),

\[ S \leq \sum_{n \leq N} \left( \sum_{d \leq w \atop d \mid n} \lambda(d) \lambda(d) \right)^2 = \sum_{d \leq w} \left( \sum_{d_1, d_2 \leq w \atop \{d_1, d_2\} = d} \lambda(d_1) \lambda(d_2) \right) \frac{N}{f(d)} \]

\[ + \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})| , \]

where \( \{d_1, d_2\} \) denotes the least common multiple of \( d_1 \) and \( d_2 \).

Now

\[ \sum_{d \leq w} \left( \sum_{d_1, d_2 \leq w \atop \{d_1, d_2\} = d} \lambda(d_1) \lambda(d_2) \right) \frac{1}{f(d)} \]

\[ = \sum_{r \leq w} f_1(r) \left( \sum_{d \leq w \atop d \equiv 0 (r)} \frac{\lambda(d)}{f(d)} \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \sum_{d \leq w \atop d \equiv 0 (r)} \epsilon_z(d) \mu(d) \frac{1}{f_1(d)} \sum_{m \leq w/d \atop (m, d) = 1} \epsilon_z(m) \frac{\mu^2(m)}{f_1(m)} \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \epsilon_z(r) \frac{\mu(r)}{f_1(r)} \sum_{n \leq w/r \atop \{n, r\} = 1} \epsilon_z(n) \frac{\mu^2(n)}{f_1(n)} \sum_{d \mid n} \mu(d) \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)} = \frac{1}{W} . \]

We thus have proved the following

**Theorem.** Under the notations and conditions described above we have

\[ S \leq \frac{N}{W} + R \]

with

\[ R = \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R(\{d_1, d_2\})| . \]

This is a generalization of [3: II, Theorem 3.1].

To evaluate the remainder term \( R \) explicitly, let us suppose that for all positive integers \( d, d_1, d_2 \)
\[ |R(d)| \leq B\omega(d), \quad \omega(\{d_1, d_2\}) \leq \omega(d_1) \omega(d_2), \]

where \( B > 0 \) is a constant independent of \( d \). These conditions imply

\[ R \leq B \left( \sum_{d \leq w} \lambda(d) \omega(d) \right)^2. \]

Then, it is not difficult to show that we have, in general,

\[ R = O\left( w^2 (\log \log w)^2 \right), \]

and, in the special case where \( \omega(p) \leq 1 \) for all primes \( p \leq z \),

\[ R = O\left( \frac{w^2}{W^2} \right), \]

where the constants implied in the symbol \( O \) depend only on the constant \( B \).

The proof of these estimates of the remainder term \( R \) can easily be carried out just in the same way as in [6], and we shall omit the details (cf. also [7]).

References


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