ON THE DISTRIBUTION OF ALMOST PRIMES
IN AN ARITHMETIC PROGRESSION

By

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1. Introduction. An almost prime is a positive integer the number of whose prime divisors is bounded by a certain constant. The purpose of this paper is to deal with an existence problem of almost primes in a short arithmetic progression of integers. We shall prove the following

Theorem. Let $k$ and $l$ be two integers with $k \geq 1$, $0 \leq l \leq k-1$, $(k, l)=1$. There exists a numerical constant $c_1 > 0$ such that for every real number $x \geq c_1 k^{3.5}$ there is at least one integer $n$ satisfying

$$x < n \leq 2x, \quad n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

where $V(n)$ denotes the total number of prime divisors of $n$. In particular, if we write $a(k, l)$ for the least positive integer $n (>1)$ satisfying

$$n \equiv l \pmod{k}, \quad V(n) \leq 2,$$

then we have

$$a(k, l) < c_2 k^{3.5}$$

with some absolute constant $c_2 > 0$.

It is of some interest to compare our results presented above, though they are not the best possible, with a recent result of T. Tatuzawa [5] on the existence of a prime number $p$ satisfying $x < p \leq 2x$, $p \equiv l \pmod{k}$ and a celebrated theorem of Yu. V. Linnik concerning the upper bound for the least prime $p \equiv l \pmod{k}$ (cf. [3: X]).

Our proof of the theorem is based essentially upon the general sieve methods due to A. Selberg. The deepest result which we shall refer to is:

$$\pi(x) = \text{li} x + O\left(x \exp\left(-c_3 (\log x)^{1/2}\right)\right)$$

with a positive constant $c_3$, where $\pi(x)$ denotes, as usual, the number of primes not exceeding $x$ (in fact, a slightly weaker result will suffice for our purpose). Apart from this, the proof is entirely elementary.

Notations. Throughout in the following, $k$ represents a fixed positive
integer, \( l \) an integer with \( 0 \leq l \leq k - 1, \ (k, l) = 1. \) The letters \( p, q \) are used to denote prime numbers and, \( d, m, n, r \) to denote positive integers. The functions \( \mu(n) \) and \( \varphi(n) \) are Möbius’ and Euler’s functions, respectively. The function \( g(n) \) is defined as follows: \( g(1) = 1 \) and for \( n > 1 \) \( g(n) \) is the greatest prime divisor of \( n \).

\( s, t, u, v, w, x, y, z \) will be used to denote real numbers, constant or variable. \( c \) represents positive constants, not depending on \( k \) and \( l \), which are not necessarily the same in each occurrence; the constants implied in the symbol \( O \) are either absolute or else uniform in \( k \) and \( l \).

2. Preliminaries. There needs the following lemma for later calculations:

**Lemma 1.** We have

\[
\sum_{p \leq t} \frac{1}{p} = \log \log t + c_4 + O\left(\frac{1}{\log t}\right),
\]

where \( c_4 \) is a constant;

\[
\sum_{p \leq t} \frac{\log p}{p} = \log t + O(1);
\]

\[
\prod_{p \leq t} \left(1 - \frac{1}{p}\right)^{-1} = e^C \log t + O(1),
\]

where \( C \) is Euler’s constant; and

\[
\varphi(m) > c \frac{m}{\log \log 3m}.
\]

These results are well known. For a proof see [3: I, Theorems 3.1, 4.1 and 5.1].

Let \( M \geq 0, N \geq 2 \) be arbitrary but fixed integers and put

\[
y = 2k(N+1), \quad w = y^{\frac{1}{2} - \epsilon},
\]

where \( 0 < \epsilon < \frac{1}{4} \); we shall fix \( \epsilon = \frac{1}{7} \) later on. Further we put

\[
z = y^{\frac{1}{a}}, \quad z_1 = y^{\frac{1}{b}}, \quad z_2 = y^{\frac{1}{r}},
\]

where \( \alpha, \beta, \gamma \) are fixed real numbers satisfying

\[
10 \geq \gamma \geq 4 \geq \alpha > 2 \geq \beta > 1.
\]
First we wish to evaluate from below the number \( S_1 \) of those integers of the form \( kn + l \) \((M < n \leq M + N)\) which are not divisible by any prime \( p \leq z \). Applying the ‘lower’ sieve of A. Selberg (see \([2]\) and \([7]\)), we find that

\[
S_1 \leq (1 - Q)N - R_1 ,
\]

where

\[
Q = \sum_{p \leq z, \gcd(p,k) = 1} \frac{1}{pZ_p}
\]

with

\[
Z_p = \sum_{\substack{n \leq \sqrt{w/p} \\ \gcd(n,p) = 1}} \frac{\mu^2(n)}{\varphi(n)},
\]

and

\[
R_1 = O \left( \omega \sum_{p \leq z} \frac{1}{pZ_p^2} \right).
\]

It will be shown later that

\[
Z_p > c \frac{\varphi(k)}{k} \log p
\]

for all \( p \leq z \), and so we have, by Lemma 1,

\[
R_1 = O \left( \omega^2 (\log \log 3k)^2 \right).
\]

We put

\[
H_p = \prod_{q < p, \gcd(q,k) = 1} \left( 1 - \frac{1}{q} \right)^{-1} \quad (p \leq z).
\]

Then it is easily verified that

\[
1 - Q = \prod_{\substack{p \leq z, \\
(p,k) = 1}} \left( 1 - \frac{1}{p} \right) - \sum_{\substack{p \leq z, \\
(p,k) = 1}} \frac{H_p - Z_p}{pH_pZ_p}.
\]

**Lemma 2.** We have

\[
S_1 \geq \frac{kN}{\varphi(k)} \prod_{p \leq z} \left( 1 - \frac{1}{p} \right) - N \sum_{\substack{p \leq z, \\
(p,k) = 1}} \frac{H_p - Z_p}{pH_pZ_p} + O \left( \frac{N (\log \log 3k)^3}{\log z} \right) + O \left( \omega^2 (\log \log 3k)^2 \right).
\]

**Proof.** We have only to prove that

\[
\prod_{\substack{p \leq z, \\
(p,k) = 1}} \left( 1 - \frac{1}{p} \right) = \frac{k}{\varphi(k)} \prod_{p \leq z} \left( 1 - \frac{1}{p} \right) + O \left( \frac{(\log \log 3k)^3}{\log z} \right).
\]
or

\[
\prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\varphi(k)} + O\left(\frac{(\log \log 3k)^3}{z}\right).
\]

Now we have

\[
0 \leq \prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right) - \frac{\varphi(k)}{k} = \prod_{p \leq z, p \mid k} \left(1 - \frac{1}{p}\right) - \prod_{p \mid k} \left(1 - \frac{1}{p}\right)
\leq \sum_{p > z, p \mid kd} \sum_{a \mid k, \equiv 0(p)} \frac{\mu^2(d)}{d} = \sum_{p > z, p \mid k} \frac{1}{p} \sum_{d \mid k/p, (d, p) = 1} \frac{\mu^2(d)}{d}
\leq O\left(\frac{\log \log 3k}{z}\right),
\]

from which follows (1) at once.

Let \( q \) be a prime number in the interval \( z < q \leq z \), with \((q, k) = 1\). We next evaluate from above the number \( S(q) \) of those integers \( kn + l \) \((M < n \leq M + N)\) which are not divisible by any prime \( p \leq z \) and are divisible by the prime \( q \). Applying the ‘upper’ sieve of A. Selberg (see the Appendix below), we find that

\[
S(q) \leq \frac{N}{q W_q} + R(q),
\]

where

\[
W_q = \sum_{\substack{n \leq z, a \mid \varphi(n) \\ a \mid \varphi(n) \leq x, (n, k) = 1}} \frac{\mu^2(n)}{\varphi(n)}
\]

with

\[
a = \frac{\alpha}{2} \left(1 - 2\varepsilon - \frac{\log q}{\log y}\right)
\]

and

\[
R(q) = O\left(\frac{z^a}{W_q^2}\right) = O\left(\frac{w^a}{q W_q^2}\right).
\]

Now, let \( r \geq 1 \) be a fixed integer and let \( S_r \) denote the number of those integers of the form \( kn + l \) \((M < n \leq M + N)\) which are not divisible by any prime \( p \leq z \) and are divisible by at least \( r+1 \) distinct primes \( q \) in the interval \( z < q \leq z \), with \((q, k) = 1\). Clearly \( S_r \) is not greater than
\[
\frac{1}{r+1} \sum_{z < q \leq z_1} S(q).
\]

Hence:

**Lemma 3.** We have

\[
S_z \leq \frac{N}{r+1} \sum_{z < q \leq z_1} \frac{1}{q W_q} + O\left(\frac{w^2 (\log \log 3k)^2}{\log^2 y}\right).
\]

**Proof.** It will later be shown that

\[
W_q > c \frac{\varphi(k)}{k} \log y \quad \text{for } z < q \leq z_1.
\]

It follows that

\[
\frac{1}{r+1} \sum_{z < q \leq z_1} R(q) = O\left(\frac{w^2 (\log \log 3k)^2}{\log^2 y} \sum_{z < q \leq z_1} \frac{1}{q}\right).
\]

since

\[
\sum_{z < q \leq z_1} \frac{1}{q} = \log \frac{\alpha}{\beta} + O(1) = O(1).
\]

3. **Some lemmas.** Here we collect some auxiliary results which will be needed in the next two sections.

**Lemma 4.** We have

\[
\sum_{d|m} \frac{\mu^2(d) \log d}{d} = O\left((\log \log 3m)^2\right).
\]

**Proof.** The left-hand side is equal to

\[
\sum_{d|m} \frac{\mu^2(d)}{d} \sum_{p|d} \log p = \sum_{p|m} \log p \sum_{d|m \atop d=\varphi(p)} \frac{\mu^2(d)}{d}
= \sum_{p|m} \log p \sum_{d|m/p \atop (d,p)=1} \frac{\mu^2(d)}{d},
\]

where we have

\[
\sum_{d|m/p \atop (d,p)=1} \frac{\mu^2(d)}{d} \leq \sum_{d|m} \frac{1}{d} = O(\log \log 3m).
\]
and
\[
\sum_{p|m} \frac{\log p}{p} = \sum_{p\leq \log m} \frac{\log p}{p} + O(1)
\]
\[
= O(\log \log 3m).
\]

This proves Lemma 4.

**Lemma 5.** We have
\[
\sum_{n \leq t} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(m)}{m} \log t + O(\log \log 3m).
\]

**Proof.** H. N. Shapiro and J. Warga [4: Appendix I] have proved that
\[
\sum_{n \leq t} \frac{\mu^2(n)}{n} = \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \log t + O(\log \log 3m).
\]
Using this inequality we obtain
\[
\sum_{n \leq t} \frac{\mu^2(n)}{\varphi(n)} = \sum_{n \leq t} \frac{\mu^2(n)}{n} \prod_{p|n} \left(1 + \frac{1}{p-1}\right)
\]
\[
= \sum_{n \leq t} \frac{\mu^2(n)}{n} \sum_{d|n} \frac{1}{\varphi(d)}
\]
\[
= \sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \sum_{n \leq t} \frac{\mu^2(n)}{n}
\]
\[
= \sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \left(\varphi(dm) \prod_{p|m}(1 - \frac{1}{p^2}) \log \frac{t}{d} + O(\log \log 3dm)\right)
\]
\[
= \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{d \leq t} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log \frac{t}{d}
\]
\[
+ O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \log \log 3dm\right)
\]
\[
= \frac{\varphi(m)}{m} \prod_{p|m} \left(1 - \frac{1}{p^2}\right) \sum_{d \leq t} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log t
\]
\[
+ O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \log \log 3dm\right)
\]
\[
+ O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1} \log d\right)
\]
\[
+ O\left(\sum_{d \leq t} \frac{\mu^2(d)}{d\varphi(d)} \log \log 3dm\right).
\]
On the Distribution of Almost Primes in an Arithmetic Progression

\[ = \frac{\varphi(m)}{m} \log t + O(\log \log 3m), \]

since

\[ \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} \prod_{p \mid d} \left( 1 - \frac{1}{p^2} \right)^{-1} = \prod_{p \mid m} \left( 1 - \frac{1}{p^2} \right)^{-1}. \]

Now, for \( u > 0, \ v \geq 2 \), let \( G(u, v) \) denote the number of positive integers \( n \leq u \) with \( g(n) \leq v \).

Define the function \( \rho(s) \) by the following properties:

\[ \rho(s) = 0 \quad (s < 0); \quad \rho(s) = 1 \quad (0 \leq s \leq 1); \quad \rho'(s) = -\rho(s-1) \quad (s > 1); \quad \rho(s) \text{ continuous for } s > 0. \]

Then the following result has been proved by N. G. de Bruijn [1]:

**Lemma 6.** Let \( u > 0, \ v \geq 2 \), and put \( t = (\log u)/\log v \). Then we have

\[ G(u, v) = O(u e^{-ct}) \]

and, more precisely,

\[ G(u, v) = u \rho(t) \left( 1 + O \left( \frac{\log(2 + t)}{\log v} \right) \right) + O(1) + O(ut^2P(v)), \]

where \( P(v) \) is a function satisfying

\[ P(v) \downarrow 0 \quad (v \to \infty), \quad P(v) > (\log v)/v \quad (v \geq 2), \]

\[ |\pi(v) - \text{li} v| < vP(v)/\log v \quad (v \geq 2). \]

As to the function \( \rho(s) \) itself, it is not difficult to prove the following result, which is known as a lemma of N. C. Ankeny:

**Lemma 7.** For \( s_1 \geq s_2 \geq 1 \) we have

\[ \rho(s_1) \leq \rho(s_2) e^{-(s_1 - s_2)}, \]

so that

\[ \int_{s}^{\infty} \rho(t) \, dt \leq \rho(s) \quad (s \geq 1). \]

For a proof of this result see [8].

**4. Evaluation of \( S_1 \).** We are now going to find an explicit lower bound for \( S_1 \) on the basis of Lemma 2.
First we have to evaluate $Z_p$ and $H_p - Z_p$ for $p \leq z$. To accomplish this we distinguish three cases on the magnitude of the prime $p$.

It is clear that

$$T_p \overset{\text{def}}{=} \sum_{n > w/\sqrt{p} \overline{p}} \frac{1}{n} \geq H_p - Z_p \geq 0.$$

**Case 1:** $2 \leq p \leq \exp(\log y)^{3/4}$. By partial summation we get

$$T_p \leq \sum_{n > w/\sqrt{p} \overline{p}} \frac{1}{n} = \sum_{n > w/\sqrt{p} \overline{p}} \frac{G(n, p)}{n^2} + O(y^{-c_1}),$$

where $c_1 = \frac{1}{2} - \epsilon - \frac{1}{2\alpha}$. By Lemma 6 we have

$$\sum_{n > w/\sqrt{p} \overline{p}} \frac{G(n, p)}{n^2} = O \left( \sum_{n > w/\sqrt{p} \overline{p}} n^{-(1+e/\log p)} \right) = O \left( (\log y)^{3/4} \exp \left( -c(\log y)^{1/4} \right) \right).$$

It follows that

$$H_p - Z_p = O \left( \frac{1}{\log^2 y} \right), \quad Z_p > c \frac{\varphi(k)}{k} \log p,$$

since, by Lemma 1,

$$H_p = \prod_{q \leq z \overline{p}} \left( 1 - \frac{1}{q} \right)^{-1} \geq \frac{\varphi(k)}{k} e^{C} \log p + O(1).$$

**Case 2:** $\exp(\log y)^{3/4} < p \leq z$. We have

$$T_p = \sum_{n > w/\sqrt{p} \overline{p}} \frac{1}{n} \sum_{d \mid (n, k)} \mu(d)$$

$$= \sum_{d \mid k \overline{p}} \frac{\mu(d)}{d} \sum_{n > w/d \sqrt{p} \overline{p}} \frac{1}{n}$$

$$= \sum_{d \mid k \overline{p}} \frac{\mu(d)}{d} \sum_{n > w/d \sqrt{p} \overline{p}} \frac{1}{n}$$

$$+ \sum_{d \mid k \overline{p}} \frac{\mu(d)}{d} \sum_{n > w/d \sqrt{p} \overline{p}} \frac{1}{n}.$$
On the Distribution of Almost Primes in an Arithmetic Progression

\[ \prod_{q|k} \left(1 - \frac{1}{q} \right) \sum_{g(n) \leq p} \frac{1}{n} + O\left((\log \log 3k)^2\right), \]

since we have, by Lemma 4,

\[ \sum_{d|k} \frac{\mu(d)}{d} \sum_{g(d) \leq p} \frac{1}{n} = O\left(\sum_{d|k} \frac{\mu^2(d) \log d}{d} \right) \]

\[ = O\left((\log \log 3k)^2\right). \]

Now, by partial summation, we have

\[ \sum_{n > \sqrt{w/p}} \frac{1}{n} = \sum_{n > \sqrt{w/p}} \frac{G(n, p)}{n^2} + O\left(y^{-c}\right). \]

Here, by Lemma 6, we find that

\[ \sum_{n > \exp(\log y)^2} \frac{G(n, p)}{n^2} = O\left(\sum_{n > \exp(\log y)^2} n^{-1+\varepsilon}\right) \]

\[ = O\left(\log y \exp(-c \log y)\right) \]

so that

\[ \sum_{n > \sqrt{w/p}} \frac{G(n, p)}{n^2} = \sum_{\exp(\log y)^2 < n < \sqrt{w/p}} \frac{G(n, p)}{n^2} + O\left(\frac{1}{\log^2 y}\right). \]

Let us write \( I \) for the interval \( \sqrt{w/p} \leq n \leq \exp(\log y)^2 \). Then, by making use of the result in Lemma 6, we obtain

\[ \sum_{n \in I} \rho\left(\frac{\log n}{\log p}\right) = \frac{1}{\log p} \int_{t_p}^{\infty} \rho(t) dt + O\left(y^{-c}\right), \]

where \( t_p = (\log w/\sqrt{p})/\log p \);
\[
\sum_{n \in I} \frac{1}{n} = O(y^{-c_{5}}); \quad \sum_{n \in I} \frac{1}{n} \left(\frac{\log n}{\log p}\right)^{2} P(p) = O\left(\frac{1}{\log^{2} y}\right),
\]

where we have taken \( P(v) = c \exp(-c (\log v)^{\frac{1}{2}}) \).

We thus have

\[
T_{p} = \log p \int_{t_{p}}^{\infty} \rho(t) dt \left(1 + O\left(\frac{\log \log y}{\log p}\right)\right) + O\left(\frac{1}{\log^{2} y}\right).
\]

Hence

\[
H_{p} - Z_{p} = \prod_{q \downarrow k, q \leq p} \left(1 - \frac{1}{q}\right) \log p \int_{t_{p}}^{\infty} \rho(t) dt \left(1 + O\left(\frac{\log \log y}{\log p}\right)\right)
+ O\left((\log \log 3k)^{2}\right),
\]

\[
Z_{p} = \prod_{q \downarrow k, q \leq p} \left(1 - \frac{1}{q}\right) \left(e^{c} - \int_{t_{p}}^{\infty} \rho(t) dt\right) \log p + O\left(\log \log y\right)
+ O\left((\log \log 3k)^{2}\right) .
\]

**Case 3:** \( x_{1} < p \leq x \). Put \( t_{p} = (\log w/\sqrt{p})/\log p \), as before. If \( 0 < t_{p} \leq 1 \) then we have

\[
Z_{p} = \sum_{n \leq w/\sqrt{p}} \frac{\mu^{2}(n)}{\varphi(n)}
= \frac{\varphi(k)}{k} \log \frac{w}{\sqrt{p}} + O(\log \log 3k)
= \frac{\varphi(k)}{k} t_{p} \log p + O(\log \log 3k),
\]

\[
H_{p} - Z_{p} = \frac{\varphi(k)}{k} \left(e^{c} - t_{p}\right) \log p + O(\log \log 3k),
\]

by Lemma 5. If \( t_{p} > 1 \) then

\[
Z_{p} \geq \sum_{n \leq w/\sqrt{p}} \frac{\mu^{2}(n)}{\varphi(n)} - \sum_{\frac{w}{\sqrt{p}} < n \leq w/\sqrt{p}} \frac{\mu^{2}(n)}{\varphi(n)}
= \sum_{n \leq w/\sqrt{p}} \frac{\mu^{2}(n)}{\varphi(n)} - \frac{1}{\varphi(q)} \sum_{\frac{w}{\sqrt{p}} < q \leq w/\sqrt{p}} \frac{\mu^{2}(n)}{\varphi(n)},
\]

where, again by Lemma 5,
On the Distribution of Almost Primes in an Arithmetic Progression

\[
\sum_{\substack{p \leq q \leq w/\sqrt{p} \\ (q, k) = 1}} \frac{1}{\varphi(q)} \sum_{\substack{n \leq w/\sqrt{p} \\ (n, qk) = 1}} \frac{\mu^2(n)}{\varphi(n)} \leq \frac{1}{q}
\]

\[
\leq \frac{\varphi(k)}{k} \sum_{p \leq q \leq w/\sqrt{p}} \frac{1}{q} \log \frac{w}{q\sqrt{p}} + O\left(\sum_{p \leq q \leq w/\sqrt{p}} \frac{\log \log 3qk}{\varphi(q)}\right)
\]

\[
= \frac{\varphi(k)}{k} \left( \log \frac{w}{\sqrt{p}} \log \frac{\log \frac{w}{\sqrt{p}}}{\log p} - \log \frac{w}{\sqrt{p}} + \log p \right) + O(\log \log y),
\]

and hence

\[
Z_p \geq \frac{\varphi(k)}{k} (2t_p - 1 - t_p \log t_p) \log p + O(\log \log y).
\]

Therefore

\[
H_p - Z_p \leq \frac{\varphi(k)}{k} \left( e^c - (2t_p - 1 - t_p \log t_p) \right) \log p + O(\log \log y).
\]

Here we have, as in the proof of Lemma 2,

\[
H_p = \prod_{q < p, q \mid k} \left( 1 - \frac{1}{q} \right) \prod_{q < p} \left( 1 - \frac{1}{q} \right)^{-1}
\]

\[
= \left( \frac{\varphi(k)}{k} + O\left( \frac{\log \log 3k}{p} \right) \right) \left( e^c \log p + O(1) \right)
\]

\[
= \frac{\varphi(k)}{k} e^c \log p + O(\log \log 3k).
\]

We are now in position to be able to evaluate the sum

\[
\sum_{\substack{p \leq w \\ (p, k) = 1}} \frac{H_p - Z_p}{pH_pZ_p}.
\]

Define:

\[
A(t) = \begin{cases} 
\frac{e^c - t}{t} & (0 < t \leq 1), \\
\frac{e^c - (2t - 1 - t \log t)}{2t - 1 - t \log t} & (1 < t < e^c),
\end{cases}
\]
where \( t = e^{\epsilon} \) is the unique solution of
\[
2t - 1 - t \log t = 0, \quad t > 1,
\]
so that \( 1.8 < c_{6} < 1.9 \); and
\[
B(t) = \frac{\int_{t}^{\infty} \rho(s) ds}{e^{\epsilon} - \int_{t}^{\infty} \rho(s) ds} \quad (t > \frac{1}{4}).
\]

Let us put, for the sake of brevity,
\[
z_{3} = \exp(\log y)^{2/3}.
\]

Then we have
\[
\sum_{z_{3} < p \leq z_{1}; (p, k) = 1} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} = O\left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^{2} y} \right)
\]
\[
= O\left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^{2} y} \right),
\]
and, noticing that
\[
\prod_{q \leq p} \left( 1 - \frac{1}{q} \right)^{-1} \leq \frac{k}{\varphi(k)}
\]
for every \( p \),
\[
\sum_{z_{3} < p \leq z_{1}} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} \leq \frac{k}{\varphi(k)} e^{-C} \sum_{z_{3} < p \leq z_{1}} \frac{B(t_{p})}{p \log p}
\]
\[
+ O\left( \frac{k}{\varphi(k)} \frac{(\log \log 3k)^{3}}{\log^{3/2} y} \right) + O\left( \frac{k}{\varphi(k)} \frac{\log \log y}{\log^{3/2} y} \right),
\]
where we have used the inequality
\[
\sum_{z_{3} < p \leq z_{1}} \frac{1}{p \log^{2} p} = O\left( \frac{1}{\log^{2} z_{3}} \right).
\]

We now assume that \( \gamma, 4 \leq \gamma \leq 10 \), be integral. Write \( J_{\gamma} \) for the interval
\[
y^{1/\gamma + 1} < p \leq y^{1/\gamma} (\nu \geq \gamma).
\]
Then, since the function \( B(t) \) is monotone decreasing,
\[
\sum_{z_{3} < p \leq z_{1}} \frac{B(t_{p})}{p \log p} = \sum_{\gamma \leq \nu < \gamma} \sum_{p \in J_{\nu}} \frac{B(t_{p})}{p \log p}
\]
On the Distribution of Almost Primes in an Arithmetic Progression

\[ \sum_{\gamma \leq \nu \leq e(\log y)^{\delta}} \left( \sum_{p \in J_{\nu}} \frac{1}{p} \right) \max_{p \in J_{\nu}} \frac{B(t_{p})}{\log p} \]

\[ \leq \sum_{\gamma \leq \nu \leq e(\log y)^{\delta}} \log \frac{\nu + 1}{\nu} \frac{\nu + 1}{\log y} B\left(\frac{1}{2} - \epsilon, \nu - \frac{1}{2}\right) \]

\[ + O\left( \sum_{\gamma \leq \nu \leq c(\log y)} \log \frac{\nu + 1}{\nu} B\left(\frac{1}{2} - \epsilon, \nu - \frac{1}{2}\right) \right) \]

\[ \leq \frac{1}{\log y} \sum_{\nu = \gamma}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} B\left(\frac{1}{2} - \epsilon, \nu - \frac{1}{2}\right) + O\left( \frac{1}{\log^{3/2} y} \right). \]

Thus we obtain

\[ \sum_{p \leq Z} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} \leq \frac{k}{\varphi(k)} e^{-C} \sum_{\iota_{2} < p \leq z} \frac{A(t_{p})}{p \log p} \]

\[ + O\left( \frac{k}{\varphi(k)} \frac{\log \log 3k}{\log^{2} y} \right) \]

We have similarly

\[ \sum_{z_{2} \leq p \leq z} \frac{H_{p} - Z_{p}}{pH_{p}Z_{p}} \leq \frac{k}{\varphi(k)} e^{-C} \sum_{z_{2} \leq p \leq z} \frac{A(t_{p})}{p \log p} \]

\[ + O\left( \frac{k}{\varphi(k)} \frac{(\log \log 3k) \log \log y}{\log^{3} y} \right). \]

Put

\[ n = \lfloor \log^{1/2} y \rfloor, \quad u_{j} = \alpha + \frac{\gamma - \alpha}{n} j \quad (j \geq 0), \]

and write \( K_{j} \) for the interval

\[ y^{1/u_{j+1}} < p \leq y^{1/u_{j}} \quad (0 \leq j \leq n - 1). \]

Now, the function \( A(t) \) is continuous, monotone decreasing in the interval \( 0 < t \leq e \) and monotone increasing in the interval \( e < t < e^{\alpha} \). Thus, if we denote by \( m \) the integer for which

\[ \left( \frac{1}{2} - \epsilon \right) u_{m} - \frac{1}{2} \leq e < \left( \frac{1}{2} - \epsilon \right) u_{m+1} - \frac{1}{2}, \]

then
\[
\sum_{\epsilon_{2} < p \leq q} \frac{A(t_{p})}{p \log p} = \sum_{j=0}^{n-1} \sum_{p \in K_{j}} \frac{A(t_{p})}{p \log p}
\]
\[
\leq \sum_{j=0}^{n-1} \left( \sum_{p \in K_{j}} \frac{1}{p} \right) \max_{\epsilon \geq x_{j}} \frac{A(t_{p})}{p \log p}
\]
\[
\leq \sum_{j=0}^{n-1} \left( \sum_{p \in K_{j}} \frac{1}{p} \right) \frac{u_{j+1}}{u_{j}} A\left(\frac{1}{2} - \epsilon, \frac{u_{j+1}}{2}\right)
\]
\[
+ \log \frac{u_{m+1}}{u_{m}} \frac{u_{m+1}}{u_{m}} \max\left(A\left(\frac{1}{2} - \epsilon, \frac{u_{m}}{2}\right), A\left(\frac{1}{2} - \epsilon, \frac{u_{m+1}}{2}\right)\right)
\]
\[
+ \sum_{j=m+1}^{n-1} \log \frac{u_{j+1}}{u_{j}} \frac{u_{j+1}}{u_{j}} A\left(\frac{1}{2} - \epsilon, \frac{u_{j+1}}{2}\right)
\]
\[
= \frac{1}{\log y} \int_{a}^{\gamma} A\left(\frac{1}{2} - \epsilon, u - \frac{1}{2}\right) du + O\left(\frac{1}{\log^{3/2} y}\right),
\]
where it should be noticed that we have uniformly
\[
u_{j+1}, \log \frac{u_{j+1}}{u_{j}} = \frac{\gamma - \alpha}{n} + O\left(\frac{1}{n^{2}}\right)
\]
Hence
\[
\sum_{\epsilon_{2} < p \leq q} \frac{H_{p} - Z_{p}}{p H_{p} Z_{p}} \leq \frac{k}{\phi(k)} \frac{e^{-c}}{\log y} \int_{a}^{\gamma} A\left(\frac{1}{2} - \epsilon, u - \frac{1}{2}\right) du
\]
\[
+ O\left(\frac{k}{\phi(k)} \frac{(\log \log y) \log \log 3k}{\log^{2} y}\right) + O\left(\frac{k}{\phi(k)} \frac{1}{\log^{3/2} y}\right).
\]
Collecting these results, we thus obtain, via Lemma 2, the following

**Lemma 8.** We have
\[
S_{i} \geq \frac{kN}{\phi(k)} \frac{e^{-c}}{\log y} \left(\alpha - \int_{a}^{\gamma} A\left(\frac{1}{2} - \epsilon, u - \frac{1}{2}\right) du\right)
\]
\[
- \sum_{\nu=1}^{\infty} (\nu + 1) \frac{\log \nu + 1}{\nu} B\left(\frac{1}{2} - \epsilon, \nu - \frac{1}{2}\right)
\]
\[
+ O\left(\frac{kN}{\phi(k)} \frac{(\log \log 3k)^{3}}{\log^{4/3} y}\right) + O\left(\frac{kN}{\phi(k)} \frac{\log \log y}{\log^{4/3} y}\right)
\]
\[
+ O\left(\frac{N(\log \log 3k)^{3}}{y^{1/4} \log y}\right) + O\left(y^{1-2\epsilon} (\log \log 3k)^{2}\right).
\]
5. **Evaluation of \( S_2 \).** By virtue of Lemma 3, our present task is only to estimate the quantity

\[
\sum_{z<q\leq z_1} \frac{1}{q W_q}.
\]

We set

\[
C(t) = \begin{cases} 
\frac{\alpha}{a} & (0<a\leq 1), \\
\frac{\alpha}{2a-1-a \log a} & (1<a\leq 2),
\end{cases}
\]

where

\[
a = \frac{\alpha}{2} \left( 1-2\varepsilon - \frac{1}{t} \right).
\]

Then, it is not difficult to verify, by Lemma 5, that, with \( t = t_q = (\log y)/\log q \)

\[
W_q = \sum_{n \leq z} \frac{\mu^2(n)}{\varphi(n)} (n,k)=1
\]

\[
\geq \frac{\varphi(k)}{k} \frac{\log y}{C(t_q)} + O(\log \log 3k) \quad (z<q\leq z_1),
\]

and consequently

\[
\sum_{z<q\leq z_1} \frac{1}{q W_q} \leq \frac{k}{\varphi(k)} \frac{1}{\log y} \sum_{z<q\leq z_1} \frac{C(t_q)}{q} + O\left( \frac{k}{\varphi(k)} \left( \frac{\log \log 3k}{\log y} \right)^3 \right).
\]

Put

\[
n = [\log^{1/2} y], \quad u_j = \beta + \frac{\alpha - \beta}{n} j \quad (0 \leq j \leq n),
\]

and write \( L_j \) for the interval

\[
y^{1/u_{j+1}} < q \leq y^{1/u_j} \quad (0 \leq j \leq n-1).
\]

Then we have

\[
\sum_{z<q\leq z_1} \frac{C(t_q)}{q} = \sum_{j=0}^{n-1} \sum_{q \in L_j} \frac{C(t_q)}{q} \leq \sum_{j=0}^{n-1} \left( \frac{1}{q} \right)_{q \in L_j} \max C(t_q)
\]

\[
\leq \sum_{j=0}^{n-1} \left( \log \frac{u_{j+1}}{u_j} + O\left( \frac{1}{\log y} \right) \right) C(u_j) = \int_{\beta}^{\alpha} C(u) \frac{du}{u} + O\left( \frac{1}{\log^{1/2} y} \right),
\]
since the function $C(t)$ is continuous and decreases monotonously for $t > (1 - 2\epsilon)^{-1}$ and since we have uniformly

$$\log \frac{u_{j+1}}{u_j} = \frac{\alpha - \beta}{n} \frac{1}{u_j} + O\left(\frac{1}{n^2}\right) \quad (0 \leq j \leq n - 1).$$

We thus have proved the following

**Lemma 9.** We have

$$S_2 \leq \frac{1}{r+1} \frac{kN}{\varphi(k)} \frac{1}{\log y} \int_{\beta}^{\alpha} \frac{C(u)}{u} du + O\left(\frac{kN}{\varphi(k)} \frac{(\log \log 3k)^2}{\log^2 y}\right) + O\left(\frac{kN}{\varphi(k)} \frac{1}{\log^{3/2} y}\right) + O\left(\frac{y^{1-2\epsilon}(\log \log 3k)^2}{\log^2 y}\right).$$

6. **Numerical computations.** We need the following easy lemma, a part of which has already been used in the proof of Lemmas 8 and 9.

**Lemma 10.** The function

$$f(s) = \frac{1}{2s-1-s \log s} \quad (1 < s < e^6)$$

is positive, convex, and monotone decreasing for $1 < s < e$ and monotone increasing for $e < s < e^e$. Putting $f_1(s) = (f(s))^{-1}$, we see that $f_1(s) > 0$, $f_1'(s) = 1 - \log s$ and $f_1''(s) = -1/s$, and the result follows at once.

We now choose $\epsilon = \frac{1}{7}$ and take

$$\alpha = 4, \quad \beta = 2, \quad \text{and} \quad \gamma = 10.$$ 

Our aim in this section is to compute numerically two integrals and a sum appearing in Lemmas 8 and 9.

(1) Computation of

$$\int_{\alpha}^{\gamma} A \left( \frac{1}{2} - \epsilon \right) u - \frac{1}{2} \right) du = \int_{4}^{10} A \left( \frac{5}{14} u - \frac{1}{2} \right) du.$$

The integral is equal to

$$\int_{4}^{4.2} A \left( \frac{5}{14} u - \frac{1}{2} \right) du + \int_{4.2}^{10} A \left( \frac{5}{14} u - \frac{1}{2} \right) du,$$

where the first integral is found to be
\[
= e^C \int_{4}^{4.2} \left(\frac{5}{14}u - \frac{1}{2}\right)^{-1} du - \int_{4}^{4.2} du
\]
while the second is
\[
= e^C \int_{4.2}^{10} F(u) du - \int_{4.2}^{10} du
\]
with \( F(u) = f(s(u)) \), where \( f(s) \) is the function defined in Lemma 10 and \( s(u) = \frac{5}{14}u - \frac{1}{2} \). To estimate the integral of \( F(u) \) over \((4.2, 10)\) we proceed as follows.

We find:

- \( F(4.2) = 1.0000 \)
- \( F(4.5) < 0.9080 \)
- \( F(5) < 0.8011 \)
- \( F(6) < 0.6803 \)
- \( F(7) < 0.6197 \)
- \( F(8) < 0.5907 \)
- \( F(9) < 0.5820 \)
- \( F(10) < 0.5896 \).

By Lemma 10, the function \( F(u) \) is convex for \( 4.2 \leq u \leq 10 \). Hence
\[
\int_{4.2}^{10} F(u) du \leq \frac{3}{20} (F(4.2) + F(4.5)) + \frac{1}{4} (F(4.5) + F(5)) + \frac{1}{2} (F(5) + F(10)) + (F(6) + F(7) + F(8) + F(9)) < 3.8817,
\]
and the second integral in (3) is less than
\[ 3.8817 e^C - 5.8 < 1.1137. \]
Thus we have
\[
\int_{4}^{10} A \left(\frac{5}{14}u - \frac{1}{2}\right)^{-1} du < 0.1696 + 1.1137 = 1.2833.
\]

(ii) Computation of
\[
\sum_{\nu=7}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} B \left(\frac{1}{2} - \epsilon\right) \nu - \frac{1}{2} \right)
\]
\[
= \sum_{\nu=10}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} B \left(\frac{5}{14} \nu - \frac{1}{2} \right).
\]
By the definition (2), the function $\rho(s)$ is positive and monotone decreasing for $s>0$, and moreover

$$\rho(s) = 1 - \log s \quad \text{for} \quad 1 \leq s \leq 2.$$ 

Put $s(\nu) = \frac{5}{14}\nu - \frac{1}{2}$. Then we have $s(10) = \frac{43}{14} > 3$ and

$$\rho\left(s(10)\right) \leq \rho(3) \leq \rho(2)e^{-1} = (1 - \log 2)e^{-1} < 0.1129,$$

by Lemma 7. Now, using Lemma 7 again, we find that for $\nu \geq 10$

$$B\left(s(\nu)\right) \leq \frac{\rho(s(\nu))}{e^\nu - \rho(s(\nu))} \leq \frac{\rho(s(10))}{e^\nu - \rho(s(10))} e^{-\frac{5}{14}(\nu - 10)}.$$ 

Since $(\nu + 1)\log((\nu + 1)/\nu)$ decreases monotonously as $\nu \to \infty$, we thus obtain

$$\sum_{\nu=10}^{\infty} (\nu + 1) \log \frac{\nu + 1}{\nu} B\left(s(\nu)\right) \leq 11 \log \frac{11}{10} \frac{\rho(s(10))}{1 - e^{-\frac{5}{14}}} < 0.2366.$$ 

(iii) Computation of

$$\int_{\beta}^{\alpha} \frac{C(u)}{u} du = \int_{2}^{4} \frac{C(u)}{u} du.$$ 

For $2 \leq u \leq 4$ we have

$$\frac{3}{7} \leq a = 2\left(\frac{5}{7} - \frac{1}{u}\right) \leq \frac{13}{14}.$$ 

Hence

$$\int_{2}^{4} \frac{C(u)}{u} du = 2\int_{2}^{4} \left(\frac{5}{7} - 1\right)^{-1} du
= \frac{14}{5} \log \frac{13}{3} < 4.1058.$$ 

7. Proof of the theorem. Let $1 \leq k < x$, $0 \leq l \leq k - 1$, $(k, l) = 1$. Take

$$M = \left[\frac{x - l}{k}\right], \quad N = \left[\frac{x}{k}\right],$$

and put
On the Distribution of Almost Primes in an Arithmetic Progression

\[ y = 2k(N+1), \quad z = y^{1/4}, \quad z_1 = y^{1/2}, \quad w = y^{5/14}. \]

Then it is clear that \( y > 2x \) and that \( M < n \leq M+N \) implies \( x < kn + l \leq 2x \).

By \( D(x; k, l) \) we denote the number of those integers of the form \( kn + l \) \((M < n \leq M+N)\) which are divisible by no primes \( p \leq z \), by at most two primes \( q \) in \( z < q \leq z_1 \), and by no integers of the form \( q^2 \), \( q \) being a prime in \( z < q \leq z_1 \): clearly such an integer \( kn + l \) \((M < n \leq M+N)\), if it exists, has at most two prime factors, i.e. \( V(kn + l) \leq 2 \).

In order to estimate \( D(x; k, l) \) from below, we apply Lemma 8 and Lemma 9 with \( r = 2 \). Let us note that we have from the data in §6

\[
e^{-c} \left( 4 - \int_4^{10} A(s(u)) \, du - \sum_{\nu=10}^{\infty} \left( \nu + 1 \right) \log \frac{\nu + 1}{\nu} B(s(\nu)) \right)
\geq e^{-c} \left( 4 - 1.2833 - 0.2366 \right) > 1.3923
\]

and

\[
\frac{1}{3} \int_2^{4} \frac{C(u)}{u} \, du < \frac{4.1058}{3} = 1.3686.
\]

Now, the number \( R_2 \) of those integers \( kn + l \) \((M < n \leq M+N)\) which are not divisible by any prime \( p \leq z \) and are divisible by some integer \( q^2 \) with \( q \) in \( z < q \leq z_1 \) does not exceed

\[
\sum_{z < q \leq z_1} \left( \frac{N}{q^2} + 1 \right) = O \left( \frac{N}{z} \right) + O(z_1).
\]

We find, therefore, that

\[
D(x; k, l) \geq S_1 - S_2 - R_2 \geq (1.3923 - 1.3686) \frac{kN}{\varphi(k)} \frac{1}{\log y}
+ O \left( \frac{kN}{\varphi(k)} \frac{\log \log 3k^3}{\log^{4/3} y} \right) + O \left( \frac{kN}{\varphi(k)} \frac{\log \log y}{\log^{4/3} y} \right)
+ O \left( \frac{N(\log \log 3k^3)}{y^{1/4} \log y} \right) + O \left( y^{5/7}(\log \log 3k^3) \right) + O \left( \frac{N}{y^{1/4}} \right).
\]

Since \( N = \frac{x}{k} + O(1) \), \( 2x < y \leq 4x \), it follows that

\[
D(x; k, l) \geq 0.0237 \frac{1}{\varphi(k)} \frac{x}{\log x}
+ O \left( \frac{1}{\varphi(k)} \frac{x(\log \log 3k^3)}{\log^{4/3} x} \right) + O \left( \frac{1}{\varphi(k)} \frac{x \log \log x}{\log^{4/3} x} \right)
\]
S. Uchiyama

\[ + O\left( \frac{1}{k} \frac{x^{3/4} \log \log 3k}{\log x} \right) + O\left( x^{5/7} \log \log 3k \right) + O\left( \frac{1}{k} x^{3/4} \right). \]

Let \( c_1 > 3.5 \) be a fixed number. If \( x \geq k^{c_1} \) and \( k \) is sufficiently large, then all the error terms on the right-hand side of the above inequality for \( D(x; k, l) \) are of negligible order of magnitude, with respect to the leading term. Thus, for all large enough \( k \), \( x \geq k^{c_1} \) implies that

\[ D(x; k, l) > 0.0236 \frac{1}{\varphi(k)} \frac{x}{\log x} > 1. \]

Hence, by continuity argument, we conclude that there is a (finite) natural number \( k_0 \) such that, if \( k \geq k_0 \) and \( x \geq k^{c_1} \) then we have \( D(x; k, l) > 0 \). Therefore there exists an absolute constant \( c_1 > 0 \) such that

\[ D(x; k, l) > 0 \quad \text{for all} \quad x \geq c_1 k^{c_1}, \quad k \geq 1. \]

This completes the proof of our theorem.

Appendix

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Here we aim at generalizing the results obtained in [6].

Let \( N > 1 \) and let \( a_1, a_2, \cdots, a_N \) be rational integers not necessarily different from each other. Let \( S \) denote the number of those integers \( a_j (1 \leq j \leq N) \) which are not divisible by any prime number \( p \leq z \), where \( z \geq 2 \). Suppose that for every positive integer \( d \)

\[ S_d \overset{\text{def}}{=} \sum_{n \leq N} 1 = \frac{\omega(d)}{d} N + R(d), \]

where \( R(d) \) is the error term for \( S_d \) and \( \omega(d) \) is a multiplicative function of \( d \). We put

\[ f(d) = \frac{d}{\omega(d)} \]

and suppose that \( f(d) > 1 \) for all \( d > 1 \).

Let \( w \) be an arbitrary but fixed real number such that \( w \geq 2 \). We define for positive integers \( m \) and \( d \)

\[ f_1(m) = \sum_{n \mid m} \mu(n) f\left( \frac{m}{n} \right), \]
On the Distribution of Almost Primes in an Arithmetic Progression

\[ W(d) = \sum_{r \leq w/d, (r, d) = 1} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)}, \quad W = W(1), \]

\[ \lambda(d) = \epsilon_z(d) \mu(d) \prod_{p|d} \left(1 - \frac{1}{f(p)}\right)^{-1} \cdot \frac{W(d)}{W}, \]

where \( \epsilon_z(n) = 0 \) or 1 according as \( n \) has or has not a prime factor \( > z \). Then we have, since \( \lambda(1) = 1 \),

\[ S \leq \sum_{n \leq N} \left( \sum_{\substack{d, d_1, d_2 \leq w \atop [d, d_1, d_2] = d}} \lambda(d_1) \lambda(d_2) \right) \frac{N}{f(d)} + \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R([d_1, d_2])|, \]

where \{d_1, d_2\} denotes the least common multiple of \( d_1 \) and \( d_2 \).

Now

\[ \sum_{\substack{d_1, d_2 \leq w \atop [d_1, d_2] = d}} \lambda(d_1) \lambda(d_2) \frac{1}{f(d)} \]

\[ = \sum_{r \leq w} f_1(r) \left( \sum_{\substack{d_1, d_2 \leq w \atop [d_1, d_2] = d}} \lambda(d) \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \sum_{d \leq w \atop d \equiv 0 (r)} \epsilon_z(d) \mu(d) \frac{1}{f_1(d)} \sum_{m \leq w/d \atop (m, d) = 1} \epsilon_z(m) \frac{\mu^2(m)}{f_1(m)} \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} f_1(r) \left( \sum_{n \leq w \atop n \equiv 0 (r)} \epsilon_z(r) \frac{\mu(r)}{f_1(r)} \sum_{d | n} \mu(d) \right)^2 \]

\[ = \frac{1}{W^2} \sum_{r \leq w} \epsilon_z(r) \frac{\mu^2(r)}{f_1(r)} = \frac{1}{W}. \]

We thus have proved the following

**Theorem.** Under the notations and conditions described above we have

\[ S \leq \frac{N}{W} + R \]

with

\[ R = \sum_{d_1, d_2 \leq w} |\lambda(d_1) \lambda(d_2) R([d_1, d_2])|. \]

This is a generalization of [3: II, Theorem 3.1].

To evaluate the remainder term \( R \) explicitly, let us suppose that for all positive integers \( d, d_1, d_2 \)
$|R(d)| \leq B\omega(d), \quad \omega\{d_1, d_2\} \leq \omega(d_1)\omega(d_2),$

where $B > 0$ is a constant independent of $d$. These conditions imply

$$R \leq B \left( \sum_{d \leq w} \lambda(d)\omega(d) \right)^2.$$  

Then, it is not difficult to show that we have, in general,

$$R = O\left( w^2(\log \log w)^2 \right),$$

and, in the special case where $\omega(p) \leq 1$ for all primes $p \leq z$,

$$R = O\left( \frac{w^2}{W^2} \right),$$

where the constants implied in the symbol $O$ depend only on the constant $B$.

The proof of these estimates of the remainder term $R$ can easily be carried out just in the same way as in [6], and we shall omit the details (cf. also [7]).

**References**


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