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<td>Author(s)</td>
<td>Uchiyama, Saburô</td>
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ON THE REPRESENTATION OF LARGE EVEN INTEGERS AS SUMS OF TWO ALMOST PRIMES. II

By

Saburō UCHIYAMA

In a previous paper [3] the writer has given with Miss A. Togashi an elementary proof for the fact that every sufficiently large even integer is representable as a sum of two almost primes, each of which has at most three prime factors, a result first obtained by A. I. Vinogradov. On the other hand, we are able to prove by a rather transcendental method that every large even integer is representable as a sum of a prime and an almost prime composed of at most four prime factors (see [4]). The aim in the present paper is to show that a somewhat weaker result than this can be obtained by an elementary argument. We shall prove the following

**Theorem.** Every sufficiently large even integer $N$ can be written in the form

$$N = n_1 + n_2,$$

where $n_1 > 1$, $n_2 > 1$, $(n_1, n_2) = 1$ and

$$V(n_1) + V(n_2) \leq 5.$$

In other words, every large even integer $N$ can be represented in the form $N = n_1 + n_2$, where $n_1 > 1$, $n_2 > 1$, $(n_1, n_2) = 1$ and either

$$V(n_1) = 1, \quad V(n_2) \leq 4,$$

or

$$V(n_1) \leq 2, \quad V(n_2) \leq 3.$$

Our method of proving this result is a refinement of that of proving the previous one, used in [3].

The writer wishes to express his gratitude to M. Uchiyama, Computation Centre, Hokkaido University, for providing various numerical data to him in

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1) Throughout in this paper, the letters $i, j, k, m, n$ (with or without indices) represent positive integers, while $p, q$ (with or without indices) represent prime numbers. We denote by $V(m)$ the total number of prime divisors of $m$. 
Let $N$ be a sufficiently large but fixed even integer. We consider as in [3] the $\varphi(N)$ integers $a_n=n(N-n) \ (1 \leq n \leq N, (n, N)=1)$.

Let $x \geq 2$ be a fixed real number satisfying

$$N^{c_1} < x < N^{c_2},$$

where $c_1$ and $c_2$ are constants with $0 < c_1 < c_2 < 1$. We denote by $P(x)$ the number of those integers $a_n \ (1 \leq n \leq N, (n, N)=1)$ which are not divisible by any prime $p \leq x$. Using the sieve method of A. Selberg (cf. [5, Appendix]) we find that

$$P(x) \leq \frac{\varphi(N)}{W} + R,$$

where

$$W = \sum_{\substack{m \leq x^a \ \text{and} \ \varphi(m) \leq x}} \frac{\mu^2(m)}{f_1(m)},$$

and

$$R = O(B_N x^{a_a}(\log \log x)^a),$$

$a > 0$ being a constant. In the following we shall suppose that $0 < a \leq 4$.

For brevity's sake we put, for $0 < a \leq 1$,

$$F(a) = F_0(a) = a^2;$$

for $1 < a \leq 2$,

$$F(a) = F_1(a) = (2a-1)^2 - 2a^2 \log a;$$

for $2 < a \leq 3$,

$$F(a) = F_1(a) + F_2(a) + F_{2,1}(a) + F_{2,2}(a),$$

where

$$F_2(a) = (a-2)^2 + 2a^2 \log^2 \frac{a}{2} - \frac{1}{2} (a-2)(7a-2) \log \frac{a}{2}$$

and $F_{2,1}(a)$ and $F_{2,2}(a)$ are defined as follows: write

2) For the definitions of $f_1(m), g(m), A_N, B_N, C_N$ we refer to [3].
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\[ a_i = 1 + \frac{a-2}{i+1} \quad (i \geq 1), \]

\[ b_i = \frac{a}{2} - \frac{a-2}{4i+2} \quad (i \geq 1), \]

and let \( k_1, k_2 \) be arbitrary but fixed positive integers. Then

\[ F_{2,1}(a) = \sum_{j=1}^{k_1} \left( 8(a_{j+1}-1)(a_j-a_{j+1}) + 4a^2 \log a_{j+1} \log \frac{a-a_{j+1}}{a-a_j} \right. \]

\[-2(a_j-a_{j+1})(2a+a_j+a_{j+1}) \log a_{j+1} \]

\[-2(a_{j+1}-1)(4a-a_{j+1}-1) \log \frac{a-a_{j+1}}{a-a_j} \), \]

\[ F_{2,2}(a) = \sum_{j=1}^{k_2} \left( 8(b_{j+1}-b_j)\left( \frac{a}{2} - b_{j+1} \right) + 4a^2 \log b_{j+1} \log \frac{2(a-b_{j+1})}{a} \right. \]

\[-\left( \frac{a}{2} - b_{j+1} \right) \left( 5a + 2b_{j+1} \right) \log \frac{b_{j+1}}{b_j} \]

\[-2(b_{j+1}-b_j)(4a-b_j-b_{j+1}) \log \frac{2(a-b_{j+1})}{a} \right). \]

Finally, we set for \( 3 < a \leq 4 \)

\[ F(a) = F_1(a) + F_2(a) + F_{2,1}(a) + F_{2,2}(a) - F_3(a), \]

where

\[ F_3(a) \overset{\text{def}}{=} \frac{4}{9} (a-3)^3 \log \frac{a}{3} \cdot \left( 3 \log^2 \frac{a}{3} \right. \]

\[ + 4 \log \frac{a}{3} \log \frac{3(a-2)}{a} + \log^2 \frac{2a-3}{a} \right). \]

(Note that the function \( F(a) \) thus defined is positive and continuous for \( 0 < a \leq 4 \).) We have then, for \( 0 < a \leq 4 \),

\[ W \geq F(a) C_N \log^2 x + O(\log N \log \log N). \]

Indeed, it is easy to verify this relation for \( 0 < a \leq 2 \) (cf. [3]). For \( 2 < a \leq 3 \) we have

\[ W = \sum_{m \leq x^a} \frac{\mu^2(m)}{f_1(m)} - \sum_{x < p \leq x^a} \sum_{m \leq x^a} \frac{\mu^2(m)}{f_1(m)} + \sum_{x < p_1 < \ldots < p_k} \sum_{m \leq x^a} \frac{\mu^2(m)}{f_1(m)}, \]
where the last double summation is found to be
\[ \geq \frac{1}{2} \sum_{z < p_1 \leq x^{a/2}} \sum_{m \leq x^{a/2} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m) + O \left( \frac{\log^2 x}{x} \right) \]

\[ + \sum_{j=1}^{k_1} \sum_{a_j < p_1 \leq x^{a/2}} \sum_{m \leq x^{a/2} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m) + \sum_{j=1}^{k_2} \sum_{b_j < p_1 \leq x^{a/2}} \sum_{m \leq x^{a/2} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m) \]

\[ = (F_1(a) + F_{21}(a) + F_{22}(a)) C_N \log^2 x + O(\log N \log \log N), \]

and for \(3 < a \leq 4\) we have
\[ W = \sum_{m \leq x^{a}} \frac{\mu^2(m)}{f_1(m)} - \sum_{m \equiv 0 (p)} \sum_{m \leq x^{a/2} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m) \]

\[ + \sum_{z < p_1 \leq x^{a/3}} \sum_{m \leq x^{a/3} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m), \]

where the last double summation is (in absolute value)
\[ \leq \left( \frac{1}{6} \sum_{z < p_1 \leq x^{a/3}} \frac{1}{2} \sum_{z < p_1 \leq x^{a/3}} \sum_{m \equiv 0 (p)} \mu^2(m) f_1(m) \right) \sum_{m \leq x^{a/3} \atop m \equiv 0 (p_1, p_2)} \mu^2(m) f_1(m) \]

\[ + O \left( \frac{\log^2 x}{x} \right) = F_3(a) C_N \log^2 x + O(\log N \log \log N). \]

This proves our assertion. Hence:

**Lemma 1.** For \(0 < a \leq 4\) we have
\[ P(x) \leq \frac{2e^c}{F(a)} A_N \frac{\varphi(N)}{\log^2 x} + O \left( \frac{\varphi(N) (\log \log N)^5}{\log^3 N} \right) + O \left( B_N x^{2a} (\log \log N)^2 \right). \]

2. We now evaluate \(P(N^{1/u})\) for some values of \(u (\geq 2)\). The result to be obtained will be of the form either
\[ P(N^{1/u}) \leq A(u) A_N \frac{\varphi(N)}{\log^2 N} + O \left( \frac{\varphi(N) (\log \log N)^5}{\log^3 N} \right) \]

or
\[ P(N^{1/u}) \geq a(u) A_N \frac{\varphi(N)}{\log^2 N} + O \left( \frac{\varphi(N) (\log \log N)^5}{\log^3 N} \right). \]

Here, it is clear that \(0 \leq a(u) \leq A(u) < \infty (u \geq 2)\): moreover, we may assume
without loss of generality that each of the coefficients $A(u)$ and $a(u)$ is, as a function of $u$, monotone non-decreasing for $u \geq 2$.

**Lemma 2.** We have

$$a(10) = 98.0.$$ 

This is in substance identical with [1, Lemma 1]. The result is obtained by simply applying the sieve method of Viggo Brun. We note a rather better result than the above, viz. $a(10)=99.9818$, is known (see [2]): but this is not necessary for our purpose.

**Lemma 3.** For $8 \leq u \leq 9$ we have

$$A(u) = 1.0541u^2.$$ 

If we apply Lemma 1 with $x = N^{\frac{1}{u}}$ ($8 \leq u \leq 9$) and $a = 3.95$, then we get

$$P(N^{\frac{1}{u}}) \leq \frac{2e^{3C}u^2}{F(3.95)} A_N \frac{\varphi(N)}{\log^2 N} + O\left(\frac{\varphi(N)(\log \log N)^5}{\log^{3} N}\right),$$

and the result follows from this at once, since we have

$$\frac{2e^{3C}}{F(3.95)} < 1.0541,$$

where we have taken $k_1=20$ and $k_2=5$.

**Lemma 4.** Suppose that $3 \leq u \leq u_1 \leq 10$. If we set

$$a_1(u) = \max\left(a(u), a(u_1) - 2\int_{u-1}^{u_1-1} A(v)\frac{v+1}{v^2}dv\right),$$

then the coefficient $a(u)$ can be replaced by the new one, $a_1(u)$.

This is [1, Theorem 1].

Now, it is known that $a(9) = 75.58$ (see [1, p. 385]), while we find using the results in Lemmas 2 and 3 that

$$a(10) - 2\int_{8}^{9} A(v)\frac{v+1}{v^2}dv = a(10) - 2 \cdot 1.0541 \int_{8}^{9} (v+1)dv$$

$$= 98.0 - 20.0279 = 77.9721.$$ 

It follows from Lemma 4 (and the definition of $a(u)$) that $a_1(9) = 77.9721$, and we thus have proved the following

**Lemma 5.** We have
\[ S_1^{\text{def}} = P(N^\frac{1}{9}) \geqq 77.9721 A_N \frac{\varphi(N)}{\log^2 N} + O\left( \frac{\varphi(N) \log \log N}{q \log^3 N} \right). \]

3. Let \( q, \langle q, N \rangle = 1 \), be a fixed prime number in the interval \( z < q \leqq z_1 \), where

\[ z = N^\frac{1}{9}, \quad z_1 = N^\frac{5}{9}, \]

and let \( S(q) \) denote the number of those integers \( a_n = n(N-n) \ (1 \leqq n \leqq N, \langle n, N \rangle = 1) \) which are not divisible by any prime \( p \leqq z \) and are divisible by the prime \( q \). Then we find as in [3] that

\[ S(q) \leqq \frac{2\varphi(N)}{q W_q} + R, \]

where

\[ W_q \geqq \sum_{m \leqq z^a, \gamma(m) \leqq z} \frac{\mu^2(m)}{f_1(m)} \quad \text{with} \quad a = 4.5 \left( 1 - 2\varepsilon - \frac{\log q}{\log N} \right) \]

and

\[ R_q = O\left( \frac{N^{1-\varepsilon} \log \log N}{q} \right), \]

\( \varepsilon \) being a sufficiently small, fixed positive real number. Clearly we have \( 0 < a < 4 \). Hence we may repeat the argument in the proof of Lemma 1 to obtain

\[ W_q \geqq F(a) C_N \log^4 z + O(\log N \log \log N), \]

so that

\[ S(q) \leqq \frac{C(t_q)}{q} A_N \frac{\varphi(N)}{\log^2 N} + O\left( \frac{\varphi(N) \log \log N}{q \log^3 N} \right), \]

where we have put \( t_q = (\log N)/\log q \) and

\[ C_t = \frac{324e^{a_2}}{F(a)} \quad \text{with} \quad a = 4.5 \left( 1 - 2\varepsilon - \frac{1}{t} \right). \]

Now, if we denote by \( S_2 \) the number of those integers \( a_n \ (1 \leqq n \leqq N, \langle n, N \rangle = 1) \) which are not divisible by any prime \( p \leqq z \) and are divisible by at least four distinct primes \( q, \langle q, N \rangle = 1 \), in the interval \( z < q \leqq z_1 \), then
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\[ S_2 \leq \frac{1}{4} \sum_{q \leq z} S(q). \]

Since we have

\[ \sum_{\frac{z}{q} \leq z, (q,N)=1} \frac{C_*(t_q)}{q} \leq \int_{1.8}^{9} \frac{C_*(t)}{t} dt + O\left( \frac{1}{\log^{1/2} N} \right), \]

we obtain the following lemma:

**Lemma 6.** We have

\[ S_2 \leq \frac{1}{4} \int_{1.8}^{9} \frac{C_*(t)}{t} dt A_N \frac{\varphi(N)}{\log^2 N} + O\left( \frac{\varphi(N) (\log \log N)^2}{\log^{3/2} N} \right). \]

4. We shall show that for some sufficiently small \( \varepsilon > 0 \) we have

\[ I(\varepsilon) \overset{\text{def}}{=} \frac{1}{4} \int_{1.8}^{9} \frac{C_*(t)}{t} dt \leq 77.5008. \]

To do this it will obviously suffice to prove that

\[ I(0) < 77.500719, \]

since the integral defining \( I(\varepsilon) \) is, as a function of \( \varepsilon \), continuous at \( \varepsilon = 0 \).

We have

\[ I(0) = \frac{1}{4} \int_{1.8}^{9} \frac{C_*(t)}{t} dt = 81e^x \int_{2}^{4} \frac{ds}{(4.5-s)F(s)} , \]

on substituting \( s = 4.5 \left( 1 - \frac{1}{t} \right) \). It is not difficult to verify—by a simple but rather tedious calculation—that the function

\[ k(s) = \frac{1}{(4.5-s)F(s)} \]

is convex on either of the intervals \( 2 \leq s \leq 3 \) and \( 3 \leq s \leq 4 \) (indeed, \( k(s) \) may be convex throughout on the interval \( 2 \leq s \leq 4 \)). Hence, in order to estimate from above the value of the integral of \( k(s) \) over the interval \( 2 \leq s \leq 4 \), we may apply the trapezoidal rule with an arbitrary set of division points including the point \( s = 3 \). Thus

\[ \int_{2}^{4} k(s) ds \leq 0.05 \left( \frac{1}{2} k(2.00) + \sum_{j=1}^{39} k(2.00 + 0.05j) + \frac{1}{2} k(4.00) \right). \]
Taking again $k_1=20$ and $k_2=5$, we find that:

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The above table was computed on the electronic digital computer, HIPAC 103. These data will give

$$\int_2^4 k(s)ds < 0.301618,$$

and consequently

$$I(0) < 81e^{\epsilon^2} \cdot 0.301618 < 77.500719.$$

This is the result which was to be shown.

5. Let us fix $\epsilon > 0$ so small as to satisfy $I(\epsilon) \leq 77.5008$, and suppose that $N \geq N_0 = N_0(\epsilon)$ be a sufficiently large even integer. Let $S$ denote the number of those integers $a_n = n(N-n)$ ($1 \leq n \leq N$, $(n, N)=1$) which are divisible by no primes $p \leq z$, by at most three primes $q$, $(q, N)=1$, in the interval $z < q \leq z_1$, and by no integers of the form $q^2$ with $q$ in the interval $z < q \leq z_1$, where, as before,

$$z = N^\frac{1}{9}, \quad z_1 = N^\frac{5}{9}.$$
Since the number $S_3$ of those integers $a_n$ ($1 \leq n \leq N$, $(n, N) = 1$) which are not divisible by any prime $p \leq z$ and are divisible by some integer $q^2$ with $q$, $(q, N) = 1$, in $z < q \leq z_1$, is of $O(N^{\frac{8}{9}})$, we have, by virtue of Lemmas 5 and 6,

$$S \geq S_1 - S_2 - S_3$$

$$\geq (77.9721 - 77.5008) A_N \frac{\varphi(N)}{\log^2 N} + O\left(\frac{\varphi(N)(\log \log N)^2}{\log^{5/2} N}\right)$$

$$> 0.4712 A_N \frac{\varphi(N)}{\log^2 N} > 2.$$

This implies the existence of at least one integer $n$ with $1 < n < N - 1$, $(n, N) = 1$, such that $V(a_n) \leq 5$, i.e.

$$V(n) + V(N - n) \leq 5,$$

which completes the proof of our theorem, since $N = n + (N - n)$.

References


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