"FOUNDATION" AND FORMALISM. II

Dedicated to Prof. K. Kunugi on his sixtieth birthday

By

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The whole speculation in this paper is an attempt to suggest that the foundations of mathematics ought not always to be studied "mainly in the framework of logico-mathematical technique" and that the discrepancies in one way or another of the standpoints on the fundations of mathematics is perhaps due to the duality of human nature, especially on knowing.

1. The Prevailing Opinion of Mathematicians.

"Studies in the foundations of mathematics tend to converge in their aims···Hilbert's formalism has taught us that we have not yet learnt how to study mathematics except under the guidance of the traditional '*esprit géométrique*'···Further, when the catchword '*the salvation of mathematics*' has lost its profound meaning and the theory of consistency has been defined as one pertaining to the formal system, that is, a problem in metamathematics, metamathematics has been liberated from the shackles of the '*finitary*' standpoint, and thus given a varied choice of standpoints. In this way, researches in the foundations of mathematics which have hitherto been very much diversified have come to fall into the following two categories; suggestions of formal systems and metamathematics concerning them".[1] This view is probably shared by most specialists today. In so far as the foundations of mathematics is regarded as a branch of mathematics, such a change in it as is described in the above quotation comes only natural from a process of narrow screening of its objects and methods. Philosophical thinking is placed out of account here. This point of view gives rise to statements like the following. "The present-day problems of the foundations of mathematics, as we see, are investigated mainly in the framework of logico-mathematical technique.—The philosophical disputation concerning the standpoint from which the logico-mathematical constructions are employed, has, therefore, been usually disregarded.—From this point of view, which may be taken for granted by almost every mathematician,

1) S. SEKI: A Report of the Symposium on the Foundations of Mathematics held in Kyoto, 20, 21 May, 1957.

a philosophical inquiry into the abovementioned question,[2] would be of little importance."[3] The present writer, for one, agrees on the whole to the prevailing opinion of mathematicians exemplified in the two quotations above. Assuredly, as history shows, it will be fruitless even in the foundations of mathematics to discuss mathematics transcendentally from a particular presumed philosophical standpoint.

2. Formalism and 'Foundation'.

Notwithstanding, it seems to the present writer that, when we speculate on the foundations of mathematics, we end by being in spite of ourselves drawn into philosophical thinking, or, at any rate, into something like it. This apparently heretical view is a product of an opinion quite of its own about the purpose of the foundations of mathematics. The present writer thinks that the purpose of the foundations of mathematics consists in '*foundation*',[4] which is, broadly speaking, "to judge whether what is supposed to be known is really known" and "to ascertain whether a seemingly self-evident matter is really an indisputable fact": in short, "to demand persistently a proof of correctness".[5] This mental attitude taking rise from man's natural desire for knowledge can be vindicated sufficiently. But it is clearly distinct from the prevailing view, which holds that the *raison d'être* of a mathematical system lies in its consistency. As stated in the previous chapter, here consistency has lost its original deeper meaning and come to be defined as a problem in metamathematics. This fact means not that this standpoint has been modified, but, on the contrary, that its has further clarified its essential character. Again, it implies that the foundations of mathematics as studied from such a standpoint is not worthy of the name in its literal sense, but is a mere branch of mathematics.

The difference between the two standpoints shows more clearly in their attitudes towards the antinomy. The orthodox view directly aims at the elimination of the antinomy and deals with it in terms of consistency. Here reflection on the foundations of mathematics is awakened by an external stimulus of the appearance of an antinomy. Whereas, with the other attitude, it is prompted by the desire to "clarify what is not yet clear", almost independently of the presence of an antinomy. This can be seen in the works of Dedekind and in Kronecker's disapproval of Cantorism. In their days no antinomy was as

2) I. e., concerning "the discrepancies between formalist and intuitionist schools of thought on the foundation".

3) H. NAGAI: The Concept of Formality in Mathematics, Annals of the Japan Association for Philosophy of Science, Vol. 1, No. 5 (1960), pp. 289–312.

4) S. NAKAZIMA: "Foundation" and Formalism, Proc. Japan Academy, Vol. 37, No. 8 (1961), pp. 452–456.

5) see 4).

yet actually present. The present writer thinks that their ideas, especially those of Dedekind, should not be dismissed summarily as antiquated. More recently, the attitude resembling above mentioned is instanced in the Intuitionism of Brouwer and his school, although it is an historical fact that Intuitionism too began to reflect in the face of the appearance of an antinomy in the set theory. Brouwer refuses to admit the correctness of a mathematical system, unless it has been evolved from one's own basic intuition, even though its consistency has been proved.

It is concluded that the standpoint of '*foundation*' necessitates what may be called philosophical thinking. Why is it, then ?

3. The scientific standpoint *vs.* The philosophical standpoint.

Thinking, in its steady pursuit of the truth, ends by calling itself in question. On other words, the subject makes itself an object. This constitutes the peculiarity of philosophy which distinguishes it from the sciences. From "*Gnõthi seauton*" inscribed over the entrance of the temple of Apollo at Delphi to Descartes' "*Cogito ergo sum*", '*Selbstbewusstsein*' was always the keystone of philosophy. In the foundations of mathematics, too, a process of thinking applied to the object is in its turn made an object. When metamathematics, which is in reality not an object but a process of thinking, is to be examined, it is only natural that philosophical thinking, or, at least something like it, should be required. This is a tentative '*formal*' conclusion of this paper. The present writer does not emphasize the improtance of philosophical thinking because he likes to increase the difficulty of the problem. Formalism has done well in eliminating philosophical humbugs from the foundations of mathematics and thus establishing itself on a scientific basis, and in this respect it can claim full appreciation. However, in so far as it is concerned with the '*Begründung der Mathematik*', it can never escape many aporias which hardly ever fail to present themselves in fundamental thinking, as is commonly the case in philosophy. "Now, our theme is the theory of real number and generally the set theory which covers much the same field as logic. Is the '*Beweisstheorie*' itself not subject to the postulates of the set theory or logic ? In other words, when an ultimate standpoint is persistently called to account, is it not likely that, so far as it is truly ultimate, having nothing but itself to provide a guarantee of its correctness, it is driven into a vicious circle ? This is why formalism seems to be far from its objective, although many interesting works have successively been published today by distinguished mathematicians under Hilbert".[6] This view will be shared by all unprejudiced students.

6) K. KUNUGI: The Essentials of Analysis p. 152.

4. The Duality of Human Nature.

Undeniably, we are confronted here with an antinomy of a higher order than one in the set theory. The standpoint of '*foundation*' deserves full justification as a manifestation of man's desire to know, but it cannot get rid of a subjective and dogmatic tendency which is undesirable in a science. Further, it is challenged to explain how subjective knowledge can ever have objectivity. (An answer to the question is given by Dr. Suetuna).[7] On the other hand, formalism tends to be objective and scientific, thereby holding the most important position in the foundations of mathematics. But where '*Begründung*' in the veritable sense is concerned, it cannot always be invulnerable to criticism.[8,9] In short, these two standpoints have each a sufficient *raison d'être*, but the further they carry their basic way of thinking, the more they seem to endanger their own footing. A perplexing phenomenon, indeed ! In philosophy proper, it finds its counterpart in the contrast between Logical-positivism and Analytic philosophy on the one hand and Existentialism on the other. Dr. K. Kunugi says, "The very structure of mathematics presents a peculiar aspect ··· Does it not embrance a fundamental contradiction in it, just as many pairs of basic concepts—such as mind and matter, or subject and object—imply an ultimate duality ?"[10] This remark originally refers to the relations between pure mathematics and applied mathematics, but it will be relevant to the relations between '*foundation*' and formalism as well. Von Neumann also speaks of '*a unique duality inherent in mathematics*'.[11] Again, this irreducible antinomy seems to be akin to the paradox of knowing, that involved relation between knowledge and ignorace, of which Plato makes Euthydemos talk ridiculously in one of his dialogues and on which Hegel subsequently makes pregnant remaks. "A man must ask a question because he does not know, but he can do so because he knows. That is, he asks a question because he knows and does not know at the same time. But is it not a contradiction that he should be knowledgeable and ignorant simultaneously ?"

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7) S. SUETUNA: Ueber die Grundlagen der Mathematik I, J. Math. Soc. Japan Vol. 3, No. 1 (1951), pp. 59-68 (especially p. 60).

8) See. 7).

9) K. JASPERS: Von der Wahrheit, s. 406.

10) K. KUNUGI: The Abstract and the Concrete in Mathematics. The Frondations of Science No. 24 (1951), pp. 1-6.

11) J. von NEUMANN: The Mathematician (The Works of the Mind, 1947, Chicago).