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ON A REPRESENTATION OF A CERTAIN LINEAR TRANSFORMATION ON VECTOR LATTICES

By

Kunio TSUKADA and Takashi ITÔ

As an example of conditionally $\sigma$-complete vector lattices\(^1\) we have various function spaces consisting in real valued measurable functions on a measure space $\Omega$. On a function space, by a given measurable transformation $\omega \to \omega'$ on $\Omega$ and a fixed measurable function $f_0(\omega)$ on $\Omega$ we can define, by the below $(\ast)$, a linear transformation $T$ which has a suitably restricted domain and preserves an orthogonality,

$$(\ast) \quad (Tf)(\omega) = f_0(\omega)f(\omega')$$

In this paper, on a conditionally $\sigma$-complete vector lattice we shall consider a certain linear transformation which preserves an orthogonality and give a representation of it, which may be considered as an abstract from of the above $(\ast)$. This is also a generalization of a representation of a dilatator which is defined by Nakano and investigated in detail in his book [1].

Briefly we shall state some fundamental facts about the theory of representations of conditionally $\sigma$-complete vector lattices, which is necessary for the description of our results. (cf. [1] Chap. I, III, VII\(^2\)). Let $R$ be a conditionally $\sigma$-complete vector lattice having an order-complete element\(^3\) $e$. For any element $p$ of $R$ we can define a projection operator\(^4\) $[p]; [p]a = \bigcup_{n=1}^{\infty} (a^{+}n|p|) - \bigcup_{n=1}^{\infty} (a^{-}n|p|)$, the set $P$ of all projection operators $[p]$ is a $\sigma$-complete Boolean lattice\(^6\). The set $E$ of all maximal ideals\(^7\) $\mathfrak{p}$ of $P$ is a totally disconnected compact Hausdorff space with the topology such that all of $U_{[p]} = \{\mathfrak{p} | \mathfrak{p} \ni [p] \}$

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1) A vector lattice in which every upper bounded countable subset has a supremum, that is, a continuous semi-ordered linear space in Nakano's terminology, cf. [1] Chap. I §6.
2) We does not necessarily use notations and terminology in [1].
3) $e$ is positive and $e \cap a = 0$ implies $a = 0$.
4) $[p]$ is called projector in [1], cf. Chap. I §7.
5) $a^+ = a \cup 0$, $a^- = (-a)^+$ and $|a| = a^+ + a^-$. 
7) A family $\mathfrak{p}$ of projection operators is called ideal, if $p \not\in 0, p \not\in [p], [p] \leq [q]$ implies $p \not\in [q]$ and $p \not\in [p]$, $[q]$ implies $p \not\in [p][q]$. An ideal is called to be maximal, if there is no other ideal including $p$. 


constitutes an open base, namely $E$ is a Stone’s representation space\(^8\) of $P$. Let $C^{\infty}(E)$ be all of almost finite\(^9\) continuous functions on $E$, then in natural way $C^{\infty}(E)$ becomes a conditionally $\sigma$-complete vector lattice\(^{10}\). The representation\(^{11}\) of $R$ is as follows: There exists an isomorphism from $R$ into $C^{\infty}(E)$, $a \rightarrow \varphi_{a}$, such that $\varphi_{e} = 1$ and all of representations of $a \in R$, denoted by $C_{R}(E)$, is a order-convex\(^{12}\) subspace of $C^{\infty}(E)$.

Let $\mathfrak{p} \rightarrow \mathfrak{p}'$ be a continuous transformation from some compact open subset $U_{0}$ of $E$ to $E$ and satisfy the following condition;

(1) for any dense $F$-open set $U$ of $E$ the inverse image $U^{-1}$ of $U$ is also dense in $U_{0}$, 

then we see easily for any almost finite continuous function $\varphi(\mathfrak{p})$ on $E$ $\varphi(\mathfrak{p}')$, putting $\varphi(\mathfrak{p}') = 0$ for $\mathfrak{p} \in U_{0}$, is also an almost finite continuous function on $E$. Furthermore, by composing the above point transformation and the multiplication by a fixed $\varphi_{0} \in C^{\infty}(E)$ we have linear transformation $\hat{T}$ on $C^{\infty}(E)$, namely

$$(\hat{T}\varphi)(\mathfrak{p}) = \varphi_{0}(\mathfrak{p})\varphi(\mathfrak{p}') .$$

Naturally $\hat{T}$ induces a linear transformation $T$ on $R$, but $T$ has the restricted domain $D$;

$$D = \{ a \mid \varphi_{0}(\mathfrak{p})\varphi_{a}(\mathfrak{p}') \in C_{R}(E) \} ,$$

and $T$ is represented in $C_{R}(E)$ as follows

$$(**) \quad \varphi_{Ta}(\mathfrak{p}) = \varphi_{0}(\mathfrak{p})\varphi_{a}(\mathfrak{p}') \quad \quad (a \in D) .$$

Especially, if $\varphi_{0}$ belongs to $C_{R}(E)$, then $T$ has the following properties;

(2) $D$ contains $[p]e$ for any projection operator $[p]$, 

(3) $T$ preserves an orthogonality, that is, $a \perp b$ for $a, b \in D$ implies $Ta \perp Tb$. 

(4) $T$ is a closed transformation with respect to the uniform convergence\(^{13}\) in order and the orthogonal sum,

In detail (4) means that if either $\{a_{n}\}$ and $\{Ta_{n}\}$ converge uniformly in order

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9) An extended real valued function which is finite valued on a dense subset is called to be almost finite. 


12) $C_{R}(E) \ni \varphi$, $|\varphi| \geq |\psi|$ implies $C_{R}(E) \ni \psi$. 

13) A sequence $\{a_{n}\}$ is said to be uniformly convergent in order to a limit $a$ if there exist a positive element $l$ and a sequence of positive numbers $\varepsilon_{n}$ converging to 0 such that $|a_{n} - a| \leq \varepsilon_{n}l \ (n = 1, 2, \cdots)$. 
to $a$ and $b$ respectively or $\{a_n\}$ is an orthogonal sequence and $a = \sum_{n=1}^{\infty} a_n$ and $b = \sum_{n=1}^{\infty} Ta_n$, then $a$ belongs to $D$ and $Ta = b$.

(2) and (3) are almost evident. (4) is as follows. Let $\chi_{[p]}$ be a characteristic function of $U_{[p]}$ and $[p]'$ be defined by $U_{[p]}' = \{p' | p' \in U_{[p]}\}$. If $\{a_n\}$ is an orthogonal sequence in $D$ and $a = \sum_{n=1}^{\infty} a_n$ and $b = \sum_{n=1}^{\infty} Ta_n$, then $\chi_{[Ta_n]}(p) \varphi_p(p) = \varphi_p(a_n(p')) = \varphi_p(p) \chi_{[a_n]}(p') \varphi_{a_n}(p') = \chi_{[a_n]}(p) \varphi_p(a_n(p'))$ on $E$. According to (1), $[a] = \bigcup [a_n]$, implies $[a]' = \bigcup [a_n]'$, so $\sum_{n=1}^{\infty} \chi_{[a_n]}(p') = \chi_{[a]}'(p)$ on some dense open set of $E^{(1)}$. Therefore we have $\varphi_p(a) = \sum_{n=1}^{\infty} \chi_{[a_n]}(p) \varphi_p(a_n) = \sum_{n=1}^{\infty} \chi_{[a_n]}(p) \varphi_a(p') = \chi_{[a]}(p) \varphi_p(a) = \varphi_{[a]}(p)$ on some dense open set of $E$, that is, $a \in D$ and $Ta = b$. Next if $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order to $a$ and $b$ respectively, especially we assume $|a_n - a| \leq \frac{1}{n} e \ (n=1,2,\cdots)$, then $|\varphi_p(a) \varphi_a(p')| \leq \frac{1}{n} |\varphi_p(p)| + |\varphi_p(a) \varphi_a(p')| = \varphi_{\frac{1}{n}[Te] + [Ta_n]}(p)$, as $C_R(E)$ is order-convex in $C^\infty(E)$, we can find $c \in R$ such that $\varphi_p(p) = \varphi_a(p) \varphi_a(p')$ on $E$, that is, $a \in D$ and $Ta = c$. Therefore we see $|Ta_n - Ta| \leq \frac{1}{n} |Te| \ (n=1,2,\cdots)$, namely $b = Ta$. For general case $|a_n - a| \leq \frac{1}{n} l \ (n=1,2,\cdots)$ for some positive element $l$, we can find an orthogonal sequence $\{p_n\}$ of projection operators such that $\bigcup [p_n] = [e]$ and all $[p_n]l$ are bounded with respect to $e$, that is, $0 \leq [p_n] l \leq r_n e$ for some positive number $r_n$. As $T[p_n]a_n = [p_n]' Ta_n \ (n=1,2,\cdots)$, applying the above argument we see $[p_n]a \in D$ and $T[p_n]a = [p_n]'b$ for all $n=1,2,\cdots$. Again, by the closedness of $T$ with respect to orthogonal sum it follows $a \in D$ and $Ta = b$.

Conversely we can obtain the following:

**Theorem.** Let $T$ be a linear transformation on $R$ with domain $D$ and satisfy the above three conditions (2), (3) and (4), then we can find a continuous transformation $p \rightarrow p'$ from some compact open subset $U_o$ of $E$ to $E$ and an almost finite continuous function $\varphi_p$ on $E$, and $T$ can be represented by the form $(**)$. Namely,

i) the transformation $t$ satisfies (1),

ii) $a$ belongs to $D$ if and only if $\varphi_p(p) \varphi_a(p')$ belongs to $C_R(E)$,

iii) $\varphi_{Ta}(p) = \varphi_p(p) \varphi_{a}(p')$ for all $a \in D$.

Proof. At first $T$ induces a transformation $[p] \rightarrow [p]'$ on $P$ defined by

$$[p]' = [T[p]e]$$

This mapping $t$ is a homomorphism from $P$ to itself as a Boolean lattice. Because, if $[p]$ and $[q]$ are mutually orthogonal, then by (2) $[p]'$ and $[q]'$ are also mutually orthogonal and hence $([p] \cup [q])' = ([p] + [q])' = [p]' + [q]' = [p]' \cup [q]'$. Therefore, shown easily, $t$ is a homomorphism, that is, $([p] \cup [q])' = [p]' \cup [q]'$ and $([p][q])' = [p]'[q]'$ for all $[p]$ and $[q]$. Furthermore, if $[p] = \bigcup_{n=1}^{\infty} [p_{n}]$ then $[p]' = \bigcup_{n=1}^{\infty} [p_{n}]'$. Because, assuming $\{[p_{n}]\}$ to be mutually orthogonal without loss of generality, $[p]e = \sum_{n=1}^{\infty} [p_{n}]e$ and $b = \sum_{n=1}^{\infty} T[p_{n}]e = \sum_{n=1}^{\infty} [p_{n}]'Te$ imply $[b] = \bigcup_{n=1}^{\infty} [p_{n}]'$ and $b = T[p]e$ by the assumption (4), so we see $\bigcup_{n=1}^{\infty} [p_{n}]' = [b] = [T[p]e]' = [p]'.

Next, this homomorphism induces a continuous transformation $p \rightarrow p'$ from $U_{[Te]}$ to $E$ defined by

$$p' = \{[p] | [p]' \in p\} \quad (p \in U_{[Te]}).$$

By the definition $p \in U_{[p]'}$ if and only if $p' \in U_{[p]}$, so the inverse image of $U_{[p]}$ by $t$ is $U_{[p']}$. As $[p] = \bigcap_{n=1}^{\infty} [p_{n}]$ implies $[p]' = \bigcap_{n=1}^{\infty} [p_{n}]'$, $U_{[p]} = \left( \bigcup_{n=1}^{\infty} U_{[p_{n}]'} \right)^{-}$ and $U_{[p]'} = \left( \bigcup_{n=1}^{\infty} U_{[p_{n}]} \right)^{-}$, and $\bigcup_{n=1}^{\infty} U_{[p_{n}]}$ is the inverse image of $\bigcup_{n=1}^{\infty} U_{[p_{n}]}$, hence $t$ satisfies the condition (1).

For an element $a = \sum_{i=1}^{n} \xi_{i}[p_{i}]e$, $[p_{i}]$ ($1 \leq i \leq n$) are mutually orthogonal, evidently $a$ belongs to $D$ and $Ta = \sum_{i=1}^{n} \xi_{i}T[p_{i}]e = \sum_{i=1}^{n} \xi_{i}[p_{i}]'Te$, so it follows $\varphi_{Te}(p)$$= \left( \sum_{i=1}^{n} \xi_{i} \varphi_{Te}([p_{i}]) \right) \varphi_{Te}(p) = \varphi_{a}(p') \varphi_{Te}(p)$. If $a$ is a bounded element with respect to $e$, $|a| \leq re$ for some $r>0$, then $\varphi_{a}(p') \varphi_{Te}(p) = \varphi_{c}(p)$ for some $c$ by the order-convexity of $C_{R}(E)$ in $C^{\alpha}(E)$, and we can find elements $a_{n}$ with the above type such that $|a_{n} - a| \leq \frac{1}{n} e$ ($n=1, 2, \cdots$), therefore $|Ta_{n} - c| \leq \frac{1}{n} |Te|$ ($n=1, 2, \cdots$) and (4) implies $a \in D$ and $Ta = c$. For any $a \in D$ we can find $[p_{n}]$ ($n=1, 2, \cdots$) which are mutually orthogonal and $\bigcup_{n=1}^{\infty} [p_{n}] = [e]$ and $[p_{n}]a$ is a bounded element with respect to $e$ for all $n=1, 2, \cdots$, then $\sum_{n=1}^{\infty} [p_{n}]a = a$, and $\sum_{n=1}^{\infty} |T[p_{n}]a| = |\sum_{n=1}^{\infty} T[p_{n}]a| \leq |Ta|$ for all $n=1, 2, \cdots$ implies that $\sum_{n=1}^{\infty} T[p_{n}]a$ converges to some $b$ in order; $b = \sum_{n=1}^{\infty} T[p_{n}]a$. So it follows $Ta = b$ from (4) and $\varphi_{b}(p) =$
Finally if $\varphi_{Te}(\mathfrak{p})\varphi_{a}(\mathfrak{p}')$ belongs to $C_{R}(E)$ and we put $\varphi_{b}(\mathfrak{p})=\varphi_{Te}(\mathfrak{p})\varphi_{a}(\mathfrak{p}')$, then similarly to the above it can be proved from (4) $a\in D$ and $b=Ta$. The uniqueness of such representation of $T$ as type (***) is almost evident.

Reference


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