# ON A REPRESENTATION OF A CERTAIN LINEAR TRANSFORMATION ON VECTOR LATTICES

By

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As an example of conditionally $\sigma$-complete vector lattices[1] we have various function spaces consisting in real valued measurable functions on a measure space $\Omega$. On a function space, by a given measurable transformation $\omega \rightarrow \omega^t$ on $\Omega$ and a fixed measurable function $f_0(\omega)$ on $\Omega$ we can define, by the below (*), a linear transformation $T$ which has a suitably restricted domain and preserves an orthogonality,

$$(*) \qquad (Tf)(\omega) = f_0(\omega) f(\omega^t)$$

In this paper, on a conditionally $\sigma$-complete vector lattice we shall consider a certain linear transformation which preserves an orthogonality and give a representation of it, which may be considered as an abstract from of the above (*). This is also a generalization of a representation of a *dilatator* which is defined by Nakano and investigated in detail in his book [1].

Briefly we shall state some fundamental facts about the theory of representations of conditionally $\sigma$-complete vector lattices, which is necessary for the description of our results. (cf. [1] Chap. I, III, VII)[2]. Let $R$ be a conditionally $\sigma$-complete vector lattice having a order-complete element[3] $e$. For any element $p$ of $R$ we can define a projection operator[4] $[p]$; $[p]a = \bigcup_{n=1}^{\infty}(a^+ \frown n|p|) - \bigcup_{n=1}^{\infty}(a^- \frown n|p|)$[5], the set $P$ of all projection operators $[p]$ is a $\sigma$-complete Boolean lattice[6]. The set $E$ of all maximal ideals[7] $\mathfrak{p}$ of $P$ is a totally disconnected compact Hausdorff space with the topology such that all of $U_{[p]} = \{\mathfrak{p} | \mathfrak{p} \ni [p]\}$

1) A vector lattice in which every upper bounded countable subset has a supremum, that is, *a continuous semi-ordered linear space* in Nakano's terminology, cf. [1] Chap. I §6.

2) We does not necessarily use notations and terminology in [1].

3) $e$ is positive and $e \cap a = 0$ implies $a = 0$.

4) $[p]$ is called *projector* in [1], cf. Chap. I §7.

5) $a^+ = a \cup 0$, $a^- = (-a)^+$ and $|a| = a^+ + a^-$.

6) Cf. [1] Chap. I §8.

7) A family $\mathfrak{p}$ of projection operators is called *ideal*, if $\mathfrak{p} \not\ni 0$, $\mathfrak{p} \ni [p]$, $[p] \leqq [q]$ implies $\mathfrak{p} \ni [q]$ and $\mathfrak{p} \ni [p]$, $[q]$ implies $\mathfrak{p} \ni [p][q]$. An ideal is called to be *maximal*, if there is no other ideal including $\mathfrak{p}$.

constitutes an open base, namely $E$ is a Stone's representation space[8] of $P$. Let $C^\infty(E)$ be all of almost finite[9] continuous functions on $E$, then in natural way $C^\infty(E)$ becomes a conditionally $\sigma$-complete vector lattice[10]. The representation[11] of $R$ is as follows: There exists an isomorphism from $R$ into $C^\infty(E)$, $a \to \varphi_a$, such that $\varphi_e = 1$ and all of representations of $a \in R$, denoted by $C_R(E)$, is a order-convex[12] subspace of $C^\infty(E)$.

Let $\mathfrak{p} \to \mathfrak{p}^t$ be a continuous transformation from some compact open subset $U_0$ of $E$ to $E$ and satisfy the following condition;

(1) *for any dense $F_\sigma$-open set $U$ of $E$ the inverse image $U^{t-1}$ of $U$ is also dense in $U_0$,*

then we see easily for any almost finite continuous function $\varphi(\mathfrak{p})$ on $E$ $\varphi(\mathfrak{p}^t)$, putting $\varphi(\mathfrak{p}^t)=0$ for $\mathfrak{p} \notin U_0$, is also an almyst finite continuous function on $E$. Furthermore, by composing the above point transformation and the multiplication by a fixed $\varphi_0 \in C^\infty(E)$ we have linear transformation $\hat{T}$ on $C^\infty(E)$, namely

$$(\hat{T}\varphi)(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi(\mathfrak{p}^t) .$$

Naturally $\hat{T}$ induces a linear transformation $T$ on $R$, but $T$ has the restricted domain $D$;

$$D = \left\{a \mid \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \in C_R(E)\right\} ,$$

and $T$ is represented in $C_R(E)$ as follows

$$(**) \qquad \varphi_{Ta}(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \qquad (a \in D) .$$

Especially, if $\varphi_0$ belongs to $C_R(E)$, then $T$ has the following properties;

(2) *$D$ contains $[p]e$ for any projection operator $[p]$,*

(3) *$T$ preserves an orthogonality, that is, $a \perp b$ for $a, b \in D$ implies $Ta \perp Tb$.*

(4) *$T$ is a closed transformation with respect to the uniform convergence[13] in order and the orthogonal sum,*

In detail (4) means that if either $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order

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8) $E$ is called the *proper space* of $R$, cf. [1] Chap. III §15, §16.

9) An extended real valued function which is finite valued on a dense subset is called to be *almost finite.*

10) Cf. [1] Chap. VII §41 Theorems 41.1–41.3.

11) Cf. [1] Chap. III the second spectral theory.

12) $C_R(E) \ni \varphi$, $|\varphi| \geqq |\psi|$ implies $C_R(E) \ni \psi$.

13) A sequence $\{a_n\}$ is said to be *uniformly convergent* in order to a limit $a$ if there exist a positive element $l$ and a sequnce of positive numbers $\varepsilon_n$ converging to 0 such that $|a_n - a| \leqq \varepsilon_n l$ $(n=1, 2, \cdots)$.

to $a$ and $b$ respectively or $\{a_n\}$ is an orthogonal sequence and $a=\sum_{n=1}^{\infty}a_n$ and $b=\sum_{n=1}^{\infty}Ta_n$, then $a$ belongs to $D$ and $Ta=b$.

(2) and (3) are almost evident. (4) is as follows. Let $\chi_{[p]}$ be a characteristic function of $U_{[p]}$ and $[p]^t$ be defined by $U_{[p]^t}=\{\mathfrak{p}|\mathfrak{p}^t\in U_{[p]}\}$. If $\{a_n\}$ is an orthogonal sequence in $D$ and $a=\sum_{n=1}^{\infty}a_n$ and $b=\sum_{n=1}^{\infty}Ta_n$, then $\chi_{[Ta_n]}(\mathfrak{p})\varphi_b(\mathfrak{p}) =\varphi_{Ta_n}(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_{a_n}(\mathfrak{p}^t)=\varphi_0(\mathfrak{p})\chi_{[a_n]}(\mathfrak{p}^t)\varphi_a(\mathfrak{p}^t)=\chi_{[a_n]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on $E$. According to (1), $[a]=\bigcup_{n=1}^{\infty}[a_n]$, implies $[a]^t=\bigcup_{n=1}^{\infty}[a_n]^t$, so $\sum_{n=1}^{\infty}\chi_{[a_n]^t}(\mathfrak{p})=\chi_{[a]^t}(\mathfrak{p})$ on some dense open set of $E$[14]. Therefore we have $\varphi_b(\mathfrak{p})=\sum_{n=1}^{\infty}\chi_{[Ta_n]}(\mathfrak{p})\varphi_b(\mathfrak{p})= \sum_{n=1}^{\infty}\chi_{[a_n]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)=\chi_{[a]^t}(\mathfrak{p})\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on some dense open set of $E$, that is, $a\in D$ and $Ta=b$. Next if $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order to $a$ and $b$ respectively, especially we assume $|a_n-a|\leqq\frac{1}{n}e$ $(n=1,2,\cdots)$, then $|\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)|\leqq\frac{1}{n}|\varphi_0(\mathfrak{p})|+|\varphi_0(\mathfrak{p})\varphi_{a_n}(\mathfrak{p}^t)|=\varphi_{\frac{1}{n}|Te|+|Ta_n|}(\mathfrak{p})$, as $C_R(E)$ is order-convex in $C^\infty(E)$, we can find $c\in R$ such that $\varphi_c(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ on $E$, that is, $a$ belongs to $D$ and $Ta=c$. Therefore we see $|Ta_n-Ta|\leqq\frac{1}{n}|Te|$ $(n=1,2,\cdots)$, namely $b=Ta$. For general case $|a_n-a|\leqq\frac{1}{n}l$ $(n=1,2,\cdots)$ for some positive element $l$, we can find an orthogonal sequence $\{[p_n]\}$ of projection operators such that $\bigcup_{n=1}^{\infty}[p_n]=[e]$ and all $[p_n]l$ are bounded with respect to $e$, that is, $0\leqq[p_n]l\leqq r_ne$ for some positive number $r_n$. As $T[p_\nu]a_n=[p_\nu]^tTa_n$ $(n=1,2,\cdots)$, applying the above argument we see $[p_\nu]a\in D$ and $T[p_\nu]a=[p_\nu]^tb$ for all $\nu=1,2,\cdots$. Again, by the closedness of $T$ with respect to orthogonal sum it follows $a\in D$ and $Ta=b$.

Conversely we can obtain the following:

**Theorem.** *Let $T$ be a linear transformation on $R$ with domain $D$ and satisfy the above three conditions* (2), (3) *and* (4), *then we can find a continuous transformation $\mathfrak{p}\to\mathfrak{p}^t$ from some compact open subset $U_0$ of $E$ to $E$ and an almost finite continuous function $\varphi_0$ on $E$, and $T$ can be represented by the form* $(**)$. *Namely,*

i) *the transformation $t$ satisfies* (1),

ii) *$a$ belongs to $D$ if and only if $\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ belongs to $C_R(E)$,*

iii) *$\varphi_{Ta}(\mathfrak{p})=\varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t)$ for all $a\in D$.*

14) Cf. [1] Chap. III §16 Theorem 16.4.

*Proof.* At first $T$ induces a transformation $[p]\to[p]^t$ on $P$ defined by

$$[p]^t = [T[p]e]$$

This mapping $t$ is a homomorphism from $P$ to itself as a Boolean lattice. Because, if $[p]$ and $[q]$ are mutually orthogonal, then by (2) $[p]^t$ and $[q]^t$ are also mutually orthogonal and hence $([p]\cup[q])^t=([p]+[q])^t=[p]^t+[q]^t=[p]^t\cup[q]^t$. Therefore, shown easily, $t$ is a homomorphism, that is, $([p]\cup[q])^t=[p]^t\cup[q]^t$ and $([p][q])^t=[p]^t[q]^t$ for all $[p]$ and $[q]$. Furthermore, if $[p]=\bigcup_{n=1}^{\infty}[p_n]$ then $[p]^t=\bigcup_{n=1}^{\infty}[p_n]^t$. Because, assuming $\{[p_n]\}$ to be mutually orthogonal without loss of generality, $[p]e=\sum_{n=1}^{\infty}[p_n]e$ and $b=\sum_{n=1}^{\infty}T[p_n]e=\sum_{n=1}^{\infty}[p_n]^tTe$ imply $[b]=\bigcup_{n=1}^{\infty}[p_n]^t$ and $b=T[p]e$ by the assumption (4), so we see $\bigcup_{n=1}^{\infty}[p_n]^t=[b]=[T[p]e]=[p]^t$. Next, this homomorphism induces a continuous transformation $\mathfrak{p}\to\mathfrak{p}^t$ from $U_{[Te]}$ to $E$ defined by

$$\mathfrak{p}^t = \{[p]\mid[p]^t\in\mathfrak{p}\} \qquad (\mathfrak{p}\in U_{[Te]}).$$

By the definition $\mathfrak{p}\in U_{[p]^t}$ if and only if $\mathfrak{p}^t\in U_{[p]}$, so the inverse image of $U_{[p]}$ by $t$ is $U_{[p]^t}$. As $[p]=\bigcup_{n=1}^{\infty}[p_n]$ implies $[p]^t=\bigcup_{n=1}^{\infty}[p_n]^t$, $U_{[p]}=\left(\bigcup_{n=1}^{\infty}U_{[p_n]}\right)^-$ and $U_{[p]^t}=\left(\bigcup_{n=1}^{\infty}U_{[p_n]^t}\right)^-$, and $\bigcup_{n=1}^{\infty}U_{[p_n]^t}$ is the inverse image of $\bigcup_{n=1}^{\infty}U_{[p_n]}$ hence $t$ satisfies the condition (1).

For an element $a=\sum_{i=1}^{n}\xi_i[p_i]e$, $[p_i]$ $(1\leqq i\leqq n)$ are mutually orthogonal, evidently $a$ belongs to $D$ and $Ta=\sum_{i=1}^{n}\xi_iT[p_i]e=\sum_{i=1}^{n}\xi_i[p_i]^tTe$, so it follows $\varphi_{Ta}(\mathfrak{p})=\left(\sum_{i=1}^{n}\xi_i\chi_{[p_i]^t}(\mathfrak{p})\right)\varphi_{Te}(\mathfrak{p})=\varphi_a(\mathfrak{p}^t)\varphi_{Te}(\mathfrak{p})$. If $a$ is a bounded element with respect to $e$, $|a|\leqq re$ for some $r>0$, then $\varphi_a(\mathfrak{p}^t)\varphi_{Te}(\mathfrak{p})=\varphi_c(\mathfrak{p})$ for some $c$ by the order-convexity of $C_R(E)$ in $C^\infty(E)$, and we can find elements $a_n$ with the above type such that $|a_n-a|\leqq\frac{1}{n}e$ $(n=1,2,\cdots)$, therefore $|Ta_n-c|\leqq\frac{1}{n}|Te|$ $(n=1,2,\cdots)$ and (4) implies $a\in D$ and $Ta=c$. For any $a\in D$ we can find $[p_n]$ $(n=1,2,\cdots)$ which are mutually orthogonal and $\bigcup_{n=1}^{\infty}[p_n]=[e]$ and $[p_n]a$ is a bounded element with respect to $e$ for all $n=1,2,\cdots$, then $\sum_{n=1}^{\infty}[p_n]a=a$, and $\sum_{\nu=1}^{n}|T[p_\nu]a|=|\sum_{\nu=1}^{n}T[p_\nu]a|\leqq|Ta|$ for all $n=1,2,\cdots$ implies that $\sum_{\nu=1}^{n}T[p_\nu]a$ converges to some $b$ in order; $b=\sum_{n=1}^{\infty}T[p_n]a$. So it follows $Ta=b$ from (4) and $\varphi_b(\mathfrak{p})=$

$\sum_{n=1}^{\infty} \varphi_{T[p_n]a}(\mathfrak{p}) = \sum_{n=1}^{\infty} \varphi_{Te}(\mathfrak{p}) \chi_{[p_n]^t}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t) = \varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$. Finally if $\varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$ belongs to $C_R(E)$ and we put $\varphi_b(\mathfrak{p}) = \varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$, then similarly to the above it can be proved from (4) $a \in D$ and $b = Ta$. The uniqueness of such representation of $T$ as type (**) is almost evident.

### Reference

[1] H. NAKANO: Modern spectral theory, Tokyo Mathematical Book Series, Vol. 2 (1950).

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