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ON A REPRESENTATION OF A CERTAIN LINEAR TRANSFORMATION ON VECTOR LATTICES

By

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As an example of conditionally σ -complete vector lattices¹⁾ we have various function spaces consisting in real valued measurable functions on a measure space Ω . On a function space, by a given measurable transformation $\omega \rightarrow \omega^t$ on Ω and a fixed measurable function $f_0(\omega)$ on Ω we can define, by the below (*), a linear transformation T which has a suitably restricted domain and preserves an orthogonality,

$$(*) \quad (Tf)(\omega) = f_0(\omega)f(\omega^t)$$

In this paper, on a conditionally σ -complete vector lattice we shall consider a certain linear transformation which preserves an orthogonality and give a representation of it, which may be considered as an abstract form of the above (*). This is also a generalization of a representation of a *dilatator* which is defined by Nakano and investigated in detail in his book [1].

Briefly we shall state some fundamental facts about the theory of representations of conditionally σ -complete vector lattices, which is necessary for the description of our results. (cf. [1] Chap. I, III, VII)²⁾. Let R be a conditionally σ -complete vector lattice having a order-complete element³⁾ e . For any element p of R we can define a projection operator⁴⁾ $[p]$; $[p]a = \bigcup_{n=1}^{\infty} (a^+ \wedge n|p|) - \bigcup_{n=1}^{\infty} (a^- \wedge n|p|)$ ⁵⁾, the set P of all projection operators $[p]$ is a σ -complete Boolean lattice⁶⁾. The set E of all maximal ideals⁷⁾ \mathfrak{p} of P is a totally disconnected compact Hausdorff space with the topology such that all of $U_{[p]} = \{\mathfrak{p} | \mathfrak{p} \ni [p]\}$

1) A vector lattice in which every upper bounded countable subset has a supremum, that is, a *continuous semi-ordered linear space* in Nakano's terminology, cf. [1] Chap. I §6.

2) We does not necessarily use notations and terminology in [1].

3) e is positive and $e \cap a = 0$ implies $a = 0$.

4) $[p]$ is called *projector* in [1], cf. Chap. I §7.

5) $a^+ = a \cup 0$, $a^- = (-a)^+$ and $|a| = a^+ + a^-$.

6) Cf. [1] Chap. I §8.

7) A family \mathfrak{p} of projection operators is called *ideal*, if $\mathfrak{p} \not\ni 0$, $\mathfrak{p} \ni [p]$, $[p] \leq [q]$ implies $\mathfrak{p} \ni [q]$ and $\mathfrak{p} \ni [p]$, $[q]$ implies $\mathfrak{p} \ni [p][q]$. An ideal is called to be *maximal*, if there is no other ideal including \mathfrak{p} .

constitutes an open base, namely E is a Stone's representation space⁸⁾ of P . Let $C^\infty(E)$ be all of almost finite⁹⁾ continuous functions on E , then in natural way $C^\infty(E)$ becomes a conditionally σ -complete vector lattice¹⁰⁾. The representation¹¹⁾ of R is as follows: There exists an isomorphism from R into $C^\infty(E)$, $a \rightarrow \varphi_a$, such that $\varphi_e = 1$ and all of representations of $a \in R$, denoted by $C_R(E)$, is a order-convex¹²⁾ subspace of $C^\infty(E)$.

Let $\mathfrak{p} \rightarrow \mathfrak{p}^t$ be a continuous transformation from some compact open subset U_0 of E to E and satisfy the following condition;

- (1) for any dense F_σ -open set U of E the inverse image $U^{t^{-1}}$ of U is also dense in U_0 ,

then we see easily for any almost finite continuous function $\varphi(\mathfrak{p})$ on E $\varphi(\mathfrak{p}^t)$, putting $\varphi(\mathfrak{p}^t) = 0$ for $\mathfrak{p} \notin U_0$, is also an almost finite continuous function on E . Furthermore, by composing the above point transformation and the multiplication by a fixed $\varphi_0 \in C^\infty(E)$ we have linear transformation \hat{T} on $C^\infty(E)$, namely

$$(\hat{T}\varphi)(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi(\mathfrak{p}^t).$$

Naturally \hat{T} induces a linear transformation T on R , but T has the restricted domain D ;

$$D = \{a \mid \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \in C_R(E)\},$$

and T is represented in $C_R(E)$ as follows

$$(**) \quad \varphi_{T a}(\mathfrak{p}) = \varphi_0(\mathfrak{p})\varphi_a(\mathfrak{p}^t) \quad (a \in D).$$

Especially, if φ_0 belongs to $C_R(E)$, then T has the following properties;

- (2) D contains $[p]e$ for any projection operator $[p]$,
 (3) T preserves an orthogonality, that is, $a \perp b$ for $a, b \in D$ implies $Ta \perp Tb$.
 (4) T is a closed transformation with respect to the uniform convergence¹³⁾ in order and the orthogonal sum,

In detail (4) means that if either $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order

8) E is called the *proper space* of R , cf. [1] Chap. III §15, §16.

9) An extended real valued function which is finite valued on a dense subset is called to be *almost finite*.

10) Cf. [1] Chap. VII §41 Theorems 41.1–41.3.

11) Cf. [1] Chap. III the second spectral theory.

12) $C_R(E) \ni \varphi, |\varphi| \geq |\psi|$ implies $C_R(E) \ni \psi$.

13) A sequence $\{a_n\}$ is said to be *uniformly convergent* in order to a limit a if there exist a positive element l and a sequence of positive numbers ε_n converging to 0 such that $|a_n - a| \leq \varepsilon_n l$ ($n=1, 2, \dots$).

to a and b respectively or $\{a_n\}$ is an orthogonal sequence and $a = \sum_{n=1}^{\infty} a_n$ and $b = \sum_{n=1}^{\infty} Ta_n$, then a belongs to D and $Ta = b$.

(2) and (3) are almost evident. (4) is as follows. Let $\chi_{[p]}$ be a characteristic function of $U_{[p]}$ and $[p]^t$ be defined by $U_{[p]^t} = \{p | p^t \in U_{[p]}\}$. If $\{a_n\}$ is an orthogonal sequence in D and $a = \sum_{n=1}^{\infty} a_n$ and $b = \sum_{n=1}^{\infty} Ta_n$, then $\chi_{[Ta_n]}(p) \varphi_b(p) = \varphi_{Ta_n}(p) = \varphi_0(p) \varphi_{a_n}(p^t) = \varphi_0(p) \chi_{[a_n]}(p^t) \varphi_a(p^t) = \chi_{[a_n]^t}(p) \varphi_0(p) \varphi_a(p^t)$ on E . According to (1), $[a] = \bigcup_{n=1}^{\infty} [a_n]$, implies $[a]^t = \bigcup_{n=1}^{\infty} [a_n]^t$, so $\sum_{n=1}^{\infty} \chi_{[a_n]^t}(p) = \chi_{[a]^t}(p)$ on some dense open set of E^{14} . Therefore we have $\varphi_b(p) = \sum_{n=1}^{\infty} \chi_{[Ta_n]}(p) \varphi_b(p) = \sum_{n=1}^{\infty} \chi_{[a_n]^t}(p) \varphi_0(p) \varphi_a(p^t) = \chi_{[a]^t}(p) \varphi_0(p) \varphi_a(p^t) = \varphi_0(p) \varphi_a(p^t)$ on some dense open set of E , that is, $a \in D$ and $Ta = b$. Next if $\{a_n\}$ and $\{Ta_n\}$ converge uniformly in order to a and b respectively, especially we assume $|a_n - a| \leq \frac{1}{n} e$ ($n = 1, 2, \dots$), then $|\varphi_0(p) \varphi_a(p^t)| \leq \frac{1}{n} |\varphi_0(p)| + |\varphi_0(p) \varphi_{a_n}(p^t)| = \varphi_{\frac{1}{n}|Te| + |Ta_n|}(p)$, as $C_R(E)$ is order-convex in $C^\infty(E)$, we can find $c \in R$ such that $\varphi_c(p) = \varphi_0(p) \varphi_a(p^t)$ on E , that is, a belongs to D and $Ta = c$. Therefore we see $|Ta_n - Ta| \leq \frac{1}{n} |Te|$ ($n = 1, 2, \dots$), namely $b = Ta$. For general case $|a_n - a| \leq \frac{1}{n} l$ ($n = 1, 2, \dots$) for some positive element l , we can find an orthogonal sequence $\{[p_n]\}$ of projection operators such that $\bigcup_{n=1}^{\infty} [p_n] = [e]$ and all $[p_n]l$ are bounded with respect to e , that is, $0 \leq [p_n]l \leq r_n e$ for some positive number r_n . As $T[p_\nu]a_n = [p_\nu]^t Ta_n$ ($n = 1, 2, \dots$), applying the above argument we see $[p_\nu]a \in D$ and $T[p_\nu]a = [p_\nu]^t b$ for all $\nu = 1, 2, \dots$. Again, by the closedness of T with respect to orthogonal sum it follows $a \in D$ and $Ta = b$.

Conversely we can obtain the following:

Theorem. *Let T be a linear transformation on R with domain D and satisfy the above three conditions (2), (3) and (4), then we can find a continuous transformation $p \rightarrow p^t$ from some compact open subset U_0 of E to E and an almost finite continuous function φ_0 on E , and T can be represented by the form (**). Namely,*

- i) *the transformation t satisfies (1),*
- ii) *a belongs to D if and only if $\varphi_0(p) \varphi_a(p^t)$ belongs to $C_R(E)$,*
- iii) *$\varphi_{Ta}(p) = \varphi_0(p) \varphi_a(p^t)$ for all $a \in D$.*

14) Cf. [1] Chap. III §16 Theorem 16.4.

Proof. At first T induces a transformation $[p] \rightarrow [p]^t$ on P defined by

$$[p]^t = [T[p]e]$$

This mapping t is a homomorphism from P to itself as a Boolean lattice. Because, if $[p]$ and $[q]$ are mutually orthogonal, then by (2) $[p]^t$ and $[q]^t$ are also mutually orthogonal and hence $([p] \cup [q])^t = ([p] + [q])^t = [p]^t + [q]^t = [p]^t \cup [q]^t$. Therefore, shown easily, t is a homomorphism, that is, $([p] \cup [q])^t = [p]^t \cup [q]^t$ and $([p][q])^t = [p]^t[q]^t$ for all $[p]$ and $[q]$. Furthermore, if $[p] = \bigcup_{n=1}^{\infty} [p_n]$ then $[p]^t = \bigcup_{n=1}^{\infty} [p_n]^t$. Because, assuming $\{[p_n]\}$ to be mutually orthogonal without loss of generality, $[p]e = \sum_{n=1}^{\infty} [p_n]e$ and $b = \sum_{n=1}^{\infty} T[p_n]e = \sum_{n=1}^{\infty} [p_n]^t Te$ imply $[b] = \bigcup_{n=1}^{\infty} [p_n]^t$ and $b = T[p]e$ by the assumption (4), so we see $\bigcup_{n=1}^{\infty} [p_n]^t = [b] = [T[p]e] = [p]^t$. Next, this homomorphism induces a continuous transformation $\mathfrak{p} \rightarrow \mathfrak{p}^t$ from $U_{[re]}$ to E defined by

$$\mathfrak{p}^t = \{[p] \mid [p]^t \in \mathfrak{p}\} \quad (\mathfrak{p} \in U_{[re]}).$$

By the definition $\mathfrak{p} \in U_{[p]^t}$ if and only if $\mathfrak{p}^t \in U_{[p]}$, so the inverse image of $U_{[p]}$ by t is $U_{[p]^t}$. As $[p] = \bigcup_{n=1}^{\infty} [p_n]$ implies $[p]^t = \bigcup_{n=1}^{\infty} [p_n]^t$, $U_{[p]} = \left(\bigcup_{n=1}^{\infty} U_{[p_n]}\right)^-$ and $U_{[p]^t} = \left(\bigcup_{n=1}^{\infty} U_{[p_n]^t}\right)^-$, and $\bigcup_{n=1}^{\infty} U_{[p_n]^t}$ is the inverse image of $\bigcup_{n=1}^{\infty} U_{[p_n]}$ hence t satisfies the condition (1).

For an element $a = \sum_{i=1}^n \xi_i [p_i]e$, $[p_i]$ ($1 \leq i \leq n$) are mutually orthogonal, evidently a belongs to D and $Ta = \sum_{i=1}^n \xi_i T[p_i]e = \sum_{i=1}^n \xi_i [p_i]^t Te$, so it follows $\varphi_{Ta}(\mathfrak{p}) = \left(\sum_{i=1}^n \xi_i \chi_{[p_i]^t}(\mathfrak{p})\right) \varphi_{Te}(\mathfrak{p}) = \varphi_a(\mathfrak{p}^t) \varphi_{Te}(\mathfrak{p})$. If a is a bounded element with respect to e , $|a| \leq re$ for some $r > 0$, then $\varphi_a(\mathfrak{p}^t) \varphi_{Te}(\mathfrak{p}) = \varphi_c(\mathfrak{p})$ for some c by the order-convexity of $C_R(E)$ in $C^\infty(E)$, and we can find elements a_n with the above type such that $|a_n - a| \leq \frac{1}{n}e$ ($n = 1, 2, \dots$), therefore $|Ta_n - c| \leq \frac{1}{n}|Te|$ ($n = 1, 2, \dots$) and (4) implies $a \in D$ and $Ta = c$. For any $a \in D$ we can find $[p_n]$ ($n = 1, 2, \dots$) which are mutually orthogonal and $\bigcup_{n=1}^{\infty} [p_n] = [e]$ and $[p_n]a$ is a bounded element with respect to e for all $n = 1, 2, \dots$, then $\sum_{n=1}^{\infty} [p_n]a = a$, and $\sum_{\nu=1}^n |T[p_\nu]a| = \left|\sum_{\nu=1}^n T[p_\nu]a\right| \leq |Ta|$ for all $n = 1, 2, \dots$ implies that $\sum_{\nu=1}^{\infty} T[p_\nu]a$ converges to some b in order; $b = \sum_{n=1}^{\infty} T[p_n]a$. So it follows $Ta = b$ from (4) and $\varphi_b(\mathfrak{p}) =$

$\sum_{n=1}^{\infty} \varphi_{T[p_n]a}(\mathfrak{p}) = \sum_{n=1}^{\infty} \varphi_{Te}(\mathfrak{p}) \chi_{[p_n]}^t(\mathfrak{p}) \varphi_a(\mathfrak{p}^t) = \varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$. Finally if $\varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$ belongs to $C_R(E)$ and we put $\varphi_b(\mathfrak{p}) = \varphi_{Te}(\mathfrak{p}) \varphi_a(\mathfrak{p}^t)$, then similarly to the above it can be proved from (4) $a \in D$ and $b = Ta$. The uniqueness of such representation of T as type (***) is almost evident.

Reference

- [1] H. NAKANO: Modern spectral theory, Tokyo Mathematical Book Series, Vol. 2 (1950).

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