ON SOME CONSIDERATIONS OF HYPERSURFACES IN CERTAIN ALMOST COMPLEX SPACES

By

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Introduction. Y. Tashiro [7][1] proved that an orientable hypersurface in an almost complex space has an almost contact structure. The purpose of the present paper is to study some properties of the induced almost contact metric structure for a hypersurface in a Kählerian space and those of the induced almost contact metric structure for a hypersurface in a K-space.

In §1 we shall give the definition of an induced almost contact metric structure for a hypersurface in an almost Hermitian space. The condition that a hypersurface in a Kählerian space be a normal contact space is already known [7]. §2 devoted to seek for a condition that a hypersurface in a Kählerian space be a contact metric space and a condition that a hypersurface in a Kählerian space be a normal almost contact metric space is given in §3. In §4 we shall show that every hypersurface in a non-Kählerian K-space can not admit a contact metric structure. §5 devoted to seek for a condition that a hypersurface in a K-space be a normal almost contact metric space. A property of a hypersurface admitting the second fundamental tensor with some conditions in a K-space will be discussed in §6.

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§1. An induced almost contact metric structure for a hypersurface in an almost Hermitian space. Let M^{2n+2} be a $(2n+2)$-dimensional almost Hermitian space with local coordinates x^i $(i=1,2,\cdots,2n+2)$ and we shall denote by $F^i{}_j$ and g_{ij} its almost complex structure and Hermitian metric tensor respectively. Then we have

$$(1.1) \qquad F^i{}_h F^h{}_j = -\delta^i{}_j\,, \qquad g_{ij}F^i{}_h F^j{}_k = g_{hk}\,.$$

By virtue of (1.1) it follows that

$$(1.2) \qquad F_{ij} = -F_{ji}\,. \qquad (F_{ij} = g_{ih}F^h{}_j)$$

Let us consider that an orientable hypersurface M^{2n+1} in the almost Her-

1) Numbers in brackets refer to the references at the end of the paper.

mitian space M^{2n+2} be defined by $2n+2$ equations involving $2n+1$ independent parameters u^α $(\alpha=1,2,\cdots,2n+1)$ such that

$$x^i = x^i(u^\alpha)\,. \qquad (i=1,2,\cdots,2n+2\,;\ \alpha=1,2,\cdots,2n+1)^{2)}$$

The set of $2n+2$ linearly independent vectors $(X^i_\alpha,\ X^i)$ determines an ennuple at each point of M^{2n+1}, where $X^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ and X^i is the contravariant component of the unit normal vector of M^{2n+1}. Now we put

$$X^\alpha_i = g^{\alpha\beta} g_{ij} X^j_\beta\,, \qquad X_i = g_{ij} X^j\,,$$

where $g^{\alpha\beta}$ is determined by the equation $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$ and $g_{\beta\gamma} = g_{ij} X^i_\beta X^j_\gamma$. Making use of these quantities we shall define $\varphi^\alpha{}_\beta$, ξ^α and η_β as follows:

$$(1.3) \qquad \varphi^\alpha{}_\beta = F^i{}_j X^\alpha_i X^j_\beta\,, \quad \xi^\alpha = -F^i{}_j X^\alpha_i X^j\,, \quad \eta_\beta = F^i{}_j X_i X^j_\alpha\,.$$

According to the relations

$$X^i_\alpha X_i = 0\,, \qquad X^i X^\alpha_i = 0\,, \qquad X^i X_i = 1\,,$$
$$X^i_\alpha X^\beta_i = \delta^\beta_\alpha\,, \qquad X^i_\alpha X^\alpha_j + X^i X_j = \delta^i_j\,,$$

we can see that the following conditions are satisfied:

$$(1.4) \qquad \begin{cases} \xi^\alpha \eta_\alpha = 1\,, & \text{rank } (\varphi^\alpha{}_\beta) = 2n\,, \\ \varphi^\alpha{}_\beta \xi^\beta = 0\,, & \varphi^\alpha{}_\beta \eta_\alpha = 0\,, \\ \varphi^\alpha{}_\beta \varphi^\beta{}_\gamma = -\delta^\alpha{}_\gamma + \xi^\alpha \eta_\gamma\,. \end{cases}$$

$$(1.5) \qquad g_{\alpha\beta}\xi^\alpha = \eta_\beta\,, \qquad g_{\alpha\beta}\varphi^\alpha{}_\gamma \varphi^\beta{}_\delta = g_{\gamma\delta} - \eta_\gamma \eta_\delta\,.$$

Therefore we may consider that the quantities $\varphi^\alpha{}_\beta$, ξ^α, η_β and $g_{\alpha\beta}$ define an almost contact metric structure for M^{2n+1}. From (1.4) and (1.5) it follows that

$$(1.6) \qquad \varphi_{\alpha\beta} = -\varphi_{\beta\alpha}\,. \qquad (\varphi_{\alpha\beta} = g_{\alpha\gamma}\varphi^\gamma{}_\beta)$$

On making use of the Gauss' equations we have from (1.3)

$$(1.7) \qquad \varphi^\alpha{}_{\beta;\gamma} = F^i_{j,k} X^\alpha_i X^j_\beta X^k_\gamma + H^\alpha_\gamma \eta_\beta - H_{\beta\gamma}\xi^\alpha\,,^{3)}$$

$$(1.8) \qquad \xi^\alpha{}_{;\beta} = -F^i{}_{j,k} X^\alpha_i X^j X^k_\beta + H^\gamma{}_\beta \varphi^\alpha{}_\gamma\,,$$

where $H^\alpha{}_\beta = g^{\alpha\gamma} H_{\beta\gamma}$ and $H_{\beta\gamma}$ denotes the covariant component of the second fundamental tensor of M^{2n+1}. There exist the following four tensors in an almost

2) Throughout the present paper the Latin indices are supposed to run over the range $1,2,\cdots,2n+2$, and the Greek indices take the values $1,2,\cdots,2n+1$.

3) In the present paper comma means the covariant differentiation with respect to the Riemann connection determined by g_{ij} and semi-colon denotes the covariant differentiation with respect to its induced connection.

contact metric space:

$$N_\alpha = \xi^\delta(\eta_{\delta;\alpha} - \eta_{\alpha;\delta}), \tag{1.9}$$

$$N_{\beta\gamma} = \varphi^\delta{}_\beta(\eta_{\gamma;\delta} - \eta_{\delta;\gamma}) - \varphi^\delta{}_\gamma(\eta_{\beta;\delta} - \eta_{\delta;\beta}), \tag{1.10}$$

$$N^\alpha{}_\beta = \xi^\delta(\varphi^\alpha{}_{\beta;\delta} - \varphi^\alpha{}_{\delta;\beta}) - \varphi^\delta{}_\beta \xi^\alpha{}_{;\delta}, \tag{1.11}$$

$$N^\alpha{}_{\beta\gamma} = \varphi^\delta{}_\gamma(\varphi^\alpha{}_{\beta;\delta} - \varphi^\alpha{}_{\delta;\beta}) - \varphi^\delta{}_\beta(\varphi^\alpha{}_{\gamma;\delta} - \varphi^\alpha{}_{\delta;\gamma}) + \xi^\alpha{}_{;\beta}\eta_\gamma - \xi^\alpha{}_{;\gamma}\eta_\beta. \tag{1.12}$$

In this paper an almost contact metric structure with condition $N^\alpha{}_{\beta\gamma}=0$ is said to be normal and if $\varphi_{\alpha\beta}$ satisfies the condition

$$\varphi_{\alpha\beta} = \frac{1}{2}(\eta_{\beta;\alpha} - \eta_{\alpha;\beta}), \tag{1.13}$$

we shall say that M^{2n+1} admits a contact metric structure.

§ 2. A condition that a hypersurface in a Kählerian space be a contact metric space. Let us consider that M^{2n+2} be a Kählerian space. Then $F^i{}_{j,k}=0$ holds good and from (1.7) and (1.8) we have

$$\varphi^\alpha{}_{\beta;\gamma} = H^\alpha{}_\gamma\eta_\beta - H_{\beta\gamma}\xi^\alpha, \tag{2.1}$$

$$\xi^\alpha{}_{;\beta} = H^\gamma{}_\beta\varphi^\alpha{}_\gamma. \tag{2.2}$$

Substituting from (2.1) in the right hand member of (1.9)–(1.12), we obtain the following expressions for tensors N_α, $N_{\beta\gamma}$, $N^\alpha{}_\beta$ and $N^\alpha{}_{\beta\gamma}$ respectively:

$$N_\alpha = H_{\varepsilon\delta}\xi^\delta\varphi^\varepsilon{}_\alpha, \tag{2.3}$$

$$N_{\beta\gamma} = H_{\varepsilon\gamma}\xi^\varepsilon\eta_\beta - H_{\varepsilon\beta}\xi^\varepsilon\eta_\gamma, \tag{2.4}$$

$$N^\alpha{}_\beta = H^\alpha{}_\delta\xi^\delta\eta_\beta - H^\alpha{}_\beta - \varphi^\delta{}_\beta\xi^\alpha{}_{;\delta}, \tag{2.5}$$

$$N^\alpha{}_{\beta\gamma} = H^\alpha{}_\delta\eta_\beta\varphi^\delta{}_\gamma - H^\alpha{}_\delta\eta_\gamma\varphi^\delta{}_\beta + \xi^\alpha{}_{;\beta}\eta_\gamma - \xi^\alpha{}_{;\gamma}\eta_\beta. \tag{2.6}$$

Now, we shall prove the following theorem:

Theorem 2.1. *In order that a hypersurface in a Kählerian space be a contact metric space, it is necessary and sufficient that $H_{\alpha\beta}$ be of the form*

$$H_{\alpha\beta} = -g_{\alpha\beta} + \phi\eta_\alpha\eta_\beta + \frac{1}{2}\left\{\varphi^\gamma{}_\alpha(\varphi_{\beta\gamma} + \eta_{\beta;\gamma}) + \varphi^\gamma{}_\beta(\varphi_{\alpha\gamma} + \eta_{\alpha;\gamma})\right\}, \tag{2.7}$$

where ϕ is a scalar function.

Proof. We shall first prove the necessity of (2.7). If a hypersurface is a contact metric space, then from $N_\alpha=0$ [5], (1.4) and (2.3) we have the relation

(2. 8) $$H_{\varepsilon\delta}\xi^{\delta}=\rho\eta_{\varepsilon}\,,$$

where ρ is a scalar function. Also from (2.1) we have

(2. 9) $$\varphi_{\alpha\gamma;\beta}+\varphi_{\beta\gamma;\alpha}=2H_{\alpha\beta}\eta_{\gamma}-H_{\beta\gamma}\eta_{\alpha}-H_{\alpha\gamma}\eta_{\beta}\,.$$

Multiplying (2.9) by ξ^{γ} and summing for γ, with the aid of (2.8) it follows that

(2. 10) $$\frac{1}{2}(\varphi_{\alpha\gamma;\beta}+\varphi_{\beta\gamma;\alpha})\xi^{\gamma}=H_{\alpha\beta}-\rho\eta_{\alpha}\eta_{\beta}\,.$$

On the other hand, by virtue of (1.4) and (1.6) it follows that

$$(\varphi_{\alpha\gamma;\beta}+\varphi_{\beta\gamma;\alpha})\xi^{\gamma}=\varphi^{\gamma}{}_{\alpha}\eta_{\gamma;\beta}+\varphi^{\gamma}{}_{\beta}\eta_{\gamma;\alpha}\,.$$

Consequently by virtue of (2.10) and the last relation it must be satisfied that

(2. 11) $$H_{\alpha\beta}=\rho\eta_{\alpha}\eta_{\beta}+\frac{1}{2}(\varphi^{\gamma}{}_{\alpha}\eta_{\gamma;\beta}+\varphi^{\gamma}{}_{\beta}\eta_{\gamma;\alpha})\,.$$

On the other hand, the condition (1.13) is satisfied. Hence we have

(2. 12) $$\eta_{\gamma;\delta}=\eta_{\delta;\gamma}+2\varphi_{\delta\gamma}\,.$$

Substituting from (2.12) in the right hand member of (2.11) we get

$$H_{\alpha\beta}=-g_{\alpha\beta}+\phi\eta_{\alpha}\eta_{\beta}+\frac{1}{2}\left\{\varphi^{\gamma}{}_{\alpha}(\varphi_{\beta\gamma}+\eta_{\beta;\gamma})+\varphi^{\gamma}{}_{\beta}(\varphi_{\alpha\gamma}+\eta_{\alpha;\gamma})\right\}\,,$$

where $\phi=\rho+1$.

Next we shall show that the relation (2.7) is the sufficient condition that a hypersurface in a Kählerian space admit a contact metric structure. If (2.7) is multiplied by ξ^{β} and β is summed from 1 to $2n+1$, by virtue of (1.4) we obtain $H_{\alpha\beta}\xi^{\beta}=\rho\eta_{\alpha}$. Therefore from (2.3) we get the condition $N_{\alpha}=0$ and we see that the condition $\eta_{\delta;\gamma}\xi^{\gamma}=0$ should be satisfied. Hence substituting from (2.7) in the right hand member of (2.2) we get

(2. 13) $$\eta_{\delta;\gamma}=2\varphi_{\gamma\delta}+\frac{1}{2}\eta_{\gamma;\delta}-\frac{1}{2}\eta_{\varepsilon;\lambda}\varphi^{\varepsilon}{}_{\delta}\varphi^{\lambda}{}_{\gamma}\,.$$

On the other hand, from the condition $N_{\alpha}=0$ we may consider that $H_{\alpha\beta}$ is of the form (2.11). If we substitute from (2.11) in the right hand member of (2.2), it follows that

(2. 14) $$\eta_{\gamma;\delta}=-\eta_{\varepsilon;\lambda}\varphi^{\varepsilon}{}_{\delta}\varphi^{\lambda}{}_{\gamma}\,.$$

Therefore by virtue of (2.13) and (2.14) we get the relation

$$\varphi_{\gamma\delta}=\frac{1}{2}(\eta_{\delta;\gamma}-\eta_{\gamma;\delta})\,.$$

Accordingly we can see that the induced almost contact metric structure for M^{2n+1} is a contact metric structure.

On making use of the result of Theorem 2.1 for a hypersurface admitting a contact metric structure with the condition $N^{\alpha}{}_{\beta\gamma}=0$, we have the following corollary:

Corollary 2. 1.[4] *In order that a hypersurface in a Kählerian space be a normal contact metric space, it is necessary and sufficient that $H_{\alpha\beta}$ be of the form*

$$H_{\alpha\beta}=-g_{\alpha\beta}+\phi\eta_{\alpha}\eta_{\beta}\,. \tag{2.15}$$

Proof. If M^{2n+1} is a normal contact metric space, then we have $\varphi_{\alpha\beta}=-\eta_{\alpha;\beta}$ [5]. Accordingly substituting from this relation in the right hand member of (2.7), we get (2.15). Conversely if $H_{\alpha\beta}$ is of the form (2.15), by virtue of (2.1) it follows that

$$\varphi_{\alpha\beta;\gamma}=\eta_{\alpha}g_{\beta\gamma}-\eta_{\beta}g_{\alpha\gamma}\,.$$

Therefore M^{2n+1} is a normal contact metric space [6].

§ 3. A condition that a hypersurface in a Kählerian space be a normal almost contact metric space. We suppose that a hypersurface in a Kählerian space admits an almost contact metric structure with the condition $N^{\alpha}_{\beta\gamma}=0$. In this case it is known that other three tensors N_{α}, $N_{\beta\gamma}$ and N^{α}_{β} also vanish [4]. By virtue of $N_{\alpha}=0$ and (2.3) we have

$$H_{\varepsilon\delta}\xi^{\delta}=\rho\eta_{\varepsilon}\,. \tag{3.1}$$

Substituting from (3.1) in the right hand member of (2.4), we can see that $N_{\beta\gamma}=0$ is satisfied identically. On making use of (3.1), from $N^{\alpha}{}_{\beta}=0$ and (2.5) we obtain

$$H_{\alpha\beta}=\rho\eta_{\alpha}\eta_{\beta}-\varphi^{\delta}{}_{\beta}\eta_{\alpha;\delta}\,.$$

By virtue of the second relation in (1.5), the above relation may be rewritten as follows:

$$H_{\alpha\beta}=-g_{\alpha\beta}+\phi\eta_{\alpha}\eta_{\beta}-\varphi^{\delta}{}_{\beta}(\varphi_{\alpha\delta}+\eta_{\alpha;\delta})\,. \tag{3.2}$$

Conversely if the second fundamental tensor $H_{\alpha\beta}$ is of the form (3.2), it is evident that the condition (3.1) is satisfied. Then from (2.3) we get $N_{\alpha}=0$

4) This result has proved by Y. Tashiro [7].

and by virtue of (1.9) it should be satisfied that $\xi^\alpha{}_{;\delta}\xi^\delta=0$. Therefore if we substitute from (3.2) in the right hand member of (2.6) we can find that $N^\alpha{}_{\beta\gamma}=0$ holds good. Consequently we have the following theorem:

Theorem 3.1. *In order that a hypersurface in a Kählerian space be a normal almost contact metric space, it is necessary and sufficient that $H_{\alpha\beta}$ be of the form* (3.2).

On making use of the result of Theorem 3.1 for a hypersurface admitting a normal almost contact metric structure with the condition (1.13), we have the following corollary:

Corollary 3.1.[4] *In order that a hypersurface in a Kählerian space be a normal contact metric space, it is necessary and sufficient that $H_{\alpha\beta}$ be of the form*

(3.3) $$H_{\alpha\beta}=-g_{\alpha\beta}+\phi\eta_\alpha\eta_\beta\,.$$

Proof. If M^{2n+1} is a normal contact metric space, then we have $\varphi_{\alpha\beta}=-\eta_{\alpha;\beta}$ [5]. Accordingly from the result of Theorem 3.1 we have (3.3). The sufficiency of the condition (3.3) is proved by the same way as in Corollary 2.1.

§4. A certain remark on a hypersurface in a K-space. Let M^{2n+2} be a K-space, then we have the following condition [1]:

(4.1) $$F_{ij,k}+F_{ik,j}=0\,.$$

Putting $F_{ij,k}X^i_\alpha X^j_\beta X^k_\gamma=A_{\alpha\beta\gamma}$, and making use of (1.2) and (4.1) we find that the covariant tensor $A_{\alpha\beta\gamma}$ is skewsymmetric with respect to all indices. Then (1.7) becomes

(4.2) $$\varphi_{\alpha\beta;\gamma}=A_{\alpha\beta\gamma}+H_{\alpha\gamma}\eta_\beta-H_{\beta\gamma}\eta_\alpha\,.$$

Multiplying (4.2) by ξ^α and summing for α, it follows that

$$\varphi_{\alpha\beta;\gamma}\xi^\alpha-H_{\gamma\alpha}\xi^\alpha\eta_\beta+H_{\beta\gamma}=A_{\alpha\beta\gamma}\xi^\alpha\,.$$

On making use of (1.6), the last relation may be rewritten as follows:

(4.3) $$\varphi^\alpha{}_\beta(\eta_{\alpha;\gamma}+H_{\varepsilon\gamma}\varphi^\varepsilon{}_\alpha)=-A_{\alpha\beta\gamma}\xi^\alpha\,.$$

On the other hand, from (1.4) it follows that

(4.4) $$\xi^\alpha\eta_\beta(\eta_{\alpha;\gamma}+H_{\varepsilon\gamma}\varphi^\varepsilon{}_\alpha)=0\,.$$

Adding (4.3) and (4.4) sides by sides, we get

(4.5) $$(\varphi^\alpha{}_\beta+\xi^\alpha\eta_\beta)(\eta_{\alpha;\gamma}+H_{\varepsilon\gamma}\varphi^\varepsilon{}_\alpha)=-A_{\alpha\beta\gamma}\xi^\alpha\,.$$

Since the matrix $(\varphi^\alpha{}_\beta+\xi^\alpha\eta_\beta)$ is non-singular, we obtain from (4.5)

$$\xi^{\alpha}{}_{;\beta} = H^{\gamma}{}_{\beta}\varphi^{\alpha}{}_{\gamma} + A^{\epsilon}{}_{\delta\beta}\xi^{\delta}\varphi^{\alpha}{}_{\epsilon}\,. \tag{4.6}$$

Substituting from (4.2) in the right hand member of (1.9)–(1.12), we obtain the following expressions for tensors N_{α}, $N_{\beta\gamma}$, $N^{\alpha}{}_{\beta}$ and $N^{\alpha}{}_{\beta\gamma}$ respectively:

$$N_{\alpha} = H_{\epsilon\delta}\xi^{\delta}\varphi^{\epsilon}{}_{\alpha}\,, \tag{4.7}$$

$$N_{\beta\gamma} = 2A_{\delta\beta\gamma}\xi^{\delta} - 2A_{\delta\epsilon\lambda}\xi^{\delta}\varphi^{\epsilon}{}_{\beta}\varphi^{\lambda}{}_{\gamma} + H_{\delta\gamma}\xi^{\delta}\eta_{\beta} - H_{\delta\beta}\xi^{\delta}\eta_{\gamma}\,, \tag{4.8}$$

$$N^{\alpha}{}_{\beta} = 2A^{\alpha}{}_{\beta\delta}\xi^{\delta} + H^{\alpha}{}_{\delta}\xi^{\delta}\eta_{\beta} - H^{\alpha}{}_{\beta} - \varphi^{\delta}{}_{\beta}\xi^{\alpha}{}_{;\delta}\,, \tag{4.9}$$

$$N^{\alpha}{}_{\beta\gamma} = 2A^{\alpha}{}_{\beta\delta}\varphi^{\delta}{}_{\gamma} - 2A^{\alpha}{}_{\gamma\delta}\varphi^{\delta}{}_{\beta} + H^{\alpha}{}_{\delta}\eta_{\beta}\varphi^{\delta}{}_{\gamma} - H^{\alpha}{}_{\delta}\eta_{\gamma}\varphi^{\delta}{}_{\beta} + \xi^{\alpha}{}_{;\beta}\eta_{\gamma} - \xi^{\alpha}{}_{;\gamma}\eta_{\beta}\,. \tag{4.10}$$

Before our theorem, we shall first prove the following lemma :

Lemma 4.1. *If $A_{\alpha\beta\gamma}=0$ holds good, then a K-space is necessarily a Kählerian space.*

Proof. Wth irespect to the ennuple (X^{i}_{α}, X^{i}), the component of the tensor $F_{ij,k}$ is expressible as follows:

$$\begin{aligned} F_{ij,k} = {} & \Phi_{\alpha\beta\gamma}X^{\alpha}_{i}X^{\beta}_{j}X^{\gamma}_{k} + \Phi^{(1)}_{\alpha\beta}X^{\alpha}_{i}X^{\beta}_{j}X_{k} + \Phi^{(2)}_{\alpha\beta}X^{\alpha}_{i}X_{j}X^{\beta}_{k} + \Phi^{(3)}_{\alpha\beta}X_{i}X^{\alpha}_{j}X^{\beta}_{k} \\ & + \Phi^{(1)}_{\alpha}X^{\alpha}_{i}X_{j}X_{k} + \Phi^{(2)}_{\alpha}X_{i}X^{\alpha}_{j}X_{k} + \Phi^{(3)}_{\alpha}X_{i}X_{j}X^{\alpha}_{k} + \Phi X_{i}X_{j}X_{k}\,. \end{aligned} \tag{4.11}$$

However, as in a K-space the tensor $F_{ij,k}$ is skewsymmetric with respect to all indices we have

$$\left\{ \begin{aligned} & \Phi_{\alpha\beta\gamma} = A_{\alpha\beta\gamma}\,, \\ & \Phi^{(1)}_{\alpha\beta} = -\Phi^{(2)}_{\alpha\beta} = \Phi^{(3)}_{\alpha\beta} = F_{ij,k}X^{i}_{\alpha}X^{j}_{\beta}X^{k}\,, \\ & \Phi^{(1)}_{\alpha} = \Phi^{(2)}_{\alpha} = \Phi^{(3)}_{\alpha} = \Phi = 0\,. \end{aligned} \right. \tag{4.12}$$

On the other hand, from (1.8) and (4.6) we have

$$F_{ij,k}X^{i}_{\alpha}X^{j}_{\beta}X^{k} = A_{\delta\epsilon\beta}\xi^{\delta}\varphi^{\epsilon}{}_{\alpha}\,. \tag{4.13}$$

In consequence of (4.11), (4.12) and (4.13), it follows that if $A_{\alpha\beta\gamma}=0$ holds good we must have $F_{ij,k}=0$. This means that the space M^{2n+2} is a Kählerian space.

By virtue of Lemma 4.1, we have the following theorem:

Theorem 4.1. *A hypersurface in a non-Kählerian K-space can not admit a contact metric structure.*

Proof. Suppose that a hypersurface in a K-space admits a contact metric structure. Then from (1.13) we have

$$\varphi_{\alpha\beta;\gamma} + \varphi_{\beta\gamma;\alpha} + \varphi_{\gamma\alpha;\beta} = 0\,.$$

On the other hand, from (4.2) and the skewsymmetric property of the tensor $A_{\alpha\beta\gamma}$, it follows that

$$\varphi_{\alpha\beta;\gamma}+\varphi_{\beta\gamma;\alpha}+\varphi_{\gamma\alpha;\beta}=3A_{\alpha\beta\gamma}\,.$$

From the last two relations we find that $A_{\alpha\beta\gamma}=0$ must be satisfied. Consequently from the result of Lemma 4.1, it is concluded that the K-space is necessarily a Kählerian space.

§5. A condition that a hypersurface in a K-space be a normal almost contact metric space. Let us consider that M^{2n+1} be a hypersurface in a K-space. Then we have

Lemma 5.1. *If $N_\alpha=0$ and $N_{\beta\gamma}=0$, then we have $A_{\alpha\beta\gamma}\xi^\alpha=0$.*

Proof. By virtue of (4.7), if $N_\alpha=0$ holds good we have

$$H_{\varepsilon\delta}\xi^\delta=\rho\eta_\varepsilon\,. \tag{5.1}$$

Moreover, if $N_{\beta\gamma}=0$ holds good, from (4.8) and (5.1) it follows that

$$A_{\delta\beta\gamma}\xi^\delta-A_{\delta\varepsilon\lambda}\xi^\delta\varphi^\varepsilon{}_\beta\varphi^\lambda{}_\gamma=0\,. \tag{5.2}$$

Multiplying (5.2) by $\varphi^\beta{}_\alpha$ and summing for β, by virtue of (1.4) we obtain

$$A_{\delta\beta\gamma}\xi^\delta\varphi^\beta{}_\alpha-A_{\delta\lambda\alpha}\xi^\delta\varphi^\lambda{}_\gamma=0\,. \tag{5.3}$$

The last relation means that $A_{\delta\varepsilon\gamma}\xi^\delta\varphi^\varepsilon{}_\alpha$ is symmetric with respect to indices α and γ. However because of the skewsymmetric property of $F_{ij,k}$ the relation (4.13) shows that $A_{\delta\varepsilon\gamma}\xi^\delta\varphi^\varepsilon{}_\alpha$ is skewsymmetric with respect to indices α and γ. Accordingly we must have

$$A_{\delta\beta\gamma}\xi^\delta\varphi^\beta{}_\alpha=0\,.$$

Multiplying the last relation by $\varphi^\alpha{}_\varepsilon$ and summing for α, we get $A_{\delta\varepsilon\gamma}\xi^\delta=0$.

By virtue of the result of Lemma 5.1, we have the following theorem:

Theorem 5.1. *In order that a hypersurface in a K-space be a normal almost contact metric space, it is necessary and sufficient that $H_{\alpha\beta}$ be of the form*

$$H_{\alpha\beta}=-g_{\alpha\beta}+\phi\eta_\alpha\eta_\beta-\varphi^\delta{}_\beta(\varphi_{\alpha\delta}+\eta_{\alpha;\delta})\,, \tag{5.4}$$

and $A_{\alpha\beta\gamma}$ satisfy the relation

$$A_{\alpha\beta\delta}\varphi^\delta{}_\gamma=A_{\alpha\gamma\delta}\varphi^\delta{}_\beta\,. \tag{5.5}$$

Proof. We suppose that a hypersurface in a K-space admits a normal almost contact metric structure. Then four tensors N_α, $N_{\beta\gamma}$, $N^\alpha{}_\beta$ and $N^\alpha{}_{\beta\gamma}$ vanish [4]. Making use of the result of Lemma 5.1, the relation (5.4) is obtained from $N_\alpha=0$, $N_{\beta\gamma}=0$ and $N^\alpha{}_\beta=0$. If we substitute from (5.4) in the right hand member of (4.10) it follows that

$$N^{\alpha}{}_{\beta\gamma} = 2A^{\alpha}{}_{\beta\delta}\varphi^{\delta}{}_{\gamma} - 2A^{\alpha}{}_{\gamma\delta}\varphi^{\delta}{}_{\beta}\,.$$

Therefore if $N^{\alpha}{}_{\beta\gamma}=0$ holds good, the relation (5.5) is satisfied.

Conversely if $H_{\alpha\beta}$ is of the form (5.4), by virtue of (4.10) and (5.5) we can easily obtain the relation $N^{\alpha}{}_{\beta\gamma}=0$. Therefore our theorem has proved.

§6. A certain property of a hypersurface in a K-space. In this section we shall seek for a property of a hypersurface in a K-space, which is equivalent to the condition

$$H_{\alpha\beta} = -g_{\alpha\beta} + \phi\eta_{\alpha}\eta_{\beta}\,. \tag{6.1}$$

where ϕ is a scalar function.

Substituting from (6.1) in the right hand member of (4.6), we have

$$\varphi_{\alpha\beta} = -\eta_{\alpha;\beta} + A_{\delta\epsilon\beta}\xi^{\delta}\varphi^{\epsilon}{}_{\alpha}\,. \tag{6.2}$$

Next, we shall show that (6.2) is the sufficient condition that $H_{\alpha\beta}$ be of the form (6.1). Multiplying (6.2) by ξ^{β} and summing for β, we have $\eta_{\alpha;\beta}\xi^{\beta}=0$. From the last relation it follows that $N_{\alpha}=0$, then from (4.7) we get the relation

$$H_{\epsilon\delta}\xi^{\delta} = \rho\eta_{\epsilon}\,. \tag{6.3}$$

Moreover, from (4.2) we have

$$\varphi_{\alpha\gamma;\beta} + \varphi_{\beta\gamma;\alpha} = 2H_{\alpha\beta}\eta_{\gamma} - H_{\beta\gamma}\eta_{\alpha} - H_{\alpha\gamma}\eta_{\beta}\,. \tag{6.4}$$

Then if we use again the process by which (2.11) is obtained from (2.9) under the condition (2.8), we can get from (6.3) and (6.4) the following expression for $H_{\alpha\beta}$:

$$H_{\alpha\beta} = \rho\eta_{\alpha}\eta_{\beta} + \frac{1}{2}(\varphi^{\gamma}{}_{\alpha}\eta_{\gamma;\beta} + \varphi^{\gamma}{}_{\beta}\eta_{\gamma;\alpha})\,. \tag{6.5}$$

Solving (6.2) for $\eta_{\alpha;\beta}$ and substituting from the solution in the right hand member of (6.5), we obtain (6.1). Consequently we have the following theorem:

Theorem 6.1. *The second fundamental tensor $H_{\alpha\beta}$ of a hypersurface in a K-space is of the form*

$$H_{\alpha\beta} = -g_{\alpha\beta} + \phi\eta_{\alpha}\eta_{\beta}\,,$$

if and only if $\varphi_{\alpha\beta}$ satisfies the relation

$$\varphi_{\alpha\beta} = -\eta_{\alpha;\beta} + A_{\delta\epsilon\beta}\xi^{\delta}\varphi^{\epsilon}{}_{\alpha}\,.$$

References

[1] K. YANO: The theory of Lie derivatives and its applications, North-Holland Publ., (1957).

[2] L. P. EISENHART: Riemannian Geometry, Princeton Univ. Press, (1960).

[3] S. SASAKI: On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Jour., 12 (1960), 456–476.

[4] S. SASAKI and Y. HATAKEYAMA: On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tôhoku Math. Jour., 13 (1961), 281–294.

[5] S. SASAKI and Y. HATAKEYAMA: On differentiable manifolds with contact metric structures, Jour. Math. Soc. Japan, 14 (1962), 249–271.

[6] Y. HATAKEYAMA, Y. OGAWA and S. TANNO: Some properties of manifolds with contact metric structure, Tôhoku Math. Jour., 15 (1963), 42–48.

[7] Y. TASHIRO: On contact structure of hypersurfaces in complex manifolds, I, Tôhoku Math. Jour., 15 (1963), 62–78.

[8] T. NAGAI and H. KÔJYÔ: On some properties of hypersurfaces with certain contact structures, Jour. Fac. Sci., Hokkaido Univ., Ser. I, 17 (1963), 160–167.

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