



Title	ON NAGAHARA'S THEOREM
Author(s)	Tominaga, Hisao
Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 18(3-4), 153-157
Issue Date	1965
Doc URL	http://hdl.handle.net/2115/56062
Type	bulletin (article)
File Information	JFSHIU_18_N3-4_153-157.pdf



[Instructions for use](#)

ON NAGAHARA'S THEOREM

By

Hisao TOMINAGA

Throughout the present note, $A = \sum De_{ij}$ will represent a simple ring ($\{e_{ij}\}$'s is a system of matrix units and $D = V_A(\{e_{ij}\})$ a division ring), B a simple subring of A containing the identity 1 of A , and \mathfrak{G} the group of all the B -(ring) automorphisms of A . And we set $V = V_A(B)$ and $H = V_A^2(B) = V_A(V_A(B))$. As to notations and terminologies used in this note, we follow [3]¹⁾ and [4].

In [3], A/B was called h -Galois if (1) B is regular, (2) A is Galois over B' and $V_A^2(B')$ is simple for every regular subring B' of A left finite over B , and if (3) $A' = V_A^2(A')$ and $[A' : H]_l = [V : V_A(A')]_r$ for every regular subring A' of A left finite over H . Recently, in his paper [1], T. Nagahara has obtained the following theorem.

Theorem 1. (i) A/B is h -Galois and left locally finite if and only if any of the following conditions (A_l) – (B_r) is satisfied:

- (A_l) (1) B is a regular subring of A and $\mathfrak{G}A_r$ is dense in $\text{Hom}_{B_l}(A, A)$.
- (2) A/B is left locally finite.
- (A_r) (1) B is a regular subring of A and $\mathfrak{G}A_l$ is dense in $\text{Hom}_{B_r}(A, A)$.
- (2) A/B is right locally finite.
- (B_l) (1) A/B is Galois and A is BV - A -irreducible.
- (2) A/B is left locally finite.
- (B_r) (1) A/B is Galois and A is A - BV -irreducible.
- (2) A/B is right locally finite.

(ii) If A/B is h -Galois and left locally finite, then $[B' : B]^{2)} \geq [V : V_A(B')] = [V_A^2(B') : H] = [B' : H \cap B']$ for every regular subring B' of A left finite over B .

And by the aid of Theorem 1, he has obtained also the next important theorem.

Theorem 2. Let A be h -Galois and left locally finite over B . If \mathfrak{S} is a $(*_f)$ -regular subgroup of \mathfrak{G} then \mathfrak{S} is f -regular.

One of the purposes of this note is to give a rather direct proof of Theorem 2. To this end, we shall prove first the following brief lemma.

1) Numbers in brackets refer to the references cited at the end of this note.
 2) If $[B' : B]_l = [B' : B]_r$, they are represented as $[B' : B]$.

Lemma 1. *Let A be left locally finite over B , and T an intermediate ring of A/B with $[T: B]_l < \infty$ such that $V_A(T)$ coincides with the center of A . If there exists an automorphism group \mathfrak{S} of A such that $J(\mathfrak{S}, A) = T$ then T is a simple ring.*

Proof. If $B' = T[\{e_{ij}\text{'s}\}]$ then $(\mathfrak{S}|B')A_r = \bigoplus_{i=1}^m (\sigma_i|B')A_r$ with some $\sigma_i \in \mathfrak{S}$. By [3, Corollary 1.1],³⁾ we have $\mathfrak{S}|B' = \{\sigma_1|B', \dots, \sigma_m|B'\}$. Accordingly, $B_0 = B'[\cup B'\sigma_i]$ is an \mathfrak{S} -invariant simple subring of A left finite over B and $\#(\mathfrak{S}|B_0) < \infty$. Since $J(\mathfrak{S}|B_0, B_0) = T$ and $V_{B_0}(T) = V_{B_0}(B_0)$, it is well-known that T is a simple ring.

Lemma 2. *Let A be h -Galois and left locally finite over B , and \mathfrak{S} a $(*_f)$ -regular subgroup of \mathfrak{G} . If $T = J(\mathfrak{S}, A)$ then $[T: H \cap T]_l < \infty$.*

Proof. Let N be a $\mathfrak{G}(H/B)$ -invariant shade of $\{d_{hk}\text{'s}\}$ (a system of matrix units of H such that the centralizer of $\{d_{hk}\text{'s}\}$ in H is a division ring), $\mathfrak{S}^* = \mathfrak{G}(N) \cap \mathfrak{S} = \mathfrak{S}(N)$, $T^* = J(\mathfrak{S}^*, A)$ ($\supseteq T[N]$), and $H^* = H \cap T^*$. Then, \mathfrak{S}^* is an invariant subgroup of \mathfrak{S} and $(\mathfrak{S}: \mathfrak{S}^*) = \#(\mathfrak{S}|N) \leq \#\mathfrak{G}(N/B) = [N: B] < \infty$. As $H^* = H \cap T^*$ is an \mathfrak{S} -invariant simple subring of H by [2, Theorem 1.1], $J(\mathfrak{S}|H^*, H^*) = H^* \cap T = H \cap T$ and $\infty > (\mathfrak{S}: \mathfrak{S}^*) \geq \#(\mathfrak{S}|H^*)$, we see that $H^*/H \cap T$ is outer Galois and $[H^*: H \cap T] < \infty$. On the other hand, in virtue of [1, Lemma 2], there holds $[T^*: H^*]_l = [H \cdot T^*: H]_l$. And further, we can see that $V_{\mathfrak{S}^*} = V_{\mathfrak{S}} = V_A(B[E])$ for some finite subset E of A . Now, let B' be a regular subring of A containing $B[E]$ such that $[B': B]_l < \infty$. If $A' = V_A^2(B')$ then $V_{\mathfrak{S}^*} = V_A(B[E]) \supseteq V_A(B')$ yields $H[T^*] \subseteq V_A(V_{\mathfrak{S}^*}) \subseteq A'$, so that $[H[T^*]: H]_l \leq [A': H]_l < \infty$. Combining this with $[T^*: H^*]_l = [H \cdot T^*: H]_l$, we obtain eventually $[T: H \cap T]_l \leq [T^*: H^*]_l \cdot [H^*: H \cap T] \leq [H[T^*]: H]_l \cdot [H^*: H \cap T] < \infty$.

The next lemma is proved essentially in [1, Lemma 3 (ii)]. However, for the sake of completeness, we shall give here a slight simplified proof.

Lemma 3. *Let A be h -Galois and left locally finite over B , and V' a simple subring of V with $[V: V']_r < \infty$. If $V_A(V_A(V')[E]) \subseteq V'$ for some finite subset E of A , then $B' = V_A(V')$ is a simple ring.*

Proof. By the light of Theorem 1, we may assume that $H = B$. If $B = \sum Kd_{hk}$ with a system of matrix units $\{d_{hk}\text{'s}\}$ such that $K = V_B(\{d_{hk}\text{'s}\})$ is a division ring, then we have $BV = K \cdot \sum Vd_{hk}$, $V_A(K) = \sum Vd_{hk}$ (simple) and $V_A(\sum Vd_{hk}) = K$. Hence, again by Theorem 1, we may assume further that B is a division ring. Since A is A - BV -irreducible, $[B': B]_l \leq [V: V']_r < \infty$ by [1, Lemma 1]. If $B'' = B'[E, \{e_{ij}\text{'s}\}]$ and $V'' = V_A(B'') (\subseteq V')$, then $V_A(V'') = B''$ and $\infty > [B'': B] = [V: V''] = [V: V_A^2(V'') \cap V]$ by Theorem 1. Noting here that

3) [3, Corollary 1.1] is valid without the assumption that B is regular.

A is V - A -irreducible by [1, Lemma 1], we obtain $\text{Hom}_{V'_i}(V, A) = (\widetilde{B}''|V)A_r = \bigoplus_i^t (\sigma_i|V)A_r$ ($\sigma_i \in \widetilde{B}''$) by [3, Lemma 3.1], where each $(\sigma_i|V)A_r$ is V_r - A_r -irreducible and $\sigma_i|V$ is linearly independent over A_r . And then, $\text{Hom}_{V'_i}(V, A)$ being a V_r - A_r -submodule of the completely reducible $\text{Hom}_{V'_i}(V, A)$, there holds $\text{Hom}_{V'_i}(V, A) = \bigoplus_i^s \mathfrak{M}_j$ with V_r - A_r -irreducible \mathfrak{M}_j . To be easily verified, each \mathfrak{M}_j is then of the form $(\sigma u_i|V)A_r$ with some σ in $\{\sigma_i\text{'s}\} (\subseteq \widetilde{B}'')$ and non-zero u , and so we may set $\mathfrak{M}_j = (b_{ji}|V)A_r$ with some non-zero b_j . Noting here that $b_{ji}|V$ is contained in $\text{Hom}_{V'_i}(V, A)$, it will be easy to see that b_j is contained in B' . Now, let $M = V'vA$ ($v \in V$) be a V' - A -submodule of A such that the length $[M|A_r]$ of the composition series of M as right A -module is minimal among non-zero V' - A -submodules of A of the form $V'xA$ ($x \in V$). Evidently, M is BV' - A -admissible. If M' is a minimal BV' - A -submodule of M then $M' = uA$ with some $u \in V$ as a direct summand of the completely reducible B - A -module A (cf. [1, Lemma 1]), and so $M \supseteq M' = V'uA$. Hence, by the minimality of $[M|A_r]$, it follows that $M = M'$, that is, M is BV' - A -irreducible. Consequently, for an arbitrary V' - A -minimal submodule VxA of M , there holds $M = BV'xA = \sum_{b \in B} V'bxA = \bigoplus_i^q V'x_iA$, where each $V'x_iA$ is V' - A -isomorphic to the V' - A -irreducible $V'xA$. Since $V'v \subseteq V$ and A is $V'_i \cdot \text{Hom}_{V'_i}(A, A)$ -irreducible, it follows that $A = v(V'_i \cdot \text{Hom}_{V'_i}(A, A)) = (V'v) \text{Hom}_{V'_i}(V, A) = \sum_j V'v\mathfrak{M}_j = \sum_{i,j} b_j(V'x_iA)$. Now, b_j being contained in B' , each $b_j(V'x_iA)$ is V' - A -homomorphic to $V'x_iA \cong V'xA$. Hence, A is homogeneously V' - A -completely reducible, and consequently B' is a simple ring.

Now, combining Lemmas 1, 2 and 3, the proof of Theorem 2 will be completed at once.

Proof of Theorem 2. We set $T = J(\mathfrak{S}, A)$. As $[V : V_\mathfrak{S}]_r < \infty$ and $V_\mathfrak{S} = V_A^2(V_\mathfrak{S})$, $V_A^2(T) = V_A(V_\mathfrak{S})$ is simple by Lemma 3. Further, by Lemma 2, there holds $[T : H \cap T]_i < \infty$. Since $A/H \cap T$ is locally finite by [1, Corollary 4], $V_{V_A^2(T)}(T)$ coincides with the center of $V_A^2(T)$ and $J(\mathfrak{S} | V_A^2(T), V_A^2(T)) = T$, Lemma 1 proves that T is a simple ring.

Next, concerning [1, Lemma 3 (i)], the method used in the proof of Lemma 3 enables us to see the following improvement.

Theorem 3. *Let A be h -Galois and left locally finite over B . If A' is a simple intermediate ring of A/H with $[A' : H]_i < \infty$, then $V' = V_A(A')$ is simple and $[A' : H] = [V : V']$.*

Proof. By Theorem 1, without loss of generality, we may assume that $B = H$. And so, by [3, Lemma 3.1], $\widetilde{V}A_r$ is dense in $\text{Hom}_{B_i}(A, A)$. Now, let M be an arbitrary minimal A' - A -submodule of A . Then, $M = eA (= A'eA)$

with some non-zero idempotent e . We set here $A'' = A'[e, \{e_{ij}\}'s]$, which is a regular subring of A left finite over B . And, as A is A'' - A -irreducible, $\text{Hom}_B(A'', A) = (\tilde{V}|A'')A_r$ is A'' - A_r -completely reducible. Accordingly, the A'' - A_r -submodule $\text{Hom}_{A'_i}(A'', A)$ is completely reducible: $\text{Hom}_{A'_i}(A'', A) = \bigoplus_j \mathfrak{M}_j$ with A'' - A_r -irreducible \mathfrak{M}_j . To be easily seen, each $\mathfrak{M}_j = (\tilde{v}_j u_{ji}|A'')A_r$ with some $\tilde{v}_j \in \tilde{V}$ and non-zero u_j , so that we may set $\mathfrak{M}_j = (a_{ji}|A'')A_r$. Recalling here that $a_{ji}|A''$ is contained in $\text{Hom}_{A'_i}(A'', A)$, it will be easy to see that a_j has to be contained in V' . Since A is $A'_i \cdot \text{Hom}_{A'_i}(A, A)$ -irreducible and $A'e \subseteq A''$, it follows $A = e(A'_i \cdot \text{Hom}_{A'_i}(A, A)) = (A'e) \text{Hom}_{A'_i}(A'', A) = \sum_j (A'e) \mathfrak{M}_j = \sum_j a_j M$. Evidently, a_j being contained in V' , each $a_j M$ is A' - A -homomorphic to M . We have proved therefore that A is homogeneously A' - A -completely reducible, and consequently V' is simple. The final assertion is then a consequence of Theorem 1.

Now, combining Theorems 2, 3 with [3, Corollary 3.3], one will readily see that if A is h -Galois and left locally finite over B with $B = V_A^2(B)$, then there exists a 1-1 dual correspondence between simple subrings of A left finite over B and closed $(*_j)$ -regular subgroups of \mathfrak{G} . Further, as another corollary to Theorem 3, we can prove the next:

Corollary. *Let A be inner Galois and locally finite over B , and V finite over the center of V . And let B' be a simple intermediate ring of A/B with $[B' : B]_i < \infty$. If B'/B is inner Galois then the center Z' of B' is contained in the center Z of B , and conversely.*

Proof. If B'/B is inner Galois then $V_{B'}^2(B) = B$ yields at once $Z' \subseteq B \cap V = Z$. Now, assume conversely $Z' \subseteq Z$. Then, V is evidently an algebra over Z' . Since A/B is h -Galois by [3, Theorems 2.2, 2.4]; Theorem 3 yields $V_A(B') \cap V_A^2(B') = V_A(B') \cap B' = Z'$, so that $V_A(B')$ is a central simple algebra of finite rank over Z' by [5, Lemma]. Hence, by Wedderburn's theorem, we obtain $V = V_A(B') \otimes_{Z'} V_V(V_A(B')) = V_A(B') \otimes_{Z'} V_{B'}(B)$. From the last relation, we see that $V_{B'}(B)$ is a simple ring. And finally, $J(\widetilde{V_{B'}(B)}|B', B') = V_A(V_{B'}(B)) \cap V_A^2(B') = V_A(V) = B$.

References

- [1] T. NAGAHARA: On Galois conditions and Galois groups of simple rings, Trans, Amer. Math. Soc., to appear.
- [2] T. NAGAHARA and H. TOMINAGA: On Galois and locally Galois extensions of simple rings, Math. J. Okayama Univ., 10 (1961), 143-166.

- [3] T. NAGAHARA and H. TOMINAGA: On Galois theory of simple rings, *Math. J. Okayama Univ.*, 11 (1963), 79-117.
- [4] T. NAGAHARA and H. TOMINAGA: Some theorems on Galois theory of simple rings, *J. Fac. Sci. Hokkaido Univ., Ser. I*, 17 (1963), 1-13.
- [5] H. TOMINAGA: On a theorem of N. Jacobson, *Proc. Japan Acad.*, 31 (1955), 653-654.

Department of Mathematics,
Hokkaido University

(Received October 3, 1964)