



Title	A CHARACTERIZATION OF MODULAR NORMS IN TERMS OF SIMILAR TRANSFORMATIONS
Author(s)	Shimogaki, Tetsuya
Citation	Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 18(3-4), 141-152
Issue Date	1965
Doc URL	http://hdl.handle.net/2115/56063
Type	bulletin (article)
File Information	JFSIU_18_N3-4_141-152.pdf



[Instructions for use](#)

A CHARACTERIZATION OF MODULAR NORMS IN TERMS OF SIMILAR TRANSFORMATIONS

*Dedicated to Professor Kinjiro Kunugi
on his 60 th birthday*

By

Tetsuya SHIMOGAKI

1. Introduction. A modulated semi-ordered linear space is a universally continuous semi-ordered linear space¹⁾ R with a non-negative functional m called a *modular* which satisfies the following conditions:

- M. 1) $|x| \leq |y|, x, y \in R$ implies $m(x) \leq m(y)$;
- M. 2) $m(\xi x) = 0$ for each $\xi > 0$ implies $x = 0$;
- M. 3) $\lim_{\xi \rightarrow 0} m(\xi x) = 0$ for each $x \in R$;
- M. 4) $m(\xi x)$ is a convex function of $\xi > 0$ for each $x \in R$;
- M. 5) $x \perp y$ ²⁾ implies $m(x+y) = m(x) + m(y)$;
- M. 6) $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $\sup_{\lambda \in \Lambda} m(x_\lambda) = m(x)$.

On a modulated space (R, m) a semi-continuous norm³⁾ $\|\cdot\|_m$ can be defined by

$$(1.1) \quad \|x\|_m = \inf \left\{ \frac{1}{|\xi|} ; m(\xi x) \leq 1 \right\}^{4)} \quad (x \in R),$$

that is, R is a normed semi-ordered linear space with the norm $\|\cdot\|_m$ at the same time. The converse of this, Every normed semi-ordered linear space $(R, \|\cdot\|)$ has an equivalent norm $\|\cdot\|_m$ defined by an appropriate modular m , is not true in general. Counter examples were constructed by the present author [7] and T. Andô [1].

L_p -spaces ($p \geq 1$) and Orlicz spaces L_ϕ ⁵⁾ on a σ -finite measure space (E, Ω, μ) , with a countably additive non-negative measure μ defined on a σ -field Ω of E ,

1) A semi-ordered linear space R is called universally continuous, if $0 \leq x_\lambda (\lambda \in \Lambda)$ implies $\cap x_\lambda \in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense.
2) $x \perp y$ means that x and y are mutually orthogonal, i.e. $|x| \cap |y| = 0$.
3) A norm $\|\cdot\|$ is called semi-continuous, if $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$ implies $\|x\| = \sup_{\lambda \in \Lambda} \|x_\lambda\|$.
4) $\|\cdot\|_m$ is termed the modular norm by m .
5) For the definition of an Orlicz space see [4].

are considered as modulated spaces with modulars $m_p(x) = \int_E |x(t)|^p d\mu(t)$ and $m_\phi(x) = \int_E \Phi(|x(t)|) d\mu(t)$ respectively, where $x \leqq y$ means $x(t) \leqq y(t)$ a.e..

A modular m on R is called *finite* if $m(x) < +\infty$ for each $x \in R$, and is called *almost finite* if m is finite on a complete semi-normal manifold⁶⁾ M of R . It is evident that the modulars of L_p -type ($1 \leqq p < +\infty$) are finite and the modulars m_ϕ of Orlicz spaces are almost finite. m_ϕ is finite if and only if Φ satisfies the so-called Δ_2 -condition.

An excellent axiomatic characterization of L_p -spaces in terms of norms on semi-ordered linear spaces was established by F. Bohnenblust in [2]. Later on, H. Nakano characterized norms of L_p -spaces as norms of unique indicatrix [5]. Since these characterizations are based on the particular structure of L_p -norms, it seems to be difficult to obtain similarly simple characterizations of general modular norms, even of modular norms of Orlicz spaces, as L_p -norms.

In this paper we shall present a necessary and sufficient condition in order that a norm $\|\cdot\|$ on R be the modular norm by a finite (almost finite) modular, in terms of the existence of a similar transformation T acting from R onto itself with the following property: *for any $x, y \in R$ with $\|x\|=1$ and $x \perp y$, $\|T(x+y)\|=1$ holds if and only if $\|y\|=1$ does* (Theorems 1, 2). According to the representation theory, this gives also an axiomatic characterization of modulated function spaces $L_{M(\xi, t)}$.⁷⁾ In 5 we shall state some supplementary remarks with concrete explanations of these results in Banach function spaces.

2. Notations and the theorems. In what follows, let $(R, \|\cdot\|)$ be a non-atomic⁸⁾ universally continuous semi-ordered linear space with a semi-continuous norm $\|\cdot\|$. A norm $\|\cdot\|$ is called *continuous* if $x, \downarrow_{\nu=1}^\infty 0$ implies $\|x, \|\downarrow_{\nu=1}^\infty 0$ always. If there exists a complete semi-normal manifold M such that $\|\cdot\|$ is continuous on M , $\|\cdot\|$ is called *almost continuous*. The modular norm $\|\cdot\|_m$ is continuous if and only if m is finite. We denote by V the unit ball and by S its surface respectively, i.e. $V = \{x: \|x\| \leqq 1\}$ and $S = \{x: \|x\| = 1\}$. We write $z = x \oplus y$, if $z = x + y$ with $x \perp y$ holds.

A one to one transformation T from R onto R is called *similar*, if it satisfies

$$(2.1) \quad T([p]x) = [p](Tx) \quad \text{for each } x \in R \text{ and projector } [p];$$

$$(2.2) \quad Tx \leqq Ty \quad \text{if and only if } x \leqq y;$$

$$(2.3) \quad T(-x) = -Tx \quad \text{for each } x \in R.$$

6) A linear lattice manifold M is called semi-normal if $|y| \leqq |x|$, $x \in M$ implies $y \in M$. A semi-normal manifold M is complete, if $M^\perp = \{0\}$.

7) For the definition of $L_{M(\xi, t)}$ see [3 or 6].

8) R is termed non-atomic, if each $0 \neq x \in R$ can be decomposed into $x = y + z$ with $y, z \neq 0$ and $y \perp z$.

We see easily from the definition that for a similar transformation T , T^{-1} is also such a one, and that T is order-continuous, i.e. $x_{\nu} \uparrow_{\nu=1}^{\infty} a$ (or $x_{\nu} \downarrow_{\nu=1}^{\infty} b$) implies $Tx_{\nu} \uparrow_{\nu=1}^{\infty} Ta$ (resp. $Tx_{\nu} \downarrow_{\nu=1}^{\infty} Tb$).

Here we consider the following condition which establishes a relation between a similar transformation T and the norm on R :

(T.C.) *For any x, y with $x \in S$ and $x \perp y$, $T(x+y) \in S$ holds if and only if $y \in S$.*

Now we can prove

Theorem 2.1. *In order that a given continuous norm $\|\cdot\|$ on R be the modular norm $\|\cdot\|_m$ by a modular m , it is necessary and sufficient that there exists a similar transformation T on R satisfying the condition (T.C.).*

If a modular m is almost finite the modular norm is almost continuous. For an almost continuous norm $\|\cdot\|$ we denote by R_c the *continuous manifold of R* with respect to $\|\cdot\|$, i.e., the totality of all continuous elements⁹⁾ of R . Evidently R_c is a complete semi-normal manifold on which $\|\cdot\|$ is continuous. Here we put $V_c = V \cap R_c$ and $S_c = S \cap R_c$. Then, for almost continuous norms we obtain

Theorem 2.2. *In order that a given almost continuous norm $\|\cdot\|$ on R be the modular norm $\|\cdot\|_m$ by a modular m , it is necessary and sufficient that there exists a similar transformation T on R_c onto R_c which satisfies the following condition:*

(T.C'.) *For any $x, y \in R_c$ with $x \in S_c$ and $x \perp y$, $T(x+y) \in S_c$ holds if and only if $y \in S_c$.*

To the proofs of these theorems the succeeding sections 3 and 4 shall be devoted.

3. Construction of orthogonal additive functional ρ . In this section, let $\|\cdot\|$ be continuous on R and T be a similar transformation satisfying the condition (T.C.). From (2.1)~(2.3) it follows that

$$(3.1) \quad T(x \oplus y) = Tx \oplus Ty \quad \text{and} \quad |Tx| = T(|x|) \quad \text{for } x, y \in R.$$

First we shall prove several auxiliary lemmas easily derived from the assumption.

Lemma 1. *We have $T(V) \subset V - S$.*

Proof. Suppose $y \in S$ with $Ty \in S$. Then we have $T(y+0) = Ty \in S$, which implies $0 \in S$ by (T.C.), a contradiction. On account of (2.2) and the semi-continuity of $\|\cdot\|$, it is now clear that $T(V) \subset V - S$ holds. Q. E. D.

9) If $\|x_{\nu}\| \downarrow 0$ for each $x_{\nu} \downarrow 0$ with $|x_{\nu}| \leq |\alpha| (1 \leq \nu)$, $\alpha \in R$ is termed a continuous element of R with respect to $\|\cdot\|$.

In the sequel, we use the following notations:

$$(3.2) \quad S_0 = S \quad \text{and} \quad S_n = TS_{n-1} \quad (n=1, 2, \dots).$$

Now we have

Lemma 2. $S_i \cap S_j = \emptyset$ holds for $i \neq j$ ($i, j = 1, 2, \dots$).

Proof. If $z \in S_i \cap S_j$ for some i, j with $i < j$, i.e., $z = T^i x = T^j y$ for some $x, y \in S$ we get $x = T^{j-i} y$. Putting $c = T^{j-i-1} y$, we obtain $x = Tc$ and $c \in V$, which is inconsistent with Lemma 1.

Lemma 3. For each $x \in S_n$ ($n = 0, 1, 2, \dots$) x can be decomposed into $x = x_1 \oplus x_2$ in such a way that $x_i \in S_{n+1}$ ($i = 1, 2$) holds.

Proof. $x \in S_n$ implies $T^{-n} x \in S$, whence $\|T^{-(n+1)} x\| > 1$. Now we put $a = T^{-(n+1)} x$. Since R contains no atomic element and $\|\cdot\|$ is continuous, we can find an element p such that $[p]a \in S$ holds. Because of $[p]a$, $Ta \in S$, it follows from (T.C.) that $(1 - [p])a \in S$ holds. Hence $x = T^{n+1} a = T^{n+1}[p]a + T^{n+1}((1 - [p])a)$ with $T^{n+1}[p]a, T^{n+1}((1 - [p])a) \in S_{n+1}$ simultaneously.

It is obvious from Lemma 3 that $x \in S$ if and only if x is represented as, for any fixed n ,

$$(3.3) \quad x = T^n \left(\bigoplus_{i=1}^{2^n} x_i \right),$$

where $x_i \in S$ ($i = 1, 2, \dots, 2^n$).

Lemma 4. Let $a, b \in S$ and $a \perp b$. Then

$$(3.4) \quad \|T^n a \oplus b\| > 1$$

stands for each $n \geq 1$.

Proof. We shall prove this lemma by induction. In case of $n=1$, $\|Ta \oplus b\| = 1$ implies $\|T(a \oplus T^{-1}b)\| = 1$, whence $T^{-1}b \in S$, contradicting Lemma 1. Thus (3.4) is valid for $n=1$. Now suppose that (3.4) holds for each $n \leq k$ and $\|T^{k+1}a \oplus b\| = 1$ for some $a, b \in S$ with $a \perp b$. Then $\|T(T^k a \oplus T^{-1}b)\| = 1$ holds and $T^{-1}b$ can be represented as $T^{-1}b = b_1 \oplus b_2$, $b_i \in S$ ($i = 1, 2$). From this and $\|T\{(T^k a \oplus b_1) + b_2\}\| = 1$, it follows that $\|T^k a \oplus b_1\| = 1$ holds on account of (T.C.), but this contradicts the induction hypothesis. Q.E.D.

Lemma 5. If $x = \bigoplus_{i=1}^n x_i = \bigoplus_{j=1}^m y_j \oplus y_0$ with $x_i \in S$, $y_j \in S$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$) and furthermore x is not a complete¹⁰⁾ element, then $n \geq m$ holds.

Proof. Suppose contrarily $n < m$. Since R is non-atomic, we can find a set of mutually orthogonal elements $\{z_i\}_{i=1}^{\rho} \subset S$ such that $z_i \perp x$ ($1 \leq i \leq \rho$) and

10) $x \in R$ is called a complete element if $\{x\}^\perp = \{0\}$ holds.

$n + \rho = 2^\mu$ for some $\mu \geq 1$. Then $T^\mu(\bigoplus_{i=1}^{\rho} z_i \oplus x) = T^\mu(z_1 \oplus \cdots \oplus z_\rho \oplus x_1 \oplus \cdots \oplus x_n) \in S$.

On the other hand, we get $T^\mu(\bigoplus_{i=1}^{\rho} z_i \oplus x) = T^\mu(z_1 \oplus \cdots \oplus z_\rho \oplus y_1 \oplus \cdots \oplus y_m \oplus y_0) = T^\mu(y_1 \oplus \cdots \oplus y_n \oplus z_1 \oplus \cdots \oplus z_\rho) + T^\mu(y_{n+1} \oplus \cdots \oplus y_m) + T^\mu(y_0)$, which implies $1 = \|T^\mu(x \oplus z)\| \geq \|w + T^\mu y_m\|$, where $w = T^\mu(y_1 \oplus \cdots \oplus y_n \oplus z_1 \oplus \cdots \oplus z_\rho)$ belongs to S . However, this is inconsistent with the preceding lemma. Q. E. D.

Lemma 6. *If x is not a complete element and $x = x_1 \oplus \cdots \oplus x_k = y_1 \oplus \cdots \oplus y_l \oplus y_0$, where $x_\nu \in S_{m_\nu}$, $y_\mu \in S_{n_\mu}$ and $y_0 \in V$ ($1 \leq \nu \leq k$, $1 \leq \mu \leq l$, $0 \leq m_\nu, n_\mu$), then $\sum_{\nu=1}^k \frac{1}{2^{m_\nu}} \geq \sum_{\mu=1}^l \frac{1}{2^{n_\mu}}$ holds.*

Proof. We put $N = \max_{\substack{1 \leq \nu \leq k \\ 1 \leq \mu \leq l}} \{m_\nu, n_\mu\}$. Then, for each ν ($1 \leq \nu \leq k$) x is decomposed into $x_\nu = x_{\nu,1} \oplus x_{\nu,2} \oplus \cdots \oplus x_{\nu,2^{N-m_\nu}}$ with $x_{\nu,i} \in S_N$ ($1 \leq i \leq 2^{N-m_\nu}$). Similarly $y_\mu = y_{\mu,1} \oplus \cdots \oplus y_{\mu,2^{N-n_\mu}}$ with $y_{\mu,j} \in S_N$ holds for each j ($1 \leq j \leq 2^{N-n_\mu}$). Hence both $x = \bigoplus_{\nu=1}^k \bigoplus_{i=1}^{2^{N-m_\nu}} x_{\nu,i}$ and $x = \bigoplus_{\mu=1}^l \bigoplus_{j=1}^{2^{N-n_\mu}} y_{\mu,j} \oplus y_0$ holds, which implies $T^{-N}x = \bigoplus_{\nu} \bigoplus_{i} T^{-N}x_{\nu,i} = \bigoplus_{\mu} \bigoplus_{j} T^{-N}y_{\mu,j} \oplus T^{-N}y_0$ with $T^{-N}x_{\nu,i} \in S$ and $T^{-N}y_{\mu,j} \in S$ for each ν, μ, i , and j . In view of the preceding lemma we find

$$\sum_{\nu=1}^k 2^{N-m_\nu} \geq \sum_{\mu=1}^l 2^{N-n_\mu}.$$

Thus we obtain $\sum_{\nu=1}^k \frac{1}{2^{m_\nu}} \geq \sum_{\mu=1}^l \frac{1}{2^{n_\mu}}$. Q. E. D.

Here we turn to define an orthogonal additive functional (i.e. $\rho(x+y) = \rho(x) + \rho(y)$ for $x \perp y$) on R from $\|\cdot\|$. Let R_0 be the set of all non-complete elements of R and \mathfrak{A} be the totality of elements of R_0 which can be represented as $x_1 \oplus \cdots \oplus x_n$ with $x_i \in S_{m_i}$ ($i = 1, 2, \dots, n$; $n = 1, 2, \dots$). On \mathfrak{A} we define a functional ρ' as follows:

$$(3.5) \quad \rho'(x) = \sum_{i=1}^n \frac{1}{2^{m_i}},$$

where $x = x_1 \oplus \cdots \oplus x_n$ with $x_i \in S_{m_i}$ ($1 \leq i \leq n$). According to Lemma 6 we see that this definition has a sense. It is evident from the definition that ρ' is orthogonally additive on \mathfrak{A} . Next, we put for each $x \in R$

$$(3.6) \quad \rho(x) = \begin{cases} \sup_{|y| \leq |x|, y \in \mathfrak{A}} \rho'(y), \\ 0, \quad \text{if there exists no element } y \in \mathfrak{A} \text{ with } |y| \leq |x|. \end{cases}$$

In the succeeding section we shall show that ρ thus defined is in fact a modular on R and that $\|\cdot\|$ is nothing but the modular norm by ρ .

4. Properties of ρ and the proofs of Theorems. In view of construction of ρ and Lemma 6 we see easily that ρ satisfies the modular conditions M. 1) and M. 2). Since R contains no atomic element, we have also

$$(4.1) \quad \rho(x) = \rho'(x) \quad \text{for each } x \in \mathfrak{A}.$$

In order to prove the remaining conditions M. 3), ~M. 6), we need some lemmas.

Lemma 7. *We have*

$$(4.2) \quad \rho(x) > \frac{1}{2^m} \text{ and } \rho(x) < \frac{1}{2^m} \text{ imply } \|T^{-m}x\| > 1 \text{ and } \|T^{-m}x\|$$

≤ 1 respectively ($m = 0, 1, 2, \dots$);

$$(4.3) \quad \rho(x) < +\infty, \quad \text{for each } x \in R;$$

$$(4.4) \quad \rho(x) = \sup_{[\rho]x \in \mathfrak{A}} \rho'([\rho]x), \quad \text{if } \rho(x) > 0.$$

Proof. (4.2) follows immediately from the definition of ρ . Since $\|\cdot\|$ is continuous, each element $x \in R$ can be represented as $x = \bigoplus_{i=1}^n x_i$ with $\|x_i\| \leq 1$ ($1 \leq i \leq n$) for some $n \geq 1$. From this we have $\rho(x) \leq n$ in view of (3.5), (4.2) and M. 2). Thus (4.3) is valid. Next, we shall show that if $\rho(x) > \frac{k}{2^m}$ x is written as $x = \bigoplus_{i=1}^k x_i \oplus x_0$ with $\|T^{-m}x_i\| > 1$ for each i ($1 \leq i \leq k$). By (3.6) there exists $0 \leq x' \in \mathfrak{A}$ such that $|x| \geq x' = \bigoplus_{i=1}^k x'_i \oplus x'_0$ with $x'_i \in S_m$ ($1 \leq i \leq k$) and $x_0 \in \mathfrak{A}$. Now we decompose x'_0 into $x'_0 = \bigoplus_{i=1}^k x''_i$ with $x''_i \in \mathfrak{A}$ for each i . On the ground of Lemma 4 $\|T^{-m}(x'_i \oplus x''_i)\| > 1$ ($1 \leq i \leq k$) must hold. Putting $x_i = [x'_i \oplus x''_i]x$ and $x_0 = x - \bigoplus_{i=1}^k x_i$, we obtain $x = \bigoplus_{i=1}^k x_i \oplus x_0$ with $\|T^{-m}x_i\| > 1$ for each i ($1 \leq i \leq k$).

From this one derives easily that if $\rho(x) > \frac{k}{2^m}$ there exist projectors $\{[\rho_\nu]\}_{\nu=1}^k$ such that $[\rho_\nu] \leq [x_i]$ and $\|T^{-m}[\rho_\nu]x_i\| = 1$ hold ($1 \leq i \leq k$), where $\{x_i\}_{i=1}^k$ satisfies the above condition. Since $[\rho_\nu]x_i \in S_m$ and $\bigoplus_{i=1}^k [\rho_\nu]x_i = (\sum_{i=1}^k [\rho_\nu])x$, $\rho'([\rho]x) \geq \frac{k}{2^m}$ follows and (4.4) is proved, where $[\rho] = \sum_{i=1}^k [\rho_\nu]$. Q. E. D.

Lemma 8. *ρ is orthogonally additive, i.e., it satisfies M. 5).*

Proof. From the definition of ρ it follows that

$$\rho(x \oplus y) \geqq \rho(x) + \rho(y)$$

holds. Now suppose $\rho(x \oplus y) > \rho(x) + \rho(y)$ for some $x, y \in R$ with $x + y \in R_0$. Then we can choose a natural number m such that $\rho(x \oplus y) > \rho(x) + \rho(y) + \frac{1}{2^m}$.

By (4.4) there exist projectors $[p], [q]$ for which $\rho(x) - \rho'([p]x) < \frac{1}{2^{m+2}}$, $\rho(y) - \rho'([q]y) < \frac{1}{2^{m+2}}$, $[p]x \in \mathfrak{A}$ and $[q]y \in \mathfrak{A}$ hold.¹¹⁾ Since $\rho((1-[p])x) \leqq \rho(x) - \rho([p]x) < \frac{1}{2^{m+2}}$ and $\rho((1-[q])y) \leqq \rho(y) - \rho([q]y) < \frac{1}{2^{m+2}}$ hold, we can find $\alpha, \beta \geqq 1$ such that both $\alpha(1-[p])x$ and $\beta(1-[q])y$ belong to S_{m+2} according to (4.2) and the fact that T is similar. Putting $x' = [p]x + \alpha(1-[p])x$ and $y' = [q]y + \beta(1-[q])y$, we obtain $x', y' \in \mathfrak{A}$ and $\rho'(x' \oplus y') = \rho'(x') + \rho'(y') = \rho'([p]x) + \rho'([q]y) + \frac{1}{2^{m+1}}$, since ρ' is orthogonally additive on \mathfrak{A} . Hence we get

$$\begin{aligned} \rho'(x' \oplus y') &\geqq \rho(x \oplus y) > \rho(x) + \rho(y) + \frac{1}{2^m} \\ &\geqq \rho'([p]x) + \rho'([q]y) + \frac{1}{2^m} = \rho'(x' \oplus y') + \frac{1}{2^{m+1}}, \end{aligned}$$

which is, however, a contradiction. Thus we see easily that ρ is orthogonally additive by virtue of Lemma 7. Q. E. D.

Lemma 9. *We have*

$$(4.5) \quad \rho(x) \leqq 1 \quad \text{if and only if } \|x\| \leqq 1.$$

Proof. The fact that $\|x\| \leqq 1$ implies $\rho(x) \leqq 1$ is obvious by virtue of Lemma 4. On the other hand, for any x with $\rho(x) \leqq 1$ we can find a sequence of projectors $\{[p_\nu]\}_{\nu=1}^\infty$ such that $[p_\nu] \uparrow_{\nu=1}^\infty [x]$, $[p_\nu]x \in \mathfrak{A}$ and $\rho([p_\nu]x) \uparrow_{\nu=1}^\infty \rho(x) \leqq 1$ on account of (4.4) and the orthogonal additivity of ρ . By (4.1) and the definition of ρ' , we now get $\|[p_\nu]x\| \leqq 1$ for each $\nu \geqq 1$, hence $\|x\| \leqq 1$ because of the semi-continuity of $\|\cdot\|$. Q. E. D.

Lemma 10. *ρ is semi-continuous, i.e., it satisfies M. 6).*

Proof. Let $0 \leqq x_\lambda \uparrow_{\lambda \in A} x$ and $\rho(x) > \frac{k}{2^m}$. As is shown in the proof of (4.4), there exists $p \in R$ such that $[p]x \in \mathfrak{A}$, $[p]x = \bigoplus_{i=1}^k w_i$ and $\|T^{-m}w_i\| > 1$ ($1 \leqq i \leqq k$). Then, since $[w_i]x_\lambda \uparrow_{\lambda \in A} [w_i]x = w_i$ holds for each i and $\|\cdot\|$ is semi-continuous, we have for a sufficiently large λ_0 that $\|T^{-m}[w_i]x_{\lambda_0}\| > 1$ stands for every i ($1 \leqq i \leqq k$). Therefore we have

11) In case of $\rho(x)=0$ (or $\rho(y)=0$), we choose $p=0$ (resp. $q=0$).

$$\rho(x_{\lambda_0}) \geq \rho([\rho]x_{\lambda_0}) \geq \frac{k}{2^m},$$

which shows the semi-continuity of ρ .

Q. E. D.

Lemma 11. ρ satisfies M.3). i.e., $\lim_{\xi \rightarrow 0} \rho(\xi x) = 0$.

Proof. If $\rho(\xi x) > \frac{1}{2^m}$ holds for each $\xi > 0$, we have $\|T^{-m}\xi x\| > 1$. Since $\bigcap_{\xi > 0} \xi|x| = 0$ stands, $\bigcap_{\xi > 0} T^{-m}\xi|x| = 0$ holds. Hence it follows that $\|T^{-m}\xi x\| \rightarrow 0$ as $\xi \rightarrow 0$, because of the continuity of $\|\cdot\|$. This is a contradiction. Q. E. D.

Summing up the above results, we see that ρ satisfies all the conditions of modular except M.4). Next lemma shall show that ρ fulfils M.4) too.

Lemma 12. $\rho(\xi x)$ is a convex function of ξ ($\xi \geq 0$) for each $x \in R$.

Proof. We shall first show that the set $B_\xi = \{x : \rho(x) \leq \xi\}$ is convex for every ξ with $0 \leq \xi \leq 1$. Let $x, y \in B_\xi$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. By virtue of semi-continuity of ρ , we may assume without loss of generality that there exists $0 \neq z \in R$ belonging to $\{x, y\}^\perp$. Furthermore we may choose z as $\rho(z) = 1 - \xi$, since ρ satisfies (4.3) and R has no atom. It follows that both $x+z$ and $y+z$ belong to V , hence $\alpha(x+z) + \beta(y+z)$ does also. Consequently, we obtain $\rho(\alpha x + \beta y) + \rho(z) \leq 1$ by Lemma 9, hence $\alpha x + \beta y \in B_\xi$. Therefore B_ξ is convex.

Next, suppose that $\rho(\xi x) \leq 1$ and $\rho(\eta x) \leq 1$ for some $x \in R$ and $\xi > \eta \geq 0$. Since ρ is finite, orthogonally additive and semi-continuous, we can find $p \in R$ for which $\rho(\xi[p]x) = \rho(\xi(1-[p])x)$ holds. If $\rho(\eta[p]x) < \rho(\eta(1-[p])x)$ stands for such $[p]$, there can be constructed a system of projectors $\{[p_\alpha]\}_{(0 \leq \alpha \leq 1)}$ and $\{[p'_\alpha]\}_{(0 \leq \alpha \leq 1)}$ such that $[p_\alpha] \downarrow ([p'_\alpha] \uparrow)$ as $\alpha \downarrow 0$, $[p_\alpha] \leq [p]$, $[p'_\alpha] \leq (1-[p])$ with $[p_1] = [p]$, $[p'_0] = (1-[p])$, and $\rho(\xi[p_\alpha]x) = \rho(\xi(1-[p])(1-[p'_\alpha])x) = \alpha \rho(\xi[p]x)$ holds for each $0 \leq \alpha \leq 1$. Putting $[q_\alpha] = [p_\alpha] + (1-[p])[p'_\alpha]$, we obtain $[q_\alpha] \leq [x]$ and $\rho(\xi[q_\alpha]x) = \rho(\xi(1-[q_\alpha])x)$ for every α . Furthermore we see easily that both $\rho(\eta[q_1]x) < \rho(\eta(1-[q_1])x)$ and $\rho(\eta[q_0]x) > \rho(\eta(1-[q_0])x)$ hold. From this it follows that $\rho(\eta[q_\alpha]x) = \rho(\eta(1-[q_\alpha])x)$ stands for some α . In consequence, we have shown that there exists $p \in R$ such that $\rho(\xi[p]x) = \frac{1}{2} \rho(\xi x)$ and $\rho(\eta[p]x) = \frac{1}{2} \rho(\eta x)$ hold simultaneously. Because $\rho(\xi[p]x + \eta(1-[p])x) = \rho(\eta[p]x + \xi(1-[p])x) = \frac{1}{2} \{\rho(\xi x) + \rho(\eta x)\} \leq 1$, we have

$$(4.6) \quad \rho\left(\frac{1}{2}(\xi x + \eta y)\right) \leq \frac{1}{2} \{\rho(\xi x) + \rho(\eta y)\}$$

by the fact shown just above.

Finally, since each x can be decomposed orthogonally into $x = \bigoplus_{i=1}^n x_i$ with $\rho(x_i) \leq 1$ ($1 \leq i \leq n$), we see that (4.6) holds for any $x \in R$, i.e. $\rho(\xi x)$ is a convex function of ξ ($\xi \geq 0$) for each $x \in R$. Q. E. D.

Here we are in position to prove the theorems stated in 2.

Proof of Theorem 1. *Sufficiency.* The functional ρ constructed in 3 is a modular satisfying (4.5), as is shown above. Hence we have $\|x\| = \inf \left\{ \frac{1}{|\xi|} ; \rho(\xi x) \leq 1 \right\}$ i.e., $\|\cdot\|$ is the modular norm by the modular ρ .

Necessity. Let $\|\cdot\|$ be the modular norm by a modular m on R . m is necessarily finite since $\|\cdot\|$ is continuous. In the same manner as in the proof of Theorem 2 in [8], we can construct a similar transformation T_0 on R satisfying

$$m(T_0 x) = \frac{1}{2} m(x) \quad \text{for every } x \in R.$$

It is now clear that T_0 satisfies the condition (T.C.).

Q. E. D.

Proof of Theorem 2. *Sufficiency.* In view of Theorem 1 we find a finite modular ρ on a complete semi-normal manifold R_C of R , for which $\|\cdot\|$ is the modular norm on R_C . We extend now ρ on the whole space R as follows:

$$(4.7) \quad \rho_0(x) = \sup_{0 \leq y \leq |x|, y \in R_C} \rho(y) \quad (x \in R).$$

ρ_0 thus defined is an almost finite modular on R , as is easily seen, and $\rho_0(x) = \rho(x)$ if $x \in R_C$. Because of the semi-continuity of $\|\cdot\|$ and ρ , $\|x\| = \inf \left\{ \frac{1}{|\xi|} ; \rho_0(\xi x) \leq 1 \right\}$ holds for each $x \in R$, that is, $\|\cdot\|$ is the modular norm by ρ_0 . The necessity is derived similarly as the proof of Theorem 1. Q. E. D.

5. Here let $(R, \|\cdot\|)$ be the same as in 3 and ρ be the modular defined, in the manner described above, from $\|\cdot\|$ and a similar transformation T on R satisfying the condition (T.C.). From the construction of ρ one derives easily

$$\rho(Tx) = \frac{1}{2} \rho(x) \quad (x \in R).$$

Also this enables us obviously to restate properties of the modular ρ in terms of similar transformations $T^{(12)}$. We describe below a few examples of such

12) Of course, we can state properties of ρ by means of $\|\cdot\|$, since there are found closed relations between modulars and their norms [1, 6, and 7].

restatements in terms of T . Being trivial, their proofs are omitted.

5. 1. ρ is simple (i.e. $\rho(x)=0$ implies $x=0$), if and only if $\bigcap_{m \geq 1} T^m x = 0$ for each $x \in R$.

5. 2. ρ is uniformly simple (i.e. $\inf_{\|x\| \geq \delta} \rho(x) > 0$ for each $\delta > 0$), if and only if for each $\varepsilon > 0$ there exists $m \geq 0$ with $\sup_{x \in S} \|T^m x\| < \varepsilon$.

5. 3. ρ is uniformly finite (i.e. $\sup_{\|x\| \leq \delta} \rho(x) < +\infty$ for each $\delta > 0$), if and only if for each $\delta > 0$ there exists $m \geq 0$ with $\inf_{x \in S} \|T^{-m} x\| > \delta$.

5. 4. ρ is upper bounded (i.e. $\rho(\alpha x) \leq r\rho(x)$ holds for every $x \in R$, where $1 < \alpha, r$ are fixed constants), if and only if $T \leq \left(\frac{1}{2}\right)^{\frac{1}{p}} I^{13)}$ for some $p \geq 1$.

Finally let (E, Ω, μ) be a σ -finite non-atomic measure space with a countably additive non-negative measure μ on a σ -field Ω of E . A modulated space (X, m) consisting of measurable functions on E is a semi-normal manifold of *modulated function space* $L_{M(\xi, t)}$ defined by a modular function $M(\xi, t)^{14)}$ on $[0, \infty) \times E$, that is, X is contained in the totality of all measurable functions f such that $\int_E M(\alpha|f(t)|, t) d\mu(t) < +\infty$ for some $\alpha > 0$, and

$$(5. 1) \quad m(f) = \int_E M(|f(t)|, t) d\mu(t)$$

holds for each $f \in X$. Conversely, it is known [6] that each modulated semi-ordered linear space R can be considered as a modulated function space $L_{M(\xi, t)}$ on a measure space (E, Ω, μ) suitably chosen, and m is represented by (5.1).

For any finite modulated function space¹⁵⁾ $(L_{M(\xi, t)}(E), \|\cdot\|)$ we can obtain a similar transformation T with the condition (T.C.) directly as follows: We define for $(\xi, t) \in [0, \infty) \times E$

$$(5. 2) \quad h(\xi, t) = \begin{cases} M_t^{-1} \left(\frac{1}{2} M(\xi, t) \right), & \text{if } M(\xi, t) > 0; \\ \xi, & \text{if } M(\xi, t) = 0, \end{cases}$$

where $M_t^{-1}(\xi)$ is the inverse of the function $M_t(\xi) = M(\xi, t)$ for each $t \in E$. Then $h(\xi, t)$ on $[0, \infty) \times E$ is a Carathéodory's function, and the transformation \mathfrak{h} defined by

13) I is the identity operator on R and 5. 4 follows from Theorem 3. 3 of [9].

14) For the definition of modular functions see [3 or 6]. Roughly speaking, $M(\xi, t)$ is a N' -function of ξ for each $t \in E$. In $L_{M(\xi, t)}$ we consider $\int_E M(|f(t)|, t) d\mu(t)$ as a modular m always.

15) m on $L_{M(\xi, t)}$ is finite, if and only if $M(2\xi, t) \leq rM(\xi, t) + a(t)$ for all $(\xi, t) \in [0, \infty) \times E$, where $r > 0$ and $a(t) \in L_1(E)$ [3]. m is almost finite if and only if $M(\xi, t) < +\infty$ a. e. in $[0, \infty) \times E$.

$$(5.3) \quad \mathfrak{h}(f(t)) = h(f(t), t) \quad (f \in \mathbf{L}_{M(\xi, t)})$$

forms a similar transformation satisfying the condition (T.C.) for the modular norm on $\mathbf{L}_{M(\xi, t)}$. Conversely, in view of Theorem 1 we have

Theorem 3. *If $(\mathbf{X}, \|\cdot\|)$ is a normed function space¹⁶⁾ with a continuous norm $\|\cdot\|$, and if a similar transformation \mathfrak{h} from \mathbf{X} onto \mathbf{X} , defined by a Carathéodory's function $h(\xi, t)$ ¹⁷⁾ on $[0, \infty) \times E$, satisfies the condition (T.C.), then there can be found a modular function $M(\xi, t)$ on $[0, \infty) \times E$ such that \mathbf{X} is a semi-normal manifold of $\mathbf{L}_{M(\xi, t)}$, $M(\xi, t)$ satisfies (5.2)¹⁸⁾ and $\|\cdot\|$ coincides with the modular norm of the space $\mathbf{L}_{M(\xi, t)}$.*

Remark 1. In this theorem if moreover, $(\mathbf{X}, \|\cdot\|)$ is monotone complete (i. e. $0 \leq f_\nu \uparrow$, $\sup_{\nu \geq 1} \|f_\nu\| < +\infty$ implies $\bigcup_{\nu=1}^{\infty} f_\nu \in \mathbf{X}$), then $\mathbf{X} = \mathbf{L}_{M(\xi, t)}$ holds.

Remark 2. In Theorem 3, if $h(\xi, t) = h(\xi)$ for all $(\xi, t) \in [0, \infty) \times E$, then $\mathbf{L}_{M(\xi, t)}$ can be replaced by an Orlicz space \mathbf{L}_M .

When $\|\cdot\|$ is almost continuous, we have a similar theorem as above on the basis of Theorem 2. In this case, \mathfrak{h} acts from $\mathbf{L}_{M(\xi, t)}^{(f)}$, the finite manifold of $\mathbf{L}_{M(\xi, t)}$ (the totality of all $f \in \mathbf{L}_{M(\xi, t)}$ with $m(\xi f) < +\infty$ for every $\xi \geq 0$), onto itself and satisfies (5.2), if $0 < M(\xi, t) < +\infty$.

On the basis of Theorems 1 and 2, a theorem characterizing the modular norms in terms of norms only can be obtained, and it shall be shown in a separate paper.

References

- [1] T. ANDÔ: *Convexity and evenness in modulated semi-ordered linear spaces*, Jour. Fac. Sci. Hokkaido Univ., **14**, No. 1, 2 (1957), 59–95.
- [2] F. BOHNENBLUST: *An axiomatic characterization of L_p -spaces*, Duke Math. J., **6** (1940), 627–640.
- [3] J. ISHII: *On the finiteness of modulated spaces*, Jour. Fac. Sci. Hokkaido Univ., **15**, No. 1, 2 (1960), 13–28.
- [4] M. A. KRASNOSELSKIĬ, J. B. RUTICKIĬ: *Convex functions and Orlicz spaces* (in Russian), Moskow, 1958.
- [5] H. NAKANO: *Stetige lineare Funktionale auf dem teilweise geordneten Modul*, Jour. Fac. Sci. Imp. Univ. Tokyo, **4** (1942), 201–382.
- [6] —————: *Modulated semi-ordered linear spaces*, Tokyo 1950.
- [7] T. SHIMOGAKI: *On the norms by uniformly finite modularity*, Proc. Japan Acad., No. 6 (1957), 304–309.
- [8] —————: *A generalization of Vainberg's theorem II*, Proc. Japan Acad., **34**,

16) We assume that X is semi-normal.

17) For $h(\xi, t)$ we assume $h(0, t) \equiv 0$ for all $t \in E$.

18) Strictly speaking, $h(\xi, t) = M_t^{-1}(\frac{1}{2}M(\xi, t))$ holds a. e. for (ξ, t) satisfying $M(\xi, t) > 0$. In general, $h(\xi, t) = \xi$ does not hold for (ξ, t) with $M(\xi, t) = 0$.

No. 10 (1958), 676-680.

- [9] S. YAMAMURO: *Exponents of modularized semi-ordered linear spaces*, Jour. Fac. Sci. Hokkaido Univ., **12**, No. 4 (1953), 211-253.

Department of Mathematics,
Hokkaido University

(Received September 28, 1964)