A CHARACTERIZATION OF MODULAR NORMS IN TERMS OF SIMILAR TRANSFORMATIONS

Dedicated to Professor Kinjiro Kunugi
on his 60th birthday

By

Tetsuya SHIMOGAKI

1. Introduction. A modulared semi-ordered linear space is a universally continuous semi-ordered linear space\textsuperscript{1) $R$} with a non-negative functional $m$ called a modular which satisfies the following conditions:

M. 1) $|x|\leq |y|$, $x, y \in R$ implies $m(x)\leq m(y)$;
M. 2) $m(\xi x) = 0$ for each $\xi > 0$ implies $x = 0$;
M. 3) $\lim_{\xi \to 0} m(\xi x) = 0$ for each $x \in R$;
M. 4) $m(\xi x)$ is a convex function of $\xi > 0$ for each $x \in R$;
M. 5) $x \perp y$\textsuperscript{2)} implies $m(x + y) = m(x) + m(y)$;
M. 6) $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $\sup_{\lambda \in \Lambda} m(x_\lambda) = m(x)$.

On a modulared space $(R, m)$ a semi-continuous norm\textsuperscript{3)} $\| \cdot \|_m$ can be defined by

\begin{equation}
\|x\|_m = \inf_{\xi} \left\{ \frac{1}{|\xi|}; \ m(\xi x) \leq 1 \right\} \quad (x \in R),
\end{equation}

that is, $R$ is a normed semi-ordered linear space with the norm $\| \cdot \|_m$ at the same time. The converse of this, Every normed semi-ordered linear space $(R, \| \cdot \|)$ has an equivalent norm $\| \cdot \|_m$ defined by an appropriate modular $m$, is not true in general. Counter examples were constructed by the present author [7] and T. Andô [1].

$L_p$-spaces ($p \geq 1$) and Orlicz spaces $L_{\Phi}^\lambda$ on a $\sigma$-finite measure space $(E, \Omega, \mu)$, with a countably additive non-negative measure $\mu$ defined on a $\sigma$-field $\Omega$ of $E$,

\textsuperscript{1)} A semi-ordered linear space $R$ is called universally continuous, if $0 \leq x_\lambda (\lambda \in \Lambda)$ implies $\bigcap_{\lambda \in \Lambda} x_\lambda \in R$, i.e. a conditionally complete vector lattice in Birkhoff's sense.

\textsuperscript{2)} $x \perp y$ means that $x$ and $y$ are mutually orthogonal, i.e. $|x| \cap |y| = 0$.

\textsuperscript{3)} A norm $\| \cdot \|$ is called semi-continuous, if $|x_\lambda| \uparrow_{\lambda \in \Lambda} |x|$ implies $\|x\| = \sup_{\lambda \in \Lambda} \|x_\lambda\|$.

\textsuperscript{4)} $\| \cdot \|_m$ is termed the modular norm by $m$.

\textsuperscript{5)} For the definition of an Orlicz space see [4].
are considered as modulared spaces with modulars $m_p(x)=\int_R|x(t)|^p\,d\mu(t)$ and $m_\phi(x)=\int_R\phi(|x(t)|)\,d\mu(t)$ respectively, where $x\leq y$ means $x(t)\leq y(t)$ a.e.

A modular $m$ on $R$ is called finite if $m(x)<+\infty$ for each $x\in R$, and is called almost finite if $m$ is finite on a complete semi-normal manifold\(^6\) $M$ of $R$. It is evident that the modulars of $L_p$-type $(1\leq p<+\infty)$ are finite and the modulars $m_\phi$ of Orlicz spaces are almost finite. $m_\phi$ is finite if and only if $\phi$ satisfies the so-called $A_\phi$-condition.

An excellent axiomatic characterization of $L_\phi$-spaces in terms of norms on semi-ordered linear spaces was established by F. Bohnenblust in [2]. Later on, H. Nakano characterized norms of $L_\phi$-spaces as norms of unique indicatrix [5]. Since these characterizations are based on the particular structure of $L_\phi$-norms, it seems to be difficult to obtain similarly simple characterizations of general modular norms, even of modular norms of Orlicz spaces, as $L_\phi$-norms.

In this paper we shall present a necessary and sufficient condition in order that a norm $\|\cdot\|$ on $R$ be the modular norm by a finite (almost finite) modular, in terms of the existence of a similar transformation $T$ acting from $R$ onto itself with the following property: for any $x,y\in R$ with $\|x\|=1$ and $x\perp y$, $\|T(x+y)\|=1$ holds if and only if $\|y\|=1$ does (Theorems 1, 2). According to the representation theory, this gives also an axiomatic characterization of modulared function spaces $L_M(\xi,t)$.\(^7\) In 5 we shall state some supplementary remarks with concrete explanations of these results in Banach function spaces.

2. Notations and the theorems. In what follows, let $(R, \|\cdot\|)$ be a non-atomic\(^8\) universally continuous semi-ordered linear space with a semi-continuous norm $\|\cdot\|$. A norm $\|\cdot\|$ is called continuous if $x_n\downarrow 0$ implies $\|x_n\|\downarrow 0$ always. If there exists a complete semi-normal manifold $M$ such that $\|\cdot\|$ is continuous on $M$, $\|\cdot\|$ is called almost continuous. The modular norm $\|\cdot\|_m$ is continuous if and only if $m$ is finite. We denote by $V$ the unit ball and by $S$ its surface respectively, i.e. $V=\{x: \|x\|\leq 1\}$ and $S=\{x: \|x\|=1\}$. We write $z=x+y$ if $z=x\oplus y$, with $x\perp y$ holds.

A one to one transformation $T$ from $R$ onto $R$ is called similar, if it satisfies

\begin{align}
(2.1) \quad & T([p]x)=[p](Tx) \quad \text{for each } x\in R \text{ and projector } [p]; \\
(2.2) \quad & Tx\leq Ty \quad \text{if and only if } x\leq y; \\
(2.3) \quad & T(-x)=-Tx \quad \text{for each } x\in R.
\end{align}

\(^6\) A linear lattice manifold $M$ is called semi-normal if $|y|\leq |x|$, $x\in M$ implies $y\in M$. A semi-normal manifold $M$ is complete, if $M^{-1}={0}$.

\(^7\) For the definition of $L_M(\xi,t)$ see [3 or 6].

\(^8\) $R$ is termed non-atomic, if each $0\neq x\in R$ can be decomposed into $x=y+z$ with $y,z\neq 0$ and $y\perp z$. 

T. Shimogaki
We see easily from the definition that for a similar transformation $T$, $T^{-1}$ is also such a one, and that $T$ is order-continuous, i.e. $x_{
u} \uparrow_{\nu=1}^{\infty} a$ (or $x_{\nu} \downarrow_{\nu=1}^{\infty} b$) implies $Tx_{\nu} \uparrow_{\nu=1}^{\infty} Ta$ (resp. $Tx_{\nu} \downarrow_{\nu=1}^{\infty} Tb$).

Here we consider the following condition which establishes a relation between a similar transformation $T$ and the norm on $R$:

(T.C.) For any $x, y$ with $x \in S$ and $x \perp y$, $T(x + y) \in S$ holds if and only if $y \in S$.

Now we can prove

**Theorem 2.1.** In order that a given continuous norm $\| \cdot \|$ on $R$ be the modular norm $\| \cdot \|_m$ by a modular $m$, it is necessary and sufficient that there exists a similar transformation $T$ on $R$ satisfying the condition (T.C.).

If a modular $m$ is almost finite the modular norm is almost continuous. For an almost continuous norm $\| \cdot \|$ we denote by $R_C$ the continuous manifold of $R$ with respect to $\| \cdot \|$, i.e., the totality of all continuous elements\(^9\) of $R$. Evidently $R_C$ is a complete semi-normal manifold on which $\| \cdot \|$ is continuous. Here we put $V_C = V \cap R_C$ and $S_C = S \cap R_C$. Then, for almost continuous norms we obtain

**Theorem 2.2.** In order that a given almost continuous norm $\| \cdot \|$ on $R$ be the modular norm $\| \cdot \|_m$ by a modular $m$, it is necessary and sufficient that there exists a similar transformation $T$ on $R_C$ onto $R_C$ which satisfies the following condition:

(T.C.' ) For any $x, y \in R_C$ with $x \in S_C$ and $x \perp y$, $T(x + y) \in S_C$ holds if and only if $y \in S_C$.

To the proofs of these theorems the succeeding sections 3 and 4 shall be devoted.

3. Construction of orthogonal additive functional $\rho$. In this section, let $\| \cdot \|$ be continuous on $R$ and $T$ be a similar transformation satisfying the condition (T.C.) From (2.1) -- (2.3) it follows that

\[(3.1) \quad T(x \oplus y) = Tx \oplus Ty \quad \text{and} \quad |Tx| = T(|x|) \quad \text{for} \quad x, y \in R.\]

First we shall prove several auxiliary lemmas easily derived from the assumption.

**Lemma 1.** We have $T(V) \subset V - S$.

*Proof.* Suppose $y \in S$ with $Ty \in S$. Then we have $T(y + 0) = Ty \in S$, which implies $0 \in S$ by (T.C.), a contradiction. On account of (2.2) and the semi-continuity of $\| \cdot \|$, it is now clear that $T(V) \subset V - S$ holds. Q. E. D.

\(^9\) If $\|x_{\nu}\| \not\to 0$ for each $x_{\nu} \not\to 0$ with $|x_{\nu}| \leq |a| (1 \leq \nu), a \in R$ is termed a continuous element of $R$ with respect to $\| \cdot \|$.
In the sequel, we use the following notations:

\[(3.2) \quad S_0 = S \quad \text{and} \quad S_n = TS_{n-1} \quad (n=1, 2, \cdots) .\]

Now we have

**Lemma 2.** \(S_i \cap S_j = \emptyset\) holds for \(i \neq j\) \((i, j = 1, 2, \cdots)\).

**Proof.** If \(z \in S_i \cap S_j\) for some \(i, j\) with \(i < j\), i.e., \(z = T^i x = T^j y\) for some \(x, y \in S\) we get \(x = T^{j-i} y\). Putting \(c = T^{j-i} y\), we obtain \(x = Tc\) and \(c \in V\), which is inconsistent with Lemma 1.

**Lemma 3.** For each \(x \in S_n\) \((n = 0, 1, 2, \cdots)\) \(x\) can be decomposed into \(x = x_1 \oplus x_2\) in such a way that \(x_i \in S_{n+1}\) \((i = 1, 2)\) holds.

**Proof.** \(x \in S_n\) implies \(T^{-n} x \in S\), whence \(\|T^{-(n+1)} x\| > 1\). Now we put \(a = T^{-(n+1)} x\). Since \(R\) contains no atomic element and \(\cdot \|\cdot\|\) is continuous, we can find an element \(p\) such that \([p] a \in S\) holds. Because of \([p] a, Ta \in S\), it follows from (T.C.) that \((1 - [p]) a \in S\) holds. Hence \(x = T^{n+1} a = T^{n+1} [p] a + T^{n+1} (1 - [p]) a\) with \(T^{n+1} [p] a, T^{n+1} (1 - [p]) a \in S_{n+1}\) simultaneously.

It is obvious from Lemma 3 that \(x \in S\) if and only if \(x\) is represented as, for any fixed \(n\),

\[(3.3) \quad x = T^n \left( \bigoplus_{i=1}^{2^n} x_i \right) ,\]

where \(x_i \in S\) \((i = 1, 2, \cdots, 2^n)\).

**Lemma 4.** Let \(a, b \in S\) and \(a \perp b\). Then

\[(3.4) \quad \|T^n a \oplus b\| > 1\]

stands for each \(n \geq 1\).

**Proof.** We shall prove this lemma by inducition. In case of \(n = 1\), \(\|Ta \oplus b\| = 1\) implies \(\|T(a \oplus T^{-1} b)\| = 1\), whence \(T^{-1} b \in S\), contradicting Lemma 1. Thus (3.4) is valid for \(n = 1\). Now suppose that (3.4) holds for each \(n \leq k\) and \(\|T^{k+1} a \oplus b\| = 1\) for some \(a, b \in S\) with \(a \perp b\). Then \(\|T (T^k a \oplus T^{-1} b)\| = 1\) holds and \(T^{-1} b\) can be represented as \(T^{-1} b = b_i \oplus b_j, b_i \in S\) \((i = 1, 2)\). From this and \(\|T \{(T^k a \oplus b_i) + b_j\}\| = 1\), it follows that \(\|T^k a \oplus b\| = 1\) holds on account of (T.C.), but this contradicts the inducition hypothesis. \(\Box\).  

**Lemma 5.** If \(x = \bigoplus_{i=1}^{m} x_i = \bigoplus_{j=1}^{m} y_j \oplus y_0\) with \(x_i \in S, y_j \in S\) \((i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)\) and furthermore \(x\) is not a complete\(^{1)}\) element, then \(n \geq m\) holds.

**Proof.** Suppose contrarily \(n < m\). Since \(R\) is non-atomic, we can find a set of mutually orthogonal elements \(\{z_i\}_{i=1}^{\rho} \subset S\) such that \(z_i \perp x\) \((1 \leq i \leq \rho)\) and

\(^{1)} \quad x \in R\) is called a complete element if \(\{x\} \perp = \{0\}\) holds.
$n + \rho = 2^\mu$ for some $\mu \geq 1$. Then $T^\mu\left(\bigoplus_{i=1}^{\rho}z_i \oplus x\right) = T^\mu\left(z_1 \oplus \cdots \oplus z_\rho \oplus y_1 \oplus \cdots \oplus y_\mu \oplus y_0\right) = T^\mu\left(y_1 \oplus \cdots \oplus y_\mu \oplus z_1 \oplus \cdots \oplus z_\rho\right) + T^\mu\left(y_0\right)$, which implies $1 = \|T^\mu(x \oplus z)\| \geq \|w + T^\mu y_\mu\|$, where $w = T^\mu(y_1 \oplus \cdots \oplus y_\mu \oplus z_1 \oplus \cdots \oplus z_\rho)$ belongs to $S$. However, this is inconsistent with the preceding lemma. Q.E.D.

**Lemma 6.** If $x$ is not a complete element and $x = x_1 \oplus \cdots \oplus x_k = y_1 \oplus \cdots \oplus y_\mu = x_1 \oplus \cdots \oplus x_n$, where $x_\nu \in S_{m_\nu}, y_\mu \in S_{n_\mu}$ and $y_0 \in V$ ($1 \leq \nu \leq k, 1 \leq \mu \leq l, 0 \leq m_\nu, n_\mu$), then $\sum_{\nu=1}^{k} \frac{1}{2^{m_\nu}} \geq \sum_{\mu=1}^{l} \frac{1}{2^{n_\mu}}$ holds.

**Proof.** We put $N = \text{Max} \{m_\nu, n_\mu\}$. Then, for each $\nu (1 \leq \nu \leq k) x$ is decomposed into $x = x_\nu, x_\nu, \ldots \oplus x_\nu, 2^{N-m_\nu}$ with $x_\nu \in S_N$ ($1 \leq \nu \leq 2^{N-m_\nu}$). Similarly $y_\mu = y_\mu, y_\mu, \ldots \oplus y_\mu, 2^{N-n_\mu}$ with $y_\mu \in S_N$ holds for each $j$ ($1 \leq j \leq 2^{N-n_\mu}$). Hence both $x = \oplus \oplus \oplus x_\nu, \mu$ and $x = \oplus \oplus y_\mu, j \oplus y_0$ holds, which implies $T^{-N}x = \oplus \oplus x_\nu, \mu = \oplus \oplus T^{-N}y_\mu, j \oplus T^{-N}y_0$ with $T^{-N}x_\nu, \mu \in S$ and $T^{-N}y_\mu, j \in S$ for each $\nu, \mu, i, j$. In view of the preceding lemma we find

$$\sum_{\nu=1}^{k} 2^{N-m_\nu} \geq \sum_{\mu=1}^{l} 2^{N-n_\mu}.$$ 

Thus we obtain

$$\sum_{\nu=1}^{k} \frac{1}{2^{m_\nu}} \geq \sum_{\mu=1}^{l} \frac{1}{2^{n_\mu}}.$$ 

Q.E.D.

Here we turn to define an orthogonal additive functional (i.e. $\rho(x + y) = \rho(x) + \rho(y)$ for $x \perp y$) on $R$ from $\|\cdot\|$. Let $R_0$ be the set of all non-complete elements of $R$ and $\mathfrak{U}$ be the totality of elements of $R_0$ which can be represented as $x_1 \oplus \cdots \oplus x_n$ with $x_i \in S_{m_i}$ ($i = 1, 2, \ldots, n; n = 1, 2, \cdots$). On $\mathfrak{U}$ we define a functional $\rho'$ as follows:

$$\rho'(x) = \sum_{i=1}^{n} \frac{1}{2^{m_i}},$$

where $x = x_1 \oplus \cdots \oplus x_n$ with $x_i \in S_{m_i}$ ($1 \leq i \leq n$). According to Lemma 6 we see that this definition has a sense. It is evident from the definition that $\rho'$ is orthogonally additive on $\mathfrak{U}$. Next, we put for each $x \in R$

$$\rho(x) = \begin{cases} \sup_{|y| \leq |x|, y \in \mathfrak{U}} \rho'(y), & |y| \leq |x|, y \in \mathfrak{U} \\ 0, & \text{if there exists no element } y \in \mathfrak{U} \text{ with } |y| \leq |x|. \end{cases}$$

In the succeeding section we shall show that $\rho$ thus defined is in fact a modular on $R$ and that $\|\cdot\|$ is nothing but the modular norm by $\rho$. 

\begin{align}
\rho'(x) &= \sum_{i=1}^{n} \frac{1}{2^{m_i}}, \\
\rho(x) &= \begin{cases} \sup_{|y| \leq |x|, y \in \mathfrak{U}} \rho'(y), & |y| \leq |x|, y \in \mathfrak{U} \\ 0, & \text{if there exists no element } y \in \mathfrak{U} \text{ with } |y| \leq |x|. \end{cases}
\end{align}
4. Properties of $\rho$ and the proofs of Theorems. In view of construction of $\rho$ and Lemma 6 we see easily that $\rho$ satisfies the modular conditions M.1) and M.2). Since $R$ contains no atomic element, we have also

$$\rho(x) = \rho'(x) \quad \text{for each } x \in \mathfrak{A}.$$  

(4.1)

In order to prove the remaining conditions M.3),~M.6), we need some lemmas.

Lemma 7. We have

$$\rho(x) > \frac{1}{2^m} \quad \text{and} \quad \rho(x) < \frac{1}{2^m} \quad \text{imply} \quad \|T^{-m}x\| > 1 \quad \text{and} \quad \|T^{-m}x\| \leq 1 \quad \text{respectively} \quad (m=0,1,2,\cdots);$$  

$$(4.2) \quad \rho(x) < +\infty, \quad \text{for each } x \in R;$$  

$$(4.3) \quad \rho(x) = \sup_{[p]x \in \mathfrak{A}} \rho'( [p]x), \quad \text{if } \rho(x) > 0.$$  

Proof. (4.2) follows immediately from the definition of $\rho$. Since $\| \cdot \|$ is continuous, each element $x \in R$ can be represented as $x = \bigoplus_{i=1}^{n} x_i$ with $\|x_i\| \leq 1$ ($1 \leq i \leq n$) for some $n \geq 1$. From this we have $\rho(x) \leq n$ in view of (3.5), (4.2) and M.2). Thus (4.3) is valid. Next, we shall show that if $\rho(x) > \frac{k}{2^m}$ $x$ is written as $x = \bigoplus_{i=1}^{k} x_i \oplus x_0$ with $\|T^{-m}x_i\| > 1$ for each $i$ ($1 \leq i \leq k$). By (3.6) there exists $0 \leq x' \in \mathfrak{A}$ such that $|x| \geq x' = \bigoplus_{i=1}^{k} x_i' \oplus x_0'$ with $x_i' \in S_m$ ($1 \leq i \leq k$) and $x_0' \in \mathfrak{A}$. Now we decompose $x_0'$ into $x_0' = \bigoplus_{i=1}^{k} x_i''$ with $x_i'' \in \mathfrak{A}$ for each $i$. On the ground of Lemma 4 $\|T^{-m}(x_i' \oplus x_i'')\| > 1$ ($1 \leq i \leq k$) must hold. Putting $x_i = [x_i' \oplus x_i'']x$ and $x_0 = x - \bigoplus_{i=1}^{k} x_i$, we obtain $x = \bigoplus_{i=1}^{k} x_i \oplus x_0$ with $\|T^{-m}x_i\| > 1$ for each $i$ ($1 \leq i \leq k$).

From this one derives easily that if $\rho(x) > \frac{k}{2^m}$ there exist projectors $\{[p_i]\}_{i=1}^{k}$ such that $[p_i] \leq [x_i]$ and $\|T^{-m}[p_i]x_i\| = 1$ hold ($1 \leq i \leq k$), where $\{x_i\}_{i=1}^{k}$ satisfies the above condition. Since $\{p_i\} x_i \in S_m$ and $\bigoplus_{i=1}^{k} [p_i] x_i = \sum_{i=1}^{k} [p_i]x$, $\rho'([p]x) \geq \frac{k}{2^m}$ follows and (4.4) is proved, where $[p] = \sum_{i=1}^{k} [p_i].$  

Q. E. D.

Lemma 8. $\rho$ is orthogonally additive, i.e., it satisfies M.5).

Proof. From the definition of $\rho$ it follows that

$$\rho(x \oplus y) \geq \rho(x) + \rho(y)$$

which completes the proof.
A Characterization of Modular Norms in Terms of Similar Transformations

holds. Now suppose $\rho(x \oplus y) > \rho(x) + \rho(y)$ for some $x, y \in R$ with $x + y \in R_\circ$.

By (4.4) there exist projectors $[p], [q]$ for which $\rho(x) - \rho'( [p] x) < \frac{1}{2^{m+2}}$, $\rho(y) - \rho'( [q] y) < \frac{1}{2^{m+2}}$, $[p] x \in \mathfrak{U}$ and $[q] y \in \mathfrak{U}$ hold. Since $\rho((1-[p])x) \leq \rho(x) - \rho'( [p] x) < \frac{1}{2^{m+2}}$ and $\rho((1-[q])y) \leq \rho(y) - \rho'( [q] y) < \frac{1}{2^{m+2}}$ hold, we can find $\alpha, \beta \geq 1$ such that both $\alpha(1-[p])x$ and $\beta(1-[q])y$ belong to $S_{m+2}$ according to (4.2) and the fact that $T$ is similar. Putting $x' = [p] x + \alpha(1-[p])x$ and $y' = [q] y + \beta(1-[q])y$, we obtain $x', y' \in \mathfrak{U}$ and $\rho'(x' \oplus y') = \rho'(x') + \rho'(y') = \rho'([p] x) + \rho'([q] y) + \frac{1}{2^{m+1}}$, since $\rho'$ is orthogonally additive on $\mathfrak{U}$. Hence we get

$$\rho'(x' \oplus y') \geq \rho(x \oplus y) > \rho(x) + \rho(y) + \frac{1}{2^{m}} \geq \rho'([p] x) + \rho'([q] y) + \frac{1}{2^{m+1}}$$

which is, however, a contradiction. Thus we see easily that $\rho$ is orthogonally additive by virtue of Lemma 7. Q. E. D.

**Lemma 9.** We have

(4.5) \[ \rho(x) \leq 1 \text{ if and only if } \|x\| \leq 1. \]

**Proof.** The fact that $\|x\| \leq 1$ implies $\rho(x) \leq 1$ is obvious by virtue of Lemma 4. On the other hand, for any $x$ with $\rho(x) \leq 1$ we can find a sequence of projectors $\{[p_\nu]\}_{\nu=1}^\infty$ such that $[p_\nu] \uparrow_{\nu=1}^\infty [x]$, $[p_\nu] x \in \mathfrak{U}$ and $\rho([p_\nu] x) \uparrow_{\nu}^\infty \rho(x) \leq 1$ on account of (4.4) and the orthogonal additivity of $\rho$. By (4.1) and the definition of $\rho'$, we now get $\|[p_\nu] x\| \leq \nu$ for each $\nu \geq 1$, hence $\|x\| \leq 1$ because of the semi-continuity of $\|\cdot\|$. Q. E. D.

**Lemma 10.** $\rho$ is semi-continuous, i.e., it satisfies M. 6.1)

**Proof.** Let $0 \leq x_i \uparrow_{i \in I} x$ and $\rho(x) > \frac{k}{2^m}$. As is shown in the proof of (4.4), there exists $p \in R$ such that $[p] x \in \mathfrak{U}$, $[p] x = \bigoplus_{i=1}^k x_i$ and $\|T^{-m} x_i\| > 1$ (1 \leq i \leq k). Then, since $[w_i] x_i \uparrow_{i \in I} [w_i] x = x_i$ holds for each $i$ and $\|\cdot\|$ is semi-continuous, we have for a sufficiently large $\lambda$ that $\|T^{-m} [w_i] x_i\| > 1$ stands for every $i$ (1 \leq i \leq k). Therefore we have

11) In case of $\rho(x) = 0$ (or $\rho(y) = 0$), we choose $p = 0$ (resp. $q = 0$).
\[ \rho(x_{\lambda_{0}}) \geqq \rho([p]x_{\lambda_{0}}) \geqq \frac{k}{2^{m}}, \]

which shows the semi-continuity of \( \rho \).

**Lemma 11.** \( \rho \) satisfies M.3). i.e., \( \lim_{\xi \to 0} \rho(\xi x) = 0 \).

**Proof.** If \( \rho(\xi x) > \frac{1}{2^{m}} \) holds for each \( \xi > 0 \), we have \( \|T^{-m}\xi x\| > 1 \). Since \( \cap_{\xi > 0} \xi|x| = 0 \) stands, \( \cap_{\xi > 0} T^{-m}\xi|x| = 0 \) holds. Hence it follows that \( \|T^{-m}\xi x\| \to 0 \) as \( \xi \to 0 \), because of the continuity of \( \|\cdot\| \). This is a contradiction. Q. E. D.

Summing up the above results, we see that \( \rho \) satisfies all the conditions of modular except M.4). Next lemma shall show that \( \rho \) fulfils M.4) too.

**Lemma 12.** \( \rho(\xi x) \) is a convex function of \( \xi (\xi \geqq 0) \) for each \( x \in R \).

**Proof.** We shall first show that the set \( B_{\xi} = \{x : \rho(x) \leqq \xi\} \) is convex for every \( \xi \) with \( 0 \leqq \xi \leqq 1 \). Let \( x, y \in B_{\xi} \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \). By virtue of semi-continuity of \( \rho \), we may assume without loss of generality that there exists \( 0 \neq z \in R \) belonging to \( \{x, y\}^{\perp} \). Furthermore we may choose \( z \) as \( \rho(z) = 1 - \xi \), since \( \rho \) satisfies (4.3) and \( R \) has no atom. It follows that both \( x + z \) and \( y + z \) belong to \( V \), hence \( \alpha(x + z) + \beta(y + z) \) also does. Consequently, we obtain \( \rho(\alpha x + \beta y) + \rho(z) \leqq 1 \) by Lemma 9, hence \( ax + by \in B_{\xi} \). Therefore \( B_{\xi} \) is convex.

Next, suppose that \( \rho(\xi x) \leqq 1 \) and \( \rho(\eta x) \leqq 1 \) for some \( x \in R \) and \( \xi > \eta \geqq 0 \). Since \( \rho \) is finite, orthogonally additive and semi-continuous, we can find \( p \in R \) for which \( \rho(\xi[p]x) = \rho(\xi(1-\[p\])x) \) holds. If \( \rho(\eta[p]x) < \rho(\eta(1-\[p\])x) \) stands for such \( [p] \), there can be constructed a system of projectors \( \{[p_{s}]\}_{(0 \leqq s \leqq 0)} \) and \( \{[p_{s}']\}_{(0 \leqq s \leqq 1)} \) such that \( [p_{s}] \downarrow ([p_{s}'] \uparrow) \) as \( a \downarrow 0 \), \( [p_{s}] \leqq [p] \), \( [p_{s}'] \leqq (1 - [p]) \) with \( [p_{1}] = [p] \), \( [p_{0}] = (1 - [p]) \), and \( \rho(\xi[p_{s}]x) = \rho(\xi(1 - [p])x) = \alpha \rho(\xi[p]x) \) holds for each \( 0 \leqq a \leqq 1 \). Putting \( [q_{s}] = [p_{s}] + (1 - [p])[p_{s}'] \), we obtain \( [q_{s}] \leqq [x] \) and \( \rho(\xi[q_{s}]x) = \rho(\xi(1 - [q_{s}])x) \) for every \( \alpha \). Furthermore we see easily that both \( \rho(\eta[q_{s}]x) < \rho(\eta(1 - [q_{s}])x) \) and \( \rho(\eta[q_{s}]x) > \rho(\eta(1 - [q_{s}])x) \) hold. From this it follows that \( \rho(\eta[q_{s}]x) = \rho(\eta(1 - [q_{s}])x) \) stands for some \( \alpha \). In consequence, we have shown that there exists \( p \in R \) such that \( \rho(\xi[p]x) = \frac{1}{2} \rho(\xi x) \) and \( \rho(\eta[p]x) = \frac{1}{2} \rho(\eta x) \) hold simultaneously. Because \( \rho(\xi[p]x + \eta(1 - [p])x) = \rho(\eta[p]x + \xi(1 - [p])x) = \frac{1}{2} \{\rho(\xi x) + \rho(\eta x)\} \leqq 1 \), we have

\[
\rho \left( \frac{1}{2}(\xi x + \eta y) \right) \leqq \frac{1}{2} \{\rho(\xi x) + \rho(\eta y)\}
\]

by the fact shown just above.
Finally, since each $x$ can be decomposed orthogonally into $x=\bigoplus_{i=1}^{n}x_{i}$ with $\rho(x_{i})\leqq 1$ ($1\leqq i\leqq n$), we see that (4.6) holds for any $x\in R$, i.e. $\rho(\xi x)$ is a convex function of $\xi$ ($\xi \geqq 0$) for each $x\in R$.

Here we are in position to prove the theorems stated in 2.

Proof of Theorem 1. Sufficiency. The functional $\rho$ constructed in 3 is a modular satisfying (4.5), as is shown above. Hence we have $\|x\| = \inf \left\{ \frac{1}{|\xi|} ; \rho(\xi x) \leqq 1 \right\}$ i.e., $\|\cdot\|$ is the modular norm by the modular $\rho$.

Necessity. Let $\|\cdot\|$ be the modular norm by a modular $m$ on $R$. $m$ is necessarily finite since $\|\cdot\|$ is continuous. In the same manner as in the proof of Theorem 2 in [8], we can construct a similar transformation $T_{0}$ on $R$ satisfying

$$m(T_{0}x) = \frac{1}{2}m(x) \quad \text{for every } x \in R.$$ 

It is now clear that $T_{0}$ satisfies the condition (T.C.). Q. E. D.

Proof of Theorem 2. Sufficiency. In view of Theorem 1 we find a finite modular $\rho$ on a complete semi-normal manifold $R_{C}$ of $R$, for which $\|\cdot\|$ is the modular norm on $R_{C}$. We extend now $\rho$ on the whole space $R$ as follows:

$$\rho_{0}(x) = \sup_{0 \leqq y|_{x} \in R_{C}} \rho(y) \quad (x \in R).$$

$\rho_{0}$ thus defined is an almost finite modular on $R$, as is easily seen, and $\rho_{0}(x) = \rho(x)$ if $x \in R_{C}$. Because of the semi-continuity of $\|\cdot\|$ and $\rho$, $\|x\| = \inf \left\{ \frac{1}{|\xi|} ; \rho_{0}(\xi x) \leqq 1 \right\}$ holds for each $x \in R$, that is, $\|\cdot\|$ is the modular norm by $\rho_{0}$. The necessity is derived similarly as the proof of Theorem 1. Q. E. D.

5. Here let $(R, \|\cdot\|)$ be the same as in 3 and $\rho$ be the modular defined, in the manner described above, from $\|\cdot\|$ and a similar transformation $T$ on $R$ satisfying the condition (T.C.). From the construction of $\rho$ one derives easily

$$\rho(Tx) = \frac{1}{2} \rho(x) \quad (x \in R).$$

Also this enables us obviously to restate properties of the modular $\rho$ in terms of similar transformations $T$:

12) Of course, we can state properties of $\rho$ by means of $\|\cdot\|$, since there are found closed relations between modulars and their norms [1, 6, and 7].
restatements in terms of $T$. Being trivial, their proofs are omitted.

5.1. $\rho$ is simple (i.e. $\rho(x)=0$ implies $x=0$), if and only if $\bigcap_{m \geq 1} T^m x = 0$
for each $x \in R$.

5.2. $\rho$ is uniformly simple (i.e. $\inf_{\|x\| \leq \delta} \rho(x) > 0$ for each $\delta > 0$), if and only
if for each $\varepsilon > 0$ there exists $m \geq 0$ with $\sup_{x \in S} \|T^m x\| < \varepsilon$.

5.3. $\rho$ is uniformly finite (i.e. $\sup_{\|x\| \leq \delta} \rho(x) < +\infty$ for each $\delta > 0$), if and
only if for each $\delta > 0$ there exists $m \geq 0$ with $\inf_{x \in S} \|T^{-m} x\| > \delta$.

5.4. $\rho$ is upper bounded (i.e. $\rho(\alpha x) \leq \rho(x)$ holds for every $x \in R$, where
$1 < \alpha, \gamma$ are fixed constants), if and only if $T \leq \left( \frac{1}{2} \right)^{\frac{1}{\nu}} I^{13}$ for some $\nu \geq 1$.

Finally let $(E, \Omega, \mu)$ be a $\sigma$-finite non-atomic measure space with a countably
additive non-negative measure $\mu$ on a $\sigma$-field $\Omega$ of $E$. A module $\rho$ of measurable functions on $E$ is a semi-normal manifold of modulared
function space $L_{M(\xi, t)}$ defined by a modular function $M(\xi, t)$ on $[0, \infty) \times E$,
that is, $X$ is contained in the totality of all measurable functions $\mathfrak{h}$ such that
$\int_{E} M(\alpha |f(t)|, t) d\mu(t) < +\infty$ for some $\alpha > 0$, and

\begin{equation}
(5.1) \quad m(f) = \int_{E} M(|f(t)|, t) d\mu(t)
\end{equation}

holds for each $f \in X$. Conversely, it is known [6] that each modulared semi
ordered linear space $R$ can be considered as a modulared function space $L_{M(\xi, t)}$ on a measure space $(E, \Omega, \mu)$ suitably chosen, and $m$ is represented by (5.1).

For any finite modulared function space $L_{M(\xi, t)}(E)$ we can obtain a similar transformation $T$ with the condition (T.C.) directly as follows: We define for $(\xi, t) \in [0, \infty) \times E$

\begin{equation}
(5.2) \quad h(\xi, t) = \begin{cases} 
M^{-1}(\xi) = \left( \frac{1}{2} \right)^{\frac{1}{\nu}} I^{13} \end{cases}
\end{equation}

where $M^{-1}(\xi)$ is the inverse of the function $M(\xi) = M(\xi, t)$ for each $t \in E$. Then $h(\xi, t)$ on $[0, \infty) \times E$ is a Carathéodory’s function, and the transformation $\mathfrak{h}$ defined by

---

13) I is the identity operator on $R$ and 5.4 follows from Theorem 3.3 of [9].

14) For the definition of modular functions see [3 or 6]. Roughly speaking, $M(\xi, t)$ is
a $N^\nu$-function of $\xi$ for each $t \in E$. In $M(\xi, t)$ we consider $\int_{E} M(|f(t)|, t) d\mu(t)$ as a modular $m$
always.

15) $m$ on $L_{M(\xi, t)}$ is finite, if and only if $M(2t, t) \leq \tau M(\xi, t) + a(t)$ for all $(\xi, t) \in [0, \infty) \times E$,
where $\tau > 0$ and $a(t) \in L_{1}(E)[3]$. $m$ is almost finite if and only if $M(\xi, t) < +\infty$ a.e. in $[0, \infty) \times E$. 

forms a similar transformation satisfying the condition (T.C.) for the modular norm on $L_{M(\xi,t)}$. Conversely, in view of Theorem 1 we have

Theorem 3. If $(X, \| \cdot \|)$ is a normed function space with a continuous norm $\| \cdot \|$, and if a similar transformation $h$ from $X$ onto $X$, defined by a Carathéodory's function $h(\xi, t)$ on $[0, \infty) \times E$, satisfies the condition (T.C.), then there can be found a modular function $M(\xi, t)$ on $[0, \infty) \times E$ such that $X$ is a semi-normal manifold of $L_{M(\xi,t)}$, $M(\xi, t)$ satisfies (5.2), and $\| \cdot \|$ coincides with the modular norm of the space $L_{M(\xi,t)}$.

Remark 1. In this theorem if moreover, $(X, \| \cdot \|)$ is monotone complete (i.e. $0 \leq f_\ast \uparrow$, sup $\| f_\ast \| < +\infty$ implies $\bigcup_{\ast=1}^\infty f_\ast \in X$), then $X=L_{M(\xi,t)}$ holds.

Remark 2. In Theorem 3, if $h(\xi, t)=h(\xi)$ for all $(\xi, t) \in [0, \infty) \times E$, then $L_{M(\xi,t)}$ can be replaced by an Orlicz space $L_M$.

When $\| \cdot \|$ is almost continuous, we have a similar theorem as above on the basis of Theorem 2. In this case, $h$ acts from $L^0_{M(\xi,t)}$, the finite manifold of $L_{M(\xi,t)}$ (the totality of all $f \in L_{M(\xi,t)}$ with $m(\xi f) < +\infty$ for every $\xi \geq 0$), onto itself and satisfies (5.2), if $0 < M(\xi, t) < +\infty$.

On the basis of Theorems 1 and 2, a theorem characterizing the modular norms in terms of norms only can be obtained, and it shall be shown in a separate paper.

References


16) We assume that $X$ is semi-normal.
17) For $h(\xi, t)$ we assume $h(0, t) = 0$ for all $t \in E$.
18) Strictly speaking, $h(\xi, t) = M_t^{-1}\left(\frac{1}{2}M(\xi, t)\right)$ holds a.e. for $(\xi, t)$ satisfying $M(\xi, t) > 0$. In general, $h(\xi, t) = \xi$ does not hold for $(\xi, t)$ with $M(\xi, t) = 0$. 


Department of Mathematics,  
Hokkaido University

(Received September 28, 1964)