NOTE ON EXTENSIONS OF DERIVATIONS IN SIMPLE RINGS

By

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Throughout the present note, we use the following conventions: A is always a simple ring, and we set $A=\sum_1^n De_{ij}$ where $\{e_{ij}\text{'s}\}$ is a system of matrix units and $D=V_A(\{e_{ij}\text{'s}\})$ a division ring. B is a simple subring of A containing the identity element 1 of A, and we set $V=V_A(B)$ and $H=V_A(V)=V_A^2(B)$. Further, $\mathfrak{A}=\mathrm{Hom}(A, A)$ (acting on the right side) and $\mathfrak{G}$ signifies the multiplicative group of all B-(ring) automorphisms of A. For subrings $A_1\supseteq A_2$ of A containing 1, $D(A_1, A/A_2)$ will denote the set of all the derivations of A_1 into A vanishing on A_2. In particular, we set $D(A_1, A)=D(A_1, A/0)$. As to other notations and terminologies used in this note, we follow [2] and [3].

As a particular case of [1, Theorem VI. 13.1], the following is well-known: *Let A be Galois and finite over B, and T an arbitrary simple intermediate ring of A/B. If δ is in $D(T, A/B)$ then there exists $v\in V$ such that $\delta=\delta_v|T$, where δ_v denotes the inner derivation v_r-v_l effected by v.* At first, we shall present an extension of the above proposition to the infinite dimensional Galois extensions, that is stated as follows:

Theorem 1. *Let A be locally h-Galois (cf. [3]) and left locally finite over B, and T a simple intermediate ring of A/B left finite over B. If δ is in $D(T, A/B)$ then $\delta=\delta_v|T$ with some $v\in V$.*

Proof. The proof will be completed by the modification of that of [1, Theorem VI. 13.1] given in [1, p. 152]. Since $D(T, A/B)$ is contained in $\mathrm{Hom}_{B_l}(T, A)$, we can find an intermediate ring A' of $A/T[T\delta]$ such that A'/B is h-Galois. Hence, we may assume from the beginning that A/B is h-Galois. Now, let e be an arbitrary primitive idempotent of T. Since $e^2=e$, $e\delta\cdot e+e\cdot e\delta=e\delta$ and so $e\cdot e\delta\cdot e=0$. If $a=[e\delta, e]=e\delta\cdot e-e\cdot e\delta$ then $[e, a]=-e\cdot e\delta-e\delta\cdot e=-e\delta$. Thus $\delta_1=\delta+\delta_a|T$ is an element of $D(T, A)$ with $e\delta_1=0$. Obviously, $A=\oplus Tea_\lambda$ $(a_\lambda\in A)$ where each $a_{\lambda r}$ induces a T-isomorphism of Te onto Tea_λ. Hence, by $(\sum t_\lambda ea_\lambda)\beta=\sum(t_\lambda e)\delta_1\cdot a_\lambda=\sum t_\lambda\delta_1\cdot ea_\lambda$ we can define $\beta\in\mathfrak{A}$. If $\alpha=\beta+a_l$ then for each $t\in T$ we have $(t_\lambda ea_\lambda)[t_l, \alpha]=t\delta\cdot(t_\lambda ea_\lambda)$, whence it follows $(t\delta)_l=[t_l, \alpha]$. Since δ is in $D(T, A/B)$, $0=(b\delta)_l=[b_l, \alpha]$ for each $b\in B$, namely, α is contained in $V_{\mathfrak{A}}(B_l)$ that is the topological closure of $\mathfrak{G}A_r$ by the hypothesis. If $T^*=T[\{e_{ij}\text{'s}\}]$ then $\mathrm{Hom}_{B_l}(T^*, A)=(\mathfrak{G}|T^*)A_r=\oplus_1^s(\sigma_i|T^*)A_r$ ($\sigma_i\in\mathfrak{G}$) and $\alpha|T^*$ has the unique representation $\sum_1^s(\sigma_i|T^*)x_{ir}$ with $x_i\in A$ ([2, Lemma 1.3 (i)]), where we may assume that $\sigma_1, \cdots, \sigma_{s'}\in\tilde{V}$ and $\sigma_{s'+1}|T^*, \cdots, \sigma_s|T^*\notin\tilde{V}|T^*$. Now, let t be an arbitrary element of T. By $x\gamma_i=(tx)\sigma_i-t(x\sigma_i)$ ($x\in T^*$) we define the homomorphisms γ_i of T^* into A ($i=1, \cdots, s$). Then, for each $y\in T^*$ we have $y_r\gamma_i=\gamma_i(y\sigma_i)_r$, and so it will be easy to see that each γ_iA_r is T_r^*-A_r-homomorphic to the irreducible module $(\sigma_i|T^*)A_r$. Hence, by [2, Lemma 1.3 (iv)], $(t\delta)_l|T^*=\sum_1^s\gamma_ix_{ir}=\sum_1^{s'}\gamma_ix_{ir}$. If we set $\sigma_i=\tilde{v}_i$ with $v_i\in V^{\cdot}$ ($i=1, \cdots, s'$) then $x\gamma_i=[v_i, t]xv_i^{-1}$, and so $(t\delta)_l|T^*=\sum_1^{s'}([v_i, t]_l|T^*)y_{ir}=\sum_1^{s'}(y_{ir}|T^*)[v_i, t]_l$ where $y_i=v_i^{-1}x_i\in A$. Now, choose an arbitrary linearly independent left $V_A(T^*)$-basis $\{u_\mu\text{'s}\}$ of A such that $u_1=1$, and represent y_i in terms of this basis. Then, $(1_r|T^*)(t\delta)_l=(t\delta)_l|T^*=\sum_\mu(u_{\mu r}|T^*)[v'_\mu, t]_l$ with some $v'_\mu\in V$ determined independently of t. It follows therefore $t\delta=[v'_1, t]=t\delta_{-v'_1}$ by the proposition symmetric to [2, Lemma 1.4 (ii)].

Next, we shall prove the following partial extension of Theorem 1, that contains [2, Theorem 4.6] as well.

Theorem 2. *Let A be locally h-Galois and left locally finite over B, and T an f-regular intermediate ring of A/B. If δ is in $D(T, A/B)$ then $\delta=\delta_v|T$ for some $v\in V$.*

Proof. There exists a simple intermediate ring B' of T/B with $[B':B]_l<\infty$ and $V_A(B')=V_A(T)$, and then $\delta|B'=\delta_v|B'$ for some $v\in V$ (Theorem 1). Since A/B' is locally h-Galois and left locally finite by [3, Corollary 1], T is an intermediate ring of $V_A^2(B')/B'$ and $\delta'=\delta-(\delta_v|T)$ is contained in $D(T, A/B')$, it suffices to prove that if T is contained in H then $\delta=0$. To see this, we set $T=\cup T_\lambda$ where T_λ runs over all the simple intermediate rings of T/B with $[T_\lambda:B]_l<\infty$. Then, by Theorem 1, $\delta|T_\lambda=\delta_{v_\lambda}|T_\lambda$ with some $v_\lambda\in V$. Hence, we have $\delta|T_\lambda=0$ for all λ, whence it follows $\delta=0$.

References

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