NOTE ON EXTENSIONS OF DERIVATIONS IN SIMPLE RINGS

By

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Throughout the present note, we use the following conventions: $A$ is always a simple ring, and we set $A=\sum e_{ij}$ where $\{e_{ij}\}$ is a system of matrix units and $D=V_{A}\{e_{ij}\}$ a division ring. $B$ is a simple subring of $A$ containing the identity element 1 of $A$, and we set $V=V_{A}(B)$ and $H=V_{A}(V)=V_{2}(B)$. Further, $\mathfrak{A}=\text{Hom}(A, A)$ (acting on the right side) and $\mathfrak{G}$ signifies the multiplicative group of all $B$-(ring) automorphisms of $A$. For subrings $A_{i}\supseteq A$, of $A$ containing 1, $D(A_{i}, A/A_{i})$ will denote the set of all the derivations of $A_{i}$ into $A$ vanishing on $A_{i}$. In particular, we set $D(A_{i}, A)=D(A_{i}, A/0)$. As to other notations and terminologies used in this note, we follow [2] and [3].

As a particular case of [1, Theorem VI. 13.1], the following is well-known: Let $A$ be Galois and finite over $B$, and $T$ an arbitrary simple intermediate ring of $A/B$. If $\delta$ is in $D(T, A/B)$ then there exists $v\in V$ such that $\delta=\delta_{v}|T$, where $\delta_{v}$ denotes the inner derivation $v_{r}-v_{l}$ effected by $v$. At first, we shall present an extension of the above proposition to the infinite dimensional Galois extensions, that is stated as follows:

Theorem 1. Let $A$ be locally $h$-Galois (cf. [3]) and left locally finite over $B$, and $T$ a simple intermediate ring of $A/B$ left finite over $B$. If $\delta$ is in $D(T, A/B)$ then $\delta=\delta_{v}|T$ with some $v\in V$.

Proof. The proof will be completed by the modification of that of [1, Theorem VI. 13.1] given in [1, p. 152]. Since $D(T, A/B)$ is contained in $\text{Hom}_{B_{T}}(T, A)$, we can find an intermediate ring $A'$ of $A/T[D\delta]$ such that $A'/B$ is $h$-Galois. Hence, we may assume from the beginning that $A/B$ is $h$-Galois. Now, let $e$ be an arbitrary primitive idempotent of $T$. Since $e^{2}=e$, $e\delta_{1}+e\cdot e\delta_{1}=e\delta$ and so $e\cdot e\delta_{1} \cdot e=0$. If $a=[e\delta, e]=e\delta \cdot e-e\cdot e\delta$ then $[e, a]=e\cdot e\delta_{1} \cdot e-e\cdot e\delta$. Thus $\delta_{v}=\delta_{v}|T$ is an element of $D(T, A)$ with $e\delta_{v}=0$. Obviously, $A=\oplus Te\alpha_{i} (\alpha_{i} \in A)$ where each $\alpha_{i}$ induces a $T$-isomorphism of $Te$ onto $Te\alpha_{i}$. Hence, by $(\sum t_{\alpha_{i}})\alpha_{i}=\sum (t_{\alpha})\delta_{i}, \alpha_{i}=\sum t_{\alpha_{i}} \delta_{i} \cdot e\alpha_{i}$ we can define $\alpha \in \mathfrak{U}$. If $\alpha=\beta+a_{t}$ then for each $t\in T$ we have $(t, e\alpha_{t})[t, \alpha]=t\delta \cdot (t, e\alpha_{t})$, whence it follows $(t\delta)_{t}=[t, \alpha]$. Since $\delta$ is in $D(T, A/B)$, $0=(b\delta)_{t}=[b_{t}, \alpha]$ for each $b_{t}\in B$, namely, $\alpha$ is contained in $V_{\mathfrak{U}}(B_{T})$ that is the topological closure of $\mathfrak{G}A_{\alpha}$ by the hypo-
thesis. If $T^* = T \{ e_{ij} 's \}$ then $\text{Hom}_{B_1}(T^*, A) = (\mathfrak{d}|T^*)A_r = \oplus i (\sigma_i|T^*)A_r$
$(\sigma_i \in \mathfrak{d})$ and $\alpha|T^*$ has the unique representation $\sum i(\sigma_i|T^*)x_{ir}$ with $x_{ir} \in A$ ([2, Lemma 1.3 (i)]), where we may assume that $\sigma_i, \cdots, \sigma_s \in \overline{\mathfrak{V}}$ and $\sigma_{s+1}|T^*, \cdots, \sigma_s|T^* \notin \overline{\mathfrak{V}}|T^*$. Now, let $t$ be an arbitrary element of $T$. By $x\gamma_i = (tx)\sigma_i - t(x\sigma_i) \ (x \in T^*)$ we define the homomorphisms $\gamma_i$ of $T^*$ into $A (i = 1, \cdots, s)$. Then, for each $y \in T^*$ we have $y\gamma_i = \gamma_i (y\sigma_i)_r$, and so it will be easy to see that each $\gamma_i A_r$ is $T^* - A_r$-homomorphic to the irreducible module $(\sigma_i|T^*)A_r$.

Hence, by [2, Lemma 1.3 (iv)], $(t\delta)|T^* = \sum i \gamma_i x_{ir} = \sum i \gamma_i x_{ir}$. If we set $\sigma_i = \gamma_i$ with $v_i \in V^* (i = 1, \cdots, s')$ then $x\gamma_i = [v_i, t] x v_i^{-1}$, and so $(t\delta)|T^* = \sum i([v_i, t]) T^* u_{ir} = \sum i(v_{ir}|T^*)[v_i, t]$, where $y_i = v_i^{-1} x_{ir} \in A$. Now, choose an arbitrary linearly independent left $V_A(T^*)$-basis $\{ u_i \}$ of $A$ such that $u_i = 1$, and represent $y_i$ in terms of this basis. Then, $(1_r|T^*) (t\delta)|T^* = \sum (u_{ir}, T^*) [v_i, t]$, with some $v_i \in V$ determined independently of $t$. It follows therefore $t\delta = [v_i, t] = t\delta_v$ by the proposition symmetric to [2, Lemma 1.4 (ii)].

Next, we shall prove the following partial extension of Theorem 1, that contains [2, Theorem 4.6] as well.

**Theorem 2.** Let $A$ be locally $h$-Galois and left locally finite over $B$, and $T$ an $f$-regular intermediate ring of $A/B$. If $\delta$ is in $D(T, A/B)$ then $\delta = \delta_v|T$ for some $v \in V$.

**Proof.** There exists a simple intermediate ring $B'$ of $T/B$ with $[B': B] \leq \infty$ and $V_A(B') = V_A(T)$, and then $\delta|B' = \delta_v|B'$ for some $v \in V$ (Theorem 1). Since $A/B'$ is locally $h$-Galois and left locally finite by [3, Corollary 1], $T$ is an intermediate ring of $V_A(B')/B'$ and $\delta = \delta - (\delta_v|T)$ is contained in $D(T, A/B')$, it suffices to prove that if $T$ is contained in $H$ then $\delta = 0$. To see this, we set $T = \cup T_i$ where $T_i$ runs over all the simple intermediate rings of $T/B$ with $[T_i : B] \leq \infty$. Then, by Theorem 1, $\delta|T_i = \delta_{v_i}|T_i$ with some $v_i \in V$. Hence, we have $\delta|T_i = 0$ for all $\lambda$, whence it follows $\delta = 0$.

**References**


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