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NOTE ON EXTENSIONS OF DERIVATIONS IN SIMPLE RINGS

By

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Throughout the present note, we use the following conventions: A is always a simple ring, and we set $A = \sum_1^n De_{ij}$ where $\{e_{ij}\}$'s is a system of matrix units and $D = V_A(\{e_{ij}\})$ a division ring. B is a simple subring of A containing the identity element 1 of A , and we set $V = V_A(B)$ and $H = V_A(V) = V_A^2(B)$. Further, $\mathfrak{A} = \text{Hom}(A, A)$ (acting on the right side) and \mathfrak{G} signifies the multiplicative group of all B -(ring) automorphisms of A . For subrings $A_1 \supseteq A_2$ of A containing 1, $D(A_1, A/A_2)$ will denote the set of all the derivations of A_1 into A vanishing on A_2 . In particular, we set $D(A_1, A) = D(A_1, A/0)$. As to other notations and terminologies used in this note, we follow [2] and [3].

As a particular case of [1, Theorem VI. 13.1], the following is well-known: *Let A be Galois and finite over B , and T an arbitrary simple intermediate ring of A/B . If δ is in $D(T, A/B)$ then there exists $v \in V$ such that $\delta = \delta_v|T$, where δ_v denotes the inner derivation $v_r - v_l$ effected by v .* At first, we shall present an extension of the above proposition to the infinite dimensional Galois extensions, that is stated as follows:

Theorem 1. *Let A be locally h -Galois (cf. [3]) and left locally finite over B , and T a simple intermediate ring of A/B left finite over B . If δ is in $D(T, A/B)$ then $\delta = \delta_v|T$ with some $v \in V$.*

Proof. The proof will be completed by the modification of that of [1, Theorem VI. 13.1] given in [1, p. 152]. Since $D(T, A/B)$ is contained in $\text{Hom}_{B_l}(T, A)$, we can find an intermediate ring A' of $A/T[T\delta]$ such that A'/B is h -Galois. Hence, we may assume from the beginning that A/B is h -Galois. Now, let e be an arbitrary primitive idempotent of T . Since $e^2 = e$, $e\delta \cdot e + e \cdot e\delta = e\delta$ and so $e \cdot e\delta \cdot e = 0$. If $a = [e\delta, e] = e\delta \cdot e - e \cdot e\delta$ then $[e, a] = -e \cdot e\delta - e\delta \cdot e = -e\delta$. Thus $\delta_1 = \delta + \delta_a|T$ is an element of $D(T, A)$ with $e\delta_1 = 0$. Obviously, $A = \bigoplus Te a_\lambda$ ($a_\lambda \in A$) where each a_λ induces a T -isomorphism of Te onto $Te a_\lambda$. Hence, by $(\sum t_\lambda e a_\lambda) \beta = \sum (t_\lambda e) \delta_1 \cdot a_\lambda = \sum t_\lambda \delta_1 \cdot e a_\lambda$ we can define $\beta \in \mathfrak{A}$. If $\alpha = \beta + a_l$ then for each $t \in T$ we have $(t_\lambda e a_\lambda) [t, \alpha] = t\delta \cdot (t_\lambda e a_\lambda)$, whence it follows $(t\delta)_l = [t, \alpha]$. Since δ is in $D(T, A/B)$, $0 = (b\delta)_l = [b, \alpha]$ for each $b \in B$, namely, α is contained in $V_{\mathfrak{A}}(B_l)$ that is the topological closure of $\mathfrak{G}A_r$ by the hypo-

thesis. If $T^* = T[\{e_{ij}'s\}]$ then $\text{Hom}_{B_t}(T^*, A) = (\mathfrak{G}|T^*)A_r = \bigoplus_i^s (\sigma_i|T^*)A_r$ ($\sigma_i \in \mathfrak{G}$) and $\alpha|T^*$ has the unique representation $\sum_i^s (\sigma_i|T^*)x_{ir}$ with $x_i \in A$ ([2, Lemma 1.3 (i)]), where we may assume that $\sigma_1, \dots, \sigma_{s'} \in \tilde{V}$ and $\sigma_{s'+1}|T^*, \dots, \sigma_s|T^* \notin \tilde{V}|T^*$. Now, let t be an arbitrary element of T . By $x\gamma_i = (tx)\sigma_i - t(x\sigma_i)$ ($x \in T^*$) we define the homomorphisms γ_i of T^* into A ($i=1, \dots, s$). Then, for each $y \in T^*$ we have $y_r\gamma_i = \gamma_i(y\sigma_i)_r$, and so it will be easy to see that each $\gamma_i A_r$ is $T_r^*-A_r$ -homomorphic to the irreducible module $(\sigma_i|T^*)A_r$. Hence, by [2, Lemma 1.3 (iv)], $(t\delta)_l|T^* = \sum_i^s \gamma_i x_{ir} = \sum_i^{s'} \gamma_i x_{ir}$. If we set $\sigma_i = \tilde{v}_i$ with $v_i \in V$ ($i=1, \dots, s'$) then $x\gamma_i = [v_i, t]xv_i^{-1}$, and so $(t\delta)_l|T^* = \sum_i^{s'} ([v_i, t]_l|T^*)y_{ir} = \sum_i^{s'} (y_{ir}|T^*)[v_i, t]_l$ where $y_i = v_i^{-1}x_i \in A$. Now, choose an arbitrary linearly independent left $V_A(T^*)$ -basis $\{u_\mu's\}$ of A such that $u_1 = 1$, and represent y_i in terms of this basis. Then, $(1_r|T^*)(t\delta)_l = (t\delta)_l|T^* = \sum_\mu (u_{\mu r}|T^*)[v'_\mu, t]_l$ with some $v'_\mu \in V$ determined independently of t . It follows therefore $t\delta = [v'_1, t] = t\delta_{-v'_1}$ by the proposition symmetric to [2, Lemma 1.4 (ii)].

Next, we shall prove the following partial extension of Theorem 1, that contains [2, Theorem 4.6] as well.

Theorem 2. *Let A be locally h -Galois and left locally finite over B , and T an f -regular intermediate ring of A/B . If δ is in $D(T, A/B)$ then $\delta = \delta_v|T$ for some $v \in V$.*

Proof. There exists a simple intermediate ring B' of T/B with $[B' : B]_l < \infty$ and $V_A(B') = V_A(T)$, and then $\delta|B' = \delta_v|B'$ for some $v \in V$ (Theorem 1). Since A/B' is locally h -Galois and left locally finite by [3, Corollary 1], T is an intermediate ring of $V_A^2(B')/B'$ and $\delta' = \delta - (\delta_v|T)$ is contained in $D(T, A/B')$, it suffices to prove that if T is contained in H then $\delta = 0$. To see this, we set $T = \cup T_\lambda$ where T_λ runs over all the simple intermediate rings of T/B with $[T_\lambda : B]_l < \infty$. Then, by Theorem 1, $\delta|T_\lambda = \delta_{v_\lambda}|T_\lambda$ with some $v_\lambda \in V$. Hence, we have $\delta|T_\lambda = 0$ for all λ , whence it follows $\delta = 0$.

References

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