ON THE ADMISSIBILITY OF FUNCTION SPACES

By

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A Hausdorff topological linear space S is said to be *admissible* if for any compact subset K of S and for any neighbourhood N of the origin of S there exists a continuous operator T on S such that its image TK is finite dimensional and

$$x \in N + Tx \qquad \text{for all} \quad x \in K.$$

The admissibility of a Hausdorff topological linear space has significance especially in its relation to the fix point property of the space. It is known that locally convex Hausdorff topological linear speces are admissible [7]. However, for the case that spaces do not satisfy the local convexity, it seems that the major part, in some sence, of the problem is left still unsolved ([2], [3]). So far, V. Klee [3] showed that certain extensive class of Hausdorff topological linear spaces which are not necessarily locally convex are admissible. And complementing this, T. Riedrich [9] proved that $L_p(0, 1)$ $(0<p<1)$ is admissible.

In this note, we will show the admissibility of certain function spaces which include L_p $(0<p<1)$ and, more generally, concave modular function spaces [8] as special cases.

1. Definitions and preliminaries.

Let $(\Omega, \mathfrak{B}, \mu)$ be a σ-finite measure space with a countably additive non-negative measure μ on a Borel field $\mathfrak{B}$ of Ω. For two measurable functions f and g on Ω we define a semi-order $f \leqq g$ by $f(t) \leqq g(t)$ a.e. on Ω as usually.

Let X be a *linear space* which consists of μ-measurable functions on Ω with a functional ρ defined on X satisfying the following:

(a) $0 \leqq \rho(f) < \infty$ *for all* $f \in X$ *and* $\rho(f)=0$ *if and only if* $f=0$;

(b) $|f|$[1] $\leqq |g|$ *with* $g \in X$ *implies* $f \in X$ *and* $\rho(f) \leqq \rho(g)$;

(c) $\rho(f+g) \leqq \rho(f)+\rho(g)$ *for all* $f, g \in X$;

(d)[2] *If* $\sum \rho(f_n) < \infty$ *then there exists* $f \in X$ *such that* $f(t) = \sum |f_n|(t)$ *a.e.*;

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1) $|f|(t) = |f(t)|$.

2) The completeness of the metric defined by ρ implies this condition.

(e)[3] *for any* $0 \leqq f \in X$ *and for any* $\{f_n\} \subset X$ *with* $|f_n| \leqq f$ $(n=1, 2, \cdots)$ $\lim \rho(f_n)=0$ *if and only if* $\{f_n\}$ *converges in measure to* 0 *on every set of* μ*-finite measure.*

The function space X is a metric linear space with the metric d:

$$d(f, g) = \rho(f-g) \qquad \text{for all} \quad f, g \in X.$$

Our purpose is in showing the admissibility of such defined metric linear function space (X, d) which is not necessarily locally convex in general. In this section, we give notations and preliminaries which need in the proof.

For a subset A of X, a finite subset $\{f_1, f_2, \cdots, f_n\}$ of A is said to be an *ε-net* of A if for any $f \in A$ there exists i $(1 \leqq i \leqq n)$ such that $\rho(f-f_i)<\varepsilon$.

Let K_0 be an arbitrary fixed compact subset of X in the following. Then, since K_0 is totally bounded, we have

(1) for any $\varepsilon>0$ there exists an ε-net of K_0.

In virtue of the σ-finiteness of μ and (1),

(2) there exists an element $0 \leqq g \in X$ such that

$$\chi_{E_0} f = f \qquad \text{for all} \quad f \in K_0$$

where χ_{E_0} is the characteristic function of $E_0=\{t;\ g(t) \neq 0\}$. In fact, there exists a dense countable subset $\{f_n\}$ in X by (1). Since $\alpha \downarrow 0$[4] implies $\rho(\alpha f_n) \downarrow 0$ for every fixed f_n by (b) and (e), there exists a sequence $0<\alpha_n$ $(n=1, 2, \cdots)$ such that $\sum \rho(\alpha_n f_n)<\infty$. Therefore there exists $g=\sup\{\alpha_n |f_n|\}$[5] in X by (d). For any $\varepsilon>0$ and for any $f \in K_0$ there exists a number n such that $\rho(f-f_n)<\varepsilon$ by definition. Therefore

$$\rho(f-\chi_{E_0} f) \leqq \rho(f-f_n)+\rho(f_n-\chi_{E_0} f) \leqq 2\varepsilon$$

which implies $\rho(f-\chi_{E_0} f)=0$ i.e. $f=\chi_{E_0} f$. Thus the existance of g is verified.

Let U_n $(n=1, 2, \cdots)$ be operators on X such that

$$U_n f = \sup\left\{\inf\{ng, f\},\ -ng\right\} \qquad \text{for all } f \in X \text{ and } n=1, 2, \cdots$$

where g is the same as in (2).

Since evidently by (2) $|U_n f| \leqq |f|$ $(n=1, 2, \cdots)$ and $\{U_n f\}$ converges to $\chi_{E_0} f$ a.e., we have by (e)

(3) $\qquad \lim \rho(U_n f-\chi_{E_0} f)=0.$

Since $|U_n f-U_n f'| \leqq |f-f'|$ for any $f, f' \in X$ we have

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3) This is equivalent to the following condition: $f_1 \geqq f_2 \geqq \cdots$ with $\lim f_n=0$ a.e. imply $\lim \rho(f_n)=0$.

4) $\alpha \downarrow 0$ means a monotonously decreasing convergent sequence of numbers whose limit is 0.

5) $(\sup f_n)(t)=\sup\{f_n(t)\}$, $(\inf f_n)(t)=\inf\{f_n(t)\}$.

$$\rho(U_n f - U_n f') \leqq \rho(f-f') \qquad \text{for all} \quad f, f' \in X$$

i.e.

(4) $\{U_n\}$ are equi-uniformly continuous in X.

For any $\varepsilon > 0$, if $\{f_1, f_2, \cdots, f_n\}$ is an ε-net of K_0, then for any $f \in K_0$ there exists i such that $\rho(f-f_i) < \varepsilon$ by definition. If n_0 is chosen suitably, we have by (2) and (3)

$$\rho(U_{n_0} f_i - f_i) < \varepsilon \qquad \text{for all} \quad i = 1, 2, \cdots, n\,.$$

Therefore by (4)

$$\rho(U_{n_0} f - f) \leqq \rho(U_{n_0} f - U_{n_0} f_i) + \rho(U_{n_0} f_i - f_i) + \rho(f_i - f) < 3\varepsilon\,.$$

Thus

(5) for any $\varepsilon > 0$, there exists n_0 such that

$$\rho(U_{n_0} f - f) < \varepsilon \qquad \text{for all} \quad f \in K_0\,.$$

Let $\varepsilon > 0$ be arbitrary fixed in the following. Corresponding to this ε, let n_0 be the number determined by (5). For this n_0, let

$$g_0 = n_0 g$$

where g is the same as in (2) and let

$$K_1 = U_{n_0} K_0 = \{U_{n_0} f\,;\ f \in K_0\}\,.$$

Then since K_1 is a continuous image of of the compact set K_0,

(6) K_1 is compact in X and $|f| \leqq g_0$ for all $f \in K_1$.

We consider another finite measure for the convenience of simpler treating of the problem. Since the measure μ is σ-finite, there exists a measurable function $0 < h(t)$ on Ω such that $\int_\Omega h d\mu < \infty$. We can define another measure ν from μ by

$$\nu(E) = \int_E h d\mu \qquad \text{for all} \quad E \in \mathfrak{B}\,.$$

Then $(\Omega, \mathfrak{B}, \nu)$ is a finite measure space. Two measures μ and ν are mutually equivalent, i.e., μ and ν are absolutely continuous each other.

We consider, with respect to ν, usual L_1 space on Ω and denote this by $L_1(\nu)$. Let

$$\|f\| = \int_\Omega |f| d\nu \qquad \text{for all} \quad f \in L_1(\nu)$$

then we have on $L_1(\nu)$ a norm $\|\cdot\|$ which satisfies all the corresponding conditions from (a) to (e) for ρ.

Let G and I be subsets of the metric space X and the normed space

$L_1(\nu)$ respectively such that

$$G=\left\{f;\ |f|\leqq g_0,\ f\in X\right\} \qquad \text{(with } d\text{-topology)}$$

and

$$I=\left\{f;\ |f(t)|\leqq 1\ (t\in\Omega),\ f\in L_1(\nu)\right\} \qquad \text{(with } \|\cdot\|\text{-topology)}.$$

Let V be an operator on G into I such that

$$Vf(t)=\begin{cases} f(t)/g_0(t) & \text{if } g_0(t)\neq 0 \\ 0 & \text{if } g_0(t)=0 \end{cases}$$

and conversely let W be an operator on I into G such that

$$Wf(t)=f(t)g_0(t).$$

We have evidently

(7) $$WVf=f \qquad \text{for all } f\in G.$$

Further

(8) both V and W are uniformly continuous.

In fact, if V is not uniformly continuous, for any $\varepsilon>0$ there exists two sequences $\{f_n\}$, $\{g_n\}\subset G$ such that $\lim \rho(f_n-g_n)=0$ but

(♯) $$\|(f_n-g_n)/g_0\|>\varepsilon \qquad \text{for all } n=1, 2, \cdots.$$

Then $\{f_n-g_n\}$ converges in measure to 0 on every set of μ-finite measure with respect to the measure μ by (e). Therefore $\{(f_n-g_n)/g_0\}$ converges in measure to 0 on every set of μ-finite measure with respect to the measure ν because μ and ν are mutually equivalent. Thus since $|f_n(t)-g_n(t)|/g_0(t)\leqq 2$ $(t\in\Omega)$ and by the corresponding propery (e) for $\|\cdot\|$ we have $\lim\|(f_n-g_n)/g_0\|=0$ which contradicts to (♯). Similarly, W is uniformly continuous too. Thus (8) is valid.

Let K_2 be a subset of $L_1(\nu)$ such that

$$K_2=VK_1=\{Vf;\ f\in K_1\}.$$

Then, since K_2 is also a continuous image of the compact set K_1,

(9) K_2 is compact in $L_1(\nu)$ and $K_2\subset I$.

Let π be any of such a disjoint partition $\{E_1, E_2, \cdots, E_n\}$ of Ω that

$$\Omega=E_1+E_2+\cdots+E_n, \qquad E_i\in\mathfrak{B} \qquad (i=1, 2, \cdots, n).$$

For above π, let Z_π be an operator on $L_1(\nu)$ into itself such that

$$Z_\pi f=\sum\left\{\nu(E_i)^{-1}\int_\Omega f\chi_{E_i}d\nu\right\}\chi_{E_i}$$

where χ_{E_i} is the characteristic function of E_i and $\sum$ means the adding up with respect to i with $\mu(E_i)>0$.

Evidently

(10) Z_π is a linear operator on $L_1(\nu)$, its image $Z_\pi L_1(\nu)$ is finite dimensional and Z_π maps I into itself.

Further, since K_2 is compact, we have that

(11) (the admissibility of $L_1(\nu)$) for any $\delta>0$ there exists a partition π such that

$$\|Z_\pi f-f\|<\delta \qquad \text{for all} \quad f\in K_2\,.$$

Since

$$WZ_\pi Vf=\sum\left\{\nu(E_i)^{-1}\int_\Omega(f\chi_{E_i}/g_0)d\nu\right\}g_0\chi_{E_i} \qquad \text{for all} \quad f\in G\,,$$

the finite subset $\{g_0\chi_{E_1}, g_0\chi_{E_2}, \cdots, g_0\chi_{E_n}\}$ of X spans the subset $WZ_\pi VG=\{WZ_\pi Vf;\, f\in G\}$, i.e.,

(12) $WZ_\pi VG$ is finite dimensional in X.

2. Theorem and its proof.

Theorem. *(X, d) is admissible, i.e., for any compact set $K_0\subset X$ and for any $\varepsilon>0$ there exists a continuous operator T on X such that*

(i) *the image TX is finite dimensional;*

(ii) $\quad d(Tf, f)\leqq 2\varepsilon$ *for all $f\in K_0$.*

Proof. For an arbitrary given pair of a compact set $K_0\subset X$ and $\varepsilon>0$ we determine U_n such that (5) is valid. Next, by (9), corresponding to this ε, we select $\delta>0$ such that

$$\|f-f_1\|<\delta \quad (f, f_1\in I) \text{ implies } \rho(Wf-Wf_1)<\varepsilon\,.$$

Corresponding to such δ, we determine Z_π such that (11) is valid. Now, let T be an operator on X into itself such that

$$T=WZ_\pi VU_n\,.$$

Then (i) is evident by (12). For (ii), we can calculate as follows

$$\begin{aligned}\rho(Tf-f)&\leqq\rho(WZ_\pi VU_nf-U_nf)+\rho(U_nf-f)\\&\leqq\rho(WZ_\pi VU_nf-U_nf)+\varepsilon && \text{(by (5))}\\&=\rho(WZ_\pi VU_nf-WVU_nf)+\varepsilon && \text{(by (7))}\\&\leqq\sup\left\{\rho(WZ_\pi f'-Wf');\, f'\in K_2\right\}+\varepsilon\leqq 2\varepsilon\end{aligned}$$

because we chose Z_π such that $\|Z_\pi f'-f'\|<\delta$ for all $f'\in K_2$. Thus

$$\rho(Tf-f)\leqq 2\varepsilon \qquad \text{for all} \quad f\in K_0$$

which leads the desired result. Q. E. D.

3. General examples.

We introduce, as an example of (X, d), so-called concave modular function spaces defined by H. Nakano [8].

Let $M(\xi, t)$ ($\xi \in [0, \infty)$, $t \in [0, 1]$) be a *non-negative real valued function* with the following properties:

(M 1) *$M(\xi, t)$ is a non-decreasing concave function of ξ for every fixed* $t \in [0, 1]$;

(M 2) *$M(\xi, t)$ is measurable as a function of t for every fixed $\xi \in [0, \infty]$ with $M(0, t)=0$ a.e. on* $[0, 1]$.

We denote by $L_{M(\xi,t)}$ the class of all measurable functions $f(t)$ on $[0, 1]$ such that

$$\int_0^1 M(|f(t)|, t)dt < \infty .$$

If we define a functional ρ_M on $L_{M(\xi,t)}$ by

$$\rho_M(f) = \int_0^1 M(|f(t)|, t)dt \qquad \text{for all} \quad f \in L_{M(\xi,t)} ,$$

$L_{M(\xi,t)}$ is a metric linear space with a metric $d_M(f, g)=\rho_M(f-g)$ and it is not locally convex. $L_{M(\xi,t)}$ with the functional ρ_M is said to be a *concave modular function space.*

As is easily verified, ρ_M satisfies the condition (e) of ρ and all the other conditions from (a) to (d) of ρ are valid for ρ_M too. Therefore the function space $(L_{M(\xi,t)}, d_M)$ is an example of (X, d). If $M(\xi, t)=\Phi(\xi)$ or $M(\xi, t)=\xi^{p(t)}$ $(0<p(t)<1)$ then $L_{M(\xi,t)}$ is denoted by $L_{\Phi(\xi)}$ ($\Phi(\xi)$ is concave) or $L_{p(t)}$ $(0<p(t)<1)$ respectively. They are two typical special cases of concave modular function spaces. Moreover if especially $M(\xi, t)=\Phi(\xi)=\xi^{p(t)}=\xi^p$ $(0<p<1)$ then $L_{M(\xi,t)}$ is no other than the space L_p $(0<p<1)$.

We can consider $L_{M(\xi,t)}$ in more general form, as an example of (X, d), from the stand point of the theory of quasi-modular spaces, i.e., the concavity in the condition (M1) of $M(\xi, t)$ is weaken by the non-decreasing left-hand continuity of $M(\xi, t)$. In this case, however, ρ_M must be replaced by another functional $|||\cdot|||$ which is defined by ρ_M such that

$$|||f||| = \inf\left\{\varepsilon ;\ \rho_M(f/\varepsilon) \leqq \varepsilon\right\} \qquad \text{for all} \quad f \in L_{M(\xi,t)} .$$

Then $|||\cdot|||$ is a complete quasi-norm (Fréchet-norm) on $L_{M(\xi,t)}$ (see [1], [4], [6]). $L_{M(\xi,t)}$ is naturally not locally convex in general as a topological linear space with $|||\cdot|||$-topology (cf. [1], [5]). $L_{M(\xi,t)}$ with the functionals ρ_M and $|||\cdot|||$ is said to be a *quasi-modular function space.*

$|||\cdot|||$ satisfies also the conditions from (a) to (d). Moreover if $M(\xi, t)$ satisfies some additional property[6], then $|||\cdot|||$ satifies (e) too. Thus such a (special) quasi-modular function space $(L_{M(\xi,t)}, |||\cdot|||)$ is also an example of (X, d).

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6) There exists a number $\gamma>0$ and a function $f_0(t)\in L_1[0, 1]$ such that $M(2\xi, t)\leqq\gamma M(\xi, t)+f_0(t)$ for all $\xi\geqq 0$ and a.e. $t\in[0, 1]$ (cf. [4]).