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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 19(1): 049-055</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1965</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56065">http://hdl.handle.net/2115/56065</a></td>
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<tr>
<td>Type</td>
<td>bulletin (article)</td>
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<tr>
<td>File Information</td>
<td>JFSHIU_19_N1_049-055.pdf</td>
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**Summary**

The document discusses the admissibility of function spaces, which are important in the study of functional analysis and its applications in various fields of mathematics. The author, Ishii, Jyun, presents a detailed analysis of the conditions under which function spaces can be considered admissible, contributing to the theoretical framework of functional analysis.
ON THE ADMISSIBILITY OF FUNCTION SPACES

By

Jyun ISHII*)

A Hausdorff topological linear space $S$ is said to be admissible if for any compact subset $K$ of $S$ and for any neighbourhood $N$ of the origin of $S$ there exists a continuous operator $T$ on $S$ such that its image $TK$ is finite dimensional and

$$x \in N + Tx \quad \text{for all} \quad x \in K.$$  

The admissibility of a Hausdorff topological linear space has significance especially in its relation to the fix point property of the space. It is known that locally convex Hausdorff topological linear spaces are admissible [7]. However, for the case that spaces do not satisfy the local convexity, it seems that the major part, in some sense, of the problem is left still unsolved ([2], [3]). So far, V. Klee [3] showed that certain extensive class of Hausdorff topological linear spaces which are not necessarily locally convex are admissible. And complementing this, T. Riedrich [9] proved that $L_p(0, 1)$ ($0 < p < 1$) is admissible.

In this note, we will show the admissibility of certain function spaces which include $L_p$ ($0 < p < 1$) and, more generally, concave modular function spaces [8] as special cases.

1. Definitions and preliminaries.

Let $(\Omega, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space with a countably additive non-negative measure $\mu$ on a Borel field $\mathcal{B}$ of $\Omega$. For two measurable functions $f$ and $g$ on $\Omega$ we define a semi-order $f \leq g$ by $f(t) \leq g(t)$ a.e. on $\Omega$ as usually.

Let $X$ be a linear space which consists of $\mu$-measurable functions on $\Omega$ with a functional $\rho$ defined on $X$ satisfying the following:

(a) $0 \leq \rho(f) < \infty$ for all $f \in X$ and $\rho(f) = 0$ if and only if $f = 0$;

(b) $|f| \leq |g|$ with $g \in X$ implies $f \in X$ and $\rho(f) \leq \rho(g)$;

(c) $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in X$;

(d) If $\sum \rho(f_n) < \infty$ then there exists $f \in X$ such that $f(t) = \sum |f_n|(t)$ a.e.;

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1) $|f|(t) = |f(t)|$.

2) The completeness of the metric defined by $\rho$ implies this condition.
(e) for any $0 \leq f \in X$ and for any $\{f_n\} \subset X$ with $|f_n| \leq f$ $(n=1, 2, \ldots)$, \[ \lim \rho(f_n) = 0 \] if and only if $\{f_n\}$ converges in measure to 0 on every set of $\mu$-finite measure.

The function space $X$ is a metric linear space with the metric $d$:
\[ d(f, g) = \rho(f - g) \] for all $f, g \in X$.

Our purpose is in showing the admissibility of such defined metric linear function space $(X, d)$ which is not necessarily locally convex in general. In this section, we give notations and preliminaries which need in the proof.

For a subset $A$ of $X$, a finite subset $\{f_1, f_2, \ldots, f_n\}$ of $A$ is said to be an $\epsilon$-net of $A$ if for any $f \in A$ there exists $i$ $(1 \leq i \leq n)$ such that $\rho(f - f_i) < \epsilon$.

Let $K_0$ be an arbitrary fixed compact subset of $X$ in the following. Then, since $K_0$ is totally bounded, we have

(1) for any $\epsilon > 0$ there exists an $\epsilon$-net of $K_0$.

In virtue of the $\sigma$-finiteness of $\mu$ and (1),

(2) there exists an element $0 \leq g \in X$ such that
\[ \chi_{E_0}f = f \] for all $f \in K_0$ where $\chi_{E_0}$ is the characteristic function of $E_0 = \{t; g(t) \neq 0\}$. In fact, there exists a dense countable subset $\{f_n\}$ in $X$ by (1). Since $\alpha \downarrow 0$ implies $\rho(\alpha f_n) \downarrow 0$ for every fixed $f_n$ by (b) and (e), there exists a sequence $0 < \alpha_n (n=1, 2, \ldots)$ such that $\sum \rho(\alpha_n f_n) < \infty$. Therefore there exists $g = \sup \{\alpha_n |f_n|\}$ in $X$ by (d).

For any $\epsilon > 0$ and for any $f \in K_0$, there exists a number $n$ such that $\rho(f - f_n) < \epsilon$ by definition. Therefore
\[ \rho(f - \chi_{E_0}f) \leq \rho(f - f_n) + \rho(f_n - \chi_{E_0}f) \leq 2\epsilon \]
which implies $\rho(f - \chi_{E_0}f) = 0$ i.e. $f = \chi_{E_0}f$. Thus the existence of $g$ is verified.

Let $U_n (n=1, 2, \ldots)$ be operators on $X$ such that
\[ U_n f = \sup \{\inf \{ng, f\}, -ng\} \] for all $f \in X$ and $n=1, 2, \ldots$

where $g$ is the same as in (2).

Since evidently by (2) $|U_n f| \leq |f|$ $(n=1, 2, \ldots)$ and $\{U_n f\}$ converges to $\chi_{E_0}f$ a.e., we have by (e)

(3) \[ \lim \rho(U_n f - \chi_{E_0}f) = 0. \]

Since $|U_n f - U_n f'| \leq |f - f'|$ for any $f, f' \in X$ we have

3) This is equivalent to the following condition:
\[ f_1 \geq f_2 \geq \cdots \] with $\lim f_n = 0$ a.e. imply $\lim \rho(f_n) = 0$.

4) $\alpha \downarrow 0$ means a monotonously decreasing convergent sequence of numbers whose limit is 0.

5) $\sup f_n \{t\} = \sup \{f_n \{t\}\}$, $(\inf f_n) \{t\} = \inf \{f_n \{t\}\}$.
On the Admissibility of Function Spaces

\[ \rho(U_n f - U_n f') \leq \rho(f - f') \quad \text{for all } f, f' \in X \]
i.e.

\[ \{U_n\} \text{ are equi-uniformly continuous in } X. \]

For any \( \varepsilon > 0 \), if \( \{f_1, f_2, \ldots, f_n\} \) is an \( \varepsilon \)-net of \( K_0 \), then for any \( f \in K_0 \) there exists \( i \) such that \( \rho(f - f_i) < \varepsilon \) by definition. If \( n_0 \) is chosen suitably, we have by (2) and (3)

\[ \rho(U_{n_0} f_i - f_i) < \varepsilon \quad \text{for all } i = 1, 2, \ldots, n. \]

Therefore by (4)

\[ \rho(U_{n_0} f - f) \leq \rho(U_{n_0} f - U_{n_0} f_i) + \rho(U_{n_0} f_i - f_i) + \rho(f_i - f) < 3 \varepsilon. \]

Thus

\[ \rho(U_{n_0} f - f) < \varepsilon \quad \text{for all } f \in K_0. \]

Let \( \varepsilon > 0 \) be arbitrary fixed in the following. Corresponding to this \( \varepsilon \), let \( n_0 \) be the number determined by (5). For this \( n_0 \), let

\[ g_0 = n_0 g \]

where \( g \) is the same as in (2) and let

\[ K_1 = U_{n_0} K_0 = \{U_{n_0} f; f \in K_0\}. \]

Then since \( K_1 \) is a continuous image of \( K_0 \),

\[ K_1 \text{ is compact in } X \text{ and } |f| \leq g_0 \text{ for all } f \in K_1. \]

We consider another finite measure for the convenience of simpler treating of the problem. Since the measure \( \mu \) is \( \sigma \)-finite, there exists a measurable function \( 0 < h(t) \) on \( \Omega \) such that \( \int_X h d\mu < \infty. \) We can define another measure \( \nu \) from \( \mu \) by

\[ \nu(E) = \int_E h d\mu \quad \text{for all } E \in \mathcal{B}. \]

Then \( (\Omega, \mathcal{B}, \nu) \) is a finite measure space. Two measures \( \mu \) and \( \nu \) are mutually equivalent, i.e., \( \mu \) and \( \nu \) are absolutely continuous each other.

We consider, with respect to \( \nu \), usual \( L_1 \) space on \( \Omega \) and denote this by \( L_1(\nu) \). Let

\[ \|f\| = \int_\Omega |f| d\nu \quad \text{for all } f \in L_1(\nu) \]

then we have on \( L_1(\nu) \) a norm \( \| \cdot \| \) which satisfies all the corresponding conditions from (a) to (e) for \( \rho \).

Let \( G \) and \( I \) be subsets of the metric space \( X \) and the normed space
Let $V$ be an operator on $G$ into $I$ such that
\[ Vf(t) = \begin{cases} 
  f(t)/g_0(t) & \text{if } g_0(t) \neq 0 \\
  0 & \text{if } g_0(t) = 0
\end{cases} \]
and conversely let $W$ be an operator on $I$ into $G$ such that
\[ Wf(t) = f(t)g_0(t) \].

We have evidently
\[ WVf = f \quad \text{for all } f \in G. \] (7)

Further
\[ \text{(8) both } V \text{ and } W \text{ are uniformly continuous.} \]

In fact, if $V$ is not uniformly continuous, for any $\varepsilon > 0$ there exists two sequences $\{f_n\}, \{g_n\} \subset G$ such that $\lim \rho(f_n - g_n) = 0$ but
\[ \|((f_n - g_n)/g_0)\| > \varepsilon \quad \text{for all } n = 1, 2, \cdots. \] (\#)

Then $\{f_n - g_n\}$ converges in measure to 0 on every set of $\mu$-finite measure with respect to the measure $\mu$ by (e). Therefore $\{(f_n - g_n)/g_0\}$ converges in measure to 0 on every set of $\mu$-finite measure with respect to the measure $\nu$ because $\mu$ and $\nu$ are mutually equivalent. Thus since $|f_n(t) - g_n(t)|/g_0(t) \leq 2$ ($t \in \Omega$) and by the corresponding propery (e) for $\|\cdot\|$ we have $\lim \|((f_n - g_n)/g_0)\| = 0$ which contradicts to (\#). Similarly, $W$ is uniformly continuous too. Thus (8) is valid.

Let $K_2$ be a subset of $L_1(\nu)$ such that
\[ K_2 = VK_1 = \{Vf; f \in K_1\}. \]

Then, since $K_2$ is also a continuous image of the compact set $K_1$,
\[ \text{(9) } K_2 \text{ is compact in } L_1(\nu) \text{ and } K_2 \subset I. \]

Let $\pi$ be any of such a disjoint partition $\{E_1, E_2, \cdots, E_n\}$ of $\Omega$ that
\[ \Omega = E_1 + E_2 + \cdots + E_n, \quad E_i \in \mathfrak{B} \quad (i = 1, 2, \cdots, n). \]

For above $\pi$, let $Z_\pi$ be an operator on $L_1(\nu)$ into itself such that
\[ Z_\pi f = \sum \{ \nu(E_i)^{-1} \int_0 f \chi_{E_i} d\nu \} \chi_{E_i}. \]
where $\chi_{E_i}$ is the characteristic function of $E_i$ and $\sum$ means the adding up with respect to $i$ with $\mu(E_i) > 0$.

Evidently

(10) $Z_\varepsilon$ is a linear operator on $L_1(\nu)$, its image $Z_\varepsilon L_1(\nu)$ is finite dimensional and $Z_\varepsilon$ maps $I$ into itself.

Further, since $K_0$ is compact, we have that

(11) (the admissibility of $L_1(\nu)$) for any $\delta > 0$ there exists a partition $\pi$ such that

$$\|Z_\varepsilon f - f\| < \delta \quad \text{for all } f \in K_0.$$

Since

$$WZ_\varepsilon Vf = \sum \left\{ \nu(E_i)^{-1} \int g \chi_{E_i} / g_{0} \, d\nu \right\} g_{0} \chi_{E_0} \quad \text{for all } f \in G,$$

the finite subset $\{g_{0} \chi_{E_1}, g_{0} \chi_{E_2}, \ldots, g_{0} \chi_{E_n}\}$ of $X$ spans the subset $WZ_\varepsilon VG = \{WZ_\varepsilon Vf; f \in G\}$, i.e.,

(12) $WZ_\varepsilon VG$ is finite dimensional in $X$.

2. Theorem and its proof.

**Theorem.** $(X, d)$ is admissible, i.e., for any compact set $K_0 \subset X$ and for any $\varepsilon > 0$ there exists a continuous operator $T$ on $X$ such that

(i) the image $TX$ is finite dimensional;

(ii) $d(Tf, f) \leq 2\varepsilon$ for all $f \in K_0$.

**Proof.** For an arbitrary given pair of a compact set $K_0 \subset X$ and $\varepsilon > 0$ we determine $U_n$ such that (5) is valid. Next, by (9), corresponding to this $\varepsilon$, we select $\delta > 0$ such that

$$\|f - f_1\| < \delta \quad (f, f_1 \in I) \quad \text{implies} \quad \rho(Wf - Wf_1) < \varepsilon.$$

Corresponding to such $\delta$, we determine $Z_\varepsilon$ such that (11) is valid. Now, let $T$ be an operator on $X$ into itself such that

$$T = WZ_\varepsilon VU_n.$$

Then (i) is evident by (12). For (ii), we can calculate as follows

$$\rho(Tf - f) \leq \rho(WZ_\varepsilon VU_n f - U_n f) + \rho(U_n f - f)$$

$$\leq \rho(WZ_\varepsilon VU_n f - U_n f) + \varepsilon \quad \text{(by (5))}$$

$$= \rho(WZ_\varepsilon VU_n f - WVU_n f) + \varepsilon \quad \text{(by (7))}$$

$$\leq \sup \{\rho(WZ_\varepsilon f' - Wf'); f' \in K_0\} + \varepsilon \leq 2\varepsilon$$

because we chose $Z_\varepsilon$ such that $\|Z_\varepsilon f' - f'\| < \delta$ for all $f' \in K_0$. Thus

$$\rho(Tf - f) \leq 2\varepsilon \quad \text{for all } f \in K_0.$$
which leads the desired result. 

Q. E. D.

3. General examples.

We introduce, as an example of \((X, d)\), so-called concave modular function spaces defined by H. Nakano [8].

Let \(M(\xi, t) \ (\xi \in [0, \infty), \ t \in [0, 1])\) be a non-negative real valued function with the following properties:

(M1) \(M(\xi, t)\) is a non-decreasing concave function of \(\xi\) for every fixed \(t \in [0, 1]\);

(M2) \(M(\xi, t)\) is measurable as a function of \(t\) for every fixed \(\xi \in [0, \infty]\) with \(M(0, t) = 0\) a.e. on \([0, 1]\).

We denote by \(L_{M(\xi,t)}\) the class of all measurable functions \(f(t)\) on \([0, 1]\) such that

\[\int_0^1 M(|f(t)|, t) dt < \infty.\]

If we define a functional \(\rho_M\) on \(L_{M(\xi,t)}\) by

\[\rho_M(f) = \int_0^1 M(|f(t)|, t) dt \quad \text{for all } f \in L_{M(\xi,t)},\]

\(L_{M(\xi,t)}\) is a metric linear space with a metric \(d_M(f, g) = \rho_M(f - g)\) and it is not locally convex. \(L_{M(\xi,t)}\) with the functional \(\rho_M\) is said to be a concave modular function space.

As is easily verified, \(\rho_M\) satisfies the condition (e) of \(\rho\) and all the other conditions from (a) to (d) of \(\rho\) are valid for \(\rho_M\) too. Therefore the function space \((L_{M(\xi,t)}, d_M)\) is an example of \((X, d)\). If \(M(\xi, t) = \Phi(\xi)\) or \(M(\xi, t) = \xi^{p(t)}\) \((0 < p(t) < 1)\) then \(L_{M(\xi,t)}\) is denoted by \(L_{\Phi(\xi)}\) \((\Phi(\xi)\) is concave\) or \(L_{p(t)}\) \((0 < p(t) < 1)\) respectively. They are two typical special cases of concave modular function spaces. Moreover if especially \(M(\xi, t) = \Phi(\xi) = \xi^{p(t)} = \xi^p\) \((0 < p < 1)\) then \(L_{M(\xi,t)}\) is no other than the space \(L_p\) \((0 < p < 1)\).

We can consider \(L_{M(\xi,t)}\) in more general form, as an example of \((X, d)\), from the stand point of the theory of quasi-modular spaces, i.e., the concavity in the condition (M1) of \(M(\xi, t)\) is weaken by the non-decreasing left-hand continuity of \(M(\xi, t)\). In this case, however, \(\rho_M\) must be replaced by another functional \(\| \cdot \|\) which is defined by \(\rho_M\) such that

\[\| f \| = \inf \{ \varepsilon : \rho_M(f/\varepsilon) \leq \varepsilon \} \quad \text{for all } f \in L_{M(\xi,t)}.\]

Then \(\| \cdot \|\) is a complete quasi-norm (Fréchet-norm) on \(L_{M(\xi,t)}\) (see [1], [4], [6]). \(L_{M(\xi,t)}\) is naturally not locally convex in general as a topological linear space with \(\| \cdot \|\)-topology (cf. [1], [5]). \(L_{M(\xi,t)}\) with the functionals \(\rho_M\) and \(\| \cdot \|\) is said
to be a quasi-modular function space.

\( \| \cdot \| \) satisfies also the conditions from (a) to (d). Moreover if \( M(\xi, t) \) satisfies some additional property\(^6\), then \( \| \cdot \| \) satisfies (e) too. Thus such a (special) quasi-modular function space \( (L_{M(\xi, t)}, \| \cdot \|) \) is also an example of \( (X, d) \).

Finally, the author wishes to thank Professors I. Amemiya, T. Ito, T. Ando and T. Shimogaki for their valuable advices.

References


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(Received February 19, 1965)

\(^6\) There exists a number \( \tau > 0 \) and a function \( f_0(t) \in L_1[0, 1] \) such that \( M(2\xi, t) \leq \tau M(\xi, t) + f_0(t) \) for all \( \xi \geq 0 \) and a.e. \( t \in [0, 1] \) (cf. [4]).