ON THE EXISTENCE OF FUNCTIONS OF EVANS'S TYPE

by

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We proved the following

Theorem[1]. *Let R be a Riemann surface with null-boundary. Then there exists a harmonic function $U(z)$ such that $U(z)$ has a negative logarithmic pole at $p \in R$ and $U(z) \to \infty$ as z tends to the ideal boundary.*

Recently M. Nakai extended the above theorem as follows:

Theorem[2]. *Let R be a Riemann surface with positive boundary. Let $G(z, p)$ be the Green's function of R with pole at p. Put $G_\delta = E[z \in R: G(z, p) > \delta]$. Then there exists a harmonic function in R such that $D(\min(M, U(z))) \leqq M\alpha < \infty$ and $U(z) \to \infty$ as z tends of the boundary of R in G_δ for any $\delta > 0$ and that any positive harmonic function $V(z)(\leqq U(z))$ must be zero, where α is a conotant.*

We consider the existence of functions of Evans's type for more general sets and obtain some results which contain the above two theorems as their special applications.

1. Generalized Green's function[3]. Let R be a Riemann surface with positive boundary. Let R_n $(n=1, 2, \cdots)$ be its exhaustion with compact relative boundary ∂R_n. Let G[4] be a subsurface in R. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i} - (G \cap (R - R_n))$ such that $w_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G$ and $w_{n,n+i}(z) = 1$ on $G \cap (R - R_n)$. We call $\lim_n \lim_i w_{n,n+i}(z)$ the harmonic measure (H.M.) of the boundary $(B \cap G)$ determined by G and denote it by $w(G \cap B, z, R)$. As for a set F in R. Let $w(F, z, R)$ be the least positive harmonic function in $R - F$ and $=1$ on F. We call $w(F, z, R)$ H.M. (harmonic measure) of F. Let $G_1 \supset G_2$ be subdomains in R. Let $\omega_{n,n+i}(z)$ be a harmonic function

1) Z. KURAMOCHI: Mass distributions on the ideal boundaries of abstract Riemann surfaces, I. Osaka Math. J., 8, 119–138 (1956).

2) M. NAKAI: Green potential of Evans type on Royden's compactification of a Riemann surface. Nagoya Math. J., 24, 205–239 (1964).

3) Z. KURAMOCHI: On harmonic functions representable by Poisson's integral. Osaka Math. J., 10, 103–117 (1958).

4) In this paper relative boundary ∂G of a subsurface G consists of enumberably infinite number of analytic curves clustering nowhere in R.

in $(G_1 \cap R_{n+i})-(G_2 \cap (R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on ∂G_1, $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}-G_2$, $\omega_{n,n+i}(z)=1$ on $G_2 \cap (R_{n+i}-R_n)$. If there exists a const. M such that $D(\omega_{n,n+i}(z)) \leqq M$. Then $\omega_{n,n+i}(z)$ [5] $\Rightarrow \omega_n(z)$ as $i \to \infty$ and $\omega_n(z) \Rightarrow$ to a function denoted by $\omega(G_2 \cap B, z, G_1)$ as $n \to \infty$ which is called C.P. (capcitary potential) of $(G_2 \cap B)$ relative to G_1.

Let $V(z)$ be a positive harmonic function in R except at most a set of capacity zero where $V(z)$ may be infinite. Put $G_M = E[z \in R: V(z) \geqq M]$. If $w(G_M \cap B, z, R)=0$ for any M and $D(\min(M, V(z)) \leqq M\alpha$ for any $M<\infty$, we call $V(z)$ a generalized Green's function, where α is a const.

We proved the following

Lemma 1[6]. *Let R be a Riemann surface with null-boundary. Let R_0 be a compact disc in R. Let G_i be a domain in which $U_i(z)$ is harmonic and $D_{G_i}(U_i(z))<\infty$ $(i=1, 2, \cdots, i_0)$. Then there exists a sequence of compact curves $\{\gamma_n\}$ such that γ_n separates the boundary B of R from R_0, $\{\gamma_n\}$ clusters at B and that $\int_{\gamma_n} \left|\frac{\partial}{\partial n} U_i(z)\right| ds$ tends to zero as $n \to \infty$, for every i.*

We shall prove

Theorem 1. *Let $V(z)$ be a G.G. (generalized Green's function) such that $D(\min(M, V(z))) \leqq M\alpha$. Then*

1) *Put $G_M = E[z \in R: V(z) > M]$ and let $w(G_M, z, R)$ be H.M. of G_M. Then $V(z) = Mw(G_M, z, R)$ in $R-G_M$ and $D(\min(M, V(z))) = M^2 D(w(G_M, z, R))$.*

2) *Let $\delta < M$. Then $\omega(G_M \cap B, z, G_\delta)=0$. Let $\hat{G}_M$ be the symmetric surface of G_M with respect to ∂G_M. Identify ∂G_M and the image $\partial \hat{G}_M$ of ∂G_M. Then we have a Riemann surface $G_M + \hat{G}_M$ called the doubled surface of G_M. Then $G_M + \hat{G}_M$ is a Riemann surface witn null-boundary.*

3) $\int_{C_M} \frac{\partial}{\partial n} V(z) ds = K$ *and* $D(\min(M, V(z))) = KM$ *for any M, where* $C_M = E[z \in R: V(z) = M]$.

4) *If $V(z)>0$, then* $\sup V(z) = \infty$.

5) *Let $V'(z)$ be a positive harmonic function except a set of capacity zero such that $V'(z) \leqq V(z)$. Then $V'(z)$ is also a G.G. and $D(\min(M, V'(z))) \leqq D(\min(M, V(z)))$.*

6) *Let $V_n(z)$ $(n=1, 2, \cdots)$ be a G.G. which is harmonic in R and $D(\min(M, V(z))) \leqq M\alpha_n$. If $V_n(z) \uparrow$ and $\alpha_n \leqq \alpha$, then the limit function $V(z)$*

5) Z. KURAMOCHI: Potentials on Riemann surfaces. J. Fac. Sci. Hokkaido Univ., 16, 5–79 (1962).

6) See (1).

of $\{V_n(z)\}$ *is a G.G..*

7) *Let* $G(z,p)$ *be a Green's function of* R *and put* $R_\delta=E[z\in R: G(z,p)>\delta]$. *Let* $U(z)$ *be a G.G. in* R_δ *such that* $D(\min(M,U(z)))\leqq M\alpha$. *Let* $U_M(z)$ *be the least positive harmonic function larger than* $U(z)$ *in* $R-G_M$, *where* $G_M=E[z\in R: U(z)\geqq M]$. *Then* $U_M(z)\uparrow\widetilde{U}(z)$ *as* $M\to\infty$ *and* $\widetilde{U}(z)$ *is a G.G. in* R *such that* $D(\min(M,\widetilde{U}(z)))\leqq M\alpha$.

Proof of 1). *Let* $G_\varepsilon=E[z\in R: V(z)>\varepsilon]$ *and* $G_M=E[z\in R: V(z)>M]$ *and let* $R'=G_\varepsilon-G_M$: $M>\varepsilon$. Let $w_n(z)$ be the least positive harmonic function in $R'\cap R_n$ such that $w_n(z)=1$ on $G_\varepsilon\cap\partial R_n$. Put $U_n(z)=w(G_M,z,R)+Mw_n(z)+\varepsilon$. *Then*

$$V(z)\leqq Mw(G_M,z,R)+Mw_n(z)+\varepsilon=U_n(z)\,.$$

Because $V(z)=\varepsilon\leqq U_n(z)$ on $\partial G_\varepsilon\cap R_n$, $V(z)\leqq M\leqq U(z)$ on $\partial R_n\cap G_\varepsilon$ and $V(z)=M=Mw(G_M,z,R)$ on G_M. Let $n\to\infty$. Then $w_n(z)\to w(G_\varepsilon\cap B,z,R')\leqq w(G_\varepsilon\cap B,z,R)=0$. Let $\varepsilon\to 0$. Then $V(z)\leqq Mw(G_M,z,R)$ in $R-G_M$. On the other hand, by the defintion $Mw(G_M,z,R)\leqq V(z)$. Thus $V(z)=Mw(G_M,z,R)$ in $R-G_M$ and $D(\min(M,V(z)))=M^2D(w(G,z,R))$.

Proof of 2). Let $V_{n+i}(z)$ be a harmonic function in $(G_\varepsilon-G_M)\cap R_{n+i}$ such that $V_{n+i}(z)=\varepsilon$ on $\partial G_\varepsilon\cap R_{n+i}$, $V_{n+i}(z)=M$ on $\partial G_M\cap R_{n+i}$ and $\dfrac{\partial}{\partial n}V_{n+i}(z)=0$ on $\partial R_{n+i}\cap(G_\varepsilon-G_M)$. Then $D(V_{n+i}(z))\leqq D(\min(M,V(z)))$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $R_{n+i}\cap G_\varepsilon-(G_M\cap(R-R_n))$ such that $\omega_{n,n+i}(z)=0$ on $\partial G_\varepsilon\cap R_{n+i}$, $\omega_{n,n+i}(z)=1$ on $G_M\cap(R_{n+i}-R_n)$ and $\dfrac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}-G_M$. Then

$$D(\omega_{n,n+i}(z))\leqq D\left(\frac{V_{n+i}(z)-\varepsilon}{M-\varepsilon}\right)\leqq\frac{D(\min(M,V(z)))}{(M-\varepsilon)^2}<\infty\,.$$

Hence $\omega_{n,n+i}(z)\Rightarrow\omega_n(z)$ as $i\to\infty$ and $\omega_n(z)\Rightarrow\omega(z)=\omega(G_MB,z,G_\varepsilon)$ as $n\to\infty$. $0\leqq\omega(z)\leqq w(G_\varepsilon\cap B,z,R)=0$. Whence $\omega(G_M\cap B,z,G_\varepsilon)=0$. Let $\omega'_n(z)$ be a harmonic function in $(G_M\cap(R_n-R_{n_0}))+(G_\varepsilon\cap R_{n_0})$ such that $\omega'_n(z)=0$ on F (where F is a compact arc on $\partial G_M\cap R_{n_0}$), $\omega'_n(z)=1$ on $G_M\cap(R-R_n)$ and $\omega'_n(z)$ has minimal Dirichlet integral. Then clearly $\dfrac{\partial}{\partial n}\omega'_n(z)=0$ on $(\partial G_\varepsilon\cap R_{n_0}-F)+(\partial G_M\cap(R_n-R_{n_0})+(\partial R_{n_0}\cap(G_\varepsilon-G_M))$ and $D(\omega'_n(z))\leqq D(\omega_n(z))$. Also it is proved that $\omega'_n(z)$ converges in mean to $\omega'(z)$. Clearly $D(\omega'(z))\leqq D(\omega(z))=0$. This implies that the doubled surface of $(G_M\cap(R-R_{n_0}))+(G_\varepsilon\cap R_n)$ is a Riemann surface with null-boundary. We shall show that $G_M+\widehat{G}_M$ is a Riemann surface with null-boundary. Let $\omega''_n(z)$ be a harmonic function in $G_M\cap R_n$ such that

$\omega_n''(z)=0$ on F', $\omega_n''(z)=1$ on $G_M\cap(R-R_n)$ and $\frac{\partial}{\partial n}\omega_n''(z)=0$ on $(\partial G_M\cap R_{n_0})-F'$, where F' is a compact arc on $\partial G_M\cap R_{n_0}$. Then clearly $\omega_n''(z)\Rightarrow\omega''(z)$ as $n\to\infty$ and $D(\omega''(z))<\infty$. Now G_M is a domain in the Riemann surface R^* with null-boundary (R^* is the doubled surface) of $(G_M\cap(R-R_{n_0}))+G_\varepsilon\cap R_{n_0})$. Hence by Lemma 1 there exists a sequence of compact surfaces R_m^* $(m=1,2,\cdots)$ of R_m^* such that $\bigcup_m R_m^*=R^*$ and $\int_{\partial R_m^*\cap G_M}\left|\frac{\partial}{\partial n}\omega''(z)\right|ds\to 0$ as $m\to\infty$. Now $D(\omega''(z))$ $=\int_{F'}\frac{\partial}{\partial n}\omega''(z)ds=\int_{\partial R_m^*\cap G_M}\frac{\partial}{\partial n}\omega''(z)ds$, whence $\omega''(z)=0$. I.e. $G_M+\hat{G}_M$ is a Riemann surface with null-boundary.

Proof of 3). Let $M<M'$. Then $\int_{(G_M-G_M)\cap\partial R_m^*}\left|\frac{\partial}{\partial n}V(z)\right|ds\to 0$ as $m\to\infty$. By $\frac{\partial}{\partial n}V(z)\geqq 0$ on $C_M+C_{M'}$ we have at once $\int_{C_M}\frac{\partial}{\partial n}V(z)ds=K=\text{const.}$ and $D(\min(M,V(z)))=MK$ for any M.

Proof of 4). Assume $M_0<\sup V(z)\leqq M$. Then by (1)

$$V(z)=M_i w(G_{M_i},z,R)\leqq Mw(G_{M_0}\cap G_{M_i},z,R)\,.$$

Let $M_i\uparrow M$. Then $G_{M_i}\to B$ or the set of capacity zero where $V(z)=\infty$. Hence

$$0\leqq V(z)\leqq Mw(G_{M_i}\cap B,z,R)=0\,.$$

Proof of 5). Put $G_M=E[z\in R:\ V(z)>M]$, $G_\varepsilon=E[z\in R:\ V(z)>\varepsilon]$ and $G'_M=E[z\in R:\ V'(z)>M]$. Then $G_\varepsilon\supset G_M\supset G'_M$. Let $V_n'(z)$ be a harmonic function in $(G_\varepsilon=G'_M)\cap R_n$ such that $V_n'(z)=\varepsilon$ on ∂G_ε, $V_n'(z)=M$ on $\partial G'_M$ and $\frac{\partial}{\partial n}V_n'(z)=0$ on $\partial R_n\cap(G_\varepsilon-G'_M)$, Then $V_n'(z)$ has M.D.I. over $R_n\cap(G_\varepsilon-G'_M)$ with value ε on ∂G_ε and M on $\partial G'_M$, whence $D(V_n'(z))\underset{G_M}{\leqq}D(V(z))\leqq M\alpha$. Hence $V_n'(z)\Rightarrow V_\varepsilon^*(z)$ as $n\to\infty$ and $D(V_\varepsilon^*(z))\leqq M\alpha$. On the other hand, $0\leqq V_n'(z)\leqq\varepsilon+Mw(G'_M,z,R)+Mw(G_\varepsilon\cap B,z,R)=\varepsilon+Mw(G'_M\cap(R-R_n),z,R)$ and $V_\varepsilon^*(z)\leqq\varepsilon+Mw(G'_M,z,R)+Mw(G_\varepsilon\cap B,z,R)=\varepsilon+Mw(G'_M,z,R)$. Let $\varepsilon\to 0$. Then also $V_\varepsilon^*(z)\Rightarrow V^*(z)$ and $V^*(z)\leqq Mw(G'_M,z,R)$ and $D(V^*(z))\leqq M\alpha$. By the definition of $w(G'_M,z,R)V^*(z)\geqq Mw(G'_M,z,R)$. Hence

$$V^*(z)=Mw(G'_M,z,R)\,. \tag{1}$$

Also $Mw(G'_M,z,R)\leqq V(z)\leqq\varepsilon+Mw(G_\varepsilon\cap B,z,R)+Mw(G'_M,z,R)$. By letting $\varepsilon\to 0$ and by (1) we have

$$V'(z)=Mw(G'_M, z, R)=V^*(z) \quad \text{in} \quad R-G'_M.$$

Next by $\underset{R-G'_M}{D}(V^*(z))<\infty$ we can prove similarly as the proof of (3) $\int_{C_l}\frac{\partial}{\partial n}V'(z)ds=K$ and $D(\min(M, V'(z))=MK$, next by $D(\min(M, V(z)))=M\alpha$ we have $K\leqq\alpha$. By $V'(z)\leqq V(z)$ we have at once $w(G'_M\cap B, z, R)\leqq w(G_M\cap B, z, R)=0$. Thus $V'(z)$ is a G.G.

Proof of 6). At first we show $V(z)<\infty$ in R. Let $V_m(z)$ be a G.G. and put $G_l=E[z\in R: V_m(z)>l]$. Then by $w(G_l\cap B, z, R)=0$ we see $V_{m,n}(z)\Rightarrow V_m(z)$ as $n\to\infty$, where $V_{m,n}(z)$ is a harmonic function in $R_n\cap(G_\varepsilon-G_M)$ such that $V_{m,n}(z)=M$ on ∂G_M, $V_{m,n}(z)=\varepsilon$ on ∂G_ε and $\frac{\partial}{\partial n}V_{m,n}(z)=0$ on $\partial R_n\cap(G_\varepsilon-G_M)$. Let D be a compact disc in R. Suppose $V(z)\geqq L$ on D. Let $\omega_n(z)$ be a harmonic function in $(R_n-D)\cap G_\varepsilon$ such that $\omega_n(z)=1$ on D, $\omega_n(z)=\varepsilon$ on ∂G_ε, and $\frac{\partial}{\partial n}\omega_n(z)=0$ on ∂R_n. Then

$$D(V_{m,n}(z)-L\omega_n(z)\,,\ \omega_n(z))\geqq 0\,.$$

Hence by mean convergency of $V_{m,n}(z)$ and $\omega_n(z)$ as $n\to\infty$ we have

$$L\alpha\geqq D(\min(L, V_m(z)))\geqq L^2D(\omega_\varepsilon(z))\,,$$

where $\omega_\varepsilon(z)=\lim_n\omega_n(z)$ and $D(\omega_\varepsilon(z))<\infty$.
Let $\varepsilon\to 0$. Then $\omega_\varepsilon(z)\Rightarrow\omega(z)$ as $\varepsilon\to 0$ and $L\alpha\geqq L^2D(\omega(z))$. Whence $\min_{z\in D}V_n(z)\leqq\frac{\alpha}{D(\omega(z))}$ and $\min_{z\in D}V(z)\leqq\frac{\alpha}{D(\omega(z))}$. Hence by Harnack's theorem $V(z)<\infty$ in R.

Put $G_M=E[z\in R: V(z)>M]$ and $G_{M,m}=E[z\in R: V_m(z)>M]$. Then by $V(z)\geqq V_m(z)$ $G_M\supset G_{M,m}$. Since $V_m(z)=Mw(G_{M,m}, z, R)$, $V_m(z)$ is the least positive harmonic function in $R-G_M$ with value $V_m(z)$ on ∂G_M. Hence $V_m(z)=\lim_n V_{m,n}(z)$, where $V_{m,n}(z)$ is the least positive harmonic function in R_n-G_M such that $V_{m,n}(z)=V_m(z)$ on ∂G_M and $V_{m,n}(z)=0$ on ∂R_n-G_M. Hence

$$V_{m,n}(z)=\frac{1}{2\pi}\int_{\partial G_M\cap R_n}V_{m,n}(\zeta)\frac{\partial}{\partial n}G_n(\zeta, z)ds\,,$$

where $G_n(\zeta, z)$ is the Green's function of R_n-G_M.
Let $n\to\infty$. Then $\frac{\partial}{\partial n}G_n(\zeta, z)\uparrow\frac{\partial}{\partial n}G(\zeta, z)$ on ∂G_M, where $G(\zeta, z)$ is the Green's function of $R-G_M$. Hence $V_m(z)=\frac{1}{2\pi}\int_{\partial G_M}V_m(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds$. Now $V_m(z)\leqq$

$V(z) \leqq M$ on ∂G_M, hence $V_m(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds$ is uniformly integrable and

$$V(z) = \frac{1}{2\pi}\int_{\partial G_M} V(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds = \frac{1}{2\pi}\lim_{m=\infty}\int_{\partial G_M} V_m(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds = \lim_m V_m(z).$$

This means that $V(z)$ is the least positive harmonic function in $R-G_M$ with value M on ∂G_M, i.e. $V(z)=Mw(G_M, z, R)$ in $R-G_M$.

Let $w_n(z)$ be the least positive harmonic function in $R-G_M-((R-R_n)\cap G_\varepsilon)$ such that $w_n(z)=1$ on $(R-R_n)\cap G_\varepsilon$. Then $w_n(z)\to w(G_\varepsilon\cap B, z, R-G_M)$ as $n\to\infty$ and $V(z)\geqq \varepsilon w(G_\varepsilon\cap B, z, R-G_M)$. Since $V(z)$ is the least positive harmonic function in $R-G_M$ larger than M on G_M, $w(G_\varepsilon\cap B, z, R-G_M)=0$. Clearly $w_n(z)+w(G_M, z, R)\geqq w(G_\varepsilon\cap B, z, R)$, because $w_n(z)+w(G_M, z, R)\geqq 1$ on $G_\varepsilon\cap(R-R_n)$. Hence

$$w(G_M, z, R) = w(G_\varepsilon\cap B, z, R-G_M)+w(G_M, z, R)\geqq w(G_\varepsilon\cap B, z, R).$$

Let $M\to\infty$. Then $w(G_M, z, R)\to 0$. This implies $w(G_\varepsilon\cap B, z, R)=0$ for any $\varepsilon>0$. Let G be a compact domain completely contained in $R-G_M$. Then there exists a number n_0 suce that $G\subset E[z\in R: V_n(z)<M]$ for $n\geqq n_0$. Then

$$\underset{G}{D}(V(z)) \leqq \underset{G}{D}(\lim V_n(z)) \leqq \underline{\lim}_n \underset{G}{D}(V_n(z)) \leqq \underline{\lim}_n D(\min(M, V_n(z)) \leqq \alpha M.$$

Let $G\to E[z\in R: V(z)<M]$. Then $D(\min(M, V(z)))\leqq M\alpha$. Thus $V(z)$ is a G.G.

Proof of 7). Let $U_n(z)$ be the least positive harmonic function in R_n-G_M such that $U_n(z)=M$ on ∂G_M, $U_n(z)=0$ on ∂R_n-R_δ, $\frac{\partial}{\partial n}U_n(z)=0$ on $(\partial R_n\cap R_\delta)-G_M$. Then $D(U_n(z))\leqq D(\min(M, U(z)))\leqq M\alpha$. Also let $U_n^*(z)$ be the least positive harmonic function in R_n-G_M larger than M on G_M. Then

$$0\leqq U_n^*(z)\leqq U_n(z)\leqq U_n^*(z)+Mw(R_\delta\cap(R-R_n), z, R).$$

Let $n\to\infty$. Then since $G(z, p)$ is a G.G., $w(R_\delta\cap(R-R_n), z, R)\to w(R_\delta\cap B, z, R)=0$. Hence $0\leqq\lim_n U_n^*(z)=\lim U_n(z)=U_M(z)$. By $U_M(z)=\lim_n U_n(z)$, we have

$$D(\min(M, U_M(z))) \leqq \lim_n D(U_n(z)) \leqq M\alpha. \tag{2}$$

By $U_M(z)=\lim_n U_n^*(z)$,

$$U_M(z) = Mw(G_M, z, R) \tag{3}$$

and $U_M(z)=Nw(G'_N, z, R)$ in $R-G'_N$, where $G'_N=E[z\in R: U_M(z)>N]+G_M$ for any $N\leqq M$ and for any $D\subset R-G_M$

$$U_M(z) = \frac{1}{2\pi}\int_{\partial D} U_M(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds, \tag{4}$$

where $G(\zeta, z)$ is the Green's function of D.
By (2) and (3) we have similarly as the proof of 2), $\omega(G'_{N'}B, z, G'_N)=0$: $M \geqq N' > N$.

This implies that the doubled surface $(G_{N'}+\hat{G}_{N'})$ is a Riemann surface with null-boundary, whence $\int_{C_l} \frac{\partial}{\partial n} U_M(z)ds = k = \text{const.}$, where by (2) $k \leqq \alpha$ for any $C_l = E[z \in R: U_M(z)=l]$, and $D(\min(N, U_M(z))) \leqq N\alpha$ for $N \leqq M$. (5)

Clearly $U_M(z) \uparrow \tilde{U}(z)$ as $M \to \infty$. Then by (5) $D(\min(N, \tilde{U}(z))) \leqq \varliminf_{M=\infty} D(\min(N, U_M(z))) \leqq N\alpha$ and $\tilde{U}(z) < \infty$. (6)

Also by (4), since $U_M(z) \uparrow \tilde{U}(z)$

$$\tilde{U}(z) = \frac{1}{2\pi} \int_{\partial D} \tilde{U}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds,$$

i.e. $U(z)$ is the least positive harmonic function in $R - \tilde{G}_M$ larger than M on $\tilde{G}_M = E[z \in R: \tilde{U}(z) > M]$, whence we have similarly as (7)

$$w(\tilde{G}_M \cap B, z, R) = 0 \qquad \text{for any } M. \qquad (7)$$

Thus by (6) and (7) $\tilde{U}(z)$ is a G.G.

2. Green's potential. Let R^* be a Riemann surface with positive boundary and let $\{R_n^*\}$ be its exhaustion with compact relative boundary ∂R_n^* $(n=1, 2, \cdots)$. Let p_0 be a fixed point in R and let $\overset{*}{G}(z, p_0)$ be the Green's function. Put $R = E[z \in R^*: \overset{*}{G}(z, p_0) > \delta]$. Then by Theorem 1), 2) the doubled surface $R + \hat{R}$ is a Riemann surface with null-boundary. Let $G(z, p)$ be the Green's function of R. Then $G(z, p)$ is a G.G. in R. Let $\{p_n\}$ be a divergent sequence in $G_\varepsilon = E[z \in R: G(z, p_0) > \varepsilon]$ such that $\{G(z, p_n)\}$ converges to a positive harmonic function. Then we say that $\{p_n\}$ determines an ideal boundary point p. We denote by B the set of all the ideal boundary points. Also we denote $\lim_n G(z, p_n)$ by $G(z, p)$: $p \in B$ simply. Let $q \in \partial R$ and let $v(q)$ be a compact neighbourhood of q in R^*. Then $G(z, p_n) \leqq M$ on $v(q) \cap R$ for $n \geqq n(q)$. Hence $G(z, p)$ $(p \in B) = 0$ on ∂R and $w(R \cap B, z, R) \leqq w(R \cap B, z, R^*) = 0$. Also by Fatou's lemma $D(\min(G(z, p), M))) \leqq 2\pi M$ for $p \in R + B$. Thus $G(z, p)$ is a G.G. for $p \in R + B$. Let $\bar{R} = R + B$. Then the distance between p_1 and p_2 in $\bar{R}$ is defined as

$$\delta(p_1, p_2) = \sup_{z \in R_0} \left| \frac{G(z, p_1)}{1 + G(z, p_1)} - \frac{G(z, p_2)}{1 + G(z, p_2)} \right|,$$

where R_0 is a compact disc in R.

Remark. *Let $p_1 \neq p_2$ in B. Then $G(z, p_1)$ may be a multiple of $G(z, p_2)$*[7]. In fact, let C be a unit circle: $|z|<1$ and let F be a closed set such that $z=0$ is contained in the closure of F and F is so thinly distributed in a neighbourhood of $z=0$ that $z=0$ is an irregular point of the Dirichlet problem in $C-F$. Then there exists only one linearly independent G.G. vanishing on $F+\partial C$ except $z=0$ and for any point p on $z=0$, $G(z, p)$ is a multiple of a fixed function.

Let $G(z, p)$: $p \in \bar{R}$ and $V_M(p)=E[G(z, p)>M]$. Let $U(z)$ be a G.G. in R and let $U_M^L(z)$ be the least positive harmonic function in $V_M(p)$ with boundary value $\min(L, U(z))$ on $\partial V_M(p)$. Then $U_M^L(z) \leqq U(z)$. Also by $w(V_M(p) \cap B, z, R)=0$, $U_M^L(z)=\lim_M U_n(z)$, where $U_n(z)$ is a harmonic function in $V_M(p) \cap R_n$ such that $U_n(z)=\min(L, U(z))$ on $\partial V_M(p) \cap R_n$ and $U_n(z)=0$ on $\partial R_n \cap V_M(p)$, where $D(U_n(z)) \leqq D(\min(L, U(z))) > \infty$. Since $R+\hat{R}$ is a Riemann surface with null-boundary, there exists an exhaustion $\{R_n'\}$ of $(R+\hat{R})$ such that

$$\int_{\partial R_n' \cap V_M(p)} \left| \frac{\partial}{\partial n} U_n^L(z) \right| ds \text{ and } \int_{\partial R_n' - V_{M'}(p)} \left| \frac{\partial}{\partial n} G(z, p) \right| ds \to 0 \text{ as } n \to \infty: \quad M' > M \text{ by}$$

Lemma 1. By Green's formula

$$\int_{\partial V_{M'}(p) \cap R_n^*} U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds + \int_{\partial R_n' \cap (V_M(p) - V_{M'}(p))} U_M^L(z) \frac{\partial}{\partial n} G(z, p)$$

$$= \int_{\partial V_M(p) \cap R_n'} U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds + \int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds$$

$$- \int_{\partial V_{M'}(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds + \int_{\partial R_n' \cap (V_M(p) - V_{M'}(p))} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds .$$

Now $\int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = M \int_{V_M(p) \cap \partial R_n'} \frac{\partial}{\partial n} U_M^L(z) ds = 0 =$

$\int_{V_{M'}(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = M' \int_{\partial V_{M'}(p) \cap R_n'} \frac{\partial}{\partial n} U_n^L(z) ds$ and other integrations, $\int U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds$ and $\int G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds$ over $\partial R_n' \to 0$ as $n \to \infty$. Hence by $\frac{\partial}{\partial n} G(z, p) \geqq 0$ on $C_M + C_{M'}$ and by letting $L \to \infty$, we have

$$\int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds = \lim_{L=\infty} \int_{\partial V_M(p)} U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds$$

7) Z. KURAMOCHI: Relations between harmonic dimensions. Proc. Japan Acad., 30, 576–580 (1954).

$$= \lim_{L=\infty} \int_{\partial V_{M'}(p)} U_M^L(z) \frac{\partial}{\partial n} G(z,p)\,ds \leqq \int_{\partial V_{M'}(p)} U(z) \frac{\partial}{\partial n} G(z,p)\,ds. \quad \text{Hence}$$

$$\int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z,p)\,ds \uparrow \text{ as } M \uparrow \infty .$$

We define the value of $U(z)$ at p by $\frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z,p)\,ds$. If $p \in R$, this definition coincides with $U(p)$ clearly. Then

Theorem 2. *$G(z,p)$: $p \in \bar{R}$ is a G.G. and the value of $G(z,p)$ is well defined on $\bar{R}$ and*

1). *$G(p,q) = G(q,p)$ and $G(p,p) = \infty$.*

2). *$G(p,q)$ is lower semicontinuous in $\bar{R} \times \bar{R}$.*

3). *If $p \in R$, $G(z,p)$ is continuous in $\bar{R}$.*

Theorem 2 can be proved (without any essential alteration) similaaly as $G(z,p)$ of $R - R_0$, where R is a Riemann surface with null-boundary and R_0 is a compact disc.[8] But in the previous paper, it was proved that $G(z,p)$ is lower semicontinuous in $\bar{R}$ for fixed p. In this paper we shall prove only 2). For any given number $\varepsilon > 0$, we can find numbers M and R_n such that

$$2\pi\big(G(p,q) - \varepsilon\big) \leqq \int_{\partial V_M(p) \cap R_n} G(\zeta, q) \frac{\partial}{\partial n} G(\zeta, p)\,ds .$$

We cover $V_M(p) \cap R_n$ by a compact domain D such that ∂D intersects $\partial V_M(p)$ orthogonally. Let $p_i \to p$ and $q_j \to q$. Then $G(z, p_i) \to G(z,p)$, $G(z, q_j) \to G(z,q)$ and $V_M(p_i) \to V_M(p)$ uniformly in D. Hence for $i > i_0$ and $j > j_0$

$$2\pi G(p_i, q_j) = \lim_{M=\infty} \int_{\partial V_M(p_i)} G(\zeta, q_j) \frac{\partial}{\partial n} G(\zeta, p_i)\,ds \geqq \int_{\partial V_M(p_i)} G(\zeta, q_j) \frac{\partial}{\partial n} G(\zeta, p_i)\,ds$$

$$\geqq \int_{\partial V_M(p) \cap D} G(\zeta, p) \frac{\partial}{\partial n} G(\zeta, q)\,ds - 2\pi\varepsilon \geqq 2\pi\big(G(p,q) - \varepsilon\big) .$$

Let $\varepsilon \to 0$. Then $\varliminf_{ij} G(p_i, q_j) \geqq G(p,q)$. Thus 2) is proved.

3. Energy integral and capacities of Green's potential. Let $R = R_\delta^* = E[z \in R^* : \overset{*}{G}(z, p_0) > \delta]$ and $\{R_n\}$ be an exhaustion of R. Let F be a compact set in R. Put $I(\mu) = \iint_F G(p,q)\,d\mu(p)\,d\mu(q)$, where μ is a positive mass distribution on $\bar{R}$. Since F is compact, $G(z,p) + \log|z-p|$ is harmonic in a neigh-

8) See (1).

bourhood of p and the continuity principle is valid in R and theorems of logarithmic potentials are also valid. Hence there exists a uniquely determined mass distribution μ_c called equilibrium distribution such that $I(\mu_c)$ is the minimal among all distributions of mass unity on F, the potential $U(z)$ of $\mu=L$ (const.) on F except a set of capacity zero and $U(z)\leqq L$ in R. Cleary $U(z)=0$ on ∂R and by $w(B\cap R, z, R)=0$, $U(z)=Lw(F, z, R)$ and $I(\mu)=L=\dfrac{D(U(z))}{2\pi}$, where L is given by $L=\dfrac{2\pi}{\int_{C_M} w(F, z, R)ds}$, $C_M=E[z\in R: w(F, z, R)=M]$, $M<1$. We define the capacity of F as $\dfrac{1}{I(\mu_c)}\left(=\dfrac{1}{L}\right)$. Let F be a closed set in $\bar{R}=R+B$. We also define $\mathrm{Cap}(F)$ by $\dfrac{1}{\inf_{\mu} I(\mu)}$, where μ is a positive mass distribution of unity on F. Put $\mathring{\mathrm{C}}\mathrm{ap}(F)=\sup_{K\subset F}\mathrm{Cap}(K)$, where K is a compact set in F. Then if F is compact, $\mathrm{Cap}(F)=\mathring{\mathrm{C}}\mathrm{ap}(F)$ and $\mathrm{Cap}(F)\geqq\mathring{\mathrm{C}}\mathrm{ap}(F)$ for closed set F (in reality it can be proved $\mathrm{Cap}(F)=\mathring{\mathrm{C}}\mathrm{ap}(F)$). In this paper we use only $\mathring{\mathrm{C}}\mathrm{ap}(F)$ for Green's potential.

Capacities of the irregular set of R of Green's function. Let $G(z, p_0)$ be Green's function of $R=E[R^*\ni z: \overset{*}{G}(z, p_0)>\delta]$. Then $G(z, p_0)$ is continuous in $\bar{R}$. Let $F_l=E\left[z\in R: G(z, p)\geqq\dfrac{1}{l}\right]$. Then F_l is closed in $\bar{R}$. $U(z)=\min\left(\dfrac{1}{l}, G(z, p_0)\right)$ is a continuous function in $\bar{R}$ such that $U(z)=0$ on ∂R, $U(z)=\dfrac{1}{l}$ on F_l and $D(U(z))\leqq\dfrac{2\pi}{l}$. Let R_n $(n=1, 2, \cdots)$ be an exhaustion of R with compact relative boundary ∂R_n. Let $\omega(F_l\cap(R-R_n), z, R)$ be a harmonic function in $R-(F_l\cap(R-R_n))$ such that $\omega(F_l\cap(R-R_n), z, R)=1$ on $F_l\cap(R-R_n)$, $=0$ on ∂R and has M.D.I.. Then by the Dirichlet principle $D(\omega(F_l\cap(R-R_n)), z, R)\leqq\dfrac{1}{l^2}D(U(z))<\infty$. It is evident $\omega(F_l\cap(R-R_n), z, R\geqq\omega(K, z, R)$ for any compact set K in $F_l\cap(R-R_n)$ and by $w(R\cap B, z, R)=0$, $\omega(F_l\cap(R-R_n), z, R)=w(F_l\cap(R-R_n), z, R)$. Let $n\to\infty$. Then $\omega(F_l\cap(R-R_n), z, R)$ converges in mean to a function $V(z)=\omega(F_l\cap B, z, R)$. Clearly $V(z)$ is a G.G., hence by Theorem 1. (5) $V(z)=0$. This implies $D(\omega(F_l\cap(R-R_n), z, R)\downarrow 0$ as $n\to\infty$ and $\lim_n \mathring{\mathrm{C}}\mathrm{ap}(F_l\cap(R-R_n))=0$ as $n\to\infty$.

Loss of mass. As usual mass $m(p)$ of $G(z, p)$ $(p\in\bar{R})$ is given by $\dfrac{1}{2\pi}\displaystyle\int_{\partial V_M(p)}\dfrac{\partial}{\partial n}G(z, p)ds$ ($m(p)$ does not depend on M by Theorem 1. (3)). It is clear

$m(p)=1$ for $p\in R$. If $p\in F_l\cap B$, $m(p)\geqq \frac{1}{kl}$, where $k=\lim_{n=\infty}\sup_{z\in R-R_n} G(z,p_0)$. In fact, $G(p,p_0)=\frac{1}{2\pi}\lim_{M=\infty}\int_{\partial V_M(p)} G(\zeta,p_0)\frac{\partial}{\partial n}G(\zeta,p)ds$. Now $\partial V(p)$ clusters at B as $M\uparrow\infty$. Hence for any number $\varepsilon>0$, there exists a number M such that $G(\zeta,p_0)-\varepsilon\leqq k$ on $\partial V_M(p)$ for $M>M_0$, whence $\frac{1}{2\pi}\int_{\partial V_M(p)}\frac{\partial}{\partial n}G(\zeta,p)ds\geqq\frac{1}{l(k+\varepsilon)}$. Hence by letting $\varepsilon\to 0$, $m(p)\geqq\frac{1}{kl}$.

4. N-Green's function and N-Martin's topology.[9] Let R be a Riemann surface with positive boundary and let $\{R_n\}$ be its exhaustion with compact relative boundary ∂R_n. Let $U(z)$ be a positive superharmonic function in $R-R_0$ such that $U(z)=0$ on ∂R_0 and $D(\min(M,U(z)))<\infty$ for any $M<\infty$. Let ${}_DU^M(z)$ be a continuous function in $R-R_0$ such that ${}_DU^M(z)=\min(M,U(z))$ on ∂D and ${}_DU^M(z)$ has M.D.I. over $R-R_0-D$, where D is a compact or non compact domain. Let ${}_DU(z)=\lim_{M=\infty}{}_DU^M(z)$. If $U(z)\geqq{}_DU(z)$ for any domain, we call $U(z)$ a superharmonic function. Let $U(z)$ be a superharmonic function. If $V(z)=\alpha U(z)$ for any superharmonic function $V(z)$ such that both $V(z)$ and $U(z)-V(z)$ are positively superharmonic, we call $U(z)$ an N-minimal function. Let $N(z,p)$: $p\in R-R_0$ be an N-Green's function in $R-R_0$ such that $N(z,p)$ is harmonic in $R-R_0-p$, $N(z,p)$ has a positive logarithmic singularity at p and $N(z,p)$ has M.D.I. (Dirichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p). We use $N(z,p)$ instead of $G(z,p)$ and define the ideal boundary points and the distance $\delta(p_1,p_2)$ is defined as

$$\delta(p_1,p_2)=\sup_{z\in R_0}\left|\frac{N(z,p_1)}{1+N(z,p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right|,$$

for p_1 and p_2 in $R-R_0+B=\bar{R}-R_0$, where B is the set of the ideal boundary points.

Let F be a closed set in $\bar{R}-R_0$ and put $F_n=\left[z\in\bar{R}:\ \delta(z,F)\leqq\frac{1}{n}\right]$. Let $\omega(F_n,z,R-R)$ be a harmonic function in $R-R_0-F$ such that $\omega(F_n,z,R-R)=1$ on $F_n\cap R$, $\omega(F_n,z,R-R)=0$ on ∂R_0 and $\omega(F_n,z,R-R)$ has M.D.I.. Let $n\to\infty$. Then $\omega(F_n,z,R-R)\Rightarrow\omega(F,z,R-R_0)$. We call $\omega(F,z,R-R_0)$ C.P. of F and define $\mathrm{Cap}(F)$ by $\frac{1}{2\pi}D(\omega(F,z,R_0-R_0))$. We state briefly the properties of B and $N(z,p)$ without proofs.[10]

9) See (5).
10) See (5).

Theorem 3. 1). *$B=B_1+B_0$ and B_0 is an F_σ set of capacity zero.*

2). *$N(z,p)$ is N-minimal if and only if $p\in B_1$.*

3). *If $\omega(p,z,R-R)>0$, $\sup_{z\in R} N(z,p)<\infty$ and $p\in B_1$. We denote the set of p such that $\omega(p,z,R-R)>0$ by B_s. Then if $p\in B_s$, $N(z,p)=k\omega(p,z,R-R)$.*

4). *For*
$$p\in \bar{R}-B_0\,,\qquad \int_{\partial V_M(p)} \frac{\partial}{\partial n} N(z,p)\,ds = 2\pi\,, \tag{10}$$
for almost M, i.e. the set of M such that (10) *is not hold is a set of measure zero. We call such $\partial V_M(p)$ a regular niveau if* (10) *is satisfied, where $V_M(p)=E[z\in R:\ N(z,p)>M]$, $M<M^*(p)=\sup_{z\in R} N(z,p)$.*

5). *For $p\in\bar{R}-B_0$, $N(z,p)=M\omega(V_M(p),z,R-R_0)$ in $R-R_0-V_M(p)$.*

6). *Let $p\in\bar{R}-B_0$. For any $M<M^*(p)$, there exists a number n such that*
$$V_M(p)\supset(R-R_0)\cap v_n(p)\,,\qquad \text{where}\quad v_n(p)=E\left[z\in\bar{R}:\ \delta(z,p)<\frac{1}{n}\right].$$

7). *Let $p\in\bar{R}-B_0$. Then for any $v_n(p)$*
$$\lim_{M\to M^*(p)} \int_{\partial V_M(p)\cap Cv_n(p)} \frac{\partial}{\partial n} N(z,p)\,ds = 0\,,\quad \lim_{M\to M^*(p)} \int_{\partial V_M(p)\cap v_n(p)} \frac{\partial}{\partial n} N(z,p)\,ds = 2\pi\,,$$
where $\partial V_M(p)$ is regular.
And $\lim_{M\to M^*(p)} N_{V_M(p)\cap Cv_n(p)}(z,p)=0.$

8). *The value of $N(z,p):p\in\bar{R}-R_0$ is given as follows:*

a). $q\in\bar{R}-B_0$, $N(q,p)=\lim_{M\to M^*(q)} \frac{1}{2\pi}\int N(\zeta,p)\frac{\partial}{\partial n}N(\zeta,q)\,ds$,

where $\partial V_M(q)$ is a regular niveau.

b). *$q\in B_0$. In this case $N(z,q)$ is represented as $\int_{B_1} N(z,r)\,d\mu_q(r)$ and $N(q,p)=\int_{B_1} N(r,p)\,d\mu_q(r)$ and the value $N(q,p)$ does not depend on particular distribution $\mu_q(r)$.*

9). *$N(p,q)=N(q,p)$ and $N(z,p)$ is lower semicontinuous in $\bar{R}-R_0$ for fixed p.*

10). *If $p\in\bar{R}-B_0$,*
$$N(p,q)\geqq\frac{1}{2\pi}\int_{\partial V_M(p)} N(\zeta,q)\frac{\partial}{\partial n}N(\zeta,p)\,ds\,.$$

Energy integral and the capacites. Let F be a closed set in $\bar{R}-R$ of positive capacity, i.e. $\omega(F,z,R-R_0)>0$. Then there exists a canonical distri-

bution of unity on F (if $\mu=0$ on B_0, μ is called canonical) such that ther energy integral $I(\mu)=\inf_{\mu_i} I(\mu_i)$[11] and the potential $U(z)$ of $\mu=\omega(F, z, R-R_0)$ and $I(\mu)=\frac{1}{2\pi}D(\omega(F, z, R-R_0))=L$, where μ_i is a canonical mass distribution of unity on F and $L=2\pi\Big/\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F, z, R-R_0)ds$. We call $\frac{1}{L}$ the capacity of F. Also define $\mathring{\mathrm{C}}\mathrm{ap}(F)$ by $\lim_{n=\infty}\mathring{\mathrm{C}}\mathrm{ap}(F_n)$, $\mathring{\mathrm{C}}\mathrm{ap}(F_n)=1/\inf_{\mu_i} I(\mu_i)$, where μ_i is a distribution of unity on $R\cap F_n$ (not on $(\bar{R}-B_0)\cap F_n$). We see at once $\mathring{\mathrm{C}}\mathrm{ap}(F)\leqq\mathrm{Cap}(F)$.

5. Transfinite diameters. We. discuss transfinite diameters and others of closed sets on Riemann surfaces. The properties of the surfaces have much influence on them. Therefore we divide types of Riemann surfaces into three types as follows: (A). R is a Riemann surface with positive boundary and N-Martin's topology is defined on $\bar{R}-R_0=R+B-R_0$. (B) $R=R_\delta=E[z\in R^*: G^*(z, p)>\delta>0]$, where R^* is a Riemann surface with positive boundary and G-Martin's topology is defined on $\bar{R}=R+B$. (C). R is a Riemann surface with null-boundary and G-Martin's topology is defined on $R-R_0+B=\bar{R}-R_0$.

Let F be a closed set in $\bar{R}$. We define transfinite diameters $\mathring{D}(F)$, $D^M(F)$ and $D(F)$ as follows:

(A). $1/D(F)=\lim_{n=\infty}\frac{1}{{}_nC_2}\inf\sum_{\substack{p_i\in F-B_0\\ p_i\neq p_j}}N(p_i, p_j)$, $1/D^M(F)=\lim_{n=\infty}\frac{1}{{}_nC_2}\inf\sum_{\substack{p_i\in F-B_0\\ p_i\neq p_j}}N^M(p_i, p_j)$ and $\mathring{D}(F)=\lim_{M=\infty}D(F)$, where $N^M(z, p)=\min(M, N(z, p))$.

(B) and (C). $1/D(F)=\lim_{n=\infty}\frac{1}{{}_nC_2}\inf\sum_{\substack{p_j, p_i\in F\\ p_i\neq p_j}}G(p_i, p_j)$, $1/D^M(F)=\lim_{n}\frac{1}{{}_nC_2}\times\inf\sum_{\substack{p_j, p_i\in F\\ p_i\neq p_j}}G^M(p_i, p_j)$ and $\mathring{D}(F)=\lim_{M=\infty}D^M(F)$, where $G^M(z, p)=\min(M, G(z, p))$.

Then clearly $\frac{1}{{}_nC_2}\inf\sum^n N^M(p_i, p_j)$ and $\frac{1}{{}_nC_2}\inf\sum^n G^M(p_i, p_j)\uparrow$ as $n\to\infty$ and $D^M(F)\downarrow\mathring{D}(F)\geqq D(F)$ as $M\uparrow\infty$ and $D^M(F)$, $\mathring{D}(F)$, $D(F)$ are increasing functions of F.

Lemma 2. *Let $p_i\,(i=1, 2)\in\bar{R}-B_0$. Let D_i be a compact or non compact domain in $V_M(p_i)=E[z\in R: N(z, p_i)>M]$. Let $\omega(D_i, z)$ be C.P. of D_i, i.e. $\omega(D_i, z)=\omega(D_i, z, \bar{R}-R_0)$. Then $M\omega(D_i, z)(\leqq N^M(z, p_i))$ can be represented by a canonical mass distribution μ_i^* on $\bar{D}_i$ such that $M\omega(D_i, z)=\int N(z, q)\times d\mu_i^*(q)$ in $\bar{R}-R_0$ and (clearly $\int d\mu_i^*\leqq 1$)*

$$\int M\omega(D_j, q)d\mu_i^*(q)\leqq N^M(p_i, p_j)\,, \qquad i, j=1, 2\,,$$

where not neccesarily $p_i\neq p_j$.

11) See (5).

(B) *and* (C). *Let* $D_i=V_M(p_i)$. *Then* $M\omega(D_i, z)=Mw(D_i, z)=\min(M, G(z, p_i))$ ($p\in\bar{R}$ *for* (B) *and* $p\in\bar{R}-R_0$ *for* (C)) *is represented by a mass* μ_i *on* R *such that* $\mu_i=0$ *on* B *and*

$$\int M\omega(D_j, q)d\mu_j(q)\leqq G^M(p_i, p_j)\,, \qquad j, i=1, 2\,,$$

where not necessarily $p_1\neq p_2$.

Proof. Case 1. $p_2\in B_1$ or $\notin D_1$. Put $D_{1,n}=D_1\cap R_n$. Then $D_{1,n}$ is compact. In this case, we can find a number M' such that $V_{M'}(p_2)\cap D_{1,n}=0$ and $\partial V_{M'}(p_2)$ is regular, because $V_M(p_2)\to B$ as $M\to M^*(p_2)$. Put $d\mu_{1,n}=\dfrac{M}{2\pi}\times\dfrac{\partial}{\partial n}\omega(D_{1,n}, z)ds$ on $\partial D_{1,n}$, i.e.

$$\omega(D_{1,n}, z)=\int N(z, r)d\mu_{1,n}(r)\,, \qquad z\in R-R_0\,. \tag{11}$$

Let $m>n$ and let $N_m(z, p_2)$ be a harmonic function in $R_m-R_0-V_{M'}(p_2)$ such that $N_m(z, p_2)=N(z, p_2)$ on $\partial R_0+\partial V_{M'}(p_2)$ and $\dfrac{\partial}{\partial n}N_m(z, p_2)=0$ on $\partial R_m-V_{M'}(p_2)$. Then by Theorem 3. (5) $N_m(z, p_2)\Rightarrow N(z, p_2)$ as $m\to\infty$ and

$$\lim_{m=\infty}\int_{\partial V_{M'}(p_2)\cap R_m} A_m(z)\frac{\partial}{\partial n}N_m(z, p_2)ds=\int_{\partial V_{M'}(p_2)} A(z)\frac{\partial}{\partial n}N(z, p_2)ds\,, \tag{12}$$

by the Lemma[12] for any $A_m(z)$ such that $0\leqq A_m(z)\leqq L<\infty$ on $\partial V_{M'}(p_2)$ and $\lim_m A_m(z)=A(z)$ on $\partial V_{M'}(p_2)$. Also let $\omega_m(D_{1,n}, z)$ be a harmonic function in $R_m-R_0-D_{1,n}$ such that $\omega_m(D_{1,n}, z)=\omega(D_{1,n}, z)$ on $\partial R_0+\partial D_{1,n}$ and $\dfrac{\partial}{\partial n}\omega_m(D_{1,n}, z)=0$ on ∂R_m. Then $\omega_m(D_{1,n}, z)\Rightarrow\omega(D_{1,n}, z)$ as $m\to\infty$. Then by

$\displaystyle\int_{\partial V_{M'}(p_2)\cap R_m}\frac{\partial}{\partial n}\omega_m(D_{1,n}, z)ds=\int_{\partial R_m\cap V_{M'}(p_2)}\frac{\partial}{\partial n}\omega(D_{1,n}, z)ds=0$ we have Green's formula

$$M\int_{\partial D_{1,n}} N(\zeta, p_2)\frac{\partial}{\partial n}\omega_m(D_{1,n}, \zeta)ds=M\int_{\partial V_{M'}(p_2)\cap R_m}\omega_m(D_{1,n}, \zeta)\frac{\partial}{\partial n}N_m(\zeta, p_2)ds\,.$$

Let $m\to\infty$. Then by (12)

$$\frac{1}{2\pi}M\int_{\partial D_{1,n}} N(\zeta, p_2)\frac{\partial}{\partial n}\omega(D_{1,n}, \zeta)ds=\frac{1}{2\pi}\int_{\partial V_{M'}(p_2)} M\omega(D_{1,n}, \zeta)\frac{\partial}{\partial n}N(\zeta, p_2)ds$$

$$\leqq\frac{1}{2\pi}\min\big(M, N(\zeta, p_2)\big)\frac{\partial}{\partial n}N(\zeta, p_2)ds$$

12) See (5).

$$\leqq \frac{1}{2\pi}\min\left(M, \lim_{M'\to M^*(p_2)} \int_{\partial V_{M'}(p_2)} N(\zeta, p_1)\frac{\partial}{\partial n}N(\zeta, p_2)ds\right) = N^M(p_2, p_1)\,. \quad (13)$$

Put $d\mu_{1,n}(\zeta)=\frac{M}{2\pi}\frac{\partial}{\partial n}\omega(D_{1,n}, z)ds$ on $\partial D_{1,n}$. Then by (13)

$$\int_{\partial D_{1,n}} N(\zeta, p_2)d\mu_{1,n}(\zeta) \leqq N^M(p_2, p_1)\,. \quad (14)$$

Next suppose $r\in R-R_0$. Then

$$\frac{M}{2\pi}\int_{\partial D_{1,n}} N(\zeta, r)\frac{\partial}{\partial n}\omega(D_{1,2}, \zeta)ds = M\omega(D_{1,n}, r)\,,$$

whence $M\omega(D_{1,n}, z)=\int N(z, \zeta)d\mu_{1,n}(\zeta)$, $z\in R-R_0$. (15)

Hence $\mu_{1,n}(\zeta)$ is the mass distribution of $M\omega(D_{1,n}, z)$ and $\int d\mu_{1,n}\leqq 1$. Let $n\to\infty$. Then there exists an werk limit μ_1 on $\bar{D}_1$ of $\{\mu_{1,n}\}$ such that

$\int N(z, \zeta)\,d\mu_{1,n}(\zeta)\to\int N(z, \zeta)\,d\mu_1(\zeta)$ for $z\in R-R_0$ as $n\to\infty$ and $M\omega(D_1, z) =\lim_n M\omega(D_{1,n}, z) =\int N(z, \zeta)d\mu_1(\zeta)$ for $z\in R-R_0$. We can find a canonical distribution μ_1^* such that $M\omega(D_1, z)=\int N(z, \zeta)d\mu_1^*(\zeta)$ for $z\in R-R_0$. On the other hand, since ${}_{D_l}\omega(D_1, z)=\omega(D_1, z)$ for any l, any canonical distribution of $\omega(D_1, z)$ has no mass on CD_1[13], where ${}_{D_l}\omega(D_1, z)$ is the harmonic function in $R-R_0-D_l$ such that ${}_{D_l}\omega(D_1, z)=\omega(D_1, z)$ on ∂D_l and ${}_{D_l}\omega(D_1, z)$ has M.D.I. and $D_l=E\left[z\in R:\ \delta(z, D_1)<\frac{1}{l}\right]$. Hence μ_1^* is a canonical distribution on $\bar{D}_1\cap(\bar{R}-B_0)$ such that $M\omega(D_1, z)=\int N(z, \zeta)d\mu_1^*(\zeta)$, $z\in R-R_0$. (16)

By (16) and the definition of $\omega(D_1, p)$ for $p\in\bar{R}-R_0-B_0$

$$M\omega(D_1, p)=\frac{M}{2\pi}\lim_{M\to M^*(p)}\int\omega(D_1, \zeta)\frac{\partial}{\partial n}N(\zeta, p)ds=\frac{M}{2\pi}\lim_{M\to M^*(p)}\int_{\partial V_M(p)} N(\zeta, \eta)\times$$

$$d\mu_1^*(\eta)\frac{\partial}{\partial n}N(\zeta, p)ds=\frac{M}{2\pi}\lim_{M\to M^*(p)}\iint_{\partial V_M(p)} N(\zeta, \eta)\frac{\partial}{\partial n}N(\zeta, p)ds\,d\mu_1^*(\eta)=\int\lim_{M\to M^*(p)}$$

$$\left(\frac{M}{2\pi}N(\zeta, \eta)\frac{\partial}{\partial n}N(\zeta, p)ds\right)d\mu_1^*(\eta)=\int N(\eta, p)d\mu_1^*(\eta)\,, \quad (17)$$

because by $\int_{\partial V_M(p)} N(\zeta, \eta)\frac{\partial}{\partial n}N(\zeta, p)ds\uparrow$ as $M\uparrow M^*(p)$ the order of integrations and letting $M\uparrow M^*(p)$ can be changed. Hence (16) is valid for $z\in\bar{R}-R_0-B_0$. Similarly we have by (11)

13) See (5).

$$\omega(D_{1,n}, p) = \int N(\zeta, p) d\mu_{1,n}(\zeta) \qquad \text{for} \quad p \in \bar{R} - R_0 - B_0 , \tag{18}$$

i.e. (11) is valid for $z \in \bar{R} - R_0 - B_0$.
Further by $\omega(D_{1,n}, z) \uparrow \omega(D_1, z)$ as $n \to \infty$, we have

$$\begin{aligned}
\omega(D_1, p) &= \frac{1}{2\pi} \lim_{M \to M^*(p)} \int \omega(D_1, z) \frac{\partial}{\partial n} N(z, p) ds \\
&= \frac{1}{2\pi} \lim_{M \to M^*(p)} \left(\int_{\partial V_M(p)} \lim_n \omega(D_{1,n}, z) \frac{\partial}{\partial n} N(z, p) ds \right) \\
&= \frac{1}{2\pi} \lim_n \left(\lim_{M \to M^*(p)} \int_{\partial V_M(p)} \omega(D_1, z) \frac{\partial}{\partial n} N(z, p) ds \right) = \lim_n \omega(D_{1,n}, p) . \qquad (19)
\end{aligned}$$

Hence by (17), (19), (15) and (14)

$$\begin{aligned}
M\omega(D_2, \eta) d\mu_1^*(\eta) &\leqq \int N(\zeta, p_2) d\mu_1^*(\eta) \\
&= M\omega(D_1, p_2) = \lim_n M\omega(D_{1,n}, p_2) \leqq N^M(p_2, p_1) .
\end{aligned} \tag{20}$$

Case 2. $p_2 \in D_1 \cap (R - R_0)$ ($j = 1$ or 2). In this case $N(z, p_1) \geqq M$ in D_1. Hence

$$N(p_2, p_1) = N(p_1, p_2) \geqq M . \tag{21}$$

There exists a number n such that $D_{1,n} \ni p_2$. Let $m > n$ and let $N_m(z, p_2)$ be a harmonic function in $R_m - R_0 - p_2$ such that $N_m(z, p_2) = 0$ on ∂R_0, $N_m(z, p_2)$ as a logarithmic singularity at p_2 and $\frac{\partial}{\partial n} N_m(z, p_2) = 0$ on ∂R_m. Then $N_m(z, p_2) \Rightarrow N(z, p_2)$ as $m \to \infty$. Let $\omega_m(D_{1,n}, z)$ be the function in case 1. Then

$$\begin{aligned}
&\frac{M}{2\pi} \int_{\partial D_{1,n}} N_m(\zeta, p_2) \frac{\partial}{\partial n} \omega_m(D_{1,n}, \zeta) ds \\
&= \frac{M}{2\pi} \int_{\partial D_{1,n}} \omega_m(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_m(z, p_2) ds = \frac{M}{2\pi} \int_{\partial D_{1,n}} \frac{\partial}{\partial n} N_m(\zeta, p_2) ds
\end{aligned}$$

$\leqq M$. Let $m \to \infty$. Then by (21)

$$\frac{M}{2\pi} \int_{\partial D_{1,n}} N(\zeta, p_2) \frac{\partial}{\partial n} \omega(D_{1,n}, \zeta) ds \leqq M = N^M(p_2, p_1) .$$

Put $d\mu_n = \frac{M}{2\pi} \frac{\partial \omega}{\partial n}(D_{1,n}, \zeta) ds$ on $\partial D_{1,n}$. Then as in case 1), there exists a canonical distribution μ_1^* such that $M\omega(D_1, z) = \int_{\bar{R} - B_0} N(z, \zeta) d\mu_1^*(\zeta)$ and

$$M\int \omega(D_2, \eta)d\mu_1^*(\eta) \leqq \int N(\eta, p_2)d\mu_1^*(\eta) \leqq N^M(p_2, p_1)\,. \tag{22}$$

If $p_2 \in R-D_1$, we have more easily (22). Thus by (21) and (22) we have the former part of Lemma 2.

(B) and (C). Suppose $p_2 \notin V_M(p_1)$. Then by Green's formula and by Lemma 1

$$\begin{aligned} G^M(p_1, p_2) = G(p_1, p_2) &= \frac{1}{2\pi}\int_{\partial V_M(p_1)} G(\zeta, p_2)\frac{\partial}{\partial n}G(\zeta, p_1)ds \\ &\geqq \frac{M}{2\pi}\int_{\partial V_M(p_1)} \omega\big(V_M(p_2), \zeta\big)\frac{\partial}{\partial n}G(\zeta, p_1)ds\,. \end{aligned} \tag{23}$$

If $p_2 \in V_M(p_1)$, $G(p_1, p_2) \geqq M$ and

$$G^M(p_1, p_2) = M \geqq \frac{M}{2\pi}\int_{\partial V_M(p_1)} \omega\big(V_M(p_2), z\big)\frac{\partial}{\partial n}G(z, p_1)ds\,. \tag{24}$$

Put $d\mu_1 = \frac{1}{2\pi}\frac{\partial}{\partial n}G(z, p_1)ds$ on $\partial V(p_1)$. Then $\mu_1 = 0$ on (B) and by (23) and (24)

$$\int M\omega\big(V_M(p_1), z\big)d\mu_1(z) \leqq G^M(p_2, p_1)\,. \tag{27}$$

Theorem 4. (A). *Let p_i $(i=1, 2, \cdots, n)$ in $\bar{R}-B_0$. Let D_i be a domain in $V_M(p_i)$. Then there exists a canonical distribution μ_i of mass $\leqq 1$ on $\bar{D}_i$ such that $M\omega(D_i, z) = \int N(z, p)d\mu_i(p)$. Put $\mu = \frac{1}{n}\sum^n \mu_i$. Then*

$$I(\mu) = \iint N(p, q)d\mu(p)d\mu(q) \leqq \frac{1}{n^2}\sum_{\substack{i=1\\ j=1}}^n N^M(p_i, p_j)\,.$$

(B). *Let p_i $(i=1, 2, \cdots, n)$ on $\bar{R}$. Then there exists a distribution μ_i on $R \cap V_M(p_i)$ ($\int d\mu_i \leqq 1$ by loss off mass) such that $M\omega(V_M(p_i), z) = \int G(z, p)d\mu_i(p)$. Put $\mu = \frac{1}{n}\sum_{i=1}^n \mu_i$. Then $I(\mu) \leqq \frac{1}{n^2}\sum_{\substack{i=1\\ j=1}}^n G^M(p_i, p_j)$.*

(C). *Let p $(i=1, 2, \cdots, n)$ on $R-R_0$. Then there exists a distribution μ_i on $R \cap V_M(p_i)$ such that $M\omega(V_M(p_i), z) = \int G(z, p)d\mu_i(p)$ and $\int d\mu_i = 1$. Put $\mu = \frac{1}{n}\sum_{i=1}^n \mu_i$. Then $I(\mu) \geqq \frac{1}{n^2}\sum_{\substack{i=1\\ j=1}}^n G^M(p_i, p_j)$.*

Proof of (A). Put $U(z) = \int N(z, p)d\mu(p)$. Then $U(z) = M\sum \omega(D_i, z)$. Then by Lemma 2, $I(\mu) = \frac{M}{n^2}\sum_{i=1}^n \omega(D_i, z)\sum_{i=1}^n d\mu_i \leqq \frac{1}{n^2}\sum_{\substack{i=1\\ j=1}}^n N^M(p_i, p_j)$.

It is proved similarly for (B) paying attention to $d\mu_i = \frac{\partial}{\partial n}G(z, p)ds$ on $\partial V_M(p)$

and $\int_{\partial V_M(p)} \frac{\partial}{\partial n} G(z,p)ds \leqq 2\pi$ (by loss of mass) i.e. $\int d\mu_i \leqq 1$ and $\mu_i=0$ on (B) and for (C) to $\int_{\partial V_M(p)} \frac{\partial}{\partial n} G(z,p)ds=2\pi$.

Lemma 3. *Let $\tilde{A}$ and A be closed sets such that $\tilde{A}\supset A$.*

(A), 1). *If there exists a number M such that $\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p)\cap\tilde{A}, z)ds>\delta>0$ for any $p\in A\cap(\bar{R}-R_0-B_0)$, then*

$$1/D^M(A)\geqq \delta^2/\mathrm{Cap}(\tilde{A}).$$

2). *If there exists a number M such that $\frac{1}{2\pi}\int_{\partial V_M(p)\subset\tilde{A}} \frac{\partial}{\partial n} N(z,p)ds\geqq\delta>0$ for any $p\in A\cap(\bar{R}-R_0-B_0)$, $1/D^M(A)\geqq\delta^2/\mathring{\mathrm{C}}\mathrm{ap}(\tilde{A})$.*

2′). (B) *and* (C). *If there exists a number M such that $\frac{1}{2\pi}\int_{\partial V_M(p)\cap\tilde{A}} \frac{\partial}{\partial n} G(z,p)ds>\delta>0$ for any $p\in A$, $1/D^M(A)\geqq\delta^2/\mathring{\mathrm{C}}\mathrm{ap}(\tilde{A})$.*

3). (A). *Let G be an open set such that $G\supset A$ and $G\supset\sum_{p\in F}V_M(p)$, then*

$$1/D^M(A)\geqq 1/\mathring{\mathrm{C}}\mathrm{ap}(G),$$

where $\mathrm{Cap}(G)=\sup_F \mathrm{Cap}(F)$, *$F$ is a closed set in G.*

3′). (C). *Let G be an open set such that $G\supset A$ and $G\supset\sum_{p\in F}V_M(p)$. Then*

$$1/D^M(A)\geqq 1/\mathring{\mathrm{C}}\mathrm{ap}(G).$$

Proof of 1). Let $D(p_i)=V_M(p_i)\cap\tilde{A}$. Let $\mu_i(p)$ be the canonical distribution of $M\omega(D(p_i),z)$ on $\bar{D}(p_i)\subset\tilde{A}$. Then by Theorem 4, putting $\mu=\frac{1}{n}\times\sum_i\mu_i(p)$, we have $\iint N(p,q)d\mu(p)d\mu(q)\leqq\frac{1}{n^2}\sum_{\substack{i=1\\j=1}}^{n} N^M(p_i,p_j)$.

Now the total mass of $\mu\geqq\delta$ and $\mu=0$ except $\tilde{A}\cap(\bar{R}-R_0-B_0)$, whence by the defintion of the capacity $I(\mu)\geqq\delta^2/\mathrm{Cap}(\tilde{A})$. On the other hand, $2(\sum_{\substack{p_i\neq p_j\\i<j}}^{n} N^M(p_i,p_j))=\sum_{\substack{i=1\\j=1}}^{n} N^M(p_i,p_j)-\sum_i^n N^M(p_i,p_i)$. Hence

$$1/D_n^M(A)=\inf\sum_{\substack{i\neq j\\i<j}}^{n} N^M(p_i,p_j)/{}_nC_2\geqq\delta^2/\mathrm{Cap}(\tilde{A})-\frac{M}{n-1}.$$

Let $n\to\infty$. Then $1/D^M(A)\geqq\delta^2/\mathrm{Cap}(\tilde{A})$.

Paying attention to $M\omega(V_M(p_i),z)=\int_{\partial V_M(p_i)} N(z,p)d\mu_i(p_i)$ $(G^M(z,p_i)=\int_{\partial V_M(p)} G(z,$

$q)d\mu_i(q)$ for (B) and (C) and $\mu>0$ only on $\partial V_M(p_i)\cap R$, we have 2), 2′) similarly as 1). Next since $N^M(z,p)=M\omega(V_M(p),z)$ $(G^M(z,p)=M\omega(V_M(p),z))$ for (C) and the mass of $M\omega(V_M(p),z)$ is unity. Hence by (1) we have 3) and 3′).

6. Activity of a point $p\in\bar{R}-R_0$ to a closed set F. Let $F_n=E\left[z\in\bar{R}:\ \delta(z,F)\leqq\frac{1}{n}\right]$. (A). Put $\lim_n \lim_{M\to M^*(p)} \frac{1}{2\pi}\int_{\partial R_0} M\omega(F_n\cap V_M(p),z)ds=\delta(p)$ *and* $\lim_n \lim_{M\to M^*(p)} \frac{1}{2\pi}\int_{\partial V_M(p)\cap F_n} M\omega(F_n\cap V_M(p),z)ds=\mathring{\delta}(p)$, *where* $\partial V_M(p)$ *is a regular niveau. We call* $\delta(p)$ *and* $\mathring{\delta}(p)$ *the weak and strong activity of* p *to* F *respectively.*

(B). $\mathring{\delta}(p)$ *is given as* $\lim_n \lim_{M=\infty} \frac{1}{2\pi}\int_{\partial V_M(p)\cap F_n} \frac{\partial}{\partial n}G(z,p)ds$.

Clearly $\delta(p)$ to $\tilde{F}\geqq\delta(p)$ to F and $\mathring{\delta}(p)$ to $\tilde{F}\geqq\delta(p)$ to F, if $\tilde{F}\supset F$.

Lemma 4. 1). $\delta(p)\geqq\mathring{\delta}(p)$.

2). *Suppose* $p\in\bar{R}-R_0-B_0$. *If* $p\in F$, $\delta(p)$ *to* $F=1$. *If* $\sup_{z\in R} N(z,p)=\infty$ *and* $p\notin F$, $\delta(p)$ *to* $F=0$.

3). *Let* F *be a closed set of capacity zero. Suppose* $U(z)=\int_{F-B_0} N(z,p)\times d\mu(p)$. *Then* $\lim_n \lim_{M\to\infty} M\omega(V_M\cap F_n,z)=U(z)$, *where* $V_M=E[z\in R:\ U(z)>M]$. *Let* $p\in B_0$. *Then* $N(z,p)=\int_{B_1} N(z,pq)d\mu(p)$. *Let* μ' *be the restriction of* μ *on* F. *Then* $N(z,p)\geqq {}_FN(z,p)=\int_{F\cap B_1} N(z,pq)d\mu'(q)$ *and* $\delta(p)$ *to* $F=\int d\mu'=\frac{1}{2\pi}\times\int_{\partial R_0}\frac{\partial}{\partial n}{}_FN(z,p)ds$.

Proof of 1). $\int_{\partial V_M(p)\cap F_n}\frac{\partial}{\partial n}\omega(F_n\cap V_M(p),z)ds\leqq\int_{\partial(F_n\cap V_M(p))}\frac{\partial}{\partial n}\omega(F_n\cap V_M(p)z)ds\leqq\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F_n\cap V_M(p),z)ds$, whence we have at once 1).

Proof of 2). Since $p\in F-B_0$, $\int_{\partial V_M(p)\cap Cv_m(p)}\frac{\partial}{\partial n}N(z,p)ds\to 0$ and $\int_{\partial V_M(p)\cap v_n(p)}\frac{\partial}{\partial n}N(z,p)ds\to 2\pi$ as $M\to M^*(p)$ by Theorem 2, where $\partial V_M(p)$ is a regular niveau. Whence $\mathring{\delta}(p)$ to F is 1. Let $p\notin F$ and $p\notin B_0$. Then there

exists a number n such that $p \notin F_n$. Put $U(z) = \lim_{M=\infty} {}_{F_n \cap V_M(p)} N(z, p)$: $m > n$. Then $N(z, p) \geqq U(z)$ and $N(z, p) - U(z)$ is superharmonic[14] by $\lim_{M=\infty} \mathrm{Cap}(V_M(p)) = 0$. On the other hand, $N(z, p)$ is N-minimal, whence $U(z) = \kappa N(z, p)$ and the mass distribution of $U(z)$ must be a point mass at $q \notin F_n$. This implies $N(z, p) = N(z, q)$: $q \notin F_n$. This is a contradiction. Hence $0 = U(z) = \lim_{M=\infty} M\omega(F_n \cap V_M(p), z)$ and $\delta(p)$ to $F = 0$.

Proof of 3). We proved the following proposition[15]. *Let F be the kernel of canonical distribution of a superharmonic function in $R - R_0$ vanishing on ∂R_0. Then the kernel of any other canonical distribution is also F.* Put $V(z) = \lim_{M=\infty} {}_{CF_n \cap V_M} U(z)$. Then since $\lim_M \mathrm{Cap}(V_M) = 0$, $V(z) = \lim_{M=\infty} {}_{V_M \cap CF_n} V(z)$ and $V(z) = \int_{\overline{CF_n} - B_0} N(z, q) d\mu(q)$ and $U(z) - V(z)$ is also superharmonic[16]. Assume $V(z) > 0$. Consider $U(z) = (U(z) - V(z)) + V(z)$. Then $U(z)$ has mass on $\overline{CF_n} - B_0$. This contradicts the above proposition. Hence $V(z) = 0$. Let $V'(z) = \lim_n {}_{F_n \cap CV_M} U(z)$, then also $U(z) - V'(z)$ is superharmonic and $\lim_n {}_{F_n \cap CV_M} V'(z) = V'(z)$. Whence the canonical distribution μ' of $V'(z) \leqq \mu$, and the kernel of μ' is contained in F. On the other hand, $\sup V'(z) \leqq M < \infty$. Assume $V'(z) > 0$, since $\mathrm{Cap}(F) = 0$, $\sup_{z \in R} V'(z) = \infty$. This is a contradiction. Hence $V(z) = 0$. Clearly by N-maximum principle[17]

$$M\omega(V_M \cap F_n, z) + {}_{CF_n \cap V_M} U(z) + {}_{CV_M \cap F_n} U(z) \geqq U(z) \geqq M\omega(V_M \cap F, z) .$$

Since $V'(z) = 0$, for any $\varepsilon > 0$ there exists a number $n_0(\varepsilon, z_0)$ such that ${}_{F_n \cap CV_M} U(z_0) < \varepsilon$ for $n > n_0$ and

$$M\omega(V_M \cap F_n, z_0) + {}_{CF_n \cap V_M} U(z_0) + \varepsilon \geqq U(z_0) \geqq M\omega(V_M \cap F_n, z_0) .$$

Let $M \to \infty$ and $\varepsilon \to 0$. Then

$$\begin{aligned} &M\omega(V_M \cap F_n, z) = U(z) \\ &\quad \text{for any } n \text{ and } \lim_n \lim_{M=\infty} M\omega(V_M \cap F_n, z) = U(z) . \end{aligned} \tag{26}$$

Let $p \in B_0$. Then $N(z, p)$ is representable by a canonical distribution μ such as $N(z, p) = \int_{B_1} N(z, q) d\mu(q)$. Let μ' be the restriction of μ on F. Then

14) See (5).
15) See (5).
16) See (5).
17) See (5).

$N(z,p)=U(z)+\sum_{m=1}^{\infty}V_m(z)$, where $V_m(z)$ is the potential of the restriction of μ on $CF_{m+1}-CF_m$. Assume ${}_FV_m(z)>0$. Then since $\mathrm{Cap}(F)=0$, ${}_F({}_FV_m(z))={}_FV_m(z)$ and ${}_FV_m(z)$ is representable by a canonical distribution on F. But any kernel of canonical distribution of $V_m(z)$ is contained in $\overline{CF_{m+1}-CF_m}$. This contradicts the above proposition. Hence ${}_FV_m(z)=0$. Put $\tilde{V}_M=E[z\in R:\ N(z,p)>M]$. Then

$$\lim_n\lim_{M=\infty}M\omega(\tilde{V}_M\cap F_n,z)$$
$$\leqq {}_FU(z)+{}_F(\sum V_m(z))={}_FU(z)+\sum{}_FV_m(z)={}_FU(z)=U(z)\,,$$

whence $2\pi\delta(p)=\lim_n\lim_M\int_{\partial R_0}\frac{\partial}{\partial n}M\omega(\tilde{V}_M\cap F_n,z)\,ds\leqq\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=2\pi\int d\mu'$. (27)

Put $V_M=E[z\in R:\ U(z)>M]$. Then $\tilde{V}_M\supset V_M$ and

$$2\pi\delta(p)\geqq\lim_n\lim_{M=\infty}\int_{\partial R_0}\frac{\partial}{\partial n}M\omega(V_M\cap F_n,z)\,ds\,.$$

Now $U(z)$ is the form $U(z)=\int_{F-B_0}N(z,q)\,d\mu'(q)$. Hence by (26)

$$2\pi\delta(p)\geqq\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=2\pi\int d\mu'\,. \tag{28}$$

Thus by (27) and (28) $\delta(p)$ to $F=\int d\mu'$.

Also $U(z)=({}_FU(z)+\sum{}_FV_m(z))={}_FN(z,p)\geqq{}_F(\int N(z,q)\,d\mu'(q))={}_FU(z)=U(z)$, hence $\delta(p)=\int d\mu'=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}{}_FN(z,p)\,ds\,.$

7. A property of irregular points of Green's functions. (A). Let $\omega(B,z)$ be C.P. of B, i.e. $\omega(B,z)=\lim_n\omega_n(z)$, where $\omega_n(z)$ is a harmonic function in R_n-R_0 such that $\omega_n(z)=0$ on ∂R_0 and $\omega_n(z)=1$ on ∂R_n. Then $\omega(B,z)=1$ on B except a set of capacity zero[18]. Let $G(z,p_0):\ p_0\in R_0$ be the Green's function of R. Then $G(z,p_0)>0$ if and only if $\omega(B,z)<1$. Now $\omega(B,z)$ is well defined on $\bar{R}-R_0$ and lower semicontinuous. Hence $S_l=E\left[z\in B:\ \omega(B,z\leqq 1-\frac{1}{l}\right]$ is closed and of capacity zero. i.e. $D(\omega(S_{l,n},z))\downarrow 0$ as $n\to\infty$, where $S_{l,n}=E\left[z\in\bar{R}:\ \delta(z,S_l)\leqq\frac{1}{n}\right]$ and $E[z\in B:\ \omega(B,z)<1]$ is an F_σ set of capacity

18) See (5).

zero. Let $p\in S_l\cap B_1$. Then by Lemma 3. (1) $\delta(p)$ to $S_l=1$. Let $p\in S_l\cap B_0$. Then $N(z,p)=\int_{B_1}N(z,q)d\mu(q)$ and the value of $\omega(B,z)$ at p is given as $\omega(B,p)=\int\omega(B,q)d(q)$. Assume $\mu(q)$ has mass $>1-\frac{1}{2l-1}$ on B_1-S_{2l}. Then $\omega(B,p)<1-\frac{1}{l}$. This is a contradiction. Hence $\mu(q)$ has its mass $\geqq\frac{1}{2l-1}$ on S_{2l} and by Lemma 4. 3) $\delta(p)$ to $S_{2l}\geqq\frac{1}{2l-1}$.

(B). Let $R=E[z\in R^*\colon G^*(z,p_0)>\delta]$. $G(z,p_0)$ of R is well defined on $\bar{R}=R+B$ and continuous in $\bar{R}$. Put $F_l=E\left[z\in\bar{R}:(Gz,p_0)\geqq\frac{1}{l}\right]$ and $S_l=B\cap F_l$. Let $\{R_n\}$ be an exhaustion of R with compact relative boundary ∂R_n. Let $S_{l,n}=F_l\cap(R-R_n)$. Then $D(\omega(S_{l,n},z))\downarrow 0$ as $n\to\infty$ (see "energy integral and capacities"). The value at $p\in S_l$ is given as $\frac{1}{l}\leqq G(p,p_0)=\frac{1}{2\pi}\lim_{M\to\infty}\int_{\partial V_M(p)}G(z,p_0)\frac{\partial}{\partial n}G(z,p)ds$ and

$$\lim_{M\to\infty}\int_{\partial V_M(p)\cap(R-R_n)\cap F_{2l}}G(z,p_0)\frac{\partial}{\partial n}G(z,p)ds=2\pi G(p,p_0)-\lim_{M=\infty}\int_{(\partial V_M(p)\cap(R-R_n)\cap CF_{2l})+(\partial V_M(p)\cap F_{2l}\cap R_n)}G(z,p_0)\frac{\partial}{\partial n}G(z,p)ds\,.$$

Let $k=\lim_n\sup_{z\in R-R_n}G(z,p_0)=\sup_{z\in B}G(z,p_0)$. Now $V_M(p)$ clusters at B as $M\uparrow\infty$. Hence for any given R_n, there exists a number $M_0(n)$ such that $V_M(p)\subset R-R_n$ for $M\geqq M_0$. Hence $\lim_{M=\infty}\frac{1}{2\pi}\int_{\partial V_M(p)\cap S_{2l,n}}\frac{\partial}{\partial n}G(z,p)ds\geqq\frac{1}{2kl}$ for any n and $\delta(p)$ *to* $S_{2l,n}\geqq\frac{1}{2kl}$ *for any* n.

(C). *Let B be the ideal boundary of R with null-boundary. Then $V_M(p)\to B$ as $M\to\infty$ for any $p\in B$, whence $\delta(p)=1$ to B for any point $p\in B$.*

Lemma 5. 1). *Let F and $\tilde{F}$ be closed sets in $\bar{R}-R_0$ such that $\tilde{F}\supset F$.* Put $\tilde{F}_n=E\left[z\in\bar{R}\colon\delta(z,F)\leqq\frac{1}{n}\right]$.

1). (A). *If* $\lim_n\varliminf_{M=\infty}\left(\inf_{p\in F}\int_{\partial R_0}\frac{\partial}{\partial n}M\omega(V_M(p)\cap\tilde{F}_n,z)ds\right)\geqq 2\pi\delta$,

$$1/D(F)\geqq 1/\mathring{D}(F)\geqq\delta^2/\mathrm{Cap}(F)\,.$$

2). *If* $\lim_n\varliminf_{M=\infty}\left(\inf_{p\in F}\int_{\partial V_M(p)}\frac{\partial}{\partial n}N(z,p)ds\right)\geqq 2\pi\delta$,

$$1/D(F) \geqq 1/\mathring{D}(F) \geqq \delta^2/\mathring{\mathrm{C}}\mathrm{ap}(\widetilde{F}).$$

2′). (B) *and* (C). *If* $\lim_n \varliminf_{M=\infty} \left(\inf_{p\in F} \int_{\partial V_M(p)} \frac{\partial}{\partial n} G(z,p)\,ds\right) \geqq 2\pi\delta$,

$$1/D(F) \geqq 1/\mathring{D}(F) \geqq \delta^2/\mathring{\mathrm{C}}\mathrm{ap}(\widetilde{F}).$$

3). (A). *Let* $M(p,n,\delta')$ *be the number such that* $\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p)\cap \widetilde{F}_n, z)\,ds \geqq 2\pi\delta'$ *for* $M \geqq M(p,n,\delta')$: $p\in F$. *Then* $M(p,n,\delta')$ *is upper semicontinuous, if* $\delta(p)$ *to* $\widetilde{F} \geqq \delta$ *for any* $p\in F$, *then*

$$1/\mathring{D}(F) \geqq \delta^2/\mathrm{Cap}(\widetilde{F}).$$

If $\mathring{\delta}(p)$ *to* $\widetilde{F} > \delta$ *for any* $p\in F$, $1/\mathring{D}(F) \geqq \delta^2/\mathring{\mathrm{C}}\mathrm{ap}(\widetilde{F})$.

3′). (B) *and* (C). *If* $\lim_n \varliminf_{M=\infty} \int_{\partial V_M(p)\cap \widetilde{F}_n} \frac{\partial}{\partial n} G(z,p)\,ds = 2\pi\delta(p) > \delta$ *for* $p\in F$,

$$1/\mathring{D}(F) \geqq \delta/\mathring{\mathrm{C}}\mathrm{ap}(\widetilde{F}).$$

Proof of 1). Since $\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p)\cap \widetilde{F}_n, z)\,ds \downarrow$ as $n\to\infty$, for any $\varepsilon>0$ there exists a number $M_0 = M(\varepsilon, p)$ such that $\frac{1}{2\pi}\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p)\cap \widetilde{F}_n, z)\,ds \geqq \delta-\varepsilon$ for $M \geqq M_0$ and for $p\in F$. Then by Lemma 3 $1/\mathring{D}(F) \geqq 1/D^M(F) \geqq (\delta-\varepsilon)^2/\mathrm{Cap}(\widetilde{F}_n)$. Let $\varepsilon\to 0$ and the $n\to\infty$. Then we have (1).

2) and 2′) can be proved similarly as (1) using Lemma 3.

Proof of 3). Let $M \geqq M(p_0, n, \delta')$. Then $\frac{1}{2\pi}\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p_0)\cap \widetilde{F}_n, z)\,ds \geqq \delta'$. For any $\varepsilon>0$, there a compact set G in $V_M(p_0)\cap \widetilde{F}_n$ such that $\frac{1}{2\pi}\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p_0)\cap G, z)\,ds \geqq \delta'-\varepsilon$. Since $N(z,p_i)\to N(z,p_0)$ in R as $p_i\to p_0$, $E[z\in R: N(z,p_i)>M] \supset G$ for $i>i_0$ and $\int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p_i)\cap \widetilde{F}_n, z)\,ds \geqq \int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p_0)\cap G, z)\,ds$. Let $\varepsilon\to 0$. Then $\frac{1}{2\pi}\lim_{p_i\to p_0} \int_{\partial R_0} \frac{\partial}{\partial n} M\omega(V_M(p_i)\cap \widetilde{F}_n, z)\,ds \geqq \delta'$. This implies $\lim_{p_i\to p_0} M(p_i, n, \delta') \leqq M(p_0, n, \delta')$ by definition of $M(p, n, \delta')$. Also for any point $p\in F$ $\varliminf_{M=\infty} \frac{M}{2\pi}\int_{\partial R_0} \frac{\partial}{\partial n}\omega(V_M(p)\cap \widetilde{F}_n, z)\,ds \geqq \delta(p) = \lim_n \varliminf_{M=\infty} \frac{1}{2\pi}\int_{\partial R_0}$

$\frac{\partial}{\partial u} M\omega(V_M(p)\cap \tilde{F}_n, z)ds$, whence $M(p, n, \delta-\varepsilon)<\infty$ and $M(p, n, \delta-\varepsilon)$ attains its maximum $M(n, \delta-\varepsilon)$ on F. Let $M>M(n, \delta-\varepsilon)$. Then $1/\mathring{D}(F)\geqq 1/D_M(F)\geqq(\delta-\varepsilon)^2/\mathrm{Cap}(\tilde{F}_n)$. Let $\varepsilon\to 0$ and then $n\to\infty$. Then $1/\mathring{D}(F)\geqq\delta^2/\mathrm{Cap}(\tilde{F}_n)$.

3′) is proved similarly as (3).

8. Existence of functions of Evans's type.

Theorem 5 (G. C. Evans)[19]. *Let F_i $(i=1, 2, \cdots)$ be a closed set in $\bar{R}-R_0$. If $D(F_i)=0$, there exists a potential $U(z)=\int_{\Sigma F_i-B_0} N(z,p)d\mu(p)$ (or $\int_{\Sigma F_i} G(z,p)\times d\mu(p)$) such that $\int d\mu d\leqq 1$ ((A) $\mu=0$ except $\Sigma F-B_0$) ((B) and (C) $\mu=0$ except ΣF_i), $U(z)=\infty$ on ΣF_i-B_0 and $\lim_{z\to F-B_0} U(z)=\infty$ ($U(z)=\infty$ on ΣF_i and $\lim_{z\to F} U(z)=\infty$ for* (B) *and* (C)).

Proof of (A). For given $p_1, p_2, \cdots, p_n$ on F_i-B_0, let $V_n(z)=\frac{1}{2\pi n}\sum^n N(z, p_i)$ and put $R(p_1, \cdots, p_n)=\inf V_n(z)$ on F_i-B_0. Let R_n be the least upper bound of the $R(p_1, p_2, \cdots, p_n)$ as $p_1, p_2, \cdots, p_n$ vary on F_i-B_0. Then for any given $\varepsilon>0$ $\left(\varepsilon<\frac{1}{2\pi n}\right)$, there exists a system $(p_1^*, p_2^*, \cdots, p_n^*)$ such that $V_n^*(z)=V(z, p_1, \cdots, p_n^*)\geqq R_n-\varepsilon$ on F_i-B_0. Next $N(p,q)=N(q,p)$, ${}_{n+1}C_2/D_{n+1}(F_i)=\inf(\sum_{i<j}^{n+1} N(p_i, p_j))\leqq\frac{1}{2}\sum_{j=1}^{n+1}(\sum_{\substack{i=1\\ j=1}}^{n+1} N(p_i, p_n))$ and we have $R_n\geqq 1/D_{n+1}(F_i)$, where $D_{n+1}(F_i)=\frac{1}{{}_nC_2}\inf(\sum_{\substack{i=1\\ i<j}}^{n} N(p_i, p_j))$. Hence

$$V_n^*(z)\geqq 1/D_n(F_i) \qquad \text{on } F_i-B_0. \tag{29}$$

Let $L>4$. Since $D(F_i)=\lim_n D_n(F_i)=0$, for any m there exists a number l'_m such that $1/D_{l_m}(F)-\frac{1}{2\pi l_m}\geqq L^m$. Let $V_{l_m}^*(z)$ (defined in (29)). Then $V_{l_m}^*(z)\geqq L^m$ on F_i-B_0. Put $U_i(z)=\sum_{m=1}^{\infty}\frac{V_{l_m}(z)}{2^m}$. Then $U_i(z)=\infty$ on F_i-B_0 and its mass $=0$ except F_i-B_0. Put $U(z)=\sum_{i=1}^{\infty}\frac{U_i(z)}{2^i}$. Then $U(z)$ is the function required. For (B) and (C) the assertion is proved similarly.

We constructed a Riemann surface with positive boundary such that R has the following properties:[20] 1). Let R_0 be a compact disc, then there exist

19) G. C. EVANS: Potentials and positively infinite singularites of harmonic functions. Monatsheft für Math. und phys., 43, 419–424 (1936).

20) Z. KURAMOCHI: Examples of singular points. J. Fac. Sci. Hokkaido Univ. 16, 149–424 (1936).

only two linearly independent positive harmonic functions vanishing on ∂R_0 in $R-R_0$. 2). $B_1=B_S=p_1+p_2$, $B_0\neq 0$, i.e. $N(z,p_i)=\alpha_i\omega(p_i,z)$ and $\sup N(z,p_i)<\infty$: $i=1,2$, and any positive harmonic function is a linear form of $\omega(p_1,z)$ and $\omega(p_2,z)$.

Let $p\in B_0$. Then $\mathrm{Cap}(p)=0$, but *there exists no harmonic function* $U(z)$ *in* $R-R_0$ *such that* $\lim_{z\to p} U(z)=\infty$ *and the Evans's theorem does not hold.* Also in this surface, there exists no superharmonic function $U(z)$ (not necessarily harmonic in $R-R_0$) such that $U(p)=\infty$. In fact, $N(z,p)=\int_{B_1} N(z,p)d\mu(p)=\alpha N(z,p_1)+\beta N(z,p_2)$: $\alpha+\beta=1$, $\alpha\geqq 0$, $\beta\geqq 0$ and $U(p)=\alpha U(p_1)+\beta U(p_2)$. Suppose $U(p)=\infty$. Then at least one of $U(p_1)$ and $U(p_2)$ must be infinite. Without loss of generality we can suppose $U(p_1)=\infty$. By the lower semicontinuity of $U(z)$, for any given $M<\infty$, there exists a number $n(M)$ such that $U(z)\geqq M$ on $v_n(p_1)$. $U(z)\geqq_{v_n(p)} U(z)\geqq M\omega(v_n(p),z)$. Let $n\to\infty$ and then $M\to\infty$. Then $U(z)\geqq_{p_1} U(z)\geqq \lim_{M=\infty} M\omega(p_1,z)=\infty$. Hence $U(z)\equiv\infty$. *Thus Evans's theorem does not hold for superharmonic function.* We shall prove the following

Theorem 6. *Let F be a closed set of capacity zero. Then there exists a positive superharmonic function in $R-R_0$ such that $U(z)=0$ on ∂R_0 and $U(z)=\infty$ on $F-B_0$. Clearly this theorem is valid for an F_σ set of capacity zero.*

Let $F_n=E\left[z\in\bar{R}:\ \delta(z,F)\leqq\frac{1}{n}\right]$ and $\omega(F_n,z)$ be CP. of F. Then $\omega(F_n,z)=1$ in $F_n\cap R$ and by $\mathrm{Cap}(F_n)\downarrow 0$, $\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F,z)ds\downarrow 0$ as $n\to\infty$. Let $L>4$ and let $n'(L)$ be the number such that $\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}\omega(F_{n'},z)ds\leqq\frac{1}{L^n}$. Put $U(z)=\sum_{n'}\omega(F_{n'},z)$. Then $U(z)$ is the function required. Because mass of $U(z)\leqq 1$ and $U(z)<\infty$. Let $p\in F-B_0$. Then $U(p)=\frac{1}{2\pi}\lim\int_{\partial V_M(p)} U(z)\frac{\partial}{\partial n}N(z,p)ds$ and $\lim_{M\to\infty}\int_{F_n\cap\partial V_M(p)}\frac{\partial}{\partial n}N(z,p)ds=2\pi$ for any F_n. Since $U(z)\geqq n$ in $F_{n'}\cap R$, $U(p)=\infty$. By the lower semicontinuity of $U(z)$. $U(z)\to\infty$ as $z\to p\in F-B_0$.

Theorem 7. (A). *Let F and $\tilde{F}$ be closed set in $\bar{R}-R_0$ such that $\tilde{F}\supset F$ and* $\mathrm{Cap}(\tilde{F})=0$. *If $\delta(p)$ to $\tilde{F}\geqq\delta_0>0$ for any $p\in F$. Then there exists a potential $U(z)=\int_{F-B_0} N(z,p)d\mu(p)$ such that $U(z)=\infty$ on $F-B_0$, $D(\min(M,U(z)))=2\pi M$ and by the lower semicontinuity of $U(z)$, $U(z)\to\infty$ as $z\to p\in F-B_0$.*

If $\delta(p)$ *to* $F>\delta_0>0$, *the above potential* $U(z)=\infty$ *on* F.

(B) *and* (C). *Let* F *and* $\tilde{F}$ *be closed sets in* $\bar{R}$ *such that* $\tilde{F}\supset F$ *and* $\mathring{\mathrm{C}}\mathrm{ap}(\tilde{F})=0$. *If* $\mathring{\delta}(p)$ *to* $F\geqq\delta_0>0$, *then there exists a potential* $U(z)=\int_F G(z,p)d\mu(p)$ *such that* $U(z)=\infty$ *on* F *and* $D(\min(M,U(z)))=2\pi M$ *and by the lower semicontinuity* $U(z)\to\infty$ *as* $z\to p\in F$.

Proof. (A). By Lemma 5 $1/D(F)\geqq\delta_0^2/\mathrm{Cap}(\tilde{F})=\infty$. Hence by Evans's theorem there exists a potential $U(z)=\int_{F-B_0} N(z,p)d\mu(p)$ such that $U(z)=\infty$ on $F-B_0$. Put $V_M=E[z\in R: U(z)>M]$. Then $V_M\supset F-B_0$ and ${}_{CV_M\cap F_n}U(z)+{}_{V_M\cap F_n}U(z)\geqq{}_{F_n}U(z)=U(z)$. $\lim_n {}_{CV_M\cap F_n}U(z)\leqq\lim M\omega(F_n,z)=0$. Hence ${}_{V_M}U(z)=U(z)$ for any M and $\min(M,U(z))=M\omega(V_M,z)$ and $D(\min(M,U(z)))=M\int_{\partial R_0}\frac{\partial}{\partial n}U(z)ds=2\pi M$. Suppose $\delta(p)$ to $F\geqq\delta_0>0$. Let $p\in F\cap B_0$. Then $N(z,p)=\int_{B_1}N(z,q)d\mu(q)$. Let μ' be the restriction of μ on $F-B_0$. Then $\int d\mu'\geqq\delta_0$ and $U(p)=\int_{F\cap B_1}U(q)d\mu(q)=\infty$, whence $U(z)=\infty$ on F.

(B) and (C) can be proved similarly.

Corollarly. (A). *Let* S *be the irregular set of Green's function. Then there exists a potential* $U(z)$ *such that* $U(z)=\int_{S-B_0}N(z,p)d\mu(p)$, $U(z)=\infty$ *on* S *and* $D(\min(M,U(z)))=2\pi M$.

(B) (*Theorem of M. Nakai*). *Let* R^* *be a Riemann surface with positive boundary. Then there exists a generalized Green's function* $U(z)$ *in* R^* *such* $U(z)\to\infty$ *as* $z\to B$ *in* $R_\delta=E[z\in R^*: G^*(z,p)>\delta]$ *for any* $\delta>0$.

Proof. Put $S=E[z\in B: G(z,p)>0]$. Then $S=E[z\in B: \omega(B,z)<1]$. Let $S_l=E\left[z\in B: \omega(B,z)\leqq 1-\frac{1}{l}\right]$. Then S_l is closed and $\mathrm{Cap}(S_{2l})=0$. Now $\delta(p)$ to $S_{2l}\geqq\frac{1}{2l}$ for any $p\in S_l$. By Theorem 7, there exists a potential $U_l(z)=\int_{S_l-B_0}N(z,p)d\mu(p)$, $\int d\mu(p)=1$ such that $U(z)=\infty$ on S_l-B_0 and $D(\min(M,U_l(z)))=2\pi M$. Let $U(z)=\sum_{l=l_0+1}^{\infty}\frac{U_l(z)}{2^{l-l_0}}$ where l_0 is a number such that $S_{l_0}\cap B_1\neq 0$. Then $U(z)=\infty$ on $S-B_0$. Let $p\in S_{2l}\cap B_0$. Then $N(z,p)=\int_{B_1}N(z,q)d\mu(q)$.

Let μ' be the restriction of μ on S_{2l}. Then $\int d'\mu(q) \geqq \frac{1}{2l}$, whence $U(p) \geqq \int_{B_1 \cap S_{2l}} U(q)\, d\mu'(q) = \infty$ and $U(z) = \infty$ on $\sum S_{2l} = S$. Put $U'_m(z) = \sum_{l=1}^{m} \frac{U_l(z)}{2^l}$. Then since $\mathrm{Cap}(\sum^m S_l) = 0$, ${}^m_{\Sigma S_l} U'_m(z) = U'_m(z)$ and $\min(M, U'_m(z)) = M\omega(V_M, z)$, where $V_M = E[z \in R : U'_m(z) > M]$. Whence $D(\min(M, U'_m(z))) = 2\pi M \left(\sum_{l=l_0+1}^{m} \frac{1}{2^l} \right)$ and $D(\min(M, U'(z))) \geqq 2\pi M$. On the other hand, by $U'_m(z) \to U(z)$ $D(\min(M, U(z))) \leqq 2\pi M$. Thus $D(\min(M, U(z))) = 2\pi M$.

(B). Let $R = E[z \in R^* : G(z, p) > \delta]$ and let $S = E[z \in R : G(z, p) > \delta] = E[z \in R^* : G(z, p) > 2\delta]$. Let $\{R_n\}$ be an exhaustion of R with compact relative boundary ∂R_n. Then $\mathrm{Cap}(S_{\frac{\delta}{2}, n}) = 0$ as $n \to \infty$, where $S_{\delta, n} = S_\delta \cap (R - R_n)$. Now $\delta(p)$ to $S_{\frac{\delta}{2}, n} \geqq \frac{\delta}{2k}$ for any $p \in S_\delta = \bigcap_n \bar{S}_{\delta, n}$ and for any n, where $k = \lim_{n=\infty} \sup_{z \in R - R_n} G(z, p)$. Whence by Lemma 5. (2) $D(S_\delta) = 0$. Hence there exists a potential $U_\delta(z) = \int_{S_\delta} G(z, p)\, d\mu(p)$, $\int d\mu(p) = 1$ such that $U(z) = \infty$ on S_δ. Now clearly $U_\delta(z)$ is a G.G. in R. Next let $U_M(z)$ be the least positive harmonic function in R^* such that $U_M(z) \geqq M$ on $E[z \in R : U_\delta(z) \geqq M]$. Then by Theorem 1. (2) $\tilde{U}_\delta(z) = \lim_{M=\infty} U_M(z) \geqq U(z)$ is a G.G. in R^* such that $D(\min(M, \tilde{U}_\delta(z))) \leqq 2\pi M$. Now clearly $\tilde{U}_\delta(z) \to \infty$ as $z \to B$ in $R = E[z \in R^* : G(z, p) > 2\delta]$. Put $\tilde{\tilde{U}}(z) = \sum \frac{U_{\frac{\delta}{2^n}}(z)}{2^n}$. Then $\tilde{\tilde{U}}(z)$ is a G.G. in R^* by Theorem 1. (6) and $D(\min(M, \tilde{\tilde{U}}(z))) \leqq 2\pi M$ and $\tilde{\tilde{U}}(z) \to \infty$ as $z \to B$ in $E[z \in R^* : G^*(z, p) > \delta]$ for any $\delta > 0$. Thus $\tilde{\tilde{U}}(z)$ is the function required.

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