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ON THE EXISTENCE OF FUNCTIONS OF EVANS'S TYPE

by

Zenjiro KURAMOCHI

We proved the following

Theorem 1. Let \( R \) be a Riemann surface with null-boundary. Then there exists a harmonic function \( U(z) \) such that \( U(z) \) has a negative logarithmic pole at \( p \in R \) and \( U(z) \to \infty \) as \( z \) tends to the ideal boundary.

Recently M. Nakai extended the above theorem as follows:

Theorem 2. Let \( R \) be a Riemann surface with positive boundary. Let \( G(z, p) \) be the Green's function of \( R \) with pole at \( p \). Put \( \omega(z, R) \) be the harmonic function in \( R \) such that \( D(\min(M, U(z))) \leq \alpha < \infty \) and \( U(z) \to \infty \) as \( z \) tends to the boundary of \( R \) in \( G \) for any \( \delta > 0 \) and that any positive harmonic function \( V(z)(\leq U(z)) \) must be zero, where \( \alpha \) is a constant.

We consider the existence of functions of Evans's type for more general sets and obtain some results which contain the above two theorems as their special applications.

1. Generalized Green's function ³. Let \( R \) be a Riemann surface with positive boundary. Let \( R_n \) (\( n = 1, 2, \cdots \)) be its exhaustion with compact relative boundary \( \partial R_n \). Let \( G^{4} \) be a subsurface in \( R \). Let \( \omega_{n, n+i}(z) \) be a harmonic function in \( R_{n+i} - (G \cap (R - R_n)) \) such that \( \omega_{n, n+i}(z) = 0 \) on \( \partial R_{n+i} - G \) and \( \omega_{n, n+i}(z) = 1 \) on \( G \cap (R - R_n) \). We call \( \lim_{n} \lim_{i} \omega_{n, n+i}(z) \) the harmonic measure (H.M.) of the boundary \( (B \cap G) \) determined by \( G \) and denote it by \( \omega(G \cap B, z, R) \). As for a set \( F \) in \( R \). Let \( w(F, z, R) \) be the least positive harmonic function in \( R - F \) and \( \omega(G \cap B, z, R) \) the harmonic measure of \( F \). Let \( G, \supseteq G, \) be subdomains in \( R \). Let \( \omega_{n, n+i}(z) \) be a harmonic function.

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4) In this paper relative boundary \( \partial G \) of a subsurface \( G \) consists of enumerably infinite number of analytic curves clustering nowhere in \( R \).
in $(G_1 \cap R_n \cap R_{n+i}) - (G_1 \cap (R_{n+i} - R_n))$ such that $\omega_{n,n+i}(z) = 0$ on $\partial G_1$, $\frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G_2$, $\omega_{n,n+i}(z) = 1$ on $G_2 \cap (R_{n+i} - R_n)$. If there exists a const. $M$ such that $D(\omega_{n,n+i}(z)) \leq M$. Then $\omega_{n,n+i}(z) \Rightarrow \omega_n(z)$ as $i \to \infty$ and $\omega_n(z) \Rightarrow$ a function denoted by $\omega(G_2 \cap (B, z, G_1)$ as $n \to \infty$ which is called C.P. (capititary potential) of $(G_2 \cap B)$ relative to $G_1$.

Let $V(z)$ be a positive harmonic function in $R$ except at most a set of capacity zero where $V(z)$ may be infinite. Put $G_M = E[z \in R : V(z) \geq M]$. If $w(G_M \cap B, z, R) = 0$ for any $M$ and $D(\min(M, V(z))) \leq M\alpha$ for any $M < \infty$, we call $V(z)$ a generalized Green’s function, where $\alpha$ is a const.

We proved the following

**Lemma 1°.** Let $R$ be a Riemann surface with null-boundary. Let $R_0$ be a compact disc in $R$. Let $G_i$ be a domain in which $U_i(z)$ is harmonic and $D_{\theta_i}(U_i(z)) < \infty$ $(i = 1, 2, \ldots, i_0)$. Then there exists a sequence of compact curves $\{\gamma_n\}$ such that $\gamma_n$ separates the boundary $B$ of $R$ from $R_0$, $\{\gamma_n\}$ clusters at $B$ and that $\frac{D}{Dn} U_i(z) ds$ tends to zero as $n \to \infty$, for every $i$.

We shall prove

**Theorem 1.** Let $V(z)$ be a G.G. (generalized Green’s function) such that $D(\min(M, V(z))) \leq M$. Then

1) Put $G_M = E[z \in R : V(z) > M]$ and let $w(G_M, z, R)$ be H.M. of $G_M$. Then $V(z) = Mw(G_M, z, R)$ in $R - G_M$ and $D(\min(M, V(z))) = M^2 D(w(G_M, z, R))$.

2) Let $\delta < M$. Then $\omega(G_M \cap B, z, G_\delta) = 0$. Let $\hat{G}_M$ be the symmetric surface of $G_M$ with respect to $\partial G_M$. Identify $\partial G_M$ and the image $\partial \hat{G}_M$ of $\partial G_M$. Then we have a Riemann surface $G_M + \hat{G}_M$ called the doubled surface of $G_M$. Then $G_M + \hat{G}_M$ is a Riemann surface with null-boundary.

3) $\int_{C_M} \frac{D}{Dn} V(z) ds = K$ and $D(\min(M, V(z))) = KM$ for any $M$, where $C_M = E[z \in R : V(z) = M]$.

4) If $V(z) > 0$, then $\sup V(z) = \infty$.

5) Let $V'(z)$ be a positive harmonic function except a set of capacity zero such that $V'(z) \leq V(z)$. Then $V'(z)$ is also a G.G. and $D(\min(M, V'(z))) \leq D(\min(M, V(z)))$.

6) Let $V_n(z)$ $(n = 1, 2, \ldots)$ be a G.G. which is harmonic in $R$ and $D(\min(M, V(z))) \leq M\alpha_n$. If $V_n(z) \uparrow$ and $\alpha_n \leq \alpha$, then the limit function $V(z)$
of \{V_n(z)\} is a G.G.

7) Let \( G(z, p) \) be a Green's function of \( R \) and put \( R_\delta=E[z\in R: G(z, p)>\delta] \). Let \( U(z) \) be a G.G. in \( R_\delta \), such that \( D(\min(M, U(z)))\leq Ma \). Let \( U_M(z) \) be the least positive harmonic function larger than \( U(z) \) in \( R-G_M \), where \( G_M=E[z\in R: U(z)\geq M] \). Then \( U_M(z) \uparrow \bar{U}(z) \) as \( M\to\infty \) and \( \bar{U}(z) \) is a G.G. in \( R \) such that \( D(\min(M, \bar{U}(z)))\leq Ma \).

Proof of 1). Let \( G_\varepsilon=E[z\in R: V(z)\geq \varepsilon] \) and \( G_M=E[z\in R: V(z)\geq M] \) and let \( R'=G_\varepsilon-G_M \): \( M>\varepsilon \). Let \( \omega_n(z) \) be the least positive harmonic function in \( R' \cap R_n \) such that \( \omega_n(z)=1 \) on \( G_\varepsilon \cap \partial R_n \). Put \( U_n(z)=\omega(G_M, z, R)+M\omega_n(z)+\varepsilon \). Then

\[ V(z) \leq M\omega(G_M, z, R)+M\omega_n(z)+\varepsilon = U_n(z). \]

Because \( V(z) = \varepsilon \leq U_n(z) \) on \( \partial G_\varepsilon \cap R_n \), \( V(z) \leq M \leq U(z) \) on \( \partial R_n \cap G_\varepsilon \), and \( V(z) = M = M\omega(G_M, z, R) \) on \( G_M \). Let \( n \to \infty \). Then \( \omega_n(z) \to \omega(G_\varepsilon \cap B, z, R') \leq \omega(G_\varepsilon \cap B, z, R) = 0 \). Let \( \varepsilon \to 0 \). Then \( V(z) \leq M\omega(G_M, z, R) \) in \( R-G_M \). On the other hand, by the definition \( M\omega(G_M, z, R) \leq V(z) \). Thus \( V(z) = M\omega(G_M, z, R) \) in \( R-G_M \) and \( D(\min(M, V(z))) = M^2D(\omega(G, z, R)) \).

Proof of 2). Let \( V_{n+\varepsilon}(z) \) be a harmonic function in \( (G_\varepsilon-G_M) \cap R_{n+\varepsilon} \) such that \( V_{n+\varepsilon}(z) = \varepsilon \) on \( \partial G_\varepsilon \cap R_{n+\varepsilon} \), \( V_{n+\varepsilon}(z) = M \) on \( \partial G_M \cap R_{n+\varepsilon} \) and \( \frac{\partial}{\partial n} V_{n+\varepsilon}(z) = 0 \) on \( \partial R_{n+\varepsilon} \cap (G_\varepsilon-G_M) \). Then \( D(V_{n+\varepsilon}(z)) \leq D(\min(M, V(z))) \). Let \( \omega_{n+\varepsilon}(z) \) be a harmonic function in \( R_{n+\varepsilon} \) such that \( \omega_{n+\varepsilon}(z) = 0 \) on \( \partial G_\varepsilon \cap R_{n+\varepsilon} \), \( \omega_{n+\varepsilon}(z) = 1 \) on \( G_M \cap (R_{n+\varepsilon}-R_n) \) and \( \frac{\partial}{\partial n} \omega_{n+\varepsilon}(z) = 0 \) on \( \partial R_{n+\varepsilon} \cap G_M \). Then

\[ D(\omega_{n+\varepsilon}(z)) \leq D\left(\frac{V_{n+\varepsilon}(z)-\varepsilon}{M-\varepsilon}\right) \leq D(\min(M, V(z))) < \infty. \]

Hence \( \omega_{n+\varepsilon}(z) \to \omega_n(z) \) as \( i \to \infty \) and \( \omega_n(z) \to \omega(z) = \omega(G_M B, z, G_\varepsilon) \) as \( n \to \infty \). Then \( \omega(z) \leq \omega(G_M B, z, G_\varepsilon) = 0 \). Hence \( \omega(G_M B, z, G_\varepsilon) = 0 \). Let \( \omega'_n(z) \) be a harmonic function in \( (G_M \cap (R_n-R_{n_0})) \) such that \( \omega'_n(z) = 0 \) on \( F \) (where \( F \) is a compact arc on \( \partial G_M \cap R_{n_0} \)), \( \omega'_n(z) = 1 \) on \( G_M \cap (R-R_{n_0}) \) and \( \omega'_n(z) \) has minimal Dirichlet integral. Then clearly \( \frac{\partial}{\partial n} \omega'_n(z) = 0 \) on \( \partial G_\varepsilon \cap R_{n_0} \cap F \) and \( \omega'_n(z) \) converges in mean to \( \omega'(z) \). Clearly \( D(\omega'(z)) \leq D(\omega(z)) = 0 \). This implies that the doubled surface of \( (G_M \cap (R-R_{n_0}))+G_M \cap R_{n_0} \) is a Riemann surface with null-boundary. We shall show that \( G_M \cup \tilde{G}_M \) is a Riemann surface with null-boundary. Let \( \omega''_n(z) \) be a harmonic function in \( G_M \cap R_n \) such that
\[ \omega''_n(z) = 0 \text{ on } F', \quad \omega''_n(z) = 1 \text{ on } G_M \cap (R - R_n) \text{ and } \frac{\partial}{\partial n} \omega''_n(z) = 0 \text{ on } (\partial G_M \cap R_n) - F', \]

where \( F' \) is a compact arc on \( \partial G_M \cap R_n \). Then clearly \( \omega''_n(z) \Rightarrow \omega''(z) \) as \( n \to \infty \) and \( D(\omega''(z)) < \infty \). Now \( G_M \) is a domain in the Riemann surface \( R^* \) with null-boundary (\( R^* \) is the doubled surface) of \( (G_M \cap (R - R_n)) + G_r \cap R_n \)). Hence by Lemma 1 there exists a sequence of compact surfaces \( R_m^* (m = 1, 2, \ldots) \) of \( R_m^* \) such that \( \bigcup_{m} R_m^* = R^* \) and \( \int_{\partial R_m^* \cap \theta_{M}} |\frac{\partial}{\partial n} \omega''(z)| ds \to 0 \) as \( m \to \infty \).

Now \( G_M \) is a domain in the Riemann surface \( R^* \) with null-boundary (\( R^* \) is the doubled surface) of \( (G_M \cap (R - R_n)) + G_r \cap R_n \)). Hence by Lemma 1 there exists a sequence of compact surfaces \( R_m^* (m = 1, 2, \ldots) \) of \( R_m^* \) such that \( \bigcup_{m} R_m^* = R^* \) and \( \int_{\partial R_m^* \cap \theta_{M}} |\frac{\partial}{\partial n} \omega''(z)| ds \to 0 \) as \( m \to \infty \).

I. \( e \).

\[ G_M + \hat{G}_M \] is a Riemann surface with null-boundary.

Proof of 3). Let \( M < M' \). Then \( \int_{(\Theta_{M'} - \theta_{M}) \cap \partial R_m^*} |\frac{\partial}{\partial n} V(z)| ds \to 0 \) as \( m \to \infty \). By \( \frac{\partial}{\partial n} V(z) \geq 0 \) on \( C_M + C_M^{\alpha} \) we have at once \( \int_{C_M} \frac{\partial}{\partial n} V(z) ds = K = \text{const.} \) and \( D(\min(M, V(z))) = MK \) for any \( M \).

Proof of 4). Assume \( M_0 < \sup V(z) \leq M \). Then \( V(z) = M_i w(G_{M_i}, z, R) \leq M w(G_{M_0} \cap G_{M_i}, z, R) \). Let \( M_i \uparrow M \). Then \( G_{M_i} \to B \) or the set of capacity zero where \( V(z) = \infty \). Hence

\[ 0 \leq V(z) \leq M w(G_{M_0} \cap B, z, R) = 0. \]

Proof of 5). Put \( G_M = E[z \in R : V(z) > M], \ G_\varepsilon = E[z \in R : V(z) > \varepsilon] \) and \( G'_M = E[z \in R : V'(z) > M] \). Then \( G_\varepsilon \supset G_M \supset G'_M \). Let \( V'_n(z) \) be a harmonic function in \( (G_{\varepsilon} \cap G_M \cap R_n) \) such that \( V'_n(z) = \varepsilon \) on \( \partial G_{\varepsilon}, \ V'_n(z) = M \) on \( \partial G'_M \) and \( \frac{\partial}{\partial n} V'_n(z) = 0 \) on \( \partial R_n \cap (G_\varepsilon - G'_M) \). Then \( V'_n(z) \) has M.D.I. over \( R_n \cap (G_\varepsilon - G'_M) \) with value \( \varepsilon \) on \( \partial G_{\varepsilon} \) and \( M \) on \( \partial G'_M \), whence \( D(V'_n(z)) \leq D(V(z)) \leq M \alpha \). Hence \( V'_n(z) \Rightarrow V'_*(z) \) as \( n \to \infty \) and \( D(V'_*(z)) \leq M \alpha \). On the other hand, \( 0 \leq V'_n(z) \leq \varepsilon + M w(G'_M, z, R) + M w(G_M \cap B, z, R) = \varepsilon + M w(G'_M \cap (R - R_n), z, R) \) and \( V'_*(z) \leq \varepsilon + M w(G'_M, z, R) + M w(G_M \cap B, z, R) = \varepsilon + M w(G'_M, z, R) \). Let \( \varepsilon \to 0 \). Then also \( V'_*(z) \Rightarrow V_*(z) \) and \( V_*(z) \leq M w(G'_M, z, R) \) and \( D(V_*(z)) \leq M \alpha \). By the definition of \( w(G'_M, z, R) V_*(z) \geq M w(G'_M, z, R) \). Hence

\[ V_*(z) = M w(G'_M, z, R). \quad (1) \]

Also \( M w(G'_M, z, R) \leq V(z) \leq \varepsilon + M w(G_\varepsilon \cap B, z, R) + M w(G'_M, z, R) \). By letting \( \varepsilon \to 0 \) and by (1) we have
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\[ V'(z) = Mw(G'_M, z, R) = V^*(z) \text{ in } R - G'_M. \]

Next by \( D(V^*(z)) < \infty \) we can prove similarly as the proof of (3)
\[
\int_{\partial G'_M} V'(z) ds = K \text{ and } D(\min(M, V'(z))) = MK,
\]
we have \( K \leq \alpha \). By \( V'(z) \leq V(z) \) we have at once \( w(G'_M \cap B, z, R) \leq w(G_M \cap B, z, R) = 0 \). Thus \( V'(z) \) is a G.G.

Proof of 6. At first we show \( V(z) < \infty \) in \( R \). Let \( V_m(z) \) be a G.G. and put \( G = \{ z \in R : V_m(z) > l \} \). Then by \( w(G, z, R) = 0 \) we see \( V_m(z) \Rightarrow V_m(z) \) as \( n \to \infty \), where \( V_m(z) \) is a harmonic function in \( R \cap (G - G'_M) \) such that \( V_m(z) = M \) on \( \partial G_M \), \( V_m(z) = \varepsilon \) on \( \partial G \), and \( \frac{\partial}{\partial n} V_m(z) = 0 \) on \( \partial R_n \cap (G - G'_M) \). Let \( D \) be a compact disc in \( R \). Suppose \( V(z) \geq L \) on \( D \). Let \( \omega_n(z) \) be a harmonic function in \( (R_n - D) \cap G \) such that \( \omega_n(z) = 1 \) on \( \partial G \), \( \omega_n(z) = \varepsilon \) on \( \partial R \), and \( \frac{\partial}{\partial n} \omega_n(z) = 0 \) on \( \partial R_n \). Then
\[
D(V_m(z) - L\omega_n(z), \omega_n(z)) \geq 0.
\]
Hence by mean convergency of \( V_{m,n}(z) \) and \( \omega_n(z) \) as \( n \to \infty \) we have
\[
L\alpha \geq D(\min(L, V_m(z))) \geq L^2D(\omega(z)),
\]
where \( \omega(z) = \lim \omega_n(z) \) and \( D(\omega(z)) < \infty \).

Let \( \varepsilon \to 0 \). Then \( \omega(z) \Rightarrow \omega(z) \) as \( \varepsilon \to 0 \) and \( L\alpha \geq L^2D(\omega(z)) \). Whence \( \min V_n(z) \leq \frac{\alpha}{D(\omega(z))} \) and \( \min V(z) \leq \frac{\alpha}{D(\omega(z))} \). Hence by Harnack's theorem \( V(z) < \infty \) in \( R \).

Put \( G = \{ z \in R : V(z) > M \} \) and \( G_{M,m} = \{ z \in R : V_m(z) > M \} \). Then by \( V(z) \geq V_m(z) \) \( G_M \supset G_{M,m} \). Since \( V_m(z) = Mw(G_{M,m}, z, R) \), \( V_m(z) \) is the least positive harmonic function in \( R - G_M \) with value \( V_m(z) \) on \( \partial G_M \). Hence \( V_m(z) = \lim V_{m,n}(z) \), where \( V_{m,n}(z) \) is the least positive harmonic function in \( R_n - G_M \) such that \( V_{m,n}(z) = V_m(z) \) on \( \partial G_M \) and \( V_{m,n}(z) = 0 \) on \( \partial R_n - G_M \). Hence
\[
V_{m,n}(z) = \frac{1}{2\pi} \int_{\partial G_M \cap R_n} V_{m,n}(\zeta) \frac{\partial}{\partial n} G_n(\zeta, z) ds,
\]
where \( G_n(\zeta, z) \) is the Green's function of \( R_n - G_M \).

Let \( n \to \infty \). Then \( \frac{\partial}{\partial n} G_n(\zeta, z) \uparrow \frac{\partial}{\partial n} G(\zeta, z) \) on \( \partial G_M \), where \( G(\zeta, z) \) is the Green's function of \( R - G_M \). Hence \( V_m(z) = \frac{1}{2\pi} \int_{\partial G_M} V_m(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds \). Now \( V_m(z) \leq
Let $V(z)$ be the least positive harmonic function in $R-G_M$ with value $M$ on $\partial G_M$, i.e. $V(z)=M\omega(G_M, z, R)$ in $R-G_M$.

This means that $V(z)$ is the least positive harmonic function in $R-G_M$ with value $M$ on $\partial G_M$, i.e. $V(z)=\frac{1}{2\pi}\int_{\partial D}V(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds = \lim_{m}V_m(z)$.

Let $M \rightarrow \infty$. Then $w(G_M, z, R) \rightarrow 0$. This implies $w(G_M, z, R) \geq w(G_M, z, R) \geq w(G_M, z, R) \geq 0$. Clearly $w_n(z)$ is the least positive harmonic function in $R-G_M$ larger than $M$ on $G_M$, $w(G_M, z, R) \geq 0$. Hence

$w(G_M, z, R) = w(G_M, z, R) + w(G_M, z, R) \geq w(G_M, z, R)$.

Let $n \rightarrow \infty$. Then $w_n(z)$ is the least positive harmonic function in $R_n-G_M$ such that $w_n(z)=M$ on $(R_n-R_n) \cap G$.

Then $w_n(z)=w(G_M, z, R) \geq w(G_M, z, R)$.

Clearly $w_n(z)+w(G_M, z, R) \geq w(G_M, z, R)$.

Let $G$ be a compact domain completely contained in $R-G_M$. Then there exists a number $n_0$ such that $G \subset \{z \in R: V_n(z)<M\}$ for $n \geq n_0$. Then

$D(V(z)) \leq D(\lim V_n(z)) \leq \lim_n D(V_n(z)) \leq D(\min(M, V_n(z))) \leq \alpha M$.

Let $G \rightarrow E[z \in R: V(z)<M]$. Then $D(\min(M, V(z))) \leq \alpha M$. Thus $V(z)$ is a G.G.

Proof of 7). Let $U_n(z)$ be the least positive harmonic function in $R_n-G_M$ such that $U_n(z)=M$ on $\partial G_M$, $U_n(z)=0$ on $\partial R_n-R_M$, $\frac{\partial}{\partial n}U_n(z)=0$ on $(\partial R_n \cap R_M)-G_M$. Then $D(U_n(z)) \leq D(\min(M, U(z))) \leq \alpha M$. Also let $U_n^*(z)$ be the least positive harmonic function in $R_n-G_M$ larger than $M$ on $G_M$. Then

$0 \leq U_n^*(z) \leq U_n(z) \leq U_n^*(z) + M\omega(R_M \cap (R-R_n), z, R)$.

Let $n \rightarrow \infty$. Then since $G(x, p)$ is a G.G., $w(R_M \cap (R-R_n), z, R) \rightarrow w(R_M \cap B, z, R)$.

Hence $0 \leq \lim_n U_n^*(z) = \lim_n U_n(z) = U_M(z)$. By $U_M(z)=\lim_n U_n(z)$, we have

$D(\min(M, U_M(z))) \leq \lim_n D(U_n(z)) \leq \alpha M$.

By $U_M(z)=\lim_n U_n^*(z)$,

$U_M(z) = M\omega(G_M, z, R)$

and $U_M(z)=N\omega(G_M, z, R)$ in $R-G_M$, where $G_M^N=E[z \in R: U_M(z)>N] + G_M$ for any $N \leq M$ and for any $D \subset R-G_M$.

$U_M(z) = \frac{1}{2\pi}\int_{\partial D}U_M(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds$, (4)
where $G(\zeta, z)$ is the Green's function of $D$.

By (2) and (3) we have similarly as the proof of 2), $\omega(G'_{n}, B, z, G'_{n})=0$: $M \geq N > N$.

This implies that the doubled surface $(G'_{N^{\prime}}, + G'_{N^{\prime}})$ is a Riemann surface with null-boundary, whence $\int \frac{\partial}{\partial n} U_{M}(z) ds=k=\text{const.}$, where by (2) $k \leq \alpha$ for any $C_{l}=E[z \in R: U_{M}(z)] \leq N \alpha$.

Clearly $U_{M}(z) \uparrow \tilde{U}(z)$ as $M \rightarrow \infty$. Then by (5) $D(\min(N, \tilde{U}(z))) \leq \varliminf_{M=\infty} D(\min(N, U_{M}(z))) \leq N \alpha$ and $\tilde{U}(z) < \infty$.

Thus by (6) and (7) $\tilde{U}(z)$ is a G.G.

2. Green's potential. Let $R^*$ be a Riemann surface with positive boundary and let $\{R_{n}^{*}\}$ be its exhaustion with compact relative boundary $\partial R_{n}^{*}$ $(n=1, 2, \cdots)$. Let $p_{0}$ be a fixed point in $R$ and let $G(z, p_{0})$ be the Green's function. Put $R=E[z \in R^{*}: G(z, p_{0}) > \delta]$. Then by Theorem 1), 2) the doubled surface $R + \hat{R}$ is a Riemann surface with null-boundary. Let $G(z, p)$ be the Green's function of $R$. Then $G(z, p)$ is a G.G. in $R$. Let $\{p_{n}\}$ be a divergent sequence in $G_{=}E[z \in R: G(z, p_{0}) > \epsilon]$ such that $\{G(z, p_{n})\}$ converges to a positive harmonic function. Then we say that $\{p_{n}\}$ determines an ideal boundary point $p$. We denote by $B$ the set of all the ideal boundary points. Also we denote $\lim_{n} G(z, p_{n})$ by $G(z, p)$: $p \in B$ simply. Let $q \in \partial R$ and let $v(q)$ be a compact neighbourhood of $q$ in $R^{*}$. Then $G(z, p_{n}) \leq M$ on $v(q) \cap R$ for $n \geq n(q)$. Hence $G(z, p)(p \in B)=0$ on $\partial R$ and $\omega(R \cap B, z, R) \leq \omega(R \cap B, z, R^{*})=0$. Also by Fatou's lemma $D(\min(G(z, p), M)) \leq 2\pi M$ for $p \in R + B$. Thus $G(z, p)$ is a G.G. for $p \in R + B$. Let $\bar{R}=R + B$. Then the distance between $p_{i}$ and $p_{i}$ in $\bar{R}$ is defined as

$$\delta(p_{i}, p_{j}) = \sup_{z \in R_{0}} \left| \frac{G(z, p_{i})}{1 + G(z, p_{i})} - \frac{G(z, p_{j})}{1 + G(z, p_{j})} \right|,$$

where $R_{0}$ is a compact disc in $R$. 
Remark. Let $p_1 \neq p_2$ in $B$. Then $G(z, p_i)$ may be a multiple of $G(z, p_2)$). In fact, let $C$ be a unit circle: $|z| < 1$ and let $F$ be a closed set such that $z = 0$ is contained in the closure of $F$ and $F$ is so thinly distributed in a neighbourhood of $z = 0$ that $z = 0$ is an irregular point of the Dirichlet problem in $C - F$. Then there exists only one linearly independent G.G. vanishing on $F + \partial C$ except $z = 0$ and for any point $p$ on $z = 0$, $G(z, p)$ is a multiple of a fixed function.

Let $G(z, p): p \in \mathbb{R}$ and $V_M(p) = E[G(z, p) > M]$. Let $U(z)$ be a G.G. in $R$ and let $U_M^L(z)$ be the least positive harmonic function in $V_M(p)$ with boundary value min$(L, U(z))$ on $\partial V_M(p)$. Then $U_M^L(z) \leq U(z)$. Also by $w(V_M(p) \cap B, z, R) = 0$, $U_M^L(z) = \lim U_n(z)$, where $U_n(z)$ is a harmonic function in $V_M(p) \cap R_n$ such that $U_n(z) = \min(L, U(z))$ on $\partial V_M(p) \cap R_n$ and $U_n(z) = 0$ on $\partial R_n \cap V_M(p)$, where $D(U_n(z)) \leq D(\min(L, U(z))) > \infty$. Since $R + \hat{R}$ is a Riemann surface with null-boundary, there exists an exhaustion $\{R_n\}$ of $(R + \hat{R})$ such that

$$\int_{\partial R_n \cap V_M(p)} |\frac{\partial}{\partial n} U_n^L(z)| ds \rightarrow 0 \text{ as } n \rightarrow \infty : M' > M$$

Lemma 1. By Green's formula

$$\int_{\partial V_M(p) \cap R_n^r} U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds + \int_{\partial V_M(p) \cap R_n^r} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = \int_{\partial V_M(p) \cap R_n^r} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds$$

Now

$$\int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = M \int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = 0$$

$$\int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds = M' \int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds$$

and other integrations,

$$\int_{\partial V_M(p) \cap R_n'} G(z, p) \frac{\partial}{\partial n} U_M^L(z) ds$$

over $\partial R_n' \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds = \lim_{L \rightarrow \infty} \int_{\partial V_M(p)} U_M^L(z) \frac{\partial}{\partial n} G(z, p) ds$$

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\[ = \lim_{L \to \infty} \int_{\partial V_M(p)} U_M(z) \frac{\partial}{\partial n} G(z, p) ds \leq \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds. \] 

Hence

\[ \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds \uparrow \text{ as } M \uparrow \infty. \]

We define the value of \( U(z) \) at \( p \) by

\[ \frac{1}{2\pi} \lim_{M \to \infty} \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds \]

this definition coincides with \( U(p) \) clearly. Then

**Theorem 2.** \( G(z, p) : p \in \overline{R} \) is a G.G. and the value of \( G(z, p) \) is well defined on \( \overline{R} \) and

1. \( G(p, q) = G(q, p) \) and \( G(p, p) = \infty \).
2. \( G(p, q) \) is lower semicontinuous in \( \overline{R} \times \overline{R} \).
3. If \( p \in R \), \( G(z, p) \) is continuous in \( \overline{\overline{R}} \).

Theorem 2 can be proved (without any essential alteration) similaaly as \( G(z, p) \) of \( R - R_0 \), where \( R \) is a Riemann surface with null-boundary and \( R_0 \) is a compact disc.\(^8\) But in the previous paper, it was proved that \( G(z, p) \) is lower semicontinuous in \( \overline{R} \) for fixed \( p \). In this paper we shall prove only 2).

For any given number \( \varepsilon > 0 \), we can find numbers \( M \) and \( R_n \) such that

\[ 2\pi (G(p, q) - \varepsilon) \leq \int_{\partial V_M(p) \cap R_n} G(\zeta, q) \frac{\partial}{\partial n} G(\zeta, p) ds. \]

We cover \( V_M(p) \cap R_n \) by a compact domain \( D \) such that \( \partial D \) intersects \( \partial V_M(p) \) orthogonally. Let \( p_i \to p \) and \( q_j \to q \). Then \( G(z, p_i) \to G(z, p) \), \( G(z, q_j) \to G(z, q) \) and \( V_M(p_i) \to V_M(p) \) uniformly in \( D \). Hence for \( i > i_0 \) and \( j > j_0 \)

\[ 2\pi G(p_i, q_j) = \lim_{M \to \infty} \int_{\partial V_M(p_i)} G(\zeta, q_j) \frac{\partial}{\partial n} G(\zeta, p_i) ds \geq \int_{\partial V_M(p_i)} G(\zeta, q_j) \frac{\partial}{\partial n} G(\zeta, p_i) ds \]

\[ \geq \int_{\partial V_M(p) \cap D} G(\zeta, p) \frac{\partial}{\partial n} G(\zeta, q) ds - 2\pi \varepsilon \geq 2\pi (G(p, q) - \varepsilon). \]

Let \( \varepsilon \to 0 \). Then \( \lim_{i, j} G(p_i, q_j) \geq G(p, q) \). Thus 2) is proved.

3. **Energy integral and capacities of Green’s potential.** Let \( R = R^*_\delta = \{ \varepsilon \in \mathbb{R}^*: G(z, p_0) > \delta \} \) and \( \{ R_n \} \) be an exhaustion of \( R \). Let \( F \) be a compact set in \( R \). Put \( I(\mu) = \int \int |G(p, q)| d\mu(p) d\mu(q) \), where \( \mu \) is a positive mass distribution on \( \overline{R} \). Since \( F \) is compact, \( G(z, p) + \log |z - p| \) is harmonic in a neigh-
bourhood of $p$ and the continuity principle is valid in $R$ and theorems of logarithmic potentials are also valid. Hence there exists a uniquely determined mass distribution $\mu_\circ$ called equilibrium distribution such that $I(\mu_\circ)$ is the minimal among all distributions of mass unity on $F$, the potential $U(z)$ of $\mu=L$ (const.) on $F$ except a set of capacity zero and $U(z) \leq L$ in $R$. Clearly $U(z)=0$ on $\partial R$ and by $w(B \cap R, z, R)=0$, $U(z)=Lw(F, z, R)$ and $I(\mu)=L=\frac{D(U(z))}{2\pi}$, where $L$ is given by $L=\frac{2\pi}{\int_{\partial R} w(F, z, R) ds}$, $C_M=E[z \in R : w(F, z, R)=M]$, $M<1$.

We define the capacity of $F$ as $\frac{1}{I(\mu_\circ)} \left(=\frac{1}{L}\right)$. Let $F$ be a closed set in $\overline{R}=R+B$. We also define $\text{Cap}(F)$ by $\frac{1}{\inf I(\mu)}$, where $\mu$ is a positive mass distribution of unity on $F$. Put $\hat{\text{Cap}}(F)=\sup_{K \subseteq F} \text{Cap}(K)$, where $K$ is a compact set in $F$. Then if $F$ is compact, $\text{Cap}(F)=\hat{\text{Cap}}(F)$ and $\text{Cap}(F) \geq \hat{\text{Cap}}(F)$ for closed set $F$ (in reality it can be proved $\text{Cap}(F)=\hat{\text{Cap}}(F)$). In this paper we use only $\hat{\text{Cap}}(F)$ for Green's potential.

**Capacities of the irregular set of $R$ of Green's function.** Let $G(z, p_0)$ be Green's function of $R=E[R^* \ni z : G(z, p_0) > \delta]$. Then $G(z, p_0)$ is continuous in $\overline{R}$. Let $F_i=E \left[ z \in R : G(z, p) \geq \frac{1}{l} \right]$. Then $F_i$ is closed in $\overline{R}$. $U(z)=\min \left( \frac{1}{l}, G(z, p_0) \right)$ is a continuous function in $\overline{R}$ such that $U(z)=0$ on $\partial R$, $U(z)=\frac{1}{l}$ on $F_t$ and $D(U(z)) \leq \frac{2\pi}{l}$. Let $R_n (n=1, 2, \cdots)$ be an exhaustion of $R$ with compact relative boundary $\partial R_n$. Let $\omega(F_t \cap (R-R_n), z, R)$ be a harmonic function in $R-(F_t \cap (R-R_n))$ such that $\omega(F_t \cap (R-R_n), z, R)=1$ on $F_t \cap (R-R_n)$, $=0$ on $\partial R$ and has M.D.I.. Then by the Dirichlet principle $D(\omega(F_t \cap (R-R_n), z, R)) \leq \frac{1}{l} D(U(z)) < \infty$. It is evident $\omega(F_t \cap (R-R_n), z, R) \geq \omega(K, z, R)$ for any compact set $K$ in $F_t \cap (R-R_n)$ and by $\omega(R \cap B, z, R)=0$, $\omega(F_t \cap (R-R_n), z, R)=\omega(F_t \cap (R-R_n), z, R)$. Let $n \to \infty$. Then $\omega(F_t \cap (R-R_n), z, R)$ converges in mean to a function $V(z)=\omega(F_t \cap B, z, R)$. Clearly $V(z)$ is a G.G., hence by Theorem 1. (5) $V(z)=0$. This implies $D(\omega(F_t \cap (R-R_n), z, R) \downarrow 0$ as $n \to \infty$ and $\lim \hat{\text{Cap}}(F_t \cap (R-R_n))=0$ as $n \to \infty$.

**Loss of mass.** As usual mass $m(p)$ of $G(z, p)$ ($p \in \overline{R}$) is given by $\frac{1}{2\pi} \int_{\partial R} \frac{\partial}{\partial n} G(z, p) ds$ ($m(p)$ does not depend on $M$ by Theorem 1. (3)). It is clear
$m(p)=1$ for $p \in R$. If $p \in F \cap B$, $m(p) \geq \frac{1}{kl}$, where $k = \lim_{n=\infty} \sup_{\varepsilon \in R - R_n} G(z, p_0)$. 
In fact, $G(p, p_0) = \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V(p)} G(\zeta, p_0) \frac{\partial}{\partial n} G(\zeta, p) d\sigma$.

Now $\partial V(p)$ clusters at $B$ as $M \uparrow \infty$. Hence for any number $\epsilon > 0$, there exists a number $M$ such that $G(\zeta, P_0) - \epsilon \leq k$ on $\partial V_M(p)$ for $M > M_0$, whence $\frac{1}{2\pi} \int_{\partial \nabla^{(p)}} \frac{\partial}{\partial n} G(\zeta, P) d\sigma \geq \frac{1}{l(k + \epsilon)}$. Hence by letting $\epsilon \rightarrow 0$, $m(p) \geq \frac{1}{kl}$.

4. *N*-Green's function and *N*-Martin's topology. Let $R$ be a Riemann manifold with positive boundary and let $\{R_n\}$ be its exhaustion with compact relative boundary $\partial R_n$. Let $U(z)$ be a positive superharmonic function in $R - R_0$ such that $U(z) = 0$ on $\partial R_0$ and $D(\min(M, U(z))) < \infty$ for any $M < \infty$. 

Let $U^M(z)$ be a continuous function in $R - R_0$ such that $U^M(z) = \min(M, U(z))$ on $\partial D$ and $U^M(z)$ has M.D.I. over $R - R_0 - D$, where $D$ is a compact or non compact domain. Let $U(z) = \lim_{M \rightarrow \infty} U^M(z)$. If $U(z) \geq U(z)$ for any domain, we call $U(z)$ a superharmonic function. Let $U(z)$ be a superharmonic function. 

If $V(z) = aU(z)$ for any superharmonic function $V(z)$ such that both $V(z)$ and $U(z) - V(z)$ are positively superharmonic, we call $U(z)$ an $N$-minimal function. 

Let $N(z, p): p \in R - R_0$ be an $N$-Green's function in $R - R_0$ such that $N(z, p)$ is harmonic in $R - R_0 - p$, $N(z, p)$ has a positive logarithmic singularity at $p$ and $N(z, p)$ has M.D.I. (Dirichlet integral is taken with respect to $N(z, p) + \log|z - p|$ in a neighbourhood of $p$). We use $N(z, p)$ instead of $G(z, p)$ and define the ideal boundary points and the distance $\delta(p_1, p_2)$ is defined as

$$\delta(p_1, p_2) = \sup_{z \in R_0} \frac{N(z, p_1)}{1 + N(z, p_1)} - \frac{N(z, p_2)}{1 + N(z, p_2)},$$

for $p_1$ and $p_2$ in $R - R_0 + B = \bar{R} - R_0$, where $B$ is the set of the ideal boundary points.

Let $F$ be a closed set in $R - R_0$ and put $F_n = \left\{ z \in \bar{R} : \delta(z, F) \leq \frac{1}{n} \right\}$. Let $\omega(F_n, z, R - R)$ be a harmonic function in $R - R_0 - F$ such that $\omega(F_n, z, R - R) = 1$ on $F_n \cap R$, $\omega(F_n, z, R - R) = 0$ on $\partial R_0$ and $\omega(F_n, z, R - R)$ has M.D.I. Let $n \rightarrow \infty$. Then $\omega(F_n, z, R - R) \Rightarrow \omega(F, z, R - R_0)$. We call $\omega(F, z, R - R_0)$ C.P. of $F$ and define $\text{Cap}(F)$ by $\frac{1}{2\pi} D(\omega(F, z, R_0 - R_0))$. We state briefly the properties of $B$ and $N(z, p)$ without proofs.\(^{9)}\)\(^{10)}\)

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9) See (5).
10) See (5).
Theorem 3. 1). \( B = B_1 + B_0 \) and \( B_0 \) is an \( F_\sigma \) set of capacity zero.

2). \( N(z, p) \) is \( N \)-minimal if and only if \( p \in B_1 \).

3). If \( \omega(p, z, R-R) > 0 \), \( \sup_{z \in R} N(z, p) < \infty \) and \( p \in B_1 \). We denote the set of \( p \) such that \( \omega(p, z, R-R) > 0 \) by \( B_0 \). Then if \( p \in B_0 \), \( N(z, p) = k_0(p, z, R-R) \).

4). For \( p \in \overline{R} - B_0 \),
\[ \int_{\partial V_M(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi , \] (10)

for almost \( M \), i.e. the set of \( M \) such that (10) is not hold is a set of measure zero. We call such \( \partial V_M(p) \) a regular niveau if (10) is satisfied, where \( V_M(p) = \{ z \in R : N(z, p) > M \}, \ M < M^*(p) = \sup_{z \in R} N(z, p) \).

5). For \( p \in \overline{R} - B_0 \), \( N(z, p) = M \omega(V_M(p), z, R-R_0) \) in \( R-R_0 - V_M(p) \).

6). Let \( p \in \overline{R} - B_0 \). For any \( M < M^*(p) \), there exists a number \( n \) such that
\[ V_M(p) \supset (R-R_0) \cap v_n(p) , \text{ where } v_n(p) = E[z \in \overline{R} : \delta(z, p) < \frac{1}{n}] . \]

7). Let \( p \in \overline{R} - B_0 \). Then for any \( v_n(p) \)
\[ \lim_{M \to M^*(p)} \int_{\partial V_M(p) \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 0 , \lim_{M \to M^*(p)} \int_{\partial V_M(p) \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi , \]

where \( \partial V_M(p) \) is regular.

And \( \lim_{M \to M^*(p)} N(z, p) = 0 \).

8). The value of \( N(z, p) : p \in \overline{R} - R_0 \) is given as follows:

a). \( q \in \overline{R} - B_0 \), \( N(q, p) = \lim_{M \to M^*(q)} \frac{1}{2\pi} \int_{\partial V_M(q)} N(\zeta, q) \frac{\partial}{\partial n} N(\zeta, q) ds \),

where \( \partial V_M(q) \) is a regular niveau.

b). \( q \in B_0 \). In this case \( N(z, q) \) is represented as \( \int_{B_0} N(z, r) d\mu_q(r) \) and \( N(q, p) = \int_{B_0} N(r, p) d\mu_q(r) \) and the value \( N(q, p) \) does not depend on particular distribution \( \mu_q(r) \).

9). \( N(p, q) = N(q, p) \) and \( N(z, p) \) is lower semicontinuous in \( \overline{R} - R_0 \) for fixed \( p \).

10). If \( p \in \overline{R} - B_0 \),
\[ N(p, q) \geq \frac{1}{2\pi} \int_{\partial V_M(p)} N(\zeta, q) \frac{\partial}{\partial n} N(\zeta, p) ds . \]

Energy integral and the capacitites. Let \( F \) be a closed set in \( \overline{R} - R \) of positive capacity, i.e. \( \omega(F, z, R-R_0) > 0 \). Then there exists a canonical distri-
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bution of unity on $F$ (if $\mu=0$ on $B_0$, $\mu$ is called canonical) such that their energy integral $I(\mu) = \inf\{I(\mu_\xi)\}$ and the potential $U(z)$ of $\mu = \omega(F, z, R - R_0)$ and $I(\mu) = \frac{1}{2\pi} D(\omega(F, z, R - R_0)) = L$, where $\mu_\xi$ is a canonical mass distribution of unity on $F$ and $L = \frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F, z, R - R_0) ds$. We call $\frac{1}{L}$ the capacity of $F$. Also define $\hat{C}ap(F)$ by $\lim_{n=\infty} \hat{C}ap(F_n) = 1/\inf I(\mu_i)$, where $\mu_i$ is a distribution of unity on $R \cap F_n$ (not on $(\overline{R} - B_0) \cap F_n$). We see at once $\hat{C}ap(F) \leq \hat{C}ap(F)$.

5. Transfinite diameters. We discuss transfinite diameters and others of closed sets on Riemann surfaces. The properties of the surfaces have much influence on them. Therefore we divide types of Riemann surfaces into three types as follows: (A). $R$ is a Riemann surface with positive boundary and $N$-Martin's topology is defined on $\overline{R} - R_0 = R + B - R_0$. (B) $R = R_\delta = E[z \in R^* : G^*(z, p) > \delta > 0]$, where $R^*$ is a Riemann surface with positive boundary and G-Martin's topology is defined on $\overline{R} = R + B$. (C). $R$ is a Riemann surface with null-boundary and G-Martin's topology is defined on $R - R_0 + B = \overline{R} - R_0$.

Let $F$ be a closed set in $\overline{R}$. We define transfinite diameters $\delta(F), D^M(F)$ and $D(F)$ as follows:

(A). $1/D(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \neq p_j, p_i \in F - B_0} N(p_i, p_j)$, $1/D^M(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \neq p_j, p_i \neq p_j} N^M(p_i, p_j)$ and $\delta(F) = \lim M=\infty D(F)$, where $N^M(z, p) = \min(M, N(z, p))$.

(B) and (C). $1/D(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \neq p_j, p_i \neq p_j} G(p_i, p_j)$, $1/D^M(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \neq p_j, p_i \neq p_j} G^M(p_i, p_j)$ and $\delta(F) = \lim M=\infty D^M(F)$, where $G^M(z, p) = \min(M, G(z, p))$.

Then clearly $\frac{1}{nC_2} \inf \sum_{p_i \neq p_j} N^M(p_i, p_j) \downarrow D(F)$ as $M \uparrow \infty$ and $D^M(F)$, $\delta(F)$, $D(F)$ are increasing functions of $F$.

Lemma 2. Let $p_i (i=1, 2) \in \overline{R} - B_0$. Let $D_i$ be a compact or non compact domain in $V_M(p_i) = E[z \in R : N(z, p_i) > M]$. Let $\omega(D_i, z)$ be C.P. of $D_i$, i.e. $\omega(D_i, z) = \omega(D_i, z, \overline{R} - R_0)$. Then $M\omega(D_i, z)(\leq N^M(z, p_i))$ can be represented by a canonical mass distribution $\mu^*_i$ on $D_i$ such that $\omega(D_i, z) = \int N(z, q) \times d\mu^*_i(q) in \overline{R} - R_0$ and (clearly $\int d\mu^*_i \leq 1$)

$$\int M\omega(D_j, q) d\mu^*_i(q) \leq N^M(p_i, p_j), \quad i, j = 1, 2,$$

where not necessarily $p_i \neq p_j$.

11) See (5).
(B) and (C). Let $D_i = V_M(p_i)$. Then $M\omega(D_i, z) = M\omega(D_i, z) = \min(M, G(z, p_i))$ ($p \in \overline{R}$ for (B) and $p \in \overline{R} - R_0$ for (C)) is represented by a mass $\mu_i$ on $R$ such that $\mu_i = 0$ on $B$ and

$$
\int M\omega(D_j, q) d\mu_j(q) \leq G^M(p_i, p_j), \quad j, i = 1, 2,
$$

where not necessarily $p_i \neq p_2$.

**Proof.** Case 1. $p_2 \in B_1$ or $p_2 \in D_1$. Put $D_{1,n} = D_1 \cap R_n$. Then $D_{1,n}$ is compact. In this case, we can find a number $M'$ such that $V_M(p_2) \cap D_{1,n} = 0$ and $\partial V_M(p_2)$ is regular, because $V_M(p_2) \rightarrow B$ as $M \rightarrow M^*(p_2)$. Put $d\mu_{1,n} = \frac{M}{2\pi} \times \frac{\partial}{\partial n}\omega(D_{1,n}, z) ds$ on $\partial D_{1,n}$, i.e.

$$
\omega(D_{1,n}, z) = \int N(z, r) d\mu_{1,n}(r), \quad z \in R - R_0. \quad (11)
$$

Let $m > n$ and let $N_m(z, p_2)$ be a harmonic function in $R_m - R_0 - V_M(p_2)$ such that $N_m(z, p_2) = N(z, p_2)$ on $\partial R_0 + \partial V_M(p_2)$ and $\frac{\partial}{\partial n}N_m(z, p_2) = 0$ on $\partial R_m - V_M(p_2)$. Then by Theorem 3. (5) $N_m(z, p_2) \Rightarrow N(z, p_2)$ as $m \rightarrow \infty$ and

$$
\lim_{m \rightarrow \infty} \int_{\partial V'(p_2) \cap R_m} A_m(z) \frac{\partial}{\partial n} N_m(z, p_2) ds = \int_{\partial V'(p_2) \cap R_m} A(z) \frac{\partial}{\partial n} N(z, p_2) ds, \quad (12)
$$

by the Lemma 12) for any $A_m(z)$ such that $0 \leq A_m(z) \leq L < \infty$ on $\partial V_M(p_2)$ and $\lim A_m(z) = \hat{A}(z)$ on $\partial V_M(p_2)$. Also let $\omega_m(D_{1,n}, z)$ be a harmonic function in $R_m - R_0 - D_{1,n}$ such that $\omega_m(D_{1,n}, z) = \omega(D_{1,n}, z)$ on $\partial R_0 + \partial D_{1,n}$ and $\frac{\partial}{\partial n}\omega_m(D_{1,n}, z) = 0$ on $\partial R_m$. Then $\omega_m(D_{1,n}, z) \Rightarrow \omega(D_{1,n}, z)$ as $m \rightarrow \infty$. Then by

$$
\int_{\partial D_{1,n}} \frac{\partial}{\partial n} \omega_m(D_{1,n}, z) ds = \int_{\partial D_{1,n}} \frac{\partial}{\partial n} \omega(D_{1,n}, z) ds = 0 \text{ we have Green's formula}
$$

$$
M \int_{\partial D_{1,n}} N(z, p_2) \frac{\partial}{\partial n} \omega_m(D_{1,n}, z) ds = M \int_{\partial D_{1,n}} \omega_m(D_{1,n}, z) \frac{\partial}{\partial n} N_m(z, p_2) ds.
$$

Let $m \rightarrow \infty$. Then by (12)

$$
\frac{1}{2\pi} M \int_{\partial D_{1,n}} N(z, p_2) \frac{\partial}{\partial n} \omega(D_{1,n}, z) ds = \frac{1}{2\pi} \int_{\partial V_M(p_2)} M\omega(D_{1,n}, z) \frac{\partial}{\partial n} N(z, p_2) ds
$$

$$
\leq \frac{1}{2\pi} \min(M, N(z, p_2)) \frac{\partial}{\partial n} N(z, p_2) ds
$$

12) See (5).
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\[ \leq \frac{1}{2\pi} \min \left( M, \lim_{M \to M^*} \frac{1}{\partial \Sigma M^*(p_i)} \int N(\zeta, p_i) \frac{\partial}{\partial n} N(\zeta, p_i) ds \right) = N^M (p_2, p_i). \quad (13) \]

Put \( d\mu_{1,n} (\zeta) = \frac{M}{2\pi} \frac{\partial}{\partial n} \omega (D_{1,n}, \zeta) ds \) on \( \partial D_{1,n} \). Then by (13)

\[ \int_{\partial D_{1,n}} N(\zeta, p_i) d\mu_{1,n} (\zeta) \leq N^M (p_2, p_i). \quad (14) \]

Next suppose \( r \in R - R_0 \). Then

\[ \frac{M}{2\pi} \int_{\partial D_{1,n}} N(r, \zeta) \frac{\partial}{\partial n} \omega (D_{1,n}, \zeta) ds = M \omega (D_{1,n}, r), \]

whence \( M \omega (D_{1,n}, z) = \int N(z, \zeta) d\mu_{1,n} (\zeta), \quad z \in R - R_0. \quad (15) \)

Hence \( \mu_{1,n} (\zeta) \) is the mass distribution of \( M \omega (D_{1,n}, z) \) and \( \int d\mu_{1,n} \leq 1 \). Let \( n \to \infty \). Then there exists an weak limit \( \mu \) on \( \bar{D}_1 \) of \( \{ \mu_{1,n} \} \) such that

\[ \int N(z, \zeta) d\mu_{1,n} (\zeta) \to \int N(z, \zeta) d\mu_{1} (\zeta) \quad \text{for} \quad z \in R - R_0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad M \omega (D_{1,n}, z) = \lim_{n \to \infty} \omega (D_{1,n}, \zeta) \quad \text{for} \quad z \in R - R_0. \]

We can find a canonical distribution \( \mu^* \) such that \( M \omega (D_{1,n}, z) = \int N(z, \zeta) d\mu^* (\zeta) \) for \( z \in R - R_0 \). On the other hand, since \( p_i \omega (D_{1,n}, z) = \omega (D_{1,n}, z) \) for any \( l \), any canonical distribution of \( \omega (D_{1,n}, z) \) has no mass on \( CD_{1,13} \), where \( p_i \omega (D_{1,n}, z) \) is the harmonic function in \( R - R_0 - D_l \) such that \( p_i \omega (D_{1,n}, z) = \omega (D_{1,n}, z) \) on \( \partial D_l \) and \( p_i \omega (D_{1,n}, z) \) has M.D.I. and \( D_l = E \left[ z \in R : \delta (z, D_l) < \frac{1}{l} \right] \). Hence \( \mu^* \) is a canonical distribution on \( \bar{D}_1 \cap (\bar{R} - B_0) \) such that \( M \omega (D_{1,n}, z) = \int N(z, \zeta) d\mu^* (\zeta), \quad z \in R - R_0. \quad (16) \)

By (16) and the definition of \( \omega (D_{1,n}, p) \) for \( p \in \bar{R} - R_0 - B_0 \),

\[ M \omega (D_{1,n}, p) = \frac{M}{2\pi} \lim_{M \to M^* \left( \frac{1}{\partial \Sigma M^*(p)} \right)} \int N(\zeta, p) \frac{\partial}{\partial n} N(\zeta, p) ds = \frac{M}{2\pi} \lim_{M \to M^* \left( \frac{1}{\partial \Sigma M^*(p)} \right)} \int N(\zeta, \eta) \times \]

\[ \frac{\partial}{\partial n} N(\zeta, p) ds = \frac{M}{2\pi} \lim_{M \to M^* \left( \frac{1}{\partial \Sigma M^*(p)} \right)} \int \int N(\zeta, \eta) \frac{\partial}{\partial n} N(\zeta, p) ds d\mu^* (\eta) = \int \int N(\eta, p) d\mu^* (\eta), \quad (17) \]

because by \( \int_{\partial \Sigma M^*(p)} \frac{\partial}{\partial n} N(\zeta, p) ds \) as \( M \uparrow M^* (p) \) the order of integrations and letting \( M \uparrow M^* (p) \) can be changed. Hence (16) is valid for \( z \in \bar{R} - R_0 - B_0 \).

Similarly we have by (11)

13) See (5).
\[ \omega(D_{1,n}, p) = \int N(\zeta, p) d\mu_{1,n}(\zeta) \quad \text{for} \quad p \in \overline{R} - R_{0} - B_{0}, \quad (18) \]

i.e. (11) is valid for \( z \in \overline{R} - R_{0} - B_{0} \).

Further by \( \omega(D_{1,n}, z) \uparrow \omega(D_{1}, z) \) as \( n \to \infty \), we have

\[
\omega(D_{1}, p) = \frac{1}{2\pi} \lim_{M \to \infty} \int_{\partial V_{M}(p)} \omega(D_{1}, z) \frac{\partial}{\partial n} N(z, p) ds
\]

\[= \frac{1}{2\pi} \lim_{n} \left( \int_{\partial V_{M}(p)} \omega(D_{1,n}, z) \frac{\partial}{\partial n} N(z, p) ds \right) = \lim_{n} \omega(D_{1,n}, p). \quad (19) \]

Hence by (17), (19), (15) and (14)

\[
M \omega(D_{2}, \eta) d\mu_{1}^{*}(\eta) \leq \int N(\zeta, p_{2}) d\mu_{1}^{*}(\eta)
\]

\[= M \omega(D_{1}, p_{2}) = \lim_{n} M \omega(D_{1,n}, p_{2}) \leq N^{M}(p_{2}, p_{1}) . \quad (20) \]

Case 2. \( p_{2} \in D_{1} \cap (R - R_{0}) \) (\( j = 1 \) or \( 2 \)). In this case \( N(z, p_{i}) \geq M \) in \( D_{i} \). Hence

\[ N(p_{2}, p_{1}) = N(p_{1}, p_{2}) \geq M . \quad (21) \]

There exists a number \( n \) such that \( D_{1,n} \ni p_{2} \). Let \( m > n \) and let \( N_{m}(z, p_{i}) \) be a harmonic function in \( R_{m} - R_{0} - p_{2} \) such that \( N_{m}(z, p_{2}) = 0 \) on \( \partial R_{m} \), \( N_{m}(z, p_{2}) \) as a logarithmic singularity at \( p_{2} \) and \( \frac{\partial}{\partial n} N_{m}(z, p_{2}) = 0 \) on \( \partial R_{m} \). Then \( N_{m}(z, p_{2}) \Rightarrow N(z, p_{2}) \) as \( m \to \infty \). Let \( \omega_{m}(D_{1,n}, z) \) be the function in case 1. Then

\[
\frac{M}{2\pi} \int_{\partial D_{1,n}} N_{m}(\zeta, p_{2}) \frac{\partial}{\partial n} \omega_{m}(D_{1,n}, \zeta) ds
\]

\[= \frac{M}{2\pi} \int_{\partial D_{1,n}} \omega_{m}(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_{m}(z, p_{2}) ds = \frac{M}{2\pi} \int_{\partial D_{1,n}} \frac{\partial}{\partial n} N_{m}(\zeta, p_{2}) ds
\]

\[\leq M. \quad \text{Let} \ m \to \infty. \quad \text{Then by (21)}
\]

\[
\frac{M}{2\pi} \int_{\partial D_{1,n}} N(\zeta, p_{2}) \frac{\partial}{\partial n} \omega(D_{1,n}, \zeta) ds \leq M = N^{M}(p_{2}, p_{1}) .
\]

Put \( d\mu_{n} = \frac{M}{2\pi} \frac{\partial}{\partial n} \omega(D_{1,n}, \zeta) ds \) on \( \partial D_{1,n} \). Then as in case 1), there exists a canonical distribution \( \mu_{1}^{*} \) such that

\[ M \omega(D_{1}, z) = \int_{R - B_{0}} N(z, \zeta) d\mu_{1}^{*}(\zeta) \]
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\[ M \int \omega(D_2, \eta) d\mu_1^*(\eta) \leq N(\eta, p_2) d\mu_1^*(\eta) \leq N^M(p_2, p_1). \]  

(22)

If \( p_2 \in R - D_i \), we have more easily (22). Thus by (21) and (22) we have the former part of Lemma 2.

(B) and (C). Suppose \( p_2 \not\in V_M(p_i) \). Then by Green's formula and by Lemma 1

\[ G^M(p_i, p_1) = G(p_i, p_1) = \frac{1}{2\pi} \int_{\partial V_M(p_i)} G(\zeta, p_2) \frac{\partial}{\partial n} G(\zeta, p_1) ds, \]  

(23)

If \( p_2 \in V_M(p_i) \), \( G(p_i, p_2) \geq M \) and

\[ G^M(p_i, p_1) = M \geq \frac{M}{2\pi} \int_{\partial V_M(p_i)} \omega(V_M(p_2), \zeta) \frac{\partial}{\partial n} G(\zeta, p_1) ds. \]  

(24)

Put \( d\mu_i = \frac{1}{2\pi} \frac{\partial}{\partial n} G(z, p_i) ds \) on \( \partial V(p_1) \). Then \( \mu_i = 0 \) on (B) and by (23) and (24)

\[ \int M \omega(V_M(p_1), \zeta) d\mu_1(\zeta) \leq G^M(p_2, p_1). \]  

(27)

**Theorem 4.** (A). Let \( p_i (i = 1, 2, \ldots, n) \) in \( R - B_0 \). Let \( D_i \) be a domain in \( V_M(p_i) \). Then there exists a canonical distribution \( \mu_i \) of mass \( \leq 1 \) on \( D_i \) such that \( M \omega(D_i, \zeta) = \int G(\zeta, p) d\mu_i(p) \). Put \( \mu = \frac{1}{n} \sum \mu_i \). Then

\[ I(\mu) = \int \int N(p, q) d\mu(p) d\mu(q) \leq \frac{1}{n^2} \sum_{i=1}^{n} N^M(p_i, p_j). \]  

(B). Let \( p_i (i = 1, 2, \ldots, n) \) on \( R \). Then there exists a distribution \( \mu_i \) on \( R \cap V_M(p_i) \) (\( d\mu_i \leq 1 \) by loss of mass) such that \( M \omega(V_M(p_i), \zeta) = \int G(z, p) d\mu_i(p) \). Put \( \mu = \frac{1}{n} \sum \mu_i \). Then

\[ I(\mu) \leq \frac{1}{n} \sum_{j=1}^{n} G^M(p_i, p_j). \]  

(C). Let \( p (i = 1, 2, \ldots, n) \) on \( R - R_0 \). Then there exists a distribution \( \mu_i \) on \( R \cap V_M(p_i) \) such that \( M \omega(V_M(p_i), \zeta) = \int G(z, p) d\mu_i(p) \) and \( \int d\mu_i = 1 \). Put

\[ \mu = \frac{1}{n} \sum \mu_i. \]  

Then

\[ I(\mu) \geq \frac{1}{n^2} \sum_{j=1}^{n} G^M(p_i, p_j). \]  

Proof of (A). Put \( U(z) = \int N(z, p) d\mu(p) \). Then \( U(z) = M \sum \omega(D_i, z) \). Then by Lemma 2,

\[ I(\mu) = \frac{M}{n^2} \sum_{i=1}^{n} \omega(D_i, z) \sum d\mu_i \leq \frac{1}{n^2} \sum_{j=1}^{n} N^M(p_i, p_j). \]  

It is proved similarly for (B) paying attention to \( d\mu_i = \frac{\partial}{\partial n} G(z, p) ds \) on \( \partial V_M(p) \).
and \( \int_{\partial V_{M}(p)} \frac{\partial}{\partial n} G(z, p) ds \leq 2\pi \) (by loss of mass) i.e. \( \int d\mu_i \leq 1 \) and \( \mu_i = 0 \) on (B) and for (C) to \( \int d\mu_{l} \leq 1 \) and \( \mu_{l} = 0 \) on (B).

**Lemma 3.** Let \( \tilde{A} \) and \( A \) be closed sets such that \( \tilde{A} \supset A \).

(A), 1). If there exists a number \( M \) such that \( \int_{\partial V_{M}(p) \cap \tilde{A}, z} ds \geq \delta > 0 \) for any \( p \in A \cap (\overline{R} - R_0 - B_0) \), then

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A})
\]

2). If there exists a number \( M \) such that \( \frac{1}{2\pi} \int_{\partial V_{M}(p) \cap \tilde{A}, z} \frac{\partial}{\partial n} G(z, p) ds \geq \delta > 0 \) for any \( p \in A \cap (\overline{R} - R_0 - B_0) \), then

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A})
\]

2'). (B) and (C). If there exists a number \( M \) such that

\[
\frac{1}{2\pi} \int_{\partial V_{M}(p) \cap \tilde{A}, z} \frac{\partial}{\partial n} G(z, p) ds \geq \delta > 0 \) for any \( p \in A \cap (\overline{R} - R_0 - B_0) \),

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A})
\]

3). (A). Let \( G \) be an open set such that \( G \supset A \) and \( G \supset \sum_{p \in F} V_{M}(p) \), then

\[
1/D^{M}(A) \geq 1/C_{\circ}(G)
\]

where \( \text{Cap}(G) = \sup_{F} \text{Cap}(F) \), \( F \) is a closed set in \( G \).

3'). (C). Let \( G \) be an open set such that \( G \supset A \) and \( G \supset \sum_{p \in F} V_{M}(p) \). Then

\[
1/D^{M}(A) \geq 1/C_{\circ}(G)
\]

**Proof of 1).** Let \( D(p_i) = V_{M}(p_i) \cap \tilde{A} \). Let \( \mu_i(p) \) be the canonical distribution of \( M\omega(D(p_i), z) \) on \( \overline{D}(p_i) \subset \tilde{A} \). Then by Theorem 4, putting \( \mu = \frac{1}{n} \times \sum \mu_i(p) \), we have \( \int \int N(p, q) d\mu(p) d\mu(q) \leq \frac{1}{n^2} \sum_{i=1}^{n} N^{M}(p_i, p_j) \).

Now the total mass of \( \mu \geq \delta \) and \( \mu = 0 \) except \( \tilde{A} \cap (\overline{R} - R_0 - B_0) \), whence by the definition of the capacity \( I(\mu) \geq \delta^{2}/\text{Cap}(\tilde{A}) \). On the other hand, \( 2( \sum_{p_i \neq p_j} N^{M}(p_i, p_j)) = \sum_{i=1}^{n} N^{M}(p_i, p_j) - \sum_{i=1}^{n} N^{M}(p_i, p_i) \). Hence

\[
1/D^{M}_{n}(A) = \inf \sum_{i < j}^{n} N^{M}(p_i, p_j) / nC_{\circ} \geq \delta^{2}/\text{Cap}(\tilde{A}) - \frac{M}{n-1}
\]

Let \( n \rightarrow \infty \). Then \( 1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A}) \).

Paying attention to \( M\omega(V_{M}(p_i), z) = \int_{\partial V_{M}(p_i)} N(z, p) d\mu_i(p_i) \) and \( G^{M}(z, p_i) = \int_{\partial V_{M}(p_i)} G(z, p) ds \).
$q)d\mu_{l}(q)$ for (B) and (C) and $\mu>0$ only on $\partial V_{M}(p_{l})\cap R$, we have 2), 2') similarly as 1). Next since $N^{M}(z, p)=M\omega(V_{M}(p), z)(G^{M}(z, p)=M\omega(V_{M}(p), z))$ for (C) and the mass of $M\omega(V_{M}(p), z)$ is unity. Hence by (1) we have 3) and 3').

6. Activity of a point $p\in\overline{R}-R_{o}$ to a closed set $F$. Let $F_{n}=E[z\in\overline{R}: \delta(z, F)\leq\frac{1}{n}]$. (A). Put $\lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial R_{o}}M\omega(F_{n}\cap V_{M}(p), z)ds=\delta(p)$ and

\[ \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial V_{M}(p)\cap F_{n}}M\omega(F_{n}\cap V_{M}(p), z)ds=\delta(p), \]

where $\partial V_{M}(p)$ is a regular niveau. We call $\delta(p)$ and $\delta(p)$ the weak and strong activity of $p$ to $F$ respectively.

(B). $\delta(p)$ is given as $\lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial V_{M}(p)\cap F_{n}}\frac{\partial}{\partial n}G(z, p)ds$.

Clearly $\delta(p)$ to $\overline{F}\geq\delta(p)$ to $F$ and $\delta(p)$ to $\overline{F}\geq\delta(p)$ to $F$, if $\overline{F}\supset F$.

Lemma 4. 1). $\delta(p)\geq\mathring{\delta}(p)$.

2). Suppose $p\in\overline{R}-R_{o}-B_{o}$. If $p\in F$, $\delta(p)$ to $F=1$. If $p\in R$, $\delta(p)$ to $F=0$.

3). Let $F$ be a closed set of capacity zero. Suppose $U(z)=\int_{F-B_{o}}N(z, p)d\mu(p)$. Then $\lim_{n\to\infty}M\omega(V_{M}(p)\cap F_{n}, z)=U(z)$, where $V_{M}=E[z\in R: U(z)>M]$.

Let $p\in F$. Then $N(z, p)=\int_{F-B_{o}}N(z, p)d\mu(p)$. Let $\mu'$ be the restriction of $\mu$ on $F$. Then $N(z, p)\geq\frac{1}{2\pi}\int_{\partial R_{o}}\frac{\partial}{\partial n}N(z, p)ds$.

Proof of 1). $\int_{\partial R_{o}}\frac{\partial}{\partial n}N(z, p)ds\leq0$.

Proof of 2). Since $p\in F-B_{o}$, $\int_{\partial R_{o}}\frac{\partial}{\partial n}N(z, p)ds\to0$ and $\int_{\partial R_{o}}\frac{\partial}{\partial n}N(z, p)ds\to2\pi$ as $M\to M_{*}(p)$ by Theorem 2, where $\partial V_{M}(p)$ is a regular niveau. Whence $\delta(p)$ to $F$ is 1. Let $p\not\in F$ and $p\not\in B_{o}$. Then there
exists a number $n$ such that $p \notin F_n$. Put $U(z) = \lim_{M=\infty} F_n \cap V_{M}(p)$, $m > n$. Then $N(z, p) \geqq U(z)$ and $N(z, p) - U(z)$ is superharmonic by $\lim \text{Cap}(V_{M}(p)) = 0$. On the other hand, $N(z, p)$ is N-minimal, whence $U(z) = \kappa N(z, p)$ and the mass distribution of $U(z)$ must be a point mass at $q \notin F_n$. This implies $N(z, p) = N(z, q)$, $q \notin F_n$. This is a contradiction. Hence $0 = U(z) = \lim_{M=\infty} M\omega(V_{M} \cap F_{n}, z)$.

**Proof of 3.** We proved the following proposition. Let $F$ be the kernel of canonical distribution of a superharmonic function in $R-R_0$ vanishing on $\partial R_0$. Then the kernel of any other canonical distribution is also $F$. Put $V(z) = \lim_{M=\infty} c_{F_n \cap V_{M}} U(z)$. Then since $\lim M \omega(V_{M} \cap F_{n}, z_{0}) + c_{F_n \cap V_{M}} U(z_{0}) + \epsilon \geqq U(z_{0}) \geqq M \omega(V_{M} \cap F_{n}, z_{0})$. Let $M \rightarrow \infty$ and $\epsilon \rightarrow 0$. Then $M \omega(V_{M} \cap F_{n}, z) = U(z)$ for any $n$ and $\lim_{n} M \omega(V_{M} \cap F_{n}, z) = U(z)$.

Let $p \in B_0$. Then $N(z, p)$ is representable by a canonical distribution $\mu$ such as $N(z, p) = \int_{B_0} N(z, q) d\mu(q)$. Let $\mu'$ be the restriction of $\mu$ on $F$. Then

\[ \int_{B_0} N(z, q) d\mu(q) \leqq \sum_{i=1}^{n} \int_{B_i} N(z, q) d\mu(q). \]

\[ \int_{B_0} N(z, q) d\mu(q) = \int_{B_0} N(z, q) d\mu(q) \leqq \sum_{i=1}^{n} \int_{B_i} N(z, q) d\mu(q). \]

\[ \int_{B_0} N(z, q) d\mu(q) \leqq \sum_{i=1}^{n} \int_{B_i} N(z, q) d\mu(q). \]

\[ \int_{B_0} N(z, q) d\mu(q) = \int_{B_0} N(z, q) d\mu(q) \leqq \sum_{i=1}^{n} \int_{B_i} N(z, q) d\mu(q). \]
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$N(z, p) = U(z) + \sum_{m=1}^{\infty} V_m(z)$, where $V_m(z)$ is the potential of the restriction of $\mu$ on $CF_{m+1} - CF_m$. Assume $\rho V_m(z) > 0$. Then since $\text{Cap}(F) = 0$, $\rho V_m(z)$ is representable by a canonical distribution on $F$. But any kernel of canonical distribution of $V_m(z)$ is contained in $CF_{m+1} - CF_m$. This contradicts the above proposition. Hence $\rho V_m(z) = 0$. Put $\overline{V}_M = E[z \in R : N(z, p) > M]$. Then $\overline{V}_M \supset V_M$ and $2\pi \delta(p) \geq \lim_{n \to \infty} \int_{\partial R} \frac{\partial}{\partial n} M \omega(V_M \cap F_n, z) ds$.

Put $V_M = E[z \in R : U(z) > M]$. Then $\overline{V}_M \supset V_M$ and $2\pi \delta(p) \geq \lim_{n \to \infty} \int_{\partial R} \frac{\partial}{\partial n} M \omega(V_M \cap F_n, z) ds$.

Now $U(z)$ is the form $U(z) = \sum_{F - B_0} N(z, q) d\mu'(q)$. Hence by (26)

$$2\pi \delta(p) \geq \int_{\partial R} \frac{\partial}{\partial n} U(z) ds = 2\pi \int d\mu'.$$

Thus by (27) and (28) $\delta(p)$ to $F = \int d\mu'$.

Also $U(z) = (\rho U(z) + \sum \rho V_m(z)) = \rho N(z, p) \geq \rho \int N(z, q) d\mu'(q) = \rho U(z) = U(z)$, hence $\delta(p) = \int d\mu' = \frac{1}{2\pi} \int_{\partial R} \frac{\partial}{\partial n} \rho N(z, p) ds$.

7. A property of irregular points of Green's functions. (A). Let $\omega(B, z)$ be C.P. of $B$, i.e. $\omega(B, z) = \lim_{n} \omega_n(z)$, where $\omega_n(z)$ is a harmonic function in $R_n - R_0$ such that $\omega_n(z) = 0$ on $\partial R_0$ and $\omega_n(z) = 1$ on $\partial R_n$. Then $\omega(B, z) = 1$ on $B$ except a set of capacity zero$. Let $G(z, p_0) : p_0 \in R_0$ be the Green's function of $R$. Then $G(z, p_0) > 0$ if and only if $\omega(B, z) < 1$. Now $\omega(B, z)$ is well defined on $\overline{R} - R_0$ and lower semicontinuous. Hence $S_l = E[z \in B : \omega(B, z) \leq 1 - \frac{1}{l}]$ is closed and of capacity zero. i.e. $D(\omega(S_{l,n}, z)) \downarrow 0$ as $n \to \infty$, where $S_{l,n} = E[z \in \overline{R} : \delta(z, S_l) \leq \frac{1}{n}]$ and $E[z \in B : \omega(B, z) < 1]$ is an $F_*$ set of capacity (5).
zero. Let $p \in S_l \cap B_1$. Then by Lemma 3. (1) $\delta(p)$ to $S_l = 1$. Let $p \in S_l \cap B_0$. Then $\omega(B, p) = \int_{\partial F} \omega(B, q) d\mu(q)$. Assume $\mu(q)$ has mass $> 1 - \frac{1}{2l-1}$ on $B_1 - S_{2l}$. Then $\omega(B, p) < 1 - \frac{1}{l}$. This is a contradiction. Hence $\mu(q)$ has its mass $\geq \frac{1}{2l-1}$ on $S_{2l}$ and by Lemma 4. 3) $\delta_p(S_l) \geq \frac{1}{2l-1}$.

(B). Let $R = E[z \in R^*: G^*(z, p_0) > \delta]$. $G(z, p_0)$ of $R$ is well defined on $\overline{R} = R + B$ and continuous in $\overline{R}$. Put $F_t = E[z \in R^*: (Gz, p_0) \geq \frac{1}{l}]$ and $S_t = B \cap F_t$.

Let $\{R_n\}$ be an exhaustion of $R$ with compact relative boundary $\partial R_n$. Let $S_{l,n} = F_t \cap (R - R_n)$. Then $D(\omega(S_{l,n}, z)) \downarrow 0$ as $n \to \infty$ (see "energy integral and capacities"). The value at $p \in S_l$ is given as $\frac{1}{l} \leq G(p, p_0) = \frac{1}{2\pi} M \lim_{\partial V} \int_{\kappa^{(p)}} G(z, p_0) \frac{\partial}{\partial n} G(z, p) ds$ and

$$\lim_{M \to \infty} \int_{\partial V_M(p) \cap (R - R_n) \cap F_t} G(z, p_0) \frac{\partial}{\partial n} G(z, p) ds = 2\pi G(p, p_0) - \lim_{M \to \infty} \int_{G(z, p_0) \frac{\partial}{\partial n} G(z, p) ds}.$$

Let $k = \limsup_n G(z, p_0) = \sup_n G(z, p_0)$. Now $V_M(p)$ clusters at $B$ as $M \uparrow \infty$.

Hence for any given $R_n$, there exists a number $M_0(n)$ such that $V_M(p) \subset R - R_n$ for $M \geq M_0$. Hence $\lim_{M \to \infty} \frac{1}{2\pi} \int_{\partial V_M(p) \cap S_{l,n}} \frac{\partial}{\partial n} G(z, p) ds \geq \frac{1}{2kl}$ for any $n$ and $\delta(p)$ to $S_{l,n} \geq \frac{1}{2kl}$ for any $n$.

(C). Let $B$ be the ideal boundary of $R$ with null-boundary. Then $V_M(p) \to B$ as $M \to \infty$ for any $p \in B$, whence $\delta(p) = 1$ to $B$ for any point $p \in B$.

Lemma 5. 1). Let $F$ and $\overline{F}$ be closed sets in $\overline{R} - R$, such that $\overline{F} \supset F$.

Put $\overline{F}_n = E[z \in \overline{F}: \delta(z, F) \leq \frac{1}{n}]$.

1. (A). If $\lim_{n \to \infty} \inf_{p \in F} \int_{\partial R_n} \omega(M_0(V_M(p) \cap \overline{F}_n, z)) ds \geq 2\pi \delta$,

$$1/D(F) \geq 1/D(F) \geq \delta^2/\text{Cap}(F).$$

2. If $\lim_{n \to \infty} (\inf_{p \in F} \int_{\partial F_n} N(z, p) ds) \geq 2\pi \delta$, 

$$1/D(F) \geq 1/D(F) \geq \delta^2/\text{Cap}(F).$$
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\[ 1/D(F) \geq 1/\hat{D}(F) \geq \delta^2/\mathring{C}(\tilde{F}) \]

2'). (B) and (C). If \( \lim_{nM} \varliminf_{=\infty} \left( \inf_{\partial V} \int_{u^{(p)}} \frac{\partial}{\partial n} G(z, p) ds \right) \geq 2\pi \delta \),

\[ 1/D(F) \geq 1/\hat{D}(F) \geq \delta^2/\mathring{C}(\tilde{F}) \]

3). (A). Let \( M(p, n, \delta') \) be the number such that \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq 2\pi \delta^\prime \) for \( M \geq M(p, n, \delta') \): \( p \in F \). Then \( M(p, n, \delta') \) is upper semicontinuous, if \( \delta(p) \) to \( \tilde{F} \geq \delta \) for any \( p \in F \), then

\[ 1/\hat{D}(F) \geq \delta'/\mathring{Cap}(\tilde{F}) \]

If \( \delta(p) \) to \( \tilde{F} \geq \delta \) for any \( p \in F \), \( 1/\hat{D}(F) \geq \delta'/\mathring{Cap}(\tilde{F}) \).

3'). (B) and (C). If \( \lim_{\kappa} nM = \infty \varliminf \int_{p\partial \nabla() \cap \tilde{F}_{n}} \frac{\partial}{\partial n} G(z, p) ds = 2\pi \delta(p) > \delta \) for \( p \in F \),

\[ 1/\hat{D}(F) \geq \delta'/\mathring{Cap}(\tilde{F}) \]

Proof of 1). Since \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \downarrow \) as \( n \to \infty \), for any \( \epsilon > 0 \)
there exists a number \( M_0 = M(\epsilon, p) \) such that \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta - \epsilon \) for \( M \geq M_0 \) and for \( p \in F \). Then by Lemma 3 \( 1/\hat{D}(F) \geq 1/D^{M}(F) \geq (\delta - \epsilon)^{2}/\mathring{Cap}(\tilde{F}) \). Let \( \epsilon \to 0 \) and the \( n \to \infty \). Then we have (1).

2) and 2') can be proved similarly as (1) using Lemma 3.

Proof of 3). Let \( M \geq M(p_0, n, \delta') \). Then \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_0) \cap \tilde{F}_{n}, z) ds \geq \delta' \). For any \( \epsilon > 0 \), there a compact set \( G \) in \( V_{M}(p_0) \cap \tilde{F}_{n} \) such that \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_0) \cap G, z) ds \geq \delta' - \epsilon \). Since \( N(z, p_0) \to N(z, p_0) \) in \( R \) as \( p_0 \to p_0 \), \( E[z \in R: N(z, p_0) > M] \to G \) for \( i > i_0 \) and \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_0) \cap \tilde{F}_{n}, z) ds \geq \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_0) \cap G, z) ds \) Let \( \epsilon \to 0 \). Then \( \frac{1}{2\pi} \lim_{p_0 \to p_0} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_0) \cap \tilde{F}_{n}, z) ds \geq \delta' \). This implies \( \lim_{p_0 \to p_0} M(p_0, n, \delta') \leq M(p_0, n, \delta') \) by definition of \( M(p, n, \delta') \).

Also for any point \( p \in F \lim_{M=\infty} \frac{M}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta(p) = \lim_{n \to M=\infty} \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} \omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta(p) \).
\[
\frac{\partial}{\partial u} M_\omega (V_M (p) \cap \bar{F}_n, z) \, ds,
\]
whence \( M(p, n, \delta - \varepsilon) < \infty \) and \( M(p, n, \delta - \varepsilon) \) attains its maximum \( M(n, \delta - \varepsilon) \) on \( F \). Let \( M > M(n, \delta - \varepsilon) \). Then \( 1/D(F) \geq 1/D_M(F) \geq (\delta - \varepsilon)^2 / \text{Cap} (\bar{F}_n) \). Let \( \varepsilon \to 0 \) and then \( n \to \infty \). Then \( 1/D(F) \geq \delta \big/ \text{Cap} (\bar{F}_n) \).

3') is proved similarly as (3).


Theorem 5 (G. C. Evans)\(^{19}\). Let \( F_i (i = 1, 2, \cdots) \) be a closed set in \( \bar{R} - R_0 \). If \( D(F_i) = 0 \), there exists a potential \( U(z) = \int_{\Sigma F_i - B_0} N(z, p) \, d\mu(p) \) (or \( \int_{\Sigma F_i} G(z, p) \times d\mu(p) \)) such that \( \sum_{F_i - B_0} U(z) = \infty \) on \( \sum F_i - B_0 \) and \( \lim_{z \to F - B_0} U(z) = \infty \) (\( U(z) = \infty \) on \( \Sigma F_i \) and \( \lim_{z \to F - B_0} U(z) = \infty \) for \( \sum F_i \)).

Proof of (A). For given \( p_1, p_2, \cdots, p_n \) on \( F_i - B_0 \), let \( V_n(z) = \frac{1}{2\pi n} \sum N(z, p_l) \), and put \( R(p_1, \cdots, p_n) = \inf V_n(z) \) on \( F_i - B_0 \). Let \( R_n \) be the least upper bound of \( R(p_1, p_2, \cdots, p_n) \) as \( p_1, p_2, \cdots, p_n \) vary on \( F_i - B_0 \). Then for any given \( \varepsilon > 0 \) \( \left( \varepsilon < \frac{1}{2\pi n} \right) \), there exists a system \( (p^*_1, p^*_2, \cdots, p^*_n) \) such that \( V_n^*(z) = V(z, p_1, \cdots, p_n^*) \geq R_n - \varepsilon \) on \( F_i - B_0 \). Next \( N(p, q) = N(q, p), n+1C_2 / D_{n+1}(F_i) = \inf (\sum_{l<j}^{n+1} N(p_l, p_j) \leq \frac{1}{2} \sum_{j=1}^{n+1} (\sum_{i=1}^{n+1} N(p_l, p_n)) \) and we have \( R_n \geq 1/D_{n+1}(F_i) \), where \( D_{n+1}(F_i) = \frac{1}{nC_2} \inf (\sum_{l<j}^{n+1} N(p_l, p_j)) \). Hence \( V_n^*(z) \geq 1/D_n(F_i) \) on \( F_i - B_0 \). (29)

Let \( L > 4 \). Since \( D(F_i) = \lim_{n} D_n(F_i) = 0 \), for any \( m \) there exists a number \( l'_m \) such that \( 1/D_{l'_m}(F_i) = \frac{1}{2\pi l'_m} \geq L^m \). Let \( V_m^*(z) \) (defined in (29)). Then \( V_m^*(z) \geq L^m \) on \( F_i - B_0 \). Put \( U_i(z) = \sum_{m=1}^{\infty} V_m(z) / 2^m \). Then \( U_i(z) = \infty \) on \( F_i - B_0 \) and its mass \( = 0 \) except \( F_i - B_0 \). Put \( U(z) = \sum_{i=1}^{\infty} U_i(z) // 2^i \). Then \( U(z) \) is the function required. For (B) and (C) the assertion is proved similarly.

We constructed a Riemann surface with positive boundary such that \( R \) has the following properties: \(^{20} \) 1). Let \( R_3 \) be a compact disc, then there exist

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only two linearly independent positive harmonic functions vanishing on \( \partial R_0 \) in \( R - R_0 \). 2). \( B_i = B_0 = p_1 + p_2, \ B_0 \neq 0 \), i.e. \( N(z, p_i) = \alpha_i \omega(p_i, z) \) and \( \sup N(z, p_i) < \infty : i = 1, 2 \), and any positive harmonic function is a linear form of \( \omega(p_i, z) \) and \( \omega(p_2, z) \).

Let \( p \in B_0 \). Then \( \text{Cap}(p) = 0 \), but there exists no harmonic function \( U(z) \) in \( R - R_0 \) such that \( \lim_{z \to p} U(z) = \infty \) and the Evans’s theorem does not hold. Also in this surface, there exists no superharmonic function \( U(z) \) (not necessarily harmonic in \( R - R_0 \)) such that \( U(p) = \infty \). In fact, \( N(z, p) = \int N(z, p) d\mu(p) = \alpha N(z, p_1) + \beta N(z, p_2) \): \( \alpha + \beta = 1 \), \( \alpha \geq 0 \), \( \beta \geq 0 \) and \( U(p) = \alpha U(p_1) + \beta U(p_2) \). Suppose \( U(p) = \infty \). Then at least one of \( U(p_1) \) and \( U(p_2) \) must be infinite. Without loss of generality we can suppose \( U(p_1) = \infty \). By the lower semicontinuity of \( U(z) \), for any given \( M < \infty \), there exists a number \( n(M) \) such that \( U(z) \geq M \) on \( v_n(p) \). \( U(z) \geq \omega_n(p) U(z) \geq M \omega(v_n(p), z) \). Let \( n \to \infty \) and then \( M \to \infty \). Then \( U(z) \geq p, U(z) \geq \lim_{M \to \infty} M \omega(p_1, z) = \infty \). Hence \( U(z) = \infty \). Thus Evans’s theorem does not hold for superharmonic function. We shall prove the following

**Theorem 6.** Let \( F \) be a closed set of capacity zero. Then there exists a positive superharmonic function in \( R - R_0 \) such that \( U(z) = 0 \) on \( \partial R_0 \) and \( U(z) = \infty \) on \( F - B_0 \). Clearly this theorem is valid for an \( F \) set of capacity zero.

Let \( F_n = E \left\{ z \in \overline{R} : \delta(z, F) \leq \frac{1}{n} \right\} \) and \( \omega(F_n, z) \) be CP. of \( F \). Then \( \omega(F_n, z) = 1 \) in \( F_n \cap R \) and by \( \text{Cap}(F_n) \downarrow 0 \), \( \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F, z) ds \downarrow 0 \) as \( n \to \infty \). Let \( L > 4 \) and let \( n'(L) \) be the number such that \( \frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F_n, z) ds \leq \frac{1}{L^n} \). Put \( U(z) = \sum_{n'} \omega(F_n, z) \). Then \( U(z) \) is the function required. Because mass of \( U(z) \leq 1 \) and \( U(z) < \infty \). Let \( p \in F - B_0 \). Then \( U(p) = \frac{1}{2\pi} \lim_{n \to \infty} \int_{\partial R_0} U(z) \frac{\partial}{\partial n} N(z, p) ds \) and

\[
\lim_{M \to \infty} \int_{F_n \cap \partial R(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \text{ for any } F_n. \quad \text{Since } U(z) \geq n \text{ in } F_n, U(p) = \infty.
\]
By the lower semicontinuity of \( U(z) \). \( U(z) \to \infty \) as \( z \to p \in F - B_0 \).

**Theorem 7.** (A). Let \( F \) and \( \overline{F} \) be closed set in \( \overline{R} - R_0 \) such that \( \overline{F} \supset F \) and \( \text{Cap}(\overline{F}) = 0 \). If \( \delta(p) \to \overline{F} \geq \delta_0 > 0 \) for any \( p \in F \). Then there exists a potential \( U(z) = \int_{F \cap \partial R_0} N(z, p) d\mu(p) \) such that \( U(z) = \infty \) on \( F - B_0 \), \( D(\min(M, U(z))) = 2\pi M \) and by the lower semicontinuity of \( U(z) \), \( U(z) \to \infty \) as \( z \to p \in F - B_0 \).
If $\delta(p)$ to $F>\delta_0>0$, the above potential $U(z)=\infty$ on $F$.

(B) and (C). Let $F$ and $\bar F$ be closed sets in $\overline{R}$ such that $\bar F \supseteq F$ and $\text{Cap}(\bar F)=0$. If $\delta(p)$ to $F \geq \delta_0>0$, then there exists a potential $U(z)=\int_{F} G(z, p) d\mu(p)$ such that $U(z)=\infty$ on $F$ and $D(\min(M, U(z)))=2\pi M$ and by the lower semicontinuity $U(z)\to \infty$ as $z \to p \in F$.

Proof. (A). By Lemma 5 $1/D(F) \geq \delta_0^2/\text{Cap}(\bar F)=\infty$. Hence by Evans’s theorem there exists a potential $U(z)=\int_{F-B_0} N(z, p) d\mu(p)$ such that $U(z)=\infty$ on $F-B_0$. Put $V_M=E[z \in R: U(z)>M]$. Then $V_M \supseteq F-B_0$ and $\int_{F-B_0} N(z, p) d\mu(p)=\infty$ and $\int_{F-B_0} N(z, p) d\mu(p)=\infty$. Hence $\delta(p)$ to $F \geq \delta_0>0$. Let $p \in F \cap B_0$. Then $N(z, p)=\int_{F-B_0} N(z, p) d\mu(p)$. Let $\mu'$ be the restriction of $\mu$ on $F-B_0$. Then $\int_{F-B_0} N(z, p) d\mu(p)=\infty$, whence $U(z)=\infty$ on $F$.

(B) and (C) can be proved similarly.

Corollary. (A). Let $S$ be the irregular set of Green’s function. Then there exists a potential $U(z)$ such that $U(z)=\int_{S-B_0} N(z, p) d\mu(p)$, $U(z)=\infty$ on $S$ and $D(\min(M, U(z)))=2\pi M$.

(B) (Theorem of M. Nakai). Let $R^*$ be a Riemann surface with positive boundary. Then there exists a generalized Green’s function $U(z)$ in $R^*$ such $U(z)\to \infty$ as $z\to B$ in $R_s=E[z \in R^*: G^*(z, p) > \delta]$ for any $\delta>0$.

Proof. Put $S=E[z \in B: G(z, p) > 0]$. Then $S=E[z \in B: \omega(B, z) < 1]$. Let $S_t=E[z \in B: \omega(B, z) \leq 1-\frac{1}{t}]$. Then $S_t$ is closed and $\text{Cap}(S_t)=0$. Now $\delta(p)$ to $S_t \supseteq \frac{1}{2t}$ for any $p \in S_t$. By Theorem 7, there exists a potential $U_t(z)=\int_{S_t-B_0} N(z, p) d\mu(p)$ such that $U(z)=\infty$ on $S_t-B_0$ and $D(\min(M, U_t(z)))=2\pi M$. Let $U(z)=\sum_{i=1}^{\infty} \frac{U_i(z)}{2^{i-1}}$ where $l_0$ is a number such that $S_{l_0} \cap B_{l_0} \neq 0$. Then $U(z)=\infty$ on $S-B_0$. Let $p \in S_{l_0} \cap B_{l_0}$. Then $N(z, p)=\int_{S_{l_0}} N(z, q) d\mu(q)$.
Let $\mu'$ be the restriction of $\mu$ on $S_{2l}$. Then $\int_{B_{1}\cap S_{zl}}U(q)d\mu'(q)\geqq\frac{1}{2l}$, whence $U(z)=\infty$ on $\sum S_{2l}=S$. Put $U_{m}'(z)=\sum_{l=1}^{m}\frac{U_{l}(z)}{2^{l}}$. Then since Cap$(\sum S_{\iota})=0$ and $m_{S_{l}}\Sigma U_{m}^{'}(z)=U_{m}^{'}(z)$ and $\min(M, U_{m}^{'}(z))=M\omega(V_{M}, z)$, where $V_{M}=E[z\in R^{*}: G(z, p)>2\delta]$. Then $\delta(F)$ to $S_{\frac{\delta}{2},n}\geqq\frac{\delta}{2k}$ for any $p\in S_{\delta}=\cap_{n}\overline{S}_{\delta,n}$ and for any $n$, where $k=\lim_{n\Rightarrow\infty}\sup_{z\in R-R_{n}}G(z, p)$. Whence by Lemma 5. (2) $\sigma_{\delta}(z)=\lim_{M=\infty}U_{M}(z)\geqq U(z)$ is a G.G. in $R$. Next let $U_{M}(z)$ be the least positive harmonic function in $R^{*}$ such that $U_{M}(z)\geqq M$ on $E[z\in R^{*}: U_{M}(z)\geqq M]$. Then by Theorem 1. (2) $\bar{U}_{s}(z)=\lim_{M=\infty}U_{M}(z)\geqq U(z)$ is a G.G. in $R^{*}$ such that $D(\min(M, \bar{U}_{s}(z)))\leqq 2\pi M$. Now clearly $\bar{U}_{s}(z)\rightarrow\infty$ as $z\rightarrow B$ in $R=E[z\in R^{*}: G(z, p)>2\delta]$. Put $\bar{U}(z)=\sum_{n=0}^{\infty}\frac{U_{\frac{\delta}{2n}}(z)}{2^{n}}$. Then $\bar{U}(z)$ is a G.G. in $R^{*}$ by Theorem 1. (6) and $D(\min(M, \bar{U}(z)))\leqq 2\pi M$ and $\bar{U}(z)\rightarrow\infty$ as $z\rightarrow B$ in $E[z\in R^{*}: G(z, p)>\delta]$ for any $\delta>0$. Thus $\bar{U}(z)$ is the function required.