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ON THE EXISTENCE OF FUNCTIONS OF EVANS'S TYPE

by

Zenjiro KURAMOCHI

We proved the following

Theorem 1). Let $R$ be a Riemann surface with null-boundary. Then there exists a harmonic function $U(z)$ such that $U(z)$ has a negative logarithmic pole at $p \in R$ and $U(z) \to \infty$ as $z$ tends to the ideal boundary.

Recently M. Nakai extended the above theorem as follows:

Theorem 2). Let $R$ be a Riemann surface with positive boundary. Let $G(z, p)$ be the Green's function of $R$ with pole at $p$. Put $G_{\delta} = \{z \in R : G(z, p) > \delta\}$. Then there exists a harmonic function in $R$ such that $D(\min(M, U(z))) \leq M\alpha < \infty$ and $U(z) \to \infty$ as $z$ tends to the boundary of $R$ in $G_{\delta}$ for any $\delta > 0$ and that any positive harmonic function $V(z)(\leq U(z))$ must be zero, where $\alpha$ is a constant.

We consider the existence of functions of Evans's type for more general sets and obtain some results which contain the above two theorems as their special applications.

1. Generalized Green's function 3). Let $R$ be a Riemann surface with positive boundary. Let $R_n (n = 1, 2, \cdots)$ be its exhaustion with compact relative boundary $\partial R_n$. Let $G^4)$ be a subsurface in $R$. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G \cap (R-R_n))$ such that $w_{n,n+i}(z) = 0$ on $\partial R_{n+i}-G$ and $w_{n,n+i}(z) = 1$ on $G \cap (R-R_n)$. We call $\lim_{n} \lim_{i} w_{n,n+i}(z)$ the harmonic measure (H.M.) of the boundary $(B \cap G)$ determined by $G$ and denote it by $w(G \cap B, z, R)$. As for a set $F$ in $R$. Let $w(F, z, R)$ be the least positive harmonic function in $R-F$ and $w(F, z, R)$ H.M. (harmonic measure) of $F$. Let $G \supset G$ be subdomains in $R$. Let $w_{n,n+i}(z)$ be a harmonic function

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4) In this paper relative boundary $\partial G$ of a subsurface $G$ consists of enumerably infinite number of analytic curves clustering nowhere in $R$. 
in \((G_i \cap R_{n+i}) - (G_i \cap (R_{n+i} - R_n))\) such that \(\omega_{n,n+i}(z) = 0\) on \(\partial G_i\), \(\frac{\partial}{\partial n}\omega_{n,n+i}(z) = 0\) on \(\partial R_{n+i} - G_i\), \(\omega_{n,n+i}(z) = 1\) on \(G_i \cap (R_{n+i} - R_n)\). If there exists a const. \(M\) such that \(D(\omega_{n,n+i}(z)) \leq M\). Then \(\omega_{n,n+i}(z) \to \omega_n(z)\) as \(i \to \infty\) and \(\omega_n(z) \Rightarrow\) to a function denoted by \(\omega(G_i \cap B, z, G_1)\) as \(n \to \infty\) which is called C.P. (capacitary potential) of \((G_i \cap B)\) relative to \(G_1\).

Let \(V(z)\) be a positive harmonic function in \(R\) except at most a set of capacity zero where \(V(z)\) may be infinite. Put \(G_M = E[z \in R: V(z) < M]\). If \(w(G_M \cap B, z, R) = 0\) for any \(M\) and \(D(\min(M, V(z)) \leq M\alpha\) for any \(M < \infty\), we call \(V(z)\) a generalized Green’s function, where \(\alpha\) is a const.

We proved the following

**Lemma 1.** Let \(R\) be a Riemann surface with null-boundary. Let \(R_0\) be a compact disc in \(R\). Let \(G_i\) be a domain in which \(U_i(z)\) is harmonic and \(D_{g_i}(U_i(z)) < \infty\) \((i = 1, 2, \cdots, i_0)\). Then there exists a sequence of compact curves \(\{r_n\}\) such that \(r_n\) separates the boundary \(B\) of \(R\) from \(R_0\), \(\{r_n\}\) clusters at \(B\) and that \(\int_{r_n} \frac{\partial}{\partial n} U_i(z) ds\) tends to zero as \(n \to \infty\), for every \(i\).

We shall prove

**Theorem 1.** Let \(V(z)\) be a G.G. (generalized Green’s function) such that \(D(\min(M, V(z))) \leq M\alpha\). Then

1) Put \(G_M = E[z \in R: V(z) > M]\) and let \(w(G_M, z, R)\) be H.M. of \(G_M\). Then \(V(z) = Mw(G_M, z, R)\) in \(R - G_M\) and \(D(\min(M, V(z))) = M^2D(w(G_M, z, R))\).

2) Let \(\delta < M\). Then \(w(G_M \cap B, z, G_0) = 0\). Let \(\hat{G}_M\) be the symmetric surface of \(G_M\) with respect to \(\partial G_M\). Identify \(\partial G_M\) and the image \(\partial \hat{G}_M\) of \(\partial G_M\). Then we have a Riemann surface \(G_M + \hat{G}_M\) called the doubled surface of \(G_M\). Then \(G_M + \hat{G}_M\) is a Riemann surface with null-boundary.

3) \(\int_{c_M} \frac{\partial}{\partial n} V(z) ds = K\) and \(D(\min(M, V(z))) = KM\) for any \(M\), where \(C_M = E[z \in R: V(z) = M]\).

4) If \(V(z) > 0\), then \(\sup V(z) = \infty\).

5) Let \(V'(z)\) be a positive harmonic function except a set of capacity zero such that \(V'(z) \leq V(z)\). Then \(V'(z)\) is also a G.G. and \(D(\min(M, V'(z))) \leq D(\min(M, V(z)))\).

6) Let \(V_n(z)\) \((n = 1, 2, \cdots)\) be a G.G. which is harmonic in \(R\) and \(D(\min(M, V(z))) \leq M\alpha_n\). If \(V_n(z) \uparrow\) and \(\alpha_n \leq \alpha\), then the limit function \(V(z)\)

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6) See (1).
of \( \{V_n(z)\} \) is a G.G.

7) Let \( G(z, p) \) be a Green's function of \( R \) and put \( R_\delta = \{z \in R: G(z, p) > \delta\} \). Let \( U(z) \) be a G.G. in \( R_\delta \), such that \( D(\min(M, U(z))) \leq Ma \). Let \( U_M(z) \) be the least positive harmonic function larger than \( U(z) \) in \( R-G_M \), where \( G_M = E[z \in R: U(z) \geq M] \). Then \( U_M(z) \uparrow \tilde{U}(z) \) as \( M \to \infty \) and \( \tilde{U}(z) \) is a G.G. in \( R \) such that \( D(\min(M, \tilde{U}(z))) \leq Ma \).

**Proof of 1.** Let \( G_* = E[z \in R: V(z) > \varepsilon] \) and \( G_M = E[z \in R: V(z) > M] \) and let \( R' = G_* - G_M: M > \varepsilon \). Let \( w_n(z) \) be the least positive harmonic function in \( R' \cap R_n \) such that \( w_n(z) = 1 \) on \( G_* \cap \partial R_n \). Put \( U_n(z) = w(G_M, z, R) + Mw_n(z) + \epsilon \). Then

\[
V(z) \leq Mw(G_M, z, R) + Mw_n(z) + \epsilon = U_n(z).
\]

Because \( V(z) = \varepsilon \leq U_n(z) \) on \( \partial G_* \cap R_n \), \( V(z) \leq M \leq U(z) \) on \( \partial R_n \cap G_* \) and \( V(z) = M = Mw(G_M, z, R) \) on \( G_M \). Let \( n \to \infty \). Then \( w_n(z) \to w(G_* \cap B, z, R') \leq w(G_* \cap B, z, R) = 0 \). Let \( \varepsilon \to 0 \). Then \( V(z) \leq Mw(G_M, z, R) \) in \( R-G_M \). On the other hand, by the definition \( Mw(G_M, z, R) \leq V(z) \). Thus \( V(z) = Mw(G_M, z, R) \) in \( R-G_M \) and \( D(\min(M, V(z))) = M^2D(w(G, z, R)) \).

**Proof of 2.** Let \( V_{n+i}(z) \) be a harmonic function in \( (G_*-G_M) \cap R_{n+i} \) such that \( V_{n+i}(z) = \varepsilon \) on \( \partial G_* \cap R_{n+i} \), \( V_{n+i}(z) = M \) on \( \partial G_M \cap R_{n+i} \) and \( \frac{\partial}{\partial n} V_{n+i}(z) = 0 \) on \( \partial R_{n+i} \cap (G_*-G_M) \). Then \( D(V_{n+i}(z)) \leq D(\min(M, V(z))) \). Let \( \omega_{n,n+i}(z) \) be a harmonic function in \( R_{n+i} \cap (G_*-G_M \cap (R-R_n)) \) such that \( \omega_{n,n+i}(z) = 0 \) on \( \partial G_* \cap R_{n+i} \), \( \omega_{n,n+i}(z) = 1 \) on \( G_M \cap (R_{n+i}-R_n) \) and \( \frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0 \) on \( \partial R_{n+i}-G_M \). Then

\[
D(\omega_{n,n+i}(z)) \leq D\left(\frac{V_{n+i}(z) - \epsilon}{M - \epsilon}\right) \leq D(\min(M, V(z))) < \infty.
\]

Hence \( \omega_{n,n+i}(z) \to \omega_n(z) \) as \( i \to \infty \) and \( \omega_n(z) = \omega(G_* B, z, G_*) \) as \( n \to \infty \).

0 \leq \omega(z) \leq w(G_* \cap B, z, G_*) = 0. \) Whence \( \omega(G_M \cap B, z, G_*) = 0 \). Let \( \omega_n'(z) \) be a harmonic function in \( (G_M \cap (R_{n+1}-R_n)) + (G_* \cap R_n) \) such that \( \omega_n'(z) = 0 \) on \( F \) (where \( F \) is a compact arc on \( \partial G_M \cap R_n \)), \( \omega_n'(z) = 1 \) on \( G_M \cap (R-R_n) \) and \( \omega_n'(z) \) has minimal Dirichlet integral. Then clearly \( \frac{\partial}{\partial n} \omega_n'(z) = 0 \) on \( \partial G_* \cap R_n - F \) and \( \frac{\partial}{\partial n} \omega_n'(z) = 1 \) on \( \partial G_M \cap (R-R_n) \). Also it is proved that \( \omega_n'(z) \) converges in mean to \( \omega'(z) \). Clearly \( D(\omega'(z)) \leq D(\omega(z)) = 0 \). This implies that the doubled surface of \( (G_M \cap (R-R_n)) + G_* \cap R_n \) is a Riemann surface with null-boundary. We shall show that \( G_M + \hat{G}_M \) is a Riemann surface with null-boundary. Let \( \omega_{\hat{M}}'(z) \) be a harmonic function in \( G_M \cap R_n \) such that
$\omega_n''(z) = 0$ on $F'$, $\omega_n''(z) = 1$ on $G_M \cap (R-R_n)$ and $\frac{\partial}{\partial n}\omega_n''(z) = 0$ on $(\partial G_M \cap R_n) - F'$, where $F'$ is a compact arc on $\partial G_M \cap R_n$. Then clearly $\omega_n''(z) \Rightarrow \omega''(z)$ as $n \to \infty$ and $D(\omega''(z)) < \infty$. Now $G_M$ is a domain in the Riemann surface $R^*$ with null-boundary ($R^*$ is the doubled surface) of $(G_M \cap (R-R_n)) + G_M \cap R_n)$. Hence by Lemma 1 there exists a sequence of compact surfaces $R_m^*$ ($m=1, 2, \cdots$) of $R^*$ such that $\bigcup_m R_m^* = R^*$ and $\int_{\partial R_m^* \cap \Theta_M} |\frac{\partial}{\partial n}\omega''(z)| ds \to 0$ as $m \to \infty$. Now $D(\omega''(z)) = \int_{\partial R_m^*} \frac{\partial}{\partial n}\omega''(z) ds = \int_{\partial R_m^* \cap \Theta_M} \frac{\partial}{\partial n}\omega''(z) ds$, whence $\omega''(z) = 0$. I.e. $G_M + \hat{G}_M$ is a Riemann surface with null-boundary.

Proof of 3). Let $M < M'$. Then $\int_{\partial R_m^* \cap \Theta_M} |\frac{\partial}{\partial n}V(z)| ds \to 0$ as $m \to \infty$. By $\frac{\partial}{\partial n}V(z) \geq 0$ on $C_M + C_M'$ we have at once $\int_{C_M} \frac{\partial}{\partial n}V(z) ds = K = \text{const.}$ and $D(\min(M, V(z))) = MK$ for any $M$.

Proof of 4). Assume $M_0 < \sup V(z) \leq M$. Then by (1) $V(z) = M_0\omega(G_M, z, R) \leq M\omega(G_M \cap G_M, z, R)$. Let $M \uparrow M$. Then $G_M \to B$ or the set of capacity zero where $V(z) = \infty$. Hence $0 \leq V(z) \leq M\omega(G_M \cap B, z, R) = 0$.

Proof of 5). Put $G_M = E[z \in R: V(z) > M], G_0 = E[z \in R: V(z) > \varepsilon]$ and $G_M' = E[z \in R: V'(z) > M]$. Then $G_M \supset G_M \supset G_M'$. Let $V_n'(z)$ be a harmonic function in $(G_0 \supset G_M \supset G_M') \cap R_n$ such that $V_n'(z) = \varepsilon$ on $\partial G_0$, $V_n'(z) = M$ on $\partial G_M'$ and $\frac{\partial}{\partial n}V_n'(z) = 0$ on $\partial R_n \cap (G_0 - G_M')$, Then $V_n'(z)$ has M.D.I. over $R_n \cap (G_0 - G_M')$ with value $\varepsilon$ on $\partial G_0$ and $M$ on $\partial G_M'$, whence $D(V_n'(z)) \leq D(V(z)) \leq M$. Hence $V_n'(z) \Rightarrow V_n^*(z)$ as $n \to \infty$ and $D(V_n^*(z)) \leq M$. On the other hand, $0 \leq V_n^*(z) \leq \varepsilon + M\omega(G_M', z, R) + M\omega(G_M \cap B, z, R) = \varepsilon + M\omega(G_M' \cap (R-R_n), z, R)$ and $V_n^*(z) \leq \varepsilon + M\omega(G_M', z, R) + M\omega(G_M \cap B, z, R) = \varepsilon + M\omega(G_M', z, R)$. Let $\varepsilon \to 0$. Then also $V_n^*(z) \Rightarrow V_n^*(z)$ and $V_n^*(z) \leq M\omega(G_M', z, R)$ and $D(V_n^*(z)) \leq M$. By the definition of $\omega(G_M', z, R) V_n^*(z) \leq M\omega(G_M', z, R)$. Hence

$$V_n^*(z) = M\omega(G_M', z, R). \quad (1)$$

Also $M\omega(G_M', z, R) \leq V(z) \leq \varepsilon + M\omega(G_M \cap B, z, R) + M\omega(G_M', z, R)$. By letting $\varepsilon \to 0$ and by (1) we have
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$V'(z) = Mw(G_M, z, R) = V^*(z)$ in $R - G_M$

Next by $D(V^*(z)) < \infty$ we can prove similarly as the proof of (3)

\[
\int_{\partial} \frac{\partial}{\partial n} V'(z) ds = K \quad \text{and} \quad D(\min(M, V'(z))) = MK,
\]

next by $D(\min(M, V(z))) = M\alpha$ we have $K \leq \alpha$. By $V'(z) \leq V(z)$ we have at once $w(G_M \cap B, z, R) \leq w(G_M \cap B, z, R) = 0$. Thus $V'(z)$ is a G.G.

Proof of 6. At first we show $V(z) < \infty$ in $R$. Let $V_m(z)$ be a G.G. and put $G_i = E[z \in R : V_m(z) > l]$. Then by $w(G_i \cap B, z, R) = 0$ we see $V_{m,n}(z) \Rightarrow V_m(z)$ as $n \rightarrow \infty$, where $V_{m,n}(z)$ is a harmonic function in $R_n \cap (G_i - G_m)$ such that $V_{m,n}(z) = M$ on $\partial G_m$, $V_{m,n}(z) = \epsilon$ on $\partial G$, and $\frac{\partial}{\partial n} V_{m,n}(z) = 0$ on $\partial R_n \cap (G_i - G_m)$. Let $D$ be a compact disc in $R$. Suppose $V(z) \geq L$ on $D$. Let $\omega_n(z)$ be a harmonic function in $(R_n - D) \cap G_i$ such that $\omega_n(z) = 1$ on $D$, $\omega_n(z) = \epsilon$ on $\partial G$, and $\frac{\partial}{\partial n} \omega_n(z) = 0$ on $\partial R_n$. Then

\[
D(V_{m,n}(z) - L\omega_n(z), \omega_n(z)) \geq 0.
\]

Hence by mean convergency of $V_{m,n}(z)$ and $\omega_n(z)$ as $n \rightarrow \infty$ we have

\[
L\alpha \geq D(\min(L, V_m(z))) \geq L^2 D(\omega(z)) = \lim_{n} \omega_{\epsilon}(z) \Rightarrow \omega(z) \quad \text{as} \quad \epsilon \rightarrow 0.
\]

Let $\epsilon \rightarrow 0$. Then $V_{m,n}(z) \Rightarrow V(z)$ as $\epsilon \rightarrow 0$ and $L\alpha \geq L^2 D(\omega(z))$. Whence $\min_{z \in D} V_{m,n}(z) \leq \frac{\alpha}{D(\omega(z))}$ and $\min_{z \in D} V(z) \leq \frac{\alpha}{D(\omega(z))}$. Hence by Harnack's theorem $V(z) < \infty$ in $R$.

Put $G_M = E[z \in R : V(z) > M]$ and $G_{M,m} = E[z \in R : V_m(z) > M]$. Then by $V(z) \geq V_m(z) G_M \supset G_{M,m}$. Since $V_m(z) = Mw(G_{M,m}, z, R)$, $V_m(z)$ is the least positive harmonic function in $R - G_M$ with value $V_m(z)$ on $\partial G_M$. Hence $V_m(z) = \lim V_{m,n}(z)$, where $V_{m,n}(z)$ is the least positive harmonic function in $R_n - G_M$ such that $V_{m,n}(z) = V_m(z)$ on $\partial G_M$ and $V_{m,n}(z) = 0$ on $\partial R_n - G_M$. Hence

\[
V_{m,n}(z) = \frac{1}{2\pi} \int_{\partial G_M \cap R_n} V_{m,n}(\zeta) \frac{\partial}{\partial n} G_n(\zeta, z) ds,
\]

where $G_n(\zeta, z)$ is the Green's function of $R_n - G_M$.

Let $n \rightarrow \infty$. Then $\frac{\partial}{\partial n} G_n(\zeta, z) \uparrow \frac{\partial}{\partial n} G(\zeta, z)$ on $\partial G_M$, where $G(\zeta, z)$ is the Green's function of $R - G_M$. Hence $V_m(z) = \frac{1}{2\pi} \int_{\partial G_M} V_m(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds$. Now $V_m(z) \leq$
$V(z) \leq M$ on $\partial G_M$, hence $V_m(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds$ is uniformly integrable and

$$V(z) = \frac{1}{2\pi} \int_{\partial G_M} V(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds = \lim_{m \to \infty} \frac{1}{2\pi} \int_{\partial G_M} V_m(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds = \lim_{m \to \infty} V_m(z).$$

This means that $V(z)$ is the least positive harmonic function in $R-G_M$ with value $M$ on $\partial G_M$, i.e. $V(z) = Mw(G_M, z, R)$ in $R-G_M$.

Let $w_n(z)$ be the least positive harmonic function in $R-G_M - ((R-R_n) \cap G_M)$ such that $w_n(z) = 1$ on $(R-R_n) \cap G_M$. Then $w_n(z) \to w(G_i \cap B, z, R-G_M)$ as $n \to \infty$ and $V(z) \geq \epsilon w(G_i \cap B, z, R-G_M)$. Since $V(z)$ is the least positive harmonic function in $R-G_M$ larger than $M$ on $G_M$, $w(G_i \cap B, z, R-G_M) = 0$. Clearly $w_n(z) + w(G_M, z, R) \geq w(G \cap B, z, R)$, because $w_n(z) + w(G_M, z, R) \geq 1$ on $G \cap (R-R_n)$. Hence

$$w(G_M, z, R) = w(G \cap B, z, R-G_M) + w(G_M, z, R) \geq w(G \cap B, z, R).$$

Let $M \to \infty$. Then $w(G_M, z, R) \to 0$. This implies $w(G_i \cap B, z, R) = 0$ for any $\epsilon > 0$. Let $G$ be a compact domain completely contained in $R-G_M$. Then there exists a number $n_0$ such that $G \subset E[z \in R: V_n(z) < M]$ for $n \geq n_0$. Then

$$D(V(z)) \leq D(\lim_{n \to \infty} V_n(z)) \leq \lim_{n \to \infty} D(V_n(z)) \leq \lim_{n \to \infty} D(\min(M, V_n(z))) \leq \alpha M.$$

Let $G \to E[z \in R: V(z) < M]$. Then $D(\min(M, V(z))) \leq M \alpha$. Thus $V(z)$ is a G.G.

**Proof of 7.** Let $U_n(z)$ be the least positive harmonic function in $R_n-G_M$ such that $U_n(z) = M$ on $\partial G_M$, $U_n(z) = 0$ on $\partial R_n - R_M$, $\frac{\partial}{\partial n} U_n(z) = 0$ on $(\partial R_n \cap R_M) - G_M$. Then $D(U_n(z)) \leq D(\min(M, U(z))) \leq M \alpha$. Also let $U_n^*(z)$ be the least positive harmonic function in $R_n-G_M$ larger than $M$ on $G_M$. Then

$$0 \leq U_n^*(z) \leq U_n(z) \leq U_n^*(z) + Mw(R_n \cap (R-R_n), z, R).$$

Let $n \to \infty$. Then since $G(z, p)$ is a G.G., $w(R_n \cap (R-R_n), z, R) \to w(R_n \cap B, z, R) = 0$. Hence $0 \leq \lim_{n \to \infty} U_n^*(z) = \lim_{n \to \infty} U_n(z) = U_M(z)$. By $U_M(z) = \lim_{n \to \infty} U_n(z)$, we have

$$D(\min(M, U_M(z))) \leq \lim_{n \to \infty} D(U_n(z)) \leq M \alpha . \tag{2}$$

By $U_M(z) = \lim_{n \to \infty} U_n^*(z)$,

$$U_M(z) = Mw(G_M, z, R) \tag{3}$$

and $U_M(z) = Nw(G_N, z, R) \in R-G_N$, where $G_N = E[z \in R: U_M(z) > N] + G_M$ for any $N \leq M$ and for any $D \subset R-G_M$

$$U_M(z) = \frac{1}{2\pi} \int_{\partial D} U_M(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds,$$
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where $G(\zeta, z)$ is the Green's function of $D$.

By (2) and (3) we have similarly as the proof of 2), $\omega(G'_{N'}, B, z, G'_{N'}) = 0$ : $M \geq N' > N$.

This implies that the doubled surface $(G_{N'}, + \hat{G}_{N'})$ is a Riemann surface with null-boundary, whence $\int_{\partial \Omega} \frac{\partial}{\partial n} U_{M}(x) ds = k = \text{const.}$, where by (2) $k \leq \alpha$ for any $\Omega$.

Clearly $U_{M}(z) \uparrow \sigma(z)\alpha$ as $M \to \infty$. Then by (5) $D(\min(N, \sigma(z))) \leq N\alpha$ and $\Omega<\delta$. Also by (4), since $U_{M}(z) \uparrow \tilde{U}(z)$ $U(z) = \frac{1}{2\pi} \int_{\partial D} U(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds$, i.e. $U(z)$ is the least positive harmonic function in $R$ larger than $M$ on $\tilde{G}_{M} = E[z \in R: \tilde{U}(z) > M]$, whence we have similarly as (7)

\[ \omega(\tilde{G}_{M} \cap B, z, R) = 0 \quad \text{for any} \ M. \]  

Thus by (6) and (7) $\Omega(z)$ is a G.G.

2. Green's potential. Let $R^{*}$ be a Riemann surface with positive boundary and let $\{R_{n}^{*}\}$ be its exhaustion with compact relative boundary $\partial R_{n}^{*}$ $(n=1, 2, \cdots)$. Let $p_{0}$ be a fixed point in $R$ and let $G(z, p_{0})$ be the Green's function. Put $R = E[z \in R^{*} : G(z, p_{0}) > \delta]$. Then by Theorem 1), 2) the doubled surface $R + \hat{R}$ is a Riemann surface with null-boundary. Let $G(z, p)$ be the Green's function of $R$. Then $G(z, p)$ is a G.G. in $R$. Let $\{p_{n}\}$ be a divergent sequence in $G_{n} = E[z \in R^{*} : G(z, p_{n}) > \varepsilon]$ such that $\{G(z, p_{n})\}$ converges to a positive harmonic function. Then we say that $\{p_{n}\}$ determines an ideal boundary point $p$. We denote by $B$ the set of all the ideal boundary points. Also we denote $\lim_{n} G(z, p_{n})$ by $G(z, p) : p \in B$ simply. Let $q \in \partial R$ and let $v(q)$ be a compact neighbourhood of $q$ in $R^{*}$. Then $G(z, p_{n}) \leq M$ on $v(q) \cap R$ for $n \geq n(q)$. Hence $G(z, p) (p \in B) = 0$ on $\partial R$ and $\omega(R \cap B, z, R) = L \omega(R \cap B, z, R^{*}) = 0$. Also by Fatou's lemma $D(\min(G(z, p), M)) \leq 2\pi M$ for $p \in R + B$. Thus $G(z, p)$ is a G.G. for $p \in R + B$. Let $\tilde{R} = R + B$. Then the distance between $p_{1}$ and $p_{2}$ in $\tilde{R}$ is defined as

\[ \delta(p_{1}, p_{2}) = \sup_{z \in R_{0}} \left| \frac{G(z, p_{1})}{1 + G(z, p_{1})} - \frac{G(z, p_{2})}{1 + G(z, p_{2})} \right|, \]

where $R_{0}$ is a compact disc in $R$. 

Remark. Let $p_1 \neq p_2$ in $B$. Then $G(z, p_1)$ may be a multiple of $G(z, p_2)$. In fact, let $C$ be a unit circle: $|z| < 1$ and let $F$ be a closed set such that $z=0$ is contained in the closure of $F$ and $F$ is so thinly distributed in a neighbourhood of $z=0$ that $z=0$ is an irregular point of the Dirichlet problem in $C-F$. Then there exists only one linearly independent G.G. vanishing on $F + \partial C$ except $z=0$ and for any point $p$ on $z=0$, $G(z, p)$ is a multiple of a fixed function.

Let $G(z, p): p \in \overline{R}$ and $V_M(p)=E[G(z, p)>M]$. Let $U(z)$ be a G.G. in $R$ and let $U_M(z)$ be the least positive harmonic function in $V_M(p)$ with boundary value $\min(L, U(z))$ on $\partial V_M(p)$. Then $U_M(z) \leq U(z)$. Also by $w(V_M(p) \cap B, z, R)=0$, $U_M(z)=\lim U_n(z)$, where $U_n(z)$ is a harmonic function in $V_M(p) \cap R_n$ such that $U_n(z)=\min(L, U(z))$ on $\partial V_M(p) \cap R_n$ and $U_n(z)=0$ on $\partial R_n \cap V_M(p)$, where $D(U_n(z)) \leq D(\min(L, U(z))) > \infty$. Since $R + \hat{R}$ is a Riemann surface with null-boundary, there exists an exhaustion $\{R_n\}$ of $(R + \hat{R})$ such that

$$\int_{\partial R_n} \frac{\partial}{\partial n} U_n(z) ds \text{ and } \int_{\partial V_M(p)} \frac{\partial}{\partial n} G(z, p) ds \rightarrow 0 \text{ as } n \rightarrow \infty: M' > M$$

Lemma 1. By Green's formula

$$\int_{V_M(p) \cap R_n} U_M(z) \frac{\partial}{\partial n} G(z, p) ds = \int_{R_n} G(z, p) \frac{\partial}{\partial n} U_M(z) ds$$

Now $\int_{V_M(p) \cap R_n} U_M(z) \frac{\partial}{\partial n} G(z, p) ds = M$ and other integrations,

$$\int_{V_M(p) \cap R_n} G(z, p) \frac{\partial}{\partial n} U_M(z) ds = M'$$

by $\frac{\partial}{\partial n} G(z, p) \geq 0$ on $C_M + C_M'$ and by letting $L \rightarrow \infty$, we have

$$\int_{V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds = \lim_{L \rightarrow \infty} \int_{V_M(p)} U_M(z) \frac{\partial}{\partial n} G(z, p) ds$$

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\[ = \lim_{L \to \infty} \int_{\partial V_M(p)} U_{M} G(z, p) ds \leq \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds. \]

Hence

\[ \int_{\partial V_M(p)} U(z) \frac{\partial}{\partial n} G(z, p) ds \uparrow \text{as} \quad M \uparrow \infty. \]

We define the value of \( U(z) \) at \( p \) by

\[ \frac{1}{2\pi} M = \infty \]

clearly. Then

\textbf{Theorem 2.} \( G(z, p) : p \in \overline{R} \) is a G.G. and the value of \( G(z, p) \) is well defined on \( \overline{R} \) and

1. \( G(p, q) = G(q, p) \) and \( G(p, p) = \infty. \)
2. \( G(p, q) \) is lower semicontinuous in \( \overline{R} \times \overline{R}. \)
3. If \( p \in R, \) \( G(z, p) \) is continuous in \( \overline{\overline{R}}. \)

Theorem 2 can be proved (without any essential alteration) similaay as \( G(z, p) \) of \( R - R_0, \) where \( R \) is a Riemann surface with null-boundary and \( R_0 \) is a compact disc.\(^8\) But in the previous paper, it was proved that \( G(z, p) \) is lower semicontinuous in \( \overline{R} \) for fixed \( p. \) In this paper we shall prove only 2). For any given number \( \varepsilon > 0, \) we can find numbers \( M \) and \( R_n \) such that

\[ 2\pi(G(p, q) - \varepsilon) \leq \int_{\partial V_M(p) \cap R_n} G(\zeta, q) \frac{\partial}{\partial n} G(\zeta, p) ds. \]

We cover \( V_M(p) \cap R_n \) by a compact domain \( D \) such that \( \partial D \) intersects \( \partial V_M(p) \) orthogonally. Let \( p_i \to p \) and \( q_j \to q. \) Then \( G(z, p_i) \to G(z, p), \) \( G(z, q_j) \to G(z, q) \) and \( V_M(p_i) \to V_M(p) \) uniformly in \( D. \) Hence for \( i > i_0 \) and \( j > j_0 \)

\[ 2\pi G(p_i, q_j) = \lim_{M \to \infty} \int_{\partial V_M(p_i)} G(\zeta, q_j) \frac{\partial}{\partial n} G(\zeta, p_i) ds \geq \int_{\partial V_M(p)} G(\zeta, p) \frac{\partial}{\partial n} G(\zeta, p) ds - 2\pi \varepsilon \geq 2\pi(G(p, q) - \varepsilon). \]

Let \( \varepsilon \to 0. \) Then \( \lim_{\varepsilon \to 0} G(p_i, q_j) \geq G(p, q). \) Thus 2) is proved.

3. \textbf{Energy integral and capacities of Green's potential.} Let \( R = R^* = E[z \in R^* : G^*(z, p) > \delta] \) and \( \{R_n\} \) be an exhaustion of \( R. \) Let \( F \) be a compact set in \( R. \) Put \( I(\mu) = \int_{\partial D} G(\zeta, p) d\mu(p) d\mu(q), \) where \( \mu \) is a positive mass distribution on \( \overline{R}. \) Since \( F \) is compact, \( G(z, p) + \log |z - p| \) is harmonic in a neigh-

\(^8\) See (1).
bourhood of \( p \) and the continuity principle is valid in \( R \) and theorems of logarithmic potentials are also valid. Hence there exists a uniquely determined mass distribution \( \mu \), called equilibrium distribution such that \( I(\mu) \) is the minimal among all distributions of mass unity on \( F \), the potential \( U(z) \) of \( \mu=L \) (const.) on \( F \) except a set of capacity zero and \( U(z)\leq L \) in \( R \). Clearly \( U(z)=0 \) on \( \partial R \) and by \( w(B\cap R, z, R)=0 \), \( U(z)=Lw(F, z, R) \) and \( I(\mu)=L=\frac{D(U(z))}{2\pi} \),

where \( L \) is given by \( L=\frac{2\pi}{\int_{\partial R} w(F, z, R)ds} \), \( C_{M}=E[z\in R: w(F, z, R)=M] \), \( M<1 \).

We define the capacity of \( F \) as \( \frac{1}{I(\mu)} \) (\( =\frac{1}{L} \)). Let \( F \) be a closed set in \( \overline{R}=R+B \). We also define \( \text{Cap}(F) \) by \( \frac{1}{\inf I(\mu)} \), where \( \mu \) is a positive mass distribution of unity on \( F \). Put \( \hat{\text{Cap}}(F)=\sup_{K} \text{Cap}(K) \), where \( K \) is a compact set in \( F \). Then if \( F \) is compact, \( \text{Cap}(F)=\hat{\text{Cap}}(F) \) and \( \text{Cap}(F)\geq \hat{\text{Cap}}(F) \) for closed set \( F \) (in reality it can be proved \( \text{Cap}(F)=\hat{\text{Cap}}(F) \)). In this paper we use only \( \hat{\text{Cap}}(F) \) for Green's potential.

**Capacities of the irregular set of \( R \) of Green's function.** Let \( G(z, p_{0}) \) be Green's function of \( R=E[R^{*}\exists z: G(z, p_{0})>\delta] \). Then \( G(z, p_{0}) \) is continuous in \( \overline{R} \). Let \( F_{l}=E[z\in R: G(z, p_{0})\geq \frac{1}{l}] \). Then \( F_{l} \) is closed in \( \overline{R} \). \( U(z)=\min \left( \frac{1}{l}, G(z, p_{0}) \right) \) is a continuous function in \( \overline{R} \) such that \( U(z)=0 \) on \( \partial R \), \( U(z)=\frac{1}{l} \) on \( F_{l} \) and \( D(U(z))\leq \frac{2\pi}{l} \). Let \( R_{n} \) \( (n=1, 2, \cdots) \) be an exhaustion of \( R \) with compact relative boundary \( \partial R_{n} \). Let \( \omega(F_{l}\cap(R-R_{n}), z, R) \) be a harmonic function in \( R-(F_{l}\cap(R-R_{n})) \) such that \( \omega(F_{l}\cap(R-R_{n}), z, R)=1 \) on \( F_{l}\cap(R-R_{n}), =0 \) on \( \partial R \) and has M.D.I.. Then by the Dirichlet principle \( D(\omega(F_{l}\cap(R-R_{n}), z, R))\leq \frac{1}{l} D(U(z))<\infty \). It is evident \( \omega(F_{l}\cap(R-R_{n}), z, R) \geq \omega(K, z, R) \) for any compact set \( K \) in \( F_{l}\cap(R-R_{n}) \) and by \( \omega(R\cap B, z, R)=0 \), \( \omega(F_{l}\cap(R-R_{n}), z, R) = \omega(F_{l}\cap(R-R_{n}), z, R) \). Let \( n\rightarrow\infty \). Then \( \omega(F_{l}\cap(R-R_{n}), z, R) \) converges in mean to a function \( V(z) = \omega(F_{l}\cap B, z, R) \). Clearly \( V(z) \) is a G.G., hence by Theorem 1. (5) \( V(z)=0 \). This implies \( D(\omega(F_{l}\cap(R-R_{n}), z, R)) \downarrow 0 \) as \( n\rightarrow\infty \) and \( \lim \frac{\hat{\text{Cap}}(F_{l}\cap(R-R_{n})))}{2\pi} \) as \( n\rightarrow\infty \).

**Loss of mass.** As usual mass \( m(p) \) of \( G(z, p) (p\in\overline{R}) \) is given by \( \frac{1}{2\pi} \int_{\partial R} \frac{\partial}{\partial n} G(z, p)ds \) \( (m(p) \) does not depend on \( M \) by Theorem 1. (3)). It is clear
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$m(p)=1$ for $p \in R$. If $p \in F \cap B$, $m(p) \geq \frac{1}{kl}$, where $k=\lim_{n=\infty} \sup_{z \in R-R_n} G(z, p_0)$. In fact, $G(p, p_0) = \frac{1}{2\pi} \lim_{M=\infty} \int_{\partial V_{\infty}} G(\zeta, p_0) \frac{\partial}{\partial n} G(\zeta, p) ds$. Hence by letting $\varepsilon \to 0$, $m(p) \geq \frac{1}{kl}$.

4. $N$-Green's function and $N$-Martin's topology. Let $R$ be a Riemannian surface with positive boundary and let $\{R_n\}$ be its exhaustion with compact relative boundary $\partial R_n$. Let $U(z)$ be a positive superharmonic function in $R-R_0$ such that $U(z)=0$ on $\partial R_0$ and $D(\min(M, U(z)))<\infty$ for any $M<\infty$. Let $\omega(F_{n}, z, R-R)=\omega(F, z, R-R_0)$ be a harmonic function in $R-R_0-F$ such that $\omega(F_n, z, R-R)=1$ on $F_n \cap R$, $\omega(F_n, z, R-R)=0$ on $\partial R_0$ and $\omega(F_n, z, R-R)$ has M.D.I. Let $n \to \infty$. Then $\omega(F_n, z, R-R) \Rightarrow \omega(F, z, R-R_0)$. We call $\omega(F, z, R-R)$ C.P. of $F$ and define $\text{Cap}(F)$ by $\frac{1}{2\pi} D(\omega(F, z, R-R_0))$. We state briefly the properties of $B$ and $N(z, p)$ without proofs.\(^{10}\)}

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9) See (5).

10) See (5).
**Theorem 3.**

1. \( B = B_1 + B_0 \) and \( B_0 \) is an \( F_\sigma \) set of capacity zero.
2. \( N(z, p) \) is \( N \)-minimal if and only if \( p \in B_1 \).
3. If \( \omega(p, z, R-R) > 0, \sup_{\zeta \in R} N(z, p) < \infty \) and \( p \in B_1 \). We denote the set of \( p \) such that \( \omega(p, z, R-R) > 0 \) by \( B_* \). Then if \( p \in B_* \), \( N(z, p) = k \omega(p, z, R-R) \).
4. For \( p \in \overline{R} - B_0 \), \( \int_{\partial V_M(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \), \( \int_{\partial V_M(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \), (10)

for almost \( M \), i.e. the set of \( M \) such that (10) is not hold is a set of measure zero. We call such \( \partial V_M(p) \) a regular niveau if (10) is satisfied, where \( V_M(p) = E[z \in R: N(z, p) > M], M < M^*(p) = \sup_{z \in R} N(z, p) \).

5. For \( p \in \overline{R} - B_0 \), \( N(z, p) = M \omega(V_M(p), z, R-R_0) \) in \( R-R_0 - V_M(p) \).
6. Let \( p \in \overline{R} - B_0 \). For any \( M < M^*(p) \), there exists a number \( n \) such that

\[
V_M(p) \supset (R-R_0) \cap v_n(p), \quad \text{where} \quad v_n(p) = E[z \in \overline{R}: \delta(z, p) < \frac{1}{n}].
\]

7. Let \( p \in \overline{R} - B_0 \). Then for any \( v_n(p) \)

\[
\lim_{M \rightarrow M^*(p)} \int_{\partial V_M(p) \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 0, \quad \lim_{M \rightarrow M^*(p)} \int_{\partial V_M(p) \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 2\pi,
\]

where \( \partial V_M(p) \) is regular.

And \( \lim_{M \rightarrow M^*(p)} \int_{\partial V_M(p) \cap v_n(p)} \frac{\partial}{\partial n} N(z, p) ds = 0 \).

8. The value of \( N(z, p) : p \in \overline{R} - R_0 \) is given as follows:

a). \( q \in \overline{R} - B_0 \), \( N(q, p) = \lim_{M \rightarrow M^*(q)} \frac{1}{2\pi} \int_{\partial V_M(q)} N(\zeta, p) \frac{\partial}{\partial n} N(\zeta, q) ds \),

where \( \partial V_M(q) \) is a regular niveau.

b). \( q \in B_0 \). In this case \( N(z, q) \) is represented as \( \int_{B_0} N(z, r) d\mu_q(r) \) and \( N(q, p) = \int_{B_0} N(r, p) d\mu_q(r) \) and the value \( N(q, p) \) does not depend on particular distribution \( \mu_q(r) \).

9. \( N(p, q) = N(q, p) \) and \( N(z, p) \) is lower semicontinuous in \( \overline{R} - R_0 \) for fixed \( p \).

10. If \( p \in \overline{R} - B_0 \),

\[
N(p, q) \geq \frac{1}{2\pi} \int_{\partial V_M(p)} N(\zeta, q) \frac{\partial}{\partial n} N(\zeta, p) ds.
\]

**Energy integral and the capacites.** Let \( F \) be a closed set in \( \overline{R} - R \) of positive capacity, i.e. \( \omega(F, z, R-R_0) > 0 \). Then there exists a canonical distri-
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bution of unity on $F$ (if $\mu=0$ on $B_o$, $\mu$ is called canonical) such that ther energy integral $I(\mu) = \inf I(\mu_i)$ and the potential $U(z)$ of $\mu = \omega(F, z, R-R_0)$ and $I(\mu) = \frac{1}{2\pi} D(\omega(F, z, R-R_0)) = L$, where $\mu_i$ is a canonical mass distribution of unity on $F$ and $L = 2\pi \int_{\partial R} \frac{\partial}{\partial n} \omega(F, z, R-R_0) ds$. We call $\frac{1}{L}$ the capacity of $F$. Also define $\hat{\text{Cap}}(F)$ by $\lim_{n=\infty} \text{Cap}(F_n) = \frac{1}{\inf I(\mu_i)}$, where $\mu_i$ is a distribution of unity on $F$ and $\text{Cap}(F) = \frac{1}{\text{Cap}(F_n)}$.

5. Transfinite diameters. We discuss transfinite diameters and others of closed sets on Riemann surfaces. The properties of the surfaces have much influence on them. Therefore we divide types of Riemann surfaces into three types as follows: (A). $R$ is a Riemann surface with positive boundary and $N$-Martin's topology is defined on $R-R_0 = R+B-R_0$. (B) $R=R_0 = E[z \in R^* : G^*(z, p) > \delta > 0]$, where $R^*$ is a Riemann surface with positive boundary and $G$-Martin's topology is defined on $R=R+B$. (C) $R$ is a Riemann surface with null-boundary and $G$-Martin's topology is defined on $R-R_0 + B = R-R_0$.

Let $F$ be a closed set in $R$. We define transfinite diameters $\tilde{D}(F)$, $D^M(F)$ and $D(F)$ as follows:

(A). $1/D(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \in F-B_0} N(p_i, p_j)$, $1/D^M(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \in F-B_0} G(p_i, p_j)$ and $\tilde{D}(F) = \lim_{M=\infty} D(F)$, where $N^M(p_i, p_j) = \min(M, N(p_i, p_j))$.

(B) and (C) $1/D(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \in F} G(p_i, p_j)$, $1/D^M(F) = \lim_{n=\infty} \frac{1}{nC_2} \inf \sum_{p_i \in F} G^M(p_i, p_j)$ and $\tilde{D}(F) = \lim_{M=\infty} D^M(F)$, where $G^M(z, p) = \min(M, G(z, p))$.

Then clearly $\frac{1}{nC_2} \inf \sum_{p_i \in F} N^M(p_i, p_j)$ and $\frac{1}{nC_2} \inf \sum_{p_i \in F} G^M(p_i, p_j)$ as $n \to \infty$ and $D^M(F)$ as $M \uparrow \infty$ and $D^M(F), \tilde{D}(F), D(F)$ are increasing functions of $F$.

Lemma 2. Let $p_i (i=1, 2) \in \overline{R}-B_o$. Let $D_i$ be a compact or non compact domain in $V_m(p_i) = E[z \in R: N(z, p_i) > M]$. Let $\omega(D_i, z)$ be C.P. of $D_i$, i.e. $\omega(D_i, z) = \omega(D_i, z, R-R_0)$. Then $M\omega(D_i, z)(\leq N^M(z, p_i))$ can be represented by a canonical mass distribution $\mu_i^*$ on $\overline{D}_i$ such that $M\omega(D_i, z) = \int N(z, q) \times d\mu_i^*(q)$ in $\overline{R}-R_0$ and (clearly $\int d\mu_i^* \leq 1$)

$$\int M\omega(D_j, q) d\mu_i^*(q) \leq N^M(p_i, p_j), \quad i, j=1, 2,$$

where not necessarily $p_i \neq p_j$.

11) See (5).
(B) and (C). Let $D_{i} = V_{M}(p_{i})$. Then $Mw(D_{i}, z) = M\omega(D_{i}, z) = \min(M, G(z, p_{i}))$ ($p \in \bar{R}$ for (B) and $p \in \bar{R} - R_{0}$ for (C)) is represented by a mass $\mu_{i}$ on $R$ such that $\mu_{i} = 0$ on $B$ and

$$\int M\omega(D_{j}, q) d\mu_{j}(q) \leq G^{M}(p_{i}, p_{j}), \quad j, i = 1, 2,$$

where not necessarily $p_{i} \neq p_{j}$.

**Proof.** Case 1. $p_{2} \in B_{1}$ or $p_{2} \notin D_{1}$. Put $D_{1,n} = D_{1} \cap R_{n}$. Then $D_{1,n}$ is compact. In this case, we can find a number $M'$ such that $V_{M'}(p_{2}) \cap D_{1,n} = 0$ and $\partial V_{M'}(p_{2})$ is regular, because $V_{M}(p_{2}) \to B$ as $M \to M^{*}(p_{2})$. Put $d\mu_{1,n} = \frac{M}{2\pi} \times \frac{\partial}{\partial n} \omega(D_{1,n}, z) ds$ on $\partial D_{1,n}$, i.e.

$$\omega(D_{1,n}, z) = \int N(z, r) d\mu_{1,n}(r), \quad z \in R - R_{0}. \quad (11)$$

Let $m \geq n$ and let $N_{m}(z, p_{2})$ be a harmonic function in $R_{m} - R_{0} - V_{M'}(p_{2})$ such that $N_{m}(z, p_{2}) = N(z, p_{2})$ on $\partial R_{0} + \partial V_{M}(p_{2})$ and $\frac{\partial}{\partial n} N_{m}(z, p_{2}) = 0$ on $\partial R_{m} - V_{M'}(p_{2})$. Then by Theorem 3. (5) $N_{m}(z, p_{2}) \to N(z, p_{2})$ as $m \to \infty$ and

$$\lim_{m \to \infty} \int_{\partial V_{M'}(p_{2}) \cap R_{m}} A_{m}(z) \frac{\partial}{\partial n} N_{m}(z, p_{2}) ds = \int_{\partial R_{m} \cap V_{M'}(p_{2})} A(z) \frac{\partial}{\partial n} N(z, p_{2}) ds = 0, \quad (12)$$

by the Lemma for any $A_{m}(z)$ such that $0 \leq A_{m}(z) \leq L < \infty$ on $\partial V_{M'}(p_{2})$ and $\lim_{m \to \infty} A_{m}(z) = A(z)$ on $\partial V_{M'}(p_{2})$. Also let $\omega_{m}(D_{1,n}, z)$ be a harmonic function in $R_{m} - R_{0} - D_{1,n}m$ such that $\omega_{m}(D_{1,n}, z) = \omega(D_{1,n}, z)$ on $\partial R_{0} + \partial D_{1,n}$ and $\frac{\partial}{\partial n} \omega_{m}(D_{1,n}, z) = 0$ on $\partial R_{m}$. Then $\omega_{m}(D_{1,n}, z) \to \omega(D_{1,n}, z)$ as $m \to \infty$. Then by

$$\int_{\partial V_{M'}(p_{2}) \cap R_{m}} \omega_{m}(D_{1,n}, z) \frac{\partial}{\partial n} \omega_{m}(D_{1,n}, \zeta) ds = \int_{\partial D_{1,n}} \omega_{m}(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_{m}(\zeta, p_{2}) ds \equiv M \int_{\partial D_{1,n}} N(\zeta, p_{2}) \frac{\partial}{\partial n} \omega_{m}(D_{1,n}, \zeta) ds = M \int_{\partial V_{M'}(p_{2}) \cap R_{m}} \omega_{m}(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_{m}(\zeta, p_{2}) ds \equiv \int_{\partial D_{1,n}} N(\zeta, p_{2}) \frac{\partial}{\partial n} N(\zeta, p_{2}) ds.$$

Let $m \to \infty$. Then by (12)

$$\int_{\partial D_{1,n}} N(\zeta, p_{2}) \frac{\partial}{\partial n} \omega(D_{1,n}, \zeta) ds = \int_{\partial D_{1,n}} \omega_{m}(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_{m}(\zeta, p_{2}) ds \equiv \frac{1}{2\pi} \min(M, N(\zeta, p_{2})) \frac{\partial}{\partial n} N(\zeta, p_{2}) ds \leq \frac{1}{2\pi} \min(M, N(\zeta, p_{2})) \frac{\partial}{\partial n} N(\zeta, p_{2}) ds$$

See (5).
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\[ \leq \frac{1}{2\pi} \min \left( M, \lim_{M \to M^*} \int_{\partial \mathcal{M}^*(p)} N(\zeta, p) \frac{\partial}{\partial \eta} N(\zeta, p) \, ds \right) = N^M(p, p). \]  

Put \( d\mu_{1,n}(\zeta) = \frac{M}{2\pi} \frac{\partial}{\partial n} \omega(D_{1,n}, z) \, ds \) on \( \partial D_{1,n} \). Then by (13)

\[ \int_{\partial D_{1,n}} N(\zeta, p) \, d\mu_{1,n}(\zeta) \leq N^M(p, p). \]

Next suppose \( r \in R - R_0 \). Then

\[ \frac{M}{2\pi} \int_{\partial D_{1,n}} N(\zeta, r) \frac{\partial}{\partial n} \omega(D_{1,n}, z) \, ds = M \omega(D_{1,n}, r), \]

whence \( M \omega(D_{1,n}, z) = \int N(z, \zeta) \, d\mu_{1,n}(\zeta), z \in \overline{R} - R_0 \).

Hence \( \mu_{1,n}(\zeta) \) is the mass distribution of \( M \omega(D_{1,n}, z) \) and \( \int d\mu_{1,n} \leq 1 \). Let \( n \to \infty \). Then there exists an weak limit \( \mu \) on \( \overline{D} \) of \( \{ \mu_{1,n} \} \) such that

\[ \int N(z, \zeta) \, d\mu_{1,n}(\zeta) \to \int N(z, \zeta) \, d\mu(\zeta) \text{ for } z \in \overline{R} - R_0 \text{ as } n \to \infty \text{ and } M \omega(D_{1,n}, z) = \lim_{n \to \infty} M \omega(D_{1,n}, z) = \int N(z, \zeta) \, d\mu_*(\zeta) \text{ for } z \in \overline{R} - R_0. \]

We can find a canonical distribution \( \mu_* \) such that \( M \omega(D_{1,n}, z) = \int N(z, \zeta) \, d\mu_*(\zeta) \) for \( z \in \overline{R} - R_0 \). On the other hand, since \( \nu_\ast \omega(D_{1,n}, z) = \omega(D_{1,n}, z) \) for any \( l \), any canonical distribution of \( \omega(D_{1,n}, z) \) has no mass on \( CD_{1} \), where \( CD_{1} \) is the harmonic function in \( R - R_0 - D_l \) such that \( \nu_\ast \omega(D_{1,n}, z) = \omega(D_{1,n}, z) \) on \( \partial D_l \) and \( \nu_\ast \omega(D_{1,n}, z) \) has M.D.I. and \( D_l = E \left[ z \in R : \delta(z, D_l) < \frac{1}{l} \right] \). Hence \( \mu_* \) is a canonical distribution on \( \overline{D} \cap (\overline{R} - B_0) \) such that

\[ M \omega(D_{1}, z) = \int N(z, \zeta) \, d\mu_*(\zeta), z \in \overline{R} - R_0. \]

By (16) and the definition of \( \omega(D_{1}, p) \) for \( p \in \overline{R} - R_0 - B_0 \)

\[ M \omega(D_{1}, p) = \frac{M}{2\pi} \lim_{M \to M^*(p)} \int N(\zeta, p) \, d\mu^* \left( \frac{M}{2\pi} \frac{\partial}{\partial n} N(\zeta, p) \, ds \right) = \frac{M}{2\pi} \lim_{M \to M^*(p)} \int N(\zeta, \eta) \, d\mu^* \left( \frac{M}{2\pi} \frac{\partial}{\partial n} N(\zeta, \eta) \, ds \right) = \frac{M}{2\pi} \lim_{M \to M^*(p)} \int N(\eta, p) \, d\mu^* \left( \frac{M}{2\pi} \frac{\partial}{\partial n} N(\zeta, \eta) \, ds \right), \]

because by \( \int N(\zeta, \eta) \, d\mu^* \left( \frac{M}{2\pi} \frac{\partial}{\partial n} N(\zeta, \eta) \, ds \right) \) as \( M \to M^*(p) \) the order of integrations and letting \( M \to M^*(p) \) can be changed. Hence (16) is valid for \( z \in \overline{R} - R_0 - B_0 \). Similarly we have by (11)

See (5).
\[ \omega(D_{1,n}, p) = \int N(\zeta, p) d\mu_{1,n}(\zeta) \quad \text{for} \quad p \in \bar{R} - R_0 - B_0, \]  

i.e. (11) is valid for \( z \in \bar{R} - R_0 - B_0 \). Further by \( \omega(D_{1,n}, z) \uparrow \omega(D_{1}, z) \) as \( n \to \infty \), we have

\[
\omega(D_{1}, p) = \frac{1}{2\pi} \lim_{M \to \mathcal{M}(p)} \int_{\partial D_{1}} \omega(D_{1}, z) \frac{\partial}{\partial n} N(z, p) ds
\]

\[
= \frac{1}{2\pi} \lim_{M \to \mathcal{M}(p)} \left( \int_{\partial V_{M}(p)} \lim_{n} \omega(D_{1,n}, z) \frac{\partial}{\partial n} N(z, p) ds \right)
\]

\[
= \frac{1}{2\pi} \lim_{n} \left( \int_{\partial V_{M}(p)} \lim_{\partial V} \int_{u^{(p)}} \omega(D_{1}, z) \frac{\partial}{\partial n} N(z, p) ds \right) = \lim_{n} \omega(D_{1,n}, p). \quad (19)
\]

Hence by (17), (19), (15) and (14)

\[
M \omega(D_{2}, \eta) d\mu_{1}^{*}(\eta) \leq \int N(\zeta, p_{2}) d\mu_{1}^{*}(\eta)
\]

\[
= M \omega(D_{1}, p_{2}) = \lim_{n} M \omega(D_{1,n}, p_{2}) \leq N^{M}(p_{2}, p_{1}).
\]

Case 2. \( p_{2} \in D_{1} \cap (R - R_0) (j = 1 \text{ or } 2) \). In this case \( N(z, p_{i}) \geq M \) in \( D_{1} \). Hence

\[
N(p_{2}, p_{1}) = N(p_{1}, p_{2}) \geq M.
\]

There exists a number \( n \) such that \( D_{1,n} \ni p_{2} \). Let \( m > n \) and let \( N_{m}(z, p_{2}) \) be a harmonic function in \( R_{m} - R_0 - p_{2} \) such that \( N_{m}(z, p_{2}) = 0 \) on \( \partial R_{0}, \) \( N_{m}(z, p_{2}) \) as a logarithmic singularity at \( p_{2} \) and \( \frac{\partial}{\partial n} N_{m}(z, p_{2}) = 0 \) on \( \partial R_{m} \). Then \( N_{m}(z, p_{2}) \Rightarrow N(z, p_{2}) \) as \( m \to \infty \). Let \( \omega_{m}(D_{1,n}, z) \) be the function in case 1. Then

\[
\frac{M}{2\pi} \int_{\partial D_{1,n}} \frac{N_{m}(\zeta, p_{2})}{\partial n} \omega_{m}(D_{1,n}, \zeta) ds
\]

\[
= \frac{M}{2\pi} \int_{\partial D_{1,n}} \omega_{m}(D_{1,n}, \zeta) \frac{\partial}{\partial n} N_{m}(z, p_{2}) ds = \frac{M}{2\pi} \int_{\partial D_{1,n}} \frac{\partial}{\partial n} N_{m}(\zeta, p_{2}) ds
\]

\[ \leq M. \]  

Let \( m \to \infty \). Then by (21)

\[
\frac{M}{2\pi} \int_{\partial D_{1,n}} N(\zeta, p_{2}) \frac{\partial}{\partial n} \omega(D_{1,n}, \zeta) ds \leq M = N^{M}(p_{2}, p_{1}).
\]

Put \( d\mu_{n} = \frac{M}{2\pi} \frac{\partial \omega}{\partial n}(D_{1,n}, \zeta) ds \) on \( \partial D_{1,n} \). Then as in case 1), there exists a canonical distribution \( \mu_{1}^{*} \) such that \( M \omega(D_{1}, \zeta) = \int_{R - B_{0}} N(z, \zeta) d\mu_{1}^{*}(\zeta) \) and
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$$M \int \omega(D_{2}, \eta) d\mu^{*}_{1}(\eta) \leqqq \int N(\eta, \mu_{2}) d\mu^{*}_{1}(\eta) \leqqq N^{V}(\mu_{2}, \mu_{1}) \quad (22)$$

If \( p_{2} \in R - D_{1} \), we have more easily (22). Thus by (21) and (22) we have the former part of Lemma 2.

(B) and (C). Suppose \( p_{2} \notin V_{M}(\mu_{1}) \). Then by Green's formula and by Lemma 1

$$G^{M}(\mu_{1}, \mu_{2}) = G(\mu_{1}, \mu_{2}) = \frac{1}{2\pi} \int_{\partial V_{M}(\mu_{1})} G(\xi, \mu_{2}) \frac{\partial}{\partial n} G(\xi, \mu_{1}) ds \quad (23)$$

If \( p_{2} \in V_{M}(\mu_{1}) \), \( G(\mu_{1}, \mu_{2}) \leqqq M \frac{1}{2\pi} \int_{\partial V_{M}(\mu_{1})} \omega(V_{M}(\mu_{2}), \xi) \frac{\partial}{\partial n} G(\xi, \mu_{1}) ds \quad (24)$$

Put \( d\mu_{i} = \frac{1}{2\pi} \frac{\partial}{\partial n} G(z, \mu_{1}) ds \) on \( \partial V(\mu_{1}) \).

Then \( \mu_{1} = 0 \) on (B) and by (23) and (24)

$$\int M \omega(V_{M}(\mu_{1}), \xi) d\mu_{1}(\xi) \leqqq G^{M}(\mu_{2}, \mu_{1}) \quad (27)$$

**Theorem 4.** (A). Let \( p_{i} (i = 1, 2, \cdots, n) \) in \( \bar{R} - B_{0} \). Let \( D_{i} \) be a domain in \( V_{M}(\mu_{1}) \). Then there exists a canonical distribution \( \mu_{i} \) of mass \( \leqqq 1 \) on \( \bar{D}_{i} \) such that \( M \omega(D_{i}, z) = \int G(z, p) d\mu_{i}(p) \).

Put \( \mu = \frac{1}{n} \sum \mu_{i} \). Then

$$I(\mu) = \int \int N(\mu, \mu) d\mu(\mu) \leqqq \frac{1}{n^{2}} \sum_{i=1}^{n} G^{M}(\mu_{i}, \mu_{j}) \quad (28)$$

(B). Let \( p_{i} (i = 1, 2, \cdots, n) \) on \( \bar{R} \). Then there exists a distribution \( \mu_{i} \) on \( R \cap V_{M}(\mu_{i}) \) (\( \sum d\mu_{i} \leqqq 1 \) by loss of mass) such that \( M \omega(V_{M}(\mu_{i}), z) = \int G(z, p) d\mu_{i}(p) \).

Put \( \mu = \frac{1}{n} \sum \mu_{i} \). Then \( I(\mu) \leqqq \frac{1}{n^{2}} \sum \sum G^{M}(\mu_{i}, \mu_{j}) \).

(C). Let \( p (i = 1, 2, \cdots, n) \) on \( \bar{R} - R_{0} \). Then there exists a distribution \( \mu_{i} \) on \( R \cap V_{M}(\mu_{i}) \) such that \( M \omega(V_{M}(\mu_{i}), z) = \int G(z, p) d\mu_{i}(p) \) and \( \int d\mu_{i} = 1 \). Put

$$\mu = \frac{1}{n} \sum \mu_{i} \quad \text{Then} \quad I(\mu) \geqq \frac{1}{n^{2}} \sum \sum G^{M}(\mu_{i}, \mu_{j}) \quad (29)$$

**Proof of (A).** Put \( U(z) = \int N(\mu, p) d\mu(\mu) \). Then \( U(z) = M \sum \omega(D_{i}, z) \). Then by Lemma 2, \( I(\mu) = \frac{M}{n^{2}} \sum \omega(D_{i}, z) \sum d\mu_{i} \leqqq \frac{1}{n^{2}} \sum \sum N^{M}(\mu_{i}, \mu_{j}) \).

It is proved similarly for (B) paying attention to \( d\mu_{i} = \frac{\partial}{\partial n} G(z, p) ds \) on \( \partial V_{M}(\mu) \).
and \( \int_{\partial V_{M}(p)} \frac{\partial}{\partial n} G(z, p) ds \leq 2\pi \) (by loss of mass) i.e. \( \int d\mu \leq 1 \) and \( \mu = 0 \) on (B)

and for (C) to \( \int_{\partial V_{M}(p)} \frac{\partial}{\partial n} G(z, p) ds = 2\pi \).

**Lemma 3.** Let \( \tilde{A} \) and \( A \) be closed sets such that \( \tilde{A} \supset A \).

(A), 1). If there exists a number \( M \) such that \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M \omega(V_{M}(p) \cap \tilde{A}, z) ds > \delta > 0 \) for any \( p \in A \cap (\bar{R} - R_{0} - B_{0}) \), then

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A}) .
\]

2). If there exists a number \( M \) such that

\[
\frac{1}{2\pi} \int_{\partial r_{u^{(p) \subset X}}} \frac{\partial}{\partial n} N(z, p) ds \geq \delta > 0
\]

for any \( p \in A \cap (\bar{R} - R_{0} - B_{0}) \), then

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A}) .
\]

2'). (B) and (C). If there exists a number \( M \) such that

\[
\frac{1}{2\pi} \int_{\partial V_{M}(p) \cap \tilde{A}} \frac{\partial}{\partial n} G(z, p) ds > \delta > 0
\]

for any \( p \in A \cap (\bar{R} - R_{0} - B_{0}) \), then

\[
1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A}) .
\]

3). (A). Let \( G \) be an open set such that \( G \supset A \) and \( G \supset \sum_{p \in F} V_{M}(p) \), then

\[
1/D^{M}(A) \geq 1/\text{Cap}(G)
\]

where \( \text{Cap}(G) = \sup_{F} \text{Cap}(F) \), \( F \) is a closed set in \( G \).

3'). (C). Let \( G \) be an open set such that \( G \supset A \) and \( G \supset \sum_{p \in F} V_{M}(p) \). Then

\[
1/D^{M}(A) \geq 1/\text{Cap}(G)
\]

**Proof of 1).** Let \( D(p_{i}) = V_{M}(p_{i}) \cap \tilde{A} \). Let \( \mu_{i}(p) \) be the canonical distribution of \( M \omega(D(p_{i}), z) \) on \( \bar{D}(p_{i}) \subset \tilde{A} \). Then by Theorem 4, putting \( \mu = \frac{1}{n} \times \sum \mu_{i}(p) \), we have

\[
\iint N(p, q) d\mu(p) d\mu(q) \leq \frac{1}{n^{2}} \sum_{i=1}^{n} N^{M}(p_{i}, p_{j}) .
\]

Now the total mass of \( \mu \geq \delta \) and \( \mu = 0 \) except \( \tilde{A} \cap (\bar{R} - R_{0} - B_{0}) \), whence by the definition of the capacity \( I(\mu) \geq \delta^{2}/\text{Cap}(\tilde{A}) \). On the other hand,

\[
2( \sum_{p_{i} \neq p_{j}}^{n} N^{M}(p_{i}, p_{j}) ) = \sum_{i=1}^{n} N^{M}(p_{i}, p_{j}) - \sum_{i=1}^{n} N^{M}(p_{i}, p_{i}) .
\]

Hence

\[
1/D^{M}_{n}(A) = \inf \sum_{i=1, j=1 \atop i < j}^{n} N^{M}(p_{i}, p_{j}) / n \geq \delta^{2}/\text{Cap}(\tilde{A}) - \frac{M}{n-1} .
\]

Let \( n \to \infty \). Then \( 1/D^{M}(A) \geq \delta^{2}/\text{Cap}(\tilde{A}) \).

Paying attention to \( M \omega(V_{M}(p_{i}), z) = \int_{\partial V_{M}(p_{i})} N(z, p) d\mu_{i}(p_{i}) \) and \( G^{M}(z, p_{i}) = \int_{\partial V(z)} G(z, p_{i}) \),
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$\mu_d\mu_{\ell}(q)$ for (B) and (C) and $\mu>0$ only on $\partial V_M(p) \cap R$, we have 2), 2' similarly as 1). Next since $N^M(z, p) = M\omega(V_M(p), z)$ $(G^M(z, p) = M\omega(V_M(p), z))$ for (C) and the mass of $M\omega(V_M(p), z)$ is unity. Hence by (1) we have 3) and 3').

6. Activity of a point $p \in \overline{R} - R_0$ to a closed set $F$. Let $F_n = E \left[ z \in \overline{R} : \frac{1}{n} \right]$. (A).

Put

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{\partial F_n} M\omega(F_n \cap V_M(p), z) \, ds = \delta(p)$$

and

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{\partial F_n} M\omega(F_n \cap V_M(p), z) \, ds = \delta(p),$$

where $\partial V_M(p)$ is a regular niveau. We call $\delta(p)$ and $\delta(p)$ the weak and strong activity of $p$ to $F$ respectively.

(B). $\delta(p)$ is given as

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{\partial F_n} \frac{\partial}{\partial n} G(z, p) \, ds.$$

Clearly $\delta(p)$ to $F \supseteq \delta(p)$ to $F$ and $\delta(p)$ to $F \supseteq \delta(p)$ to $F$, if $F \subseteq F$.

Lemma 4. 1). $\delta(p) \geq \delta(p)$.

2). Suppose $p \in \overline{B} - B_0$. If $p \in F$, $\delta(p)$ to $F = 1$. If $\sup_{\mathbb{R}} N(z, p) = \infty$ and $p \notin F$, $\delta(p)$ to $F = 0$.

3). Let $F$ be a closed set of capacity zero. Suppose $U(z) = \int_{F-B_0} N(z, p) \times d\mu(p)$. Then

$$\lim_{n \to \infty} M\omega(V_M \cap F_n, z) = U(z),$$

where $V_M = \{ z \in R : U(z) > M \}$.

Let $p \in B_0$. Then $N(z, p) = \int_{\partial F_n} \frac{\partial}{\partial n} N(z, p) \, ds$. Let $\mu'$ be the restriction of $\mu$ on $F$. Then

$$N(z, p) \geq \int_{F-B_0} N(z, pq) \, d\mu'(q)$$

and $\delta(p)$ to $F = \int_{F-B_0} \frac{1}{2\pi} \int_{\partial F_n} \frac{\partial}{\partial n} N(z, p) \, ds$.

Proof of 1). $\int_{\partial F_n} \frac{\partial}{\partial n} \omega(F_n \cap V_M(p), z) \, ds \leq \int_{\partial F_n} \frac{\partial}{\partial n} \omega(F_n \cap V_M(p), z) \, ds$

$\leq \int_{\partial F_n} \omega(F_n \cap V_M(p), z) \, ds$, whence we have at once 1).

Proof of 2). Since $p \in F - B_0$, $\int_{\partial F_n} \frac{\partial}{\partial n} N(z, p) \, ds \to 0$ and

$$\int_{\partial F_n} \frac{\partial}{\partial n} N(z, p) \, ds \to 2\pi$$

as $M \to M^*(p)$ by Theorem 2, where $\partial V_M(p)$ is a regular niveau. Whence $\delta(p)$ to $F$ is 1. Let $p \notin F$ and $p \notin B_0$. Then there
exists a number $n$ such that $p \notin F_n$. Put

$$U(z) = \lim_{M=\infty \atop \infty} F_n \cap V_M(p) \cap U(z) : m > n.$$ 

Then $N(z, p) \geq U(z)$ and $N(z, p) - U(z)$ is superharmonic

$$\sup_{\partial R_0} \text{harmonic}^{14)}$$

by $\lim_{M=\infty} \text{Cap} (V_M(p)) = 0$. On the other hand, $N(z, p)$ is N-minimal, whence $U(z) = \epsilon N(z, p)$ and the mass distribution of $U(z)$ must be a point mass at $q \notin F_n$. This implies $N(z, p) = N(z, q) : q \notin F_n$. This is a contradiction. Hence $0 = U(z) = \lim_{M=\infty} M \omega (F_n \cap V_M(p), z)$ and $\delta(p)$ to $F=0$.

Proof of 3). We proved the following proposition

$$V(z) = \lim_{M=\infty \atop \infty} c_{F_n \cap V_M} U(z).$$

Then since $\lim\limits_{n} F_n \cap V_{M} U(z) = U(z) \geq \lim\limits_{n} F_n \cap V_{M} U(z)$ and

$$V(z) = \int_{\overline{F_n} - B_0} N(z, q) d\mu(q)$$

and $U(z) - V(z)$ is also superharmonic

$$\sup_{\partial R_0} V(z) \leq M < \infty$$

Assume $V(z) > 0$, consider $U(z) = (U(z) - V(z)) + V(z)$. Then $U(z)$ has mass on $\overline{CF_n} - B_0$. This contradicts the above proposition. Hence $V(z) = 0$. Let $V'(z) = \lim\limits_{\infty} F_n \cap V_M U(z)$, then also $U(z) - V'(z)$ is superharmonic and $\lim\limits_{\infty} F_n \cap V_M V'(z) = V'(z)$. Whence the canonical distribution $\mu'$ of $V'(z) \leq \mu$, and the kernel of $\mu'$ is contained in $F$. On the other hand, $\sup_{\partial R_0} V'(z) = \infty$. Assume $V'(z) > 0$, since $\text{Cap} (F) = 0$, $\sup_{\partial R_0} V'(z) = \infty$. This is a contradiction. Hence $V'(z) = 0$.

Clearly by $N$-maximum principle

$$M \omega (V_M \cap F_n, z) + c_{F_n \cap V_M} U(z) + c_{F_n \cap V_M} U(z) \geq U(z) \geq M \omega (V_M \cap F, z).$$

Since $V'(z) = 0$, for any $\epsilon > 0$ there exists a number $n_\epsilon (z)$ such that $F_n \cap V_M U(z) < \epsilon$ for $n > n_\epsilon$ and

$$M \omega (V_M \cap F_n, z) + c_{F_n \cap V_M} U(z) + \epsilon \geq U(z) \geq M \omega (V_M \cap F_n, z).$$

Let $M \to \infty$ and $\epsilon \to 0$. Then

$$M \omega (V_M \cap F_n, z) = U(z)$$

for any $n$ and $\lim\limits_{n} \lim\limits_{M=\infty} M \omega (V_M \cap F_n, z) = U(z).$ 

(26)

Let $p \in B_0$. Then $N(z, p)$ is representable by a canonical distribution $\mu$ such as $N(z, p) = \int_{\overline{B_0}} N(z, q) d\mu(q)$. Let $\mu'$ be the restriction of $\mu$ on $F$. Then

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14) See (5).
15) See (5).
16) See (5).
17) See (5).
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Let 

\[ N(z, p) = U(z) + \sum_{m=1}^{\infty} V_m(z) \]

where \( V_m(z) \) is the potential of the restriction of \( \mu \) on \( CF_{m+1} - CF_m \). Assume \( \mu V_m(z) > 0 \). Then since \( \operatorname{Cap}(F) = 0 \), \( \mu \left( \int \frac{\partial}{\partial n} N(z, q) d\mu'(q) \right) = U(z) \), hence \( \delta(p) = \int_{\partial R_0} \frac{\partial}{\partial n} N(z, p) d\mu' \).

Put \( V_M = E[z \in R : N(z, p) > M] \). Then

\[ 2\pi \delta(p) \geq \lim_{n \to \infty} \frac{1}{2\pi} \int_{\partial R_n} \frac{\partial}{\partial n} \mu N(z, p) d\mu' \leq 0 \text{ as } n \to \infty \]

where \( S_{l,n} = E[z \in \overline{R} : \delta(z, S_l) \leq \frac{1}{n}] \) is closed and of capacity zero. i.e. \( D(\omega(S_{l,n}, z)) \downarrow 0 \) as \( n \to \infty \), where \( \omega(B, z) = \lim_{n} \omega_n(z) \) is a harmonic function in \( R_n - R_0 \) such that \( \omega_n(z) = 0 \) on \( \partial R_0 \) and \( \omega_n(z) = 1 \) on \( \partial R_n \). Then \( \omega(B, z) = 1 \) on \( B \) except a set of capacity zero. Let \( G(z, p_0) : p_0 \in R_0 \) be the Green’s function of \( R \). Then \( G(z, p_0) > 0 \) if and only if \( \omega(B, z) < 1 \). Now \( \omega(B, z) \) is well defined on \( \overline{R} - R_0 \) and lower semicontinuous. Hence \( S_l = E[z \in B : \omega(B, z) \leq 1 - \frac{1}{n}] \) is closed and of capacity zero. i.e. \( D(\omega(S_{l,n}, z)) \downarrow 0 \) as \( n \to \infty \), where

\[ \omega(B, z) = \lim_{n} \omega_n(z) \]

is a harmonic function in \( R_n - R_0 \) such that \( \omega_n(z) = 0 \) on \( \partial R_0 \) and \( \omega_n(z) = 1 \) on \( \partial R_n \). Then \( \omega(B, z) = 1 \) on \( B \) except a set of capacity zero. Let \( G(z, p_0) : p_0 \in R_0 \) be the Green’s function of \( R \). Then \( G(z, p_0) > 0 \) if and only if \( \omega(B, z) < 1 \). Now \( \omega(B, z) \) is well defined on \( \overline{R} - R_0 \) and lower semicontinuous. Hence \( S_l = E[z \in B : \omega(B, z) \leq 1 - \frac{1}{n}] \) is closed and of capacity zero. i.e. \( D(\omega(S_{l,n}, z)) \downarrow 0 \) as \( n \to \infty \), where

\[ S_{l,n} = E[z \in \overline{R} : \delta(z, S_l) \leq \frac{1}{n}] \] and \( E[z \in B : \omega(B, z) < 1] \) is an \( F \) set of capacity zero. 18) See (5).
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zero. Let \( p \in S_l \cap B_1 \). Then by Lemma 3. (1) \( \delta(p) \) to \( S_l = 1 \). Let \( p \in S_l \cap B_0 \). Then \( \delta(F) \) to \( S_l = 1 \).

Assume \( \mu(q) \) has mass \( > 1 - \frac{1}{2l-1} \) on \( B_1 - S_{ll} \). Then \( \omega(B, p) < 1 - \frac{1}{l} \). This is a contradiction. Hence \( \mu(q) \) has its mass \( \geq \frac{1}{2l-1} \) on \( S_{2l} \) and by Lemma 4. 3) \( \delta(p) \) to \( S_{2l} \geq \frac{1}{2l-1} \).

Let \( R = E[z \in R^*: G^*(z, p_0) > \delta] \). \( G(z, p_0) \) of \( R \) is always defined on \( \overline{R} = R + B \) and continuous in \( \overline{R} \). Put \( F_i = E[z \in \overline{R}: (Gz, p_0) \geq \frac{1}{l}] \) and \( S_i = B \cap F_i \).

Let \( \{R_n\} \) be an exhaustion of \( R \) with compact relative boundary \( \partial R_n \). Then \( D(\omega(S_{i,n}, z)) \downarrow 0 \) as \( n \to \infty \) (see “energy integral and capacities”). The value at \( p \in S_i \) is given as \( \frac{1}{l} \leq G(p, p_0) = \frac{1}{2\pi} M \lim_{\partial V} \int G(z, p_0) \frac{\partial}{\partial n} G(z, p) ds \).

Let \( k = \lim_{n \to \infty} \sup_{R \in B} G(z, p_0) = \sup_{p \in B} G(z, p_0) \). Now \( V_M(p) \) clusters at \( B \) as \( M \uparrow \infty \). Hence for any given \( R_n \), there exists a number \( M_0(n) \) such that \( V_M(p) \subset R - R_n \) for \( M \geq M_0 \). Hence \( \lim_{M \to \infty} \frac{1}{2\pi} \int_{\partial V_M(p) \cap S_{i,n}} \frac{\partial}{\partial n} G(z, p) ds \geq \frac{1}{2kl} \) for any \( n \) and \( \delta(p) \) to \( S_{i,n} \). For any \( n \).

(C). Let \( B \) be the ideal boundary of \( R \) with null-boundary. Then \( V_M(p) \to B \) as \( M \to \infty \) for any \( p \in B \), whence \( \delta(p) = 1 \) to \( B \) for any point \( p \in B \).

Lemma 5. 1). Let \( F \) and \( \overline{F} \) be closed sets in \( \overline{R} - R \), such that \( \overline{F} \supset F \).

Put \( \tilde{F}_n = E[z \in \bar{R} : \delta(z, F) \leq \frac{1}{n}] \).

1. (A). If \( \lim_{n \to \infty} \inf_{p \in \partial \tilde{R}_n} \frac{\partial}{\partial n} M \omega(V_M(p) \cap \tilde{F}_n, z) ds \geq 2\pi \delta \),
   \[ 1/D(F) \geq 1/D(F) \geq \delta^2/\text{Cap}(F) \).

2. If \( \lim_{n \to \infty} \inf_{p \in \partial \tilde{R}_n} \frac{\partial}{\partial n} N(z, p) ds \geq 2\pi \delta \),
\[ 1/D(F) \geq 1/\tilde{D}(F) \geq \delta/\text{Cap}(\tilde{F}). \]

2'}. (B) and (C). If \( \lim \lim_{n} \inf_{p \in F} \int_{\partial V_{M}(p)} \frac{\partial}{\partial n} G(z,p) ds \geq 2\pi \delta \),

\[ 1/D(F) \geq 1/\tilde{D}(F) \geq \delta/\text{Cap}(\tilde{F}). \]

3). (A). Let \( M(p, n, \delta') \) be the number such that \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq 2\pi \delta' \) for \( M \geq M(p, n, \delta') : p \in F \). Then \( M(p, n, \delta') \) is upper semicontinuous, if \( \delta(p) \) to \( \tilde{F} \geq \delta \) for any \( p \in F \), then

\[ 1/\tilde{D}(F) \geq \delta'/\text{Cap}(\tilde{F}). \]

If \( \delta(p) \) to \( \tilde{F} \geq \delta \) for any \( p \in F \), \( 1/\tilde{D}(F) \geq \delta'/\text{Cap}(\tilde{F}) \).

3'). (B) and (C). If \( \lim_{\kappa} nM = \infty \liminf_{n} \int_{p \in F} \frac{\partial}{\partial n} G(z,p) ds = 2\pi \delta(p) > \delta \) for \( p \in F \),

\[ 1/D^{o}(F) \geq \delta/\text{Cap}(\tilde{F}). \]

Proof of 1). Since \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \downarrow \) as \( n \rightarrow \infty \), for any \( \epsilon > 0 \) there exists a number \( M_{0} = M(\epsilon, p) \) such that \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta - \epsilon \) for \( M \geq M_{0} \) and for \( p \in F \). Then by Lemma 3 \( 1/\tilde{D}(F) \geq 1/D^{M}(F) \geq (\delta - \epsilon)'/\text{Cap}(\tilde{F}_{n}) \). Let \( \epsilon \rightarrow 0 \) and the \( n \rightarrow \infty \). Then we have (1).

2) and 2') can be proved similarly as (1) using Lemma 3.

Proof of 3). Let \( M \geq M(p_{0}, n, \delta') \). Then \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_{0}) \cap \tilde{F}_{n}, z) ds \geq \delta' \). For any \( \epsilon > 0 \), there a compact set \( G \) in \( V_{M}(p_{0}) \cap \tilde{F}_{n} \) such that \( \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_{0}) \cap \tilde{F}_{n}, z) ds \geq \delta' - \epsilon \). Since \( N(z, p_{i}) \rightarrow N(z, p_{0}) \) in \( R \) as \( p_{i} \rightarrow p_{0} \), \( E[z \in R: N(z, p_{i}) > M] \supset G \) for \( i > i_{0} \) and \( \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p_{i}) \cap \tilde{F}_{n}, z) ds \geq \delta' \). This implies \( M(p_{i}, n, \delta') \leq M(p_{0}, n, \delta') \) by definition of \( M(p, n, \delta') \).

Also for any point \( p \in F \lim_{n} \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta(p) = \lim_{n} \frac{1}{2\pi} \int_{\partial R_{0}} \frac{\partial}{\partial n} M\omega(V_{M}(p) \cap \tilde{F}_{n}, z) ds \geq \delta(p) \).
\[
\frac{\partial}{\partial u} M_\omega (V_M(p) \cap \tilde{F}_n, z) ds,
\]
whence \( M(p, n, \delta - \epsilon) < \infty \) and \( M(p, n, \delta - \epsilon) \) attains its maximum \( M(n, \delta - \epsilon) \) on \( F \). Let \( M > M(n, \delta - \epsilon) \). Then \( 1/D(F) \geq 1/D_M(F) \geq (\delta - \epsilon)^2 / \text{Cap}(\tilde{F}_n) \). Let \( \epsilon \to 0 \) and then \( n \to \infty \). Then \( 1/D(F) \geq \delta^2 / \text{Cap}(\overline{F}_n) \).


**Theorem 5** (G. C. Evans). Let \( F_i \) \((i = 1, 2, \cdots)\) be a closed set in \( \overline{R} - R_0 \). If \( D(F_i) = 0 \), there exists a potential \( U(z) = \int_{\Sigma F_i - B_0} N(z, p) d\mu(p) \) (or \( \int_{\Sigma F_i} G(z, p) \times d\mu(p) \)) such that \( \mu = 0 \) except \( \Sigma F_i - B_0 \). Then \( 1/L(F) \geq \delta^2 / \text{Cap}(\overline{F}_n) \) for \( \epsilon \to 0 \) and then \( n \to \infty \).

3') is proved similarly as (3).


**Theorem 5** (G. C. Evans). Let \( F_i \) \((i = 1, 2, \cdots)\) be a closed set in \( \overline{R} - R_0 \). If \( D(F_i) = 0 \), there exists a potential \( U(z) = \int_{\Sigma F_i - B_0} N(z, p) d\mu(p) \) such that \( \mu = 0 \) except \( \Sigma F_i - B_0 \). Then \( 1/D(F_i) \geq \delta^2 / \text{Cap}(\overline{F}_n) \) for \( \epsilon \to 0 \) and then \( n \to \infty \).

3') is proved similarly as (3).

Proof of (A). For given \( p_1, p_2, \cdots, p_n \) on \( F_i - B_0 \), let \( V_n(z) = \frac{1}{2\pi} \sum N(z, p_l) \) and put \( R(p_1, \cdots, p_n) = \inf V_n(z) \) on \( F_i - B_0 \). Then for any given \( \epsilon > 0 \) \( \left( \epsilon < \frac{1}{2\pi n} \right) \), there exists a system \( (p_1^*, p_2^*, \cdots, p_n^*) \) such that \( V_n^*(z) = \inf V(z, p_1, \cdots, p_n^*) \geq R_n - \epsilon \) on \( F_i - B_0 \). Next \( N(p, q) = N(q, p) \), \( n+1C_2 \frac{1}{D_{n+1}(F_i)} \geq \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} N(p, q) \). Hence

\[
V_n^*(z) \geq 1/D_n(F_i), \quad \text{on } F_i - B_0. \tag{29}
\]

Let \( L > 4 \). Since \( D(F_i) = \lim_{m} D_n(F_i) = 0 \), for any \( m \) there exists a number \( l_m \) such that \( 1/D_{l_m}(F_i) = \frac{1}{2\pi l_m} \geq L^m \). Let \( V_{m}^*(z) \) (defined in (29)). Then \( V_{m}^*(z) \geq L^m \) on \( F_i - B_0 \). Put \( U_4(z) = \sum_{m=1}^{\infty} \frac{V_m(z)}{2^m} \). Then \( U_4(z) = \infty \) on \( F_i - B_0 \) and its mass \( = 0 \) except \( F_i - B_0 \). Put \( U(z) = \sum_{t=1}^{\infty} \frac{U_t(z)}{2^t} \). Then \( U(z) \) is the function required. For (B) and (C) the assertion is proved similarly.

We constructed a Riemann surface with positive boundary such that \( R \) has the following properties: \( 1) \). Let \( R_3 \) be a compact disc, then there exist


only two linearly independent positive harmonic functions vanishing on \( \partial R_0 \) in \( R - R_0 \). 2) \( B_i = B_0 = p_1 + p_2, \ B_0 \neq 0, \) i.e. \( N(z, p_i) = \alpha_i \omega(p_i, z) \) and \( \sup N(z, p_i) < \infty : \ i = 1, 2, \) and any positive harmonic function is a linear form of \( \omega(p_i, z) \) and \( \omega(p_2, z) \).

Let \( \rho \in B_0 \). Then \( \operatorname{Cap}(\rho) = 0, \) but there exists no harmonic function \( U(z) \) in \( R - R_0 \) such that \( \lim_{z \to \rho} U(z) = \infty \) and the Evans's theorem does not hold. Also in this surface, there exists no superharmonic function \( U(z) \) (not necessarily harmonic in \( R - R_0 \)) such that \( U(\rho) = \infty \). In fact, \( N(z, \rho) = \int N(z, \rho) d\mu(\rho) = \alpha N(z, p_1) + \beta N(z, p_2): \ \alpha + \beta = 1, \ \alpha \geq 0, \ \beta \geq 0 \) and \( U(\rho) = \alpha U(p_1) + \beta U(p_2) \). Suppose \( U(\rho) = \infty \). Then at least one of \( U(p_1) \) and \( U(p_2) \) must be infinite. Without loss of generality we can suppose \( U(p_1) = \infty \). By the lower semicontinuity of \( U(z) \), for any given \( M < \infty \), there exists a number \( n(M) \) such that \( U(z) \geq M \) on \( v_n(p) \). \( U(z) \geq v_n(p) U(z) \geq M \omega(v_n(p), z) \). Let \( n \to \infty \) and then \( M \to \infty \). Then \( U(z) \geq M, U(z) \geq \lim_{M=\infty} M \omega(p_1, z) = \infty \). Hence \( U(z) = \infty \). Thus Evans's theorem does not hold for superharmonic function. We shall prove the following

**Theorem 6.** Let \( F \) be a closed set of capacity zero. Then there exists a positive superharmonic function in \( R - R_0 \) such that \( U(z) = 0 \) on \( \partial R_0 \) and \( U(z) = \infty \) on \( F - B_0 \). Clearly this theorem is valid for an \( F \), set of capacity zero.

Let \( F_n = E \left[ z \in \bar{R} : \delta(z, F) \leq \frac{1}{n} \right] \) and \( \omega(F_n, z) \) be CP. of \( F \). Then \( \omega(F_n, z) = 1 \) in \( F_n \cap R \) and by \( \operatorname{Cap}(F_n) \downarrow 0, \) \( \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F, z) ds \downarrow 0 \) as \( n \to \infty \). Let \( L > 4 \) and let \( n'(L) \) be the number such that \( \frac{1}{2\pi} \int_{\partial R_0} \frac{\partial}{\partial n} \omega(F_n', z) ds \leq \frac{1}{L^n} \). Put \( U(z) = \sum_{n'} \omega(F_n', z) \). Then \( U(z) \) is the function required. Because mass of \( U(z) = 1 \) and \( U(z) < \infty \). Let \( \rho \in F - B_0 \). Then \( U(\rho) = \frac{1}{2\pi} \lim \int_{\partial R_0} U(z) \frac{\partial}{\partial n} N(z, \rho) ds \) and

\[
\lim_{M \to \infty} \int_{F_n \cap \partial V_M(\rho)} \frac{\partial}{\partial n} N(z, \rho) ds = 2\pi \text{ for any } F_n. \]  

Since \( U(z) \geq n \) in \( F_n' \cap R, \) \( U(\rho) = \infty. \) By the lower semicontinuity of \( U(z) \), \( U(z) \to \infty \) as \( z \to \rho \in F - B_0. \)

**Theorem 7.** (A). Let \( F \) and \( \bar{F} \) be closed set in \( \bar{R} - R_0 \) such that \( \bar{F} \supset F \) and \( \operatorname{Cap}(\bar{F}) = 0. \) If \( \delta(\rho) \) to \( \bar{F} \), \( \delta_0 > 0 \) for any \( \rho \in F. \) Then there exists a potential \( U(z) = \int_{F - B_0} N(z, p) d\mu(p) \) such that \( U(z) = \infty \) on \( F - B_0, \) \( D(\min(M, U(z))) = 2\pi M \) and by the lower semicontinuity of \( U(z) \), \( U(z) \to \infty \) as \( z \to \rho \in F - B_0. \)
If $\delta(p)$ to $F>\delta_{0}>0$, the above potential $U(z)=\infty$ on $F$.

(B) and (C). Let $F$ and $\bar{F}$ be closed sets in $\mathbb{R}$ such that $\bar{F} \supset F$ and $\text{Cap}(\bar{F})=0$. If $\delta(p)$ to $F \geq \delta_{0}>0$, then there exists a potential $U(z)=\int_{F} G(z, p) d\mu(p)$ such that $U(z)=\infty$ on $F$ and $D(\min(M, U(z)))=2\pi M$ and by the lower semicontinuity $U(z) \to \infty$ as $z \to p \in F$.

Proof. (A). By Lemma 5 $1/D(F) \geq \delta_{0}/\text{Cap}(\bar{F})=\infty$. Hence by Evans's theorem there exists a potential $U(z)=\int_{F-B_{0}} N(z, p) d\mu(p)$ such that $U(z)=\infty$ on $F-B_{0}$.

Put $V_{M}=E(z \in R: U(z)>M)$. Then $V_{M} \supset F-B_{0}$ and $\sup \cap F_{n} U(z)+V_{M} \cap F_{n} U(z) \geq F_{n} U(z)=U(z)$. Put $V_{M}=E(z \in R: U(z)>M)$. Then $V_{M} \supset F-B_{0}$.

Let $S$ be the irregular set of Green's function. Then there exists a potential $U(z)$ such that $U(z)=\int_{S-B_{0}} N(z, p) d\mu(p)$, $U(z)=\infty$ on $S$ and $D(\min(M, U(z)))=2\pi M$.

(B) (Theorem of M. Nakai). Let $R^{*}$ be a Riemann surface with positive boundary. Then there exists a generalized Green's function $U(z)$ in $R^{*}$ such $U(z) \to \infty$ as $z \to B$ in $R_{\delta}=E[z \in R^{*}: G^{*}(z, p)>\delta]$ for any $\delta>0$.

Proof. Put $S=E[z \in B: G(z, p)>0]$. Then $S=E[z \in B: \omega(B, z)<1]$. Let $S_{t}=E[z \in B: \omega(B, z) \leq 1-\frac{1}{t}]$. Then $S_{t}$ is closed and $\text{Cap}(S_{t})=0$. Now $\delta(p)$ to $S_{\delta} \supseteq \frac{1}{2l}$ for any $p \in S_{t}$. By Theorem 7, there exists a potential $U_{t}(z)=\int_{S_{t}-B_{0}} N(z, p) d\mu(p)$, $U_{t}(z)=1$ such that $U(z)=\infty$ on $S_{t}-B_{0}$ and $D(\min(M, U_{t}(z)))=2\pi M$. Let $U(z) = \sum_{l_{0}+1}^{\infty} \frac{U_{l}(z)}{2^{l_{0}+l}}$ where $l_{0}$ is a number such that $S_{t} \cap B_{l_{0}} \neq 0$. Then $U(z)=\infty$ on $S-B_{0}$. Let $p \in S_{t} \cap B_{0}$. Then $N(z, p)=\int_{S_{t}} N(z, q) d\mu(q)$.
Let \( \mu' \) be the restriction of \( \mu \) on \( S_{2l} \). Then \( \int d^r \mu(q) = \frac{1}{2l} \), whence \( U(p) \geq \frac{1}{2l} \), and \( U(z) = \infty \) on \( \sum S_{2l} = S \). Put \( U'_m(z) = \sum_{l=1}^{m} \frac{U_l(z)}{2^l} \).

Then since \( \text{Cap}(\sum S_l) = 0 \), \( \sum S_{2l} \subseteq S \), and \( \min(M, U'_m(z)) = M \omega(V_M, z) \), where \( V_M = E[\exists z \in R : U'_m(z) > M] \). Whence \( D(\min(M, U'_m(z))) = 2\pi M \left( \sum_{l=l+1}^{m} \frac{1}{2^l} \right) \) and \( D(\min(M, U'(z))) \geq 2\pi M \). On the other hand, by \( U'_m(z) \rightarrow U(z) \) \( D(\min(M, U(z))) \leq 2\pi M \).

(B). Let \( R = E[\exists z \in R^* : G(z, p) > \delta] \) and let \( S = E[\exists z \in R : G(z, p) > \delta] = E[z \in R^* : G(z, p) > 2\delta] \). Let \( \{R_n\} \) be an exhaustion of \( R \) with compact relative boundary \( \partial R_n \). Then \( \text{Cap}(S_{\frac{\delta}{2}, n}) = 0 \) as \( n \rightarrow \infty \), where \( S_{\frac{\delta}{2}, n} = S_{\frac{\delta}{2}} \cap (R - R_n) \). Now \( \delta(p) \) to \( S_{\frac{\delta}{2}, n} \geq \frac{\delta}{2k} \) for any \( p \in S_s = \cap S_{\frac{\delta}{2}, n} \) and for any \( n \), where \( k = \lim_{n \rightarrow \infty} \sup_{z \in R - R_n} G(z, p) \). Whence by Lemma 5. (2) \( D(S_s) = 0 \). Hence there exists a potential \( U_s(z) = \int G(z, p) d\mu(p) \), \( \int d\mu(p) = 1 \) such that \( U(z) = \infty \) on \( S_s \). Now clearly \( U_s(z) \) is a G.G. in \( R \). Next let \( U_M(z) \) be the least positive harmonic function in \( R^* \) such that \( U_M(z) \geq M \) on \( E[z \in R : U_s(z) \geq M] \). Then by Theorem 1. (2) \( U_M(z) = \lim_{M \rightarrow \infty} U_s(z) \geq U(z) \) is a G.G. in \( R^* \) such that \( D(\min(M, U(z))) \leq 2\pi M \). Now clearly \( U_M(z) \rightarrow \infty \) as \( z \rightarrow B \) in \( R = E[z \in R^* : G(z, p) > 2\delta] \). Put \( U(z) = \sum \frac{U_s(z)}{2^n} \).

Then \( U(z) \) is a G.G. in \( R^* \) by Theorem 1. (6) and \( D(\min(M, U(z))) \leq 2\pi M \) and \( U(z) \rightarrow \infty \) as \( z \rightarrow B \) in \( E[z \in R^* : G(z, p) > \delta] \) for any \( \delta > 0 \). Thus \( U(z) \) is the function required.

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