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EXAMPLES OF NON MINIMAL POINTS ON RIEMANN SURFACES OF PLANER CHARACTER.

by

Zenjiro Kuramoto

The Martin's topologies on Riemann surfaces have been discussed by many authors and some examples of boundary points have been given. Prof. M. Brelot\(^1\) gave a domain \(D\) in the \(z\)-plane such that there exist two sequences \(q_{i,n}(n=1,2, \ldots)(i=1,2)\) in \(D\) tending to \(q_{i}\) as \(n\to\infty\), \(q_{2}\neq q_{1}\) which determine the same \(K\)-Martin's boundary point to show that the \(K\)-Martin's topology is not necessarily finer than the euclidean topology. Also we constructed examples in the \(z\)-plane to show that \(N\)-Martin's topology is neither finer than the \(K\)-Martin's topology and \(K\)-Martin's topology is nor finer than the \(N\)-Martin's topology\(^2\). As for non minimal points, R. S. Martin presented an example of \(K\)-non minimal point in 3-dim. euclidean space\(^3\) and we gave a Riemann surface of infinite genus contained in the class\(H.2.P.\)\(^4\) in which there exists atleast one \(K\)- and \(N\)-non minimal point\(^5\). But the examples of non minimal point in a domain of planer character have not been given. Mr. Ikegami proposed the following problem:

\textit{Does there exist a non minimal point on Riemann surfaces of planer character?}

The purpose of the present paper is to discuss the relation between the classes of positive harmonic functions of some classes in \(R\) and \(R'\) when \(R\) varies to \(R'\) and also is to give examples of non minimal points of Riemann surfaces of planer character.

\textbf{Lemma 1.} (An estimation of the harmonic measure of an arc on}


Examples of Non Minimal Points on Riemann Surfaces of Planer Character

Let $C$ be a unit circle, $C: |z|<1$ and let $S'$ be a straight in $C$ with endpoints, $A'$ and $B'$ such that $S'$: \( \text{Im } z=0, a' \leq \text{Re } z \leq b' \), where $a' \leq -\frac{1}{3}$ and $b' \geq \frac{1}{3}$. Let $\Gamma$ be a circle in $C$: \( |z|=\frac{1}{12} \). Let $T$ be a straight on $S'$ such that $T: \text{Im } z=0, -\delta \leq \text{Re } z \leq \delta$. Suppose $\delta \leq \frac{\pi}{18}$. Then $w(T, z)$, \( H.M. \) (harmonic measure) of $T$ with respect to $C-S'$ satisfies

$$w(T, z) \leq \frac{0.664}{\pi} |\sin \theta|, \quad z = e^{i\theta}.$$

**Proof.** Let $S$ be a straight on the real axis, $S: \text{Im } z=0, -\frac{1}{3} \leq \text{Re } z \leq \frac{1}{3}$. Then $S' \supset S$. Let $\Omega_z$ be the complementary set of $S$ in the $z$-plane. By brief consideration, when $\delta$ is sufficiently small, we see $w^*(T, z)$ has almost same value as $w(T, z)$ on $\Gamma$, where $w^*(T, z)$ is $H.M.$ of $T$ with respect to $\Omega_z$. Clearly $w(T, z) \leq w^*(T, z)$. Map $\Omega_z$ by

$$w=\frac{z+\frac{1}{3}}{z-\frac{1}{3}}$$
on $\Omega_z$ onto $\Omega_w$, where $\Omega_w$ is the complementary domain of the straight: $\text{Im } w=0, 0 \leq \text{Re } w \leq \infty$. Then $\Gamma \to$ a circle: $\left| w-\frac{6}{5} \right| < \frac{7}{15}$.

Map $\Omega_w$ by $\zeta=\sqrt{w}$ onto $\Omega_\zeta$: $\text{Im } \zeta>0$. Also map $\Omega_z$ by $\xi=\frac{\zeta-i}{\zeta+i}$ onto $|\xi|<1$. Then $T$ is mapped onto $T_1+T_2$ and $\Gamma$ is mapped onto $\Gamma_1+\Gamma_2$ respectively, where $T_1: e^{i(-\frac{h}{2})}$ for $-\frac{3\delta}{2} \leq \varepsilon \leq \frac{3\delta}{2}$, $T_2: e^{i(\frac{h}{2}+i)}$ for $-\frac{3\delta}{2} \leq \varepsilon \leq \frac{3\delta}{2}$, $\Gamma_1$ and $\Gamma_2$ are curves in the lower and upper semicircles respectively. Let $w^*(T_\xi, \xi)$ be $H.M.$ of $T_\xi$. Then $w(T, z) \leq \sum_{\xi} w^*(T_\xi, \xi)$. We consider $w^*(T_\xi, \xi)$ on $\Gamma_1$. Then \( w(T, \xi) \leq \frac{\frac{\Omega-3-\delta}{\pi}}{\frac{2}{3-\delta}} \),

where
\[ \Theta = \arg \frac{e^{i\left(-\frac{\pi}{2} + \frac{3\delta}{2}\right)} - \xi}{e^{i\left(-\frac{\pi}{2} - \frac{3\delta}{2}\right)} - \xi}. \]  

(1)

We denote by \( p_t \), the point \( \xi \) on \( \Gamma_t \). We shall express \( p_t \) by \( z \). Let \( p_z = \frac{e^{i\theta}}{12} \) on \( \Gamma \) in the upper half plane. Then \( p_w \), the image of \( p_z \) is given by Fig. 3 as

\[ p_w = re^{i\varphi}, \]

where \( r = \sqrt{\frac{17 + 8 \cos \theta}{17 - 8 \cos \theta}} \) and

\[ \varphi = \cos^{-1} \frac{15}{\sqrt{17 - 8 \cos \theta}}. \]

Let \( p_t \) be the image of \( p_w \). Then \( p_t = \rho e^{i\phi} \),

where \( \rho = r^{\frac{1}{2}}, \phi = \frac{\varphi}{2} \).

Let \( p_t \) be the image of \( p_t \). Then \( p_t = Re^{i\theta} \),

(2)

where \( R = \sqrt{\frac{1 + \rho^2 - 2\rho \sin \phi}{1 + \rho^2 + 2\rho \sin \phi}} \), \( \sin \phi = \frac{2\rho \cos \varphi}{\sqrt{1 + \rho^2 + 2\rho \cos 2\varphi}} \geq 0 \),

(3)

\[ \cos \Phi = \frac{\rho^2 - 1}{\sqrt{1 + \rho^2 + 2\rho^2 \cos 2\varphi}}. \]

By the shape of \( \Gamma_t \), we see that \( R \) is minimal, when \( \Phi = -\frac{\pi}{2} \), i.e. \( \theta = \frac{\pi}{2} \), \( \rho = 1 \), \( \cos \varphi = \frac{15}{17} \), \( \sin \phi = \frac{1}{\sqrt{17}} \) and \( R = \frac{\sqrt{17} - 1}{4} \).

(4)

Now by (1) \( \Theta = \tan^{-1} \frac{2R \sin \Phi \sin \frac{3\delta}{2} + \sin 3\delta}{R^2 + 2R \sin \cos \frac{3\delta}{2} + \cos 3\delta} \). Put \( \Psi = \Theta - \frac{3\delta}{2} \). Then

\[ \tan \Psi = \frac{(1 - R^2) \sin \frac{3\delta}{2}}{(1 + R^2) \cos \frac{3\delta}{2} + 2R \sin \Phi + \cos \frac{3\delta}{2}} \leq \frac{(1 - R^2) \sin \frac{3\delta}{2}}{(2 + R^2) \cos \frac{3\delta}{2}}. \]

We have by (2) and (3) \( (1 - R^2) = \frac{2 \sin \theta}{(17^2 - 8^2 \cos^2 \theta)^{\frac{1}{4}} ((17^2 - 8^2 \cos^2 \theta) + 15)^{\frac{1}{4}}} \leq \frac{\sqrt{2}}{5} \sin \theta \). Hence

by \( \delta \leq \frac{\pi}{18} \) and by (4)
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\[ \Psi \leq \tan \Psi \leq \frac{2 \sin \theta \sin \frac{3}{2} \delta}{15(2 + R^2)} \leq \frac{8 \sqrt{2} \sin \theta \sin \frac{3}{2} \delta}{15(25 - \sqrt{17}) \cos \frac{3}{2} \delta} \]

\[ \leq \frac{12 \sqrt{2} \delta \sin \theta}{15(25 - \sqrt{17}) \cos \frac{\pi}{12}} < 0.28 \delta \sin \theta. \]

Hence \( w^*(T, \xi) \leq \frac{0.332}{\pi} \delta \sin \theta, \ z = \frac{e^{i\theta}}{12} \) on \( \Gamma_1 \). Next clearly

\[ w(T, z) \leq w^*(T, \xi) \] on \( \Gamma_1 \), whence

\[ w(T, z) \leq \frac{0.664 \delta \sin \theta}{\pi}, \quad z = \frac{e^{i\theta}}{12}, \quad \pi \geq \theta \geq 0. \quad (6) \]

Similar result is obtained for \( \pi \leq \theta \leq 2\pi \). Thus we have Lemma 1.

**Lemma 2.** (An estimation of harmonic measure of an arc). Let \( \Omega \) be the semicircle: \( |z| \leq 1, \ \text{Im} \ z \geq 0 \). Let \( S \) be an arc on \( |z| = 1 \) with endpoints \( A \) and \( B, A = 1 - \sqrt{2} + 2\sqrt{2} \sqrt{2 - 1} i \) and \( B = \sqrt{2} - 1 + 2\sqrt{2} \sqrt{2 - 1} i \). Let \( \Gamma \) be a semicircle: \( z = \frac{e^{i\theta}}{2} \), \( 0 \leq \theta \leq \pi \). Let \( w(S, z) \) be H.M. of \( S \) with respect to \( \Omega \). Then

\[ w(S, z) \geq \frac{2}{3\pi} \min \left( \frac{\pi}{12}, \frac{24}{25} \sin \theta \right) \] on \( \Gamma \).

Map \( \Omega \) by \( w = \frac{z + 1}{1 - z} \) onto \( \Omega_w \), map \( \Omega_w \) by \( \zeta = w^2 \) onto \( \Omega \); and map \( \Omega \) by \( z = \frac{\zeta - i}{\zeta + i} \) onto \( |\xi| < 1 \). Then A and B are mapped onto \( e^{\frac{3}{\pi}i} \) and \( e^{\frac{\pi}{2}i} \) of \( \xi \)-plane respectively. Let \( p = \frac{e^{i\theta}}{2} : \pi \geq \theta \geq 0 \) and let \( p_w, p_\zeta \) and \( p_\xi \) be the images of \( p \) in \( w, \zeta \) and \( \xi \)-plane. Then we have \( p_w = r e^{\phi} \), where

\[ \cos \phi = \frac{3}{\sqrt{25 - 16 \cos^2 \theta}}, \quad r = \sqrt{\frac{5 + 4 \cos \theta}{5 - 4 \cos \theta}}, \]

\[ p_\zeta = \rho e^{i\phi}, \]

where \( \rho = r^2 \) and \( \phi = 2\phi \), \( \sin \phi = \frac{24 \sin \theta}{25 - 16 \cos^2 \theta} \leq \frac{24 \sin \theta}{25} \) and \( \frac{1}{9} \leq \rho \leq 9. \quad (7) \)

\[ p_\xi = \rho e^{i\phi}, \]

where \( R = \sqrt{\frac{1 + \rho^2 - 2\rho \sin \phi}{1 + \rho^2 + 2\rho \sin \phi}}, \quad \sin \Phi = \frac{2 \cos \phi}{\sqrt{1 + \rho^2 + 2\rho \cos \phi}} \geq 0 \quad (8) \)
and $R$ is minimal, when $\theta = 0$, i.e. $r = 1$, $\rho = 1$, $\sin \phi = \frac{24}{25}$ and $R = \frac{1}{7}$.

Put $\Theta = \arctan \frac{e^{\frac{z}{4}} - p}{1 - e^{\frac{z}{4}} - p}$. Then by (8) $\Theta = \left(1 + \frac{1 - R^{2}}{R^{2} - \sqrt{2} R \sin \Phi \cdot \frac{1 - R^{2}}{2 \phi}}\right)$. On the other hand, we see easily

$$\tan\left(\frac{\pi}{4} + \frac{s}{4}\right) \leq 1 + s \quad \text{for} \quad 0 \leq s \leq \frac{\pi}{3},$$

whence

$$\Theta \geq \frac{\pi}{4} + \min\left(\frac{\pi}{12}, \frac{24}{25} \sin \theta\right).$$

By (8) and (7) $(1 - R^{2}) = \frac{4 \rho \sin \phi}{1 + \rho^{2} + 2 \rho \sin \phi} \geq \frac{4 \rho \sin \phi}{(1 + \rho)^{2}} \geq \frac{48}{25} \sin \theta$, because

$$\frac{4 \rho}{1 + \rho^{2}} \leq 1 \quad \text{for} \quad \frac{1}{9} \leq \rho \leq 9.$$ 

Also by (8) $\Theta \geq \frac{\pi}{4} + \min\left(\frac{\pi}{12}, \frac{24}{25} \sin \theta\right)$. Hence

$$w(T, z) \geq \frac{\Theta - \frac{\pi}{4}}{\frac{3}{4} \pi} = \frac{4}{3 \pi} \min\left(\frac{\pi}{12}, \frac{24}{25} \sin \theta\right), \quad z = \frac{e^{i\theta}}{2}.$$

**Lemma 3.** Let $C, C', C''$ and $C'''$ be circles, $C: |z| < 1$, $C': |z| < \frac{1}{3}$, $C'': |z| < \frac{1}{6}$, $C''': |z| < \frac{1}{12}$. Let $\Gamma_{1}$ and $\Gamma_{2}$ be straights on the real axis such that $\Gamma_{1}: a \leq Re z \leq -\delta$, $\Gamma_{2}: -\delta \leq Re z \leq b$, where $a \leq -\frac{1}{3}$, $b \geq -\frac{1}{3}$ and let $\gamma$ be a closed set in $C - C'$. Let $T: \text{Im} z = 0$, $-\delta \leq Re z \leq \delta$ and let $U(z)$ be a positive harmonic function in $C' - \Gamma_{1} - \Gamma_{2}$ vanishing on $\Gamma_{1} + \Gamma_{2}$ and let $w(T, z)$ be H.M. of $T$ with respect to $C - T$. Then there exists a constant $M$ such that

$$M \delta U(z) \geq (\sup_{e \in \partial C'''} U(z)) w(T, z) \quad \text{on} \quad \partial C''' \cup \Gamma_{1} - \Gamma_{2},$$

Let $F = F_{1} + F_{2}$ and $F_{1}: e^{i\theta}, \quad \frac{3 \pi}{4} \leq \theta \leq \frac{\pi}{4}$, $F_{2}: e^{i\theta}, \quad \frac{7 \pi}{4} \leq \theta \leq \frac{5 \pi}{4}$. Map $C' - \Gamma_{1} - \Gamma_{2}$ onto $|\xi| < 1$ so that $z = 0 \rightarrow \xi = 0$ and let $\Gamma'_{1}, F'_{1}$ and $C'''_{t}$ be the images of $\Gamma_{1} + \Gamma_{2}, F_{1} + F_{2}$ and $C'''$ respectively. Let $C'_{t}$
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be the complementary periphery of \( \Gamma_t \). Since \( \text{dist}(C\Gamma_t,F_t) > 0 \), there exists a const. \( K_1 \) such that
\[
\frac{1-r^2}{1-2r \cos(\theta-\varphi)+r^2} \geq K_1 \quad \text{for} \quad \xi \in F_t, \quad \xi = re^{i\theta},
\]
whence
\[
\min_{z \in F_1+F_2} U(z) = \min_{\xi \in F_t} U(\xi) \geq K_1 U(0).
\]
Also since \( \text{dist}(\partial C''\xi, C\Gamma_t) = 0 \), there exists a const. \( K_2 \) such that
\[
\frac{1}{1-2r \cos(\theta-\varphi)+r^2} \leq K_2 \quad \text{for} \quad \xi \in \partial C''\xi,
\]
whence
\[
\sup_{\epsilon \in \partial C''\xi} U(\xi) \leq K_2 (1-r^2) U(0).
\]
Hence putting \( M^*=\frac{K_1}{K_2} \) we have
\[
M^* \min_{z \in F_1+F_2} U(z) \geq \sup_{z \in \partial C''} U(z).
\]
Put \( A = \sup_{z \in \partial C''} U(z) \) and \( B = \min_{z \in F_1+F_2} U(z) \).
Then
\[
B \geq \frac{A}{M^*}.
\]
Clearly
\[
U(z) \geq B w(F, z) \quad \text{in} \quad C''
\]
and by Lemma 2 \( U(z) \geq BC_\sin \theta \) for \( z = \frac{e^{i\theta}}{12} \), where \( w(F,z) \) is H.M. of \( F \) with respect to \( C''-\Gamma_1-\Gamma_2-T \).

Let \( R \) be a Riemann surface with positive boundary and let \( \{R_n\} \) be its exhaustion with compact relative boundary \( \partial R_n \) (n=1,2, \ldots). Let \( G \) be a sub-domain (in this paper we suppose the relative boundary \( \partial G \) of \( G \) consists of enumerably infinite number of analytic curves clustering nowhere in \( R \)). Let P.H.(G) and P.H.(G) be the sets of positive harmonic function (is abbreviated to P.H.) in \( G \) and P.H in \( G \) vanishing on \( \partial G \). Let \( U(z) \in \text{P.H.}(G) \) and let \( U_n(z) \) be the least positive harmonic function (is abbreviated to L.P.H.) in \( R-(G\cap(R-R_n)) \) such that \( U_n(z) = U(z) \) on \( G \cap (R-R_n) \). Then \( U_n(z) \uparrow \) a limit function denoted by \( EU(z) \). Let \( V(z) \in \text{P.H.}(R) \) and let \( V_n(z) \) be the L.P.H. in \( G-(R-R_n) \) such that \( V_n(z) \geq V(z) \) on \( G \cap (R-R_n) \). Then \( V_n(z) \downarrow \) \( IV(z) \). Then we have the following

**Theorem 1.**

a). If \( EU(z) < \infty \), \( IEU(z) = U(z)^\alpha \).

b). Let \( V(z) \in \text{P.H.}(R) \). Let \( D \) be a domain in \( R \) and let \( V_D(z) \) be L.P.H. in \( R-D \) such that \( V_D(z) = V(z) \) on \( D \). Let \( U(z) \in \text{P.H.}(G) \) and \( EU(z) < \infty \).

Then
\[
\lim_{n \to \infty} \frac{\partial U(z)}{U_n(z)} = EU(z) = 0.
\]

This means $EU(z)$ tends to zero as $z \to B \cap CG$ ($B \cap CG$ means the ideal boundary determined by a domain $CG$ except a set of harmonic measure zero).

Let $V(z) \in P.H.(R)$. If $\lim_{n \to \infty} V_{\theta \cap (R - R_n)}(z) = V(z)$, we say $V(z) = 0$ a.e. on $(B \cap CG)$. Next let $\omega(z)$ be L.P.H. in $G$ such that $\omega(z) \geq V(z)$ on $\partial G$. Let $\tilde{\omega}_n(z)$ be L.P.H. in $R - (G \cap (R - R_n))$ such that $\tilde{\omega}_n(z) = \omega(z)$ on $G \cap (R - R_n)$. If $\lim \tilde{\omega}_n(z) = 0$, we say $V(z)$ is regular relative to $G$. Then we have

c). Let $V(z) \in P.H.(R)$. Suppose $V(z) = 0$ a.e. on $CG \cap B$ and $V(z)$ is regular relative to $G$. Then if $IV(z) > 0$, then $EIV(z) = V(z)$.

d). If $EU(z) < \infty$, then $EU(z)$ is regular relative to $G$.

e). Let $V(z) \in P.H.(R)$. If there exists at least one $U(z)$ in $P.H.(G)$ such that $V(z) \leq EU(z)$ and if $IV(z) > 0$, then $EIV(z) = V(z)$.

Proof of a) is given in the previous paper.

Proof of b). Let $\tilde{V}_{n, n+i}(z)$ be a P.H. in $R_n + ((R_{n+i} - R_n) \cap G)$ such that $\tilde{V}_{n, n+i}(z) = EU(z)$ on $\partial R_n + \partial G$, $\tilde{V}_{n, n+i}(z) = 0$ on $(\partial R_n - G) + (\partial G \cap (R_{n+i} - R_n))$. Let $\check{V}_{n, n+i}(z)$ be a P.H. in $R_n + ((R_{n+i} - R_n) \cap G)$ such that $\check{V}_{n, n+i}(z) = EU(z)$ on $(\partial R_n \cap CG) + (\partial G \cap (R_{n+i} - R_n))$ and $\check{V}_{n, n+i}(z) = 0$ on $\partial R_{n+i} \cap G$. Then

$$\hat{V}_{n, n+i}(z) + \check{V}_{n, n+i}(z) = EU(z).$$

where $G^\sum_{n=0}^{\infty} \Sigma_{n} (z, p)$ is the Green's function of $R'$, because $R'' - \sum_{n=0}^{\infty} \Sigma_{n} = R + \sum_{n=1}^{\infty} \Sigma_{n} = R'$.

In the following we suppose $K$-Martin's topologies are defined in $R''$ and $R'$ by using $K''(z, p)$ and $K'(z, p)$, where $K''(z, p) = \frac{G''(z, p)}{G'(z_0, p)}$, $K'(z, p) = \frac{G'(z, p)}{G'(z_0, p)}$ and $z_0$ is a fixed point in $R - \sum_{n=0}^{\infty} C_n$. Put $R' = R' + B'$ and $R'' = R'' + B''$, where $B'$ and $B''$ are ideal boundaries of $R'$ and $R''$ respectively.

Let $\{p_i\}$ be a sequence determining a point $p' \in B'$ such that $p_i \notin \sum_{n=0}^{\infty} C_n$.

Then by (10)

$$\frac{1}{2} K''(z, p_i) \leq K'(z, p_i) \leq 2K''(z, p_i) \quad \text{for} \quad z \notin \sum_{n=0}^{\infty} C_n'. \quad (11)$$

Then we can find a subsequence $\{p'_i\}$ of $\{p_i\}$ such that $\{p'_i\}$ converges to a point $p'' \notin B''$. Then

$$\frac{1}{2} K''(z, p'') \leq K'(z, p') \leq 2K''(z, p'') \quad \text{for} \quad z \notin \sum_{n=0}^{\infty} C_n', \quad (12)$$
Proof of d). Let \( U(z) \in \hat{\text{P.H}}(G) \) and \( EU(z) < \infty \). Let \( \omega_n(z) \) be a P.H. in \( G \cap R_n \) such that \( \omega_n(z) = 0 \) on \( \partial R_n \cap G \), \( \omega_n(z) = EU(z) \) on \( \partial G \cap R_n \). Then \( \omega_n(z) + U(z) \leq EU(z) \) on \( (\partial G \cap R_n) + (\partial R_n \cap G) \). Let \( n \to \infty \). Then \( \lim \omega_n(z) = \omega(z) \) is L.P.H. in \( G \) such that \( \omega(z) \geq EU(z) \) on \( \partial G \) and \( \omega(z) + U(z) \leq EU(z) \) in \( G \). Let \( \tilde{\omega}_{n,n+1}(z) \) be a P.H. in \( R_{n+1} - (G \cap (R_{n+1} - R_n)) \) such that \( \tilde{\omega}_{n,n+1}(z) = \omega(z) \) on \( \partial R_n \cap G \), \( \tilde{\omega}_{n,n+1}(z) = 0 \) on \( (\partial R_n \cap (R_{n+1} - R_n)) + \partial R_{n+1} - G \). Let \( T_{n,n+1}(z) \) be a P.H. in \( R_{n+1} - (G \cap (R_{n+1} - R_n)) \) such that \( T_{n,n+1}(z) = U(z) \) on \( \partial R_n \cap G \), \( T_{n,n+1}(z) = 0 \) on \( (\partial G \cap (R_n - R_{n+1})) + (\partial R_{n+1} - G) \). Then \( \tilde{\omega}_{n,n+1}(z) + T_{n,n+1}(z) \leq \omega(z) + U(z) \) on \( (\partial R_{n+1} \cap G) + (\partial G \cap (R_{n+1} - R_n)), \tilde{\omega}_{n,n+1}(z) + T_{n,n+1}(z) \leq EU(z) \). Let \( n \to \infty \). Then \( \tilde{\omega}_{n,n+1}(z) + EU(z) \leq EU(z) \) on \( G \cap R_n \), and \( \tilde{\omega}_{n,n+1}(z) = 0 \). Hence \( EU(z) \) is L.P.H. in \( G \).

Proof of e). By \( V(z) \leq EU(z) \), \( \lim_{n} V_{\partial G \cap (R-R_n)}(z) = 0 \) by (b). Clearly

\[
\lim_{n} V_{\partial G \cap (R-R_n)}(z) + \lim_{n} V_{\partial G \cap (R-R_n)}(z) \geq \lim_{n} V_{R-R_n}(z) = V(z) \geq \lim_{n} V_{\partial G \cap (R-R_n)}(z).
\]

Hence \( V(z) = \lim_{n} V_{\partial G \cap (R-R_n)}(z) \), i.e. \( V(z) = 0 \) a.e. on \( CG \cap B \). By \( V(z) \leq EU(z) \) and by (d) \( V(z) \) is regular relative to \( G \). Hence by (c) we have (e).

We denote by \( \hat{\text{P.H}}(R) \) the subclass of \( V(z) \) in \( \text{P.H.}(R) \) such that \( V(z) = 0 \) a.e. on \( CG \cap B \), \( V(z) \) is regular relative to \( G \) and \( IV(z) > 0 \) and let \( \hat{\text{P.H}}(G) \) be the subclass of \( U(z) \) in \( \text{P.H.}(G) \) such that \( EU(z) < \infty \). Then by a) and c) we have at once the following

Corollary. \( \hat{\text{P.H.}}(R) \) and \( \hat{\text{P.H.}}(G) \) are isomorphic with respect to \( E \) and \( I \) operations.

Let \( G \subsetneq G' \) be subdomains and let \( I' \) and \( I'' \) be sets consisting of arcs in \( \partial G \) and \( \partial G' \) respectively, where \( I'' \subset I' \). Let \( \hat{\text{P.H.'}}(G) \) be the set of P.H.'s \( U(z) \) in \( G \) such that \( U(z) = 0 \) on \( \partial G - I', \frac{\partial}{\partial n} U(z) = 0 \) on \( I' \). Let \( U_n(z) \) be L.P.H. in \( G' \) such that \( U_n(z) \geq U(z) \) on \( G' \cap (R-R_n), \frac{\partial}{\partial n} U_n(z) = 0 \) on \( I' \). Put \( \hat{E} U(z) = \lim_{n} U_n(z) \). Similarly we define \( \hat{I} V(z) \) from \( V(z) \) in \( \hat{\text{P.H.'}}(G) \). Let \( \hat{G} \) be the symmetric surface of \( G \) with respect to \( \partial G \) and identify \( I' \) and \( \hat{I} \), the symmetric image of \( I' \). Then we have a doubled surface \( G + \hat{G} \) which has its relative boundary \( (\partial G - I') + (\partial G - \hat{I}) \), where \( \hat{G} \) is the symmetric image of \( G \). Let \( \hat{z} \) be the symmetric image of \( z \) and put \( U(\hat{z}) = U(z) \) in \( \hat{G} \). Then \( \hat{U}(z) \) ( = \( U(z) \) in \( G \) and = \( U(\hat{z}) \) in \( \hat{G} \)\)) \( \in \text{P.H.}(G + \hat{G}) \). Let \( \hat{G}' \) to the symmetric image
of $G'$ with respect to $\partial G$ and identify $\Gamma'$ and $\hat{\Gamma}'$. Then we have $(G' + \hat{G}')$. Now $(G' + \hat{G}')$ is contained in $(G + \hat{G})$ by $G' \subset G$ and $\Gamma'' \subset \Gamma$, $(G' + \hat{G}')$ has relative boundary $(\partial G' - \Gamma' + \partial \hat{G}' - \hat{\Gamma}')$ in $(G + \hat{G})$. Hence Theorem 1 is valid for $\hat{E}$ and $\hat{\Gamma}$. To avoid repetition we do not state the Theorem 1 for $\hat{E}$ and $\hat{\Gamma}$ and we denote it simply by $N$-Theorem 1.

**Lemma 4.** Let $R^*$ be a Riemann surface with positive boundary. Let $R$ be a subsurface in $R^*$ and let $\sum_{n=1}^{n_0} \gamma_n$ be a compact set on $\partial R$, which is compact in $R^*$. Put $R' = R + \sum_{n=1}^{n_0} \gamma_n$. Then $\mathcal{P}.H.(R)$ and $\mathcal{P}.H.(R')$ are isomorphic.

In fact, let $U(z) \in \mathcal{P}.H.(R)$. Since $\sum_{n=1}^{n_0} \gamma_n$ is compact, there exists a number $n'$ such that $\sum_{n=1}^{n_0} \gamma_n \subset R^*_n$, where $R^*_n$ is an exhaustion of $R^*$ with compact relative boundary $\partial R^*_n$. Put $M = \sup_{z \in \sum_{n=1}^{n_0} \gamma_n} U(z)$. Then $M < \infty$. Let $\omega(\sum_{n=1}^{n_0} \gamma_n, z, R)$ be H.M. of $\sum_{n=1}^{n_0} \gamma_n$. Then $U(z) + M \omega(\sum_{n=1}^{n_0} \gamma_n, z, R)$ is superharmonic in $R'$ and $\geq U(z)$, whence $EU(z) \leq U(z) + M \omega(\sum_{n=1}^{n_0} \gamma_n, z, R) < \infty$, where $E$ is from $R$ to $R'$. Let $V(z) \in \mathcal{P}.H.(R')$. Let $V'(z) = \mathcal{P}.H. in R$ such that $V'(z) \geq V(z)$. Then $V'(z) - V(z) > 0$, because if $V'(z) = V(z)$, then $V'(z)$ is harmonic in $R'$ and by the maximum principle $V'(z) = V(z) = 0$. Whence $IV(z) \geq V(z) - V'(z) > 0$, where $I$ is from $R'$ to $R$. Next we see easily that any $V(z)$ is regular relative to $R$ and $V(z) = 0$ a.e. on $(CR \cap B)$ by $R \cap (R^* - R^*_n) = R' \cap (R^* - R^*_n)$ for $n > n'$, where $R^*_n \supset \sum_{n=1}^{n_0} \gamma_n$.

**Theorem 2.** Let $R^*$ be a Riemann surface with positive boundary. Let $R$ be a sub Riemann surface of $R^*$ with relative boundary $\partial R$ in $R^*$. Let $C_n \supset C'_n \supset C''_n$ be discs in $R^*$ such that $\partial C_n + \partial C'_n$ and $\partial C''_n$ may intersect $\partial R$ and $C_{n,1}$ are disjoint each other. Let $\gamma_n$ be a continuum in $\partial R \cap C''_n$. Suppose there exists a const. $M_n$ such that

$$
(\sup_{z \in \partial C''_n} U(z)) \omega(\gamma_n, z) \leq M_n U(z) \text{ on } \partial C''_n
$$

for any P.H. $U(z)$ in $C_n \cap R$, $V^\gamma_n$ where $\omega(\gamma_n, z)$ is H.M. of $\gamma_n$ with respect to $R'' = R + \sum_{n=1}^{\infty} \gamma_n$.

If $\sum_{n=1}^{\infty} M_n < \infty$,

then $\mathcal{P}.H.(R)$ and $\mathcal{P}.H.(R'')$ are isomorphic, where $R''$ is the surface obtained from $R$ by
adding \( \sum \gamma_{n} \) to \( R \), i.e. \( R'' = R = \sum \gamma_{n} \) and \( \partial R'' = \partial R - \sum \gamma_{n} \).

**Proof.** Let \( n \) be a number such that \( \prod_{n} (1-2M_{n}) \geq \frac{1}{4} \). Let \( R' = R + \sum \gamma_{n} \) and \( R'' = R + \sum \gamma_{n} \). Then \( R \subset R' \subset R'' \). Since \( R' - R = \sum \gamma_{n} \) is compact, \( P.H.(R) \) and \( P.H.(R') \) are isomorphic by Lemma 4. We shall prove \( P.H.(R') \) and \( P.H.(R'') \) are isomorphic. Let \( G''(z, p) \) be the Green's function of \( R'' \). Suppose \( p \notin \sum C_{n} \). Let \( R' = R + \sum_{n=1}^{n_{0}} \gamma_{n} \) and \( R'' = R + \sum \gamma_{n} \). Then \( R \subset R' \subset R'' \).

Since \( R' - R = \sum_{n=1}^{n_{0}} \gamma_{n} \) is compact, \( P.H.(R) \) and \( P.H.(R') \) are isomorphic. We shall prove \( P.H.(R') \) and \( P.H.(R'') \) are isomorphic. Let \( G''(z, p) \) be the Green's function of \( R'' \). Consider \( G''(z, p) - 2M \omega(\gamma_{n}, z) \) in \( C' \). Since \( G''(z, p) \) is a P.H. in \( C', \) vanishing on \( \partial R' - \gamma_{n} \), \( G''(z, p) - 2M \omega(\gamma_{n}, z) \) is a P.H. except endpoints of \( \gamma_{n} \) and \( p \), \( G''(z, p) \) is subharmonic except \( p \) and \( G''(z, p) = 0 \) on \( \gamma_{n} \). Let \( G'(z, p) \) be the Green's function of \( R'' - \gamma_{n} \). Then \( G'(z, p) \) is L.P.H. except \( p \) where \( G'(z, p) \) has a logarithmic singularity. Hence

\[
G''(z, p) \geq G'(z, p) \geq G''(z, p) - 2M \omega(\gamma_{n}, z) .
\]

Also \( G''(z, p) \geq \frac{M_{n}}{M} \omega(\gamma_{n}, z) \) on \( \partial C' \). By the maximum principle \( 2M \omega(\gamma_{n}, z) \leq 2M_{n} G''(z, p) \) for \( z \in C' \). Hence

\[
G''(z, p) \geq G'(z, p) \geq G''(z, p) (1 - 2M_{n}) \quad \text{for} \quad z \notin C'.
\]

Replace \( G''(z, p) \) and \( G'(z, p) \) by \( G'(z, p) \) and \( G'(z, p) + G'(z, p) \), where \( G'(z, p) \) is the Green's function of \( R'' - n_{n+1} \). Then as above

\[
G''(z, p) \geq G'(z, p) \geq G'(z, p) + G'(z, p) (1 - 2M_{n+1}) \geq G''(z, p) (1 - 2M_{n})(1 - 2M_{n+1}) \quad \text{for} \quad z \notin C' + C'' .
\]

In this way we have

\[
G''(z, p) \geq G'(z, p) \geq G'(z, p) \prod_{n_{0}} (1 - 2M_{n}) \geq \frac{G''(z, p)}{4}
\]

for \( z \notin \sum_{n_{0}} C' \),
where \( G_{\Sigma_{n}}^{\gamma_{n}}(z, p) \) is the Green's function of \( R' \), because \( R'' - \sum_{n}^{\infty} \gamma_{n} = R' + \sum_{n}^{\infty} \gamma_{n} \).

In the following we suppose K-Martin's topologies are defined in \( R'' \) and \( R' \) by \( K''(z, p) \) and \( K'(z, p) \), where \( K''(z, p) = \frac{G''(z, p)}{G''(z_{0}, p)} \), \( K'(z, p) = \frac{G'(z, p)}{G'(z_{0}, p)} \) and \( z_{0} \) is a fixed point in \( R - \sum C_{n} \). Put \( R' = R' + B' \) and \( R'' = R'' + B'' \), where \( B' \) and \( B'' \) are ideal boundaries of \( R' \) and \( R'' \) respectively.

Let \( \{ p_{i} \} \) be a sequence determining a point \( p' \in B' \) such that \( p_{i} \notin \sum C_{n} \). Then by (10)
\[
\frac{1}{2} K''(z, p_{i}) \leq K'(z, p_{i}) \leq 2K''(z, p_{i}) \quad \text{for} \quad z \notin \sum C_{n}.
\] (11)

Then we can find a subsequence \( \{ p'_{i} \} \) of \( \{ p_{i} \} \) such that \( \{ p'_{i} \} \) converges to a point \( p'' \in B'' \). Then
\[
\frac{1}{2} K''(z, p'') \leq K'(z, p') \leq 2K''(z, p'') \quad \text{for} \quad z \notin \sum C_{n},
\] (12)
i.e. there exists at least one point \( p'' \in B'' \) corresponding to any \( p' \in B' \).

Suppose \( p_{i} \in C_{m} \). Then \( K'_{C_{m}}(z, p_{i}) = K'(z, p_{i}) \) for \( z \notin C_{m} \) and \( K'_{C_{m}}(z, p) \) is representable by a positive mass distribution \( \mu_{p_{i}}(q) \) on \( \partial C_{m} \) such that \( \int_{\partial C_{m}} d\mu_{p_{i}}(q) = 1 \) by \( K'(z_{0}, p_{i}) = 1 \) and \( K'_{C_{m}}(z, p_{i}) = \int_{\partial C_{m}} K(z, p) d\mu_{p_{i}}(q) \) for \( z \notin C_{m} \), where \( K'_{C_{m}}(z, p) \) is L.P.H. in \( R' \) larger than \( K'(z_{0}, p) \) on \( C_{m} \). Now since \( q \notin C_{m} \), by (10) \( \frac{1}{2} K''(z, q) \leq K'(z, q) \leq 2K''(z, q) \), whence
\[
\frac{1}{2} \int K''(z, q) d\mu_{p_{i}}(q) \leq K'(z, p_{i}) = \int K'(z, q) d\mu_{p_{i}}(q) \leq 2 \int K''(z, q) d\mu_{p_{i}}(q)
\] for \( z \notin C_{m} \). (13)

Let \( \{ p_{i} \} \) be a sequence in \( R' \) determining a point \( p' \in B' \). Then by (11) and (13) we can find an weak limit \( \mu_{p'}(q) \) on \( B' \) of \( \{ \mu_{p_{i}}(q) \} \) such that
\[
\frac{1}{2} \int K(z, q) d\mu_{p'}(q) \leq K'(z, p') \leq 2 \int K''(z, q) d\mu_{p'}(q) \quad \text{for} \quad z \notin \sum C_{n}.' \] (14)

Let \( U(z) \in \hat{P}.H.(R') \) and \( U(z_{0}) = 1 \). Then \( U(z) \) is representable by a positive mass \( \mu \) on \( B' \) such that \( \mu = 0 \) on \( \partial R' = \partial R'' + \sum \gamma_{n} \), \( \int d\mu = 1 \) and
\[
U(z) = \int_{B'} K(z, p') d\mu(p').
\]
Hence by (14) there exists a function $V(z) \in \mathcal{P}.H.(R'')$ such that
\[
\frac{1}{2} V(z) \leq U(z) \leq 2V(z) \quad \text{for } z \notin \sum C_n, \tag{15}
\]
where

\[
V(z) = \left( \int_{B''} K''(z, q)d\mu_{p''}(q) \right)d\mu(p') \quad \text{and} \quad V(z_0) = 1.
\]
Similarly for any $V(z) \in \check{\mathcal{P}}.H.(R'')$, $V(z_0) = 1$ we can find a $U(z) \in \mathcal{P}_0.H.(R')$, $U(z_0) = 1$ such that
\[
\frac{1}{2} U(z) \leq V(z) \leq 2U(z) \quad \text{for } z \notin \sum C_n. \tag{16}
\]

**Proof of the theorem.** As for non constant positive function $A(z)$, we can suppose without loss of generality $A(z_0) = 1$. To define $E$ (from $R'$ to $R''$) and $I$ (from $R''$ to $R'$) operations we can use decreasing sequence $\{v_n\} : v_n = (R^* - R_n^* - \sum_{m=n}^{\infty} C_m)$ instead of $(R^* - R_n^*)$ by the maximum principle, where $\{R_n^*\}$ is an exhaustion of $R^*$. For example $EU(z) = \lim U_n(z)$ for $U(z) \in \check{\mathcal{P}}.H.(R')$, where $U_n(z)$ is L.P.H. in $R' - v_n$ such that $U_n(z) \leq U(z)$ on $v_n$. $IV(z)$ is defined similarly.

Let $U(z) \in \mathcal{P}.H.(R')$. Then by (15) there exists a $V(z) \in \check{\mathcal{P}}.H.(R'')$ such that $U(z) \leq 2V(z)$ on $v_n$. Hence
\[
EU(z) \leq 2V(z) < \infty. \tag{a}
\]
Let $V(z) \in \check{\mathcal{P}}_0.H.(R'')$. Then by (16) there exists an $U(z) \in \check{\mathcal{P}}.H.(R')$ such that
\[
\frac{U(z)}{2} \leq V(z) \leq U(z) \quad \text{on } v_n,
\]
and hence
\[
0 < \frac{U(z)}{2} \leq IV(z). \tag{b, 1}
\]
Also by (15) there exists another $\check{V}(z) \in \check{\mathcal{P}}.H.(R'')$ depending on $U(z)$ such that $2\check{V}(z) \geq U(z)$. Hence $V(z) \leq 2U(z) \leq 4\check{V}(z)$, whence $V(z) \leq 2EU(z) \leq 4\check{V}(z) < \infty$ and
\[
V(z) \leq 2EU(z). \tag{b, 2}
\]
Thus the conditions, a), b, 1) and b, 2) are verified. Hence by the corollarly of Theorem 1 $\check{\mathcal{P}}.H.(R')$ and $\check{\mathcal{P}}.H.(R'')$ are isomorphic and $\mathcal{P}.H.(R)$ and $\check{\mathcal{P}}.H.(R'')$ are isomorphic.

**Corollary.** Under the condition of Theorem 2 any positive minimal harmonic function in $R''$ vanishing on $\partial R''$ is the image of a uniquely determined minimal function in $R$ vanishing on $\partial R$ by the operation $E$ and
conversely any minimal function in $R$ vanishing on $\partial R$ is the image of a uniquely determined minimal function in $R''$ vanishing on $\partial R''$ and the correspondence is one-to-one manner.

**Example 1.** Let $R^*$ be a unit circle, $R^*: |z|<1$ and let $I$ be a straight on the real axis, $I: \text{Im } z=0, 0\leq \text{Re } z\leq 1$. Put $R=R^*-I$. Let $C_n (n=1, 2, \cdots)$ be a hyperbolic circle with centre at $q_n: 1 - \frac{1}{4^n}$ and with hyperbolic radius

$$\frac{1}{3}, \text{ i.e. } C_n: \left| \frac{z-1 + \frac{1}{4^n}}{1 - \left(1 - \frac{1}{4^n}\right)z} \right| < \frac{1}{3}. \text{ Then } C_n \text{ intersects } I \text{ at } 1-$

$$\frac{4}{3} \quad \text{and} \quad 1- \frac{2}{3} \quad \text{and } \{C_{n,t}\} \text{ are disjoint each other.}$

Let $s_n$ be a straight on $I$ of hyperbolic length $2\delta_n < \frac{1}{12}$ with its middle point at $q_n, s_n: \text{Im } z=0, 1- \frac{1+\delta_n}{4^n(1-\frac{\delta_n}{4^n})} \leq \text{Re } z \leq 1- \frac{1-\delta_n}{4^n(1+\frac{\delta_n}{4^n})}$. Let $R''=R+\sum_{n=1}^{\infty}s_n$. Suppose K-Martin’s topology is defined in $R''$ and

$$\sum_{n=1}^{\infty}\delta_n < \infty.$$

Then $R''$ has the following properties.

1). There exist only two minimal points $p^v$ and $p^c$ on $z=1$.

2). Let $\{p^v_i\} (i=1, 2, \cdots) (\{p^c_i\})$ be a sequence in $R'' - \sum_{n=1}^{\infty}C_n$ tending to $z=0$ such that the imaginery part of $p^v_i$, $\text{Im } p^v_i > 0$ ($\text{Im } p^c_i < 0$). Then $K''(z, p^v_i) \rightarrow K''(z, p^v)$ and $K''(z, p^c_i) \rightarrow K''(z, p^c)$ as $i \rightarrow \infty$, where $K''(z, p^v)$ and $K''(z, p^c)$ are minimal functions corresponding to $p^v$ and $p^c$ and $K''(z, p_i)$ are normalized as $K''\left(-\frac{1}{2}, p_i\right) = 1$.

3). Let $p_i \in \sum s_n$. Then $K(z, p_i) \rightarrow \frac{1}{2}(K''(z, p^v)+K''(z, p^c))$, i.e. any sequence $\{p_i\}$ on $\sum s_n$ determines a non minimal point. Let $D_n (n=1, 2, \cdots)$ be a domain $\exists q_n$ and $\{(D_n - q_n)\}$ are all conformally equivalent. Let $\{p_i\}$
be a fundamental sequence in $\sum_{n}^{\infty}D_{n}$. Then $\{p_{i}\}$ determines a non minimal boundary point on $z=1$.

Map $C_{n}$ conformally onto $|w|<\frac{1}{3}$. Then by Lemma 3), there exists a const. $M$ such that for any positive harmonic function $U(w)$ in $|w|<\frac{1}{3}$ vanishing on the image of $1-s_{n}$ and $M\delta_{n}U(w)\geqslant\sup U(w)w(s'_{n},w)$, where $w(s'_{n},w)$ is H.M. of the image of $s_{n}$ relative to $|w|<1$. Put $M_{n}=M\delta_{n}$, then $\sum_{n=1}^{\infty}s_{n}$ and $M\delta_{n}U(w)\geqslant suU(w)w(s_{n}'\frac{1}{2})$ where $w(s_{n}'\frac{1}{2})$ is H.M. of the image of $s_{n}$ relative to $|w|<1$.

Let $v$ be a sufficiently small euclidean neighbourhood of $z=1$. Then since $v\cap R'$ consists of two simply connected domains, there exist only two points $p^{L}$ and $p^{U}$ on $z=1$ which are minimal and the corresponding functions, $K'(z, p^{U})$ and $K'(z, p^{L})$ have the properties. 1). $K'(z, p^{U})=K'(-\frac{1}{2}, p^{L})=1$. 2). $K'(z, p^{L})\rightarrow\infty$ as $z\rightarrow 1$ in the upper half plane and $K'(z, p^{L})<L<\infty$ in the lower half plane. Let $v_{n}=E\left[z:|z|>1-\frac{1}{n}\right]$. Then $EU(z)<\infty$ and $IEU(z)=U(z)$ for $U(z)\in\hat{P}.H.(R')$, $IV(z)>0$ and $EV(z)=V(z)$ for $V(z)\in\hat{P}.H.(R'')$.

We see at once 1). $EK'(z, p^{U})=EK'(\bar{z}, p^{L})$ and by (19) $EK'(z, p^{U})\rightarrow\infty$ as $z\rightarrow 1$ outside of $\sum C_{n}$ in the upper half plane and $EK'(z, p)<2L$ for $z\in\sum C_{n}$ and $\Im z<0$. 2). $EK'(z, p^{U})$ and $EK'(z, p^{L})$ are minimal by the minimality of $K'(z, p^{U})$ and $K'(z, p^{L})$. Let $\{p_{i}\}$ be a sequence in $R''$ determining an ideal
boundary point on \( z = 1 \), i.e. \( K''(z, p_i) \) converges to a \( K(z, p) \) as \( p_i \to z = 1 \). Then clearly \( K''(z, p) = 0 \) on \( I - \sum_{n=0}^{\infty} s_n \) and \( |z| = 1 \) except \( z = 1 \). By (18) and since there exist only two points of the boundary points of \( R' \) on \( z = 1 \), \( IK''(z, p) \) has the form

\[
IK''(z, p) = \alpha K'(z, p^U) + \beta K'(z, p^L)
\]

and

\[
K''(z, p) = EIK''(z, p) = E(\alpha K'(z, p^U) + \beta K'(z, p^L)).
\]

Hence there exist only two minimal boundary points of \( R'' \) on \( z = 1 \).

Let \( \{p_i\} \) be any sequence in \( R'' - \sum_{n=0}^{\infty} C_n \) such that \( p_i \to z = 1 \) and \( p_i \) lies in the upper half plane. Let \( \{p'_i\} \) be a subsequence of \( \{p_i\} \) such that \( K''(z, p'_i) \to K''(z, p) \). Then by (19) \( K''(z, p) < 2L \) in the lower half plane outside of \( \sum_{n=0}^{\infty} C_n \). Whence the representation (20) of \( K''(z, p) \) has the form \( K''(z, p) = \alpha EK'(z, p^U) \), where \( \alpha \) is given by \( K''(-\frac{1}{2}, p) = \alpha EK'(-\frac{1}{2}, p^U) \) and does not depend on the subsequence \( \{p'_i\} \). Hence \( \{p_i\} \) converges to \( p \) on \( z = 1 \) of \( R'' \) (\( p \) corresponds to \( K''(z, p^U) \)). Similarly for any sequence \( \{p_i\} \) in \( R'' - \sum_{n=0}^{\infty} C_n \) in the lower half plane, \( K''(z, p_i) \to K''(z, p^U) \). Thus we have the property (i).

Let \( \{p_i\} \) be a sequence on \( \sum_{n=0}^{\infty} s_n \). Then by \( K''(z, p_i) = K''(\bar{z}, p_i) \), \( K''(z, p_i) \to \frac{1}{2} (K''(z, p^U) + K''(z, p^L)) \) as \( i \to \infty \). Let \( \{p'_i\} \) be a sequence of \( \{p_i\} \) in \( \sum_{n=0}^{\infty} D_n \) such that \( p_i \to z = 1 \) and \( K''(z, p_i) \) converges to \( K''(z, p) \) as \( i \to \infty \). Let \( q_n \) be the middle point of \( s_n \). Then as above \( K''(z, q_i) \to \frac{1}{2} (K''(z, p^U) + K''(z, p^L)) \) as \( i \to \infty \). Since \( G''(z, p) \) of \( R'' \) is a harmonic function of \( p \) for fixed \( z \notin \sum_{n=n_0}^{\infty} D_n \), there exists by Harnack’s theorem a const. \( M \) depending on \( D_n \) but non on \( n \) by the conformal equivalency of \( D_n \) such that

\[
\frac{1}{M} \leq \frac{G''(z, p_i)}{G''(z, q_i)} \leq M \quad \text{and} \quad \frac{1}{M} \leq \frac{G''(-\frac{1}{2}, p_i)}{G''(-\frac{1}{2}, q_i)} \leq M.
\]

Hence

\[
\frac{1}{M^2} \leq \frac{K''(z, p_i)}{K''(z, q_i)} \leq M^2 \quad \text{for} \quad z \notin \sum_{n=n_0}^{\infty} C_n.
\]

Let \( i \to \infty \). Then \( K''(z, p_i) \to K''(z, p) \) and
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\[ \frac{1}{M^2} \leq \frac{\frac{1}{2}(K''(z, p^U) + K''(z, p^L))}{K'(z, p) e^{\frac{1}{2}}} \leq M^2, \quad z \notin \sum C_n. \quad (21) \]

Consider \( K''(z, p) \) and \( \frac{1}{2}(K''(z, p^U) + K''(z, p^L)) \) in the upper half plane outside of \( \sum C_n \). Then since \( K''(z, p) \leq 2L, \frac{M^2}{2} \geq \alpha \geq \frac{1}{2M^2} > 0 \) in the form, \( K''(z, p) = \alpha K''(z, p^U) + \beta K''(z, p^L) \). Similarly we have \( \beta \geq \frac{1}{2M^2} \). Hence \( K''(z, p) \) is non minimal and \( \{ p_i^J \} \) determines a non minimal point of \( R'' \). Thus we have the property (2).

Remark. In Example 1, we can take a circle \( C_n \) of hyperbolic radius \( r_n \) with centre at \( q_n \) (\( r_n \) and \( q_n \) depend on \( n \)) instead of \( C_n \). Let \( I_n = E[z : 1+ r_n e^{i\theta_n}] \) (\( n = 1, 2, \cdots \)), \( 0 < r_n < -2l_n \cos \theta_n, \pi > \theta_n > \frac{\pi}{2} \). Let \( s_{n,m} \) be a straight on \( I_n \). Then we can choose \( s_{n,m} \) so small that there exist infinitely many \( K \)-Martin's minimal boundary points of \( R''' \) on \( z = 1 \), where \( R''' = \text{unit circle} - \sum_{n=1}^\infty I_n - \sum_{n,m}^\infty s_{n,m} \).

In Example 1 we discussed \( \hat{PH}(R'') \) by \( \hat{PH}(R) \), when \( R \) increased to \( R'' \). We show an example to consider \( \hat{PH}(R'') \) by \( PH(R) \), when \( R \) decreases to \( R'' \).

Example 2. Let \( R : |z| < 1 \) and let \( C_n \) be a hyperbolic circle with hyperbolic radius \( \frac{1}{3} \) and its centre at \( q_n \) also let \( S_n \) be a concentric hyperbolic circle with hyperbolic radius \( \delta_n \). Suppose \( C_n \)'s are disjoint each other and

\[ \sum_{n=1}^\infty \frac{1}{-\log \delta_n} < \infty. \]

Put \( R'' = R - \sum_{n=1}^\infty S_n \). Let \( \hat{PH}(R) \) be the class of \( PH \) functions in \( |z| < 1 \) and let \( \hat{PH}(R'') \) be of \( PH \) functions vanishing on \( \sum S_n \). Then \( \hat{PH}(R) \) and \( \hat{PH}(R'') \) are isomorphic. Especially there exist only one \( K \)-Martin's boundary point of \( R'' \) which is minimal at \( e^{i\theta} \).

In fact, map \( C_n \) onto \( |\zeta| < \frac{1}{3} \) and let \( U(\zeta) \) be a \( PH \) in \( |\zeta| < \frac{1}{6} \). Then there exists a const. \( M \) such that \( \max_{|\zeta| = \frac{1}{12}} U(\zeta) \leq M \min_{|\zeta| = \frac{1}{6}} U(\zeta) \). Let \( C_n \) and \( C_n'' \) be concentric circles as \( C_n \) with hyperbolic radius \( \frac{1}{6} \) and \( \frac{1}{12} \) respectively. Then \( \omega(S_n, z) = \frac{\log 12}{-\log \delta_n} \) on \( \partial C_n'' \), where \( \omega(S_n, z) \) is H.M. of \( S_n \) relative to \( |z| < 1 \).
Put $M_n = M \frac{\log 12}{-\log \delta_n}$. Then $M_n \min_{z \in \partial C_n} U(z) \geq \max_{z \in \partial C_n} U(z) \omega(S_n, z)$. Hence by Theorem 1, $\mathbb{P} \mathbb{H}(R)$ and $\mathbb{P} \mathbb{H}(R'')$ are isomorphic and there exists only one boundary point $R$ at $e^{\theta}$ which is minimal.

Let $R$ be a Riemann surface with positive boundary and let $R_0$ be a compact disc. Let $N(z, p)$ be an $N$-Green’s function in $R - R_0$ such that $N(z, p) = 0$ on $\partial R_0$, $N(z, p)$ has a logarithmic singularity at $p$ and $N(z, p)$ has minimal Dirichlet integral (Dirichlet integral is taken about $N(z, p) + \log |z - p|$ in a neighbourhood of $p$). Suppose the $N$-Martin’s topology is defined on $R - R_0$. We shall construct a Riemann surface of planer character in which there exist non $N$-minimal points.

**Example 3.** Let $C$ be a unit circle, $C: |z| < 1$. Let $s_n (n = 1, 2, \cdots)$ be a straight, $s_n$: Im $z = 0$, $0 < a_n \leq \text{Re} z \leq b_n$, $a_n < b_n \cdots < a_i < b_i = 1$ in $C$ and let $t_n$: Im $z = 0$, $b_{n+1} \leq \text{Re} z \leq a_n$. Let $\hat{s}_n$ and $\check{t}_n$ be symmetric images of $s_n$ and $t_n$ with respect to the imaginary axis. Let $I$ be a straights such that $I: \text{Re} z = 0$, $0 < \text{Im} z < \frac{1}{2}$ and $0 > \text{Im} z > -\frac{1}{2}$.

**Condition.** $z = 0$ is contained in the closure of $\sum s_n$ and $\sum s_n$ is so thinly distributed as $z = 0$ is an irregular point for the Dirichlet problem in $C - \sum (s_n + \hat{s}_n)$. Put $R - R_0 = C - I - \sum (t_n + \check{t}_n)$. Then there exist four $N$-minimal points and non $N$-minimal points on $z = 0$.

Let $p_i = r_i e^{i\theta_i}$. Suppose $r_i \to 0$ as $i \to \infty$ and $\frac{\pi}{2} > \theta_i > \delta > 0$. Then $p_i$ determines an $N$-minimal point. Similar fact occurs in each sector $S_j$: $0 < |z| < 1$, $\frac{\pi(j-1)}{2} \leq \arg z \leq \frac{\pi j}{2}$, ($j = 1, 2, 3, 4$).

Let $N(z, p)$ be an $N$-Green’s function of $R - R_0$. Then $N(z, p) = 0$ on $\partial R_0$, $\frac{\partial}{\partial n} N(z, p) = 0$ on $\sum (t_n + \check{t}_n) + I$.

Let $G(z, p)$ be the Green’s function of $C$. Then

$$N(z, p_i) + N(z, \overline{p}_i) + N(z, \hat{p}_i) + N(z, \check{p}_i) = G(z, p_i) + G(z, \overline{p}_i) + G(z, \hat{p}_i) + G(z, \check{p}_i),$$

where $\overline{p}$ and $\hat{p}$ are the conjugate of $p$ and the symmetric image of $p$ with respect to $I$.

Let $p_i \to p$ on $z = 0$. Then

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9) See (8).
$N(z, p) + N(z, \overline{p}) + N(z, \hat{p}) + N(z, \xi) = -4 \log |z|.$

Let $C'$ be a circle, $C': |z| < \frac{2}{3}$ and put $\Omega_s = C' - \sum_{\infty}(s_n + \hat{s}_n)$. Let $G'(z, p)$ be the Green's function of $\Omega_s$. By the condition, $z=0$ is an irregular point for also the Dirichlet problem in $\Omega_s$. Hence there exists a sequence $\{p_i\}$ such that $\lim_i G'(z, p_i) = G'(z, p) > 0$. Let $v_n = \left[ z : |z| < \frac{1}{n} \right]$. We define $E$ and $I$ operations between $C$ and $\Omega_s$ with respect to $\{v_n\}$. Then by $G(z, p_i) \geq G'(z, p_i), \quad E G'(z, p_i) < \infty$. Now $E G'(z, p)$ is harmonic in $\Omega_s$ except $z=0$, whence $E G'(z, p)$ must be $\alpha(-\log|z|)$ ($\alpha$ is a positive constant $\leq 1$). By $I E G'(z, p) = G'(z, p), \quad I(-\log|z|)$ is a harmonic function in $\Omega_s$ such that $I(-\log|z|) \leq (-\log|z|), \quad I(-\log|z|) = 0$ on $\sum_{\infty}(s_n + \hat{s}_n), \quad \frac{\partial}{\partial n} I(-\log|z|) = 0$ on $I + \sum_{\infty}(t_n + \hat{t}_n)$. Hence for any $p$ on $z=0: \quad p = \lim_i p_i$.

$-4 \log |z| = N(z, p) + N(z, \overline{p}) + N(z, \hat{p}) + N(z, \xi) \geq 4I(-\log |z|) > 0. \quad (22)$

Let $R'$ be the part of $\overline{R-R_0}$ over $|z| < \frac{2}{3}$. Put $R_s = R' - \sum_{\infty}(s_n + \hat{s}_n)$. Then $R_s$ consists of four components, $R'_j (j=1, 2, 3, 4)$, where $R'_j \subset S_j$. Suppose $p_i \in R'_j$. Let $N^s(z, p_i)$ be an $N$-Green's function in $R_s$ such that $N^s(z, p_i)$ is harmonic in $R_s, \quad N^s(z, p_i)$ has a logarithmic singularity at $p_i, \quad N^s(z, p_i) = 0$ on $\sum_{\infty}(s_n + \hat{s}_n)$ and on $|z| = \frac{2}{3}, \quad \frac{\partial}{\partial n} - N^s(z, p_i) = 0$ on $\sum_{\infty}(t_n + \hat{t}_n) + I$ and $N^s(z, p_i) = 0$ in $R'_j$ for $j' \neq j$. Then

$-4 \log |z| \geq \lim_i (N^s(z, p_i) + N^s(z, \overline{p}_i) + N^s(z, \hat{p}_i) + N^s(z, \xi))$

$= \lim_i (G'(z, p_i) + G(z, \overline{p}_i) + G(z, \hat{p}_i) + G(z, \xi))$

$= 4\alpha I(-\log |z|), \quad (23)$

$\alpha$ depends on $\{p^i\}$ and $1 \geq \alpha \geq 0$.

We shall use $I^{N, \Gamma}$ and $E^{N, \Gamma}$ operations between $R-R_s$ and $R_s$, where $\Gamma = I + \sum(t_n + \hat{t}_n)$. Since $I(-\log |z|) = 0$ on $\sum_{\infty}(s_n + \hat{s}_n), \quad \frac{\partial}{\partial n} I(-\log |z|) = 0$ on $I + \sum_{\infty}(t_n + \hat{t}_n), \quad I^{N, \Gamma}$ can be defined and we have by (22)

$I^{N, \Gamma}(\lim_i N(z, p_i)) > 0 \quad \text{for any } p_i \to p \text{ on } z=0. \quad (24)$

Suppose $\lim_i N^s(z, p_i)$ exists and $>0$. Then by (23)

$E^{N, \Gamma}(\lim_i N^s(z, p_i)) > 0. \quad (25)$
We can find a sequence \( \{ p_i \} \) such that \( \lim G'(z, p_i) = \beta I(-\log |z|) > 0 \). Then by (23) \( E^N_r \sum_i (N^s(z, p_i) + N^s(z, \bar{p}_i) + N^s(z, \hat{p}_i) + N^s(z, \check{p}_i) + N^s(z, \tilde{p}_i)) = 4\beta EI(-\log |z|) \). On the other hand, since \( \sum_i (t_n + \hat{t}_n) \), \( E^N_r I(-\log |z|) \)

\[ = EI(-\log |z|) = -\log |z|. \]

Put \( U(z) = \frac{4}{\beta} \lim_i (N^s(z, p_i) + N^s(z, \bar{p}_i) + N^s(z, \hat{p}_i) + N^s(z, \check{p}_i) + N^s(z, \tilde{p}_i)) > 0. \)

Then \( U(z) > 0 \) on \( \sum (s_n + \hat{s}_n) \) and on \( |z| = \frac{2}{3} \), \( \frac{\partial}{\partial n} U(z) = 0 \) on \( I + \sum (t_n + \hat{t}_n) \), \( E^N_r U(z) = -4 \log |z| \) and for any \( p_j \rightarrow p \) on \( z = 0 \)

\[ E^N_r U(z) = -4 \log |z| = N(z, p) + N(z, \bar{p}) + N(z, \hat{p}) + N(z, \check{p}). \]

Hence by Theorem 1, \( N \), a) and e) and (24)

\[ E^N_r I^N_r N(z, p) = N(z, p) > 0 \quad \text{for any } p \text{ on } z = 0. \]

Let \( \hat{P}.H.N.(R) \) be the set of positive harmonic function \( U(z) \) of the form \( N(z, p) \) in \( R - R_0 \) except \( z = 0 \) such that \( U(z) = 0 \) on \( \partial R_0 \), \( \frac{\partial}{\partial n} U(z) = 0 \) on \( \sum (t_n + \hat{t}_n) + I \) and let \( \hat{P}.H.N.(R_0) \) be the set of positive harmonic function \( V(z) \) of the form \( N^s(z, p) \) in \( R \) except \( z = 0 \) such that \( V(z) = 0 \) on \( |z| = \frac{2}{3} \) and \( \sum (s_n + \hat{s}_n) \) and \( \frac{\partial}{\partial n} V(z) = 0 \) on \( I + \sum (t_n + \hat{t}_n) \). Then \( \hat{P}.H.N.(R) \) and \( \hat{P}.H.N.(R_0) \) are isomorphic by (25) and (26).

Let \( N^s(z, p) = \lim_i (N^s(z, p_i) > 0; p_i \in R_j' \) and \( p_i \rightarrow p \) on \( z = 0 \). We shall show that \( N^s(z, p) \) is \( N \)-minimal in \( R \). Let \( U(z) \) be a superharmonic function in \( R \) such that \( N^s(z, p) - U(z) > 0 \) is also superharmonic in \( R \). Now \( \frac{\partial}{\partial n} N^s(z, p) = 0 \) on \( I + \sum (t_n + \hat{t}_n) \). This means \( N^s(z, p) \) has no mass on \( I + \sum (t_n + \hat{t}_n) \), whence by the superharmonicity of \( N^s(z, p) - U(z) \), \( \frac{\partial}{\partial n} U(z) = 0 \) on \( I + \sum (t_n + \hat{t}_n) \).

Define \( \bar{U}(z) \) similarly from \( U(z) \) into \( R - R_0 \). Then \( \bar{U}(z) \) and \( \hat{V}(z) \) is harmonic in \( C' - \sum (s_n + \hat{s}_n) \), where \( C' : |z| < \frac{2}{3} \). Now there exists only one linearly independent harmonic function \( I(-\log |z|) \) in this surface. Whence \( \bar{U}(z) = \alpha \hat{V}(z) = \beta I(-\log |z|) \) and \( U(z) = \alpha N^s(z, p) \) and \( N^s(z, p) \) is \( N \)-minimal in \( R \). On the other hand, in every \( R_j' \), \( N^s(z, p) \) exists denoted by \( N^s(z, p') \). Then clearly \( N^s(z, p') \) are linearly independent. Because \( N^s(z, p_j) = 0 \) in \( R_j' \) for \( j' \neq j \). Let \( N(z, p) = \lim_i N(z, p_i) \). Then by (26) \( I^N_r N(z, p) > 0 \), whence \( I^N_r (z, p) \).
Examples of Non Minimal Points on Riemann Surfaces of Planer Character

Next also by (26) $N(z, p) = \sum_{j=1}^{r} \alpha_j E^{N,r} N^s(z, p^j)$ and there exist exact four $N$-minimal points of $R-R_0$ on $z=0$.

**Property of $E^{N,r} N^s(z, p^j)$.** $N^s(z, p^j) = 0$ in $R'_j$ for $j \neq j'$. Hence

$E^{N,r} N^s(z, p^j) = E^{N,r}_{R'_j} N^s(z, p^j),$

where $E^{N,r}_{R'_j}$ is from $R'_j$ to $R-R_0$ and $E^{N,r}$ is from $R$ to $R-R_0$.

By Theorem N.1. b)

$I^N_{R'_j} E^{N,r} N^s(z, p^j) \leq E^{N,r}_{R'_j} E^{N,r} N^s(z, p^j) \leq E^{N,r}_{R'_j} E^{N,r}_{R'_j} N^s(z, p^j) = 0$ and

$I^N_{R'_j} E^{N,r}_{R'_j} N^s(z, p^j) = N^s(z, p^j)$ for $k \neq j$.

We shall show $p_i = r_i e^{i\theta_i} \left(\frac{\pi}{2} \geq \theta_i > 0\right) \in S$, determines an $N$-minimal point $p_i$ of $R-R_0$. $N(z, p_i) + N(z, \hat{p}_i) = N(z, \overline{p}_i) + N(z, \overline{p}_i) = U(z, p_i)$ for $\text{Im} z = 0$, where $U(z, p_i) = \frac{1}{2} (G(z, p_i) + G(z, \overline{p}_i) + G(z, \hat{p}_i) + G(z, \hat{p}_i))$ and $G(z, q) = \log \frac{|1-qz|}{|z-q|}.

$N(z, p_i)$ is harmonic in $S_i$ (has no singularity) such that $\frac{\partial}{\partial n} N(z, p_i) = 0$ on $I + \sum s_n + I^\prime$ and $N(z, p_i) \leq U(z, p_i)$ on $\sum s_n + I^\prime$ and $\sum s_n + I^\prime + I^\prime$.

Put $p_i = r_i e^{i\theta_i}$. Then by $\frac{\pi}{2} > \theta_i > 0, \frac{5}{4} > |1 - p_i \zeta| > \frac{3}{4}$ for $r_i < \frac{1}{4}$ and $|\zeta - p_i| \geq \zeta \sin \zeta$. Hence there exists a const. $A$ such that

$U(\zeta, p_i) \leq -2 \log |\zeta| + A$ for $|r| \leq \frac{1}{4},$

whence $N(z, p_i) \frac{\partial}{\partial n} N^s(\zeta, z) = 0$ is uniformly integrable on $\sum s_n + I^\prime$ for $|p_i| \leq \frac{1}{4}$.

Hence

$N(z, p^j) = \frac{1}{2\pi} \int_{\sum s_n + I^\prime} N(\zeta, p^j) \frac{\partial}{\partial n} N^s(\zeta, z) ds = U(z)$ in $S_i$,

where $N(z, p^j) = \lim \ N(z, p_i)$.

$I^N_{R'_j} N^s(z, p^j) = \lim V_n(z)$, where $V_n(z)$ is a harmonic function in $R'_j \cap E \left[|z| > \frac{1}{n}\right]$ such that $V_n(z) = U(z)$ on $|z| = \frac{1}{n}$, $\frac{\partial}{\partial n} V_n(z) = 0$ on $I + \sum t_n, V_n(z) = 0$ on
\[ I + \sum_{n} s_{n} \text{ and on } |z| = \frac{2}{3}. \]

Let \( w_{n}(z) \) be a harmonic function in \( R_{k}^{l} \cap E \left[ |z| > \frac{1}{n} \right] \) such that \( w_{n}(z) = 0 \) on \( |z| = \frac{1}{n} \) and \( \frac{\partial}{\partial n} w_{n}(z) = 0 \) on \( I \) and \( w_{n}(z) = U(z) \) on \( \sum_{n} s_{n} + I' \) and on \( |z| = \frac{2}{3} \). Then \( w_{n}(z) = \frac{1}{2\pi} \int_{\sum_{n} s_{n} + I'} N(\zeta, p) \frac{\partial}{\partial n} N_{n}^{s}(\zeta, z) ds \), \( U(z) = w_{n}(z) + V_{n}(z) \) and \( \lim_{n} w_{n}(z) = U(z) \) by \( \frac{\partial}{\partial n} N_{n}^{s}(\zeta, z) \uparrow \frac{\partial}{\partial n} N^{s}(\zeta, z) \), where \( N_{n}^{s}(\zeta, z) \) is a function in \( R_{k}^{l} \cap E \left[ |z| > \frac{1}{n} \right] \) such that \( N_{n}^{s}(\zeta, z) = 0 \) on \( I + \sum_{n} s_{n} \) and \( |z| = \frac{1}{n} \) and \( \frac{\partial}{\partial n} N_{n}^{s}(\zeta, z) = 0 \) on \( I + \sum_{n} t_{n} \).

Whence \( \lim_{n} V_{n}(z) = I_{R_{k}^{l}}^{N, \Gamma} N(z, p) = 0 \). Similarly we have
\[ I_{R_{k}^{l}}^{N, \Gamma} N(z, p^i) = 0 \quad \text{for } k \neq 1. \quad (28) \]

Hence in the form
\[ N(z, p) = \sum_{j} \alpha_{j} E^{N, \Gamma} N^{s}(z, p^j) \cdot \alpha_{k} = 0 : \quad k \neq 1 \text{ and } N(z, p^i) = \alpha_{i} E^{N, \Gamma} N^{s}(z, p^i). \]

Now by \( \int_{|z|=1} \frac{\partial}{\partial n} N(z, p) ds = 2\pi \), \( \alpha_{i} \) does not depend on special sequence \( \{p_{i}\} \) if \( \frac{\pi}{2} \geq \arg p_{i} > \delta > 0 \). Hence any \( \{p_{i}\} \) determines an \( N \)-minimal point on \( z=0 \) as \( p_{i} \rightarrow z=0 \), if \( \frac{\pi}{2} > \arg p_{i} > \delta > 0 \). Similar fact occurs in \( S_{k} \), \( k=2, 3, 4 \).

Let \( p_{i} \in \sum_{n} s_{n} \). Then we have at once \( \lim_{i} N(z, p_{i}) = \frac{1}{2} (N(z, p^1) + N(z, p^4)) \) and \( \{p_{i}\} \) determines a non \( N \)-minimal point on \( z=0 \).

**Remark.** It is easy to construct a boundary point \( z=0 \) on which infinitely many \( N \)-minimal points exist as the remark of Example 1.

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