ON A DIFFERENTIAL-DIFFERENCE EQUATION

By

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In connexion with the study of certain incomplete sums of multiplicative functions, N. G. de Bruijn and J. H. van Lint [3] have introduced the function $f_s(x)$ $(s\geqq 0)$ satisfying the set of conditions:

(i) $f_s(x)=0$ for $x<0$,

(ii) $f_s(x)$ is continuous for $x>0$,

(iii) $f_s(x)=x^{s-1}$ for $0<x\leqq 1$,

(iv) $xf_s'(x)=(s-1)f_s(x)-sf_s(x-1)$ for $x>1$.

(The function $f_s(x)$ is originally defined in [3; II, §2] only for $x>0$; it will be convenient, however, to define $f_s(x)=0$ for $x<0$ for our purpose.)

On the other hand, N. G. de Bruijn [1 and 2] has investigated in detail the property and behaviour of $f_s(x)$ for $s=1$. In particular, there he obtained an explicit formula for $f_1(x)$:

$$f_1(x)=\frac{e^C}{2\pi i}\int_{-i\infty}^{i\infty}\exp\left(-xt+\int_0^t\frac{e^z-1}{z}dz\right)dt \qquad (x>0),$$

where C is Euler's constant,

$$C=\lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m}-\log n\right).$$

In the present note we shall prove an analogous formula for $f_s(x)$ with general $s>0$.

Remark. For $s=0$ it is easy to see that $f_s(x)=f_0(x)=x^{-1}$ $(x>0)$. We may suppose, therefore, that $s>0$ throughout in the following.

1. Lemmata. We require two lemmas independent of one another.

Lemma 1. *If $\phi(s)$ is a (complex valued) continuous function defined for $s>0$ and satisfying the functional equation*

$$\phi(s+r)=\phi(s)\phi(r) \qquad (s>0,\ r>0),$$

then there is an integer A independent of s such that

$$\phi(s)=e^{2\pi iAs}\left(\phi(1)\right)^s \qquad (s>0).$$

Proof. We may assume without loss of generality that $\phi(s)$ does not vanish for $s>0$ and hence that $\phi(1)\neq 0$. Consider the continuous function

$$\psi(s)=\frac{\phi(s)}{(\phi(1))^s}\qquad (s>0),$$

where $z^s=\exp(s\log z)$ and the branch of $\log z$ is taken in such a way that $\log z$ is real for real $z>0$. We have for any $s>0$

$$\psi(s+1)=\frac{\phi(s+1)}{(\phi(1))^{s+1}}=\frac{\phi(s)}{(\phi(1))^s}=\psi(s).$$

Thus, if we put

$$\alpha(s)=\frac{1}{1+|\log|\psi(s)||},$$

then

$$\int_0^1\alpha(s)\,ds=\int_0^1\alpha(2s)\,ds. \tag{1}$$

Indeed, we have

$$\begin{aligned}\int_0^1\alpha(s)ds&=\int_0^1\alpha(s+1)ds=\int_1^2\alpha(s)ds\\&=2\int_{1/2}^1\alpha(2s)ds=2\int_0^1\alpha(2s)ds-\int_0^1\alpha(s)ds,\end{aligned}$$

which is equivalent to (1). Since $\psi(2s)=(\psi(s))^2$, we deduce from (1) that

$$\int_0^1\frac{|\log|\psi(s)||}{(1+|\log|\psi(s)||)(1+2|\log|\psi(s)||)}ds=0,$$

and this implies that $\log|\psi(s)|=0$ almost everywhere on $(0,1)$. It follows that $|\psi(s)|=1$ *everywhere* on $(0,\infty)$. This means that, if we set

$$\frac{\phi(s)}{(\phi(1))^s}=e^{2\pi i\,\theta(s)}, \tag{2}$$

then $\theta(s)$ is a real valued continuous function of $s>0$ satisfying the congruence

$$\theta(s+r)\equiv\theta(s)+\theta(r)\pmod 1\qquad (s>0,\ r>0).$$

Hence, there is a constant $c\equiv 0\pmod 1$ such that

$$\theta(s+r)=\theta(s)+\theta(r)+c\qquad (s>0,\ r>0),$$

and it follows from this that the limit

$$\lim_{s\to+0}\theta(s)=-c$$

exists. Thus, if we put

$$\theta^*(s) = \theta(s) + c\,,$$

then $\theta^*(s)$ satisfies the equation

$$\theta^*(s+r) = \theta^*(s) + \theta^*(r) \qquad (s>0,\ r>0).$$

Since $\theta^*(s)$ is continuous for $s>0$, we find by a well-known theorem (which is in fact easy to prove) that $\theta^*(s) = As$ $(s>0)$ for some real constant A, so that $\theta(s) = \theta^*(s) - c = As - c$. But, in view of (2), we may take $c=0$. Finally, the constant A must be integral, since $e^{2\pi i A} = 1$. This completes the proof of the lemma.

Lemma 2. *We have*

$$f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)} \int_0^x f_s(y) f_r(x-y)\,dy \qquad (s>0,\ r>0).$$

Proof. Put

$$f(x) = \int_0^x f_s(y) f_r(x-y)\,dy\,.$$

Apparently, $f(x)=0$ for $x<0$ and $f(x)$ is continuous for $x>0$. For $0<x\leqq 1$ we have

$$\begin{aligned} f(x) &= x^{s+r-1} \int_0^1 z^{s-1}(1-z)^{r-1}\,dz \\ &= f_{s+r}(x) \frac{\Gamma(s)\,\Gamma(r)}{\Gamma(s+r)}\,. \end{aligned}$$

Suppose now that $x>1$ and write

$$\begin{aligned} xf(x) &= \int_0^x y f_s(y) f_r(x-y)\,dy + \int_0^x f_s(y)(x-y) f_r(x-y)\,dy \\ &= \int_0^x (x-y) f_s(x-y) f_r(y)\,dy + \int_0^x f_s(y)(x-y) f_r(x-y)\,dy \\ &= I_1 + I_2\,, \end{aligned}$$

say. We have

$$\begin{aligned} \frac{dI_1}{dx} &= \int_0^x \Big((x-y) f_s'(x-y) + f_s(x-y)\Big) f_r(y)\,dy \\ &= \int_0^x (x-y) f_s'(x-y) f_r(y)\,dy + \int_0^x f_s(x-y) f_r(y)\,dy \\ &= (s-1) f(x) - s f(x-1) + f(x) \\ &= s f(x) - s f(x-1)\,, \end{aligned}$$

and, by symmetry,

$$\frac{dI_2}{dx} = rf(x) - rf(x-1) .$$

Since $\bigl(xf(x)\bigr)' = xf'(x) + f(x)$, we thus obtain

$$xf'(x) = (s+r-1)f(x) - (s+r)f(x-1) .$$

Hence the function $\dfrac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)} f(x)$ satisfies all the conditions (i)—(iv) with $s+r$ in place of s, and, since these conditions uniquely determine the function $f_{s+r}(x)$, it follows that

$$f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)} f(x) ,$$

which is the required result.

Remark. The substance of Lemma 2 is a particular case of a slightly more general result. Thus, let $h(x) = h(x\,;\,c)$ be a function defined by the following conditions, $c = (c_0, c_1, \cdots, c_n)$ being an $(n+1)$-tuple of constants ($n \geqq 0$ fixed):

(i) $h(x) = 0$ for $x < 0$,

(ii) $h(x)$ is continuous for $x > 0$,

(iii) $\lim\limits_{x \to +0} xh(x) = 0$,

(iv) $xh'(x) = (c_0 - 1)h(x) + \sum\limits_{j=1}^{n} c_j h(x-j)$ for all $x > 0$, $x \neq m$ $(1 \leqq m \leqq n)$.

(Obviously the above conditions for $h(x) = h(x\,;\,c)$ imply that $c_0 > 0$ and $h(x) = Bx^{c_0 - 1}$ for $0 < x \leqq 1$ with B a constant and, for $x > 1$, $h(x)$ is uniquely determined once B is fixed. We shall be concerned with those functions $h(x)$ only which are not identically zero.) Then we have

$$h(x\,;\,a+b) = K \int_0^x h(y\,;\,a) h(x-y\,;\,b)\, dy \qquad (x \neq 0) ,$$

where K is a constant and where we set

$$a + b = (a_0 + b_0,\ a_1 + b_1, \cdots, a_n + b_n)$$

if

$$a = (a_0, a_1, \cdots, a_n) \quad \text{and} \quad b = (b_0, b_1, \cdots, b_n).$$

2. The Explicit Formula. We now consider the Laplace transform of $f_s(x)$,

$$F_s(\xi) = \int_0^\infty e^{-\xi x} f_s(x)\, dx\,,$$

where ξ is a complex variable. The integral defining $F_s(\xi)$ is absolutely convergent on the line $\mathrm{Re}\,\xi=0$ (cf. §3 below). Also, $F_s(\xi)$ is, as a function of s, continuous for $s>0$, ξ ($\mathrm{Re}\,\xi \geqq 0$) being fixed.

In view of Lemma 2 we have

$$F_{s+r}(\xi) = \frac{\Gamma(s+r)}{\Gamma(s)\,\Gamma(r)} F_s(\xi) F_r(\xi) \qquad (s>0,\ r>0)$$

or

$$\frac{F_{s+r}(\xi)}{\Gamma(s+r)} = \frac{F_s(\xi)}{\Gamma(s)} \frac{F_r(\xi)}{\Gamma(r)} \qquad (s>0,\ r>0).$$

Hence, by applying Lemma 1 to $\phi(s)=F_s(\xi)/\Gamma(s)$, we get for $s>0$

$$\frac{F_s(\xi)}{\Gamma(s)} = e^{2\pi i A s}\left(\frac{F_1(\xi)}{\Gamma(1)}\right)^s$$

or

$$F_s(\xi) = \Gamma(s)\left(F_1(\xi)\right)^s,$$

the constant A being necessarily zero since for any $s>0$ $F_s(\xi)$ has a positive real value for real $\xi \geqq 0$.

We see from the explicit formula for $f_1(x)$ ([2; §1]) that

$$F_1(\xi) = e^C \exp\left(\int_0^{-\xi} \frac{e^z-1}{z}\, dz\right).$$

Therefore,

$$F_s(\xi) = \Gamma(s) e^{Cs} \exp\left(s\int_0^{-\xi} \frac{e^z-1}{z}\, dz\right) \qquad (s>0). \tag{3}$$

By a standard inversion formula for the Laplace transform, we thus obtain the following result.

Theorem. *We have for* $s>0$

$$f_s(x) = \lim_{T\to\infty} \frac{\Gamma(s)e^{Cs}}{2\pi i}\int_{-iT}^{iT} \exp\left(-xt+s\int_0^t \frac{e^z-1}{z}dz\right)dt \qquad (x\neq 0).$$

3. Notes. 1) We note that for $s=1$ the right-hand side of the equality in the theorem is equal to $\frac{1}{2}$ at $x=0$ and for $s>1$ it is equal to 0 at $x=0$. Also, if $s=1$ then we have

$$f_1(x) = \frac{e^C}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(-xt+\int_0^t \frac{e^z-1}{z}\,dz\right)dt$$

for all $x \neq 0$, and if $s>1$ then

$$f_s(x) = \frac{\Gamma(s)e^{Cs}}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left(-xt+s\int_0^t \frac{e^z-1}{z}\,dz\right)dt$$

for all x, $-\infty<x<\infty$, $f_s(0)$ being defined to be equal to 0.

2) de Bruijn and van Lint [3; I] have also considered the function $g_s(x)$ ($s\geqq 0$) defined by the conditions:

(i) $g_s(x)=0$ for $x<0$,

(ii) $g_s(x)$ is continuous for $x\geqq 0$,

(iii) $g_s(x)=x^s$ for $0\leqq x\leqq 1$,

(iv) $xg_s'(x)=sg_s(x)-sg_s(x-1)$ for $x>1$.

As is noted in [3; II, §2], we have for $s>0$

$$g_s(x) = s\int_0^x f_s(y)\,dy\ .$$

They showed in [3; I, §2] that if $s=0$ then $g_s(x)=g_0(x)=1$ for all $x>0$ and if $s>0$ then $g_s(x)$ is a positive, monotone increasing function of x for $x>0$.

It is also proved there that we have

$$\lim_{x\to\infty} g_s(x) = \Gamma(s+1)e^{Cs} \qquad (s>0)$$

and this implies at once that

$$\int_0^\infty f_s(x)\,dx = \Gamma(s)e^{Cs} \qquad (s>0),$$

which clearly agrees with (3). And in the course of its proof they found a formula which is essentially the same as (3). (In fact, using the relation between $f_s(x)$ and $g_s(s)$, we can show that $F_s(\xi)$ ($s>0$) satisfies as a function of ξ the differential equation

$$\xi F_s'(\xi) = s(e^{-\xi}-1)F_s(\xi)\ ,$$

and, by integrating this equation, we get the formula (3).) Thus, our main interest of this note is in deriving the explicit formula for $f_s(x)$ from a somewhat different point of view, that is, on the basis of Lemma 2 which shows an interesting interrelation existent among the functions $f_s(x)$ ($s>0$).

References

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