<table>
<thead>
<tr>
<th>Title</th>
<th>ON A DIFFERENTIAL-DIFFERENCE EQUATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Uchiyama, Saburô</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics = 北海道大学理学部紀要, 19(2): 059-065</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1966</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56070">http://hdl.handle.net/2115/56070</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSHIU_19_N2_059-065.pdf</td>
</tr>
</tbody>
</table>

Hokkaido University Collection of Scholarly and Academic Papers: HUSCAP
ON A DIFFERENTIAL-DIFFERENCE EQUATION

By

Saburô UCHIYAMA

In connexion with the study of certain incomplete sums of multiplicative functions, N. G. de Bruijn and J. H. van Lint [3] have introduced the function $f_s(x)$ $(s \geq 0)$ satisfying the set of conditions:

(i) $f_s(x) = 0$ for $x < 0$,

(ii) $f_s(x)$ is continuous for $x > 0$,

(iii) $f_s(x) = x^{n-1}$ for $0 < x \leq 1$,

(iv) $xf_s'(x) = (s-1)f_s(x) - sf_s(x-1)$ for $x > 1$.

(The function $f_s(x)$ is originally defined in [3; II, §2] only for $x > 0$; it will be convenient, however, to define $f_s(x) = 0$ for $x < 0$ for our purpose.)

On the other hand, N. G. de Bruijn [1 and 2] has investigated in detail the property and behaviour of $f_s(x)$ for $s = 1$. In particular, there he obtained an explicit formula for $f_1(x)$:

$$f_1(x) = \frac{e^C}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left( -xt + \int_0^t \frac{e^z - 1}{z} \, dz \right) \, dt \quad (x > 0),$$

where $C$ is Euler’s constant,

$$C = \lim_{n \to \infty} \left( \sum_{m=1}^n \frac{1}{m} - \log n \right).$$

In the present note we shall prove an analogous formula for $f_s(x)$ with general $s > 0$.

Remark. For $s = 0$ it is easy to see that $f_s(x) = f_0(x) = x^{-1}$ $(x > 0)$. We may suppose, therefore, that $s > 0$ throughout in the following.

1. Lemmata. We require two lemmas independent of one another.

Lemma 1. If $\phi(s)$ is a (complex valued) continuous function defined for $s > 0$ and satisfying the functional equation

$$\phi(s + r) = \phi(s) \phi(r) \quad (s > 0, \ r > 0),$$

then there is an integer $A$ independent of $s$ such that

$$\phi(s) = e^{2\pi i A s} \left( \phi(1) \right)^s \quad (s > 0).$$
Proof. We may assume without loss of generality that $\phi(s)$ does not vanish for $s > 0$ and hence that $\phi(1) \neq 0$. Consider the continuous function

$$\phi(s) = \frac{\phi(s)}{(\phi(1))^s} \quad (s > 0),$$

where $z^s = \exp(s \log z)$ and the branch of $\log z$ is taken in such a way that $\log z$ is real for real $z > 0$. We have for any $s > 0$

$$\phi(s + 1) = \frac{\phi(s + 1)}{(\phi(1))^{s+1}} = \frac{\phi(s)}{(\phi(1))^s} = \phi(s).$$

Thus, if we put

$$\alpha(s) = \frac{1}{1 + |\log |\phi(s)||},$$

then

$$\int_0^1 \alpha(s) ds = \int_0^1 \alpha(2s) ds.$$

Indeed, we have

$$\int_0^1 \alpha(s) ds = \int_0^1 \alpha(s + 1) ds = \int_1^2 \alpha(s) ds = 2 \int_{1/2}^1 \alpha(2s) ds = 2 \int_0^1 \alpha(2s) ds - \int_0^1 \alpha(s) ds,$$

which is equivalent to (1). Since $\phi(2s) = (\phi(s))^2$, we deduce from (1) that

$$\int_0^1 \frac{|\log |\phi(s)||}{(1 + |\log |\phi(s)||)(1 + 2|\log |\phi(s)||)} ds = 0,$$

and this implies that $\log |\phi(s)| = 0$ almost everywhere on $(0, 1)$. It follows that $|\phi(s)| = 1$ everywhere on $(0, \infty)$. This means that, if we set

$$\phi(s) = e^{\theta(s)} \quad \frac{\phi(s)}{(\phi(1))^s},$$

then $\theta(s)$ is a real valued continuous function of $s > 0$ satisfying the congruence

$$\theta(s + r) \equiv \theta(s) + \theta(r) \quad (\text{mod } 1) \quad (s > 0, \ r > 0).$$

Hence, there is a constant $c \equiv 0 \ (\text{mod } 1)$ such that

$$\theta(s + r) = \theta(s) + \theta(r) + c \quad (s > 0, \ r > 0),$$

and it follows from this that the limit

$$\lim_{s \to +0} \theta(s) = -c.$$
On a Differential-Difference equation

exists. Thus, if we put

$$\theta^*(s) = \theta(s) + c,$$

then $\theta^*(s)$ satisfies the equation

$$\theta^*(s + r) = \theta^*(s) + \theta^*(r) \quad (s>0, \ r>0).$$

Since $\theta^*(s)$ is continuous for $s>0$, we find by a well-known theorem (which is in fact easy to prove) that $\theta^*(s) = As$ ($s>0$) for some real constant $A$, so that $\theta(s) = \theta^*(s) - c = As - c$. But, in view of (2), we may take $c=0$. Finally, the constant $A$ must be integral, since $e^{2\pi i A} = 1$. This completes the proof of the lemma.

**Lemma 2.** We have

$$f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s) \Gamma(r)} \int_{0}^{x} f_s(y) f_r(x-y) dy \quad (s>0, \ r>0).$$

**Proof.** Put

$$f(x) = \int_{0}^{x} f_s(y) f_r(x-y) dy .$$

Apparently, $f(x) = 0$ for $x<0$ and $f(x)$ is continuous for $x>0$. For $0 < x \leq 1$ we have

$$f(x) = x^{s+r-1} \int_{0}^{1} z^{s-1} (1-z)^{r-1} dz = f_{s+r}(x) \frac{\Gamma(s) \Gamma(r)}{\Gamma(s+r)} .$$

Suppose now that $x>1$ and write

$$xf(x) = \int_{0}^{x} yf_s(y)f_r(x-y) dy + \int_{0}^{x} f_s(y)(x-y)f_r(x-y) dy$$

$$= \int_{0}^{x} (x-y)f_s(x-y)f_r(y) dy + \int_{0}^{x} f_s(y)(x-y)f_r(x-y) dy$$

$$= I_1 + I_2 ,$$

say. We have

$$\frac{dI_1}{dx} = \int_{0}^{x} \left( (x-y)f'_s(x-y) + f_s(x-y) \right) f_r(y) dy$$

$$= \int_{0}^{x} (x-y)f'_s(x-y)f_r(y) dy + \int_{0}^{x} f_s(x-y)f_r(y) dy$$

$$= (s-1)f(x) - sf(x-1) + f(x)$$

$$= sf(x) - sf(x-1) ,$$
and, by symmetry,
\[ \frac{dI_s}{dx} = rf(x) - rf(x-1).\]

Since \((xf(x))' = xf'(x) + f(x),\) we thus obtain
\[ xf'(x) = (s + r - 1)f(x) - (s + r)f(x-1).\]

Hence the function \( \frac{\Gamma(s + r)}{\Gamma(s) \Gamma(r)} f(x) \) satisfies all the conditions \((i)-(iv)\) with \(s + r\) in place of \(s,\) and, since these conditions uniquely determine the function \(f_{s+r}(x),\) it follows that
\[ f_{s+r}(x) = \frac{\Gamma(s + r)}{\Gamma(s) \Gamma(r)} f(x), \]
which is the required result.

**Remark.** The substance of Lemma 2 is a particular case of a slightly more general result. Thus, let \(h(x) = h(x; c)\) be a function defined by the following conditions, \(c=(c_0, c_1, \ldots, c_n)\) being an \((n+1)\)-tuple of constants \((n \geq 0\) fixed):

1. \(h(x) = 0\) for \(x < 0,\)
2. \(h(x)\) is continuous for \(x > 0,\)
3. \(\lim_{x \to 0} xh(x) = 0,\)
4. \(xh'(x) = (c_0 - 1)h(x) + \sum_{j=1}^{n} c_j h(x-j)\) for all \(x > 0,\)
   \[ x \neq m (1 \leq m \leq n).\]

(Obviously the above conditions for \(h(x) = h(x; c)\) imply that \(c_0 > 0\) and \(h(x) = Bx^{c_0-1}\) for \(0 < x \leq 1\) with \(B\) a constant and, for \(x > 1,\) \(h(x)\) is uniquely determined once \(B\) is fixed. We shall be concerned with those functions \(h(x)\) only which are not identically zero.) Then we have
\[ h(x; a+b) = K \int_{0}^{x} h(y; a) h(x-y; b) \, dy \quad (x \neq 0), \]
where \(K\) is a constant and where we set \(a + b = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n)\) if
\[ a = (a_0, a_1, \ldots, a_n) \quad \text{and} \quad b = (b_0, b_1, \ldots, b_n). \]

2. **The Explicit Formula.** We now consider the Laplace transform of \(f_s(x),\)
$F_{s}(\xi) = \int_{0}^{\infty} e^{-\xi x} f_{s}(x) \, dx$,

where $\xi$ is a complex variable. The integral defining $F_{s}(\xi)$ is absolutely convergent on the line $\text{Re} \, \xi = 0$ (cf. § 3 below). Also, $F_{s}(\xi)$ is, as a function of $s$, continuous for $s > 0$, $\xi$ ($\text{Re} \, \xi \geq 0$) being fixed.

In view of Lemma 2 we have

$$F_{s+r}(\xi) = \frac{\Gamma(s+r)}{\Gamma(s) \Gamma(r)} F_{s}(\xi) F_{r}(\xi) \quad (s > 0, \, r > 0)$$

or

$$\frac{F_{s+r}(\xi)}{\Gamma(s+r)} = \frac{F_{s}(\xi)}{\Gamma(s)} \frac{F_{r}(\xi)}{\Gamma(r)} \quad (s > 0, \, r > 0).$$

Hence, by applying Lemma 1 to $\phi(s) = F_{s}(\xi)/\Gamma(s)$, we get for $s > 0$

$$\frac{F_{s}(\xi)}{\Gamma(s)} = e^{\pi i A s} \left( \frac{F_{1}(\xi)}{\Gamma(1)} \right)^{s}$$

or

$$F_{s}(\xi) = \Gamma(s) \left( F_{1}(\xi) \right)^{s},$$

the constant $A$ being necessarily zero since for any $s > 0$ $F_{s}(\xi)$ has a positive real value for real $\xi \geq 0$.

We see from the explicit formula for $f_{1}(x)$ ([2; § 1]) that

$$F_{1}(\xi) = e^{x} \exp \left( \int_{0}^{-\xi} \frac{e^{z} - 1}{z} \, dz \right).$$

Therefore,

$$F_{s}(\xi) = \Gamma(s) e^{Cs} \exp \left( s \int_{0}^{-\xi} \frac{e^{z} - 1}{z} \, dz \right) \quad (s > 0).$$

By a standard inversion formula for the Laplace transform, we thus obtain the following result.

**Theorem.** We have for $s > 0$

$$f_{s}(x) = \lim_{T \to \infty} \frac{\Gamma(s) e^{Cs}}{2\pi i} \int_{-iT}^{iT} \exp \left( -x t + s \int_{0}^{t} \frac{e^{z} - 1}{z} \, dz \right) \, dt \quad (x \neq 0).$$

3. **Notes.** 1) We note that for $s = 1$ the right-hand side of the equality in the theorem is equal to $\frac{1}{2}$ at $x = 0$ and for $s > 1$ it is equal to 0 at $x = 0$. Also, if $s = 1$ then we have
for all \( x \neq 0 \), and if \( s > 1 \) then

\[
f_s(x) = \frac{\Gamma(s)e^{\xi s}}{2\pi i} \int_{-i\infty}^{i\infty} \exp(-xt+s\int_{0}^{t} \frac{e^{t}-1}{z} \, dz) \, dt
\]

for all \( x, -\infty < x < \infty \), \( f_s(0) \) being defined to be equal to 0.

2) de Bruijn and van Lint [3; I] have also considered the function \( g_s(x) \) \((s \geq 0)\) defined by the conditions:

(i) \( g_s(x) = 0 \) for \( x < 0 \),
(ii) \( g_s(x) \) is continuous for \( x \geq 0 \),
(iii) \( g_s(x) = x^s \) for \( 0 \leq x \leq 1 \),
(iv) \( xg'_s(x) = sg_s(x) - sg_s(x-1) \) for \( x > 1 \).

As is noted in [3; II, §2], we have for \( s > 0 \)

\[
g_s(x) = s \int_0^x f_s(y) \, dy.
\]

They showed in [3; I, §2] that if \( s = 0 \) then \( g_s(x) = g_0(x) = 1 \) for all \( x > 0 \) and if \( s > 0 \) then \( g_s(x) \) is a positive, monotone increasing function of \( x \) for \( x > 0 \). It is also proved there that we have

\[
\lim_{x \to \infty} g_s(x) = \Gamma(s+1)e^{\xi s} \quad (s > 0)
\]

and this implies at once that

\[
\int_0^\infty f_s(x) \, dx = \Gamma(s)e^{\xi s} \quad (s > 0),
\]

which clearly agrees with (3). And in the course of its proof they found a formula which is essentially the same as (3). (In fact, using the relation between \( f_s(x) \) and \( g_s(s) \), we can show that \( F_s(\xi) \) \((s > 0)\) satisfies as a function of \( \xi \) the differential equation

\[
\xi F'_s(\xi) = s(e^{-\xi} - 1)F_s(\xi),
\]

and, by integrating this equation, we get the formula (3).) Thus, our main interest of this note is in deriving the explicit formula for \( f_s(x) \) from a somewhat different point of view, that is, on the basis of Lemma 2 which shows an interesting interrelation existent among the functions \( f_s(x) \) \((s > 0)\).
References


Department of Mathematics, Hokkaidō University

(Received August 10, 1965)