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ON A DIFFERENTIAL-DIFFERENCE EQUATION

By

Saburō UCHIYAMA

In connexion with the study of certain incomplete sums of multiplicative functions, N. G. de Bruijn and J. H. van Lint [3] have introduced the function $f_s(x) (s \geq 0)$ satisfying the set of conditions:

(i) $f_s(x) = 0$ for $x < 0$,
(ii) $f_s(x)$ is continuous for $x > 0$,
(iii) $f_s(x) = x^{s-1}$ for $0 < x \leq 1$,
(iv) $xf_s'(x) = (s-1)f_s(x) - sf_s(x-1)$ for $x > 1$.

(The function $f_s(x)$ is originally defined in [3; II, §2] only for $x > 0$; it will be convenient, however, to define $f_s(x) = 0$ for $x < 0$ for our purpose.)

On the other hand, N. G. de Bruijn [1 and 2] has investigated in detail the property and behaviour of $f_s(x)$ for $s = 1$. In particular, there he obtained an explicit formula for $f_1(x)$:

$$f_1(x) = \frac{e^C}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left(-xt + \int_0^t \frac{e^z - 1}{z} dz\right) dt \quad (x > 0),$$

where $C$ is Euler's constant,

$$C = \lim_{n \to \infty} \left( \sum_{m=1}^n \frac{1}{m} - \log n \right).$$

In the present note we shall prove an analogous formula for $f_s(x)$ with general $s > 0$.

Remark. For $s = 0$ it is easy to see that $f_s(x) = f_0(x) = x^{-1} (x > 0)$. We may suppose, therefore, that $s > 0$ throughout in the following.

1. Lemmata. We require two lemmas independent of one another.

Lemma 1. If $\phi(s)$ is a (complex valued) continuous function defined for $s > 0$ and satisfying the functional equation

$$\phi(s+r) = \phi(s) \phi(r) \quad (s > 0, \ r > 0),$$

then there is an integer $A$ independent of $s$ such that

$$\phi(s) = e^{2\pi i A s} \left( \phi(1) \right)^s \quad (s > 0).$$
Proof. We may assume without loss of generality that \( \phi(s) \) does not vanish for \( s > 0 \) and hence that \( \phi(1) \neq 0 \). Consider the continuous function

\[
\phi(s) = \frac{\phi(s)}{(\phi(1))^s} \quad (s > 0),
\]

where \( z^s = \exp(s \log z) \) and the branch of \( \log z \) is taken in such a way that \( \log z \) is real for real \( z > 0 \). We have for any \( s > 0 \)

\[
\phi(s + 1) - \frac{\phi(s + 1)}{(\phi(1))^{s+1}} = \frac{\phi(s)}{(\phi(1))^s} = \psi(s).
\]

Thus, if we put

\[
\alpha(s) = \frac{1}{1 + |\log |\psi(s)||},
\]

then

\[
(1) \quad \int_0^1 \alpha(s) \, ds = \int_0^1 \alpha(2s) \, ds.
\]

Indeed, we have

\[
\int_0^1 \alpha(s) \, ds = \int_0^1 \alpha(s + 1) \, ds = \int_0^1 \alpha(s) \, ds = 2 \int_{1/2}^1 \alpha(2s) \, ds = 2 \int_0^1 \alpha(2s) \, ds - \int_0^1 \alpha(s) \, ds,
\]

which is equivalent to (1). Since \( \phi(2s) = (\phi(s))^2 \), we deduce from (1) that

\[
\int_0^1 \frac{|\log |\psi(s)||}{(1 + |\log |\psi(s)||)(1 + 2|\log |\psi(s)||)} \, ds = 0,
\]

and this implies that \( \log |\psi(s)| = 0 \) almost everywhere on \( (0, 1) \). It follows that \( |\phi(s)| = 1 \) everywhere on \( (0, \infty) \). This means that, if we set

\[
(2) \quad \frac{\phi(s)}{(\phi(1))^s} = \theta(s),
\]

then \( \theta(s) \) is a real valued continuous function of \( s > 0 \) satisfying the congruence

\[
\theta(s + r) \equiv \theta(s) + \theta(r) \pmod{1} \quad (s > 0, r > 0).
\]

Hence, there is a constant \( c \equiv 0 \pmod{1} \) such that

\[
\theta(s + r) = \theta(s) + \theta(r) + c \quad (s > 0, r > 0),
\]

and it follows from this that the limit

\[
\lim_{s \to +0} \theta(s) = -c.
\]
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exists. Thus, if we put

\[ \theta^*(s) = \theta(s) + c, \]

then \( \theta^*(s) \) satisfies the equation

\[ \theta^*(s + r) = \theta^*(s) + \theta^*(r) \quad (s > 0, \ r > 0). \]

Since \( \theta^*(s) \) is continuous for \( s > 0 \), we find by a well-known theorem (which is in fact easy to prove) that \( \theta^*(s) = A s \quad (s > 0) \) for some real constant \( A \), so that \( \theta(s) = \theta^*(s) - c = A s - c \). But, in view of (2), we may take \( c = 0 \). Finally, the constant \( A \) must be integral, since \( e^{2\pi i A} = 1 \). This completes the proof of the lemma.

Lemma 2. We have

\[ f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s) \Gamma(r)} \int_{0}^{x} f_s(y) f_r(x-y) dy \quad (s > 0, \ r > 0). \]

Proof. Put

\[ f(x) = \int_{0}^{x} f_s(y) f_r(x-y) dy. \]

Apparently, \( f(x) = 0 \) for \( x < 0 \) and \( f(x) \) is continuous for \( x > 0 \). For \( 0 < x \leq 1 \) we have

\[ f(x) = x^{s+r-1} \int_{0}^{1} z^{s-1}(1-z)^{r-1} dz = f_{s+r}(x) \frac{\Gamma(s) \Gamma(r)}{\Gamma(s+r)}. \]

Suppose now that \( x > 1 \) and write

\[ xf(x) = \int_{0}^{x} y f_s(y) f_r(x-y) dy + \int_{0}^{x} f_s(y) (x-y) f_r(x-y) dy \\
= \int_{0}^{x} (x-y) f_s(x-y) f_r(y) dy + \int_{0}^{x} f_s(y) (x-y) f_r(x-y) dy = I_1 + I_2, \]

say. We have

\[ \frac{dI_1}{dx} = \int_{0}^{x} ((x-y) f'_s(x-y) + f_s(x-y)) f_r(y) dy \\
= \int_{0}^{x} (x-y) f'_s(x-y) f_r(y) dy + \int_{0}^{x} f_s(x-y) f_r(y) dy = (s-1)f(x) - sf(x-1) + f(x) = sf(x) - sf(x-1), \]
and, by symmetry,

\[ \frac{dI_2}{dx} = rf(x) - rf(x-1). \]

Since \((xf(x))' = xf'(x) + f(x)\), we thus obtain

\[ xf'(x) = (s + r - 1)f(x) - (s + r)f(x-1). \]

Hence the function \( \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} f(x) \) satisfies all the conditions (i)—(iv) with \( s + r \) in place of \( s \), and, since these conditions uniquely determine the function \( f_{s+r}(x) \), it follows that

\[ f_{s+r}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} f(x), \]

which is the required result.

Remark. The substance of Lemma 2 is a particular case of a slightly more general result. Thus, let \( h(x) = h(x; c) \) be a function defined by the following conditions, \( c = (c_0, c_1, \ldots, c_n) \) being an \((n+1)\)-tuple of constants \((n \geq 0 \) fixed):

(i) \( h(x) = 0 \) for \( x < 0 \),

(ii) \( h(x) \) is continuous for \( x > 0 \),

(iii) \( \lim_{x \to 0} x h(x) = 0 \),

(iv) \( x h'(x) = (c_0 - 1) h(x) + \sum_{j=1}^{n} c_j h(x-j) \) for all \( x > 0 \), for \( x \neq m \) \((1 \leq m \leq n)\).

(Obviously the above conditions for \( h(x) = h(x; c) \) imply that \( c_0 > 0 \) and \( h(x) = B x^{c_0-1} \) for \( 0 < x \leq 1 \) with \( B \) a constant and, for \( x > 1 \), \( h(x) \) is uniquely determined once \( B \) is fixed. We shall be concerned with those functions \( h(x) \) only which are not identically zero.) Then we have

\[ h(x; a+b) = K \int_{0}^{x} h(y; a) h(x-y; b) \, dy \quad (x \neq 0), \]

where \( K \) is a constant and where we set

\[ a + b = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n) \]

if

\[ a = (a_0, a_1, \ldots, a_n) \quad \text{and} \quad b = (b_0, b_1, \ldots, b_n). \]

2. The Explicit Formula. We now consider the Laplace transform of \( f_s(x) \),

\[ f_s(x).

\]
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\[ F_s(\xi) = \int_0^\infty e^{-tx} f_s(x) \, dx , \]

where \( \xi \) is a complex variable. The integral defining \( F_s(\xi) \) is absolutely convergent on the line \( \text{Re} \, \xi = 0 \) (cf. \( \S 3 \) below). Also, \( F_s(\xi) \) is, as a function of \( s \), continuous for \( s > 0 \), \( \xi (\text{Re} \, \xi \geq 0) \) being fixed.

In view of Lemma 2 we have

\[ F_{s+r}(\xi) = \frac{\Gamma(s+r)}{\Gamma(s) \Gamma(r)} F_s(\xi) F_r(\xi) \quad (s > 0, \ r > 0) \]

or

\[ \frac{F_{s+r}(\xi)}{\Gamma(s+r)} = \frac{F_s(\xi)}{\Gamma(s)} \frac{F_r(\xi)}{\Gamma(r)} \quad (s > 0, \ r > 0) . \]

Hence, by applying Lemma 1 to \( \phi(s) = F_s(\xi)/\Gamma(s) \), we get for \( s > 0 \)

\[ \frac{F_s(\xi)}{\Gamma(s)} = e^\pi i A s \left( \frac{F_1(\xi)}{\Gamma(1)} \right)^s \]

or

\[ F_s(\xi) = \Gamma(s) \left( F_1(\xi) \right)^s , \]

the constant \( A \) being necessarily zero since for any \( s > 0 \) \( F_s(\xi) \) has a positive real value for real \( \xi \geq 0 \).

We see from the explicit formula for \( f_1(x) \) ([2; \( \S 1 \)]) that

\[ F_1(\xi) = e^\xi \exp \left( \int_0^{\xi} \frac{e^z - 1}{z} \, dz \right) . \]

Therefore,

\[ F_s(\xi) = \Gamma(s) e^{c_s} \exp \left( s \int_0^{\xi} \frac{e^z - 1}{z} \, dz \right) \quad (s > 0) . \]

By a standard inversion formula for the Laplace transform, we thus obtain the following result.

**Theorem.** We have for \( s > 0 \)

\[ f_s(x) = \lim_{T \to \infty} \frac{\Gamma(s) e^{c_s}}{2\pi i} \int_{-iT}^{iT} \exp \left( -zt + s \int_0^t \frac{e^z - 1}{z} \, dz \right) dt \quad (x \neq 0) . \]

**3. Notes.**

1) We note that for \( s = 1 \) the right-hand side of the equality in the theorem is equal to \( \frac{1}{2} \) at \( x = 0 \) and for \( s > 1 \) it is equal to 0 at \( x = 0 \). Also, if \( s = 1 \) then we have
\[ f_i(x) = \frac{e^t}{2\pi i} \int_{-i\infty}^{i\infty} \exp(-xt + \int_0^t \frac{e^z - 1}{z} \, dz) \, dt \]

for all \( x \neq 0 \), and if \( s > 1 \) then

\[ f_s(x) = \frac{\Gamma(s)e^{C_s}}{2\pi i} \int_{-i\infty}^{i\infty} \exp(-xt + s\int_0^t \frac{e^z - 1}{z} \, dz) \, dt \]

for all \( x, -\infty < x < \infty \), \( f_s(0) \) being defined to be equal to 0.

2) de Bruijn and van Lint \([3; I]\) have also considered the function \( g_s(x) \) \((s \geq 0)\) defined by the conditions:

(i) \( g_s(x) = 0 \) for \( x < 0 \),
(ii) \( g_s(x) \) is continuous for \( x \geq 0 \),
(iii) \( g_s(x) = x^s \) for \( 0 \leq x \leq 1 \),
(iv) \( xg_s'(x) = sg_s(x) - sg_s(x - 1) \) for \( x > 1 \).

As is noted in \([3; II, \S 2]\), we have for \( s > 0 \)

\[ g_s(x) = s \int_0^x f_s(y) \, dy. \]

They showed in \([3; I, \S 2]\) that if \( s = 0 \) then \( g_s(x) = g_0(x) = 1 \) for all \( x > 0 \) and if \( s > 0 \) then \( g_s(x) \) is a positive, monotone increasing function of \( x \) for \( x > 0 \).

It is also proved there that we have

\[ \lim_{x \to \infty} g_s(x) = \Gamma(s + 1)e^{C_s} \quad (s > 0) \]

and this implies at once that

\[ \int_0^\infty f_s(x) \, dx = \Gamma(s)e^{C_s} \quad (s > 0), \]

which clearly agrees with (3). And in the course of its proof they found a formula which is essentially the same as (3). (In fact, using the relation between \( f_s(x) \) and \( g_s(s) \), we can show that \( F_s(\xi) \) \((s > 0)\) satisfies as a function of \( \xi \) the differential equation

\[ \xi F_s'(\xi) = s(e^{-\xi} - 1)F_s(\xi), \]

and, by integrating this equation, we get the formula (3).) Thus, our main interest of this note is in deriving the explicit formula for \( f_s(x) \) from a somewhat different point of view, that is, on the basis of Lemma 2 which shows an interesting interrelation existent among the functions \( f_s(x) \) \((s > 0)\).
References


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