NOTE ON \( q \)-GALOIS EXTENSIONS OF SIMPLE RINGS

By

Hisa\(\text{ }\)o Tominaga

In the previous paper [2], the notions of \( q \)-Galois extensions and \( h \cdot q \)-Galois extensions of simple rings were introduced with a view to unifying all the earlier works cited at the end of [2] under the assumption that the simple ring extension in question is left locally finite and \( h \cdot q \)-Galois. Recently, in his paper [1], T. Nagahara has obtained some useful \( q \)-Galois conditions and proved that any left locally finite \( q \)-Galois extension is necessarily \( h \cdot q \)-Galois ([1, Th. 8]).

The purpose of this note is to present a direct proof to the last. In fact, the sharpening of [2, Th. 1] together with the modification of the proof of [2, Cor. 1] enables us to complete our proof. It will be noteworthy that all the results are obtained without the assumption that \( V \) is simple.

As to notations and terminologies used in this note, we follow the previous paper [2]. Moreover, as in [1], by \( \mathbb{R} \) we denote the set of all regular intermediate rings of \( A/B \), and we shall use the following notations: \( \mathbb{R}^p = \{ A' \in \mathbb{R}; [A'|A'] = [A|A] \} \), \( \mathbb{R}_{i.f.} = \{ A' \in \mathbb{R}; [A':B]_l < \infty \} \), \( \mathbb{R}^0_{i.f.} = \mathbb{R}^p \cap \mathbb{R}_{i.f.} \), and for any subset \( S \) of \( A \) we set \( \mathbb{R}/S = \{ A' \in \mathbb{R}; A' \supseteq S \} \), \( \mathbb{R}^0/S = \mathbb{R}^0 \cap \mathbb{R}/S \), \( \mathbb{R}_{i.f.}/S = \mathbb{R}^0_{i.f.} \cap \mathbb{R}/S \).

At first, to our end, [2, Th. 1] (that was useful in [1], too) has to be improved as follows:

**Theorem 1.** Assume that (a simple ring) \( A \) is left locally finite over (a unital simple subring) \( B \) and \( \text{Hom}_{B_l}(T, A) = \mathfrak{G}(T, A/B) A_r \) for each \( T \in \mathbb{R}_{i.f.} \).

(a) If \( B' \) is a simple intermediate ring of \( A/B \) with \( [B':B]_l < \infty \), then for any finite subset \( F \) of \( A \) there exists \( T' \in \mathbb{R}^0_{i.f.}/B'[F] \) such that each intermediate ring \( A' \) of \( A/T' \) is \( B'-A' \)-completely reducible.

(b) If \( B' \) is in \( \mathbb{R}_{i.f.} \), then for any finite subset \( F \) of \( A \) there exists \( T' \in \mathbb{R}^0_{i.f.}/B'[F] \) such that each intermediate ring \( A' \) of \( A/T' \) is homogeneously \( B'-A' \)-completely reducible with \( [A'|B'_l \cdot A'_r] = [V_A(B')|V_A(B')] \) (and \( [V_{A'}(B): V_{A'}(B')_l]_l < \infty \)).

**Proof.** (a) Let \( M \) be a minimal \( B'-A \)-submodule of \( A \) such that the composition series of \( M \) as \( A \)-module is of the shortest length among minimal \( B'-A \)-submodules of \( A \). Then, \( M = eA \) with a non-zero idempotent \( e \). In virtue of [2, Lemma 1], we can find \( T^* \in \mathbb{R}^0_{i.f.}/B'[e, F] \) with \( [eT^*|T^*] = [M|A] \).
One may remark here that $eT^*$ is a $B'-T^*$-submodule of $T^*$, for $B'e \subseteq B'eA \cap T^* = eA \cap T^* = eT^*$. Since $\text{Hom}_{B_{r}}(T^*, A) = \mathfrak{G}(T^*, A/B)A_r = \sum \tau_i a_iA_r$ with some $\tau_i \in \mathfrak{G}(T^*, A/B)$, there exists $T' \in \mathfrak{G}_{i,f}/T^*$ such that $\text{Hom}_{B_{r}}(T^*, A') = \mathfrak{G}(T^*, A'/B)A_r'$ for each intermediate ring $A'$ of $A/T'$. By the above remark $B'eT^* = eT^*$ and [2, Lemma 1], one will easily see that $M' = eA'$ is a (minimal) $B'-A'$-submodule of $A'$ such that the length of the composition series of $M'$ as $A'$-module coincides with $[M|A'] = [eT^*|T^*]$ (and so $[M'|A'] \leq [N'|A']$ for each non-zero $B'-A'$-submodule $N'$ of $A'$). Since $\text{Hom}_{B_{r}}(T^*, A')$ is $T^* - A'$-completely reducible by [2, Lemma 2 (a)], the $T^* - A'$-submodule $\text{Hom}_{B_{r}}(T^*, A') = \oplus \mathfrak{M}_j$ with some irreducible $\mathfrak{M}_j$. By [2, Lemma 2 (b)], $\mathfrak{M}_j = \sigma_j \mu_j A_{r}'$ with some $\sigma_j \in \mathfrak{G}(T^*, A'/B)$ and non-zero $u_j \in V_{A_r}(B)$. Since $\mathfrak{M}_j$ is contained in $\text{Hom}_{B_{r}}(T^*, A')$ and $B'e \subseteq eT^* \subseteq T^*$, each $M_j = (B'e)\mathfrak{M}_j$ is a $B'-A'$-submodule of $A'$. Further, there holds $M_j = u_j(B'e)\sigma_j A' = u_j(B'eT^*)\sigma_j A' = u_j(eT^*)\sigma_j A' = u_j \mathfrak{G}(T^*, A')$ whence it follows $[M_j|A'] = [u_j \mathfrak{G}(T^*, A')|A'] \leq [\mathfrak{G}(T^*, A')|A'] = [\mathfrak{G}(T^*, A')|A']$ by [2, Lemma 1]. Recalling here that $[M'|A']$ is the least, the we see that each $M_j$ is either $0$ or $B'-A'$-irreducible. Finally, noting that $A'$ is $B_{r}' \text{Hom}_{B_{r}}(A', A')$-irreducible, there holds $A' = e(B_{r}' \text{Hom}_{B_{r}}(A', A')) = (B'e)\text{Hom}_{B_{r}}(T^*, A') = (B'e)\sum \mathfrak{M}_j = \sum \mathfrak{M}_j$, which proves evidently the complete reducibility of $A'$ as $B'-A'$-module.

(b) Let $V' = V_{A}(B') = \sum U'g'_{pq}$, where $\tau' = \{g'_{pq}\}$ is a system of matrix units and $U' = V_{V'}(\tau')$ a division ring. Let $M = eA$ be chosen as in (a), and $T^*$ a member of $\mathfrak{G}_{i,f}/B'[e, F, T']$ such that $[eT^*|T^*] = [M|A]$]. Then, there exists $T' \in \mathfrak{G}_{i,f}/T^*$ such that $\text{Hom}_{B_{r}}(T^*, A') = \mathfrak{G}(T^*, A'/B)A_{r}'$ for each intermediate ring $A'$ of $A/T'$. Since $\text{Hom}_{B_{r}}(T^*, A')$ coincides with the simple ring $(V_{A}(B'))$, the proof of (a) enables us to see that $A'$ is homogeneously $B'-A'$-completely reducible and $[A'|B'; A_{r}] = [V_{A}(B')|V_{A}(B')] = [V'|V']$. Accordingly, $A'$ is $B' \cdot V_{A}(B')-A'$-irreducible, and hence $[V_{A}(B') : V_{A}(B')]_{r} \leq [B : B_{r}] < \infty$ by [2, Prop. 1 (b)].

**Lemma 1.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_{r}}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in \mathfrak{G}_{i,f}$.

(a) If $\rho$ is a $B$-ring homomorphism of an intermediate ring $A_r$ of $A/B$ with $[A_{r}:B_{r}] < \infty$ into $A$ such that $A$ is $A_{r}\rho$-irreducible, then $\rho$ is contained in $\mathfrak{G}(A_{r}, A/B)A_r$ for each $A_{r} \in \mathfrak{G}_{i,f}/A_{r}$.

(b) If $B'$ is in $\mathfrak{G}_{i,f}$, then $\mathfrak{G}(T, A/B)B' \subseteq \mathfrak{G}(B', A/B)$ for each $T \in \mathfrak{G}_{i,f}/B'$, and so $\text{Hom}_{B_{r}}(B', A) = \mathfrak{G}(B', A/B)A_r$.

**Proof.** The proof of (a) will be obvious by that of [2, Lemma 4]. In what follows, we shall prove (b). Let $V' = V_{A}(B') = \sum U'g'_{pq}$, where $\tau' = \{g'_{pq}\}$ is a system of matrix units and $U' = V_{V'}(\tau')$ a division ring. We set $\overline{B} = B'[\tau']$.
and $T = T[I^n]$. Now, let $\sigma$ be an arbitrary element of $B(T, A/B)$. By (a), $\sigma = \overline{\sigma}|T$ for some $\sigma \in B(T, A/B)$. Obviously, $V_{\sigma}(B') = V_{\sigma}(B')$ contains $T^* \overline{\sigma}$ as a system of matrix units and the centralizer of $T^* \overline{\sigma}$ in $A(B')$ coincides with $V_{\sigma}(B')$. If $a$ is an arbitrary non-zero element of $V_{\sigma}(B')$ and $T^* = (T \overline{\sigma})$ then $\sigma^{-1} = \tau^* \overline{\sigma}$ for some $\tau^* \in B(T, A/B)$ again by (a). For any $x \in B$, the equality $x \sigma \cdot a = a \cdot x \sigma$ yields $x \cdot \alpha \sigma = \alpha \sigma \cdot x$. Hence, $a \tau^*$ is contained in the division ring $V_{\sigma}(B) = U'$. Accordingly, $a \tau^*$ is a regular element of $T^* \overline{\sigma}$, and so $a$ is a regular element. We have proved thus $V_{\sigma}(B')$ is a division ring, which implies that $V_{\sigma}(B') = \sum V_{\sigma}(B') \cdot \varphi \sigma$ is a simple ring.

**Theorem 2 ([1, Th. 2]).** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = B(T, A/B)A_r$ for each $T \in R_{i.f}$. If $B_i \supseteq B_j$ are in $R_{i.f}$, then $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)A_r$. If $B_i \supseteq B_j$ are in $R_{i.f}$, then $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)A_r$. Hence, $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)$.

**Proof.** By Lemma 1 (b), the argument in the proof of [2, Th. 3] applies to obtain $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)$. If $T$ is an arbitrary member of $R_{i.f}/B_i$, it follows obviously $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)$. Accordingly, again by Lemma 1 (b), we obtain $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)$ for each $B_i \supseteq B_j$ in $R_{i.f}$. However, Th. 2 implies that $A/B$ is left locally finite and $q$-Galois if and only if $B$ is regular, $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = B(T, A/B)A_r$ for each $T \in R_{i.f}$. This fact has been pointed out in [1]. Moreover, [1, Th. 8] asserts that if $A/B$ is left locally finite and $q$-Galois then $A/B$ is $h$-Galois. This fact is obviously contained in the following theorem.

**Theorem 3.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = B(T, A/B)A_r$ for each $T \in R_{i.f}$. If $B'$ is in $R_{i.f}$, then $\text{Hom}_{B_i}(B', A) = B(T, A/B)A_r$ for each $B' \in R_{i.f}/B'$. 

**Proof.** Let $T$ be an arbitrary member of $R_{i.f}/B'$. Evidently, $\text{Hom}_{B_i}(T, A)$ is a $T_r - A_r$-submodule of $\text{Hom}_{B_i}(T, A) = B(T, A/B)A_r$ and then $\text{Hom}_{B_i}(T, A) = B(T, A/B)A_r$ with some $\sigma \in B(T, A/B)$ and non-zero $u \in V$. By Th. 1 (b), there exists $T' \in R_{i.f}/B'$ such that each intermediate ring $A'$ of $A/T'$ is homogeneously $B' - A'$-completely reducible with $A'|B'|A' = (V_{A'}(B')|V_{A'}(B'))$ and $\text{Hom}_{B_i}(T, A') = B(T, A')A_r$. Now, let $\sigma u \in B(T, A/r)A_r$ be an arbitrary $\sigma u \in B(T, A/r)A_r$. By Th. 2, there exists $\sigma \in B(T, A/B)$ such that $\sigma = \sigma|T'$. If $A'' = (T'|A'T')$, then $\sigma = \tau''|T'|A''|T''$ with some $\tau'' \in B(T'', A/B)$ again by Th. 2. We set here $A'' = A''|T''$ (in $R_{i.f}/T''$) and $\sigma'' = \tau''$. Let $M_o$ be a minimal $B' - A'$-submodule of $A'$ such that $N_o = M_o \sigma u$ is non-zero (and so $B'$-
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Let $A''$-irreducible. Obviously, $A' = M_0 \oplus M_1 \oplus \cdots \oplus M_t$ and $A'' = N_0 \oplus N_1 \oplus \cdots \oplus N_t$ with some $B'$-isomorphic $M_i$'s and $B''$-$A''$-isomorphic $N_i$'s, where $t + 1 = [V_d(B')]_t [V_d(B')]$. Then, we can easily find such a $B'$-isomorphism $\nu$ of $A'$ onto $A''$ that $a' = \nu(a')$, for each $a' \in A'$. As $\nu^{-1}(a')$ is contained in $\text{Hom}_{B_i}(A', A'') = (V_{A'}(B'))_t$ (simple), $a' u_i = \nu u_i + \cdots + \nu u_{mi}$ with some regular elements $v_j$'s in $V_{A'}(B')$. Noting that $T$ is in $\mathfrak{R}_{l.f}$, one will easily see that every $(\nu v_{jl}|T)A_r$ is a $T,-A''$-irreducible submodule of $\text{Hom}_{B_j}(T, A'' \subseteq \text{Hom}_{B_i}(T, A'') = \mathfrak{G}(T, A''|B)A_r$. It follows therefore $(\nu v_{jl}|T)A'' = \tau w_{jl} A''$ with some $\tau \in \mathfrak{G}(T, A''|B)$ and non-zero $w_j \in V_{A''}(B)$. We have then $A'' = w_j A'' = v_j A' = \nu_{jl}(T \cdot A' \nu = v_j \cdot T \cdot A'' = T(\nu v_{jl}|T)A'' = w_j \cdot T \cdot A'' = w_j A''$, whence it follows that $w_j$ is a regular element of $V_{A''}(B)$. Hence, $\tau w_j$ is contained in $\mathfrak{G}(T, A''|B) \cap \text{Hom}_{B_j}(T, A''|B')$. We have proved thus $\sigma u_i = \sigma' u_i|T$ is contained in $\mathfrak{G}(T, A'/B)A_r$, and so $\text{Hom}_{B_j}(T, A) = \mathfrak{G}(T, A'/B)A_r$. Then, our assertion is a consequence of Lemma 1 (b).

Now, [2, Cor. 1] can be restated as follows:

\textbf{Corollary 1.} Let $A$ be left locally finite over $B$, and $\mathfrak{S}$ a subset of $\mathfrak{S}$ with $\mathfrak{S}V = \mathfrak{S}$. If $\mathfrak{S}A_r$ is dense in $\text{Hom}_{B_j}(A, A)$ then $\mathfrak{S}(B', A/B) = \mathfrak{S}|B'$, $\mathfrak{S}(B')A_r$ is dense in $\text{Hom}_{B_j}(A, A)$ and $T(\mathfrak{S}(B'), A) = B'$ for each $B' \in \mathfrak{R}_{l.f}$.

Proof. If $T$ is in $\mathfrak{R}_{l.f}$, then $\mathfrak{S}(T, A/B) \subseteq \text{Hom}_{B_j}(T, A) = (\mathfrak{S}|T)A_r$ yields $\text{Hom}_{B_j}(T, A) = \mathfrak{S}(T, A/B)A_r$ and $\mathfrak{S}(T, A/B) = \mathfrak{S}|T$ [2, Lemma 2 (b)]. Accordingly, for any $A' \in \mathfrak{R}_{l.f}/B'$ there holds $\mathfrak{S}(B', A/B) = \mathfrak{S}(A', A/B)|B' = (\mathfrak{S}|A')|B' = \mathfrak{S}|B'$ (Th. 2). Now, our assertion will be easy by Ths. 2, 3 and [2, Remark 1].

Let $B'$ be an intermediate ring of $A/B$ such that $A$ is $B''$-irreducible and $[B':B]<\infty$. Then $A = B' V A = \sum a \in A'$ for an arbitrary minimal left ideal $V A$ of $B'$, which means that $A$ is homogeneously $B'$-completely reducible. Hence, $B'$ is also homogeneously $B'$-completely reducible, and so $B'$ is contained in $\mathfrak{R}_{l.f}$. Combining this fact with [4, Lemma 1.1], we readily see that if $A$ is left locally finite over $B$ and $B$-$A$-irreducible then every intermediate ring of $A/B$ in $\mathfrak{R}$.

The first assertion of the following corollary is a direct consequence of Th. 1 (b), [2, Remark 4] and the last remark. Moreover, Th. 3 and [2, Remark 1] yield the second one.

\textbf{Corollary 2} (cf. [1, Ths. 10, 7 and Lemma 16]). Assume that $A/B$ is left locally finite and $\text{Hom}_{B_j}(T, A) = \mathfrak{S}(T, A/B)A_r$ for each $T \in \mathfrak{R}_{l.f}$.

(a) If $V$ is a division ring then every intermediate ring of $A/B$ is simple, and conversely.

(b) If $B'$ is in $\mathfrak{R}_{l.f}$ then $J(\mathfrak{S}(B'', A/B'), B'') = B'$ for each $B'' \in \mathfrak{R}_{l.f}/B$. In particular, if $B'$ is an intermediate ring of $A/B$ such that $A$ is $B''$-$A$-irreducible and $[B':B]<\infty$ then $J(\mathfrak{S}(B'', A/B'), B'') = B'$ for each $B'' \in \mathfrak{R}_{l.f}/B$. 
Patterning after the proof of [3, Th. 1], one will readily see that [3, Th. 1] is still valid under the same assumption as in Th. 3. Moreover, taking the validity of Th. 3 into the mind, it will be obvious that the proof of [3, Th. 2] is efficient even under the same assumption as in Th. 3. Thus, we obtain the following:

Theorem 4. Assume that \( A/B \) is left locally finite and \( \text{Hom}_{B_{l}}(T, A) = \mathfrak{G}(T, A/B)A_{r} \) for each \( T \in \Re_{l.f}^{0} \). Let \( B' \) be a simple intermediate ring of \( A/B \), and \( \delta \) a derivation of \( B' \) into \( A \) vanishing on \( B \). If \( [B' : B]_{l} < \infty \) or \( B' \) is \( f \)-regular then \( \delta \) can be extended to an inner derivation of \( A \).

References


Department of Mathematics, Hokkaido University

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