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NOTE ON \textit{q-Galois Extensions of Simple Rings}

By

Hisao TOMINAGA

In the previous paper [2], the notions of \textit{q-Galois} extensions and \textit{h-q-Galois} extensions of simple rings were introduced with a view to unifying all the earlier works cited at the end of [2] under the assumption that the simple ring extension in question is left locally finite and \textit{h-q-Galois}. Recently, in his paper [1], T. Nagahara has obtained some useful \textit{q-Galois} conditions and proved that any left locally finite \textit{q-Galois} extension is necessarily \textit{h-q-Galois} ([1, Th. 8]).

The purpose of this note is to present a direct proof to the last. In fact, the sharpening of [2, Th. 1] together with the modification of the proof of [2, Cor. 1] enables us to complete our proof. It will be noteworthy that all the results are obtained without the assumption that \(V\) is simple.

As to notations and terminologies used in this note, we follow the previous paper [2]. Moreover, as in [1], by \(\Re\) we denote the set of all regular intermediate rings of \(A/B\), and we shall use the following notations: \(\Re^{0} = \{A'\in\Re; [A'|A'] = [A|A]\}\), \(\Re_{l.f.} = \{A'\in\Re; [A':B]_{l}<\infty\}\), \(\Re_{l.f.}^{0} = \Re^{0}\cap\Re_{l.f.}\). and for any subset \(S\) of \(A\) we set \(\Re/S = \{A'\in\Re; A' \supseteq S\}\), \(\Re^{0}/S = \Re^{0}\cap\Re/S\), \(\Re_{l.f.}/S = \Re_{l.f.}\cap\Re/S\) and \(\Re_{l.f.}^{0}/S = \Re_{l.f.}^{0}\cap\Re/S\).

At first, to our end, [2, Th. 1] (that was useful in [1], too) has to be improved as follows:

\textbf{Theorem 1.} Assume that (a simple ring) \(A\) is left locally finite over (a unital simple subring) \(B\) and \(\text{Hom}_{B_{t}}(T, A) = \emptyset\) (\(T, A/B\)) \(A_{r}\) for each \(T\in\Re_{l.f.}\).

(a) If \(B'\) is a simple intermediate ring of \(A/B\) with \([B':B]_{l}<\infty\), then for any finite subset \(F\) of \(A\) there exists \(T\in\Re_{l.f.}/B'[F]\) such that each intermediate ring \(A'\) of \(A/T\) is \(B'\)-\(A'\)-completely reducible.

(b) If \(B'\) is in \(\Re_{l.f.}\), then for any finite subset \(F\) of \(A\) there exists \(T\in\Re_{l.f.}/B'[F]\) such that each intermediate ring \(A'\) of \(A/T\) is homogeneously \(B'\)-\(A'\)-completely reducible with \([A'|B';A'] = [V_{A}(B')\mid V_{A}(B')]\) (and \([V_{A'}(B):V_{A'}(B')]<\infty\)).

\textbf{Proof.} (a) Let \(M\) be a minimal \(B'\)-\(A\)-submodule of \(A\) such that the composition series of \(M\) as \(A\)-module is of the shortest length among minimal \(B'\)-\(A\)-submodules of \(A\). Then, \(M = eA\) with a non-zero idempotent \(e\). In virtue of [2, Lemma 1], we can find \(T^{*}\in\Re_{l.f.}/B'[e, F]\) with \([eT^{*}\mid T^{*}] = [M|A]\).
One may remark here that $eT^*$ is a $B^*-T^*$-submodule of $T^*$, for $B'e \subseteq B'e A \cap T^* = eA \cap T^* = eT^*$. Since $\text{Hom}_{B_1}(T^*, A) = \mathfrak{G}(T^*, A,B_1) = \sum \tau_\ell A_\ell$ with some $\tau_\ell \in \mathfrak{G}(T^*, A,B_1)$, there exists $T^* \in \mathfrak{R}_{i,f}/T^*$ such that $\text{Hom}_{B_1}(T^*, A') = \mathfrak{G}(T^*, A'/B)A_{r}'$ for each intermediate ring $A'$ of $A/T'$. By the above remark $B'eT^* = eT^*$ and [2, Lemma 1], one will easily see that $M'=eA'$ is a (minimal) $B'-A'$-submodule of $A'$ such that the length of the composition series of $M'$ as $A'$-module coincides with $[M|A']=[eT^*|T^*]$ (and so $[M'|A'] \leq [N'|A']$ for each non-zero $B'-A'$-submodule $N'$ of $A'$). Since $\text{Hom}_{B_1}(T^*, A')$ is $T^*-A'$-completely reducible by [2, Lemma 2 (a)], the $T^*-A'$-submodule $\text{Hom}_{B_1}(T^*, A') = \oplus \mathfrak{M}_j$ with some irreducible $\mathfrak{M}_j$'s. By [2, Lemma 2 (b)], $\mathfrak{M}_j = \sigma_j u_j A'$ with some $\sigma_j \in \mathfrak{G}(T^*, A'/B)$ and non-zero $u_j \in V_{A'}(B)$. Since $\mathfrak{M}_j$ is contained in $\text{Hom}_{B_1}(T^*, A')$ and $B'e \subseteq eT^* \subseteq T^*$, each $M_j = (B'e)\mathfrak{M}_j$ is a $B'-A'$-submodule of $A'$. Further, there holds $M_j = u_j (B'e)\sigma_j A' = u_j (B'eT^*)\sigma_j A' = u_j e \sigma_j A'$, whence it follows $[M_j|A'] = [u_j e \sigma_j A'|A'] \leq [e \sigma_j A'|A'] \leq [e \sigma_j A'|A'] = [e \sigma_j A'|A'] = [eT^*|T^*] = [M'|A']$ by [2, Lemma 1]. Recalling here that $[M'|A']$ is the least, we see that each $M_j$ is either 0 or $B'-A'$-irreducible. Finally, noting that $A'$ is $B_1 \cdot \text{Hom}_{B_1}(A', A')$-irreducible, there holds $A' = B \cdot \text{Hom}_{B_1}(A', A') = (B'e) \text{Hom}_{B_1}(T^*, A') = (B'e) \mathfrak{M}_j = \mathfrak{M}_j$, which proves evidently the complete reducibility of $A'$ as $B'-A'$-module.

(b) Let $V' = V_{A'}(B') = \sum U' g'_{pq}$, where $I' = \{g'_{pq}\}$ is a system of matrix units and $U' = V_{A'}(I')$ a division ring. Let $M = eA$ be chosen as in (a), and $T^*$ a member of $\mathfrak{R}_{i,f}/B'[e,F,I']$ such that $[eT^*|T^*] = [M|A]$. Then, there exists $T^* \in \mathfrak{R}_{i,f}/T^*$ such that $\text{Hom}_{B_1}(T^*, A') = \mathfrak{G}(T^*, A'/B)A_{r}'$ for each intermediate ring $A'$ of $A/T'$. Since $\text{Hom}_{B_1}(T^*, A')$ coincides with the simple ring $(V_{A'}(B'))$, the proof of (a) enables us to see that $A'$ is homogeneously $B'-A'$-completely reducible and $[A'|B'; A'] = [V_{A}(B')|V_{A}(B')] = [V'|V']$. Accordingly, $A'$ is $B' \cdot V_{A'}(B')^{-1} A'$-irreducible, and hence $[V_{A}(B): V_{A}(B')]_{r} \leq [B': B], < \infty$ by [2, Prop. 1 (b)].

**Lemma 1.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_1}(T,A) = \mathfrak{G}(T,A/B)A_{r}$ for each $T \in \mathfrak{R}_{i,f}$. 

(a) If $\rho$ is a $B$-ring homomorphism of an intermediate ring $A$, of $A/B$ with $[A_{1}, B], < \infty$ into $A$ such that $A$ is $A_{1}\rho$-irreducible, then $\rho$ is contained in $\mathfrak{G}(A_{1}, A/B)A_{r}$ for each $A_{1} \in \mathfrak{R}_{i,f}/A_{1}$.

(b) If $B'$ is in $\mathfrak{R}_{i,f}$, then $\mathfrak{G}(T,A/B)B' \subseteq \mathfrak{G}(B', A/B)$ for each $T \in \mathfrak{R}_{i,f}/B'$, and so $\text{Hom}_{B_1}(B',A) = \mathfrak{G}(B', A/B)A_{r}$.

**Proof.** The proof of (a) will be obvious by that of [2, Lemma 4]. In what follows, we shall prove (b). Let $V' = V_{A}(V') = \sum U' g'_{pq}$, where $I' = \{g'_{pq}\}$ is a system of matrix units and $U' = V_{A'}(I')$ a division ring. We set $B = B'[I']$
and $T = T[1^\prime]$. Now, let $\sigma$ be an arbitrary element of $\mathfrak{G}(T, A/B)$. By (a), $\sigma = \overline{\sigma}|T$ for some $\overline{\sigma} \in \mathfrak{G}(\overline{T}, A/B)$. Obviously, $V_A(B'\sigma) = V_A(B'\overline{\sigma})$ contains $1^\prime \overline{\sigma}$ as a system of matrix units and the centralizer of $1^\prime \overline{\sigma}$ in $V_A(B'\sigma)$ coincides with $V_A(B\overline{\sigma})$. If $a$ is an arbitrary non-zero element of $V_A(B\overline{\sigma})$ and $T^* = (T\overline{\sigma})$ \([a] \in \Re_{l.f.}^0\), then $\overline{\sigma}^{-1} = \tau^* |\overline{T}\overline{\sigma}$ for some $\tau^* \in \mathfrak{G}(T^*, A/B)$ again by (a). For any $x \in \overline{B}$, the equality $x\overline{\sigma} \cdot a = a \cdot x\overline{\sigma}$ yields $x \cdot \alpha^* = \alpha^* \cdot x$. Hence, $\alpha^\ast$ is contained in the division ring $V_A(B) = U'$. Accordingly, $\alpha^\ast$ is a regular element of $T^* \overline{\sigma}$, and so $a$ is a regular element. We have proved thus $V_A(B\overline{\sigma})$ is a division ring, which implies that

$$V_A(B'\sigma) = \sum V_A(B\overline{\sigma}) \cdot g_{pq} \overline{\sigma}$$

is a simple ring.

**Theorem 2** ([1, Th. 2]). Assume that $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in \Re_{l.f.}^0$. If $B_i \supseteq B_i$ are in $\Re_{l.f.}$ for each $T \in \Re_{l.f.}^0$. If $B_i \supseteq B_i$ are in $\Re_{l.f.}$, then $\mathfrak{G}(B_i, A/B)|B_i = \mathfrak{G}(B_i, A/B)$.

**Proof.** By Lemma 1 (b), the argument in the proof of [2, Th. 3] applies to obtain $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(B_i, A/B)|B_i$. If $T$ is an arbitrary member of $\Re_{l.f.}^0/B_i$, it follows obviously $\mathfrak{G}(B_i, A/B) \subseteq \mathfrak{G}(T, A/B)|B_i$. Accordingly, again by Lemma 1 (b), we obtain $\mathfrak{G}(B_i, A/B)|B_i \subseteq \mathfrak{G}(T, A/B)|B_i = \mathfrak{G}(T, A/B)|B_i \subseteq \mathfrak{G}(B_i, A/B)$, which completes our proof.

In [2], $A/B$ was defined to be $q$-Galois if (1) $B$ is regular, (2) $\text{Hom}_{B_i}(B', A) = \mathfrak{G}(B', A/B)A_r$ for each $B' \in \Re_{l.f.}$, and (3) $\mathfrak{G}(B_i, A/B)|B_i \subseteq \mathfrak{G}(B_i, A/B)$ for each $B_i \supseteq B_i$ in $\Re_{l.f.}$. However, Th. 2 implies that $A/B$ is left locally finite and $q$-Galois if and only if $B$ is regular, $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in \Re_{l.f.}^0$. This fact has been pointed out in [1]. Moreover, [1, Th. 8] asserts that if $A/B$ is left locally finite and $q$-Galois then so is $A/B'$ for each $B' \in \Re_{l.f.}$, namely, $A/B$ is $h$-$q$-Galois. This fact is obviously contained in the following theorem.

**Theorem 3.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in \Re_{l.f.}$. If $B'$ is in $\Re_{l.f.}$ then $\text{Hom}_{B_i}(B'', A) = \mathfrak{G}(B'', A/B)A_r$ for each $B'' \in \Re_{l.f.}/B'$.

**Proof.** Let $T$ be an arbitrary member of $\Re_{l.f.}^0/B'$. Evidently, $\text{Hom}_{B_i}(T, A)$ is a $T', A_r$-submodule of $\text{Hom}_{B_i}(T, A)|B_r = \mathfrak{G}(T, A/B)A_r$ and then $\text{Hom}_{B_i}(T, A)$ is $\oplus_i \sigma \cdot u_i A_r$ with some $\sigma_i \in \mathfrak{G}(T, A/B)$ and non-zero $u_i \in V$ [2, Lemma 2 (a), (b)]. By Th. 1 (b), there exists $T' \in \Re_{l.f.}^0/T$ such that each intermediate ring $A'$ of $A/T'$ is homogeneously $B' - A'$-completely reducible with $[A'|B_i'.A'] = [V_A(B')|V_A(B')]$ and $\text{Hom}_{B_i}(T, A') = \mathfrak{G}(T, A'/B)A_r'$. Now, let $\sigma u_i$ be an arbitrary $\sigma u_i A_r$. By Th. 2, there exists $\sigma^* \in \mathfrak{G}(T', A/B)$ such that $\sigma = \sigma^* |T'$. If $A'' = (T'' \sigma^*) \cdot (T', u_i) \in \Re_{l.f.}/T'$, then $\sigma^* = \tau'' |T'' \sigma^*$ with some $\tau'' \in \mathfrak{G}(A'', A/B)$ again by Th. 2. We set here $A'' = A'' \tau'' (\in \Re_{l.f.}/T')$ and $\sigma^* = \tau''^{-1}$. Let $M_0$ be a minimal $B' - A'$-submodule of $A'$ such that $N_0 = M_0 \sigma u_i$ is non-zero (and so $B'$-
A''-irreducible). Obviously, $A'=M_0 \oplus M_1 \oplus \cdots \oplus M_t$ and $A''=N_0 \oplus N_1 \oplus \cdots \oplus N_t$ with some $B'$-$A'$-isomorphic $M_i$'s and $B'$-$A''$-isomorphic $N_i$'s, where $t+1=[V_d(B')|V_d(B')]$. Then, we can easily find such a $B'$-isomorphism $\nu$ of $A'$ onto $A''$ so that $\nu_1'=\nu_1(a' \sigma')$, for each $a' \in A'$. As $\nu^{-1} \sigma' u_i$ is contained in $\text{Hom}_{B_i}(A'', A'') = (V_{A''}(B'))_t$ (simple), $\sigma' u_i = \nu v_{1i} + \cdots + \nu v_{mt}$ with some regular elements $v_{ji}$'s in $V_{A''}(B')$. Noting that $T$ is in $\Re_{l.f.}$, one will easily see that every $\{\nu v_{1i} T\} A''$ is a $T$-$A''$-irreducible submodule of $\text{Hom}_{B_i}(T, A'') \subseteq \text{Hom}_{B_i}(T, A'') = \mathfrak{G}(T, A, A''/B')$. It follows therefore that $\{\nu_1' T\} A'' = \tau \nu T A''$ with some $\tau \in \mathfrak{G}(T, A''/B')$ and non-zero $w_j \in V_{A''}(B)$ ([2, Lemma 2 (b)]). We have then $A'' = v_j A'' = v_j \cdot (T \cdot A') = \tau w_j A''$ whence it follows that $w_j$ is a regular element of $V_{A''}(B)$. Hence, $\tau w_j$ is contained in $\mathfrak{G}(T, A''/B') \cap \text{Hom}_{B_i}(T, A''/B')$. We have proved thus $\sigma u_i = \sigma' u_i$ is contained in $\mathfrak{G}(T, A'/B') A_r$, and so $\text{Hom}_{B_i}(T, A) = \mathfrak{G}(T, A'/B') A_r$. Then, our assertion is a consequence of Lemma 1 (b).

Now, [2, Cor. 1] can be restated as follows:

**Corollary 1.** Let $A$ be left locally finite over $B$, and $\mathfrak{S}$ a subset of $\mathfrak{S}$ with $\mathfrak{S} \mathfrak{V} = \mathfrak{S}$. If $\mathfrak{S} A_r$ is dense in $\text{Hom}_{B_i}(A, A)$ then $\mathfrak{S}(B', A/B) = \mathfrak{S}|B'$, $\mathfrak{S}(B') A_r$ is dense in $\text{Hom}_{B_i}(A, A)$ and $J(\mathfrak{S}(B'), A) = B'$ for each $B' \in \Re_{l.f.}$.

**Proof.** If $T$ is in $\Re_{l.f.}$, then $\mathfrak{S}(T, A/B) \subseteq \text{Hom}_{B_i}(T, A) = (\mathfrak{S}|T) A_r$ yields $\text{Hom}_{B_i}(T, A) = \mathfrak{S}(T, A/B) A_r$ and $\mathfrak{S}(T, A/B') = \mathfrak{S}|T$ ([2, Lemma 2 (c)]). Accordingly, for any $A' \in \Re_{l.f./B'}$ there holds $\mathfrak{S}(B', A/B) = \mathfrak{S}(A', A/B)|B' = (\mathfrak{S}|A')|B' = \mathfrak{S}|B'$ (Th. 2). Now, our assertion will be easy by Ths. 2, 3 and [2, Remark 1].

Let $B'$ be an intermediate ring of $A/B$ such that $A$ is $B'$-$A$-irreducible and $[B':B]<\infty$. Then $A = B'V A = \sum a_i v_i a$ for an arbitrary minimal left ideal $V$ of $B'$, which means that $A$ is homogeneously $B'$-completely reducible. Hence, $B'$ is also homogeneously $B'$-completely reducible, and so $B'$ is contained in $\Re_{l.f.}$. Combining this fact with [4, Lemma 1.1], we readily see that if $A$ is left locally finite over $B$ and $B$-$A$-irreducible then every intermediate ring of $A/B$ is in $\Re$. The first assertion of the following corollary is a direct consequence of Th. 1 (b), [2, Remark 4] and the last remark. Moreover, Th. 3 and [2, Remark 1] yield the second one.

**Corollary 2** (cf. [1, Ths. 10, 7 and Lemma 16]). Assume that $A/B$ is left locally finite and $\text{Hom}_{B_i}(T, A) = \mathfrak{S}(T, A/B) A_r$ for each $T \in \Re_{l.f.}$.

(a) If $V$ is a division ring then every intermediate ring of $A/B$ is simple, and conversely.

(b) If $B'$ is in $\Re_{l.f.}$ then $J(\mathfrak{S}(B'', A/B'), B'') = B'$ for each $B'' \in \Re_{l.f./B'}$. In particular, if $B'$ is an intermediate ring of $A/B$ such that $A$ is $B'$-$A$-irreducible and $[B':B]<\infty$ then $J(\mathfrak{S}(B'', A/B'), B'') = B'$ for each $B'' \in \Re_{l.f./B'}$. 

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Patterning after the proof of [3, Th. 1], one will readily see that [3, Th. 1] is still valid under the same assumption as in Th. 3. Moreover, taking the validity of Th. 3 into the mind, it will be obvious that the proof of [3, Th. 2] is efficient even under the same assumption as in Th. 3. Thus, we obtain the following:

**Theorem 4.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B,T}(T, A) = \mathfrak{G}(T, A/B)A_{r}$ for each $T \in \mathcal{R}_{l.f}^{0}$. Let $B'$ be a simple intermediate ring of $A/B$, and $\delta$ a derivation of $B'$ into $A$ vanishing on $B$. If $[B':B]<\infty$ or $B'$ is $f$-regular then $\delta$ can be extended to an inner derivation of $A$.

**References**


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