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NOTE ON $q$-GALOIS EXTENSIONS OF SIMPLE RINGS

By

Hisao TOMINAGA

In the previous paper [2], the notions of $q$-Galois extensions and $h$-$q$-Galois extensions of simple rings were introduced with a view to unifying all the earlier works cited at the end of [2] under the assumption that the simple ring extension in question is left locally finite and $h$-$q$-Galois. Recently, in his paper [1], T. Nagahara has obtained some useful $q$-Galois conditions and proved that any left locally finite $q$-Galois extension is necessarily $h$-$q$-Galois ([1, Th. 8]).

The purpose of this note is to present a direct proof to the last. In fact, the sharpening of [2, Th. 1] together with the modification of the proof of [2, Cor. 1] enables us to complete our proof. It will be noteworthy that all the results are obtained without the assumption that $V$ is simple.

As to notations and terminologies used in this note, we follow the previous paper [2]. Moreover, as in [1], by $\mathcal{R}$ we denote the set of all regular intermediate rings of $A/B$, and we shall use the following notations: $\mathcal{R}^0 = \{A' \in \mathcal{R}; [A'|A'] = [A|A]\}$, $\mathcal{R}_{l.f.} = \{A' \in \mathcal{R}; [A':B]_l < \infty\}$, $\mathcal{R}_{l.f.}^0 = \mathcal{R}^0 \cap \mathcal{R}_{l.f.}$, and for any subset $S$ of $A$ we set $\mathcal{R}/S = \{A' \in \mathcal{R}; A' \supseteq S\}$, $\mathcal{R}^0/S = \mathcal{R}^0 \cap \mathcal{R}/S$, $\mathcal{R}_{l.f.}/S = \mathcal{R}_{l.f.} \cap \mathcal{R}/S$ and $\mathcal{R}_{l.f.}^0/S = \mathcal{R}_{l.f.}^0 \cap \mathcal{R}/S$.

At first, to our end, [2, Th. 1] (that was useful in [1], too) has to be improved as follows:

**Theorem 1.** Assume that (a simple ring) $A$ is left locally finite over (a unital simple subring) $B$ and $\text{Hom}_{B_l}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in \mathcal{R}_{l.f.}$.

(a) If $B'$ is a simple intermediate ring of $A/B$ with $[B':B]_l < \infty$, then for any finite subset $F$ of $A$ there exists $T' \in \mathcal{R}_{l.f.}^0/B'[F]$ such that each intermediate ring $A'$ of $A/T'$ is $B'$-$A'$-completely reducible.

(b) If $B'$ is in $\mathcal{R}_{l.f.}$, then for any finite subset $F$ of $A$ there exists $T' \in \mathcal{R}_{l.f.}^0/B'[F]$ such that each intermediate ring $A'$ of $A/T'$ is homogeneously $B'$-$A'$-completely reducible with $[A'|B'_l \cdot A'] = [V_A(B')|V_A(B')]$ (and $[V_{A'}(B): V_{A'}(B')], < \infty$).

**Proof.** (a) Let $M$ be a minimal $B'$-$A$-submodule of $A$ such that the composition series of $M$ as $A$-module is of the shortest length among minimal $B'$-$A$-submodules of $A$. Then, $M=eA$ with a non-zero idempotent $e$. In virtue of [2, Lemma 1], we can find $T^* \in \mathcal{R}_{l.f.}^0/B'[e, F]$ with $[eT^*|T^*] = [M|A]$. 


One may remark here that $eT^{*}$ is a $B'-T^{*}$-submodule of $T^{*}$, for $B'e \subseteq B'eA \cap T^{*} = eA \cap T^{*} = eT^{*}$. Since $\text{Hom}_{B_{l}'}(T^{*}, A) = \mathfrak{G}(T^{*}, A/B)A_{r} = \sum \tau_{i}A_{r}$ with some $\tau_{i} \in \mathfrak{G}(T^{*}, A/B)$, there exists $T' \in \mathfrak{N}_{i,f}/T^{*}$ such that $\text{Hom}_{B_{l}'}(T^{*}, A') = \mathfrak{G}(T^{*}, A'/B)A_{r}'$ for each intermediate ring $A'$ of $A/T'$. By the above remark $B'eT^{*} = eT^{*}$ and [2, Lemma 1], one will easily see that $M' = eA'$ is a (minimal) $B'-A'$-submodule of $A'$ such that the length of the composition series of $M'$ as $A'$-module coincides with $[M|A] = [eT^{*}|T^{*}]$ (and so $[M'|A'] \leq [N'|A']$ for each non-zero $B'-A'$-submodule $N'$ of $A'$). Since $\text{Hom}_{B_{l}'}(T^{*}, A') = T_{r}^{*}-A_{r}'$-completely reducible by [2, Lemma 2 (a)], the $T_{r}^{*}-A_{r}'$-submodule $\text{Hom}_{B_{l}'}(T^{*}, A') = \bigoplus \mathfrak{M}_{j}$ with some irreducible $\mathfrak{M}_{j}$'s. By [2, Lemma 2 (b)], $\mathfrak{M}_{j} = \sigma_{j}u_{j}A_{r}'$ with some $\sigma_{j} \in \mathfrak{G}(T^{*}, A'/B)$ and non-zero $u_{j} \in V_{A}(B)$. Since $\mathfrak{M}_{j}$ is contained in $\text{Hom}_{B_{l}'}(T^{*}, A')$ and $B'e \subseteq eT^{*} \subseteq T^{*}$, each $M_{j} = (B'e)\mathfrak{M}_{j}$ is a $B'-A'$-submodule of $A'$. Further, there holds $M_{j} = u_{j}(B'e)\sigma_{j}A' = u_{j}(B'eT^{*})\sigma_{j}A' = u_{j}(eT^{*})\sigma_{j}A'$, whence it follows $[M_{j}|A'] = [u_{j}eA_{r}'|A'] \leq [eA_{r}'|A'] \leq [eA_{r}'|A']$ by [2, Lemma 1]. Recalling here that $[M'|A']$ is the least, we see that each $M_{j}$ is either 0 or $B'-A'$-irreducible. Finally, noting that $A'$ is $B_{l}'\cdot \text{Hom}_{B_{l}'}(A', A')$-irreducible, there holds $A' = e(B_{l}'\cdot \text{Hom}_{B_{l}'}(A', A')) = (B'e)\text{Hom}_{B_{l}'}(T^{*}, A') = (B'e)\bigoplus \mathfrak{M}_{j} = \bigoplus \mathfrak{M}_{j}$, which proves evidently the complete reducibility of $A'$ as $B'-A'$-module.

(b) Let $V' = V_{A}(B') = \sum U'g_{pq}'$, where $I' = \{g_{pq}'\}$ is a system of matrix units and $U' = V_{A}(F')$ a division ring. Let $M = eA$ be chosen as in (a), and $T^{*}$ a member of $\mathfrak{N}_{i,f}/B'[e, F, I']$ such that $[eT^{*}|T^{*}] = [M|A]$. Then, there exists $T' \in \mathfrak{N}_{i,f}/T^{*}$ such that $\text{Hom}_{B_{l}'}(T^{*}, A') = \mathfrak{G}(T^{*}, A'/B)A_{r}'$ for each intermediate ring $A'$ of $A/T'$. Since $\text{Hom}_{B_{l}'}(A', A')$ coincides with the simple ring $(V_{A}(B'))$, the proof of (a) enables us to see that $A'$ is homogeneously $B'-A'$-completely reducible and $[A'|B'; A_{r}'] = [V_{A}(B')|V_{A}(B')] = [V'|V']$. Accordingly, $A'$ is $B'\cdot V_{A}(B')$-irreducible, and hence $[V_{A}(B): V_{A}(B')] < [B': B_{l}] < \infty$ by [2, Prop. 1 (b)].

**Lemma 1.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_{l}'}(T, A) = \mathfrak{G}(T, A/B)A_{r}$ for each $T \in \mathfrak{N}_{i,f}$.

(a) If $\rho$ is a $B'$-ring homomorphism of an intermediate ring $A_{i}$ of $A/B$ with $[A_{i}: B_{l}] < \infty$ into $A$ such that $A_{i}, \rho-\text{A}-\text{irreducible}$, then $\rho$ is contained in $\mathfrak{G}(A_{i}, A/B)A_{r}$ for each $A_{i} \in \mathfrak{N}_{i,f}/A_{i}$.  

(b) If $B'$ is in $\mathfrak{N}_{i,f}$, then $\mathfrak{G}(T, A/B)|B' \subseteq \mathfrak{G}(B', A/B)$ for each $T \in \mathfrak{N}_{i,f}/B'$, and so $\text{Hom}_{B_{l}'}(B', A) = \mathfrak{G}(B', A/B)A_{r}$.

**Proof.** The proof of (a) will be obvious by that of [2, Lemma 4]. In what follows, we shall prove (b). Let $V' = V_{A}(B') = \sum U'g_{pq}'$, where $I' = \{g_{pq}'\}$ is a system of matrix units and $U' = V_{A}(F')$ a division ring. We set $B = B'[I']$.
and $T=T[T']$. Now, let $\sigma$ be an arbitrary element of $G(T, A/B)$. By (a), $\sigma=\overline{\sigma}|T$ for some $\overline{\sigma}\in G(T, A/B)$. Obviously, $V_A(B'^{\sigma})=V_A(B'^{\overline{\sigma}})$, so $T'\overline{\sigma}$ is a system of matrix units and the centralizer of $T'\overline{\sigma}$ in $V_A(B'^{\sigma})$ coincides with $V_A(B'^{\sigma})$. If $a$ is an arbitrary non-zero element of $V_A(B'^{\sigma})$ and $T^*=(T\overline{\sigma})$ 
\[a]\] (in $R_{l.f.}$), then $\overline{\sigma}^{-1}=\tau^{*}|T\overline{\sigma}$ for some $\tau^{*}\in G(T, A/B)$ again by (a). For any $x\in B$, the equality $x\overline{\sigma}\cdot a=a\cdot x\overline{\sigma}$ yields $x\cdot a\tau^{*}=a\tau^{*}\cdot x$. Hence, $a\tau^{*}$ is contained in the division ring $V_A(B)=U'$. Accordingly, $a\tau^{*}$ is a regular element of $T'\tau^{*}$, and so $a$ is a regular element. We have proved that $V_A(B')$ is a division ring, which implies that $V_A(B'^{\sigma})=\sum V_A(B'^{\sigma})\cdot q\overline{q}\sigma$ is a simple ring.

**Theorem 2** ([1, Th. 2]). Assume that $A/B$ is left locally finite and $\text{Hom}_{B_{l}'}(T, A)=G(T, A/B)A_{r}$ for each $T\in R_{l.f.}$. If $B_{l}\supseteq B_{2}$ are in $R_{l.f.}$, then $G(B_{1}, A/B)|B_{1}=G(B_{2}, A/B)|B_{2}=G(B_{1}, A/B)$.

**Proof.** By Lemma 1 (b), the argument in the proof of [2, Th. 3] applies to obtain $G(B_{1}, A/B)\subseteq G(B_{1}, A/B)|B_{2}$. If $T$ is an arbitrary member of $R_{l.f.}^{|B_{l}'|/B_{l}}$, it follows obviously $G(B_{1}, A/B)\subseteq G(T, A/B)|B_{2}$. Accordingly, again by Lemma 1 (b), we obtain $G(B_{1}, A/B)|B_{2}\subseteq G(T, A/B)|B_{2}=G(B_{1}, A/B)|B_{2}\subseteq G(B_{1}, A/B)$, which completes our proof.

In [2], $A/B$ was defined to be $q$-Galois if (1) $B$ is regular, (2) $\text{Hom}_{B_{l}'}(B', A)=G(T, A/B)A_{r}$ for each $B'\in R_{l.f.}$, and (3) $G(B_{1}, A/B)|B_{1}\subseteq G(B_{2}, A/B)$ for each $B_{l}\supseteq B_{2}$ in $R_{l.f.}$. However, Th. 2 implies that $A/B$ is left locally finite and $q$-Galois if and only if $B$ is regular, $A/B$ is left locally finite and $\text{Hom}_{B_{l}'}(T, A)=G(T, A/B)A_{r}$ for each $T\in R_{l.f}$. This fact has been pointed out in [1]. Moreover, [1, Th. 8] asserts that if $A/B$ is left locally finite and $q$-Galois then so is $A/B'$ for each $B'\in R_{l.f.}$, namely, $A/B$ is $h$-$q$-Galois. This fact is obviously contained in the following theorem.

**Theorem 3.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_{l}'}(T, A)=G(T, A/B)A_{r}$ for each $T\in R_{l.f}$. If $B'$ is in $R_{l.f.}$, then $\text{Hom}_{B_{l}'}(B'', A)=G(B'', A/B')A_{r}$ for each $B''\in R_{l.f.}/B'$. 

**Proof.** Let $T$ be an arbitrary member of $R_{l.f.}^{|B'|/B'}$. Evidently, $\text{Hom}_{B_{l}'}(T, A)$ is a $T_{l}A_{r}$-submodule of $\text{Hom}_{B_{l}'}(T, A)=G(T, A/B)A_{r}$ and then $\text{Hom}_{B_{l}'}(T, A)=G(T, A/B)A_{r}$ with some $\sigma\in G(T, A/B)$ and non-zero $u_{l}\in V$ [2, Lemma 2 (a), (b)]. By Th. 1 (b), there exists $T^{*}\in R_{l.f.}^{|T'|/T}$ such that each intermediate ring $A'$ of $A/T'$ is homogeneously $B'-A'$-completely reducible with $[A'|B'|A'_{r}]=\left[V_A(B')|V_A(B')\right]$ and $\text{Hom}_{B_{l}'}(T, A')=G(T, A'/B')A_{r}$. Now, let $\sigma u_{l}$ be an arbitrary $\sigma u_{l}$. By Th. 2, there exists $\sigma^{*}\in G(T', A')$ such that $\sigma^*=\sigma^{*}|T$. If $A''=(T''\sigma^{*})(T'', \sigma)(\in R_{l.f.}^{|T'|/T'})$, then $\sigma^{-1}=\tau''|T''\sigma^{*}$ with some $\tau''\in G(A'', A/B)$ again by Th. 2. We set here $A''=A''\tau''(\in R_{l.f.}^{|T'|/T'})$ and $\sigma^{*}=\tau''$. Let $M_{0}$ be a minimal $B'-A'$-submodule of $A'$ such that $N_{0}=M_{0}\sigma u_{l}$ is non-zero (and so $B'$-
A''-irreducible). Obviously, $A' = M_0 \oplus M_1 \oplus \cdots \oplus M_t$ and $A'' = N_0 \oplus N_1 \oplus \cdots \oplus N_t$ with some $B'-A'$-isomorphic $M_i$'s and $B''-A''$-isomorphic $N_i$'s, where $t+1 = [V_d(B')|V_d(B')]$. Then, we can easily find such a $B'$-isomorphism $\nu$ of $A'$ onto $A''$ that $a' \nu = \nu(a')$, for each $a' \in A'$. As $\nu^{-1}\sigma u_i$ is contained in $\text{Hom}_{B_l^t}(A'', A'') = \mathfrak{G}(A'', A'') = (V_{A''}(B'))t$ (simple), $\sigma u_i = v_{i1} + \cdots + v_{im}$ with some regular elements $v_j$'s in $V_{A''}(B')$. Noting that $T$ is in $R_{l.f.}$, one will easily see that every $(\nu v_{jl}|T)A''_r$ is a $T,-A''$-irreducible submodule of $\text{Hom}_{B_l^t}(T, A'') \subseteq \text{Hom}_{B_t}(T, A'') = \mathfrak{G}(T, A'', B''|B)A''_r$. It follows therefore $(\nu v_{jl}|T)A''_r = \tau w_{jl}A''_r$ with some $\tau \in \mathfrak{G}(T, A''|B)$ and non-zero $w_j \in V_{A''}(B')$. We have then $A'' = v_j A'' = v_j A' = (T.A) v = v_j, T \nu A'' = T(\nu v_{jl}|T)A''_r = w_j \cdot T \tau A''_r = w_j A'', \tau \in \mathfrak{G}(T, A''|B)$, whence it follows that $w_j$ is a regular element of $V_{A''}(B')$. Hence, $\tau w_j$ is contained in $\mathfrak{G}(T, A''|B) \cap \text{Hom}_{B_l^t}(T, A''|B')$. Let $B'$ be a left ring of ring $A/B$ such that $A$ is $B'-A$-irreducible and $B':B]<\infty$. Then $A = B' \cap A = \sum a_i B'$ for an arbitrary minimal left ideal $I'$ of $B'$, which means that $A$ is the intersection of $I'$-completely reducible. Hence, $B'$ is also homogeneously $B'$-completely reducible, and so $B'$ is contained in $R_{l.f.}$. Combining this fact with [4, Lemma 1.1], we readily see that if $A$ is left locally finite over $B$ and $B-A$-irreducible then every intermediate ring of $A/B$ in $R$. The first assertion of the following corollary is a direct consequence of Th. 1 (b), [2, Remark 4] and the last remark. Moreover, Th. 3 and [2, Remark 1] yield the second one.

**Corollary 2** (cf. [1, Ths. 10, 7 and Lemma 16]). Assume that $A/B$ is left locally finite and $\text{Hom}_{B_l^t}(T, A) = \mathfrak{G}(T, A/B)A_r$ for each $T \in R_{l.f.}$.  

(a) If $V$ is a division ring then every intermediate ring of $A/B$ is simple, and conversely.

(b) If $B'$ is in $R_{l.f.}$, then $J(\mathfrak{G}(B'', A'B'), B'') = B'$ for each $B'' \in R_{l.f.}|B$. In particular, if $B'$ is an intermediate ring of $A/B$ such that $A$ is $B''-A'$-irreducible and $B':B]<\infty$ then $J(\mathfrak{G}(B'', A'B'), B'') = B'$ for each $B'' \in R_{l.f.}|B''$. 


Patterning after the proof of [3, Th. 1], one will readily see that [3, Th. 1] is still valid under the same assumption as in Th. 3. Moreover, taking the validity of Th. 3 into the mind, it will be obvious that the proof of [3, Th. 2] is efficient even under the same assumption as in Th. 3. Thus, we obtain the following:

**Theorem 4.** Assume that $A/B$ is left locally finite and $\text{Hom}_{B_{\tau}}(T, A) = \otimes(T, A/B)A_{\tau}$ for each $T \in \mathcal{R}_{l.f}$. Let $B'$ be a simple intermediate ring of $A/B$, and $\delta$ a derivation of $B'$ into $A$ vanishing on $B$. If $[B':B]_{l} < \infty$ or $B'$ is f-regular then $\delta$ can be extended to an inner derivation of $A$.

**References**


Department of Mathematics, Hokkaido University

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