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QUASI-PROJECTIVE MODULES, PERFECT MODULES, 
AND A THEOREM FOR MODULAR LATTICES

By

Yôichi MIYASHITA

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§ 0. Introduction. Throughout the present paper, $R$ is a ring with 1, and $M$ a unital $R$-left module. “Submodule” and “homomorphism” will mean “$R$-submodule” and “$R$-homomorphism”, respectively. For any $R$-submodule $A$ of $M$, we denote by $\nu(M\rightarrow M/A)$ and $\iota(A\rightarrow M)$ the projection of $M$ onto $M/A$ and the injection of $A$ into $M$, respectively. $M$ is called $R$-perfect if for any pair of submodules $A$, $B$ of $M$ with $A+B=M$ there exists a submodule $B_0$ of $B$ that is minimal with respect to the property that $A+B_0=M$. In this case, $B_0$ is called a d-complement of $A$ (in $M$). A submodule $D$ of $M$ is said to be d-dense in $M$, if $D+X\neq M$ for all proper submodule $X$. Then, $\mathfrak{R}(R^\mathfrak{M})$ (radical of $M$, i.e. the meet of all maximal submodules of $M$) = the sum of all d-dense submodules (Prop. 1.4). From this point of view, we should like to have another look at radicals of modules (Th. 2.11, Th. 2.12 and Th. 4.3). $M$ is called $R$-quasi-projective if for any submodule $B$ of $M$, $\text{Hom}(R^M, R^M/B) = \text{Hom}(R^M, R^M) \cdot \nu(M\rightarrow M/B)$ holds. If $M$ is $R$-quasi-projective and $A$, $B$ are d-complements of each other then $M=A\oplus B$. By the light of this fact, we shall prove the following: If $\mathfrak{P}$ is an $R$-projective module, then the following conditions for $P$ are equivalent. (i) $P$ is $R$-perfect. (ii) Every homomorphic image of $P$ has a projective cover. (iii) $\mathfrak{R}(\mathfrak{P})$ is d-dense in $P$, and $P$ is a direct sum of sum-irreducible submodules, where a sum-irreducible module is a module such that the sum of its two proper submodules is always proper (Th. 3.3 and Th. 3.7).

$R$ is called a semi-perfect ring, if $R^R$ is perfect, and $R$ is called h-central
if \( R \) is quasi-projective as an \( R-R \)-module. As a generalization of Wedderburn’s structure theorem for semi-simple rings, we shall obtain the following: \( R \) is a semi-perfect h-central ring if and only if \( R \) is a direct sum of a finite number of finite dimensional matrix rings over sum-irreducible h-central rings (Th. 4.5).

In §5, we shall generalize the uniqueness theorem of maximal independent sets of uniform submodules which was given in the preceding paper [8], namely, Th. 5.9 is valid for any modular lattice with 0. In virtue of this generalization, we can present another proof to Azumaya’s generalization of Krull-Remak-Schmidt’s theorem.

§ 1. Preliminary results and perfect modules.

A submodule \( D \) of \( M \) is said to be \((R)\) \( d \)-dense in \( M \), if \( D + X \neq M \) for any proper submodule \( X \) of \( M \). If \( D \) is \( d \)-dense in \( M \neq 0 \), then \( D \neq 0 \). Evidently \( \{0\} \) is \( d \)-dense in \( M \).

**Proposition 1.1.** (1) If \( A \) and \( B \) are \( d \)-dense submodules of \( M \), then so is \( A + B \). (2) Let \( A \) and \( B \) be submodules of \( M \) with \( A \supseteq B \). If \( B \) is \( d \)-dense in \( A \) then \( B \) is \( d \)-dense in \( M \). (3) Let \( \varphi \) be a homomorphism from \( M \) to an \( R \)-left module \( N \). If \( B \) is a \( d \)-dense submodule of \( M \), then so is \( B \varphi \) in \( N \).

**Proof.** (1) will be easily seen. (2) If \( B + X = M \) for some submodule \( X \), then \( A = A \cap (B + X) = B + (A \cap X) \). Since \( B \) is \( d \)-dense in \( A \), we have \( A \cap X = A \), that is, \( A \subseteq X \). Therefore, as \( B \supseteq A \supseteq X \), \( M = B + X = X \). (3) By (2), we may assume that \( M \varphi = N \). If \( B \varphi + Y = N \) for some submodule \( Y \subseteq N \), then \( M = B + Y \varphi^{-1} \), where \( Y \varphi^{-1} = \{m \in M; m \varphi \in Y\} \). Since \( B \) is \( d \)-dense in \( M \), we have \( M = Y \varphi^{-1} \), and therefore \( N = M \varphi = (Y \varphi^{-1}) \varphi = Y \).

The sum of all \( d \)-dense submodules of \( M \) is called the **radical** of \( R(M) \), and denoted by \( R(R) \). The following is a direct consequence from Prop. 1.1.

**Proposition 1.2.** (1) \( R(R) \cdot \text{Hom}(R(R), R(M)) \subseteq R(R) \). (2) Every finitely generated submodule of \( R(R) \) is \( d \)-dense in \( M \). (3) For any submodule \( A \) of \( M \), \( R(R(A)) \subseteq R(R(M)) \).

Let \( M = A + B \), where \( A \) and \( B \) are submodules of \( M \). If \( B \) is minimal with respect to the property \( A + B = M \), then \( B \) is called a \( d \)-complement of \( A \) (in \( M \)). A submodule \( B \) is called a \( d \)-complemented submodule of \( M \) if \( B \) is a \( d \)-complement of some \( A \).

**Proposition 1.3.** Let \( M = A + B \), where \( A \) and \( B \) are submodules of \( M \). If \( B \) is a \( d \)-complement of \( A \) then \( A \cap B \) is \( d \)-dense in \( B \), and conversely. If \( B \) is a \( d \)-complement of \( A \), \( D \cap B \) is \( d \)-dense in \( B \) for any \( d \)-dense submodule \( D \) of \( M \), and \( R(R) \) coincides with \( R(R) \cap B \).
Proof. Let $B$ be a d-complement of $A$. If $(A \cap B) + X = B$ for some submodule $X$ of $B$, then $M = A + B = A + X$. The minimality of $B$ implies $X = B$. Hence $A \cap B$ is d-dense in $B$. Conversely, assume that $A \cap B$ is d-dense in $B$. Let $A + B_0 = M$ for some submodule $B_0$ of $B$. Then $B = B \cap (A + B_0) = (A \cap B) + B_0$. Since $A \cap B$ is d-dense in $B$, we have $B = B_0$. Hence $B$ is a d-complement of $A$. Let $D$ be a d-dense submodule of $M$, and let $B = (D \cap B) + X$ for some submodule $X$. Then $M = A + B = A + (D \cap B) + X$. Since $D \cap B(\subseteqq D)$ is d-dense in $M$, we have $M = A + X$. Then, the minimality of $B$ implies $X = B$. Hence $D \cap B$ is d-dense in $B$. By Prop. 1.2, $\Re(RM) \subseteq \Re(RM) \cap B$. Conversely, for any $x \in \Re(RM) \cap B$, $Rx$ is d-dense in $M$, and therefore $Rx = Rx \cap B$ is d-dense in $B$.

Corollary. If $M = \sum_{i \in A} \oplus M_i$ with submodules $M_i$ then $\Re(RM) = \sum_{i \in A} \oplus \Re(RM_i)$.

Proof. Since $\Re(RM) \cdot \Hom(RM, RM) \subseteq \Re(RM)$, we have $\Re(RM) = \sum_{i \in A} \oplus (\Re(RM) \cap M_i)$. Since any direct summand is a d-complemented submodule, we have $\Re(RM) \cap M_i = \Re(RM_i)$ by Prop. 1.3, and so $\Re(RM) = \sum_{i \in A} \oplus \Re(RM_i)$.

Proposition 1.4. $\Re(RM) \neq M$ if and only if $M$ has a maximal submodule. When this is the case, $\Re(RM)$ coincides with the meet of all maximal submodules of $M$.

Proof. Assume $\Re(RM) \neq M$. Evidently every d-dense submodule of $M$ is contained in all maximal submodules of $M$. Let $x$ be any element of $M$ with $x \in \Re(RM)$, and $Rx + X = M$ for some proper submodule $X$. By Zorn's lemma, $X$ is contained in some maximal submodule $X_i$ with $x \in X_i$. Hence $\Re(RM) \cap M_i$ coincides with the meet of all maximal submodules of $M$. Conversely, we assume that $M$ has a maximal submodule $A$. If $x \in M$ and $x \notin A$, then $Rx + A = M$, and therefore $Rx$ is not d-dense in $M$. Hence $\Re(RM) \neq M$, by Prop. 1.2 (2).

Corollary. $\Re(RM/\Re(RM)) = 0$.

Remark. (1) The dual of Prop. 1.4 is also true: The meet of all dense submodules (cf. [8]) of $M$ coincides with the sum of all minimal $R$-submodules of $M$. (2) For a ring $R(\oplus 1)$, $\Re(R)$ coincides with the Jacobson radical of $R$.

The following lemma is evident.

Lemma 1.5. Let $A, B, C$ be submodules of $M$ such that $A + C = B + C$ and $A \cap C = B \cap C$. If $A \subseteq B$ then $A = B$.

Proposition 1.6. If a submodule $A (\subseteqq M)$ satisfies the descending chain condition for its submodules, then for any submodule $B$ of $M$ with $A + B = M$ there exists a submodule $B_0$ of $B$ that is minimal with respect to the property $A + B_0 = M$.  

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Proof. This is evident from Lemma 1.5.

Theorem 1.7. If a submodule $A$ of $M$ satisfies the descending chain condition for submodules and $A \cap \Re(RM) = 0$, then $A$ is a direct sum of a finite number of minimal submodules, and a direct summand of $M$.

Proof. Let $B$ be a d-complement of $A$ (Prop. 1.6). Then, since $A \cap B \subseteq \Re(RB) \subseteq \Re(RM)$ by Prop. 1.3, we have $A \cap B \subseteq A \cap \Re(RM) = 0$, and so $M = A \oplus B$. Since $A$ satisfies the descending chain condition for submodules and $\Re(RA) = \Re(RM) \cap A = 0$ (Prop. 1.3), the above shows that every submodule of $A$ is a direct summand of $A$. Hence, as is well known, $A$ is completely reducible.

$M$ is called $(R)$ perfect, if for any pair of submodules $A, B$ of $M$ with $A + B = M$, there exists a submodule $B_{0}$ of $B$ that is minimal with respect to $M = A + B_{0}$.

Proposition 1.8. If a submodule $B$ of $M$ is a d-complement of $A$, then for any d-dense submodule $D$ of $M$, $(B + D)/D$ is a d-complement of $(A + D)/D$ in $M/D$. Therefore, if $M$ is perfect, then so is $M/D$.

Proof. Let $X$ be a submodule with $B + D \supseteq X \supseteq D$ and $(A + D)/D + X/D = M/D$. Then, as $X = X \cap (B + D) = (X \cap B) + D$, we have $M = A + D + X = A + D + (X \cap B)$, and so $A + (X \cap B) = M$, because $D$ is d-dense in $M$. Therefore, $X \cap B = B$ by the minimality of $B$, that is, $X \cap B = B$. Hence $X = B + D$, that is, $X/D = (B + D)/D$, which means that $(B + D)/D$ is a d-complement of $(A + D)/D$ in $M/D$.

Proposition 1.9. Let $B$ be a d-complement of $A$ in $M$. If $M$ is perfect, then so is $B$. In particular, every direct summand of a perfect module is perfect.

Proof. Let $B = X + Y$, where $X$ and $Y$ are submodules of $B$. Then $M = A + B = A + X + Y$. Let $Y_{0}$ be a d-complement of $A + X$ in $M$ such that $Y_{0} \subseteq Y$. As $M = A + (X + Y_{0})$, the minimality of $B$ implies that $X + Y_{0} = B$. If $Y_{1}$ is a submodule of $B$ with $X + Y_{1} = B$ and $Y_{1} \subseteq Y_{0}$, then $M = A + X + Y_{1}$. Since $Y_{0}$ is a d-complement of $A + X$ in $M$, we have $Y_{1} = Y_{0}$. Hence $Y_{0}$ is a d-complement of $X$ in $B$ such that $Y_{0} \subseteq Y$. Thus $B$ is perfect.

Proposition 1.10. Let $A$ be a submodule of $M$, $D$ a d-dense submodule of $M$ with $D \subseteq A$. Then, $A$ is d-dense in $M$ if and only if $A/D$ is d-dense in $M/D$.

Proof. If $A$ is d-dense in $M$, $A/D$ is d-dense in $M/D$ by Prop. 1.1 (3). Conversely, assume that $A/D$ is d-dense in $M/D$. If $A + B = M$ for some submodule $B$, then $A/D + (B + D)/D = M/D$, and so $(B + D)/D = M/D$ by as-
assumption. Hence $B + D = M$. Since $D$ is d-dense in $M$, we have $B = M$.

**Corollary.** If $D$ is a d-dense submodule of $M$, then $\mathcal{R}(\mathcal{R}(M)/D) = \mathcal{R}(\mathcal{R}(M) / D)$.

**Proposition 1.11.** Let $A$ be a submodule of $M$, $B$ a d-complement of $A$ (in $M$). Then $\mathcal{R}(\mathcal{R}(M) = (\mathcal{R}(\mathcal{R}(M) \cap A) + (\mathcal{R}(\mathcal{R}(M) \cap B)$.

**Proof.** Put $\mathcal{R}(\mathcal{R}(M) = N$. Then, $A \cap B$ is d-dense in $B$ and $\mathcal{R}(\mathcal{R}(B) = N \cap B$ by Prop. 1.3. Therefore, by Cor. to Prop. 1.10, $\mathcal{R}(\mathcal{R}(B) = (N \cap B) / (A \cap B)$. Now, $B / (A \cap B) \cong (A + B) / A = M / A$ canonically, and $\mathcal{R}(\mathcal{R}(B) = (N \cap B) / (A \cap B) \cong (N \cap B + A) / A \cong (N + A) / A \subseteq \mathcal{R}(\mathcal{R}(M) / A)$ under the above isomorphism. Hence $(N \cap B + A) / A = (N + A) / A$, that is, $(N \cap B) + A = N + A$. Thus we have $N = (N \cap A) + (N \cap B)$. 

**Theorem 1.12.** If $M$ is perfect, then so is $M / A$ for any submodule $A$ of $M$.

**Proof.** If $B$ is a d-complement of $A$, then $B$ is perfect, and $A \cap B$ is d-dense in $B$ by Prop. 1.9 and Prop. 1.3. Therefore, by Prop. 1.8, $B / (A \cap B) \cong M / A$ is perfect.

**Proposition 1.13.** If $M$ is perfect and $\mathcal{R}(\mathcal{R}(M) = 0$, then $M$ is $R$-completely reducible.

**Proof.** For any submodule $A$, we take a d-complement $B$ of $A$. Then $A \cap B \subseteq \mathcal{R}(\mathcal{R}(B) \subseteq \mathcal{R}(\mathcal{R}(M) = 0$ by Prop. 1.3, and so $M = A \oplus B$. Hence, every submodule of $M$ is an $R$-direct summand of $M$, and hence, as is well known, $M$ is completely reducible.

**Proposition 1.14.** If $M \neq 0$ is $R$-projective, then $M \supseteq \mathcal{R}(\mathcal{R}(M) = \mathcal{R}(R) \cdot M \supseteq \mathcal{R}(M_{K})$, where $K = \text{Hom}(\mathcal{R}(M), \mathcal{R}(M))$. (See Bass [3; p. 474]).

**Proof.** The first important assertion $M \neq \mathcal{R}(\mathcal{R}(M)$ is proved in Bass [3]. For an $R$-free module $F = _{R}R^{(\Lambda)}$, we have $\mathcal{R}(\mathcal{R}(F) = \mathcal{R}(R)^{(\Lambda)} = \mathcal{R}(R) \cdot F$ by Cor. to Prop. 1.3. As is well known, $\text{Hom}(\mathcal{R}(F), \mathcal{R}(F)) = (R_{T})$, the set of all raw-finite matrices over $R$. We set $T = (R_{T})$, and take an idempotent $e \in T$. Then, $\mathcal{R}(\mathcal{R}(Fe)) = \mathcal{R}(\mathcal{R}(F) \cap Fe = \mathcal{R}(R) \cdot Fe$ by Prop. 1.3, and as is easily seen, $\text{Hom}(\mathcal{R}(Fe), \mathcal{R}(Fe)) = eTe$. Since $F_{T}$ is $T$-projective, a symmetric argument to the above yields $\mathcal{R}(F_{T}) = F \cdot \mathcal{R}(T)$. As is well known, $\mathcal{R}(T) = (\mathcal{R}(R))_{T}$, and so $\mathcal{R}(F_{T}) = F \cdot \mathcal{R}(T) \subseteq F \cdot (\mathcal{R}(R))_{T} \subseteq \mathcal{R}(\mathcal{R}(F))$. Let $X$ be any $eTe$-d-dense submodule of $Fe$, and let $X \cdot T + Y = F$ for some $T$-submodule $Y$ of $F$. Then $Fe = X \cdot Te + Ye = X + Ye$, and so $Ye = Fe \supseteq X$, because $X$ is $eTe$-d-dense in $Fe$. Hence $Y \supseteq X \cdot T$, and we have $Y = X \cdot T + Ye$ for some $T$-submodule $Y$ of $F$. This implies that $X \subseteq \mathcal{R}(F_{T})$. Thus

---

1) The proof will proceed as that of the case $\# A < \aleph_{0}$.

2) Cf. [10; p. 113].
we have $\mathfrak{R}(F_{c,e}) \subseteq \mathfrak{R}(F)$. Therefore $\mathfrak{R}(F_{c,e}) \subseteq \mathfrak{R}(F) e \subseteq \mathfrak{R}(F) e = \mathfrak{R}(F e)$. Noting that the projective module $M$ can be written as $Fe$ with some $F$ and $e$, the proof is now complete.

**Proposition 1.15.** If every proper submodule of $M$ contained in a maximal submodule, then $\mathfrak{R}(K)M$ is $R$-d-dense in $M$. In particular, if $M$ is $R$-finitely generated, then $\mathfrak{R}(K)M$ is $R$-d-dense in $M$.

**Proof.** If $\mathfrak{R}(K)M + X = M$ for some proper submodule $X$ of $M$ then, by assumption, $X$ is contained in a maximal submodule $X_i$, so that $\mathfrak{R}(K)M + X_i = M$. On the other hand, $\mathfrak{R}(K)M \subseteq X_i$ by Prop. 1.14, and we have a contradiction $X_i = M$.

§ 2. Quasi-projective modules.

$M$ is called $R$-quasi-projective if $\text{Hom} (K)M, K)M / A) = \text{Hom} (K)M, K)M)$.\[\nu(M \rightarrow M/A)\] for any submodule $A$ of $M$, or equivalently, $\text{Hom} (K)M, K)M) = \text{Hom} (K)M, K)M) \varphi$ for any $R$-epimorphism $\varphi : M \rightarrow N$, where $\nu(M \rightarrow M/A)$ is the projection of $M$ on $M/A$. $M$ is called $R$-quasi-injective if $\text{Hom} (K)A, K)M) = \iota(A \rightarrow M) \cdot \text{Hom} (K)M, K)M)$ for any submodule $A$ of $M$, where $\iota(A \rightarrow M)$ is the injection of $A$ into $M$.

**Proposition 2.1.** The following are equivalent: (i) $M$ is $R$-quasi-projective. (ii) For any submodules $A, B$ of $M$ and any epimorphism $\varphi : A \rightarrow M/B$, there exists a homomorphism $\phi : M \rightarrow A$ with $\phi \varphi = \nu(M \rightarrow M/B)$.

**Proof.** (i)$\Rightarrow$(ii) $\varphi : A \rightarrow M/B$ induces an isomorphism $\bar{\varphi} : A/Ker \varphi \cong M/B$. Further, we have a homomorphism $\varphi^* = \nu(M \rightarrow M/B) \bar{\varphi}^{-1} : M \rightarrow M/Ker \varphi$. Then, by assumption, there exists a homomorphism $\phi : M \rightarrow M$ with $\varphi^* = \phi \cdot \nu(M \rightarrow M/Ker \varphi)$. Then $\varphi \varphi = \nu(M \rightarrow M/B)$. (ii)$\Rightarrow$(i) Let $\varphi$ be an epimorphism from $M$ to $B/A$, where $A$ and $B$ are submodules of $M$ with $A \subseteq B$. $\varphi$ induces an isomorphism $\rho : B/A \cong M/Ker \varphi$. By assumption there exists a homomorphism $\psi : M \rightarrow M/B$ with $\phi \cdot \nu(M \rightarrow M/A) \rho = \nu(M \rightarrow M/Ker \varphi)$. Then $\phi \cdot \nu(M \rightarrow M/A) = \varphi$.

**Proposition 2.2.** The following are equivalent: (i) $M$ is $R$-quasi-injective. (ii) For any submodules $A, B$ of $M$ and any monomorphism $\varphi : A \rightarrow M/B$ there exists a homomorphism $\phi : M/B \rightarrow M$ with $\phi \varphi = 1_A$.

**Proof.** (i)$\Rightarrow$(ii) Let $\varphi : A \cong C/B$, where $B$ and $C$ are submodules of $M$ with $B \subseteq C$. Then we have a homomorphism $\varphi^* = \nu(C \rightarrow C/B) \varphi^{-1} : C \rightarrow M$. By assumption, there exists a homomorphism $\phi : M \rightarrow M$ with $\phi | C$ (restriction to $C) = \varphi^*$. As $B \varphi = 0$, $\phi$ induces a homomorphism $\hat{\phi} : M/B \rightarrow M$. Then, $\varphi \hat{\phi} = 1_A$.  

3) The symbol “$\rightarrow$” means an epimorphism.

4) The symbol “$\Rightarrow$” means a monomorphism.
(ii) \(\Rightarrow\) (i) Let \(\varphi : A \to M\), where \(A\) is a submodule of \(M\). Then, \(\varphi\) induces a monomorphism \(\sigma : A\varphi \to M/\text{Ker} \varphi\). By assumption, there exists a homomorphism \(\varphi : M/\text{Ker} \varphi \to M\) with \(\sigma \varphi = 1_{A\varphi}\). We put \(\varphi^* = \nu(M \to M/\text{Ker} \varphi)\varphi\), then \(\varphi^* : M \to M\) and \(\varphi^* | A = \varphi\).

**Theorem 2.3.** Let \(M\) be \(R\)-quasi-projective. If \(A\) and \(B\) are \(d\)-complements of each other, then \(M = A \oplus B\).

**Proof.** Since \(\nu(M \to M/A)|B : B \to M/A\) is an epimorphism, by Prop. 2.1 there exists a homomorphism \(\sigma : M \to B\) with \(\sigma(\nu(M \to M/A)|B) = \nu(M \to M/A)\). Then \(M \sigma + A = M\), \(M \sigma \subseteq B\), and therefore the minimality of \(B\) implies that \(B = M \sigma = A \sigma + B \sigma\). Since \(m \sigma + A = m + A\) for all \(m \in M\), we have \(A \sigma \subseteq A\). Therefore \(M = A + A \sigma + B \sigma = A + B \sigma\), \(B \sigma \subseteq B\). Again, by the minimality of \(B\), we have \(B \sigma = B\), that is, \(B + \text{Ker} \sigma = M\). Since \(\text{Ker} \sigma \subseteq A\), the minimality of \(A\) implies that \(\text{Ker} \sigma = A\). From this fact, we know that \(\nu(M \to M/A)|B\) is \(1 - 1\), that is, \(A \cap B = 0\).

**Remark.** As is easily seen from the above proof, if \(M\) is \(R\)-quasi-projective and \(B\) is a \(d\)-complemented submodule of \(M\) then there exists a homomorphism \(\sigma : M \to M\) with \(M \sigma = B \sigma = B\). Further, if \(B\) satisfies the ascending chain condition for \(R\)-submodules then \(B \cap \text{Ker} \sigma = 0\), that is, \(M = B \oplus \text{Ker} \sigma\). In particular, if \(M\) is \(R\)-finitely generated and projective, and \(R\) is left Noetherian, then every \(d\)-complemented submodule of \(M\) is an \(R\)-direct summand of \(M\).

**Theorem 2.4.** Let \(M\) be \(R\)-quasi-projective.

1. Every \(R\)-direct summand of \(M\) is \(R\)-quasi-projective.
2. If \(M_0\) is an \(R\)-\(K\)-submodule of \(M\), then \(M|M_0\) is \(R\)-quasi-projective, where \(K = \text{End}_R(M)\) acting on the right.
3. \(M\) is \(Q\)-quasi-projective for any intermediate ring \(Q\) between \(R\) and \(\text{End}_R(M_K)\), where \(R\) is the image of \(R\) in \(\text{End}_R(M)\) acting on the left.

**Proof.** (1) Let \(\rho\) be an idempotent in \(K\), and let \(\varphi\) be an epimorphism from \(M_\rho\) to an \(R\)-module \(B\). Then \(M(\rho \varphi) = B\). Therefore \(\text{Hom}_R(K M, B) = \text{Hom}_K(K M, K M) \rho \varphi\), and so \(\text{Hom}_R(K M_\rho, B) = \iota(M \rho \to M) \cdot \text{Hom}_R(K M, B) = \text{Hom}_R(K M \rho, K M) \rho \varphi\). Hence \(M_\rho\) is \(R\)-quasi-projective. (2) Let \(M_0\) be an \(R\)-\(K\)-submodule of \(M\), and \(\sigma\) an epimorphism from \(M\) to an \(R\)-module \(B\). Then \(\nu(M \to M|M_0)\sigma\) is onto. Therefore, for any homomorphism \(\varphi : M|M_0 \to B\), there exists a homomorphism \(\phi : M \to M\) with \(\phi \cdot \nu(M \to M|M_0)\sigma = \nu(M \to M|M_0)\varphi\). If we define \(\phi^* : M|M_0 \to M|M_0\) by \((m + M_0)\phi^* = m\phi + M_0\) \((m \in M)\), then \(\phi^* \sigma = \varphi\). (3) Noting that \(\text{Hom}(\nu M, \nu M) = \text{Hom}(K M, K M)\), (3) will be easily seen.

**Remark.** Every completely reducible module is quasi-injective and quasi-projective.
Proposition 2.5. If $M$ is $R$-perfect, then the following are equivalent:
(i) $M$ is $R$-quasi-projective.  (ii) For each d-dense submodule $D$ of $M$, Hom $(\kappa M, \kappa M/D)$ = Hom $(\kappa M, \kappa M) \cdot \nu(M\rightarrow M/D)$.

Proof. (i) $\Rightarrow$ (ii) is trivial.  (ii) $\Rightarrow$ (i) For any submodule $A$ of $M$, we take a d-complement $B$ of $A$.  Then $A \cap B$ is d-dense (in $B$, and so) in $M$ (Prop. 1.3).  Let $\phi$ be arbitrary homomorphism from $M$ to $M/A$, and let $\alpha$ be the isomorphism: \[ B/(A \cap B) \cong M/A \] given by \[ (b + (A \cap B)) \alpha = b + A \] \[ (b \in B). \] Then $\phi \alpha^{-1}$: $M\rightarrow M/(A \cap B)$.  Since $A \cap B$ is d-dense in $M$, there exists a homomorphism $\phi$: $M\rightarrow M$ with $\phi \cdot \nu(M\rightarrow M/(A \cap B)) = \phi \alpha^{-1}$.  Then $\phi \cdot \nu(M\rightarrow M/A) = \phi \cdot \nu(M\rightarrow M/(A \cap B))\alpha = \phi$.

The following fact was found by Johnson and Wong [6]: $M$ is $R$-quasi-injective if and only if $M \cdot \text{Hom} (\kappa M^\wedge, \kappa M^\wedge) \subseteq M$, where $M^\wedge$ is the injective envelope of $M$.  By making use of this fact we can prove the following:

Proposition 2.6. Let $n$ be a natural number.  If $M$ is $R$-quasi-injective then so is $M^n$, and conversely.

Proof. Evidently, $(M^n)^\wedge = (M^\wedge)^n$ and Hom $(\kappa (M^\wedge)^n, \kappa (M^\wedge)^n)$ = Hom $(\kappa M^\wedge, \kappa M^\wedge)^n$, and therefore $M^n \cdot (\text{Hom} (\kappa M^\wedge, \kappa M^\wedge))^n \subseteq (M \cdot \text{Hom} (\kappa M^\wedge, \kappa M^\wedge))^n \subseteq M^n$.  Hence $M^n$ is $R$-quasi-injective.  Conversely, if $M^n$ is $R$-quasi-injective, then $M$ $(\equiv (M, 0, \cdots, 0))$ is $R$-quasi-injective by [5; Cor. 2.2] or [8; Th. 4.3].

The next is the dual of Johnson and Wong [6; 1.1. Th.].

Theorem 2.7. Let $\kappa P$ be an $R$-projective module, and $D$ an $R$-d-dense submodule of $P$.  Then the following are equivalent:

(i) $P/D$ is $R$-quasi-projective.
(ii) $D \cdot \text{Hom} (\kappa P, \kappa P) \subseteq D$.

Proof. (ii) $\Rightarrow$ (i) is a direct consequence of Th. 2.4 (2).  (i) $\Rightarrow$ (ii) Assume that there exists a homomorphism $\varphi$: $P\rightarrow P$ with $D \varphi \not\subseteq D$.  Then $\varphi$ induces a homomorphism $\bar{\varphi}$: $P/D\rightarrow P/(D\varphi + D)$.  If $\sigma$: $P/D\rightarrow P/(D\varphi + D)$ is defined by $(x + D)\sigma = x + (D\varphi + D)$, by assumption there exists a homomorphism $\phi$: $P/D\rightarrow P/D$ with $\phi \sigma = \bar{\varphi}$.  Furthermore, since $P$ is projective, there exists a homomorphism $\rho$: $P\rightarrow P$ with $x \rho + D = (x + D)\phi$ for all $x \in P$.  Then $x \rho + (D\varphi + D) = x \varphi + (D\varphi + D)$, that is, $x (\varphi - \rho) \in D\varphi + D$ for all $x \in P$.  Since $D\varphi \not\subseteq D$ and $D \rho \subseteq D$, we have $D(\varphi - \rho) \not\subseteq D$, and so $D(\varphi - \rho)^{-1} \not\subseteq D$.  Therefore $D(\varphi - \rho)^{-1} + D \not\subseteq P$.  But this is impossible.  In fact, if $a$ is an element of $P$ not contained in $D(\varphi - \rho)^{-1} + D$, and if $a (\varphi - \rho) = d \rho + d' (d, d' \in D)$, then $(a - d)(\varphi - \rho) = d' + d \rho \in D$, that is, $a - d \in D(\varphi - \rho)^{-1}$, whence it follows $a \in D(\varphi - \rho)^{-1} + D$.

Corollary. Let $\kappa P$ be an $R$-projective module, $D$ a d-dense submodule of $P$, and $n$ a natural number.  Then the following are equivalent: (i) $P/D$ is $R$-quasi-projective.  (ii) $(P/D)^n$ is $R$-quasi-projective.
Proof. (ii) $\Rightarrow$ (i) is a consequence of Thm. 2.4 (1). (i) $\Rightarrow$ (ii) $\text{Hom}_{R}(P_{n}, P_{n}) = (\text{Hom}_{R}(P_{n}, P_{n}))_{\text{n}}$. By Prop. 1.1, $\mathbb{D}$ is $R$-dense in $P_{n}$, and $\mathbb{D} \cdot (\text{Hom}_{R}(P, P))_{\text{n}} \subseteq (D \cdot \text{Hom}_{R}(P, P))_{\text{n}} \subseteq D_{n}$ by Th. 2.7. Hence $P_{n}/D_{n}$ is $R$-quasi-projective by Th. 2.4 (2).

Let $N$ be a $T$-right module, where $T$ is a ring with $1$ $(\neq 0)$. Then $N_{n}=\{(x_{1}, \cdots, x_{n}) \mid x_{i} \in N\}$ may be regarded naturally as a $(T)_{n}$-right module. We set $S=\text{Hom}(N_{(T)_{n}}, N_{(T)_{n}})$ (acting on the left). Then $N_{n}$ is as usual an $S$-left, $(T)_{n}$-right module, and, to be easily verified, the correspondence $X \rightarrow X_{n}$ is an order isomorphism between the $T$-submodules of $N$ and the $(T)_{n}$-submodules of $N_{n}$. Now, we shall prove the following:

**Theorem 2.8.** (1) $N$ is $T$-quasi-projective if and only if $N_{n}$ is $(T)_{n}$-quasi-projective.

(2) $N$ is $T$-quasi-injective if and only if $N_{n}$ is $(T)_{n}$-quasi-injective.

**Proof.** (1) Suppose that $N$ is $T$-quasi-projective. If $X$ is a $T$-submodule of $N$, then $(N/X)^{n}_{(T)_{n}} \cong N^{n}/X^{n}_{(T)_{n}}$. Hence we may consider $(N/X)^{n}_{(T)_{n}}$ and $N^{n}/X^{n}_{(T)_{n}}$. Let $\Phi$ be arbitrary $(T)_{n}$-homomorphism from $N_{n}$ to $(N/X)^{n}_{(T)_{n}}$. Then, noting that $(\Phi(x)e_{ij})=\Phi(xe_{ij})$ for each $x$ in $N_{n}$ and each matrix unit $e_{ij}$ of $(T)_{n}$, we can easily see that there exists a $T$-homomorphism $\varphi$ from $N$ to $N/X$ such that $\Phi(x_{1}, \cdots, x_{n})=(\varphi x_{1}, \cdots, \varphi x_{n})$ for all $(x_{1}, \cdots, x_{n}) \in N^{n}$. By assumption, $\varphi=s \cdot \nu(N\rightarrow N/X)$ for some $s \in S$. Since $\text{Hom}(N^{n}_{(T)_{n}}, N^{n}_{(T)_{n}})=S$, the last implies that $N_{n}$ is $(T)_{n}$-quasi-projective. Next, suppose that $N_{n}$ is $(T)_{n}$-quasi-projective. If $X$ is a $T$-submodule of $N$ and $\varphi$ a $T$-homomorphism from $N$ to $N/X$, then $\varphi$ induces a $(T)_{n}$-homomorphism $\Phi: N^{n}_{(T)_{n}} \rightarrow (N/X)^{n}_{(T)_{n}}$. By assumption, there exists an element $s$ of $S$ $\in \text{Hom}(N^{n}_{(T)_{n}}, N^{n}_{(T)_{n}})$ with $\Phi(x_{1}, \cdots, x_{n})=(sx_{1}+X_{1}, \cdots, sx_{n}+X)$ for all $(x_{1}, \cdots, x_{n}) \in N_{n}$. Then $\varphi x_{i}=sx_{i}+X$ for all $x_{i} \in N$. (2) The proof of this part is quite symmetric to the above. Therefore we omit it.

**Corollary.** Let $T$ be a ring with $1$ $(\neq 0)$. Then $T_{\tau}$ is injective if and only if $(T)_{n(T)_{n}}$ is injective.

**Proof.** One may remark here a ring is injective if and only if it is quasi-injective. $T_{\tau}$ is quasi-injective if and only if $T^{n}_{(T)_{n}}$ is quasi-injective by the above theorem. And this is equivalent to that $(T)_{n(T)_{n}}$ is quasi-injective by Prop. 2.6.

A ring $R$ is called $h$-central, if $R$ is quasi-projective as an $R$-$R$-module.

**Proposition 2.9.** $R$ is an $h$-central ring if and only if for any ideal $I$ of $R$, $Z(R/I)=Z(R)+I/I$, where $Z(*)$ is the center of *. If $R$ is $R$-$R$-perfect, then $R$ is $h$-central if and only if $Z(R/I)=Z(R)+I/I$ for each ideal $I$ of $R$ contained in $R_{(x,Rk)}$.

**Proof.** Evidently, $(Z(R)+I/I) \subseteq Z(R/I)$. Suppose that $R$ is $h$-central, and
let \( a+I \) be any element of \( Z(R/I) \). If \( \varphi : R \rightarrow R/I \) is an \( R-R \)-homomorphism defined by \( x\varphi = xa+I \), there exists an element \( c \) of \( Z(R) \) with \( xa+I = xc+I \) for all \( x \in R \). Then \( a+I = c+I \). Hence \( (Z(R)+I)/I = Z(R/I) \). Conversely, we assume that \( (Z(R)+I)/I = Z(R/I) \) for each ideal \( I \) of \( R \). Let \( \varphi \) be any \( R-R \)-homomorphism from \( R \) to \( R/I \). Then \( I\varphi = I(1\varphi) = 0 \), so that \( \varphi \) induces an \( R-R \)-homomorphism \( \bar{\varphi} : R/I \rightarrow R/I \). Then there exists an element \( c+I \) of \( Z(R/I) \) with \( x\varphi = xc+I \) for all \( x \in R \). By assumption, we may assume that \( c \in Z(R) \). Then \( x\varphi = (xc) \cdot \nu(R \rightarrow R/I) \) for all \( x \) in \( R \). Hence \( R \) is \( h \)-central.

The second half is evident from the above proof and Th. 2.5.

**Corollary.** Let \( n \) be a natural number. If \( R \) is \( h \)-central then so is \((R)_n\), and conversely.

**Proof.** As is easily seen, for any ideal \( \bar{I} \) of \((R)_n\) there exists an ideal \( I \) of \( R \) with \( (I)_n = \bar{I} \). Then there exists a canonical ring isomorphism from \((R/I)_n\) to \((R)_n/(I)_n\), and \( ((Z(R)+I)/I)_n \cong (Z(R)+I)/I_n = (Z(R))_n + (I)_n/(I)_n = (Z((R)_n))_n + (I)_n/(I)_n \), \( Z((R/I)_n) \cong Z((R/I))_n = Z((R)/I)_n \) under the above isomorphism. Therefore \((Z((R)_n) + (I)_n)/(I)_n = Z((R)_n/I_n)\) if and only if \((Z(R)+I)/I_n = (Z(R/I))_n\), or equivalently, \((Z(R)+I)/I = Z(R/I)\).

We set \( J = \{ \alpha \in K = \text{Hom}(R, M), M\alpha \text{ is } R-d \text{-dense in } M \} \), that is an ideal of \( K \) by Prop. 1.1.

**Proposition 2.10.** If \( M \) is \( R \)-quasi-projective, then \( J \subseteq \Re(K) \).

**Proof.** Let \( \bar{\gamma} \in J, A \) a left ideal of \( K \) with \( K\bar{\gamma} + A = K \). Then, there exists an element \( \alpha \) of \( A \) with \( K\bar{\gamma} + K\alpha = K \). Therefore, \( M\bar{\gamma} + M\alpha = M \), and so \( M\alpha = M \), because \( M\bar{\gamma} \) is \( R-d \)-dense in \( M \). By Prop. 2.1, there exists an element \( \beta \) of \( K \) with \( \beta\alpha = 1 \). Thus we have \( K\alpha = K \), and so \( A = K \). Hence \( K\bar{\gamma} \) is \( d \)-dense in \( xK \), that is, \( \bar{\gamma} \in \Re(K) \).

**Theorem 2.11.** If \( M \) is \( R \)-projective and \( \Re(R,M) \) is \( R-d \)-dense in \( M \), then \( J = \Re(K) \).

**Proof.** By Prop. 1.13, \( \Re(R,M) \cong \Re(M\Re) \). Since \( \Re(M\Re) \cong M \cdot \Re(K) \) by Prop. 1.1, we have \( \Re(K) \subseteq J \). On the other hand, \( J \subseteq \Re(K) \) by Prop. 2.10, and hence we have \( J = \Re(K) \).

**Remark.** In Th. 2.11, \( \text{End}(R,M/\Re(R,M)) \) is naturally isomorphic to \( K/\Re(K) \). The following is the dual of Faith and Utumi [5; Th. 3.1].

**Theorem 2.12.** If \( M \) is \( R \)-quasi-projective and \( R \)-perfect, then \( J = \Re(K) \) and \( K/J \) is a regular ring.

**Proof.** By Prop. 2.10, \( J \subseteq \Re(K) \). Since a regular ring is semi-simple, it suffices to prove that \( K/J \) is a regular ring. Take any element \( \bar{\gamma} \) of \( K \) not contained in \( J \), and let \( A \) be a \( d \)-complement of \( M\bar{\gamma} \). Then \( M\bar{\gamma} \cap A \) is \( d \)-dense.
in $A$ by Prop. 1.3. For the epimorphism $\varphi = \nu(M \to M/A) : M \to M/A$, there exists a homomorphism $\epsilon : M \to M$ with $\epsilon \varphi = \nu(M \to M/A) \equiv \nu(M \to M/A)$ (Prop. 2.1). Then, $(\gamma - \gamma \epsilon \gamma) \cdot \nu(M \to M/A) = \nu(1 - \epsilon \gamma) \cdot \nu(M \to M/A) = 0$, and hence $M / (\gamma - \gamma \epsilon \gamma) \cong A \cap M \gamma$. As $A \cap M \gamma$ is $d$-dense in $M$, so is $M / (\gamma - \gamma \epsilon \gamma)$. Hence $\gamma - \gamma \epsilon \gamma \in J$, as desired.

Remark. Further, if $\mathfrak{r}(R)$ is $R$-$d$-dense in $M$, then $M / \mathfrak{r}(R)$ is $R$-completely reducible, and $\text{End}(M / \mathfrak{r}(R)) \cong K / \mathfrak{r}(K)$ naturally.

§ 3. Perfect projective modules.

The projective cover of $R$-$M$ is an epimorphism $\varphi : R \to M$ from an $R$-projective module $R \to M$ to $M$ such that $\text{Ker} \varphi$ is $R$-$d$-dense in $P$. The following proposition (the uniqueness of projective cover) is well known.

**Proposition 3.1.** Let $\varphi : P \to M$ and $\varphi' : P' \to M$ be projective covers of $M$. If $\sigma$ is a homomorphism from $P$ to $P'$ with $\sigma \varphi = \varphi$, then $\sigma$ is an isomorphism.

**Proposition 3.2** (Shukla [11]). If $M$ has a projective cover, and $\varphi$ is an epimorphism from an $R$-module $R \to M$ with $S \varphi = M$, where $S$ is an $R$-submodule of $L$, then exists an $R$-submodule $S_\sigma$ of $S$ that is minimal with respect to the property $S_\sigma \varphi = M$.

**Proof.** We may assume that $L = S$. Let $\varphi : P \to M$ be a projective cover of $M$, and let $\sigma$ be a homomorphism from $P$ to $L$ with $\sigma \varphi = \varphi$. Then $P \sigma$ is a desired one.

**Theorem 3.3.** Let $R \to P$ be an $R$-projective module. The following are equivalent:

(i) Every homomorphic image of $P$ has a projective cover.

(ii) $P$ is $R$-perfect.

**Proof.** (i) $\Rightarrow$ (ii) Let $A, B$ be submodules of $P$ with $A + B = P$. By assumption, $P/A$ has a projective cover. As $B \to (B + A)/A = P/A$, by Prop. 3.2 there exists a submodule $B_0$ of $B$ that is minimal with respect to the property $B_0 + A = P$. Hence $P$ is perfect. (ii) $\Rightarrow$ (i) Let $A$ be any submodule of $M$, and let $B$ be a $d$-complement of $A$ in $M$. Then, by Th. 2.3, $B$ is a direct summand of $P$, so that $B$ is projective. If $\varphi = \nu(P \to P/A) \mid B$, then $B \varphi = P/A$ and $\text{Ker} \varphi = A \cap B$ is $d$-dense in $B$ by Prop. 1.3. Hence $\varphi : B \to P/A$ is a projective cover of $P/A$.

**Proposition 3.4.** If an $R$-left module $P$ is perfect and projective, then $\mathfrak{r}(R \to P)$ is $R$-$d$-dense in $P$.

**Proof.** We may assume $P \neq 0$, then $\mathfrak{r}(R \to P) \neq P$ by Prop. 1.14. If $\mathfrak{r}(R \to P)$
is not d-dense in $P$, $R(P)$ contains a non-zero direct summand $X$ of $P$ by Th. 2.3. Since $X$ is a direct summand, $X$ is projective, and so $R(X)\neq X$ by Prop. 1.14. On the other hand, Prop. 1.3 yields a contradiction $R(X) = R(P) \cap X = X$.

$M \neq 0$ is called $R$-sum-irreducible, if for any pair of proper submodules $A, B$ of $M$, $A + B$ is also a proper submodule. The following is evident.

**Proposition 3.5.** Let $R(M) \neq M$. Then $M$ is $R$-sum-irreducible if and only if $R(M)$ is maximal in $M$. When it is the case, $R(M)$ is the unique maximal submodule of $M$, and $M = Rx$ for each $x \in M$ not contained in $R(M)$.

In this paper, a ring $T (\ni 1)$ is called a sum-irreducible ring, if $TT$ is sum-irreducible. As is well known, $TT$ is sum-irreducible if and only if $T$ is sum-irreducible.

**Theorem 3.6.** Let $M$ be $R$-quasi-projective. Then the following are equivalent:

(i) $M$ is $R$-sum-irreducible.

(ii) $K = \text{Hom}_{R}(R, M)$ is a sum-irreducible ring.

**Proof.** (i) $\Rightarrow$ (ii) Let $I + I^\prime = K$, where $I$ and $I^\prime$ are left ideals of $K$. Then, $K\alpha + K\alpha^\prime = K$ for some $\alpha \in I$, $\alpha^\prime \in I^\prime$. Then $M = M\alpha + M\alpha^\prime$, and so $M = M\alpha$ or $M = M\alpha^\prime$. If $M = M\alpha$, there exists an element $\beta$ of $K$ with $\beta\alpha = 1$, so that $I \supseteq K\alpha = K$. Hence $K$ is a sum-irreducible ring. (ii) $\Rightarrow$ (i) Let $A$ and $B$ be submodules of $M$ with $M = A + B$. Then $M/(A \cap B) = A/(A \cap B) + B/(A \cap B)$. Let $\overline{\alpha}$ be the projection of $M/(A \cap B)$ to $A/(A \cap B)$. Then, by assumption we can easily obtain $\omega$ in $K$ such that $(A \cap B)\omega \subseteq (A \cap B)$ and $(x + (A \cap B))\overline{\omega} = x\omega + (A \cap B)$ ($x \in M$). Since $K = K\omega + K(1 - \omega)$, $K = K\omega$ or $K = K(1 - \omega)$. Therefore, $M = M\omega$ or $M = M(1 - \omega)$. As $M\omega \subseteq A$ and $M(1 - \omega) \subseteq B$, we have eventually $M = A$ or $M = B$.

**Theorem 3.7.** If an $R$-left module $P \neq 0$ is projective, the following are equivalent:

(i) $P$ is $R$-perfect.

(ii) $R(P)$ is $R$-d-dense in $P$, and $P$ is a direct sum of $R$-sum-irreducible submodules.

**Proof.** (i) $\Rightarrow$ (ii) We put $R(P) = N$. Then, $N$ is $R$-d-dense in $P$ by Prop. 3.4, and $P/N$ is $R$-completely reducible by Th. 1.12 and Prop. 1.13. Let $P/N = \sum_{r \in \Gamma} A_r/N$, where each $A_r/N$ is minimal. As $P \rightarrow P/N \rightarrow A_r/N$, each $A_r/N$ has a projective cover $\varphi_r : P_r \rightarrow A_r/N$ by Th. 3.3. Since $\text{Ker} \varphi_r$ is d-dense in $P_r$ and $P_r/\text{Ker} \varphi_r \cong A_r/N$ is minimal, $P_r$ is $R$-sum-reducible. Let $P'$ be the external direct sum of $\{ P_r ; r \in \Gamma \}$, and $\varphi : P' \rightarrow P/N$ the homomor-
phism defined by $\langle x_i \rangle \varphi = \sum x_i \varphi_i \langle x_i \rangle \ni P'$. Then, $P'$ being projective, there exists a homomorphism $\tau : P' \rightarrow P$ with $\tau \cdot \nu(P \rightarrow P/N) = \varphi$. Since $N$ is d-dense in $P$, $\tau$ is an epimorphism. Let $\{\tau_1, \cdots, \tau_n\}$ be any finite subset of $\Gamma$, and let $P'' = \langle x_i \rangle \in P'; x_i = 0$ for each $i \in \left\{ \tau_1, \cdots, \tau_n \right\}$. We put $\sum \langle A_{\tau_i} \rangle/N = B$.

$P'' \xrightarrow{\varphi} B$

Then, we have a commutative diagram $\tau \downarrow \pi \uparrow \phi$, where $\phi = \nu(P \rightarrow P/N)$, $P \xrightarrow{\phi} P/N$.

and $\pi$ is the projection of $P/N$ to $B$. Since $\varphi|P'' : P'' \rightarrow B$ is a projective cover of $B$ and $(P''\tau)\varphi_\tau = B$, by Prop. 3.2 there exists a submodule $P_0$ of $P''\tau$ that is minimal with respect to the property $P_0 + \ker \phi_\tau = P$. Then, by Th. 2.3, $P_0$ is an $R$-direct summand of $P$, so that $P_0$ is projective. Since $P_0 \subseteq P''\tau$, we have $(P_\tau^{-1} \cap P'')\varphi = (P_\tau^{-1} \cap P'')\tau \varphi_\tau = P\phi_\tau P = B$. Recalling here that $\varphi|P''$ is a projective cover of $B$, it follows $P_\tau^{-1} \cap P'' = P''$, that is, $P''\tau = P_\tau$. Thus we have $P''\tau = P$. Since $\ker \varphi \cap P''$ is d-dense in $P''$, $(\ker \varphi \cap P'')\tau = \ker \phi_\tau \cap P_0$ is d-dense in $P_0$. Hence $\phi_\tau|P_0 : P_0 \rightarrow B$ is a projective cover of $B$. Therefore, Prop. 3.1 yields that $\tau|P''$ is an isomorphism. Thus we conclude that $\tau$ is an isomorphism. (ii) $\Rightarrow$ (i) Let $P = \sum \oplus P_\tau$, where each $P_\tau$ is sum-irreducible. Then, $P/N = \sum \oplus (P_\tau + N)/N$, and $(P + N)/N \cong P_\tau/(P_\tau \cap N) = P_\tau/\ker \rho_\tau$ is $R$-minimal. Now, take an arbitrary proper submodule $A$ of $P$. Since $P/N (\rightarrow P/(A + N))$ is $R$-completely reducible, there exists an $R$-isomorphism $\varphi : (P' + N)/N = \sum \oplus \oplus (P_\tau + N)/N \cong P/(A + N)$, where $P''$ is a subset of $\Gamma$ and $P'' = \sum \oplus P_\tau$. Let $\sigma : P' \rightarrow (P' + N)/N$ and $\rho : P/A \rightarrow P/(A + N)$ be canonical homomorphisms. Then, there exists a homomorphism $\phi : P' \rightarrow P/A$ with $\phi_\rho = \sigma_\varphi$, because $P'$ is $R$-projective. Since $N$ is d-dense in $P$, $\ker \rho = (A + N)/A$ is d-dense in $P/A$. As $P'\phi_\rho = P'\sigma_\varphi = P/(A + N)$, we have $P'\phi + \ker \rho = P/A$, so that $P'\phi = P/A$. Since $\ker \phi$ is contained in $\ker \sigma_\varphi = \ker \sigma = P' \cap N$ that is d-dense in $P'$ (Prop. 1.3), $\phi : P' \rightarrow P/A$ is a projective cover of $P/A$. Hence every homomorphic image of $P$ has a projective cover and therefore $P$ is $R$-perfect by Th. 2.3.

**Corollary.** Let $_R P$ be an $R$-projective module, and let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Then $P$ is $R$-perfect if and only if every $P_i$ is $R$-perfect.

In the proof (ii) $\Rightarrow$ (ii) of Th. 3.7, one may remark that the minimality of $A_i/N$ is not needed to prove that $\tau$ is an isomorphism. We obtained therefore the following:

**Proposition 3.8.** Let $_R P$ be projective and perfect, and let $P/N = \sum \oplus A_\lambda/N$, where $N = \mathfrak{R}(R_P)$. If $\varphi_\lambda : P_\lambda \rightarrow A_\lambda/N$ (for $\lambda \in \Lambda$) is a projective cover of $A_\lambda/N$, and $P'$ the external direct sum of $\{P_\lambda; \lambda \in \Lambda\}$, then any homo-
morphism $\alpha: P' \to P$ with $\alpha \cdot \psi(P \to P/N) = \varphi$ is an isomorphism, where $\varphi: P' \to P/N$ is defined by $((x_i)) \varphi = \sum x_i \varphi_i ((x_i) \in P')$.

**Theorem 3.9.** Let $\mathcal{P}$ be projective and perfect, $\{Q_r; r \in \Gamma\}$ a set of projective submodules such that each $Q_r \cap \Re(\mathcal{P})$ is $R$-d-dense in $Q_r$. Let $P = \sum_{x \in \mathcal{A}} \oplus P_x$, where each $P_x$ is sum-irreducible. If $\{Q_r; r \in \Gamma\}$ is independent modulo $\Re(\mathcal{P})$, $\{Q_r; r \in \Gamma\}$ is independent and $P = \sum_{x \in \mathcal{A}} \oplus P_x \oplus (\sum_{r \in \Gamma} \oplus Q_r)$ for some subset $\Lambda'$ of $\Lambda$.

**Proof.** We put $N = \Re(\mathcal{P})$, and for any element $x$ of $P$ and any submodule $A$ of $P$ we denote $x + N$ and $(A + N)/N$ by $\bar{x}$ and $\overline{A}$, respectively. Since $\bar{P} = \sum_{x \in \mathcal{A}} \oplus \bar{P}_x$ is completely reducible, as is easily seen, $\bar{P} = (\sum_{x \in \mathcal{A}} \oplus \bar{P}_x) \oplus (\sum_{r \in \Gamma} \oplus \bar{Q}_r)$ for some subset $\Lambda'$ of $\Lambda$. Then $P = \sum_{x \in \mathcal{A}} \oplus \bar{P}_x + \sum_{r \in \Gamma} \oplus \bar{Q}_r$, and so $P = \sum_{x \in \mathcal{A}} \oplus \bar{P}_x + \sum_{r \in \Gamma} \oplus \bar{Q}_r$ (Prop. 3.4). Now, $\rho_x: P_x \to \bar{P}_x$ and $\sigma_r: Q_r \to \bar{Q}_r$ are projective covers of $\bar{P}_x$ and $\bar{Q}_r$, respectively. Let $P'$ and $Q'$ be the external direct sum of $P_x$'s ($x \in \Lambda'$) and $Q_r$'s ($r \in \Gamma$), respectively. If $\alpha: (P', Q') \to P$ and $\varphi': (P', Q') \to \bar{P}$ are respectively defined by $((p_x), (q_r)) \alpha = \sum p_x + \sum q_r$, and $((p_x), (q_r)) \varphi' = \sum \rho_x + \sum \sigma_r$, then the following diagram is commutative:

$$
\begin{array}{ccc}
(P', Q') & \xrightarrow{\varphi'} & \bar{P} \\
\alpha \downarrow & & \downarrow \nu(P \to P/N) \\
P & & P \\
\end{array}
$$

By Prop. 3.8, $\alpha$ is then an isomorphism, and so $P = (\sum_{x \in \mathcal{A}} \oplus \bar{P}_x) \oplus (\sum_{r \in \Gamma} \oplus \bar{Q}_r)$, as desired.

For any non-zero idempotent $e$ of $K = \text{Hom}(\mathcal{P}, \mathcal{P})$, $\text{Hom}(\mathcal{P}, \mathcal{P}) \cong e\mathcal{P} \cong \text{Hom}(eK, eK)$. Consequently, $Ke$ is a sum-irreducible left ideal if and only if $eK$ is a sum-irreducible right ideal (Th. 3.6). A ring $T \ni 1$ is called a semi-perfect ring (Bass [3]), if $T$ is a direct sum of sum-irreducible left ideals. By the above remark, this is equivalent to that $T$ is a direct sum of sum-irreducible right ideals.

**Theorem 3.10.** The following are equivalent:

(i) $M$ is $R$-finitely generated, projective and perfect.

(ii) $M$ is a direct sum of a finite number of $R$-projective, sum-irreducible submodules.

(iii) $M$ is $R$-finitely generated and projective, and $K$ is a semi-perfect ring.

**Proof.** The equivalence (i) $\iff$ (ii) will be easily seen by Th. 3.7 and Prop. 3.5 (and Prop. 1.14). Next, let $M$ be $R$-finitely generated and projective. For an idempotent $e$ of $K$, $\text{End}(\mathcal{P}) \cong e\mathcal{P} \cong \text{End}(eK)$. Therefore, $Me$ is $R$-sum-irreducible if and only if $Ke$ is a sum-irreducible left ideal of $K$ (Th. 3.6). From this, as is easily seen, $M$ is a direct sum of a finite number of
sum-irreducible submodules if and only if $K$ is a direct sum of (a finite number of) sum-irreducible left ideals. Accordingly, Th. 3.7 proves at once (i) $\iff$ (iii).

**Proposition 3.11.** If $P$ is a perfect $R$-projective module, then $P/\alpha P$ is a perfect $R/\alpha$-projective module for any ideal $\alpha$ of $R$.

**Proof.** As is well known, $P/\alpha P$ is $R/\alpha$-projective. Since $P/\alpha P$ is $R$-perfect (Th. 1.12), $P/\alpha P$ is $R/\alpha$-perfect.

**Proposition 3.12.** If $P_{R} \neq 0$ is projective and perfect, then $\mathfrak{R}(\mathcal{P})$ is the unique maximal submodule of $P$ that contains no non-zero $R$-direct summand of $P$.

**Proof.** We set $N = \mathfrak{R}(\mathcal{P})$. Then, since $N$ is $R$-d-dense in $P$, $N$ does not contain a non-zero $R$-direct summand. If a non-zero submodule $X$ of $P$ is not contained in $N$, there exists a proper submodule $Y$ of $P$ with $P = X + Y$. By Th. 2.3, $X$ contains a non-zero $R$-direct summand of $P$.

An element $c$ of $R$ is called a root element of $R$ if $cR$ does not contain a non-zero idempotent (see Azumaya [1]). We denote the set of all root elements of $R$ by $c$. Azumaya called $R$ a strongly semi-primary ring if $c$ is an ideal and $R/c$ satisfies the descending chain condition for right ideals, and proved the following ([10]): For a ring $R \ni 1$, the following are equivalent: (i) $R$ is strongly semi-primary, (ii) $c$ is an ideal and $R$ is a direct sum of mutually orthogonal idempotents $e_{1}, \ldots, e_{n}$ such that $\sum_{i} e_{i} = 1$ and each $e_{i} Re_{i}$ is a completely primary ring (or a sum-irreducible ring).

By Th. 3.6, (iii) is equivalent to that $R$ is a semi-perfect ring, namely, a strongly semi-primary ring is nothing but a semi-perfect ring.

**Proposition 3.13.** Let $D$ be a $d$-dense submodule of $M$. (1) $M$ has a projective cover if and only if so does $M/D$. (2) Every homomorphic image of $M$ has a projective cover if and only if so does every homomorphic image of $M/D$.

**Proof.** (1) Let $\varphi : P \to M$ be a projective cover of $M$. If $D \varphi^{-1} + X = P$ for some $R$-submodule $X$ of $P$, then $D + X \varphi = M$, and so $X \varphi = M$, that is, $X + \text{Ker} \varphi = P$. Since $\text{Ker} \varphi$ is $d$-dense in $P$, we have $X = P$, whence it follows that $D \varphi^{-1}$ is $d$-dense in $P$. Hence $\varphi \cdot \nu(M \to M/D) : P \to M/D$ is a projective cover of $M/D$. Conversely, if $\varphi : Q \to M/D$ is a projective cover of $M/D$, there exists an $R$-homomorphism $\sigma : Q \to M$ with $\sigma \cdot \nu(M \to M/D) = \varphi$. Then, $Q \sigma + D = M$, and therefore $Q \sigma = M$. As $\text{Ker} \sigma \subseteq \text{Ker} \varphi$, $\text{Ker} \sigma$ is $d$-dense in $Q$. Hence, $\sigma$ is a projective cover of $M$. (2) For any submodule $A$ of $M$ we have a canonical homomorphism $\sigma_{A} : M/A \to M/(A + D)$, and $\text{Ker} \sigma_{A} = (A + D)/A$. 

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is d-dense in $M/A$. Therefore, by (1), $M/A$ has a projective cover if and only if $M/(A+D)\cong (M/D)/(D+A)/D$ has a projective cover. Hence, every homomorphic image of $M$ has a projective cover if and only if every homomorphic image of $M/D$ has a projective cover.


**Proposition 4.1.** (1) Let $\mathfrak{l}$ be a left ideal of $R$. If $\mathfrak{l}'$ and $\mathfrak{l}''$ are d-complements of $\mathfrak{l}$, then $\mathfrak{l}'=\mathfrak{l}'' \cdot \mathfrak{l}'$. (2) Let $\mathfrak{a}$ be an ideal of $R$. If $\mathfrak{b}$ is a d-complement of $\mathfrak{a}R$, then $\mathfrak{b}\mathfrak{c}=\mathfrak{b}$ for any ideal $\mathfrak{c}$ with $\mathfrak{a}+\mathfrak{c}=R$, and hence $\mathfrak{b}$ is the unique d-complement of $\mathfrak{a}$.

**Proof.** (1) Since $\mathfrak{l}'' \subseteq \mathfrak{l}'' \cdot \mathfrak{l}'=\mathfrak{l}' \cdot \mathfrak{l}'' \cdot \mathfrak{l}'$, we have $R=I+I''=I+I'' \cdot \mathfrak{l}'$, and the minimality of $\mathfrak{l}'$ implies $\mathfrak{l}'=\mathfrak{l}'' \cdot \mathfrak{l}'$. (2) Let $R=\mathfrak{a}+\mathfrak{c}$ for some ideal $\mathfrak{c}$ of $R$. Then, as $\mathfrak{b}=\mathfrak{b}R=\mathfrak{b}\mathfrak{a}+\mathfrak{b}\mathfrak{c}$, we have $R=\mathfrak{a}+\mathfrak{b}=\mathfrak{a}+\mathfrak{b}\mathfrak{c}$, and the minimality of $\mathfrak{b}$ implies $\mathfrak{b}=\mathfrak{b}\mathfrak{c}$.

**Theorem 4.2.** (1) Let $R=\mathfrak{b}_1+\cdots+\mathfrak{b}_n$ be a sum of sum-irreducible ideals $\mathfrak{b}_i$ ($i=1,\cdots,n$), and $\mathfrak{a}$ an ideal of $R$. Then, $\sum_{\mathfrak{b}_i \subset \mathfrak{a}} \mathfrak{b}_i$ is the unique d-complement of $\mathfrak{a}R$, and $R$ is a direct sum of directly indecomposable ideals.

(2) $R$ is $R$-R-perfect if and only if $R$ is a sum of a finite number of sum-irreducible ideals.

**Proof.** (1) Evidently $R=\mathfrak{a}+\sum_{\mathfrak{b}_i \subset \mathfrak{a}} \mathfrak{b}_i$. If $R=\mathfrak{a}+\mathfrak{b}$ for some ideal $\mathfrak{b}$, then $\mathfrak{b}_i=\mathfrak{b}_i \cdot \mathfrak{a}+\mathfrak{b}_i \cdot \mathfrak{b}$, and so $\mathfrak{b}_i=\mathfrak{b}_i \cdot \mathfrak{a} \subseteq \mathfrak{a}$ or $\mathfrak{b}_i=\mathfrak{b}_i \cdot \mathfrak{b} \subseteq \mathfrak{b}$. Hence $\mathfrak{b} \supseteq \sum_{\mathfrak{b}_i \subset \mathfrak{a}} \mathfrak{b}_i$, which means that $\sum_{\mathfrak{b}_i \subset \mathfrak{a}} \mathfrak{b}_i$ is the unique d-complement of $\mathfrak{a}R$. Since any $R$-R-direct summand of $R$ is an $R$-R-d-complemented submodule, the latter is evident. (2) The “if” part is contained in (1). Assume that $R$ is $R$-R-perfect. We put $\mathfrak{R}(R)=I$. Then $R/I$ is $R$-R-perfect and $\mathfrak{R}(R/(I))=0$ (Th. 1.12). Hence $R/I$ is $R$-R-completely reducible by Prop. 1.13. Therefore there are ideals $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$ of $R$ such that $R/I=\sum_{i=1}^n \mathfrak{a}_i/I$ and $\mathfrak{a}_i/I$ is $R$-R-minimal. Then, $R=\mathfrak{a}_1+\cdots+\mathfrak{a}_n$, and evidently this is an irredundant sum. Let $\mathfrak{b}_i$ be an $R$-R-d-complement of $\mathfrak{a}_i+\cdots+\mathfrak{a}_i+\cdots+\mathfrak{a}_n$ with $\mathfrak{b}_i \subseteq \mathfrak{a}_i$. If $\mathfrak{b}_i \subseteq I$, then $R=\mathfrak{a}_1+\cdots+\mathfrak{a}_i+\cdots+\mathfrak{a}_n+I$, and so $R=\mathfrak{a}_1+\cdots+\mathfrak{a}_i+\cdots+\mathfrak{a}_n$ (Prop. 1.15), a contradiction.

Hence, $\mathfrak{b}_i+I=\mathfrak{a}_i$ by the minimality of $\mathfrak{a}_i/I$, and so $R=\mathfrak{b}_i+\cdots+\mathfrak{b}_n+I$, whence it follows $R=\mathfrak{b}_i+\cdots+\mathfrak{b}_n$. Since $\mathfrak{b}_i$ is an $R$-R-complemented submodule, $R_{(\mathfrak{b}_i)R}=\mathfrak{b}_i \cap I$ by Prop. 1.3, and hence $\mathfrak{b}_i/\mathfrak{R}(\mathfrak{b}_iR)$ is $R$-R-isomorphic to the minimal $\mathfrak{a}_i/I$. We have proved thus each $\mathfrak{b}_i$ is a sum-irreducible ideal (Prop. 3.5).

**Theorem 4.3.** Let $R$ be an $h$-central ring. Then the following are equivalent:
(i) \( R \) is \( R-R \)-perfect.

(ii) \( R \) is a direct sum of a finite number of sum-irreducible ideals.

(iii) The center of \( R \) is a direct sum of sum-irreducible ideals.

\textbf{Proof.} The equivalence (ii) \( \Leftrightarrow \) (iii) is easy by Th. 2.4 and Th. 3.6. (ii) \( \Rightarrow \) (i) is trivial by Th. 4.2 (2). (i) \( \Rightarrow \) (ii) By Th. 4.2, \( R \) is an irredundant sum of some sum-irreducible ideals \( b_i \) \((i=1, \ldots, n)\). Since \( b_i \) and \( b_1+\cdots+b_n \) are \( R-R \)-complements of each other (Th. 4.2), \( b_i \cap (b_1+\cdots+b_n)=0 \) by Th. 2.3. Hence \( R=b_1\oplus\cdots\oplus b_n \), as desired.

\textbf{Proposition 4.4.} Let \( R \) be a semi-perfect ring, \( \mathfrak{a} \) an ideal of \( R \). If \( I \) and \( \pi \) are respective \( d \)-complements of \( _{R}\mathfrak{a} \) and \( \mathfrak{a}_{R} \), then \( \pi=IR=R\pi \), which coincides with the unique \( d \)-complement of \( _{R}\mathfrak{a}_{R} \). Therefore, \( R \) is perfect as an \( R-R \)-module.

\textbf{Proof.} As \( \pi \subseteq R\pi=\mathfrak{a}R+R\pi \), we have \( R=\mathfrak{a}+R\pi \). Let \( b \) be an ideal with \( R=\mathfrak{a}+b \). Then \( \pi R=\mathfrak{a}R+\pi b \), and so \( R=\mathfrak{a}+\pi b \). The minimality of \( \pi \) implies \( \pi=\mathfrak{a}R \subseteq b \), so that \( R\pi \subseteq b \). Similarly we have \( IR \subseteq b \), in particular, if we set \( b=\mathfrak{a}R \), then \( IR=\pi R \). Thus we know that \( \pi R=\pi R=IR \), and this is the unique \( d \)-complement of \( _{R}\mathfrak{a}_{R} \).

\textbf{Proposition 4.5.} If \( R \) is a semi-perfect ring, then \( \mathfrak{N}(R)=\mathfrak{N}(_{R}R_{R}) \).

\textbf{Proof.} Since \( R/\mathfrak{N}(R) \) is \( R-R \)-completely reducible, we have \( \mathfrak{N}(_{R}R_{R}) \subseteq \mathfrak{N}(R) \), and evidently \( \mathfrak{N}(R) \subseteq \mathfrak{N}(_{R}R_{R}) \). Hence \( \mathfrak{N}(R)=\mathfrak{N}(_{R}R_{R}) \).

\textbf{Theorem 4.6.} The following conditions for \( R \) are equivalent:

(i) \( R \) is a semi-perfect \( h \)-central ring.

(ii) There are \( \text{sum-irreducible} \) \( h \)-central rings \( R_1, \ldots, R_s \), and natural numbers \( m_1, \ldots, m_s \) such that \( R\cong(R_1)_{m_1}\oplus\cdots\oplus(R_s)_{m_s} \) (external direct sum) as rings.

\textbf{Proof.} (ii) \( \Rightarrow \) (i) Since each \( R_i \) is a \( \text{sum-irreducible} \) \( h \)-central ring, \( (R_i)_{m_i} \) \((\cong\text{End}(R_i))_{m_i} \) is a semi-perfect \( h \)-central ring by Cor. to Prop. 2.9 and Th. 3.10. Hence, as is easily seen, \( R \) is a semi-perfect \( h \)-central ring. (i) \( \Rightarrow \) (ii) By Th. 4.3, \( R=\mathfrak{a}_1\oplus\cdots\oplus\mathfrak{a}_n \) with some \( \text{sum-irreducible} \) ideals \( \mathfrak{a}_i \). Since each \( \mathfrak{a}_i \) is evidently a \( \text{semi-perfect} \) \( h \)-central ring, we may assume that \( R \) is \( R-R \)-sum-irreducible. Let \( R=I_1\oplus\cdots\oplus I_s \), where each \( I_i \) is a \( \text{sum-irreducible} \) left ideal (Th. 3.7). We put \( N=\mathfrak{N}(R) \). Then \( R/N=\sum I_i+N/N \) and \( I_i/(I_i\cap N)=I_i/\mathfrak{N}(I_i) \) is minimal (Prop. 1.3). Since \( R/N \) is \( R-R \)-minimal, \( I_i/\mathfrak{N}(I_i) \cong I_i/\mathfrak{N}(I_i) \) for all \( i \). Hence \( I_i \cong I_i \) for all \( i \), because \( I_i \cong I_i/\mathfrak{N}(I_i) \) is a \( \text{projective} \) \( \text{cover} \) of \( I_i/\mathfrak{N}(I_i) \) (Prop. 3.1). Thus \( R\cong(\text{End}(I_i))_{m_i} \). Since \( I_i \) is \( \text{sum-irreducible} \) and \( R \) is \( h \)-central, \( \text{End}(I_i) \) is a \( \text{sum-irreducible} \) \( h \)-central ring by Th. 3.6 and Cor. to Prop. 2.9.
Proposition 4.7. Let \( R \) be a semi-perfect ring, and \( R = b_1 + \cdots + b_n \) the irredundant sum of sum-irreducible ideals \( b_i \) \( (i = 1, \cdots, n) \) of \( R \). Then, \( b_i = R e R \) for any non-zero idempotent \( e \) in \( b_i + R \).

Proof. We put \( R(R) = (R(R_R)) = N \). Since \( Re \subseteq b_1 + N \), we have \( Re = b_1 e + Ne \), so that \( Ne \) is \( R \)-dense in \( Re \) (Prop. 1.1). Hence \( R e R \subseteq b_1 \). Since \( b_1 \) is the unique \( R-R \)-complement of \( b_2 + \cdots + b_n \), there holds \( \mathfrak{R}_e b_i R = b_1 \cap N \) (Prop. 1.3), and \( b_i / (b_i \cap N) = (b_i + N) / N \) is \( R-R \)-minimal (Prop. 3.5). As \( e \in N \), the minimality of \( (b_i + N) / N \) implies \( (R e R + N) / N = (b_i + N) / N \), or equivalently, \( R e R + N = b_i + N \). Hence \( b_i = R e R + (b_i \cap N) \), so that \( b_i = R e R \), because \( b_i \cap N \) is \( R-R \)-dense in \( b_i \) (Prop. 1.3).

§ 5. A theorem for modular lattices.

In this section, we shall restate the fundamental theorem which was given in the preceding paper[8]. The theorem is valid for any modular lattice with 0. Throughout the section, \( \mathfrak{S} \) will means a modular lattice with 0, and we shall use conveniently “+” instead of “\( \vee \)”. A non-empty subset \( \mathfrak{S} \) is called independent, if each finite subset of \( \mathfrak{S} \) is independent, namely, \( a_i \wedge (a_2 + \cdots + a_n) = 0 \) for any different \( a_i, \cdots, a_n \) in \( \mathfrak{S} \).

Proposition 5.1. If \( \{a_1, \cdots, a_n\} (n \geq 3) \) is an independent subset of \( \mathfrak{S} \), and \( a_i \wedge (a_2 + \cdots + a_n) = 0 \) for \( a_i \in \mathfrak{S} \), then \( (a_i + a_j) \wedge (a_3 + \cdots + a_n) = 0 \).

Proof. \( a_i \wedge (a_2 + \cdots + a_n) = 0 \) yields \( (a_2 + \cdots + a_n) \wedge (a_i + a_j) = a_2 \), and so \( 0 = a_2 \wedge (a_3 + \cdots + a_n) = (a_2 + \cdots + a_n) \wedge (a_i + a_j) \wedge (a_3 + \cdots + a_n) = (a_i + a_j) \wedge (a_3 + \cdots + a_n) \).

Proposition 5.2. Let \( \{a_1, \cdots, a_n\} \) be an independent subset of \( \mathfrak{S} \), and \( a \) an element of \( \mathfrak{S} \). If \( a \wedge (a_1 + \cdots + a_n) = 0 \), then \( \{a, a_1, \cdots, a_n\} \) is independent (and conversely).

Proof. We may assume that \( n \geq 2 \). Since \( (a + a_1) \wedge (a_2 + \cdots + a_n) = 0 \) by Prop. 5.1, we have \( (a + a_1) \wedge (a_3 + \cdots + a_n) = 0 \). Therefore \( a_1 \wedge (a_2 + \cdots + a_n) = a_1 \wedge (a_2 + \cdots + a_n) = a = 0 \).

Proposition 5.3. Let \( \{a_i; \lambda \in \Lambda\} \) be an independent finite subset of \( \mathfrak{S} \), and let \( \Lambda_1, \Lambda_2 \) be non-empty subsets of \( \Lambda \). If \( \Lambda_1 \cap \Lambda_2 = \emptyset \), then \( \sum_{i \in \Lambda_1} a_i \wedge \sum_{i \in \Lambda_2} a_i = 0 \). If \( \Lambda_1 \cap \Lambda_2 \neq \emptyset \), then \( \sum_{i \in \Lambda_1} a_i \wedge \sum_{i \in \Lambda_2} a_i = \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i \).

Proof. Let \( \{a_i; \lambda \in \Lambda\} = \{a_1, \cdots, a_n\} \). Then, by Prop. 5.1, \( a_i \wedge (a_2 + \cdots + a_n) = 0 \) yields \( (a_2 + \cdots + a_n) \wedge (a_3 + \cdots + a_n) = 0 \), so that \( \{a_1 + a_2, a_3, \cdots, a_n\} \) is independent by Prop. 5.2. Repeating the same procedure, we know that \( (a_i + \cdots + a_n) \wedge (a_{i+1} + \cdots + a_n) = 0 \) for all \( i \). Thus we obtain the first assertion. Next we assume that \( \Lambda_1 \cap \Lambda_2 \neq \emptyset \). Then \( \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i \leq \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i + \sum_{i \in \Lambda_1 \cap \Lambda_2} a_1 \leq \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i \). Therefore \( \sum_{i \in \Lambda_1} a_i \wedge \sum_{i \in \Lambda_1} a_i = \sum_{i \in \Lambda_1} a_i \wedge (\sum_{i \in \Lambda_1 \cap \Lambda_2} a_i + \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i) = \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i + \sum_{i \in \Lambda_1 \cap \Lambda_2} a_i \).
\[(\sum_{i \in \mathfrak{L}} a_i \wedge \sum_{i \in 1 - a_i} a_i) = \sum_{i \in a_i \cap a_i} a_i, \text{ as desired.}\]

**Proposition 5.4.** Let \(\{a\} \cup \{a_i; \lambda \in \Lambda\} \) be an independent subset of \(\mathfrak{L}\). If \(\{b_i; \bar{r} \in \Gamma\}\) is an independent subset of \(\mathfrak{L}\) such that \(b_i \leq a\) for all \(\bar{r} \in \Gamma\), then \(\{a_i; \lambda \in \Lambda\} \cup \{b_i; \bar{r} \in \Gamma\}\) is independent.

**Proof.** The proof will be easy, and may be omitted.

Let \(a, b, c\) be elements of \(\mathfrak{L}\). If \(a \wedge c = b \wedge c = 0\) and \(a \wedge (b + c) \neq 0\), then \(b \wedge (a + c) \neq 0\) (Prop. 5.2). In this case, we write \(a \sim b\). Evidently, if \(a, b \in \mathfrak{L}\) and \(0 \neq a \leq b\), then \(a \sim b\). Let \(x, y\) be in \(\mathfrak{L}\). If there exists a subset \(\{c_1, \ldots, c_n, x_1, \ldots, x_{n-1}\}\) with \(x \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim y\), we write \(x \sim y\). Obviously, \(\sim\) is an equivalence relation.

A non-zero element \(u\) is called uniform, if \(x \wedge y \neq 0\) for all non-zero elements \(x, y\) of \(\mathfrak{L}\) with \(x, y \leq u\). In the rest of this section, we assume that \(\mathfrak{L}\) contains at least one uniform element.

**Proposition 5.5.** Let \(u, v\) be uniform elements of \(\mathfrak{L}\) with \(u \wedge v \neq 0\), and \(x\) an element of \(\mathfrak{L}\). If \(x \wedge u = 0\) then \(x \wedge v = 0\) (and conversely).

**Proof.** If \(x \wedge v \neq 0\), then \(0 \neq (x \wedge v) \wedge (u \wedge v) = x \wedge v \wedge u\), and so \(x \wedge u \neq 0\).

**Corollary.** Let \(\{u_\lambda; \lambda \in \Lambda\}\) and \(\{v_\lambda; \lambda \in \Lambda\}\) be subsets of uniform elements of \(\mathfrak{L}\) such that \(u_\lambda \wedge v_\lambda \neq 0\) for all \(\lambda \in \Lambda\). If \(\{u_\lambda; \lambda \in \Lambda\}\) is independent, then so is \(\{v_\lambda; \lambda \in \Lambda\}\).

**Proof.** By the validity of Prop. 5.2 and Prop. 5.5, the proof will proceed in the same way as that of [8; Prop. 1.1] does.

**Proposition 5.6.** Let \(\{a_1, \ldots, a_n\} (n \geq 2)\) be an independent subset of \(\mathfrak{L}\), and \(a\) an element of \(\mathfrak{L}\). If \(a \wedge (a_1 + \cdots + a_n) \neq 0\) and \(a \wedge (a_2 + \cdots + a_n) = 0\), then \(a \wedge (a_1 + c) \sim (a + c) \wedge a_i\), where \(c = a_2 + \cdots + a_n\).

**Proof.** We set \(a' = a \wedge (a_1 + c)\). Evidently \(a' \wedge c = (a_1 + c) \wedge a \wedge c = 0\) and \((a + c) \wedge a_i) \wedge c = a_i \wedge c = 0\). Moreover, \((a + c) \wedge a_i) + c = (a + c) \wedge (a_1 + c)\) and \(a' \wedge (a + c) \wedge (a_1 + c) = a' \neq 0\).

**Corollary.** Under the assumption of Prop. 5.6, if \(a\) and \(a_1\) are uniform, then \(a \sim a_1\).

Let \(\mathfrak{U}\) be the set of all uniform elements of \(\mathfrak{L}\). A non-empty subset \(\mathfrak{U}_0\) of \(\mathfrak{U}\) is called a \((*)\)-subset of \(\mathfrak{U}\), if \(\mathfrak{U}_0\) satisfies the following condition:

\((*)\) if \(v \in \mathfrak{U}_0\) and \(0 \neq u \leq v\), then \(u \in \mathfrak{U}_0\).

Evidently, if \(\mathfrak{U}_1\) and \(\mathfrak{U}_2\) are \((*)\)-subsets with \(\mathfrak{U}_1 \cap \mathfrak{U}_2 \neq \emptyset\), then so is \(\mathfrak{U}_1 \cap \mathfrak{U}_2\).

**Proposition 5.7.** Let \(\mathfrak{U}_0\) be a \((*)\)-subset of \(\mathfrak{U}\), \(\{u_\lambda; \lambda \in \Lambda\}\) an independent
subset of \( \mathcal{U} \), and \( u \) an element of \( \mathcal{U} \) such that \( \{u\} \cup \{u_{\lambda}; \lambda \in \Lambda\} \) is dependent. Then, there exists a unique minimal finite subset \( \{u_{i};\lambda \in \Lambda\} \) of \( \{u_{i}; \lambda \in \Lambda\} \) such that \( u \wedge (u_{1} + \cdots + u_{n}) \neq 0 \). For such \( u_{i} \), there holds \( u \sim u_{i} \) (\( i = 1, \ldots, n \)). Accordingly, if \( \{u_{i}; \lambda \in \Lambda\} \) is a maximal independent subset of \( \mathcal{U} \), then \( \{u_{i}; u_{i} \notin \mathcal{U}\} \) is a maximal independent subset of \( \mathcal{U}_{0} \cap \mathcal{U}_{u} \), where \( \mathcal{U}_{u} = \{v \in \mathcal{U}; u \sim v\} \).

**Proof.** This is a direct consequence from Prop. 5.3 and Cor. to Prop. 5.6.

**Proposition 5.8.** If \( \{u_{i}; \lambda \in \Lambda\} \) and \( \{v_{i}; r \in \Gamma\} \) are maximal independent subsets of a \( (*) \)-subset \( \mathcal{U}_{0} \), then \( \#A = \# \Gamma \), where \( \#A \) means the cardinal number of \( A \).

**Proof.** We shall distinguish here between two cases.

Case 1. \( \#A < \aleph_{0} \) or \( \# \Gamma < \aleph_{0} \). We may assume \( \# \Gamma \leq \#A \). We set \( \{v_{i}; r \in \Gamma\} = \{v_{1}, \ldots, v_{s}\} \), and take \( \lambda_{0} \in \Lambda \). If \( \{v_{i}\} \cup \{u_{i}; \lambda \neq \lambda_{0}\} \) is dependent for all \( i \), then for each \( v_{i} \) there exists a unique minimal finite subset \( \{u_{i}'s\} \) of \( \{u_{i}; \lambda \neq \lambda_{0}\} \) with \( v_{i}' = v_{i} \wedge \sum u_{i,j} \neq 0 \). Evidently \( \{v_{1}', \ldots, v_{s}'\} \) is independent, and further, as \( u_{i} \wedge (v_{1}' + \cdots + v_{s}') \leq u_{i} \wedge \sum u_{i,j} = 0 \), \( \{u_{i}\} \cup \{v_{i}', \ldots, v_{s}'\} \) is independent (Prop. 5.2), so that \( \{u_{i}\} \cup \{v_{i}, \ldots, v_{s}\} \) is independent (Cor. to Prop. 5.5), a contradiction. Hence, for some \( v_{s} \), \( \{v_{k} \} \cup \{u_{i}; \lambda = \lambda_{0}\} \) is independent. Now, \( u_{s}' = u_{s} \wedge (v_{k} + u_{1} + \cdots + u_{n}) \neq 0 \) for some \( \{u_{i}, \ldots, u_{n}\} = \{u_{i}; \lambda = \lambda_{0}\} \). We assume that \( \{c, v_{k}\} \cup \{u_{i}; \lambda = \lambda_{0}\} \) is independent for some \( c \in \mathcal{U}_{0} \). Then, since \( \{u_{i}; \lambda = \lambda_{0}\} \) is a maximal independent subset of \( \mathcal{U}_{0} \), \( c \wedge (u_{i}' + u_{n} + \cdots + u_{m}) \neq 0 \) for some \( \{u_{i}, \ldots, u_{n}\} \subseteq \{u_{i}; \lambda = \lambda_{0}\} \). On the other hand, \( c \wedge (u_{i}' + u_{n} + \cdots + u_{m}) \leq c \wedge (v_{k} + u_{1} + \cdots + u_{n}) + u_{n} + \cdots + u_{m} = 0 \), a contradiction. Hence \( \{v_{k}\} \cup \{u_{i}; \lambda = \lambda_{0}\} \) is a maximal independent subset of \( \mathcal{U}_{0} \). Continuing the same argument step by step, we obtain eventually \( s = \#A \).

Case 2. \( \#A \geq \aleph_{0} \) and \( \# \Gamma \geq \aleph_{0} \). For each \( v \in \{v_{i}; r \in \Gamma\} \), there exists a unique minimal finite subset \( \{u_{1}, \ldots, u_{n}\} \) of \( \{u_{i}; \lambda \in \Lambda\} \) such that \( v \wedge (u_{1} + \cdots + u_{n}) \neq 0 \). We shall prove that \( \cup v \{u_{1}, \ldots, u_{n}\} = \{u_{i}; \lambda = \lambda_{0}\} \). For each \( u \in \{u_{i}; \lambda = \lambda_{0}\} \), there exists a unique minimal finite subset \( \{v_{1}, \ldots, v_{m}\} \) of \( \{v_{i}; r \in \Gamma\} \) such that \( u \wedge (v_{1} + \cdots + v_{m}) \neq 0 \). Now, if \( \{u_{i}'s\} \) is the unique minimal finite subset of \( \{u_{i}; \lambda \in \Lambda\} \) with \( v_{i}' = v_{i} \wedge \sum_{i} u_{i,j} = 0 \) (\( i = 1, \ldots, m \)), then \( u \wedge (v_{1} + \cdots + v_{m}) \neq 0 \) by Cor. to Prop. 5.5. Therefore, we have \( u \wedge \sum_{i,j} u_{i,j} = 0 \), which implies \( u \in \{u_{i,j}; i, j\} \). Thus we have \( \cup v \{u_{1}, \ldots, u_{n}\} = \{u_{i}; \lambda = \lambda_{0}\} \). Hence \( \#A \leq \# \Gamma \cdot \aleph_{0} = \# \Gamma \). And, we have symmetrically \( \# \Gamma \leq \#A \). Hence \( \#A = \# \Gamma \).

Combining Prop. 5.7 with Prop. 5.8, we obtain the following fundamental theorem.

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5) This is evidently a \( (*) \)-subset of \( \mathcal{U} \).
Theorem 5.9. Let \( \mathcal{U}_o \) be a \((\ast)\)-subset of \( \mathcal{U} \). If \( \{ u_i; \lambda \in \Lambda \} \) and \( \{ v_i; \gamma \in \Gamma \} \) are maximal independent subsets of \( \mathcal{U}_o \), then there exists a 1-1 mapping \( f \) from \( \Lambda \) onto \( \Gamma \) such that \( u_\lambda \sim v_{f(\lambda)} \) for all \( \lambda \in \Lambda \).

Corresponding to [8; Prop. 1.9], we obtain the following whose proof is easy by Prop. 5.7.

Proposition 5.10. Let \( \{ \mathcal{U}_i; \lambda \in \Lambda \} \) be a family of \((\ast)\)-subsets of \( \mathcal{U} \) such that if \( \lambda \neq \lambda' \) then \( u \nsim u' \) for all \( u \in \mathcal{U}_i, u' \in \mathcal{U}_{i'} \), and let \( \{ u_i; \gamma \in \Gamma_i \} \) be a maximal independent subset of \( \mathcal{U}_i \). Then \( \bigcup_{i \in \Lambda} \{ u_i; \gamma \in \Gamma_i \} \) is a maximal independent subset of \( \bigcup_{i \in \Lambda} \mathcal{U}_i \).

The set of all \( R \)-submodules of \( M \) forms a modular lattice \( \mathcal{L}(RM) \) with respect to dual order\(^6\). A subset \( \mathcal{B} \) of \( R \)-submodules is called \( d \)-independent if \( B_i \cap \bigcap_{i \neq j} B_j = M \) for each subset \( \{ B_1, \cdots, B_n \} \) of \( \mathcal{B} \) \((n \geq 2)\). A submodule \( B \) \((\neq M)\) is called \( d \)-uniform, if \( M/B \) is \( R \)-sum-irreducible. Evidently, every maximal submodule is \( d \)-uniform.

Let \( B, C \) be \( d \)-uniform submodules of \( M \). If \( M/B' \cong M/C' \) for some proper submodules \( B', C' \) of \( M \) with \( B \subseteq B' \) and \( C \subseteq C' \) we write \( B \approx C \) (d-similar)\(^7\). As is easily seen, this is an equivalence relation concerning the \( d \)-uniform submodules of \( M \).

Let \( A, B, C \) be submodules with \( A + C = B + C = M \) and \( A + (B \cap C) \neq M \). Then, \( C \rightarrow (A + C)/((A + (B \cap C) = M/(A + (B \cap C)) \), and therefore \( C/((A \cap C) + (B \cap C)) \cong M/(A + (B \cap C)). \) Similarly, we have \( C/((A \cap C) + (B \cap C)) \cong M/(A \cap C) + B). \) Hence \( M/(A + (B \cap C)) \cong M/(A \cap C) + B). \) If moreover \( A \) and \( B \) are \( d \)-uniform, then \( A \cong B \). Thus, Th. 5.9 yields at once the following:

Theorem 5.11. Let \( \{ A_i; \lambda \in \Lambda \} \) and \( \{ B_i; \gamma \in \Gamma \} \) be maximal \( d \)-independent subsets of the \( d \)-uniform submodules of \( M \). Then, there exists a 1-1 mapping \( f \) from \( \Lambda \) onto \( \Gamma \) with \( M/A_i \cong M/B_{f(i)} \) for all \( \lambda \in \Lambda \). Accordingly we may set \( \text{max-dim } \mathcal{L}(RM) = \#\Lambda \).

Theorem 5.12. Let \( \{ A_i; \lambda \in \Lambda' \} \) and \( \{ B_i; \gamma \in \Gamma' \} \) be maximal \( d \)-independent subsets of the maximal submodules of \( M \). Then, there exists a 1-1 mapping \( g \) from \( \Lambda' \) onto \( \Gamma' \) with \( M/A_i \cong M/B_{g(i)} \) for all \( \lambda \in \Lambda' \). Accordingly we may set \( \text{max-dim } \mathcal{L}(RM) = \#\Lambda' \).

Proof. Let \( \mathcal{L} = \mathcal{L}(RM) \) the modular lattice of all submodules of \( M \) defined above, \( \mathcal{U} \) the set of all \( d \)-uniform submodules of \( M \), and \( \mathcal{U}_0 \) the set of all maximal submodules of \( M \). Then Th. 5.9 applies to obtain our assertion.

Evidently, \( \text{max-dim } \mathcal{L}(RM) \leq \text{d-dim } \mathcal{L}(RM) \). If every proper submodule of \( M \) is contained in a maximal submodule, then \( \text{max-dim } \mathcal{L}(RM) \) by Cor. to

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6) For \( R \)-submodules \( A, B \) of \( M, A \geq B \) if and only if \( A \subseteq B \).

7) "d-similar" is the dual concept of "similar" defined by [8].
Prop. 5.5.

**Remark.** (1) We consider $R$ as an $R$-module. Then, the set of all maximal ideals is $d$-independent, because maximal ideals are prime ideals. Hence $(\max\dim R R =) d$-$\dim R R = \text{the cardinal number of the set of all maximal ideals of } R$. (2) Let $M = A_1 \oplus \cdots \oplus A_n$, where each $A_\ell$ is $R$-sum-irreducible. Put $B_\ell = A_1 \oplus \cdots \oplus \hat{A}_\ell \oplus \cdots \oplus A_n$. Then $\{B_1, \cdots, B_n\}$ is a maximal $d$-independent subset of the $d$-uniform submodules of $M$. Hence, in this case, $d$-$\dim M = n$. (3) In Th. 5.12, if every proper submodule of $M$ is contained in a maximal submodule, then $\cap_\ell A_\ell = \Re(R_M)$. Because, if $\cap_\ell A_\ell \not\supseteq \Re(R_M)$, then $A + \cap_\ell A_\ell = M$ for some maximal submodule $A$ of $M$. Then, $\{A\} \cup \{A_\ell; \ell \in A'\}$ is $d$-independent, a contradiction.

Proposition 5.13. If $D$ is a $d$-dense submodule of $M$, then $d$-$\dim \frac{RM}{D}$ and $\max\dim \frac{RM}{D} = \max\dim \frac{RM}{D}$.

**Proof.** Let $\{A_\ell; \ell \in A\}$ be a maximal $d$-independent subset of the set of $d$-uniform (resp. maximal) submodules of $M$. If $A_\ell' = A_\ell + D$, then $M/A_\ell \rightarrow M/A_\ell' \cong (M/D)/(A_\ell/D)$, and so $\{A_\ell'/D; \ell \in A\}$ is a $d$-independent subset of the set of $d$-uniform (resp. maximal) submodules of $M/D$. Let $X$ be a submodule of $M$ such that $X \supseteq D$ and $(M/D)/(X/D)$ ($\cong M/X$) is $R$-sum-irreducible (resp. minimal). Then, since $\{A_\ell; \ell \in A\}$ is a maximal $d$-independent subset of the set of $d$-uniform (resp. maximal) submodules (Cor. to Prop. 5.5), $X + (A_\ell' \cap \cdots \cap A_\ell'') \neq M$ for some finite subset $\{A_\ell', \cdots, A_\ell''\}$ of $\{A_\ell'; \ell \in A\}$, and so $X/D + \bigcap_\ell A_\ell'/D \neq M/D$. Hence $\{A_\ell'/D; \ell \in A\}$ is a maximal $d$-independent subset of the set of $d$-uniform (resp. maximal) submodules of $M/D$.

Proposition 5.14. Assume that every proper submodule of $M$ is contained in a maximal submodule of $M$. Then, $d$-$\dim \frac{RM}{M}$ if and only if $M/\Re(R_M)$ is a direct sum of a finite number of $R$-minimal submodules (or equivalently, if $M/\Re(R_M)$ satisfies the descending chain condition for $R$-submodules (Prop. 1.13)). When it is the case, $M$ is $R$-finitely generated.

**Proof.** As $\Re(R_M)$ is $R$-dense in $M$ by Prop. 1.14, we obtain $\max\dim \frac{RM}{\Re(R_M)}$ by Prop. 5.13. If $M/\Re(R_M)$ is $R$-finitely generated then $M$ is $R$-finitely generated (and conversely). Hence it suffices to prove that if $d$-$\dim \frac{RM}{\Re(R_M)} = 0$ then $M$ is a direct sum of a finite number of minimal submodules. Let $\{A_1, \cdots, A_n\}$ be a maximal $d$-independent subset of maximal submodules. Then $A_1 \cap \cdots \cap A_n = 0$ (Remark (3) to Th. 5.10). As is well known, this implies that $M$ is a direct sum of a finite number of minimal submodules, so that $M$ is $R$-finitely generated.

**Corollary.** If $d$-$\dim \frac{R}{\Re(R)}$, then $d$-$\dim \frac{R}{R_d} = \text{the length of}$
an $R$-composition series of $\frac{R}{\mathfrak{R}(R)}$ (or $\frac{R}{\mathfrak{R}(R)}_R$).

Finally, as an application of Th. 5.9, we can derive Azumaya's generalization of Krull-Remak-Schmidt's theorem [2]. In what follows, we shall give only the facts which are needed to see the last.

(1) Let $e$ and $f$ be idempotents of $K=\text{Hom}(\kappa M, \kappa M)$. Then $\text{Hom}(\kappa Me, \kappa Mf)\cong eKf\cong \text{Hom}(fK_e, eK_f)$, canonically. Therefore, $Me\cong Mf$ as $R$-modules if and only if $fK\cong eK$ as right ideals.

(2) Let $e$ and $f$ be idempotents of $K$ different from 1 such that $eK$ and $fK$ are $d$-uniform right ideals of $K$. Then, $eK\approx fK$ ($d$-similar) if and only if $K/eK\cong K/fK$, or equivalently $(1-e)K\cong (1-f)K$. For $(1-e)K(\cong K/eK)$ and $(1-f)K(\cong K/fK)$ are projective (Prop. 3.1).

(3) We put $\mathfrak{U}$ the modular lattice $\mathfrak{U}(K_K)$ of all right ideals of $K$ with respect to dual order, $\mathfrak{U} =$ the set of all $d$-uniform right ideals of $K$ and $\mathfrak{U}_d =$ the set of all $d$-uniform right ideals which contain $d$-uniform direct summands of $K$. Then $\mathfrak{U}_d$ is a $(\ast)$-subset of $\mathfrak{U}$.

(4) Let $M=\sum_{i\in I} M_i$, where each $\text{Hom}(\kappa M_i, \kappa M_j)$ is a sum-irreducible ring (a completely primary ring in [2]), and let $e_i$ be the projection to $M_i$. Then $\{(1-e_i)K; \lambda \in \Lambda\}$ is a maximal $d$-independent subset of the set of all $d$-uniform right ideals which contain $d$-uniform direct summands of $K$. In fact, the $d$-independence of $\{(1-e_i)K; \lambda \in \Lambda\}$ will be easily seen. To prove the maximality, we take any idempotent $f$ of $K$ such that $fK$ is $d$-uniform. As $1-f\neq 0$, $M(1-f)\neq 0$. Let $x$ be arbitrary non-zero element of $M(1-f)$. Then there exists a finite subset $\{M_{i_1}, \cdots, M_{i_n}\}$ of $\{M_i; \lambda \in \Lambda\}$ with $x \in M_{i_1} + \cdots + M_{i_n}$. Since $xf=0$ and $xk=0$ for each $k \in (1-e_i)K$, $fK + (1-e_i)K \neq K$.

§. 6. Lemma on radical and perfectness.

Let $\mathfrak{L}$ be a lattice with 0. $\mathfrak{L}$ is called perfect if, for any pair of elements $x, y$ of $\mathfrak{L}$ with $x \wedge y = 0$ there exists an element $y'$ of $\mathfrak{L}$ that is maximal with respect to the property $y \leq y'$ and $x \wedge y' = 0$. A non-zero element $d$ of $\mathfrak{L}$ is said to be dense in $\mathfrak{L}$, if $d \wedge y = 0$ for each non-zero $y$ in $\mathfrak{L}$.

**Lemma 6.1.** Let $\mathfrak{L}, \mathfrak{K}$ be lattices with 0, and $\varphi$ an order preserving mapping from $\mathfrak{L}$ to $\mathfrak{K}$, $\phi$ an order preserving mapping from $\mathfrak{K}$ to $\mathfrak{L}$. We assume that the pair $(\varphi, \phi)$ satisfies the following conditions:

\begin{enumerate}
  \item[(a)] $a \wedge b = 0 \ (a, b \in \mathfrak{L}) \Rightarrow a\varphi \wedge b\phi = 0$.
  \item[(b)] $x \wedge y = 0 \ (x, y \in \mathfrak{K}) \Rightarrow x\phi \wedge y\phi = 0$.
  \item[(c)] $a \leq a\varphi\phi$ and $x \leq x\varphi\phi$ for any $a \in \mathfrak{L}$, $x \in \mathfrak{K}$.
\end{enumerate}

Then there hold the following:

\begin{enumerate}
  \item $(\mathfrak{L}$ is perfect if and only if $\mathfrak{K}$ is perfect.
  \item If $d$ is a dense element of $\mathfrak{L}$, then $d\varphi$ is a dense element of $\mathfrak{K}$, and
conversely.

Proof. (1) Let $\mathcal{L}$ be perfect, and let $a, b$ be elements of $\mathcal{L}$ with $a \wedge b = 0$. Then $ap \wedge b \varphi = 0$ by assumption. If $y$ is an element that is maximal with respect to the property $b \varphi \leq y$ and $a \varphi \wedge y = 0$, then $a \varphi \wedge y \varphi = 0$, and so $a \wedge y \varphi = 0$, because $a \leq a \varphi \varphi$. If $b'$ is an element of $\mathcal{L}$ with $y \varphi \leq b'$ and $a \wedge b' = 0$, then $a \varphi \wedge b' \varphi = 0$. As $y \leq y \varphi \varphi \leq b' \varphi$, we have $y = b' \varphi$, and so $b' \leq b' \varphi \varphi = y \varphi$. Hence $y \varphi$ is maximal with respect to the property $b \leq y \varphi$ and $a \wedge y \varphi = 0$. Thus $\mathcal{L}$ is perfect. (2) If $d \varphi \wedge x = 0 \ (x \in \mathcal{I})$ then $d \varphi \wedge x \varphi = 0$. As $d \leq d \varphi$, $d \wedge x \varphi = 0$, and so $x \varphi = 0$. Since $x \leq x \varphi = 0$, we have $x = 0$. Hence $d \varphi$ is dense in $\mathcal{I}$. Conversely, assume that $d \varphi$ is dense in $\mathcal{I}$. If $d \wedge a = 0 \ (a \in \mathcal{L})$ then $d \varphi \wedge a \varphi = 0$, and so $a \varphi = 0$. As $a \leq a \varphi \varphi = 0$, we have $a = 0$. Thus $d$ is dense in $\mathcal{L}$.

Let $\mathcal{L}, \mathcal{I}$ be complete lattices. By $s(\mathcal{L})$ (resp. $s(\mathcal{I})$), we denote the meet of all dense elements of $\mathcal{L}$ (resp. $\mathcal{I}$).

Lemma 6.2. Let $\mathcal{L}, \mathcal{I}$ be complete lattices, and let $\varphi$ be a mapping from $\mathcal{L}$ to $\mathcal{I}$, and $\psi$ a mapping from $\mathcal{I}$ to $\mathcal{L}$ satisfying the following conditions:

(α′) $0 \varphi = 0$ and $(\bigcap_{\gamma \in \Gamma} a_{\gamma}) \varphi = \bigcap_{\gamma \in \Gamma} (a_{\gamma} \varphi)$ for each subset $\{a_{\gamma} ; \gamma \in \Gamma \}$ of $\mathcal{L}$.
(b′) $0 \varphi = 0$ and $(\bigcap_{\lambda \in \Lambda} a_{\lambda}) \varphi = \bigcap_{\lambda \in \Lambda} (a_{\lambda} \varphi)$ for each subset $\{a_{\lambda} ; \lambda \in \Lambda \}$ of $\mathcal{I}$.
(r′) $a = a \varphi \varphi$ and $x \varphi \leq x \varphi \varphi$ for each $a \in \mathcal{L}$, $x \in \mathcal{I}$.

Then there hold the following:

(1) $s(\mathcal{L}) = s(\mathcal{I}) \varphi$
(2) Let $s(\mathcal{L})$ be dense in $\mathcal{L}$. If $x \varphi \geq s(\mathcal{L}) \ (x \in \mathcal{I})$ then $s(\mathcal{I}) \leq x$.

Proof. $s(\mathcal{I}) \leq s(\mathcal{L}) \varphi$ and $s(\mathcal{L}) \leq s(\mathcal{I}) \varphi$ implies that $s(\mathcal{I}) \varphi \leq s(\mathcal{L})$, which proves (1). If $s(\mathcal{L})$ is dense in $\mathcal{L}$, and $x \varphi \geq s(\mathcal{L})$, then $x \varphi$ is dense in $\mathcal{I}$, so that $x$ is dense in $\mathcal{I}$ by Lemma 6.1 (2). Hence $x \geq s(\mathcal{I})$.

Now, let $N$ be a unital $R$-left module. We set $L = \text{End} \ (nN)$, which acts on the right. Then, $A = \text{Hom} \ (nM, nN)$ is a $K$-left, $L$-right module, and $B = \text{Hom} \ (nN, nM)$ is an $L$-left, $K$-right module, where $K = \text{End} \ (nM)$. In what follows, our attention will be restricted to the important case $A \cdot B = K$. We consider then the modular lattices $\mathcal{L}(K_{x}), \mathcal{L}(\mathcal{K}), \mathcal{L}(A_{L}), \mathcal{L}(R_{a}), \mathcal{L}(B_{K}), \mathcal{L}(B \cdot A_{L})$ and $\mathcal{L}(A_{L} \cdot B)$. Among them, we can define several pairs of mappings $(\varphi, \psi)$ satisfying the conditions (α′), (b′), (r′) in Lemma 6.2:

(1) For $\mathcal{L}(K_{x})$ and $\mathcal{L}(A_{L})$, we define $X \varphi = X \cdot A$ and $Y \psi = Y \cdot B \ (X \in \mathcal{L}(K_{x}), Y \in \mathcal{L}(A_{L}))$.
(2) For $\mathcal{L}(\mathcal{K})$ and $\mathcal{L}(B_{K})$, we define $X \varphi = B \cdot X$ and $Y \psi = A \cdot Y \ (X \in \mathcal{L}(\mathcal{K}), Y \in \mathcal{L}(B_{K}))$.
(3) For $\mathcal{L}(R_{a})$ and $\mathcal{L}(B \cdot A_{L})$, we define $X \varphi = B \cdot X$ and $Y \psi = A \cdot Y \ (X \in \mathcal{L}(R_{a}), Y \in \mathcal{L}(B \cdot A_{L}))$.
$\mathfrak{L}(\lambda A), \ Y \in \mathfrak{L}(\lambda B \cdot A)$.

(4) For $\mathfrak{L}(B_X)$ and $\mathfrak{L}(B \cdot A_L)$, we define $X\varphi = X \cdot A$ and $Y\psi = Y \cdot B(\ X \in \mathfrak{L}(B_X), \ Y \in \mathfrak{L}(B \cdot A_L))$.

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