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RELATIONS BETWEEN TWO MARTIN TOPOLOGIES
ON A RIEMANN SURFACE

By

Zenjiro Kuramochi

Let $R$ be a Riemann surface. Let $G$ be a domain in $R$ with relative boundary $\partial G$ of positive capacity. Let $U(z)$ be a positive superharmonic function in $G$ such that the Dirichlet integral $D(\min(M, U(z))) < \infty$ for every $M$. Let $D$ be a compact domain in $G$. Let $\partial U^M(z)$ be the lower envelope of superharmonic functions $\{U_n(z)\}$ such that $U_n(z) \geq \min(M, U(z))$ on $D + \partial G$ except a set of capacity zero, $U_n(z)$ is harmonic in $G - D$ and $U_n(z)$ has M.D.I. (minimal Dirichlet integral) $\leq D(\min(M, U(z))) < \infty$ over $G - D$ with the same value as $U_n(z)$ on $\partial G + \partial D$. Then $\partial U^M(z)$ is uniquely determined. Put $\partial U(z) = \lim_{M=\infty} \partial U^M(z)$. If for any compact domain $D \partial U(z) = U(z)$ or $\partial U(z) \leq U(z)$, we call $U(z)$ a full harmonic (F.H.) or a full superharmonic (F.S.H.) in $G$ respectively. If $U(z)$ is an F.S.H. in $G$ and $U(z) = 0$ on $\partial G$ except a set of capacity zero, $U(z)$ is called an F.s.H. in $G$. Let $U(z)$ be an F.S.H. in $G$. Then $\partial U(z)$ is an exhaustion of $G$ with compact relative boundary $\partial G_n$ ($n = 0, 1, 2 \cdots$).

$\mathfrak{M}(U(z))$ of an F.s.H. $U(z)$ in $G$. Let $D$ be a domain in $G$. Suppose there exists at least one $C_1$-function $V(z)$ in $G - D$ such that $V(z) = 1$ on $D$, $= 0$ on $\partial G$ except a set of capacity zero and $D(V(z)) < \infty$. Let $\omega(D, z, G)$ be a harmonic function in $G - D$ such that $\omega(D, z, G) = 1$ on $D$, $= 0$ on $\partial G$ except a set of capacity zero and $\omega(D, z, G)$ has M.D.I. over $G - D$. We call $\omega(D, z, G)$ a C.P. (capacitary potential) of $D$. Let $U(z)$ be an F.s.H. in $G$. Then $D(\omega_m U(z)) = M D(\omega(g_m, z, G))$ as $M \to 0$, where $g_m = E[z: U(z) > M]$. Put $\mathfrak{M}(U(z)) = \lim_{M=\infty} \frac{1}{2\pi} D(\omega_m U(z))$.

$\mathfrak{M}(U(z))$ of an F.S.H. $U(z)$ in $G$. For any compact domain $D$ in $G$, if we can define functions $U_n(z)$ such that $U_n(z)$ is superharmonic in $G$, $U_n(z)$ is harmonic in $G - D$, $U_n(z) \geq \min(M, U(z))$ on $D$, $U_n(z) = 0$ on $\partial G$ except a set of capacity zero and $U_n(z)$ has M.D.I. over $G - D$. Let $\omega U^M(z)$ be the lower envelope of $\{U_n(z)\}$. Put $\partial U(z) = \lim_{M=\infty} \partial U^M(z)$ (clearly $\partial U(z) \leq \partial U(z)$).

Relations between Two Martin Topologies on a Riemann Surface

Since $D$ is compact, $\mathfrak{g}_n U(z) = 0$ on $\partial G$ except a set of capacity zero. For a noncompact domain $D$, $\mathfrak{g}_n U(z)$ is defined as $\mathfrak{g} U(z)$. For $U(z)$, put $\mathfrak{W}(U(z)) = \lim \mathfrak{W}^\prime (\mathfrak{a}_n U(z))$, where $\{G_n\}$ is an exhaustion of $G$ with compact relative boundary $\partial G_n$.

Let $\{R_n\}$ with compact relative boundary $\partial R_n$ ($n = 0, 1, 2, \cdots$) Let $U(z)$ be an F$_0$.S.H. in $R - R_0$ such that $U(z) = 0$ on $\partial R_0$. Consider $R - R_0$ as $G$. Then $\mathfrak{g} U(z)$ is defined. In this case we say that $\mathfrak{g} U(z)$ is defined relative to $R - R_0$. It is clear that the mapping $U(z) \rightarrow \mathfrak{g} U(z)$ depends on the domain ($G$ or $R - R_0$) in which $\mathfrak{g} U(z)$ is defined. In the following we use $\mathfrak{g} U(z)$ relative to $R - R_0$ which will be denoted by $\mathfrak{g}_n U(z)$ to distinguish from $\mathfrak{g} U(z)$ (relative to $G$). We understand $\mathfrak{g} U(z)$ (without $R$ on $D$) means $\mathfrak{g} U(z)$ of $U(z)$ relative to $G$.

**Martin topologies on $R - R_0$ and on a subdomain $G \subset (R - R_0)$**. Let $N(z, p)$ be an $N$-Green’s function of $G$ such that $N(z, p)$ is positively harmonic in $G - p$, $N(z, p) = 0$ on $\partial G$ except a set of capacity zero, $N(z, p)$ has a logarithmic singularity at $p$ and $N(z, p)$ has M.D.I. (where Dirichlet integral is taken with respect to $N(z, p) + \log|z - p|$ in a neighbourhood of $p$). We suppose $N$-Martin topology is defined on $G + B$ using $N(z, p)$,s and the distance between $p_1$ and $p_2$ is given as

$$\delta(p_1, p_2) = \sup_{z \in D} \left| \frac{N(z, p_1)}{1 + N(z, p_1)} - \frac{N(z, p_2)}{1 + N(z, p_2)} \right|,$$

where $D$ is a fixed compact domain and $B$ is the set of the ideal boundary. Let $L(z, p)$ be an $N$-Green’s function of $R - R_0$ with pole at $p$. Then also $N$-Martin topology is introduced on $R - R_0 + B^L$ with metric:

$$\delta(p_1, p_2) = \sup_{z \in R, L} \left| \frac{L(z, p_1)}{1 + L(z, p_1)} - \frac{L(z, p_2)}{1 + L(z, p_2)} \right|,$$

where $B^L$ is the set of the ideal boundary points.

In the following for simplicity we call above two topogies $L$ and $N$-topologies.

Let $p \in R - R_0 + B^L$ ($B^L$ is the set of minimal boundary points of $R - R_0$). If $c_0^L L(z, p) < L(z, p)$ ($CG$ is thin at $p$), we denote by $p \subset G$. Then

**Theorem 1.** Suppose $p \in R - R_0 + B^L$ and $p \subset G$. Then $U(z, p) = L(z, p) - N(z, p)$ is an F$_0$.S.H. in $G$ with $D(\min(M, U(z))) \leq 2\pi M$, whence $\mathfrak{W'}(U(z, p)) \leq 1$.

**Proof.** $N(z, p) : p \in R - R_0 + B^L$ is continuous on $\partial G$ except $p$. Hence $c_0^L L(z, p) = L(z, p)$ on $\partial G$ and $U(z, p) = 0$ on $\partial G$ except a set of capacity zero.

Case 1. $p \in G$. In this case, clearly $U(z, p) = N(z, p)$ and $D(\min(M, U(z, p))) \leq 2\pi M$.

Case 2. $p \in \partial G$. Put $G_n = G + v_n(p)$, Then $CG_n \uparrow CG$ and $c_0^L L(z, p) \uparrow c_0^L L(z, p)$
as $n \to \infty$, where $v_n(p) = E\left[ z : \text{dist}(z, p) < \frac{1}{n} \right]$. By $p \in G_n$, we have

$$D(\min(M, U(z, p))) \leq \lim_{n} D(\min(M, L(z, p) - c_\Theta L(z, p))) \leq 2\pi M$$

Case 3. $p \in B_1^\ell - B_1^L$. In this case it was proved\(^2\) $D(\min(M, U(z, p))) \leq 2\pi M$, where $B_1^L$ is the set of singular points, i.e. set of point $p$ such that $\omega(p, z, R - R_0) > 0$ and $B_1^L$ is the set of minimal boundary points of $R - R_0$.

Case 4. $p \in B_1^L$. It was proved only $D(U(z, p)) < \infty$ but as case 3 it can be proved $D(\min(M, U(z, p))) \leq 2\pi M$.

Hence $\frac{\delta}{4} U(z, p)$ can be defined. Now $\delta_0 L(z, p) = \delta_0 L(z, p)$ by $CG + D) \subset CG$ and $\delta_0 L(z, p) \leq L(z, p)$. Hence $U(z, p) = \delta_0 L(z, p) - \delta_0 L(z, p) \leq L(z, p) - \delta_0 L(z, p) = U(z, p)$. By $D(\min(M, U(z, p))) \leq 2\pi M$ we have at once $\mathfrak{M}(U(z, p)) \leq 1$. Thus $U(z, p)$ is an $F_0$. S.H. in $G$ with $\mathfrak{M}(U(z, p)) \leq 1$.

**Lemma 1.** 1). Let $p_i \in R - R_0$ and $p_i \to p \in R - R_0 + B_1^\ell$ ($p_i$ tends to $p$ relative to $L$-topology). Then $L(z, p) - \lim_{i} L(z, p_i) \leq L(z, p) - \delta_0 L(z, p)$.

2). Let $p_i \to p^a \in R - R_0 + B_1^\ell$ and $p_i \to p^a \in G + B : p_i \in G$. Then

$$N(z, p^a) = (1 - a) (L(z, p^a) - \delta_0 L(z, p^a)): 1 \geq a \geq 0.$$  

**Proof of 1.** For any $\varepsilon > 0$ we can find a number $n_0$ such that $\delta_0 L(z, p) \leq \delta_0 L(z, p^a) \leq \delta_0 L(z, p^a) + \varepsilon$ for $n \geq n_0$. Since $L(z, p_i) \to L(z, p^a)$ on $CG \cap R_n$, $\lim_{i} \delta_0 L(z, p_i) \geq \lim_{i} \delta_0 L(z, p_i) \geq \delta_0 L(z, p^a) - \varepsilon$. Let $\varepsilon \to 0$. Then we have (1).

**Proof of 2.** $L(z, p_i) - \lim_{i} \delta_0 L(z, p_i) = N(z, p_i)$ in $G$ for $p_i \in G$. By the assumption $\lim_{i} L(z, p_i)$ and $\lim_{i} N(z, p_i)$ exist, whence $\lim_{i} \delta_0 L(z, p_i)$ exists. We denote this limit by $U$. Let $\mu$ be a canonical mass distribution\(^4\) of $U(z)$ on $R - R_0 + B_1^L$. Assume $\mu$ has a positive mass in int $(G \cap CV_n(p^a))$ (int $G$ means the interior of $G$ relative to $L$-topology and $v_n(p^a)$ is a neighbourhood of $p^a$ relative to $L$-topology). Then we can find a number $n_0$ such that $G_{n_0}$ has a positive mass on $\overline{G}_{n_0} \cap CV_n(p^a)$, where $G_n = E\left[ z \in R - R_0 + B_1^\ell : \text{dist}(z, CG) > \frac{1}{n} \right]$. Since $\text{dist}(CG + v_n(p^a), G_n - v_n(p^a)) > 0$.


\(^3\) If $p \in G$, $U(z, p) = N(z, p)$, we suppose $p \in B^\ell$. Then $L(z, p)$ is harmonic in $R - R_0$, whence $\sup L(z, p) < \infty$ on a compact domin $D$ and it is clear $D U(z) = \delta_0 L(z, p) - \delta_0 L(z, p)$. If $D$ is non compact, consider $D \cap G_n$ and let $n \to \infty$.

Relations between Two Martin Topologies on a Riemann Surface

$\mathcal{R}^{\star}_{\alpha+n+i}(p^\alpha)L(z, p^\alpha) = L(z, p^\alpha)$ (for $p^\alpha \in R - R + B_i^L$) we have

\[ N(z, p^\alpha) = L(z, p^\alpha) - U(z) > \mathcal{R}^{\star}_{\alpha+n+i}(p^\alpha)L(z, p^\alpha) - \mathcal{R}^{\star}_{\alpha+n+i}(p^\alpha)U(z) = \mathcal{R}^{\star}_{\alpha+n+i}(p^\alpha)(L(z, p^\alpha) - U(z)). \quad (1) \]

On the other hand, $L(z, p^\alpha) - U(z) = N(z, p^\beta)$ is an $F_{0}$ S.H. in $G$, whence

\[ v_{n+i}(p^\alpha)(L(z, p^\alpha) - U(z)) \leq L(z, p^\alpha) - U(z). \quad (2) \]

(1) contradicts (2). Hence $\mu = 0$ on $Cv_{n}(p) \cap \text{int} G$. Let $n \to \infty$. Then $\mu = 0$ except on $p + CG$. Put $V(z) = \int L(z, p)d\mu'(p)$, where $\mu'$ is the restriction of $\mu$ on $CG$. Let $a$ be the mass of $\mu$ at $p$. Then $1 \geq a \geq 0$, $c_{\theta}^{R}V(z) = V(z)$ and $U(z) = V(z) + aL(z, p^\alpha)$. Now $V(z) = (1 - a)L(z, p^\alpha)$ on $\partial G$ except a set of capacity zero. Hence $V(z) = c_{\theta}^{R}V(z) = (1 - a)c_{\theta}^{R}L(z, p^\alpha)$. Thus $U(z) = (1 - a)c_{\theta}^{R}L(z, p^\alpha) + aL(z, p^\alpha)$ and

\[ N(z, p^\alpha) = L(z, p^\alpha) - \lim_{i}c_{\theta}^{R}eL(z, p_{i}) = (1 - a)((L(z, p^\alpha) - L(z, p_{i})) \quad (3) \]

We denote by $B(G)$ the set of points $p$ such that $p \in R - R_{0} + B_{1}^L$, $p \in B$ and $p \in G$. Clearly $B(G)$ is an $F_{0}$ set relative to $L$-topology by the upper semicontinuity of $L(z, p) - c_{\theta}^{R}L(z, p)$ and if $p \in \partial G$, $p \in B(G)$ if and only if $p$ is an irregular point for the Dirichlet problem in $G$ by Lemma 1. (2).

**Lemma 2.** Let $p_{i} \in B(G) + G$ and $p_{1} \neq p_{2}$. Then $L(z, p_{1}) - c_{\theta}^{R}L(z, p_{1}) \neq L(z, p_{2}) - c_{\theta}^{R}L(z, p_{2})$.

Assume $L(z, p_{1}) - c_{\theta}^{R}L(z, p_{1}) = L(z, p_{2}) - c_{\theta}^{R}L(z, p_{2}) = U(z)$. Let $n$ be a number such that $\text{dist}(v_{n}(p_{1}), v_{n}(p_{2})) > 0$, where $v_{n}(p_{i})$ is a neighbourhood of $p_{i}$ relative to $L$-topology. Now $p_{2} \in v_{n}(p_{2})$ imply

\[ (G \cap v_{n}(p_{2})) \ni p_{2}. \quad (6) \]

Let $V_{n} = G - v_{n}(p_{1})$. Then $V_{n} \supset (G \cap v_{n}(p_{2})) \ni p_{2}$. Whence

\[ c_{\theta}^{R}L(z, p) < L(z, p). \]

By $CV_{n} \supset CG$ we have $c_{\theta}^{R_{n}}(c_{\theta}^{R}L(z, p_{i})) = c_{\theta}^{R}L(z, p_{i}) : i = 1, 2$. Now $c_{\theta}^{R_{n}}L(z, p) \downarrow$ as $n \to \infty$ by $CV_{n}.$ Hence there exist a point $z_{0}$ in $V_{n}$, a number $n_{0}$ and a const. $\delta > 0$ such that $c_{\theta}^{R_{n}}L(z_{0}, p_{2}) < L(z_{0}, p_{2}) - \delta$ for $n \geq n_{0}$. Hence

\[ c_{\theta}^{R_{n}}(U(z_{0})) = c_{\theta}^{R_{n}}L(z_{0}, p_{2}) > c_{\theta}^{R}L(z_{0}, p_{2}) - c_{\theta}^{R}L(z_{0}, p_{2}) - \delta = U(z_{0}) - \delta : n \geq n_{0}. \quad (3) \]

5) See page 60 of 4).

6) See page 99 of 2).
By $CV_{n}+(v_{n}(p_{1})\cap CG)\supset v_{n}(p_{1})$, we have

$$c_{n}^{R}L(z, p_{1}) + v_{n}(p_{1}) \supset CG \Rightarrow v_{n}(p_{1}) = L(z, p_{1}).$$

We proved if a domain $\Omega \in p$, $\lim_{n}v_{\mathcal{R}}(p)\cap c^{R}QL(z, p) = 0$.  
Hence for any $\varepsilon > 0$ there exists a number $n'$ such that

$$CV_{n}RL(z_{0}, p_{1}) \geq L(z_{0}, p_{1}) - \varepsilon \text{ for } n \geq n'.$$

By (3) and (4) $U(z_{0}) - \delta \geq U(z_{0}) - \varepsilon$. This is a contradiction. Hence $L(z, p_{1}) - c_{\Omega}^{R}L(z, p_{1}) = L(z, p_{1}) - \varepsilon$.

Let $p^{*}$ be a point in $G + B(G)$. If there exists a sequence $\{p_{i}\}$ such that $p_{i}^{L} \rightarrow p^{*}$ and $p_{i}^{M} \rightarrow p^{*} \in G + B$, we say that $p^{*}$ lies on $p^{*}$. We denote the set of points $p$ lying on $p^{*}$ by $\mathfrak{p}(p^{*})$. Then

**Lemma 3.** Let $p^{*} \in G + B(G)$. Then $\mathfrak{p}(p^{*})$ contains only one point $p^{*}$ of $G + B_{1}$ and $L(z, p^{*}) - c_{\Omega}^{R}L(z, p) = N(z, p^{*})$, where $B_{1}$ is the set of minimal boundary points of $G$ relative to $L$-topology. We denote such $p^{*}$ by $f(p_{n}).$

Let $p_{n}^{L} \rightarrow p^{*}$ and $p_{n}^{M} \rightarrow p^{*}$. Then by Lemma 1.2 $N(z, p^{*}) = (1 - a_{\beta})(L(z, p^{*}) - c_{\Omega}^{R}L(z, p^{*})).$ Hence any function $N(z, p^{*})$ corresponding to $p^{*}$ is a submultiple of a fixed function and there exists at most one minimal or inner point $p^{*}$ of $G + B_{1}$ in $\mathfrak{p}(p^{*})$ such that $\mathfrak{M}(p^{*}) = 1$ ($\mathfrak{M}(p^{*}) = \mathfrak{M}(N(z, p^{*})) = 1$ is a necessary condition for $p^{*}$ to be minimal). Let $p^{*} \in G + B(G)$ and $v_{n}(p^{*})$ be a neighbourhood of $p^{*}$ relative to $L$-topology and $\bar{v}_{n}(p^{*})$ be the closure of $v_{n}(p^{*})$ relative by $M$-topology. Then by $p \in G + B(G)$ $L(z, p^{*}) - c_{\Omega}^{R}L(z, p^{*}) = \delta_{p}N(z, p^{*})$:

$$\delta_{p} = \frac{1}{1 - a_{\beta}}$$

and by $v_{n}(p^{*})L(z, p^{*}) = L(z, p)$ and $CG + v_{n}(p^{*}) \supset CG$ we have

$$\delta_{p}N(z, p^{*}) = L(z, p^{*}) - c_{\Omega}^{R}L(z, p^{*}) = \delta_{p}N(z, p^{*}) = L(z, p^{*}).$$

Let $n \rightarrow \infty$. Then $N(z, p^{*}) = \inf_{n > 0}N(z, p^{*}) > 0$, where $F = \cap \bar{v}_{n}(p^{*})$ is a $M$-closed set, whence $N(z, p^{*})$ is representable by a canonical mass distribution on $F$. This implies $\mathfrak{p}(p^{*})$ contains at least one point in $G + B_{1}$. Thus $\mathfrak{p}(p^{*})$ contains only one point $p^{*}$ in $G + B_{1}$ and $(1 - a^{*})(L(z, p^{*}) - c_{\Omega}^{R}L(z, p^{*})) = N(z, p^{*})$. On the other hand, $\mathfrak{M}(L(z, p^{*}) - c_{\Omega}^{R}L(z, p^{*})) \leq 1$ by Theorem 1 and $\mathfrak{M}(N(z, p^{*})) = 1$.

7) See 6).
8) See Lemma 4 of 1).
9) See 5).
Hence $a^\ast = 0$ and $L(z, p^\ast) - c_0^L L(z, p^\ast) = N(z, p^\ast)$.

**Theorem 2.** Let $p^\delta$ be a point in $G + B_1$. Let $f^{-1}(p^\delta)$ be the set of points $p$ in $R - R_0 + B^L$ (not only in $G + B_1$) such that $L(z, p) - c_0^L L(z, p) = N(z, p^\delta)$. Then $f^{-1}(p^\delta)$ consists of only one point $p \in G + \tilde{B}(G)$. Hence the mapping $f(p^\delta)$: $p^\delta \in G + \tilde{B}(G)$ is one-to-one manner between $G + \tilde{B}(G)$ and $G + B_1$ and further $f^{-1}(p^\delta)$ is a continuous function of $p^\delta$ in $G + B_1$, but $f(p^\alpha)$ is not necessarily continuous in $G + \tilde{B}(G)$.

Let $p \in f^{-1}(p^\delta)$. Then $L(z, p) - c_0^L L(z, p)$ is minimal in $G$ and is equal to $L(z, p^\beta) : p \in G + B_1$. There exists a canonical distribution $\mu(p^\delta)$ on $R - R_0 + B^L$ and $L(z, p)$ such that $L(z, p^\delta) - RL(z, p) = N(z, q)$.

Let $p \in f^{-1}(p^\delta)$. Then $L(z, p) - c_0^L L(z, p)$ is minimal in $G$, where $p^\delta \in G + \tilde{B}(G)$ and $q = f(p^\delta)$. Clearly $L(z, p^\delta) - c_0^L L(z, p^\delta) = 0$ for $p^\delta \in G + \tilde{B}(G)$. Since $N(z, p^\delta)$ is minimal $\mu(p^\delta)$ must be a point mass at $p^\delta \in G + \tilde{B}(G)$ and clearly $p^\delta \in G + \tilde{B}(G)$. Hence $N(z, p^\delta) = a(L(z, p^\delta) - c_0^L L(z, p)): a > 0$. But $\mathfrak{M}(N(z, p^\delta)) = 1$ and $\mathfrak{M}(L(z, p^\delta) - c_0^L L(z, p)) \leq 1$ by Theorem 1, hence $a = 1$ and $N(z, p^\delta) = L(z, p^\delta) - c_0^L L(z, p^\delta): p^\delta \in G + \tilde{B}(G)$.

Suppose there exist two points $p_i$ and $p_2$ in $G + \tilde{B}(G)$ such that $L(z, p_i) - c_0^L L(z, p_i) = N(z, p^\delta)$: $i = 1, 2$. Then by Lemma 2 $p_1 = p_2$. Thus $f^{-1}(p^\delta)$ is uniquely determined and $f^{-1}(p^\delta) \in G + \tilde{B}(G)$.

We show $f^{-1}(p^\delta)$ is continuous in $G + B_1$. Let $p_i \in G + B_1$ and $p_j \rightarrow p^\delta \in G + B_1$ as $i \rightarrow \infty$ and let $p_i = f^{-1}(p^\delta)$. Then $\{p_i\}$ has at least one limiting point $p$ in $R - R_0 + B^L$, since $R - R_1 + B^L$ is compact. Let $\{p_j\}$ be a subsequence of $\{p_i\}$ such that $p_j \rightarrow p$ and $p_j \rightarrow p^\delta$.

Let $L(z, p_j) = L(z, p)$, $\lim N(z, p_j) = N(z, p^\delta)$ and $\lim c_0^L L(z, p_j)$ exists, i.e. $L(z, p) - \lim c_0^L L(z, p_j) = N(z, p^\delta)$. Let $p^\delta = f^{-1}(p^\delta)$. Then $L(z, p^\delta) - c_0^L L(z, p^\delta) = N(z, p^\delta)$ and $p^\delta \in G + \tilde{B}(G)$. By lim $c_0^L L(z, p_j) = c_0^L L(z, p)$, we have

$L(z, p) - c_0^L L(z, p) \geq L(z, p) - \lim c_0^L L(z, p_j) = N(z, p^\delta)$.

Let $\mu(q)$ be a canonical mass distribution of $L(z, p)$ on $R - R_0 + B^L$. Then

$L(z, p) = \int L(z, q) d\mu(q)$ and $\int d\mu(q) = 1$ by $\mathfrak{M}(L(z, p)) = \frac{1}{2\pi} \int_{S_{R_0}} \frac{\partial}{\partial n} L(z, p) ds$.
\[
\int d\mu(p) = 1. \quad \text{Now}
\]
\[
L(x, p) - c_0^L(z, p) = \int (L(x, q) - c_0^L(z, q)) d\mu(q) = \int N(z, q^\prime) \delta(q) d\mu(q),
\]
where \(\delta(q) = 1\) or \(0\) according as \(q \in G + B(G)\) or not and \(q^\prime = f(q) \subset G + B_1\).

Hence \(\mathfrak{M}'(L(x, p) - c_0^L(z, p)) = \int \delta(q) d\mu(q) \) by Theorem 6. On the other hand, by \(N(z, p^\prime) \leq L(x, p) - c_0^L(z, p)\), \(\mathfrak{M}'(N(z, p^\prime)) = 1 \leq \mathfrak{M}'(L(x, p) - c_0^L(z, p)) \leq 1\) by Theorem 1. Hence \(\delta(q) = 1\) if \(\mu(q) > 0\) and \(\int d\mu(q) = 1 = \mathfrak{M}'(L(x, p) - c_0^L(z, p))\).

Both \(L(x, p) - c_0^L(x, p)\) and \(N(z, p^\prime)\) are F.S.H.s in \(G\). Let \(V_M = E[z : L(x, p) - c_0^L(z, p) > M]\) and \(V_M' = E[z : N(z, p) > M]\). Then \(V_M \supset V_M'\). \(\mathfrak{M}'(N(z, p^\prime)) = 1\) for any \(M\), since \(N(z, p^\prime)\) is minimal and \(\mathfrak{M}'(V_M, z, G) = 1\).

Also \(1 = \mathfrak{M}'(L(x, p) - c_0^L(z, p)) \geq \frac{MD(\omega(V_M, z, G))}{2\pi} \geq \frac{MD(\omega(V_M', z, G))}{2\pi}\) = 1, because \(MD(\omega(V_M, z, G)) \uparrow as M \to 0\). Hence \(V_M \supset V_M'\) and \(\omega(V_M, z, G) = \omega(V_M', z, G)\) for any \(M\). This implies \(L(x, p) - c_0^L(z, p) = N(z, p^\prime)\) and \(p = f^{-1}(p^\prime) \in G + B(G)\). Since any subsequence \(p_i \to p'\), \(\{p_i\}\) converges to \(f^{-1}(p_0)\) as \(p_i \to p'\).

We show \(f(p^\prime)\) is not necessarily continuous. Let \(R - R_0\) be \(E[0 < |z| < 1]\) = \(\Omega\), and \(F\) be a closed set on the real axis such that \(z_0 = 0\) is an irregular point for the Dirichlet problem of \(G = \Omega - F\), where \(F = \sum_{K=0}^{\infty} F_K\) and \(F_K\) is a segment. Then \(L(x, p)\) of \(\Omega\) and \(N(z, p)\) of \(G\) are Green's functions \(G(x, p)\) and \(G'(x, p)\) of \(\Omega\) and \(G\) respectively. Then by Lemma 3 there exists a sequence \(\{p_i\}\) such that \(G(x, p_i)\) converges to a function \(G'(x, p^\prime)\) with \(\mathfrak{M}'(G'(x, p^\prime)) = 1\) and \(p_i \to z_0\). Hence \(p^\prime = f(z_0)\). Let \(p_0\) be a fixed point in \(G\). Let \(q_i\) be a point such that \(q_i\) is so near \(F_i\) that \(G(p_0, q_i) \leq \frac{1}{i}\). Then \(lim G(z, q_i) = 0\). For any \(i\) we can find \(G'(x, p_i)\) such that \(p_i\) lies on a curve connecting \(p_i\) and \(q_i\) and that \(G(p_0, p_i) \to \omega(G'(p_0, p_i) as i \to \infty, where 0 < a < 1\). Also we choose a subsequence \(\{p_i^\prime\}\) from \(\{p_i\}\) so that \(p_i^\prime \to z_0\) (relative to \(T\)-topology) and \(G'(x, p_i^\prime) \to p^\prime\) (relative to \(M\)-topology): \(p^\prime \neq p^\prime\). Then since \(p_i^\prime \in G', p_i^\prime = f(p_i^\prime)\) and \(p_i^\prime \to z_0\) but \(f(p_i^\prime) \to p^\prime \neq p^\prime = f(z_0)\). Hence \(f(p)\) is not continuous at \(z_0\).

We call the harmonic dimension of \(p \in (\partial G + B)\) relative to \(G\) and \(R - R_0\) the number of linearly independent \(F_0\).S.H.s with finite \(\mathfrak{M}'\) in \(G\) and \(R - R_0\) which are harmonic in \(G\) and \(R - R_0\) respectively. Then by Lemma 1 we have the following

11) If \(\mu\) is canonical, \(\mathfrak{M}'(U(x)) = \int d\mu(p)\). See Theorem 6 of 1).
Corollary. Harmonic dimension of \( p \) relative to \( R-R_0 \) is equal to that of \( p \) relative to \( G \).

Applications to extremisations. Let \( U(z) \) be an \( F_0 \)-S.H. in \( R-R_0 \) with \( \mathbb{W}'(U(z)) < \infty \). Then there exists a canonical distribution \( \mu \) such that \( U(z) = \int L(z, p)d\mu(p) \) and \( \int d\mu(p) = \mathbb{W}'(U(z)) \). Put \( V(z) = U(z) - c_0 U(z) \). Then \( V(z) = \int (L(z, p) - c_0 L(z, p))d\mu(p) = \int N(z, q)\delta(p)d\mu(p) \), where \( q = f(p) \) and \( \delta(p) \) = 1 or 0 according as \( p \in G + B(G) \) or not. Hence \( V(z) \) is an F.S.H. in \( G \) with \( \mathbb{W}'(V(z)) \leq \int \delta(p)d\mu(p) < \infty \) and \( U(z) - V(z) = c_0 U(z) \) is full harmonic in \( G \). We denote \( V(z) \) by \( \text{in}_n U(z) \). Let \( V'(z) \) be an F.S.H. in \( G \) with \( \mathbb{W}'(V(z)) < \infty \). Then \( V(z) \) is a potential such that \( V(z) = \int N(z, q)d\mu(q) \) and \( \int d\mu(q) = \mathbb{W}'(V'(z)) \). Put \( U'(z) = \int L(z, p)d\mu(q) \), where \( p = f^{-1}(q) \). Then \( U'(z) \) is an \( F_0 \)-S.H. in \( R-R_0 \) with \( \mathbb{W}'(U'(z)) \leq \int d\mu(q) \) and \( U'(z) - V(z) \) is full harmonic in \( G \). We denote \( U'(z) \) by \( \text{ex} V'(z) \). Then \( U'(z) - c_0 U'(z) = V'(z) \).

Let \( \{G_n\} \) be an exhaustion of \( G \) with compact relative boundary \( \partial G_n \). Since \( a_n V'(z) \) is full harmonic in \( G - \overline{G}_n \), the solution of Neumann's problem (to obtain an \( F_0 \)-S.H. \( W(z) \) in \( R-R_0 \) such that \( W(z) - a_n V'(z) \) is full harmonic in \( G_{n+\delta} - \partial G_{n+\delta} \) and \( W(z) \) is full harmonic in \( G - G_{n+\delta} \) can be obtained by smoothing process by dist \( \partial G_{n+\delta}, \partial G_{n+\delta} > 0 \) for a given singularity of \( a_n V(z) \) in \( G_n \) and its solution is unique. It is evident that this solution coincides with \( \text{ex} (a_n V'(z)) \). Clearly \( \text{ex} (a_n V'(z)) \uparrow \) as \( n \to \infty \). On the other hand, \( f^{-1}(p) : p \in G + B_1 \) is continuous, we have \( \text{ex} V'(z) = \lim_{n \to \infty} (a_n V'(z)) \). Hence \( \text{ex} V'(z) \) is the least \( F_0 \)-S.H. in \( R-R_0 \) such that \( \text{ex} V'(z) - V(z) \) is full harmonic in \( G \). We have easily the following

Theorem 3. 1). Let \( U(z) \) be an \( F_0 \)-S.H. in \( R-R_0 \) with \( \mathbb{W}'(U(z)) < \infty \). Then \( \text{ex} (\text{in}_n U(z)) \leq U(z) \) and \( \text{ex} (\text{in}_n U(z)) = U(z) \) if and only if the canonical distribution of \( U(z) \) has no mass on \( CG \).

2). Let \( V(z) \) be an F.S.H. in \( G \) with \( \mathbb{W}'(V(z)) < \infty \). Then
\[
\text{in}_n (\text{ex} V(z)) = V(z).
\]

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