ON THE LIMITS OF RIEMANN SUMS OF FUNCTIONS IN BANACH SPACES

By

Michiko NAKAMURA and Ichiro AMEMIYA

Let $f(t)$ be a bounded function defined on a real interval $[a, b]$ taking values in a Banach space E. If f is continuous, then the Riemann integral $\int_a^b f(t)dt$ of f can be defined as the limit (by the norm of E) of the Riemann sums.

$$\sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1}), \qquad (a = t_0 < t_1 < \cdots < t_n = b,\ t_{i-1} < \tau_i < t_i)$$

Where the limit is taken making $\operatorname{Max}_i |t_i - t_{i-1}|$, *the order of the partition* $\{t_i\}$, tend to 0, A sequence of the Riemann sums of f with the order of partitions tending to 0 can converge even if f is not continuous. The limit of such a sequence is said to be a *Riemann limit* of f; we denote the set of all Riemann limits of f by $R.L.(f)$.

The question whether $R.L.(f)$ is always a convex set or not was asked by I. Halperin who, with Miller in [3], has given the affirmative answer in the case where E is a Hilbert space, generalizing the result of finite dimensional case by P. Hartman [1] and R. L. Jeffery [2].

Here we give (i) a condition for E which is weaker than uniform convexity and sufficient for $R.L.(f)$ to be convex for every f, (ii) an example of a non-convex $R.L.(f)$, and (iii) a proof of the non-emptiness of $R.L.(f)$ for separable E.

1. We consider the following property (*) of Banach space E:

(*) *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every finite sequence* $x_1, x_2 \cdots, x_n \in E$ *with* $\|x_i\| \leqq \delta$ $(i = 1, 2, \cdots, n)$ *and* $\sum_{i=1}^{n} \|x_i\| \leqq 1$ *there exists a subset* J *of* $\{1, 2, \cdots, n\}$ *for which* $\|\sum_{i \in J} x_i - \sum_{i \notin J} x_i\| < \varepsilon$.

We show, by an argument which is essentially due to [2], that for every bounded function f, $R.L.(f)$ is convex, if E satisfies (*). We suppose that $\|f(t)\| \leqq 1$ for every t in the interval.

Suppose $x, y \in R.L.(f)$, then there exist Riemann sums $x' = \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$ and $y' = \sum_{j=1}^{m} f(\sigma_j)(s_j - s_{j-1})$ with $\|x - x'\| < \varepsilon$ and $\|y - y'\| < \varepsilon$.

We can suppose, moreover, that for the intersection

$$\{r_0, r_1, \cdots, r_l\} = \{t_0, t_1, \cdots, t_n\} \frown \{s_0, s_1, \cdots, s_m\} ,$$
$$\operatorname*{Max}_k (r_k - r_{k-1}) < \delta ,$$

δ being chosen for ε as to satify the condition of (*). For every $k=1,2,\cdots,l$, we put

$$x_k = \sum_{r_{k-1} \leqq t_{i-1} < r_k} f(\tau_i)(t_i - t_{i-1}) \quad \text{and} \quad y_k = \sum_{r_{k-1} \leqq s_{j-1} \leqq r_k} f(\sigma_j)(s_j - s_{j-1}) ,$$

then we have $\quad x' = \sum_{k=1}^{l} x_k \quad y' = \sum_{k=1}^{l} y_k$ and $\|x_k\|, \|y_k\| \leqq \delta$.

Now we can apply the conclusion of (*) to the sequence $\frac{x_k - y_k}{2}$, $k=1,2,\cdots,l$, since $\left\|\frac{x_k - y_k}{2}\right\| \leqq \delta$ and $\sum_{k=1}^{l}\left\|\frac{x_k - y_k}{2}\right\| \leqq 1$. So there exists $J \subset \{1,2,\cdots,l\}$ such that $\left\|(\sum_{k\in J} x_k + \sum_{k\notin J} y_k) - \frac{x'+y'}{2}\right\| = \left\|\sum_{k\in J}\frac{x_k - y_k}{2} + \sum_{k\notin k}\frac{y_k - x_k}{2}\right\| < \varepsilon$, and hence a Rieman sum $\sum_{k\in J} x_k + \sum_{k\notin J} y_k$ obtained by "mixing" the two Rieman sums x' and y' lies at a distance less than 2ε from $\frac{x+y}{2}$. This shows that

$$\frac{x+y}{2} \in R.L.(f) .$$

Since $R.L.(f)$ is closd, it is convex.

In order that E has the property (*), it is sufficient that δ exists for an ε with $1>\varepsilon>0$, because then δ^2 (we suppose $\delta<1$) satisfies the condition for ε^2. (if $\|x_i\| \leqq \delta^2$ and $\sum_{i=1}^{n}\|x_i\| \leqq 1$, then for a decomposition $\{1,2,\cdots,n\} = N_1 \smile N_2 \smile \cdots \smile N_m$ such that $\sum_{i\in N_k}\|x_i\| \leqq \delta$, there exists $M_k \subset N_k$ such that $\|\sum_{i\in M_k} x_i - \sum_{i\in N_k - M_k} x_i\| < \varepsilon \sum_{i\in N_k}\|x_i\|$ and then, applying the condition *to* $\frac{1}{\varepsilon} X_k$ where $X_k = \sum_{i\in M_k} x_i - \sum_{i\in N_k - M_k} x_i$, there exists $J \subset \{1,2,\cdots,n\}$ such that $\|\sum_{k\in J} X_k - \sum_{k\notin J} X_k\| < \varepsilon^2$.)

By virtue of the above remark, we can see easily that E has the property (*) if E satifies the condition:

(**) *there exists* ε *with* $1>\varepsilon>0$ *such that for every* $x, y \in E$ *with* $\|x\|$, $\|y\| \leqq 1$, Min $\{\|x+y\|, \|x-y\|\} < 2\varepsilon$.

It is obvious that if E is *uniformly convex*, then E has the property (**). So our reesults generalize that of [3].

By the way, we will show that $R.L.(f)$ *is not empty if* E *is reflexive and has the property* (*). Let B_δ be the set of all Riemann sums

$$\sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$$

wite $\mathrm{Max}(t_i - t_{i-1}) \leqq \delta$, then $R.L.(f)$ is the intersection of all the closure of B_δ for $\delta > 0$. We have seen that for $\varepsilon > 0$ there sxists $\delta > 0$ such that $x, y \in B_\delta$ implies $\frac{x+y}{2} \in B_\delta + \varepsilon U$ where U is the unit ball of E.

It is not difficult to see that, making δ smaller, if necessary, the convex hull $\Gamma(B_\delta)$ of B_δ is included in $B_\delta + \varepsilon U$. So $R.L.(f)$ is the intersection of the closure of $\Gamma(B_\delta)$ and it is not empty since every closed convex bounded set is weakly compact.

We remark that $R.L.(f)$ is convex whenever the range of f is relatively compact. This is an immediate consequence of the facts that the set of Riemann limits is convex for finite dimensional spaces and that in the convex hull of the range of f, the norm topology coincides with the weak topology.

The separability of the range of f, however, does not give such an advantage; if $R.L.(f)$ is not convex, then we can find a countably valued function g (suitably modifying f) for which $R.L.(g)$ is also not convex.

2. Here we give an example of f for which $R.L.(f)$ consists of exactly two different elements.

Let E be $l^1(R)$, R being the set of all real numbers considered as a discrete space; an element x of E is a function of $t \in R$ such that $\sum_{t \in R} |x(t)| < +\infty$, and the norm of x is defined by this sum. $e(t)$ denotes the characteristic function of the set consisting of one point t.

We put

$$x_n = \frac{1}{2_n} \sum_{k=1}^{2^n} e\left(\frac{k}{2^n}\right)$$

$$y_n = \frac{1}{2^n} \sum_{k=1}^{2^n} e\left(\frac{k}{2^n} - \varepsilon_n\right)$$

$$z_n = \frac{1}{2_n} \sum_{k=1}^{2^n} e\left(\frac{k}{2^n} + \varepsilon_n\right)$$

where ε_n are chosen so that $\frac{k}{2^n}, \frac{k}{2^n} - \varepsilon, \frac{k}{2^n} + \varepsilon_n$ are all different for all possible n and k, $\varepsilon_n < \frac{1}{2_n}$, and $\sum_n 2^n \varepsilon_n < +\infty$. Let S_n $(n = 1, 2, \cdots)$ be mutually disjoint subsets of $(1, 2]$ such that every S_n is dense in $(1, 2]$ and $\bigcup_n S_n = (1, 2]$.

For an arbitrary element $a \in E$, a function f_a of $[0, 2]$ into E is defined as follows:

$$f_a(t) = \begin{cases} e(t) & \text{if } 0 \leqq t \leqq 1 \\ -x_n & \text{if } t \in S_{2n} \\ -\dfrac{y_n + z_n}{2} + a & \text{if } t \in S_{2n-1}\,. \end{cases}$$

We claim that $R.L.(f_a) = \{0, a\}$.

Consider a sequence of Riemann sum of f_a which converges to $b \in E$ and for which the order of partitions tend to 0. Without loss of generality, we can suppose that $t=1$ is one of the partition point for each of the Riemann sum in the sequence. We denote by s an arbitrary member of the sequence; we can write

$$s = s_1 + s_2$$

$$s_1 = \sum_{i=0}^{l} e(\tau_i)(t_i - t_{i-1})$$

$$s_2 = -\sum_{n=1}^{N} (\alpha_n x_n + \beta_n y_n + \gamma_n z_n) + \delta a$$

Where $\{t_i\}$ is a partition of $[0, 1]$, $\alpha_n, \beta_n, \gamma_n, \delta \geqq 0$, $\sum_{n=1}^{N} (\alpha_n + \beta_n + \gamma_n) = 1$, $\beta_n = \gamma_n$ and $\sum_{n=1}^{N} (\beta_n + \gamma_n) = \delta$.

Replacing the sequence by a suitable subsequence, we can suppose that δ converges to λ with $0 \leqq \lambda \leqq 1$ according to the convergence of s to b. If we apply the same formation of Riemann sums to f_0 in place of f_a, we obtain a sequence of Riemann sums of f_0 which converges to $b - \lambda a$. So if we prove (i) $R.L.(f_0) = \{0\}$ and (ii) for a converging sequence of Riemann sums of f_0, the numbers corresponding to $\sum_{n=1}^{N} (\beta_n + \gamma_n)$ in the above expression of s converges to either 0 or 1, then we can conclude, from (i), that $R.L.(f_a) \subset \{\lambda a;\ 0 \leqq \lambda \leqq 1)$ and, from (ii), that λ with $\lambda a \in R.L.(f_a)$ is either 0 or 1.

We continue to make use of s with the detailed expressions given before (putting $a=0$) to denote any Riemann sum of f_0 which we are considering of. If s converges to b with the orders of partition tending to 0, then s_1, if considerd as a function on R, converges point-wise to 0, and s_2 converges point-wise to b; b must be non-positive as a function. If we define a linear functional L on E as $L(x) = \sum_{t \in R} x(t)$, then we have $L(s) = 0$ since $L(s_1) = 1$ and $L(s_2) = -1$, and hence $L(b) = 0$ and, since b is non-positive, $b = 0$. Thus we have proved that $R.L.(f_0) = \{0\}$.

We make use of the following notations for

$$s_1 = \sum_{i=1}^{l} e(\tau_i)(t_i - t_{i-1}):$$

$$M = \{\tau_i;\quad i=0,1,2,\cdots,l\};$$

$$a_n = \frac{1}{2^n}\left\{\textit{the number of } k \textit{ with } \frac{k}{2^n} \notin M\right\},$$

$$b_n = \frac{1}{2^n}\left\{\textit{the number of } k \textit{ with } \frac{k}{2^n} - \varepsilon_n \notin M\right\},$$

$$c_n = \frac{1}{2^n}\left\{\textit{the number of } k \textit{ with } \frac{k}{2^n} + \varepsilon_n \notin M\right\}.$$

We can see easily, by the definition, that

$$(1) \qquad a_n \leqq 2a_{n+1}, \quad b_n \leqq 2b_{n+1} \quad \text{and} \quad c_n \leqq 2c_{n+1}.$$

If $\frac{k}{2^n} - \varepsilon_n$, $\frac{k}{2^n}$ and $\frac{k}{2^n} + \varepsilon_n$ are all in M and $f(\tau_i) = \frac{k}{2^n}$, then we have obviously $t_i - t_{i-1} \leqq 2\varepsilon_n$. So the sum of $t_i - t_{i-1}$ for all such i does not exceeds

$$(2) \qquad 2^m\delta + \sum_{n>m} 2^{n+1}\varepsilon_n$$

for every $m=0,1,2,\cdots$, where δ is the order of s; if δ is sufficiently small, then (2) can be arbitrarily small, and hence we can modify the Riemann sum s to obtain s' which is arbitrarily close to s and satisfy the condition:

($\sharp$) One of $\left\{\frac{k}{2^n} - \varepsilon_n, \frac{k}{2^n}, \frac{k}{2^n} + \varepsilon_n\right\}$ does not belong to M for every n.

Hereafter we consider the Riemann sum s with this property. By ($\sharp$) we have

$$(3) \qquad a_n + b_n + c_n \geqq 1.$$

For our purpose, it is sufficient to prove the following:

For every s with the property ($\sharp$) and for every integer m,

$$(4) \qquad \operatorname{Min}\left\{\sum_{n=1}^{N} \alpha_n, \sum_{n=1}^{N} \beta_n, \sum_{n=1}^{N} \gamma_n\right\} \leqq 2^{m+4}\|s\| + \frac{1}{2^m}.$$

Let σ_n be the sum of $t_i - t_{i-1}$ for all i such that $\tau_i = \frac{k}{2^n}$ for some $k=1,2,\cdots,2^n$ and suppose

$$\sigma_{r-1} < \frac{1}{2^m} \quad \text{nda} \quad \sigma_r \geqq \frac{1}{2^m}.$$

If, for any interval I of length λ, $\frac{k}{2^n} \in I$ for just p values of k's, then we have

$$(p+1)\frac{1}{2^n} \geqq \lambda$$

or

$$\frac{p}{2^n} \geqq \lambda - \frac{1}{2^n}.$$

Therefore if A is the union of q intervals, where the sum of their length is λ and $\frac{k}{2^n} \in I$ for just p values of k's, then we have $\frac{p}{2^n} \geqq \lambda - \frac{q}{2^n}$.

Applying this to the above set of the intervals $t_i - t_{i-1}$ for which $\tau_i = \frac{k}{2^r}$, we have

$$a_n \geqq \sigma_r - \frac{2^r}{2^n} \geqq \frac{1}{2^m} - \frac{2^r}{2^n}$$

and hence $a_n \geqq \frac{1}{2^{m+1}}$ for every $n > r+m$. By a similar argument, we have also b_n, $c_n \geqq \frac{1}{2^{m+2}}$ for every $n > r+m+1$. (Here the difference is due to the fact that the distance of any subsequent $\frac{k}{2^r} - \varepsilon_r$, $\frac{k+1}{2^r} - \varepsilon_r$ is not necessarily $\frac{1}{2^r}$ but only less than $\frac{1}{2^{r-1}}$).

Now if a linear functional φ on E is defined as $\varphi(x) = \sum_{t \in M} x(t)$, then we have

$$\|s\| \geqq |\varphi(s)| = |\varphi(s_2)| = \sum_{n=1}^{N} (a_n \alpha_n + b_n \beta_n + c_n \gamma_n), \tag{5}$$

and hence

$$\|s\| \geqq \sum_{n > r+m+1} (a_n \alpha_n + b_n \beta_n + c_n \gamma_n) \geqq \frac{1}{2^{m+2}} \sum_{n > r+m+1} (\alpha_n + \beta_n + \gamma_n),$$

that is,

$$2^{m+2}\|s\| \geqq \sum_{n > r+m+1} (\alpha_n + \beta_n + \gamma_n). \tag{6}$$

Since, by (1), we have

$$a_r, a_{r+1}, \cdots, a_{r+m+1} \geqq \frac{1}{2^{m+1}} a_r$$

and the same inequality for b and c, and, by (3), one of a_r, b_r and c_r, say a_r, is not less than $\frac{1}{3}$, we have, by (5),

$$\|s\| \geqq \sum_{n=r}^{r+m+1} a_n\alpha_n \geqq \frac{1}{3\cdot 2^{m+1}} \sum_{n=r}^{r+m+1} \alpha_n$$

and hence, by (6),

$$(7) \qquad \sum_{n\geqq r} \alpha_n \leqq (2^{m+2}+3\cdot 2^{m+1})\|s\| = 5\cdot 2^{m+1}\|s\|.$$

Finally we will estimate $\sum_{n=1}^{r-1}\alpha_n$. Let ϕ be a linear functional on E defined as

$$\phi(x) = \sum_{k=1}^{2^{r-1}} x\left(\frac{k}{2^{r-1}}\right)$$

then

$$\phi(s_1) = \sigma_{r-1} < \frac{1}{2^m}$$

and hence

$$\|s\| \geqq |\phi(s)| \geqq |\phi(s_2)| - |\phi(s_1)| \geqq \sum_{n=1}^{N} \alpha_n \phi(x_n) - \frac{1}{2^m}$$

Since

$$\phi(x_n) = \frac{1}{2^{n-r+1}},$$

We have

$$\|s\| \geqq \sum_{n=1}^{N} \frac{1}{2^{n-r+1}}\alpha_n - \frac{1}{2^m} \geqq \sum_{n=1}^{r-1} \alpha_n - \frac{1}{2^m},$$

that is,

$$\sum_{n=1}^{r-1} \alpha_n \leqq \|s\| + \frac{1}{2^m},$$

and hence, combining with (7),

$$\sum_{n=1}^{N} \alpha_n \leqq (5\cdot 2^{m+1}+1)\|s\| + \frac{1}{2^m}$$

Thus we have proved our final (4).

We remark that $P\circ f_a$ is an example of separable valued function with the non convex set of Riemann limits, where P is the projection of E to the closed subspace generated by $\left\{e\left(\frac{k}{2^n}-\varepsilon_n\right),\ e\left(\frac{k}{2^n}\right),\ e\left(\frac{k}{2^n}+\varepsilon_n\right);\ n, k=1, 2, \cdots\right\}$ and $a=e(1)$. Here $R.L.(P\circ f_a)$ contains elemnts other than 0 and a, but does not contain λa for any λ with $0<\lambda<1$.

3. A class of spaces for which every $R.L.(f)$ is not empty was given

in 1, that is, reflexive spaces with the property (*). On the other hand, an example of f with $R.L.(f)=\phi$ is provided by the restrition of f_a given in 2 to the interval [0, 1], that is, the function defined by $f(t)=e(t)$. Here we will prove that *$R.L.(f)$ is not empty, if the range of f is separable.*

Such an f is the uniform limit of a sequence of countably valued functions f_n $n=1,2,\cdots$. So it is sufficient to show

(i) The existence of Riemann limits of a certain type for countably valued functions.

(ii) If x is a Riemann limit of f with the type refered in (i) and if another countably valued function g is given, then we can find a Riemann limit y of g of the same type with

$$\|x-y\| \leqq \|f-g\|,$$

where the norm $\|f\|$ of f is defined as

$$\|f\| = \sup_{t\in I} \|f(t)\|,$$

I being the interval for which Riemann sums are considered. In fact, if countably valued f_n converges uniformly to f, then, by (ii), we can choose a Riemann limit x_n of f_n successively so that we have

$$\|x_n - x_{n+1}\| \leqq \|f_n - f_{n+1}\|$$

for every $n=1,2,\cdots$, then x_n form a Cauchy sequence and the limit x of x_n is obviously a Riemann limit of the limit function f.

We need some preparations. For a subset A of R, $m^+(A)$ and $m^-(A)$ denote the (Lebesgue) outer measure and inner measure of A respectively; $m(A)$ denotes the Lebesgue measure of A, in case A is measureable.

For a mutually disjoint family of subsets. A_n $n=1,2,\cdots$, a mutually disjoint family of measurable sets B_n $n=1,2,\cdots$, is said to be subordinate to $\{A_n\}$ if $m^-(B_n-A_n)=0$ and

$$m(\bigcup_n B_n) = m^+(\bigcup_n A_n).$$

The existence of a family $\{B_n\}$ subordinate to a given family $\{A_n\}$ can be proved as follows:

Choose a measureable set B_1, with

$$B_1 \supset A_1 \quad \textit{and} \quad m(B_1) = m^+(A_1);$$

if $B_1, B_2, \cdots, B_n$ are mutually disjoint and

$$m\{B_1 \cup B_2 \cup \cdots \cup B_n\} = m^+\{A_1 \cup A_2 \cup \cdots \cup A_n\},$$

then B_{n+1} can be given as

$$B_{n+1} = B - B_1 \cup B_2 \cup \cdots \cup B_n$$

where B is a measurable set with

$$B \supset A_1 \cup A_2 \cup \cdots \cup A_{n+1}$$

and

$$m(B) = m^+ \{A_1 \cup A_2 \cup \cdots \cup A_{n+1}\}.$$

Since

$$m^- \{B_1 \cup B_2 \cup \cdots \cup B_n - A_1 \cup A_2 \cup \cdots \cup A_{n+1}\} = 0,$$

we have

$$m^- \{B_{n+1} - A_{n+1}\} \leqq m^- \{B - A_1 \cup A_2 \cup \cdots \cup A_{n+1}\} = 0.$$

We have also

$$m \{B_1 \cup B_2 \cup \cdots \cup B_{n+1}\} = m^+ \{A_1 \cup A_2 \cup \cdots \cup A_{n+1}\}$$

and hence, for the fomily $\{B_n\}$ thus obtained,

$$m(\bigcup_n B_n) = m^+ (\bigcup_n A_n).$$

If a function f takes a constant value a_n on A_n for a decomposition $\{A_n\}$ of I, and if $\{B_n\}$ is a measurable decomposition subordinate to $\{A_n\}$, then $\sum_{n=1}^{\infty} m(B_n) a_n$ belongs to $R.L.(f)$, in other words, the usual Lebesgue intergral $\int_I f^*(t)dt$ where f^* is the function taking constant values a_n *on* B_n is a Riemann limit of f. This can be seen if we show

$$(8) \qquad \int_I f^*(t)dt \in R.L.(f^*),$$

because we can modify B_n within the difference of measure 0 so as to be included in the closure of A_n and, for f^* defined by these B_n, we have $R.L.(f^*) \subset R.L.(f)$.

On the other hand, (8) can be proved easily from the fact that for every measurable set B, there exist finite number of open intervals $I_1, I_2, \cdots, I_m$ such that the measure of the symmetric difference of B and $\bigcup_{i=1}^{m} I_i$ is arbitrarily small and every I_i contains a point in B.

Thus we have proved (i), that is, the existence of a Riemann limit of a countably valued function f as the Lebesgue integral of a certain measurable function obtained by modifying f.

Let f and g be any two countably values functions. We can suppose, without loss of generality, that f takes constant values a_n in A_n, $\{A_n\}$ being a decomposition of I, and g takes $a_{n,m}$ in $A_{n,m}$ where $\{A_{n,m};\ n, m = 1, 2, \cdots\}$

is also a decomposition of I satisfying $A_n=\bigcup_{m=1}^{\infty} A_{n,m}$. Suppose a measurable decomposition $\{B_n\}$ subordinate to $\{A_n\}$ is given. We will show that there exists a measurable decomposition $\{B_{n,m}\}$ which is subordinate to $\{A_{n,m}\}$ and satisfies $B_n=\bigcup_{m=1}^{\infty} B_{n,m}$. This gives (ii) completing the rest of our proof, because we have then

$$\sum_n m(B_n)a_n \in R.L.(f),$$

$$\sum_{n,m} m(B_{n,m})a_{n,m} \in R.L.(g)$$

and

$$\|\sum_n m(B_m)a_n - \sum_{n,m} m(B_{n,m})a_{n,m}\|$$
$$= \|\sum_{n,m} m(B_{n,m})(a_n - a_{n,m})\| \leqq \sup_{n,m}\|a_n - a_{n,m}\| \leqq \|f-g\|.$$

Now put

$$A'_{n,m} = B_n \frown A_{n,m},$$

and

$$A'_n = \bigcup_m A'_{n,m} = B_n \frown A_n.$$

For a fixed n, let $\{B'_{n,m}\}$ be a measurable family subordinate to $\{A'_{n,m}\}$, then, for $B'_n = \bigcup_m B'_{n,m}$,

We have

$$m(B'_n) = m^+(\bigcup_m A'_{n,m}) = m^+(B_n \frown A_n) = m^+\{B_n - (B_n - A_n)\}$$
$$= m(B_n) - m^-(B_n - A_n) = m(B_n)$$

and, since

$$m(B'_n - B_n) \leqq m^-(B'_n - A'_n) = 0,$$

the symmetric difference $B_n \varDelta B'_n$ of B_n and B'_n is measure 0. Therefore we can replace $B'_{n,m}$ by a suitable $B_{n,m}$ for which we have

$$m(B_{n,m} \varDelta B'_{n,m}) = 0$$

and

$$B_n = \bigcup_m B_{n,m}.$$

$\{B_{n,m}\}$ is obviously subordinate to $\{A_{n,m}\}$.

Thus our proof is completed.

References

[1] P. HARTMAN, On the limits of Riemann approximating sums, Quarterly J. Math., 18 (1947), 124–127.

[2] R. L. JEFFERY, Limit points of Riemann sums, Trans, Royal Soc. of Canada, Series III, 44 (1950), Sec. III, 43–49.

[3] I. HALPERIN and N. MILLER, An ineauality of Steinitz and the limits of Riemann sums, Transactions of the Royal Society of Canada 48 (1954), p. 27–29.

Tokyo College of Science,
Hokkaido University

(Received June 17, 1966)