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ON THE LIMITS OF RIEMANN SUMS OF
FUNCTIONS IN BANACH SPACES

By

Michiko NAKAMURA and Ichiro AMEMIYA

Let $f(t)$ be a bounded function defined on a real interval $[a, b]$ taking values in a Banach space $E$. If $f$ is continuous, then the Riemann integral $\int_{a}^{b} f(t) \, dt$ of $f$ can be defined as the limit (by the norm of $E$) of the Riemann sums.

$$\sum_{i=1}^{n} f(\tau_i) (t_i - t_{i-1}), \quad (a = t_0 < t_1 < \cdots < t_n = b, \ t_{i-1} < \tau_i < t_i)$$

Where the limit is taken making $\text{Max} |t_i - t_{i-1}|$, the order of the partition $\{t_i\}$, tend to 0. A sequence of the Riemann sums of $f$ with the order of partitions tending to 0 can converge even if $f$ is not continuous. The limit of such a sequence is said to be a Riemann limit of $f$; we denote the set of all Riemann limits of $f$ by $R.L.(f)$.

The question whether $R.L.(f)$ is always a convex set or not was asked by I. Halperin who, with Miller in [3], has given the affirmative answer in the case where $E$ is a Hilbert space, generalizing the result of finite dimensional case by P. Hartman [1] and R. L. Jeffery [2].

Here we give (i) a condition for $E$ which is weaker than uniform convexity and sufficient for $R.L.(f)$ to be convex for every $f$, (ii) an example of a non-convex $R.L.(f)$, and (iii) a proof of the non-emptiness of $R.L.(f)$ for separable $E$.

1. We consider the following property (*) of Banach space $E$:

(*) For every $\epsilon > 0$, there exists $\delta > 0$ such that for every finite sequence $x_1, x_2, \ldots, x_n \in E$ with $\|x_i\| \leq \delta$ ($i = 1, 2, \ldots, n$) and $\sum_{i=1}^{n} \|x_i\| \leq 1$ there exists a subset $J$ of $\{1, 2, \ldots, n\}$ for which $\|\sum_{i \in J} x_i - \sum_{i \not\in J} x_i\| < \epsilon$.

We show, by an argument which is essentially due to [2], that for every bounded function $f$, $R.L.(f)$ is convex, if $E$ satisfies (*). We suppose that $\|f(t)\| \leq 1$ for every $t$ in the interval.

Suppose $x, y \in R.L.(f)$, then there exist Riemann sums $x' = \sum_{i=1}^{n} f(\tau_i) (t_i - t_{i-1})$ and $y' = \sum_{j=1}^{m} f(\sigma_j) (s_j - s_{j-1})$ with $\|x - x'\| < \epsilon$ and $\|y - y'\| < \epsilon$. 
We can suppose, moreover, that for the intersection
\[
\{r_0, r_1, \ldots, r_l\} = \{t_0, t_1, \ldots, t_n\} \cap \{s_0, s_1, \ldots, s_m\},
\]
\[\max_k (r_k - r_{k-1}) < \delta,
\]
\(\delta\) being chosen for \(\epsilon\) as to satisfy the condition of (*)
For every \(k = 1, 2, \ldots, l\), we put
\[
x_k = \sum_{r_{k-1} \leq t_{i-1} < r_k} f(t_i)(t_i - t_{i-1})
\]
and
\[
y_k = \sum_{r_{k-1} \leq s_{j-1} < r_k} f(s_j)(s_j - s_{j-1})
\]
then we have
\[
x' = \sum_{k=1}^l x_k, \quad y' = \sum_{k=1}^l y_k
\]
and
\[\|x_k\|, \|y_k\| \leq \delta.
\]
Now we can apply the conclusion of (*) to the sequence \(\frac{x_k - y_k}{2}\), \(k = 1, 2, \ldots, l\), since
\[
\left\| \frac{x_k - y_k}{2} \right\| \leq \delta
\]
and
\[\sum_{k=1}^l \left\| \frac{x_k - y_k}{2} \right\| \leq 1.
\]
So there exists \(J \subseteq \{1, 2, \ldots, l\}\) such that
\[
\left\| \sum_{k \in J} x_k + \sum_{k \notin J} y_k \right\| - \frac{x' + y'}{2}\left\| \sum_{\kappa J} x_k - \sum_{k \notin k} y_k \right\| < \epsilon,
\]
and hence a Riemann sum \(\sum_{j \in J} x_k + \sum_{j \notin J} y_k\) obtained by “mixing” the two Riemann sums \(x'\) and \(y'\) lies at a distance less than \(2\epsilon\) from \(\frac{x+y}{2}\). This shows that
\[
\frac{x+y}{2} \in R.L.(f).
\]
Since \(R.L.(f)\) is closed, it is convex.

In order that \(E\) has the property (\(*\)), it is sufficient that \(\delta\) exists for an \(\epsilon\) with \(1 > \epsilon > 0\), because then \(\delta^2\) (we suppose \(\delta < 1\)) satisfies the condition for \(\epsilon^2\). (if \(\|x_i\| \leq \delta\) and \(\sum_{i=1}^n \|x_i\| \leq 1\), then for a decomposition \(\{1, 2, \ldots, n\} = N_1 \cup N_2 \cup \ldots \cup N_m\) such that \(\sum_{i \in N_k} \|x_i\| \leq \delta\), there exists \(M_k \subseteq N_k\) such that
\[
\left\| \sum_{i \in M_k} x_i - \sum_{i \notin M_k} x_i \right\| < \epsilon \sum_{i \in M_k} \|x_i\|
\]
and then, applying the condition to \(\frac{1}{\epsilon} X_k\) where \(X_k = \sum_{i \in M_k} x_i - \sum_{i \notin M_k} x_i\), there exists \(J \subseteq \{1, 2, \ldots, n\}\) such that \(\|\sum_{k \in J} X_k - \sum_{k \notin J} X_k\| < \epsilon^2\).

By virtue of the above remark, we can see easily that \(E\) has the property (\(*\)) if \(E\) satisfies the condition:

\[**\] there exists \(\epsilon\) with \(1 > \epsilon > 0\) such that for every \(x, y \in E\) with \(\|x\|, \|y\| \leq 1\), \(\min \{\|x + y\|, \|x - y\|\} < 2\epsilon\).

It is obvious that if \(E\) is uniformly convex, then \(E\) has the property (\(**\)). So our results generalize that of [3].

By the way, we will show that \(R.L.(f)\) is not empty if \(E\) is reflexive and has the property (\(*\))

Let \(B_\delta\) be the set of all Riemann sums.
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$\sum_{i=1}^{n} f(\tau_i) (t_i - t_{i-1})$

wite $\max (t_i - t_{i-1}) \leq \delta$, then $R.L. (f)$ is the intersection of all the closure of $B_\delta$ for $\delta > 0$. We have seen that for $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in B_\delta$ implies $\frac{x + y}{2} \in B_\delta + \varepsilon U$ where $U$ is the unit ball of $E$.

It is not difficult to see that, making $\delta$ smaller, if necessary, the convex hull $\Gamma(B_\delta)$ of $B_\delta$ is included in $B_\delta + \varepsilon U$. So $R.L. (f)$ is the intersection of the closure of $\Gamma(B_\delta)$ and it is not empty since every closed convex bounded set is weakly compact.

We remark that $R.L. (f)$ is convex whenever the range of $f$ is relatively compact. This is an immediate consequence of the facts that the set of Riemann limits is convex for finite dimensional spaces and that in the convex hull of the range of $f$, the norm topology coincides with the weak topology.

The separability of the range of $f$, however, does not give such an advantage; if $R.L. (f)$ is not convex, then we can find a countably valued function $g$ (suitably modifying $f$) for which $R.L. (g)$ is also not convex.

2. Here we give an example of $f$ for which $R.L. (f)$ consists of exactly two different elements.

Let $E$ be $l^1 (R)$, $R$ being the set of all real numbers considered as a discrete space; an element $x$ of $E$ is a function of $t \in R$ such that $\sum_{t \in R} |x(t)| < +\infty$, and the norm of $x$ is defined by this sum. $e(t)$ denotes the characteristic function of the set consisting of one point $t$.

We put

$x_n = \frac{1}{2^n} \sum_{k=1}^{2^n} e\left( \frac{k}{2^n} \right)$

$y_n = \frac{1}{2^n} \sum_{k=1}^{2^n} e\left( \frac{k}{2^n} - \varepsilon_n \right)$

$z_n = \frac{1}{2^n} \sum_{k=1}^{2^n} e\left( \frac{k}{2^n} + \varepsilon_n \right)$

where $\varepsilon_n$ are chosen so that $\frac{k}{2^n}, \frac{k}{2^n} - \varepsilon, \frac{k}{2^n} + \varepsilon_n$ are all different for all possible $n$ and $k$, $\varepsilon_n < \frac{1}{2^n}$, and $\sum_{n} 2^n \varepsilon_n < +\infty$. Let $S_n (n=1, 2, \cdots)$ be mutually disjoint subsets of $(1, 2]$ such that every $S_n$ is dense in $(1, 2]$ and $\bigcup S_n = (1, 2]$.

For an arbitrary element $a \in E$, a function $f_a$ of $[0, 2]$ into $E$ is defined as follows:
\[ f_a(t) = \begin{cases} 
  e(t) & \text{if } 0 \leq t \leq 1 \\
  -x_n & \text{if } t \in S_{2n} \\
  -\frac{y_n+z_n}{2}+a & \text{if } t \in S_{2n-1} 
\end{cases} \]

We claim that \( R.L.(f_a) = \{0, a\} \).

Consider a sequence of Riemann sum of \( f_a \) which converges to \( b \in E \) and for which the order of partitions tend to 0. Without loss of generality, we can suppose that \( t=1 \) is one of the partition point for each of the Riemann sum in the sequence. We denote by \( s \) an arbitrary member of the sequence; we can write

\[
  s = s_1 + s_2 \\
  s_1 = \sum_{i=0}^{l} e(\tau_i)(t_{i}-t_{i-1}) \\
  s_2 = -\sum_{n=1}^{N} (\alpha_n x_n + \beta_n y_n + \gamma_n z_n) + \delta a
\]

Where \( \{\tau_i\} \) is a partition of \([0, 1]\), \( \alpha_n, \beta_n, \gamma_n, \delta \geq 0, \sum_{n=1}^{N} (\alpha_n + \beta_n + \gamma_n) = 1, \beta_n = \gamma_n \) and \( \sum_{n=1}^{N} (\beta_n + \gamma_n) = \delta \).

Replacing the sequence by a suitable subsequence, we can suppose that \( \delta \) converges to \( \lambda \) with \( 0 \leq \lambda \leq 1 \) according to the convergence of \( s \) to \( b \). If we apply the same formation of Riemann sums to \( f_0 \) in place of \( f_a \), we obtain a sequence of Riemann sums of \( f_0 \) which converges to \( b-\lambda a \). So if we prove (i) \( R.L.(f_0) = \{0\} \) and (ii) for a converging sequence of Riemann sums of \( f_0 \), the numbers corresponding to \( \sum_{n=1}^{N} (\beta_n + \gamma_n) \) in the above expression of \( s \) converges to either 0 or 1, then we can conclude, from (i), that \( R.L.(f_a) \subset \{\lambda a ; 0 \leq \lambda \leq 1\} \) and, from (ii), that \( \lambda \) with \( \lambda a \in R.L.(f_a) \) is either 0 or 1.

We continue to make use of \( s \) with the detailed expressions given before (putting \( a=0 \)) to denote any Riemann sum of \( f_0 \) which we are considering of. If \( s \) converges to \( b \) with the orders of partition tending to 0, then \( s_i \), if considered as a function on \( R \), converges point-wise to 0, and \( s_2 \) converges point-wise to \( b \); \( b \) must be non-positive as a function. If we define a linear functional \( L \) on \( E \) as \( L(x) = \sum_{t \in \mathbb{R}} x(t) \), then we have \( L(s) = 0 \) since \( L(s_1) = 1 \) and \( L(s_2) = -1 \), and hence \( L(b) = 0 \) and, since \( b \) is non-positive, \( b = 0 \). Thus we have proved that \( R.L.(f_0) = \{0\} \).

We make use of the following notations for
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\[ s_i = \sum_{i=1}^{l} e(\tau_i)(t_i - t_{i-1}) : \]
\[ M = \{ \tau_i ; \ i = 0, 1, 2, \cdots, l \} ; \]
\[ a_n = \frac{1}{2^n} \{ \text{the number of } k \text{ with } \frac{k}{2^n} \notin M \} , \]
\[ b_n = \frac{1}{2^n} \{ \text{the number of } k \text{ with } \frac{k}{2^n} - \varepsilon_n \notin M \} , \]
\[ c_n = \frac{1}{2^n} \{ \text{the number of } k \text{ with } \frac{k}{2^n} + \varepsilon_n \notin M \} . \]

We can see easily, by the definition, that

\begin{equation}
(1) \quad a_n \leq 2a_{n+1}, \quad b_n \leq 2b_{n+1} \quad \text{and} \quad c_n \leq 2c_{n+1}.
\end{equation}

If \( k/2^n - \varepsilon_n \), \( k/2^n \), and \( k/2^n + \varepsilon_n \) are all in \( M \) and \( f(\tau_i) = k/2^n \), then we have obviously \( t_i - t_{i-1} \leq 2\varepsilon_n \). So the sum of \( t_i - t_{i-1} \) for all such \( i \) does not exceed

\begin{equation}
(2) \quad 2^m\delta + \sum_{n>m} 2^{n+1}\varepsilon_n
\end{equation}

for every \( m = 0, 1, 2, \cdots \), where \( \delta \) is the order of \( s \); if \( \delta \) is sufficiently small, then \( (2) \) can be arbitrarily small, and hence we can modify the Riemann sum \( s \) to obtain \( s' \) which is arbitrarily close to \( s \) and satisfy the condition:

\begin{equation}
(\#) \quad \text{One of } \left\{ \frac{k}{2^n} - \varepsilon_n, \frac{k}{2^n}, \frac{k}{2^n} + \varepsilon_n \right\} \text{ does not belong to } M \text{ for every } n.
\end{equation}

Hereafter we consider the Riemann sum \( s \) with this property. By \( (\#) \) we have

\begin{equation}
(3) \quad a_n + b_n + c_n \geq 1.
\end{equation}

For our purpose, it is sufficient to prove the following:

For every \( s \) with the property \( (\#) \) and for every integer \( m \),

\begin{equation}
(4) \quad \text{Min} \left\{ \sum_{n=1}^{N} a_n, \sum_{n=1}^{N} b_n, \sum_{n=1}^{N} c_n \right\} \leq 2^{m+4}\|s\| + \frac{1}{2^m}.
\end{equation}

Let \( \sigma_n \) be the sum of \( t_i - t_{i-1} \) for all \( i \) such that \( \tau_i = k/2^n \) for some \( k = 1, 2, \cdots, 2^n \) and suppose

\[ \sigma_{r-1} < \frac{1}{2^m} \quad \text{nda} \quad \sigma_r \geq \frac{1}{2^m} . \]

If, for any interval \( I \) of length \( \lambda \), \( k/2^n \in I \) for just \( p \) values of \( k \)'s, then we have
\[(p + 1) \frac{1}{2^n} \geq \lambda\]
or
\[\frac{p}{2^n} \geq \lambda - \frac{1}{2^n}.\]

Therefore if \(A\) is the union of \(q\) intervals, where the sum of their length is \(\lambda\) and \(\frac{k}{2^n} \in I\) for just \(p\) values of \(k\)'s, then we have \(\frac{p}{2^n} \geq \lambda - \frac{q}{2^n}\).

Applying this to the above set of the intervals \(t_i - t_{i-1}\) for which \(\tau_i = \frac{k}{2^r}\), we have
\[a_n \geq \sigma_r - \frac{2^r}{2^n} \geq \frac{1}{2^m} - \frac{2^r}{2^n}\]
and hence \(a_n \geq \frac{1}{2^{m+1}}\) for every \(n > r + m\). By a similar argument, we have also \(b_n, c_n \geq \frac{1}{2^{m+2}}\) for every \(n > r + m + 1\). (Here the difference is due to the fact that the distance of any subsequent \(\frac{k}{2^r} - \varepsilon, \frac{k+1}{2^r} - \varepsilon\) is not necessarily \(\frac{1}{2^r}\) but only less than \(\frac{1}{2^{r-1}}\).

Now if a linear functional \(\varphi\) on \(E\) is defined as \(\varphi(x) = \sum_{t \in M} x(t)\), then we have
\[(5)\]
\[\|s\| \geq |\varphi(s)| = |\varphi(s_2)| = \sum_{n=1}^{N} (a_n \alpha_n + b_n \beta_n + c_n \gamma_n),\]
and hence
\[\|s\| \geq \sum_{n > r + m + 1} (a_n \alpha_n + b_n \beta_n + c_n \gamma_n) \geq \frac{1}{2^{m+2}} \sum_{n > r + m + 1} (\alpha_n + \beta_n + \gamma_n),\]
that is,
\[(6)\]
\[2^{m+2} \|s\| \geq \sum_{n > r + m + 1} (\alpha_n + \beta_n + \gamma_n).\]

Since, by (1), we have
\[a_r, a_{r+1}, \ldots, a_{r+m+1} \geq \frac{1}{2^{m+1}} a_r\]
and the same inequality for \(b\) and \(c\), and, by (3), one of \(a_r, b_r\) and \(c_r\), say \(a_r\), is not less than \(\frac{1}{3}\), we have, by (5),
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\[
\|s\| \geq \sum_{n=r}^{r+m+1} a_n \alpha_n \geq \frac{1}{3 \cdot 2^{m+1}} \sum_{n=r}^{r+m+1} \alpha_n
\]

and hence, by (6),

\[
\sum_{n \geq r} \alpha_n \leq (2^{m+2} + 3 \cdot 2^{m+1}) \|s\| = 5 \cdot 2^{m+1} \|s\|.
\]

Finally we will estimate \(\sum_{n=1}^{r-1} \alpha_n\). Let \(\phi\) be a linear functional on \(E\) defined as

\[
\phi(x) = \sum_{k=1}^{2^{r-1}} x\left(\frac{k}{2^{r-1}}\right)
\]

then

\[
\phi(s_1) = \sigma_{r-1} < \frac{1}{2^m}
\]

and hence

\[
\|s\| \geq |\phi(s)| \geq |\phi(s_2)| - |\phi(s_1)| \geq \sum_{n=1}^{N} \alpha_n \phi(x_n) - \frac{1}{2^m}
\]

Since

\[
\phi(x_n) = \frac{1}{2^{n-r+1}},
\]

We have

\[
\|s\| \geq \sum_{n=1}^{N} \frac{1}{2^{n-r+1}} \alpha_n - \frac{1}{2^m} \geq \sum_{n=1}^{r-1} \alpha_n - \frac{1}{2^m},
\]

that is,

\[
\sum_{n=1}^{r-1} \alpha_n \leq \|s\| + \frac{1}{2^m},
\]

and hence, combining with (7),

\[
\sum_{n=1}^{N} \alpha_n \leq (5 \cdot 2^{m+1} + 1) \|s\| + \frac{1}{2^m}
\]

Thus we have proved our final (4).

We remark that \(P \circ f_a\) is an example of separable valued function with the non convex set of Riemann limits, where \(P\) is the projection of \(E\) to the closed subspace generated by \(\{e\left(\frac{k}{2^n} - \epsilon_n\right), e\left(\frac{k}{2^n}\right), e\left(\frac{k}{2^n} + \epsilon_n\right); n, k=1, 2, \cdots\}\) and \(a=e(1)\). Here \(R.L.(P \circ f_a)\) contains elements other than 0 and \(a\), but does not contain \(\lambda a\) for any \(\lambda\) with \(0 < \lambda < 1\).

3. A class of spaces for which every \(R.L.(f)\) is not empty was given
in 1, that is, reflexive spaces with the property (*). On the other hand, an example of \( f \) with \( R.L.(f) = \phi \) is provided by the restriction of \( f_a \) given in 2 to the interval \([0, 1]\), that is, the function defined by \( f(t) = e(t) \). Here we will prove that \( R.L.(f) \) is not empty, if the range of \( f \) is separable.

Such an \( f \) is the uniform limit of a sequence of countably valued functions \( f_n \), \( n = 1, 2, \cdots \). So it is sufficient to show

(i) The existence of Riemann limits of a certain type for countably valued functions.

(ii) If \( x \) is a Riemann limit of \( f \) with the type referred in (i) and if another countably valued function \( g \) is given, then we can find a Riemann limit \( y \) of \( g \) of the same type with

\[
\|x - y\| \leq \|f - g\|
\]

where the norm \( \|f\| \) of \( f \) is defined as

\[
\|f\| = \sup_{t \in I} \|f(t)\|
\]

where \( I \) is the interval for which Riemann sums are considered. In fact, if countably valued \( f_n \) converges uniformly to \( f \), then, by (ii), we can choose a Riemann limit \( x_n \) of \( f_n \) successively so that we have

\[
\|x_n - x_{n+1}\| \leq \|f_n - f_{n+1}\|
\]

for every \( n = 1, 2, \cdots \), then \( x_n \) form a Cauchy sequence and the limit \( x \) of \( x_n \) is obviously a Riemann limit of the limit function \( f \).

We need some preparations. For a subset \( A \) of \( R \), \( m^+(A) \) and \( m^-(A) \) denote the (Lebesgue) outer measure and inner measure of \( A \) respectively; \( m(A) \) denotes the Lebesgue measure of \( A \), in case \( A \) is measurable.

For a mutually disjoint family of subsets. \( A_n \), \( n = 1, 2, \cdots \), a mutually disjoint family of measurable sets \( B_n \), \( n = 1, 2, \cdots \), is said to be subordinate to \( \{A_n\} \) if \( m^-(B_n - A_n) = 0 \) and

\[
m_n(\bigcup B_n) = m^+(\bigcup A_n).
\]

The existence of a family \( \{B_n\} \) subordinate to a given family \( \{A_n\} \) can be proved as follows:

Choose a measureable set \( B_1 \), with

\[
B_1 \supset A_1 \quad \text{and} \quad m(B_1) = m^+(A_1);
\]

if \( B_1, B_2, \cdots, B_n \) are mutually disjoint and

\[
m \left\{ B_1 \supset B_2 \supset \cdots \supset B_n \right\} = m^+ \left\{ A_1 \supset A_2 \supset \cdots \supset A_n \right\},
\]

then \( B_{n+1} \) can be given as
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\[ B_{n+1} = B - B_1 \cup B_2 \cup \cdots \cup B_n \]
where \( B \) is a measurable set with
\[ B \supset A_1 \cup A_2 \cup \cdots \cup A_{n+1} \]
and
\[ m(B) = m^+ \{ A_1 \cup A_2 \cup \cdots \cup A_{n+1} \} \]
Since
\[ m^- \{ B_1 \cup B_2 \cup \cdots \cup B_n - A_1 \cup A_2 \cup \cdots \cup A_{n+1} \} = 0 \]
we have
\[ m^- \{ B_{n+1} - A_{n+1} \} \leq m^- \{ B - A_1 \cup A_2 \cup \cdots \cup A_{n+1} \} = 0 \]
We have also
\[ m \{ B_1 \cup B_2 \cup \cdots \cup B_{n+1} \} = m^+ \{ A_1 \cup A_2 \cup \cdots \cup A_{n+1} \} \]
and hence, for the family \( \{ B_n \} \) thus obtained,
\[ m( \bigcup_n B_n ) = m^+ ( \bigcup_n A_n ) \]
If a function \( f \) takes a constant value \( a_n \) on \( A_n \) for a decomposition \( \{ A_n \} \) of \( I \), and if \( \{ B_n \} \) is a measurable decomposition subordinate to \( \{ A_n \} \), then \( \sum_{n=1}^{\infty} m(B_n) a_n \) belongs to \( R.L.(f) \), in other words, the usual Lebesgue integral \( \int_{I} f^*(t) dt \) where \( f^* \) is the function taking constant values \( a_n \) on \( B_n \) is a Riemann limit of \( f \). This can be seen if we show
\[ \left( 8 \right) \quad \int_{I} f^*(t) dt \in R.L.(f^*) , \]
because we can modify \( B_n \) within the difference of measure 0 so as to be included in the closure of \( A_n \) and, for \( f^* \) defined by these \( B_n \), we have \( R.L.(f^*) \subset R.L.(f) \).

On the other hand, (8) can be proved easily from the fact that for every measurable set \( B \), there exist finite number of open intervals \( I_1, I_2, \cdots, I_m \) such that the measure of the symmetric difference of \( B \) and \( \bigcup_{i=1}^{m} I_i \) is arbitrarily small and every \( I_i \) contains a point in \( B \).

Thus we have proved (i), that is, the existence of a Riemann limit of a countably valued function \( f \) as the Lebesgue integral of a certain measurable function obtained by modifying \( f \).

Let \( f \) and \( g \) be any two countably valued functions. We can suppose, without loss of generality, that \( f \) takes constant values \( a_n \) in \( A_n \), \( \{ A_n \} \) being a decomposition of \( I \), and \( g \) takes \( a_{n,m} \) in \( A_{n,m} \) where \( \{ A_{n,m} ; n,m=1,2,\cdots \} \)
is also a decomposition of \( I \) satisfying \( A_n = \bigcup_{m=1}^{\infty} A_{n,m} \). Suppose a measurable decomposition \( \{B_n\} \) subordinate to \( \{A_n\} \) is given. We will show that there exists a measurable decomposition \( \{B_{n,m}\} \) which is subordinate to \( \{A_{n,m}\} \) and satisfies \( B_n = \bigcup_{m=1}^{\infty} B_{n,m} \). This gives (ii) completing the rest of our proof, because we have then

\[
\sum_{n} m(B_n) a_n \in R.L. (f),
\]
\[
\sum_{n, m} m(B_{n,m}) a_{n,m} \in R.L. (g)
\]

and

\[
\| \sum_{n} m(B_n) a_n - \sum_{n, m} m(B_{n,m}) a_{n,m} \|
\]
\[
= \| \sum_{n, m} m(B_{n,m}) (a_n - a_{n,m}) \| \leq \sup_{n, m} \|a_n - a_{n,m}\| \leq \|f - g\|.
\]

Now put

\[
A'_{n,m} = B_n \setminus A_{n,m},
\]

and

\[
A'_{n} = \bigcup_{m} A'_{n,m} = B_n \setminus A_n.
\]

For a fixed \( n \), let \( \{B'_{n,m}\} \) be a measurable family subordinate to \( \{A'_{n,m}\} \), then, for \( B'_n = \bigcup_{m} B'_{n,m} \),

We have

\[
m(B'_n) = m^+(\bigcup_{m} A'_{n,m}) = m^+(B_n \setminus A_n) = m^+ \{B_n - (B_n - A_n)\}
\]

\[
= m(B_n) - m^- (B_n - A_n) = m(B_n)
\]

and, since

\[
m(B'_n - B_n) \leq m^- (B'_n - A'_n) = 0,
\]

the symmetric difference \( B_n \Delta B'_n \) of \( B_n \) and \( B'_n \) is measure 0. Therefore we can replace \( B'_{n,m} \) by a suitable \( B_{n,m} \) for which we have

\[
m(B_{n,m} \Delta B'_{n,m}) = 0
\]

and

\[
B_n = \bigcup_{m} B_{n,m}.
\]

\( \{B_{n,m}\} \) is obviously subordinate to \( \{A_{n,m}\} \).

Thus our proof is completed.
On the limits of Riemann sums of functions in Banach spaces

References


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