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FINITE OUTER GALOIS THEORY OF NON-COMMUTATIVE RINGS

By

Yôichi MIYASHITA

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§ 0. Introduction. It is the purpose of this paper to extend the Galois theory of commutative rings given by S. U. Chase, D. K. Harrison and A. Rosenberg [4] to non-commutative case. In what follows, for the sake of simplicity, we shall state main results for directly indecomposable rings: Let $A
\triangleright 1$ be a directly indecomposable ring, $G$ a finite group of automorphisms of $A$, and $B=A^{G}=\{x\in A; \sigma(x)=x \text{ for all } \sigma \in G\}$. We call $A/B$ a $G$-Galois extension if there are elements $a_{1}, \ldots, a_{n}$; $a_{1}^{*}, \ldots, a_{n}^{*}$ in $A$ such that $\sum a_{t}\cdot \sigma(a_{i}^{*})=\delta_{1,\sigma}(\sigma \in G)$, where $\delta_{1,\sigma}$ means Kronecker’s delta. If $V_{A}(B)=C$ (the center of $A$), then $A/B$ is a $G$-Galois extension if and only if the mapping $x\otimes y \rightarrow xy$ from $A\otimes_{B}A$ to $A$ splits as an $A-A$-homomorphism (Th. 1.5). Let $A/B$ be a $G$-Galois extension, and $A'$ a $G$-invariant subring of $A$, i.e., $\sigma(A')=A'$ for all $\sigma \in G$, and put $B'=A'^{G}$. If $A'/B'$ is a $G$-Galois extension and $B''$ is a direct summand of $A''$, then there hold the following: (1) For any subgroup $H$ of $G$, $A''=B\otimes_{B'}A''=A''\otimes_{B'}B$. (2) Let $\{T\}$ be the set of all $G$-invariant intermediate rings of $A/A'$, and $\{T\}$ the set of all intermediate rings of $B/B'$ such that $A'T=TA'$. Then, $T\rightarrow T \cap B$ and $T\rightarrow A'T=TA'$ are mutually converse order isomorphisms between $\{T\}$ and $\{T\}$, and $T/(T \cap B)$ is a $G$-Galois extension (Th. 5.1).

Let $A/B$ be a $G$-Galois extension, $V_{A}(B)=C$, and $B_{B}$ a direct summand of $A_{B}$. Then there hold the following: (1) $G$ coincides with the set of all $B$-automorphisms of $A$ (Th. 4.2). (2) For any subgroup $H$ of $G$, $\{\sigma \in G; \sigma|A''=A''\otimes_{B'}B\}$. (3) If $T$ is an intermediate ring of $A/B$, the following are
equivalent: (a) $T=A^H$ for some subgroup $H$ of $G$. (b) The mapping $x \otimes y \rightarrow xy$ from $T \otimes A$ to $A$ splits as a $T$-$T$-homomorphism (Th. 2.6). (c) $A/T$ is a projective Frobenius extension (in the sense of Kasch), and $T_T$ is a direct summand of $A_T$ (Th. 3.2). In case $bB_B$ is a direct summand of $bA_B$, the next is also equivalent to (a). (b') The mapping $x \otimes y \rightarrow xy$ from $T \otimes bT$ to $T$ splits as a $T$-$T$-homomorphism (Th. 2.9). (4) For any subgroup $H$ of $G$, every $B$-isomorphism from $A^H$ to $A$ can be extended to a $B$-ring automorphism of $A$ (Th. 4.2). (5) If $A_B$ is finitely generated and free, and $B$ is a semi-primary ring (i.e. $B/R(B)$ satisfies the minimum condition for left ideals, where $R(B)$ means the Jacobson radical of $B$), then $A$ has a normal basis (Th. 1.7).

Let $A=\Delta(A, G)=\sum_{\sigma \in \theta} + Au_{\sigma}$ be the trivial crossed product of $A$ with $G$. $G$ is said to be completely outer if $\Delta A u_{\sigma}$ and $\Delta A u_{\tau}$ have no isomorphic non-zero subquotients provided $\sigma \neq \tau$. If $G$ is completely outer, then $A/B$ is a $G$-Galois extension and $V_A(B)=C$ (Prop. 6.4). If $A$ is commutative, then $A/B$ is a $G$-Galois extension if and only if $G$ is completely outer (Th. 6.6). In case $A$ is two-sided simple, $G$ is completely outer if and only if $A/B$ is a $G$-Galois extension and $V_A(B)=C$ (Cor. to Prop. 6.4).

The author wishes to express his best thanks to Dr. H. Tominaga for helpful suggestions.

§ 1. Galois extension and normal basis.

Throughout the present paper, all rings have identities, modules are unitary. A subring of a ring will mean one containing the same identity. By a ring homomorphism, we mean always a ring homomorphism such that the image of 1 is 1. Let $A$ be a ring, $C$ the center of $A$, $G$ a finite group of automorphisms of $A$ which acts on the left side, and $B=A^G=\{x \in A ; \sigma(x)=x \text{ for all } \sigma \text{ in } G\}$. For any subgroup $H$ of $G$, $\delta_{H, \sigma}$ means the mapping from $G$ to \{1, 0\} (\subseteq A) such that $\delta_{H, \sigma}=1$ if and only if $\sigma \in H$.

Let $B'$ and $T$ be subrings of a ring $A'$ such that $B' \subseteq T$. $A'$ is said to be $(B', T)$-projective, if the mapping $\sum_{j} x_{j} \otimes y_{j} \rightarrow \sum_{j} x_{j} y_{j}$ from $T \otimes A'$ to $A'$ splits as a $T$-$T$-homomorphism. As is easily seen, $A'$ is $(B', T)$-projective if and only if there are elements $t_{1}, \cdots, t_{n} \in T$ and $a_{1}', \cdots, a_{n}' \in A'$ such that $\sum_{i} t_{i} a_{i}'=1$ and $\sum_{i} x t_{i} \otimes a_{i}' = \sum_{i} x t_{i} \otimes a_{i}' x \ (\in T \otimes A')$ for all $x \in T$. When this is the case, \{(t_{i}, a_{i}') \}; i=1, \cdots, n\} is called a $(B', T)$-projective coordinate system for $A'$. If $A'$ is $(B', A')$-projective, then we call $A'/B'$ a separable extension.

Let $f$ and $g$ be ring homomorphisms from a ring $A'$ to a ring $A''$. $f$ and $g$ are called strongly distinct if, for any non-zero central idempotent $e$ of $A''$, there is an element $x$ in $A'$ such that $f(x)e \neq g(x)e$. Let $\Sigma$ be a set of
ring homomorphisms from $A'$ to $A''$. $\mathcal{E}$ is called strongly distinct if any distinct $f$, $g$ in $\mathcal{E}$ are strongly distinct.

$\mathcal{A}=\mathcal{A}(A, G)$ denotes the trivial crossed product of $A$ with $G$: $\mathcal{A} = \sum_{\sigma \in G} A u_{\sigma}, \ u_{\sigma} u_{\tau} = u_{\sigma \tau} (\sigma, \tau \in G), \ u_{\sigma} x = \sigma(x) u_{\sigma}$ ($x \in A$). By $j$, we denote the ring homomorphism from $\mathcal{A}$ to $\text{Hom} (A_B, A_B)$ defined by $j(xu_{\sigma}) (y) = x \cdot \sigma(y)$ for $x, y$ in $A$ and $\sigma$ in $G$.

$A/B$ is called a $G$-Galois extension if there are elements $a_1, \ldots, a_n$; $a_1^* \ldots, a_n^*$ in $A$ such that $\sum_i t_i a_i \sigma(a_i^*) = \delta_{1,i}$ for all $\sigma$ in $G$. When this is the case, $\{ (a_i, a_i^*) : i = 1, \ldots, n \}$ is called a $G$-Galois coordinate system for $A/B$. Then the following is known: $A/B$ is a $G$-Galois extension if and only if $A_B$ is finitely generated and projective and $j$ is an onto isomorphism (cf. [6]). When this is the case we identify $\mathcal{A}$ with $\text{Hom} (A_B, A_B)$: $\mathcal{A} = A_B G = AG$, where $A_t$ means the set of all left multiplications by elements of $A$. If $A/B$ is $G$-Galois and $C = V_{A}(B)$ (the centralizer of $B$ in $A$), it is called outer $G$-Galois. If $A/B$ is $G$-Galois (resp. outer $G$-Galois) and $H$ is a subgroup of $G$, then $A/A^H$ is evidently $H$-Galois (resp. outer $H$-Galois).

**Proposition 1.1.** Let $A'$ and $A''$ be rings, $T$ a subring of $A'$, $f$ a ring homomorphism from $T$ to $A''$, and $g$ a ring homomorphism from $A'$ to $A''$. If there are elements $t_1, \ldots, t_n \in T$ and $a_1, \ldots, a_n \in A'$ such that $\sum_i t_i a_i = 1$ and $\sum_i f(t_i) g(a_i) = 0$, then $f$ and $g|T$ (the restriction of $g$ to $T$) are strongly distinct.

**Proof.** Let $e$ be a central idempotent of $A''$ such that $f(x)e = g(x)e$ for all $x$ in $T$. Since $\sum_i t_i a_i = 1$, we have $\sum_i g(t_i) g(a_i) = 1$, and therefore $e = e_1 = \sum_i e g(t_i) g(a_i) = \sum_i e f(t_i) g(a_i) = 0$. Thus, $f$ and $g|T$ are strongly distinct.

**Proposition 1.2.** Let $B'$ and $T$ be subrings of a ring $A'$ such that $B' \subseteq T$, and $A''$ an extension ring of $B'$ such that $V_{A''}(B') = V_{A''}(A'')$, where $V_{A''}(B')$ means the centralizer of $B'$ in $A''$. Let $A'$ be $(B', T)$-projective, and $\{(t_i, a_i) : i = 1, \ldots, n\}$ a $(B', T)$-projective coordinate system for $A'$. Let $f$ be a $B'$-ring homomorphism from $T$ to $A''$, and $g$ and $g'$ $B'$-ring homomorphisms from $A'$ to $A''$. We set $e = \sum_i f(t_i) g(a_i)$ and $e' = \sum_i f(t_i) g'(a_i)$. Then there hold the following:

1. $e$ is a central idempotent in $A''$.
2. $f(x)e = g(x)e$ for all $x$ in $T$.
3. $ee' = e \sum_i g(t_i) g'(a_i)$.
4. $f$ and $g|T$ are strongly distinct if and only if $e = 0$.
5. If $g|T$ and $g'|T$ are strongly distinct, then $ee' = 0$.

**Proof.** Since $\sum_i xt_i \otimes a_i = \sum_i t_i \otimes a_i x$ ($\in T \otimes_{B'} A'$) for all $x$ in $T$, $\sum_i f(xt_i) \otimes g(a_i) = \sum_i f(t_i) \otimes g(a_i x)$ ($\in A'' \otimes_{B'} A''$) for all $x$ in $T$. Therefore,
$f(x)e=e\cdot g(x)$ for all $x$ in $T$, in particular, $ye=ey$ for all $y$ in $B'$. Hence, by assumption, $e$ is contained in the center of $A''$. Since $\sum f(t_j)(\sum f(t_i)\otimes g(a_i))g'(a_j) = (\sum f(t_i)g(a_i)) \sum f(t_j)g'(a_j)$, we obtain $ee' = \sum f(t_j)e\cdot g'(a_j) = e\sum f(t_j)g'(a_j)$.

If we put $g=g'$, then we have $e^2=e$, and so $e$ is a central idempotent of $A''$ such that $f(x)e=e\cdot g(x)$ for all $x$ in $T$. Therefore $f$ and $g|T$ are strongly distinct if and only if $e=0$ (Prop. 1.1). Now, it is left only to prove (5). If $g|T$ and $g'|T$ are strongly distinct, then $\sum f(t_j)g'(a_j)=0$ by (4), and so $ee'=e\sum f(t_j)g'(a_j)=0$.

Evidently, the mapping $x\otimes y\rightarrow x\sum u_{\sigma}y$ from $A\otimes_{B}A$ to $A$ is an $A$-$A$-homomorphism. We denote this homomorphism by $h$. One may remark here that $h$ is a $A$-$A$-homomorphism. In fact, $u_{\sigma}x\sum u_{\sigma}y=\tau(x)u_{\sigma}\sum u_{\sigma}y=\tau(x)\sum u_{\sigma}y$.

**Proposition 1.3.** Let $A/B$ be a $G$-Galois extension, and let $\{(a_i, a_i^\ast); i=1, \cdots, n\}$ be a $G$-Galois coordinate system for $A/B$. Then $h$ is a $A$-$A$-isomorphism, $h^{-1}(\sum_{i}x_{i}u_{i})=\sum_{i}\sum_{j}x_{i}\sigma(a_i)\otimes a_j^\ast$ for every $\sum_{i}x_{i}u_{i}$ in $A$, and $\{(a_i, a_i^\ast); i=1, \cdots, n\}$ is a $(B, A)$-projective coordinate system for $A$.

**Proof.** To be easily seen, $h(\sum_{i}\sum_{j}x_{i}\sigma(a_i)\otimes a_j^\ast)=\sum_{i}x_{i}u_{i}$, and hence $h$ is onto. Let $x, y$ be in $A$. Then $\sum_{i}\sum_{j}x_{i}\sigma(y)\sigma(a_i)\otimes a_j^\ast=x\otimes\sum_{i}\sum_{j}\sigma(y)\sigma(a_i)a_j^\ast=x\otimes y$, whence we can easily see that $h$ is $1$-$1$. Hence, $h$ is a $A$-$A$-isomorphism. Since $h(\sum_{i}a_i\otimes a_i^\ast)=u_i$ and $h$ is an $A$-$A$-isomorphism, $\sum_{i}x_{i}a_i\otimes a_i^\ast=\sum_{i}a_i\otimes a_i^\ast x$ for any $x$ in $A$.

**Proposition 1.4.** Assume $V_A(B)=C$ (the center of $A$), and let $a_i, a_i^\ast$ $(i=1, \cdots, n)$ be elements of $A$. Then the following conditions are equivalent:

(i) $\{a_i, a_i^\ast\}; i=1, \cdots, n$ is a $G$-Galois coordinate system for $A/B$. (ii) $\{a_i, a_i^\ast\}; i=1, \cdots, n$ is $(B, A)$-projective coordinate system for $A/B$ and $G$ is strongly distinct.

**Proof.** (i)$\Rightarrow$(ii) follows from Prop. 1.3 and Prop. 1.1. (ii)$\Rightarrow$(i) follows from Prop. 1.2 (4).

Restating the above proposition we obtain the following theorem.

**Theorem 1.5.** (Cf. [4; Th. 1.3].) Let $V_A(B)=C$. Then following conditions are equivalent:

(i) $A/B$ is a $G$-Galois extension.

(ii) $A/B$ is a separable extension and $G$ is strongly distinct.

**Remark.** To prove the part (i)$\Rightarrow$(ii) we do not need the condition $V_A(B)=C$.

**Proposition 1.6.** (Cf. [4; Th. 4.2].) If $A/B$ is a $G$-Galois extension and $B\cong B^m$ for some natural number $m$, then $B\cong B^m$.

**Proof.** Let $A=\sum_i\oplus Bd_i$ $(i=1, \cdots, n)$, and $B\cong B^m$ by the correspondence
y \rightarrow yd_i \ (y \in B). \ Then \ \Delta = \sum_e \sum u_e A = \sum_e \sum u_e B d_i = \sum_e (\sum u_e) d_i \ and \ \sum_e (Bu_e) d_i = \sum_e Bu_e \ as \ \sum_e Bu_e \ - \ \text{left modules}. \ Hence, \ \Re(BA) \cong \Re(Be^A). \ On \ the \ other \ hand, \ \Delta \cong \Delta A \otimes \Delta B \cong \Delta \otimes \Delta (B^m) \cong \Delta A^m \ (\text{Prop. 1.3}). \ We \ obtain \ therefore \ \Re(BG) \cong \Re(BA).

**Theorem 1.7.** Let $A/B$ be a $G$-Galois extension and $\Re(BA) \cong \Re(BA^m)$ for some natural number $m$. If $B$ is semi-primary (i.e., $B/\Re(B)$ satisfies the minimal condition for left ideals, where $\Re(B)$ means the Jacobson radical of $B$), then $\Re(BG) \cong \Re(BA)$, that is, $A$ has a normal basis.

**Proof.** By Prop. 1.6, $\Re(BG) \cong \Re(BA)$ since $\Re(BG) = (\Re(BA))^m$ under the above isomorphism, $BG/\Re(B)B \cong (A/\Re(BA))^m$ as $BG/\Re(B)B$-left modules. Since $BG/\Re(B)B$ is $B/\Re(B)$-left finitely generated and $B$ is semi-primary, $BG/\Re(B)B$ satisfies the minimal condition (and the maximal condition) for left ideals. Hence, by Krull-Remak-Schmidt’s theorem, we have $BG/\Re(B)B \cong A/\Re(BA)$ as $BG$-left modules. Since $\Re(BG)$ and $\Re(BA)$ are finitely generated and projective and $\Re(BA) \subseteq \Re(BG)$ and $\Re(BA) \subseteq \Re(BG)$, $BG \cong A$ as $BG$-left modules by the uniqueness of projective cover (cf. [11]).

§ 2. The first characterization of fixed-subrings.

For any subgroup $H$ of $G$, the mapping $x \rightarrow \sum_{e \in H} \tau(x)$ from $A$ to $A^H$ is evidently an $A^H$-$A^H$-homomorphism. We denote this by $t_H$.

Let $A/B$ be a $G$-Galois extension. Then $(\sum_u u) A \cong \text{Hom} (A_B, B_B)$ by $j$ (cf. [6]). From this fact, one will easily see that $B_B$ is a direct summand of $A_B$ if and only if there exists an element $c$ in $A$ such that $t_0(c) = 1$. Further, since $j((\sum_u u) V_A(B)) = \text{Hom} (B_B, B_B), B_B$ is a direct summand of $B_B$ if and only if there exists an element $c$ in $V_A(B)$ such that $t_0(c) = 1$.

Let $c$ be an element of $A$ such that $t_0(c) = 1$, $H$ a subgroup of $G$, and $G = H \sigma_1 \cup \cdots \cup H \sigma_r$ the right coset decomposition of $G$. If we set $\sum \sigma_i(c) = d$, then $t_H(d) = 1$. Hence, if $A/B$ is $G$-Galois and $B_B$ is a direct summand of $A_B$, then $A_B^H$ is a direct summand of $A_A^H$.

For any $G$-left module $A$ and any subgroup $H$ of $G$, we denote by $M^H$ \{u \in M; \ \tau(u) = u \ for \ all \ \tau \ in H\}. \ If \ A/B$ is a $G$-Galois extension, then $h: A_B \otimes_{B_B} A_A \cong A_A$ (Prop. 1.3), and evidently $(A \otimes A)^H \cong A^H$ under $h$.

**Proposition 2.1.** Let $A/B$ be a $G$-Galois extension. If $H$ is a subgroup of $G$, then $A^H = \{ \sum_{e \in H} u_e x_e; \ if \ Ha = H \sigma \ then \ x_e = x_e \}$ and $(A \otimes A)^H = A^H \otimes A$.

**Proof.** The first assertion is evident. We shall prove the second one. Evidently $A^H \otimes A \subseteq (A \otimes A)^H$. Let \{(a_i, a^*_i); i = 1, \cdots, n\} be a $G$-Galois coordinate system for $A/B$. If $\rho$ is an element of $G$, then $\sum_{e \in H} u_e \in A^H$ and $h^{-1}(\sum_{e \in H} u_e) = \sum_{e \in H} \sum \tau(\rho(a_i)) \otimes a^*_i = \sum_i (\sum_{e \in H} \tau(\rho(a_i)) \otimes a^*_i \in A^H \otimes A$. Noting that $h$
is an $A$-right isomorphism, we have $(A \otimes A)^u \subseteq A^u \otimes A$. Thus $(A \otimes A)^u = A^u \otimes A$.

**Proposition 2.2.** Let $A/B$ be $G$-Galois. If $H$ is a subgroup of $G$, then there are elements $t_1, \ldots, t_n \in A^u$ and $a_1^*, \ldots, a_n^* \in A$ such that $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$ for all $\sigma$ in $G$, and $\{\sigma \in G; \sigma|A^u = 1_{A^u}\} = H$.

**Proof.** Let $\{(a_i, a_i^*); i = 1, \ldots, n\}$ be a $G$-Galois coordinate system for $A/B$. If we put $t_i = t_{H}(a_i)$, then $t_i \in A^u$ and $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$. If $\sigma|A^u = 1_{A^u}$, then $1 = \sum_i \sigma(t_i) \sigma(a_i^*) = \sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$. Hence $\sigma \in H$.

**Theorem 2.3.** Let $A/B$ be $G$-Galois, and $B_0$ a direct summand of $A_B$. If $H$ is a subgroup of $G$ and $T$ is an intermediate subring of $A/B$ such that $T \subseteq A^u$, then the following conditions for $T$ are equivalent.

(i) $T = A^u$.

(ii) There are elements $t_1, \ldots, t_n \in T$ and $a^*_1, \ldots, a^*_n \in A$ such that $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$ for all $\sigma$ in $G$.

(iii) $T \otimes A = A^u \otimes A$ in $A \otimes_{B} A$.

**Proof.** (i) $\Rightarrow$ (ii) follows from Prop. 2.2. (ii) $\Rightarrow$ (iii) Evidently $T \otimes A \subseteq A^u \otimes A$ in $A \otimes_{B} A$. If $\rho$ is in $G$, then $\sum_i t_i \otimes \rho^{-1}(a_i^*) \in T \otimes A$ and $h(\sum_i t_i \otimes \rho^{-1}(a_i^*)) = \sum_{\sigma \in H} u_{\sigma}$. Noting that $h$ is an $A$-right homomorphism, we know that $h(T \otimes A) = A^u$, and hence $T \otimes A = A^u \otimes A$ (Prop. 2.1). (iii) $\Rightarrow$ (i) There is an element $c$ of $A$ such that $t_0(c) = 1$. For any $x$ in $A^u$, $x \otimes c \in A^u \otimes A = T \otimes A$. Therefore, there are elements $x'_j s \in T$, $y'_j s \in A$ such that $x \otimes c = \sum_j x'_j \otimes y'_j$. By making use of the mapping $1 \otimes t_0$, we can easily see $x = x \cdot t_0(c) = \sum_j x'_j \cdot t_0(y'_j) \in T \cdot B = T$. Hence $T = A^u$.

**Proposition 2.4.** Let $A/B$ be a $G$-Galois extension. If $H$ is a subgroup of $G$, then $G|A^u$ is strongly distinct and the mapping $x \otimes y \rightarrow xy$ from $A^u \otimes B$ to $A$ splits as an $A^u$-$A^u$-homomorphism (i.e., $A$ is $(B, A^u)$-projective).

**Proof.** Let $\{(a_i, a_i^*); i = 1, \ldots, n\}$ be a $G$-Galois coordinate system for $A/B$. If we set $t_i = t_{H}(a_i)$, then $t_i \in A^u$ and $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$ for every $\sigma$ in $G$. Therefore, by Prop. 1.1, $G|A^u$ is strongly distinct. Now, $t_{H} \otimes 1$ is an $A^u$-$A^u$-homomorphism from $A \otimes_{B} A$ to $A^u \otimes_{B} A$. Since $\sum_i x a_i \otimes a_i^* = \sum_i a_i \otimes a_i^* x$ ($\in A \otimes_{B} A$) for all $x$ in $A$ (Prop. 1.3), $\sum_i x t_i \otimes a_i^* = \sum_i t_i \otimes a_i^* x$ ($\in A^u \otimes_{B} A$) for all $x$ in $A^u$. Hence the mapping $x \rightarrow \sum_i x t_i \otimes a_i^* x$ from $A$ to $A^u \otimes_{B} A$ is an $A^u$-$A^u$-homomorphism, and $\sum_i t_i a_i^* x = x$. Hence the mapping $x \otimes y \rightarrow xy$ from $A^u \otimes_{B} A$ to $A$ splits as an $A^u$-$A^u$-homomorphism.

**Proposition 2.5.** Let $A/B$ be outer $G$-Galois, and $T$ an intermediate ring of $A/B$. If $G|T$ is strongly distinct, and $A$ is $(B, T)$-projective then there are elements $t_1, \ldots, t_n \in T$ and $a_1^*, \ldots, a_n^* \in A$ such that $\sum_i t_i \cdot \sigma(a_i^*) = \delta_{H, \sigma}$.
for all \( \sigma \) in \( G \), where \( H = \{ \sigma \in G; \sigma|T = 1_T \} \).

Proof. Let \( \{(t_i, a_i^*); i = 1, \cdots, n\} \) be a \((B, T)\)-projective coordinate system for \( A \). Then, by Prop. 1.2, \( \sum t_i \sigma(a_i^*) = 0 \) for every \( \sigma \notin H \). Whereas, if \( \sigma \in H \), then \( 1 = \sum t_i \sigma(t_i) \sigma(a_i^*) = \sum t_i \sigma(a_i^*) \).

Combining Props 2.4, 2.5 with Th. 2.3, we readily obtain the following:

**Theorem 2.6.** Let \( A/B \) be outer \( G\)-Galois, and \( B_B \) a direct summand of \( A_B \). If \( T \) is an intermediate ring of \( A/B \), then the following conditions are equivalent:

(i) There is a subgroup \( H \) of \( G \) such that \( T = A^H \).

(ii) \( A \) is \((B, T)\)-projective and \( G|T \) is strongly distinct.

**Lemma 2.7.** Let \( S \) and \( T \) be subrings of a ring \( R \) such that \( S \supseteq T \).

1. If \( R/T \) is separable, then so is \( R/S \).
2. If \( S/T \) is separable, then \( R \) is \((T, S)\)-projective.
3. If both \( R/S \) and \( S/T \) are separable, then so is \( R/T \).

Proof. (1) will be easily seen. (2) Since \( S \otimes_T S \supseteq S \otimes_T R \) and \( S \otimes_T R \supseteq R \), this is obvious. (3) Since the mapping \( s \otimes s' \rightarrow ss' \) from \( S \otimes_T S \) to \( S \) splits as an \( S-S \)-homomorphism, the mapping \( r \otimes r' \rightarrow r \otimes r' \) from \( R \otimes_T R \) to \( R \otimes_S R \) splits as an \( R-R \)-homomorphism. Since \( R/S \) is separable, the mapping \( r \otimes r' \rightarrow rr' \) from \( R \otimes_T R \) to \( R \) splits as an \( R-R \)-homomorphism.

**Proposition 2.8.** Let \( A/B \) be outer \( G\)-Galois, and \( _BB \) a direct summand of \( _BA_B \). If \( H \) is a subgroup of \( G \), then \( A^H \) is an \( A^H-A^H \)-direct summand of \( A \), and \( A^H/B \) is a separable extension.

Proof. Since \( _BB \) is a direct summand of \( _BA_B \), there is an element \( c \) of \( C \) such that \( t_0(c) = 1 \). Let \( G = H \sigma_1 \cup \cdots \cup H \sigma_r \) be the right coset decomposition of \( G \). If we set \( d = \sum_k \sigma_k(c) \), then \( t_H(d) = 1 \) and \( d \in C \). Hence \( A^H \) is an \( A^H \)-direct summand of \( A \). Let \( \{(a_i, a_i^*); i = 1, \cdots, n\} \) be a \((B, A)\)-projective coordinate system for \( A/B \). Then, \( \{(a_i, a_i^*); i = 1, \cdots, n\} \) is a \( G \)-Galois coordinate system for \( A/B \) (Prop. 1.4). The mapping \( x \rightarrow t_H(dx) \) from \( A \) to \( A^H \) is an \( A^H-A^H \)-homomorphism. We denote this by \( t' \). Then, the mapping \( t_H \otimes t' \) from \( A \otimes_B A \) to \( A^H \otimes_B A^H \) is evidently an \( A^H-A^H \)-homomorphism, and therefore the mapping \( y \rightarrow \sum t_H(ya_i) \otimes t'(a_i^*) = \sum t_H(a_i) \otimes t'(a_i^*)y \) from \( A^H \) to \( A^H \otimes_B A^H \) is an \( A^H-A^H \)-homomorphism. Since \( \sum t_H(a_i) t'(a_i^*) = \Sigma \sum e_H \sigma(a_i) \tau(a_i^*) \tau(d)y = \sum e_H \Sigma \sigma(a_i) \tau(a_i^*) \tau(d)y = \sum e_H \tau(d)y \) for all \( y \) in \( A^H \), \( A^H/B \) is a separable extension.

By Th. 2.6, Lemma 2.7 and Prop. 2.8, we obtain at once the following:

**Theorem 2.9.** (Cf. [4; Th. 2.2]). Let \( A/B \) be outer \( G\)-Galois, and \( _BB \) a direct summand of \( _BA_B \). If \( T \) is an intermediate ring of \( A/B \), then the
following conditions are equivalent:

(i) There is a subgroup $H$ of $G$ such that $T = A^u$.

(ii) $T/B$ is a separable extension and $G|T$ is strongly distinct.

§3. The second characterization of fixed-subrings.

Let $R$ be a ring, $S$ a subring of $R$. $R/S$ is called a projective Frobenius extension if $R_S$ is finitely generated and projective and $sR_S \cong \pi \Hom(R_S, S_S)_R$ (cf. [10]). If $A/B$ is a $G$-Galois extension, then $(\pi_0A) \cong \pi \sum_i u_i A \cong \pi \Hom(A_B, B_B)_A$ by $j$. Hence, $A/B$ is a projective Frobenius extension. Now, we shall state the next lemma without proof.

**Lemma 3.1.** Let $R/S$ be a projective Frobenius extension, and $1 \leftarrow t$ under an isomorphism $sR_S \cong \pi \Hom(R_S, S_S)_R$. Then $t \in \Hom(sR_S, S_S)_R$ and $\Hom(R_S, S_S) = tR$ and $G|T$ is strongly distinct.

**Theorem 3.2.** Let $A/B$ be outer $G$-Galois, and $B_B$ a direct summand of $A_B$. If $T$ is an intermediate ring of $A/B$, then the following conditions are equivalent.

(i) There is a subgroup $H$ of $G$ such that $A^u = T$.

(ii) $A/T$ is a projective Frobenius extension, $T_T$ is a direct summand of $A_T$, and $G|T$ is strongly distinct.

**Proof.** It suffices to prove that (ii) $\Rightarrow$ (i) (cf. §2). We identify $\Hom(A_B, A_B)$ with $\mathcal{A}$, and set $\mathcal{A}_0 = \Hom(A_T, A_T)$, which is a subring of $\mathcal{A}$. Let $t = \sum_i c_i u_i$ be the image of $1$ under the isomorphism $\pi \mathcal{A} \cong \pi \Hom(A_T, T_T)_A$. Then, $tA = \Hom(A_T, T_T)$. $A\otimes_\pi A = \mathcal{A}_0$ and $t \in \Hom(\pi A_T, T_T)$ (Lemma 3.1). Since $xt = tx$ for all $x$ in $T$, we have $xc_i = c_i \sigma(x)$ for all $x$ in $T$ and $\sigma$ in $G$, in particular, $yc_i = c_i y$ for $y$ in $B$. Therefore, by assumption, each $c_i$ is an element of $C$. Since $A\otimes_\pi A = \mathcal{A}_0$, there are elements $c_i'$'s, $d_i$'s in $A$ such that $\sum_i c_i d_i = u_i$. From this fact, $c_1$ is an invertible element of $C$. Now, the mapping $\alpha: \delta \mapsto \delta c_i^{-1}$ is a $A_0$-A-homomorphism from $A_0$ to $A$, and the mapping $\beta: \sum_i x_i u_i \mapsto \sum_i x_i c_i u_i$ is evidently an $A$-A-endomorphism of $\mathcal{A}$. For any $y$ in $A$ and $z$ in $T$, we have $\sum_i x_i c_i u_i (yz) = \sum_i x_i c_i \sigma(y) \sigma(z) = \sum_i x_i \sigma(y) c_i \sigma(z) = \sum_i x_i \sigma(y) z c_i = \sum_i x_i c_i \sigma(y) z$, which means $\beta(\mathcal{A}) \subseteq \mathcal{A}_0$. If $x \otimes y$ is in $A \otimes_B A$, then $\beta h(x \otimes y) = \beta(x(x \otimes u) y) = \beta(\sum_i x_i \sigma(y) u_i) = \sum_i x_i \sigma(y) c_i u_i = x \sum_i c_i u_i y = xty$. For any $\delta_0$ in $\mathcal{A}_0$ and any $z$ in $A$, we have $\delta_0 xty(z) = \delta_0(x(t(yz)) = \delta_0(x\cdot t(yz)) = \delta_0(x) t(yz)$. Thus, $\beta h$ is a $A_0$-A-homomorphism from $A \otimes_B A$ to $\mathcal{A}_0$, and so is $A_0$-A-homomorphism from $\mathcal{A}$ to $\mathcal{A}_0$. Since $\beta \alpha(u_i) = \beta(u_i c_i^{-1}) = u_i$, it holds that $\alpha = 1_{\mathcal{A}_0}$. Thus, we have $\mathcal{A} = \Im \alpha \oplus \Ker \beta = \mathcal{A}_0 \oplus (\sum_i \oplus \Ann_A(c_i) \cdot u_i)$, where $\Ann_A(c_i) = \{x \in A; xc_i = 0\}$. Now, let $\{(a_i, a_i^*)\}; i = 1, \ldots, n$ be a $G$-Galois coordinate system for $A/B$. If $\tau$ is in $G$, then $\mathcal{A}_0 = \mathcal{A} \oplus \sum_i \tau(a_i) a_i^* = c_i u_i$, and so $\delta_0 = \sum A \cdot c_i u_i$, whence it follows that $A = A \cdot c_i \oplus \Ann_A(c_i)$. Let $A \cdot c_i = A e_i$ with a
central idempotent $e_\sigma$ in $A$. Then, $e_\sigma\cdot\sigma(y)=e_\sigma y$ for any $y$ in $T$. By assumption, if $\sigma|T\neq 1_T$ then $e_\sigma=0$, and so $A_0=\sum_{\tau\in H}Au_\tau$, where $H=\{\tau\in G; \tau|T=1_T\}$. Since $T_\tau$ is a direct summand of $A_T$, $\text{End}\langle A_\tau \rangle=T_\tau$ the set of all right multiplications by elements of $T$ (see [1; Th. A. 2]). On the other hand, since $A_0=\sum_{\tau\in H}Au_\tau$, $\text{End}\langle A_\tau \rangle=(A^H)_\tau$. Hence, $T=A^H$.

\section{Extension of isomorphisms.}

\textbf{Theorem 4.1.} Let $A/B$ be $G$-Galois, and $A'$ an extension ring of $B$ such that $V_A(B)=V_{A'}(A')$. Assume that there exists at least one $B$-ring homomorphism from $A$ to $A'$.

(1) If $H$ is a subgroup of $G$ such that $A^H_H$ is a direct summand of $A_{A^H}$. Then every $B$-ring homomorphism from $A^H$ to $A'$ can be extended to a ($B$-)ring homomorphism from $A$ to $A'$.

(2) If $f$ and $g$ are $B$-ring homomorphisms from $A$ to $A'$. Then $A'$ contains orthogonal central idempotents $e_\sigma(\sigma\in G)$ such that $\sum_\sigma e_\sigma=1$ and $f(x)=\sum_\sigma g\sigma(x)e_\sigma$ for all $x$ in $A$. (Cf. [4; Th. 3.1].)

\textbf{Proof.} There are elements $a_i$, $a_i^* (i=1, \cdots, n)$ in $A$ such that $\sum_i xa_i \otimes a_i^* = \sum_i a_i \otimes a_i^* x (\in A \otimes B A)$ for all $x$ in $A$ and $\sum_i a_i \cdot \sigma(a_i^*)=\delta_i, \sigma$ for all $\sigma$ in $G$ (Prop. 1.3). If we set $t_i=t_H(a_i)$, then $t_i \in A^H$, $\sum_i t_i \cdot \sigma(a_i^*)=\delta_{i, \sigma}$ (\sigma\in G$) and $\sum_i x t_i \otimes a_i^* = \sum_i t_i \otimes a_i^* x (\in A^H \otimes B A)$ for all $x$ in $A^H$. Let $f$ be a $B$-ring homomorphism from $A^H$ to $A'$, and $g$ a $B$-ring homomorphism from $A$ to $A'$. If we set $e_\sigma=\sum_i f(t_i) g\sigma(a_i^*)$, then each $e_\sigma$ is a central idempotent in $A'$ (Prop. 1.2). By Prop. 1.2 (3), $e_\sigma=e_\sigma g(\sum_i \sigma(t_i) \tau(a_i^*))$ for any $\sigma$, $\tau$ in $G$. Therefore, if $\sigma^{-1}\tau \in H$ then $e_\sigma=e_\tau$, and if $\sigma^{-1}\tau \notin H$ then $e_\sigma=e_\sigma$. Recalling that $A^H_H$ is a direct summand of $A_{A^H}$ there is an element $d$ of $A$ such that $t_H(d)=1$. Since $\sum_i \sum_j t_i \otimes \sigma(a_j^*)=\sum_i t_i \otimes \sigma(a_j^*)=\delta_i, \sigma$ for all $\sigma$ in $G$ (Prop. 1.3), and therefore $\sum_i f(t_i) g\sigma(a_j^*)=1 (\in A')$. Let $G=\sigma, H \cup \cdots \cup \sigma, H$ be the left coset decomposition of $G$. Then, $1=\sum_i \sum_j f(t_i) g\sigma(a_j^*)=\sum_k \sum_{\tau\in H} e_\tau g_k \sigma_\tau t_H(d)=\sum_k e_\tau g_k \sigma_\tau t_H(d)=\sum_k e_\tau g_k (x) e_\tau$ for all $x$ in $A^H$ (Prop. 1.2), we have $f(x)=\sum_k f(x) g_k (x) e_\tau$ for all $x$ in $A^H$. Evidently, the mapping $x \mapsto \sum_k g_k (x) e_\tau$ is a $B$-ring homomorphism from $A$ to $A'$, and an extension of $f$.

Now, the following theorem will follow at once from Th. 4.1.

\textbf{Theorem 4.2.} Let $A/B$ be an out $G$-Galois extension, and let $A$ be directly indecomposable. If $H$ is a subgroup of $G$ such that $A^H_H$ is a direct summand of $A_{A^H}$, then every $B$-ring homomorphism from $A^H$ to $A$ can be extended to an element of $G$. In particular, $G$ is the set of all $B$-ring automorphisms of $A$. 

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Theorem 5.1. Let $A|B$ be $G$-Galois, $A'$ a $G$-invariant subring of $A$, and $B'=A'^m$. Assume that there are elements $a_1, \cdots, a_n; a_1^*, \cdots, a_n^*$ and $c$ in $A'$ such that $\sum_i a_i^* \sigma(a_i^*) = \delta_{1, a}$, and $t_0(c) = 1$.

1. $A'|B'$ is a $G$-Galois extension, and $A'' = B \otimes_B A'^m = A'^{m} \otimes_B B$ for any subgroup $H$ of $G$, in particular, $A = B \otimes_B A' = A' \otimes_B B$.

2. Let $\{\overline{X}\}$ be the set of all $A'G$-left submodules of $A$, and $\{X\}$ the set of all $B'$-left submodules of $B$. Then, $\overline{X} \rightarrow \overline{X} \cap B$ and $X \rightarrow A'X = A' \otimes_B X$ are mutually converse order isomorphisms between $\{\overline{X}\}$ and $\{X\}$.

3. Let $\{\overline{Y}\}$ be the set of all $G$-invariant intermediate rings of $A|A'$, and $\{Y\}$ the set of all intermediate rings of $B|B'$ such that $A'Y = YA'$. Then, $\overline{Y}/(\overline{Y} \cap B)$ is $G$-Galois, and $\overline{Y} \rightarrow \overline{Y} \cap B$ and $Y \rightarrow A'Y = YA'$ are mutually converse order isomorphisms between $\{\overline{Y}\}$ and $\{Y\}$.

Proof. (1) Evidently, $G \cong G|A'$, and $G$ may be regarded as a finite group of automorphisms of $A'$. Hence, $A'|B'$ is $G$-Galois. Let $G = H_0 \cup \cdots \cup H_r$ be the right coset decomposition of $G$. If we put $d = \sum_i a_i \sigma(c)$ and $t = t_0(a_1)$, then $t_0(d) = 1$ and $\sum_i t_i \sigma(a_i^*) = \delta_{H, a} (a \in G)$. If $x$ is in $A'$, then $A'' \cdot B \supseteq \sum_i t_i \sigma(a_i^*) \sigma(dx) = t_0(dx) = t_0(d)x = x$. Thus, we obtain $A'' = A'' \cdot B$. To be easily seen, the mapping $\sum_j x_j \otimes y_j \rightarrow \sum_j x_j y_j$ from $A'' \otimes_B B$ to $A'' \cdot B = A''$ is well-defined and $\sum_i t_i \otimes t_0(a_i^* d \sum_j x_j y_j) = \sum_j x_j \otimes y_j$. Hence, $A'' \otimes_B B \cong A''$ by the mapping $\sum_j x_j \otimes y_j \rightarrow \sum_j x_j y_j$. Symmetrically, it follows $A'' = B \otimes_B A''$. (2) Let $X$ be an $A'G$-left submodule of $A$. Evidently, $\overline{X} \supseteq A'\overline{(X \cap B)}$, and $\overline{X} \cap B$ is a $B'$-left submodule of $B$. If $x$ is in $\overline{X}$, then $t_0(a_1^* x)$ is in $\overline{X} \cap B$, and hence $x = \sum_i a_i \otimes t_0(a_i^* x) \in A'\overline{(X \cap B)}$. Hence, $\overline{X} = A'(\overline{X} \cap B)$, and the mapping $\sum_j x_j \otimes y_j \rightarrow \sum_j x_j y_j$ from $A' \otimes_B (\overline{X} \cap B)$ to $A' \overline{(X \cap B) = \overline{X}}$ is onto. Moreover, to be easily seen, $\sum_i a_i \otimes t_0(a_i^* \sum_j x_j y_j) = \sum_j x_j \otimes y_j$. Hence, $\overline{X} = A' \otimes_B (\overline{X} \cap B)$. Now, let $X$ be a $B'$-left submodule of $B$. Then, $A'X$ is an $G$-left submodule of $A$, and $A'X \cap B \supseteq X$. If $\sum_j x_j y_j (x_j \in A', y_j \in X)$ is in $A'X \cap B$, then $\sum_j x_j y_j = t_0(c)(\sum_j x_j y_j) = \sigma(c) \sum_j \sigma(x_j) y_j = \sum_j t_0(c x_j) y_j \in X$. Hence, $A'X \cap B \subseteq X$, namely, $A'X \cap B = X$. (3) Evidently, $(\overline{Y}/\overline{Y} \cap B)$ is $G$-Galois. Hence $\overline{Y} = A'\overline{(Y \cap B)} = (\overline{Y} \cap B)A'$ by (1), and then our assertion is an easy consequence of (2).

Corollary. Let $A|B$ be $G$-Galois, and $B' = V_B(B)$. Assume that there are elements $a_i, a_i^*$ $(i = 1, \cdots, n)$ in $V_A(B)$ such that $\sum_i a_i \sigma(a_i^*) = \delta_{1, a}$.

1. $V_A(B)|B'$ is $G$-Galois, $A'' = B \otimes_B V_A(B)|B''$ for any subgroup $H$ of $G$, and the center of $A''$ coincides with the center of $V_A(B)|B''$. In particular, $A = B \otimes_B V_A(B)$, and $B' \subseteq C$.

2. Let $\{\overline{Y}\}$ be the set of all $G$-invariant intermediate rings of $A|V_A(B)$,
and \( \{ Y \} \) the set of all intermediate rings of \( B/B' \). Then \( Y \rightarrow Y \cap B \) and \( Y \rightarrow V_A(B)Y = V_A(B) \otimes_B Y \) are mutually converse order isomorphisms between \( \{ Y \} \) and \( \{ Y \} \).

(3) \( A/V_A(B) \) is separable if and only if \( B \) is a separable \( B' \)-algebra.

Proof. If remains to prove (3). If \( B/B' \) is separable, then \( A/B' \) is separable, because both \( A/B \) and \( B/B' \) are separable (Lemma 2.7). Hence \( A/V_A(B) \) is separable. Conversely, assume that \( A/V_A(B) \) is separable. Then, since both \( A/V_A(B) \) and \( V_A(B)/B' \) are separable, \( A/B' \) is separable, or equivalently, \( A \) is a separable \( B' \)-algebra (Lemma 2.7). Since \( A = B \otimes_B V_A(B) \), by [2; Prop. 1.7 and its Remark], \( B \) is a separable \( B' \)-algebra.

Remark. The above corollary contains Kanzaki [8; Th. 5].

Let \( A, A' \) be \( R \)-algebras over a commutative ring \( R \) such that \( A \otimes_R A' \neq 0 \). Assume that \( A/B \) is a \( G \)-Galois extension such that \( R \cdot 1 \subseteq B \) and \( B \) is a direct summand of \( A \), and assume that \( A'/B' \) is a \( G' \)-Galois extension such that \( R \cdot 1 \subseteq B' \) and \( B' \) is a direct summand of \( A' \). Let \( \{ a_i, a_i^* \}; i = 1, \ldots, n \) and \( \{ (d_j, d_j^*); j = 1, \ldots, m \} \) be a \( G \)-Galois coordinate system for \( A/B \) and a \( G' \)-Galois coordinate system for \( A'/B' \), respectively. For any \( \sigma \times \tau \) in \( G \times G' \), we can define \( \sigma \times \tau \cdot \sum_j x_j \otimes y_j = \sum_j \sigma(x_j) \otimes \tau(y_j) (x_j \in A, y_j \in A') \). Then, since \( \sum_i a_i \otimes d_i = (-1)^{\sigma} \sum_i a_i \otimes d_i \), \( (A \otimes A')/(A \otimes A')^{G \times G'} \) is a \( G \times G' \)-Galois extension.

Proof. By assumption, there is an element \( c \) in \( A \) such that \( t_{H \cap K}(c) = 1 \).
Evidently, $A^{H_{i}K} \supseteq A^{H_{i}}A^{K}$. Let $\{(a_{i}, a_{i}^{*}) ; i=1, \cdots, n\}$ be a $G$-Galois coordinate system for $A/B$. If $x$ is in $A^{H_{i}K}$, then $A^{H_{i}}A^{K} \ni \sum_{i} r_{i}(a_{i})t_{i}(a_{i}^{*}cx) = \sum_{i \in H} \sum_{j \in K} \sum_{i} r_{i}(a_{i}) \sigma(a_{i}^{*}) \sigma(cx) = t_{H_{i}K}(c)x = x$. Hence $A^{H_{i}K} = A^{H_{i}}A^{K}$. Symmetrically we have $A^{H_{i}K} = A^{H_{i}}A^{K}$.

**Corollary.** Let $A/B$ be a $G$-Galois extension. If $H$ and $K$ are subgroups of $G$ such that $H \cap K = \{1\}$, then $A = A^{H_{i}}A^{K} = A^{H_{i}}A^{K}$.

**Theorem 5.4.** Let $A/B$ be a $G$-Galois extension, and $B_{B}$ a direct summand of $A_{B}$. If $G = KH$ and $K \cap H = \{1\}$ for a normal subgroup $K$ and a subgroup $H$, then there hold the following:

1. $A = A^{K} \otimes_{B} A^{H} = A^{H} \otimes_{B} A^{K}$.
2. $A^{K}/B$ is an $H$-Galois extension.
3. For any subgroup $H_{0}$ of $H$ and any subgroup $K_{0}$ of $K$ such that $N(K_{0}) \supseteq H$ (where $N(K_{0})$ means the normalizer of $K_{0}$ in $G$), $A^{K_{0}H} = A^{K_{0}} \otimes_{B} A^{K_{0}H_{0}} = A^{K_{0}H_{0}} \otimes_{B} A^{K_{0}H}$ and $A^{K_{0}H} \otimes_{B} A^{K_{0}H_{0}} = A^{H_{0}} \otimes_{B} A^{K_{0}H_{0}}$ is an $H$-Galois extension.

**Proof.** Let $\{(a_{i}, a_{i}^{*}) ; i=1, \cdots, n\}$ be a $G$-Galois coordinate system for $A/B$. Since $B_{B}$ is a direct summand of $A_{B}$, there is an element $c$ in $A$ such that $t_{0}(c) = 1$. Put $t_{i} = t_{K}(a_{i})$, $t_{i}^{*} = t_{K}(a_{i}^{*})$, and $d = t_{K}(c)$. Then, $t_{0}(d) = 1$ and $\sum_{i} t_{i} \tau(t_{i}) = \delta_{1}$, for $\tau$ in $H$. $N(K_{0}) \supseteq H$ implies that $\tau(A^{K_{0}}) = A^{K_{0}}$ for all $\tau$ in $H$. Hence, by Th. 5.1, $A^{K_{0}H} = A^{K_{0}H_{0}} \otimes_{B} A^{K_{0}H}$ is an $H$-Galois extension. By Th. 5.1, $A^{H} = A^{K} \otimes_{B} A^{K_{0}H} = A^{K} \otimes_{B} A^{K_{0}H_{0}} \otimes_{B} A^{K_{0}H}$. Since $K_{0}H_{0} = K_{0}H \cap KH_{0}$, $A^{K_{0}H} = A^{K_{0}H_{0}} \otimes_{B} A^{K_{0}H_{0}} = A^{K_{0}H_{0}} \otimes_{B} A^{K_{0}H}$ (Prop. 5.3). Since $A^{H} \supseteq A^{K_{0}H}$ and $A^{K_{0}H}$ is an $A^{K_{0}H}$-right direct summand of $A$, and so of $A$, $A^{K_{0}H} = A^{K_{0}H} \otimes_{B} A^{K_{0}H_{0}}$. Similarly, we have $A^{K_{0}H} = A^{K_{0}H} \otimes_{B} A^{K_{0}H}$.

**Corollary.** Let $A/B$ be a $G$-Galois extension, $B_{B}$ a direct summand of $A_{B}$, and $G = N_{1} \times \cdots \times N_{r}$. If $H = N_{1} \times \cdots \times \check{N}_{i} \times \cdots \times N_{r} (i=1, \cdots, r)$, then $A^{H}/B$ is $N_{i}$-Galois, $A = A^{H_{i}} \otimes_{B} \cdots \otimes_{B} A^{H_{r}}$, and $A^{K_{1}, \cdots, K_{r}} = A^{H_{1}} \otimes_{B} \cdots \otimes_{B} A^{H_{r}}$ for each subgroup $K_{i}$ of $N_{i}$.

**Proposition 5.5.** Let $A/B$ be outer $G$-Galois. $B_{B}$ a direct summand of $A_{B}$, and $A$ directly indecomposable. Let $T$ and $T'$ be intermediate rings of $A/B$ such that $A = T \otimes_{B} T'$. If $H = \{\sigma \in G ; \delta|T = 1_{T}\}$ and $H' = \{\sigma \in G ; \sigma|T' = 1_{T'}\}$, then $T = A^{H}$ and $T' = A^{H'}$.

**Proof.** Since $T \otimes_{B} T' = A$, we have $T \otimes_{B} A^{H} \equiv T \otimes_{B} A_{A}$. Since $A/T'$ is a separable extension, $A$ is $(B, T')$-projective. Hence, by Th. 2.6, $T = A^{H}$. Symmetrically we have $T' = A^{H'}$.

Let $A/B$ be a $G$-Galois extension, $B_{B}$ a direct summand of $A_{B}$, and $A$ a $G$-invariant proper ideal of $A$. Let $\{(a_{i}, a_{i}^{*}) ; i=1, \cdots, n\}$ be a $G$-Galois coordinate system for $A/B$. For any $x$ in $A$ we denote $x + A$ ($x + A$) by $\bar{x}$. If we define $\sigma(\bar{x}) = \bar{\sigma(x)}$, then $\sum_{i} a_{i} \sigma(a_{i}^{*}) = \delta_{1}$, for $\sigma$ in $G$, and therefore
($A/\mathfrak{M}$)/($A/\mathfrak{M}$)$^g$ is a $G$-Galois extension. By assumption, for any subgroup $H$ of $G$ there is an element $c$ in $A$ such that $t^H(c)=1$. If $\bar{x}$ is in $(A/\mathfrak{M})^H$, then $\bar{x}=x\Sigma_{\in H} \tau(\bar{c})=\Sigma_{\in H} \tau(\bar{x}c)\in (A^H+\mathfrak{M})/\mathfrak{M}$. Thus, we prove the following:

**Theorem 5.6.** Let $A/B$ be a $G$-Galois extension, $B_B$ a direct summand of $A_B$, and $\mathfrak{M}$ a $G$-invariant proper ideal of $A$. Then $(A/\mathfrak{M})/(B+\mathfrak{M})/\mathfrak{M}$ is a $G$-Galois extension, and $(A/\mathfrak{M})^H=(A^H+\mathfrak{M})/\mathfrak{M}$ for any subgroup $H$ of $G$.

**Corollary.** Let $A/B$ be a $G$-Galois extension, and $B_B$ a direct summand of $A_B$. If $B$ contains a non-zero central idempotent $e$ of $A$, then $Ae/Be$ is a $G$-Galois extension, and $(Ae)^H=A^He$ for any subgroup $H$ of $G$.

**Proposition 5.7.** Let $A/B$ be a $G$-Galois extension. If $N$ is a normal subgroup of $G$ such that $A^N$ is an $A^N$-right direct summand of $A$, then $A^N/B$ is a $G/N$-Galois extension.

**Proof.** Let $\{(a_i, a_i^*); i=1, \ldots, n\}$ be a $G$-Galois coordinate system for $A/B$. By assumption, there is an element $c$ in $A$ such that $t^A(c)=1$. If we put $t_N(a_i)=t_i$ and $t_N(a_i^*c)=t_i^*$, then $t_i$ and $t_i^*$ are $A^N$, and $\Sigma_i t_i \sigma(t_i^*)=\delta_{N,s}$ for all $\sigma$ in $G$. Hence, $A^N/B$ is a $G/N$-Galois extension (Prop. 2.2).

Let $A/B$ be a $G$-Galois extension, and $m$ a natural number. Then, every $\sigma$ in $G$ induces a ring automorphism in the $m \times m$ complete matrix ring $(A)_m$. Accordingly, $G$ may be regarded as a finite group of automorphisms of $(A)_m$ such that $((A)_m)^G=(B)_m$. Let $E$ be the identity of $(A)_m$, and let $\{(a_i, a_i^*); i=1, \ldots, n\}$ be a $G$-Galois coordinate system for $A/B$. Then $\Sigma_i a_i E \sigma(a_i^* E) = \delta_{1,s}$ for all $\sigma$ in $G$. Thus $(A)_m/(B)_m$ is a $G$-Galois extension. (Remark. This may be considered as a special case of Th. 5.2).

**Theorem 5.8.** Let $A/B$ be a $G$-Galois extension, and $\{e_{ij}; i, j=1, \ldots, m\}$ a system of matrix units contained in $B$. If $A_0=V_A(\{e_{ij}\})$, then $A_0/A_0^g$ is a $G$-Galois extension, and $B=\Sigma i \oplus A_0^g e_{ij}$.

**Proof.** Obviously, $G$ induces an automorphism group of $A_0$ and $B=\Sigma i \oplus A_0^g e_{ij}$. Let $\{(A_i, A_i^*); i=1, \ldots, n\}$ be a $G$-Galois coordinate system for $A/B$. Let $A_i=\Sigma_{j,k} a_{ijk}^g e_{jk}$, $A_i^*=\Sigma_{j,k} d_{ijk} e_{jk}$ ($a_{ijk}, d_{ijk} \in A_0$). Then, $\sigma(A_i^*)=\Sigma_{j,k} \sigma(d_{ijk}) e_{jk}$ and therefore $\Sigma_i a_{ik} \sigma(d_{ik})=\delta_{1,s}$ for $\sigma$ in $G$. Thus $A_0/A_0^g$ is a $G$-Galois extension.

§ 6. Completely outer case.

Let $R$ be a ring. If non-zero $R$-left modules $M$ and $N$ have no non-zero isomorphic subquotients, we say that $R M$ and $R N$ are unrelated.

**Proposition 6.1.** Let $M$ be a non-zero $R$-left module, and $M=M_1 \oplus \cdots \oplus M_s$ with non-zero $R$-submodules $M_i$'s of $M$.

1. If $M_i$'s are unrelated to each other, then each $M_i$ is $\text{End}(R M)$-
admissible and $X=\sum_{i}(X \cap M_{i})$ for every submodule $X$ of $_{\kappa}M$.

(2) If $X=\sum_{i}(X \cap M_{i})$ for every submodule $X$ of $_{\kappa}M$, then $M_{i}$'s are unrelated to each other.

Proof. (1) will be rather familiar. We shall prove here (2). To our end, it suffices to prove that if $M=M_{1}\oplus M_{2}$ and $X=(X \cap M_{1})+(X \cap M_{2})$ for every submodule $X$ of $_{\kappa}M$ then $M_{1}$ and $M_{2}$ are unrelated. Let $M_{i}/N_{i}$ and $M_{j}/N_{j}$ be non-zero subquotients of $M_{i}$ and $M_{j}$, respectively. If there exists an $R$-isomorphism $\alpha$; $M_{i}/N_{i} \cong M_{j}/N_{j}$, we can define an $R$-homomorphism $\varphi$; $M_{i}\oplus M_{j} \rightarrow M_{j}/N_{j}$ by the following rule: $(m_{i}+m_{j})\varphi=(m_{i}+N_{i})\alpha+(m_{j}+N_{j})$. Then, our assumption yields $\text{Ker}\ \varphi=(M_{i}\cap \text{Ker}\ \varphi)+(M_{j}\cap \text{Ker}\ \varphi)$, and so $(M_{i}+M_{j})\varphi=M_{i}\varphi \oplus M_{j}\varphi=M_{i}/N_{i}\oplus M_{j}/N_{j}$, which is a contradiction.

$G$ is said to be completely outer, if each $A$-$A$-modules $Au_{\sigma}$, $Au_{\tau}$ ($\sigma \neq \tau$) are unrelated.

To be easily seen, $Au_{\sigma}$ and $Au_{\tau}$ ($\sigma, \tau \in G$) are $A$-$A$-isomorphic if and only if $\sigma^{-1}$ is an inner automorphism of $A$, and every $A$-$A$-submodule of $Au_{\sigma}$ is written as $\mathfrak{H}u_{\sigma}$, with some ideal $\mathfrak{H}$ of $A$. Therefore, if $G$ is completely outer, then $G$ contains no inner automorphism of $A$, and in case $A$ is two-sided simple, the converse is true. Now, for $\sigma$ in $G$ we set $J_{\sigma}=\{a \in A; \sigma(x)\alpha=ax \ \text{for all} \ \ x \in A\}$. Then each $J_{\sigma}$ is a $C$-submodule of $A$, and $J_{1}=C$.

In his paper [9], T. Kanzaki proved the following: Let $A/B$ be a $G$-Galois extension. Then $V_{A}(B)=\sum_{\sigma}J_{\sigma}$. Therefore, if $A/B$ is $G$-Galois, then $V_{A}(B)=C$ if and only if $J_{\sigma}=0$ for all $\sigma$ in $G$ such that $\sigma \neq 1$.

**Proposition 6.2.** $J_{\sigma}=0$ if and only if $\text{Hom}(\mathfrak{A}u_{\sigma}, \mathfrak{A}u_{A})=0$.

**Proof.** Assume $J_{\sigma}=0$. If $f$ is in $\text{Hom}(\mathfrak{A}u_{\sigma}, \mathfrak{A}u_{A})$, then $\sigma(x)f(u_{\sigma})=f(xu_{\sigma})=f(ux)=f(u_{\sigma})x$ for $x$ in $A$. Hence $f(u_{\sigma})=0$, and so $f=0$. Conversely, assume that $\text{Hom}(\mathfrak{A}u_{\sigma}, \mathfrak{A}u_{A})=0$. If $a$ is in $J_{\sigma}$, then we can easily see that the mapping $xu_{\sigma}\rightarrow xa$ ($x \in A$) is an $A$-$A$-homomorphism from $Au_{\sigma}$ to $A$. Hence, by assumption, $a=0$.

Prop. 6.2 together with Kanzaki's result cited above yields at once the following:

**Proposition 6.3.** If $A/B$ is a $G$-Galois extension, then the following are equivalent. (i) $V_{A}(B)=C$. (ii) $\text{Hom}(\mathfrak{A}u_{\sigma}, \mathfrak{A}u_{A})=0$ for every $\sigma \neq 1$ in $G$.

The following proposition will play a fundamental role in our study.

**Proposition 6.4.** If $G$ is completely outer, then $A/B$ is a $G$-Galois extension and $V_{A}(B)=C$.

**Proof.** At first, $V_{A}(B)=C$ is evident by Prop. 6.3. Since $u_{1} \in A(\sum_{\sigma}u_{\sigma})A$ (Prop. 6.1.), there are elements $a_{i}, a_{i}^{*}$ ($i=1, \cdots, n$) in $A$ such that $u_{1}=$
\[ \sum_{t}a_{t}(\sum_{u}u_{t})a_{t}^{*} = \sum_{\sigma}(\sum_{t}a_{t}\sigma(a_{t}^{*}))u_{\tau}. \] Hence \( \sum_{t}a_{t}\sigma(a_{t}^{*}) = \delta_{1,\tau} \) for \( \sigma \) in \( G \).

**Corollary.** If \( A \) is two-sided simple, then the following conditions are equivalent: (i) \( G \) is completely outer. (ii) \( G \) contains no inner automorphisms. (iii) \( A/B \) is an outer \( G \)-Galois extension.

**Proposition 6.5.** If there are elements \( a_{i}, a_{i}' \) \( (i=1, \cdots, n) \) in \( A \) such that \( \sum_{t}a_{t}x_{t}\sigma(a_{t}) = \delta_{1,\tau}x \) for each \( x \) in \( A \) \((\sigma \in G)\), then \( G \) is completely outer.

*Proof.* Let \( X \) be any \( A \)-submodule of \( A \). If \( \sum_{\tau}x_{\tau}u_{\tau} \in X \), then \( X \ni \sum_{\tau}a_{t}(\sum_{\tau}x_{\tau}u_{\tau})\tau^{-1}(a_{t}) = x_{\tau}u_{\tau} \) for each \( \tau \) in \( G \). Hence, by Prop. 6.1, \( G \) is completely outer.

Combining Prop. 6.4 with Prop. 6.5, we readily obtain the following:

**Theorem 6.6.** Let \( A \) be a commutative ring. If \( A/B \) is \( G \)-Galois, then \( G \) is completely outer, and conversely.

**Proposition 6.7.** Let \( A/B \) be a \( G \)-Galois extension, \( H \) a subgroup of \( G \), and \( a \) an element of \( A \). If \( \sigma_{0} \in G \) is not contained in \( H \), and \( ax = a\cdot\sigma_{0}(x) \) for all \( x \) in \( A^{H} \), then \( a=0 \).

*Proof.* There are elements \( t_{1}, \cdots, t_{n} \in A^{H} \) and \( a_{1}^{*}, \cdots, a_{n}^{*} \in A \) such that \( \sum_{t}t_{i}\cdot\sigma(a_{i}^{*}) = \delta_{1,\sigma} \) for any \( \sigma \) in \( G \) (Prop. 2.2). Hence, \( a = a\sum_{t}t_{i}a_{i}^{*} = \sum_{\ell}a\cdot\sigma_{0}(t_{\ell})a_{i}^{*} = \sigma_{0}(a^{-1}(a))\sum_{t}t_{i}\sigma(a_{i}^{*}) = 0 \).

**Lemma 6.8.** Let \( S \) be a subring of a ring \( R \). If \( R_{S} \) is finitely generated and projective, then \( \text{End}(R_{S}) \) is an \( \text{End}(R) \)-left direct summand of \( \text{End}(R) \), where \( \text{End}(R_{S}) \) and \( \text{End}(R) \) act on the left side.

*Proof.* As is well known, there are elements \( a_{i} \in R, f_{i} \in \text{Hom}(R_{S}, S_{S}) \) \( (i=1, \cdots, n) \) such that \( \sum_{i}a_{i}f_{i}(x) = x \) for every \( x \) in \( R \) (cf. [3]). If \( g \) is in \( \text{End}(R) \), then \( \sum_{i}g(a_{i})f_{i} \) is in \( \text{End}(R_{S}) \), and so the mapping \( g \rightarrow \sum_{i}g(a_{i})f_{i} \) is an \( \text{End}(R) \)-left homomorphism from \( \text{End}(R) \) to \( \text{End}(R_{S}) \). To be easily seen, if \( g \) is in \( \text{End}(R_{S}) \) then \( \sum_{i}g(a_{i})f_{i} = g \). This implies that \( \text{End}(R_{S}) \) is an \( \text{End}(R) \)-left direct summand of \( \text{End}(R) \).

Let \( T \) be an intermediate ring of \( A/B \). \( G/T \) is said to be \( \ast \)-strongly distinct if, for any non-zero idempotent \( e \) in \( A \) such that \( eA \subseteq Ae \) and any distinct \( \sigma, \tau \) in \( G \), there is at least an element \( x \) in \( T \) such that \( e\cdot\sigma(x) \neq e\cdot\tau(x) \).

If \( A/B \) is a \( G \)-Galois extension, then \( G/A^{H} \) is \( \ast \)-strongly distinct for any subgroup \( H \) of \( G \) (Prop. 6.7).

**Theorem 6.9.** Let \( G \) be completely outer, \( B_{n} \) a direct summand of \( A_{n} \), and \( T \) an intermediate ring of \( A/B \). Then the following conditions are equivalent.

(i) \( T = A^{H} \) for some subgroup \( H \) of \( G \).

(ii) \( A_{T} \) is finitely generated and projective, and \( T_{\sigma} \) is a direct summand
of $A_T$, and $G | T$ is* strongly distinct.

Proof. Since $A | A^\pi$ is $H$-Galois, it remains to prove $(ii) \Rightarrow (i)$. If we put $A_0 = \text{End}(A_T)$, then $A_0$ is a subring of $A$. Since $A_0$ is an $A$-$A$-submodule of $A$, $A_0 = \sum \oplus \mathfrak{U}_s u_s$ with some ideals $\mathfrak{U}_s$ of $A$. By Lemma 6.8, $A_0$ is a direct summand of $A$, so that each $\mathfrak{U}_s u_s$ is a direct summand of $A$. Therefore each $\mathfrak{U}_s u_s$ is a direct summand of $A_A u_s$. Hence $\mathfrak{U}_s$ is a direct summand of $A$. Let $\mathfrak{U}_s = A e_s$ with an idempotent $e_s$ in $A$. Then, since $e_s u_s$ is in $A_0$, $e_s \cdot \sigma(x y) = e_s \cdot \sigma(x) y$ for each $x$ in $A$ and $y$ in $T$; in particular, $e_s \cdot \sigma(y) = e_s y$ for each $y$ in $T$. Therefore, if we set $H = \{ \sigma \in G ; \sigma | T = 1_T \}$, then $e_s = 0$ for $\sigma$ not contained in $H$. Evidently $\mathfrak{U}_s = A$ for each $\sigma$ in $H$. We obtain therefore $A_0 = \sum e_s u_s$, and hence $\text{End}_{(A, A)}(A^\pi)$. On the other hand, since $T_T$ is a direct summand of $A_T$, $\text{End}_{(A, A)}(A^\pi) = T_T$ (cf. [1]). Hence we obtain $T = A^\pi$.

Now, if $A$ is a semi-prime ring (i.e., $A$ has no nilpotent ideals) and $e$ is an idempotent in $A$ such that $e A \subseteq A e$, then $e A = A e$ so that $e$ is a central idempotent in $A$. Noting this fact, Th. 6.9 yields at once the following:

**Theorem 6.10.** Let $A$ be a semi-prime ring. If $G$ is completely outer, $B_b$ a direct summand of $A_B$, and $T$ an intermediate ring of $A | B$, then the following conditions are equivalent:

(i) $T = A^\pi$ for some subgroup $H$ of $G$.

(ii) $A_T$ is finitely generated and projective, and $T_T$ is a direct summand of $A_T$, $G | T$ is strongly distinct.

**Proposition 6.11.** The following are equivalent:

(i) $G$ is completely outer.

(ii) For any $x, y$ in $A$ and any $\sigma$ in $G$ such that $\sigma \neq 1$, there are elements $a_i, a'_i \ (i = 1, \cdots, n)$ in $A$ such that $\sum_i a_i x a'_i = x$ and $\sum_i a_i y \cdot \sigma(a'_i) = 0$.

Proof. (i) $\Rightarrow$ (ii) Let $x, y$ be in $A$, and $\sigma$ any element of $G$ such that $\sigma \neq 1$. We set $X = A(x u_1 + y u_1)A$, which is an $A$-$A$-submodule of $A u_1 + A u_1$. By Prop. 6.1, $x u_1 \in X$, and hence there are elements $a_i, a'_i \ (i = 1, \cdots, n)$ in $A$ such that $\sum_i a_i(x u_1 + y u_1) a'_i = x u_1$. Then, $\sum_i a_i x a'_i = x$ and $\sum_i a_i y \cdot \sigma(a'_i) = 0$.

(ii) $\Rightarrow$ (i) Let $\sigma, \tau$ be distinct elements in $G$, and $X$ any $A$-$A$-submodule of $A u_1 + A u_1$. Let $x u_1 + y u_1$ be any element of $X$. For $\sigma^{-1}(x)$ and $\sigma^{-1}(y)$, there are elements $a_i, a'_i \ (i = 1, \cdots, n)$ in $A$ such that $\sum_i a_i \cdot \sigma^{-1}(x) a'_i = \sigma^{-1}(x)$ and $\sum_i a_i \cdot \sigma^{-1}(y) \cdot \sigma^{-1}(a'_i) = 0$. Then, $\sum_i \sigma(a_i) x \cdot \sigma(a'_i) = x$ and $\sum_i \sigma(a_i) y \cdot \tau(a'_i) = 0$, and so $X \ni \sum_i \sigma(a_i)(x u_1 + y u_1) a'_i = x u_1$. Thus, by Prop. 6.1, $A u_1$ and $A u_1$ are unrelated.

**Theorem 6.12.** Let $G$ be completely outer, and $N$ a proper normal subgroup of $G$ such that $A^\pi$ is an $A^\pi$-right direct summand of $A$. Then,
G/N is completely outer as an automorphism group of A^N.

Proof. Let x, y be in A^N. Since xu_1 \in A(\sum_{\tau \in N}xu_\tau + \sum_{\tau \in G \backslash N}yu_\tau)A (Prop. 6.1), there are elements x_i, y_i (i = 1, \cdots, n) in A such that \sum_i x_i(\sum_{\tau \in N}xu_\tau + \sum_{\tau \in G \backslash N}yu_\tau)y_i = xu_1. Then \sum_i x_i x \cdot \tau(y_i) = \delta_{i,1}x (\tau \in N) and \sum_i x_i y \cdot \sigma(y_i) = 0 \sigma \in G \backslash N). By assumption, there is an element c in A such that t_N(c) = 1. We set t_N(x_i) = x'_i and t_N(y_i) = y'_i, then x'_i, y'_i (i = 1, \cdots, n) are in A^N. To be easily seen, \sum_i x'_i x'_i = x and \sum_i x'_i y'_i = 0 for any \rho \in G \backslash N. Thus, by Prop. 6.11, G/N is completely outer as an automorphism group of A^N.

§ 7. Several results.

The following lemma is well known.

Lemma 7.1. Let S be a subring of a ring R. If S is a direct summand of R, then R \cap S = 1 for any left ideal I of S.

Lemma 7.2. Let S be a subring of a ring R such that S is a direct summand of R and sR is finitely generated. If R satisfies the minimal condition (resp. the maximal condition) for left ideals, then so does S, and conversely.

Proof. If R satisfies the minimal condition (resp. the maximal condition) for left ideals, then so does S (Lemma 7.1). Conversely, if S satisfies the minimal condition (resp. the maximal condition) for left ideals then sR satisfies the minimal condition (resp. the maximal condition) for S-left submodules, so that R satisfies the minimal condition (resp. the maximal condition) for left ideals.

A ring R is called a semi-primary ring if R/\Re(R) satisfies the minimal condition for left ideals, where \Re(R) means the Jacobson radical of R. If R is semi-primary, then (R)_n and eRe are semi-primary rings, where n is a natural number and e is a non-zero idempotent in R (cf. [7]). Therefore, in case R is semi-primary, if an R-right module M is finitely generated and projective then \End(M_R) is semi-primary. As to notations and terminologies used in below, we follows [11].

Proposition 7.3. (1) Let R be a semi-primary ring, and S a subring of R. If S is a direct summand of R, then S is a semi-primary ring.

(2) Let R be a ring, and S a subring of R such that R is finitely generated and projective. If S is semi-primary, then so is R.

Proof. (1) Let \{I_i; i = 1, \cdots, n\} be a d-independent set of maximal left ideals of S (cf. [11]). Then, \{RI_i; i = 1, \cdots, n\} is a d-independent set of proper left ideals of R (Lemma 7.1). Since each RI_i is contained in a maximal left ideals of R, \leq \max \dim R = \max \dim sR (cf. [11]). Thus d-dim sS \leq d-dim sR < \aleph_0, and hence S is semi-primary ([11; Prop. 5.14]. (2) Since S
is semi-primary, \( \text{End}(R_S) \) is semi-primary. By Lemma 6.8, \( R'_i R_l \) (the set of all left multiplications by elements of \( R \)) is a direct summand of \( R'_i \text{End}(R_S) \). Hence, by (1), \( R \cong R_i \) is semi-primary.

**Remark.** Let \( A/B \) be a \( G \)-Galois extension, and \( B_B \) a direct summand of \( A_B \). If \( A \) is a semi-primary ring, then so is \( B \), and conversely (cf. Th. 1.7).

Let \( R \) be a ring, and \( S \) a subring of \( R \). \( R/S \) is called a free Frobenius extension if \( R_S \) is finitely generated and free and \( R^R_S \cong S_{S} \text{Hom}(R_S, S_R) \) (Kasch [10]).

**Lemma 7.4.** Let \( R/S \) be a free Frobenius extension.

1. \( \text{End}(R_S)/R_i \) is a free Frobenius extension.
2. If \( R_R \) is injective, then so is \( S_S \), and conversely.

**Proof.** (1) and the if part of (2) are given in [10]. Assume that \( R_R \) is injective. By (1) and the if part, we can easily see that \( \text{End}(R_S) \) is \( \text{End}(R_S) \)-right injective. Let \( R_S \cong R_S^S \). Then, \( \text{End}(R_S) \cong (S)_m \), and hence we readily see that \( S_S \) is injective (cf. [11]).

**Proposition 7.5.** Let \( R \) be a ring, and \( S \) a subbing of \( R \). If \( S_S \) is a direct summand of \( R_S \), then \( \Re(R) \cap S \subseteq \Re(S) \).

**Proof.** If \( \Re(R) \cap S \not\subseteq \Re(S) \), then \( \Re(R) \cap S + I = S \) for some maximal left ideal \( I \) of \( S \). Since \( R(\Re(R) \cap S) + RI = R \) and \( R(\Re(R) \cap S) \subseteq \Re(R) \), we have \( RI = R \). If follows then a contradiction \( I = R \cap S = S \) (Lemma 7.1).

**Proposition 7.6.** The set of all maximal \( A \)-submodules of \( A \) coincides with \( \{ \cap x(\mathfrak{P}) ; \mathfrak{P} \text{ ranges over all maximal ideals of } A \} \).

**Proof.** Let \( X \) be a maximal \( A \)-submodule of \( A \). Take a maximal ideal \( \mathfrak{P}_1 \) such that \( \mathfrak{P}_1 \supseteq X \). Then, \( \cap x(\mathfrak{P}_1) \supseteq X \), and so \( \cap x(\mathfrak{P}_1) = X \). Now, let \( \mathfrak{P} \) be a maximal ideal of \( A \), and \( Y \) a maximal \( A \)-submodule of \( A \) such that \( Y \supseteq \cap x(\mathfrak{P}) \). Then \( Y = \cap x(\mathfrak{P}_2) \) for some maximal ideal \( \mathfrak{P}_2 \) of \( A \). If \( \cap x(\mathfrak{P}_2) \supseteq \cap x(\mathfrak{P}_1) \), then \( \mathfrak{P} \not\subseteq \cap x(\mathfrak{P}_2) \), and so \( \mathfrak{P} + \cap x(\mathfrak{P}_2) = A \), whence it follows a contradiction \( \cap x(\mathfrak{P}) + \cap x(\mathfrak{P}_2) = A \).

**Proposition 7.7.** Let \( A/B \) be a \( G \)-Galois extension, and \( B_B \) a direct summand of \( A_B \). Let \( \{ X \} \) be the set of all \( A \)-submodules of \( A \) and \( \{ X \} \) be the set of all left ideals of \( B \). Then \( X \to X \cap B \) and \( X \to AX = A \otimes_B X \) are mutually converse order isomorphisms between \( \{ X \} \) and \( \{ X \} \).

**Proof.** This is a special case of Th. 5.1 (2).

**Proposition 7.8.** Let \( A/B \) be a \( G \)-Galois extension, and \( B_B \) a direct summand of \( A_B \). If \( A \cdot \Re(B) \) is an ideal of \( A \), then \( \Re(A) = A \cdot \Re(B) \).

**Proof.** By Prop. 7.7 and Prop. 7.5, \( \Re(A) = A(\Re(A) \cap B) \subseteq A \cdot \Re(B) \).
Since $A_{\mathcal{B}}$ is finitely generated, $A \cdot \Re(B)$ is d-dense in $A_{\mathcal{B}}$, and so d-dense in $A_{A}$ (cf. [11]). Hence $A \cdot \Re(B) \subseteq \Re(A)$.

**Theorem 7.9.** Let $A/B$ be a G-Galois extension such that $B \subseteq C$. If $A'$ is a $B$-algebra, then $\Re(A' \otimes_{B} A) = \Re(A') \otimes A$.

**Proof.** By Cor. to Th. 5.2, $(A' \otimes_{B} A)/(A' \otimes 1)$ is a $G$-Galois extension. Since $(A' \otimes A) (\Re(A') \otimes 1) = \Re(A') \otimes A$ is an ideal of $A' \otimes A$, $\Re(A' \otimes A) = \Re(A') \otimes A$ by Prop. 7.8.

Now, assume that $G$ is completely outer and $B_{\mathcal{B}}$ is a direct summand of $A_{\mathcal{B}}$. If $A$ is an $A'$-$A$-submodule (resp. $A'$-$A$-submodule) of $\Delta$, then $A = \sum u_{\mathcal{A}} \mathcal{A}$, for some ideals $\mathcal{A}$ of $A$ (resp. $A = \sum u_{\mathcal{A}} \mathcal{A}$ for some ideal $\mathcal{A}$ of $A$), and conversely. In particular, if $A$ is an ideal of $\Delta$, then $A = \Delta \mathcal{A} = \mathcal{A} \Delta$ for some $G$-invariant ideal $\mathcal{A}$ of $A$, and conversely (cf. §6 and [13]). Now, let $\{\Delta\}$ be the set of all ideals of $\Delta$, $\{a\}$ the set of all ideals of $B$, and $\{\mathcal{A}\}$ the set of all $G$-invariant ideals of $A$. Then, there exists an order isomorphism $\Delta \rightarrow a$ between $\{\Delta\}$ and $\{a\}$ such that $\Delta(A) = aA$ (cf. [1; Prop. A. 5]). Consequently, there exists an order isomorphism $\mathcal{A} \rightarrow \Delta$ between $\{\mathcal{A}\}$ and $\{\Delta\}$ (cf. Th. 5.1 (2)). Accordingly, if $A$ is semi-prime, (prime, two-sided simple) then so is $B$. Since $A \cdot \Re(B) = \Re(B) A$ is an ideal of $A$, Prop. 7.8 implies $\Re(A) = A \cdot \Re(B) = \Re(B) A$. Next, we shall consider $\Re(\Delta)$. There exists $\mathcal{A}' \in \{\mathcal{A}\}$ such that $\Re(\Delta) = \mathcal{A}' \Delta = \Delta \mathcal{A}'$. Since $\mathcal{A}' u_{\mathcal{A}} = \Re(\Delta) \cap A u_{\mathcal{A}} \subseteq \Re(A) u_{\mathcal{A}} = \Re(A) u_{\mathcal{A}}$ by Prop. 7.5, we obtain $\Re(\Delta) = \sum \mathcal{A}' \Re(A) = \Re(A) \Delta$. On the other hand, noting that $\Delta_{\mathcal{A}}$ is finitely generated and $A \cdot \Re(A)$ is an ideal of $\Delta$, we see that $\Delta \cdot A \Re(A) \subseteq \Re(\Delta)$ (cf. the proof of Prop. 7.8). Hence, we have $\Re(\Delta) = \Delta \Re(A) = \Re(A) \Delta$. Since $\Re(\Delta A_{\mathcal{A}}) = \Re(\Delta A_{\mathcal{A}}) = (\Re(\Delta A_{\mathcal{A}}) \Delta) (A) = \Re(\Delta) (A) = A \cdot \Re(B_{\mathcal{B}})$ by Prop. 7.6, we have $\Re(\Delta A_{\mathcal{A}}) = A \cdot \Re(B_{\mathcal{B}}) = \Re(B_{\mathcal{B}}) A$ and $\Re(A A_{\mathcal{A}}) \cap B = \Re(B_{\mathcal{B}})$. Summarizing the above, we state the following theorem.

**Theorem 7.10.** If $G$ is completely outer and $B_{\mathcal{B}}$ a direct summand of $A_{\mathcal{B}}$, then $\Re(A) = A \cdot \Re(B) = \Re(B) A$, $\Re(A) \cap B = \Re(B)$, $\Re(\Delta A_{\mathcal{A}}) = \Re(B_{\mathcal{B}}) A$, $\Re(\Delta A_{\mathcal{A}}) \cap B = \Re(B_{\mathcal{B}})$, $\Re(\Delta) = \Delta \cdot \Re(A) = \Re(A) \Delta$, and $\Re(\Delta A_{\mathcal{A}}) = \Delta \cdot \Re(A_{\mathcal{A}})$ = $\Re(\Delta A_{\mathcal{A}}) \Delta$.

**Proposition 7.11.** Let $B$ be directly indecomposable, and let $A = \mathcal{A}_{1} \oplus \cdots \oplus \mathcal{A}_{n}$ be a direct sum of minimal ideals. If $\mathcal{A}$ is a minimal ideal of $A$, then $\{\mathcal{A} \sigma(\mathcal{A}); \sigma \in G\} = \{\mathcal{A}_{1}, \cdots, \mathcal{A}_{n}\}$, and $n$ divides $(G : 1)$. If $\mathcal{B}$ is a maximal ideal of $A$, $\{\mathcal{B} \sigma(\mathcal{B}); \sigma \in G\}$ coincides with the set of all maximal ideals of $A$. For any $\mathcal{A}_{\mathcal{L}}$, we set $\sum \mathcal{A}_{\mathcal{L}} = \mathcal{B}$. Then, $\mathcal{B} = A \mathcal{L}$ with some non-zero
central idempotent $e$ of $A$. Since $\sigma(\mathfrak{B})=\mathfrak{B}$ for all $\sigma$ in $G$, $\sigma(e)=e$ for all $\sigma$ in $G$, so that $e \in B$, which means $e=1$. Hence $\mathfrak{B}=A$, which implies that 
$
\{\sigma(\mathfrak{U}_i); \sigma \in G\} = \{\mathfrak{U}_1, \ldots, \mathfrak{U}_n\}.
$
If we set $H = \{\sigma \in G; \sigma(\mathfrak{U}_i) = \mathfrak{U}_i\}$, then $\#\{\sigma(\mathfrak{U}_i); \sigma \in G\} = (G: H)$, which divides $(G: 1)$. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be maximal ideals of $A$. Then $A = \mathfrak{A} \oplus \mathfrak{P} = \mathfrak{A}' \oplus \mathfrak{P}'$ with some minimal ideals $\mathfrak{A}, \mathfrak{A}'$ of $A$. There is an element $\sigma$ in $G$ such that $\sigma(\mathfrak{A}) = \mathfrak{A}'$. Then $A = \mathfrak{A}' \oplus \sigma(\mathfrak{P}) = \mathfrak{A}' \oplus \mathfrak{P}'$, so that $\sigma(\mathfrak{P}) = \mathfrak{P}'$.

**Corollary 1.** Let $G$ be completely outer, and $B_\mathfrak{U}$ a direct summand of $A_\mathfrak{U}$. If $B$ is a two-sided simple rings, then $A$ is a direct sum of isomorphic two-sided simple rings, and the number of components divides $(G: 1)$.

**Proof.** Let $\mathfrak{P}$ be a maximal ideal of $A$. Then $\cap_\sigma \sigma(\mathfrak{P})$ is a $\Delta$-$A$-submodule of $\mathfrak{A}$. As we remarked above, $A$ is $\Delta$-$A$-simple, and so we have $\cap_\sigma \sigma(\mathfrak{P}) = 0$. Hence $A$ is a direct sum of two-sided simple rings.

**Corollary 2.** Let $A/B$ be a $G$-Galois extension, and $B$ a division ring. Then $A$ is a direct sum of isomorphic (Artinian) simple rings.

**Proof.** Let $\mathfrak{P}$ be a maximal left ideal of $A$. Then $\cap_\sigma \sigma(\mathfrak{P})$ is a $\Delta$-submodule of $A$. Since $\Delta A$ is simple (Prop. 7.7), $\cap_\sigma \sigma(\mathfrak{P}) = 0$. Hence, as is easily seen, $\Delta A$ is completely reducible, so that $A$ is a direct sum of simple rings.

Let $A/B$ be a $G$-Galois extension, $A$ a commutative ring, and $A'$ a $B$-algebra. Then, by Prop. 6.5 and Th. 5.2, $(A' \otimes_B A)/(A' \otimes 1)$ is $G$-Galois and $G$ is completely outer (as an automorphism group of $A' \otimes A$). Further, if $A'$ is two-sided simple, then $A' \otimes_B A$ is a direct sum of isomorphic two-sided simple rings (Cor. 1. to Prop. 7.11). Thus we have the following:

**Theorem 7.12.** Let $A/B$ be a $G$-Galois extension, $A$ commutative, and $A'$ a $B$-algebra. If $A'$ is two-sided simple, then $A' \otimes_B A$ is a direct sum of isomorphic two-sided simple rings, and the number of components divides $(G: 1)$.

References

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