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ON DIMENSIONS OF SIMPLE RING EXTENSIONS

By

Takasi NAGAHARA and Hisao TOMINAGA

Let $A$ be a division ring, and $B$ a division subring of $A$. If $T$ is an intermediate ring of $A/V_{A}^{2}(B)$ then $[T:V_{A}^{2}(B)]=\left[V_{A}(B):V_{A}(T)\right]_{r}$, provided we do not distinguish between two infinite dimensions ([8, Lemma 2]). If $A/B$ is left locally finite then so is $A/V_{A}^{2}(B)$ ([8, Theorem 1]). Moreover, if $A/B$ is Galois and $A/V_{A}^{2}(B)$ is left locally finite then for any intermediate ring $B'$ of $A/B$ left finite over $B$ there holds $[B':B]_{r}=[B':B]$ ([7, Corollary 2]).

The purpose of the present paper is to extend those results stated above to simple ring extensions. As one will see later, our extension of [7, Corollary 2] is especially satisfactory (Theorem 3).

Throughout the present paper, we use the following conventions: $A$ will represent a ring with 1, $B$ a unital subring of $A$ (i.e. a subring of $A$ containing 1), $V$ the centralizer $V_{A}(B)$ of $B$ in $A$, and $H$ the double centralizer $V_{A}^{2}(B)=V_{A}(V)$ of $B$ in $A$. Moreover, $\mathfrak{A}$ and $\mathfrak{B}$ will denote the absolute endomorphism ring $\text{Hom}(A,A)$ of $A$ and the group of all $B$-ring automorphisms of $A$, respectively. If $X$ is a subset of $A$, then $X_{r}$, $X$, and $\overline{X}$ will mean the sets of all left multiplications effected by elements of $X$, of all the right multiplications effected by elements of $X$ and of all inner automorphisms effected by regular elements of $A$ contained in $X$, respectively.

The following easy lemma will play an essential role in our subsequent consideration.

**Lemma 1.** Let $A$ be a right Artinian ring, and $\mathfrak{B}$ a subring of $\mathfrak{A}$ such that $A$ is $V_{\mathfrak{A}}(\mathfrak{B})$-$\mathfrak{B}$-irreducible. If $\mathfrak{B}A_{r}=\mathfrak{B}$ and $A$ is $\mathfrak{B}$-unital (i.e. $x\mathfrak{B}\neq 0$ for every non-zero $x \in A$), then $V_{\mathfrak{A}}(\mathfrak{B})$ is an (Artinian) simple subring of $A_{r}$ and $\overline{V_{\mathfrak{A}}(\mathfrak{B})}$ is the closure of $\mathfrak{B}$ (in the finite topology).

**Proof.** Noting that every $x\mathfrak{B} \ (x \in A)$ is a right $A$-submodule of $A$ and $A$ is $\mathfrak{B}$-unital, one will easily see that $A$ contains a minimal $\mathfrak{B}$-submodule $M$. Since $A=MV_{\mathfrak{A}}(\mathfrak{B})=\sum_{\alpha \in r_{\mathfrak{A}}(\mathfrak{B})} M\alpha$ and each $M\alpha$ is either $\mathfrak{B}$-isomorphic to $M$ or 0, $A$ is homogeneously $\mathfrak{B}$-completely reducible. Hence, $V_{\mathfrak{A}}(\mathfrak{B})$ is simple, and $\mathfrak{B}$ is dense in $V_{\mathfrak{A}}(\mathfrak{B}) (\exists \ 1)$ by [2, Theorem VI. 2.2]. Now, the rest of our assertion will be easily seen.

The converse of Lemma 1 will be rather familiar: Let $B$ be a direct
summand of the left $B$-module $A$. If a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_{\mathfrak{A}}(B_{l})$ then it is known that $B_{l}=V_{\mathfrak{A}}(\mathfrak{B})^{1)}$. In particular, if $\mathfrak{B}'A_{r}$ is dense in $V_{\mathfrak{A}}(B_{l})$ then $J(\mathfrak{B}', A)=\{a\in A; a\sigma=a$ for every $\sigma\in \mathfrak{B}'\}$ coincides with $B$, where $\mathfrak{B}'$ is a group of $B$-ring automorphisms of $A$. Further, if $B$ is a simple ring and a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_{\mathfrak{A}}(B_{l})$ then it turns out that $A$ is $V_{\mathfrak{A}}(\mathfrak{B})\cdot \mathfrak{B}'$-irreducible. Accordingly, combining the above with Lemma 1, we obtain the following:

Corollary 1. If $A$ is a right Artinian ring, then $B'\rightarrow V_{\mathfrak{A}}(B_{l})$ and $\mathfrak{B}'\rightarrow (1)V_{\mathfrak{A}}(\mathfrak{B}')$ are mutually converse 1–1 dual correspondences between unital simple subrings $B'$ of $A$ and closed intermediate rings $\mathfrak{B}'$ of $\mathfrak{A}/A_{r}$ such that $A$ is $V_{\mathfrak{A}}(\mathfrak{B}')(\mathfrak{B}')$-irreducible.

Remark 1. In Corollary 1, $\mathfrak{B}'$ can be characterized as a closed subring of $\mathfrak{A}$ such that $\mathfrak{B}'A_{r}=\mathfrak{B}'$ and that $A$ is $\mathfrak{B}'$-unital and $V_{\mathfrak{A}}(\mathfrak{B}')(\mathfrak{B}')$-irreducible (Lemma 1). Moreover, in case $A$ is a division ring, the $V_{\mathfrak{A}}(\mathfrak{B}')(\mathfrak{B}')$-irreducibility of $A$ is an easy consequence of the assumption that $\mathfrak{B}'A_{r}=\mathfrak{B}'$ and $A$ is $\mathfrak{B}'$-unital. Hence, Corollary 1 contains essentially [1, Satz V, 1].

The next lemma stated without proof is [3, Lemma 1], in particular, the first assertion is an immediate consequence of Lemma 1.

Lemma 2. Let $A$ be a simple ring, $W$ a unital subring of $V$, and let $A$ be $B\cdot W\cdot A$-irreducible.

(a) $V$ and $V_{\sigma}(W)$ are simple rings, and $A$ is homogeneously completely reducible as $B\cdot A$-module and $[A|B_{l}\cdot A_{r}] = [V|V]^{2)}$.

(b) If $S$ is a unital simple subring of $B$ such that $[B:S]<\infty$, then $[V_{\sigma}(S): V]_{l} \leq [B:S]_{l}$. If moreover $A/S$ is left locally finite, then $[V_{\sigma}(S): V]_{l} \leq [B:S]_{l}$.

If $A$ is right Artinian and $B\cdot V\cdot A$-irreducible, then $A$ is obviously a simple ring. In what follows, we shall often treat with a simple ring extension $A/B$ such that $A$ is $B\cdot V\cdot A$-irreducible. Such an extension is, we believe, not so extraordinary. In fact, as was shown in [4] and [10], if $A/B$ is $q$-Galois and left locally finite then $A$ is $B\cdot V\cdot A$-irreducible.

Corollary 2. Let a simple ring $A$ be $B\cdot V\cdot A$-irreducible.

(a) If $\mathfrak{G}$ is a subset of $\mathfrak{A}$ containing $\mathfrak{V}$ such that $V_{\mathfrak{A}}(A_{r}[\mathfrak{G}])=B_{l}$, then $B$ is regular and $A_{r}[\mathfrak{G}]$ is dense in $V_{\mathfrak{A}}(B_{l})$.


2) $[A|B_{l}\cdot A_{r}]$ means the length of the composition series of $A$ as $B_{l}\cdot A_{r}$-module, and $[V|V]$ does the capacity of the simple ring $V$ (=length of the composition series of $V$ as one-sided $V$-module).
(b) If $\mathfrak{S}$ is a subgroup of $\mathfrak{G}$ containing $\overline{V}$ and $J(\mathfrak{S}, A) = B$, then $B$ is regular, $\mathfrak{S}A_r$ is dense in $V_{\mathfrak{G}}(B_i)$ and $(\mathfrak{S}|H)H_r$ is dense in $\text{Hom}_{H_r}(H, H)$. 

Proof. By Lemma 2 (a), $V$ and $H$ are simple rings.

(a) Since $A_r[\mathfrak{S}] \ni \overline{V}A_r = V_r \cdot A_r$, $A$ is $V_{\mathfrak{G}}(A_r[\mathfrak{S}]) \cdot A_r[\mathfrak{S}]$-irreducible. Hence, the assertion is clear by Lemma 1.

(b) By the validity of (a), it suffices to prove the last assertion. Let $h$ be an arbitrary non-zero element of $H$. Then, $( Bh ) \mathfrak{S}H_r = eH$ with some non-zero idempotent $e$. Since $A = ( Bh ) \mathfrak{S}A_r = ( ( Bh ) \mathfrak{S}H_r ) A_r = eA$, $e$ has to be 1. Hence, $H$ is $B_r \cdot ( \mathfrak{S}|H)H_r$-irreducible. Consequently, again by Lemma 1, $( \mathfrak{S}|H)H_r$ is dense in $\text{Hom}_{H_r}(H, H)$.

Obviously, [8, Lemma 2] and [8, Theorem 1] are contained in the following theorem.

Theorem 1. Let a simple ring $A$ be $B \cdot V \cdot A'$-irreducible.

(a) If $T$ is an intermediate ring of $A/H$ such that $A$ is $T \cdot A'$-irreducible then $[V : V_A(T)]_l = [T : H]_l$, provided we do not distinguish between two infinite dimensions (cf. Lemma 2 (a)).

(b) Let $B$ be simple. If $B'$ is an intermediate ring of $A/B$ left finite over $B$ such that $A$ is $B' \cdot A'$-irreducible, then $[V_A^2(B') : H]_l = [V : V_A(B')]_l < \infty$ and $V_A^2(B') = H[B']$.

(c) Let $B$ be simple. If $B'$ is an intermediate ring of $A/B$ right finite over $B$ such that $A$ is $A' \cdot B'$-irreducible, then $[V_A^2(B') : H]_r = [V : V_A(B')]_r < \infty$ and $V_A^2(B') = H[B']$.

(d) Let $B$ be simple. If $A/B$ is left (or right) locally finite, then $A$ is $h$-Galois and (two-sided) locally finite over $H$ and then $A/A'$ is inner Galois and $[A' : H]_l = [A' : H]_r = [V : V_A(A')]_l = [V : V_A(A')]_r$ for every simple intermediate ring $A'$ of $A/H$ left (or right) finite over $H$.

Proof. (a) Since $V_r \cdot A_r$ is dense in $\text{Hom}_{H_r}(A, A)$ by Lemma 1, one will easily see that $[ (V_r \cdot T) A_r : A_r ]_l = [T : H]_l$, provided we do not distinguish between two infinite dimensions (cf. [6, Lemma 1.4]). On the other hand, by [6, Lemma 1.4], $[(V_r \cdot T) A_r : A_r]_l = [V : V_A(T)]_l$. Combining those, it follows at once our assertion.

(b) Obviously, $H \subseteq H[B'] \subseteq V_A^2(B')$ and $A$ is $H[B']$-$A$-irreducible. Since $V_A(H[B']) = V_A(B') = V_A(V_A^2(B'))$ and $\infty > [B' : B]_l \geq [V : V_A(B')]_r$ by Lemma 2 (b), (a) implies that $[V_A^2(B') : H]_l = [V : V_A(B')]_l = [H[B'] : H]_l$.

(c) By Lemma 2 (b), we obtain $\infty > [B' : B]_l \geq [V : V_A(B')]_r \geq [V_A^2(B') : H]_l \geq [H[B'] : H]_l \geq [V : V_A(B')]_l$, namely, $[V : V_A(B')]_l = [H[B'] : H]_l = [V_A^2(B') : H]_l$.

(d) By (b) (or (c)) and [3, Theorem 1], $A/H$ is $h$-Galois and locally finite. Since $V_r \cdot A_r$ is dense in $\text{Hom}_{H_r}(A, A)$ (Lemma 1), $A/A'$ is inner Galois.
On Dimensions of Simple Ring Extensions

by [11, Proposition 4], and then \([A':H]_{r}=[A':H]_{l}=[V:V_{A}(A')]_{r}=[V:V_{A}(A')]_{l}\) by [3, Theorem 1] or [9, Theorem 8].

Now, let \(A/B\) be a left locally finite simple ring extension. We consider the following conditions\(^3\):

1. \(B\) is regular.
2. \(A\) is \(B\cdot V\cdot A\)-irreducible.
3. \(A\) is \(B\cdot V\cdot A\)-irreducible and \(\mathfrak{G}(A', A/B)|H=\mathfrak{G}(H, A/B)\) for every \(A'\in \mathcal{R}^{0}/H\) left finite over \(H\).
4. \(A\) is \(A\cdot B\cdot V\)-irreducible.
5. \(A\) is \(A\cdot B\cdot V\)-irreducible and \(\mathfrak{G}(A', A/B)|H=\mathfrak{G}(H, A/B)\) for every \(A'\in \mathcal{R}^{0}/H\) left finite over \(H\).
6. \(\mathfrak{G}(B_{1}, A/B)|B_{2}=\mathfrak{G}(B_{2}, A/B)\) for every \(B_{1}\supseteq B_{2}\) in \(\mathcal{R}_{l.f}\).
7. \(H/B\) is Galois.
8. \(H/B\) is Galois and \([V_{A}'(T):H]=[V:V_{A}(T)]\) for every \(T\in \mathcal{R}^{0}_{l.f}\).
9. \((T\cap H)\mathfrak{G}(T, A/B)\subseteq H\) for every \(T\in \mathcal{R}^{0}_{l.f}\).

In [4, Theorems 3, 4 and 5], one of the present authors has given several useful conditions those which are equivalent to the condition that \(A/B\) be \(q\)-Galois. Now, we shall add other equivalent ones to those.

**Theorem 2.** Let a simple ring \(A\) be left locally finite over a simple ring \(B\). In order that \(A/B\) be \(q\)-Galois, it is necessary and sufficient that any of the following equivalent conditions be satisfied:

1. \((\text{ii})+(\text{vi})+(\text{vii})\).
2. \((\text{iv})+(\text{vi})+(\text{vii})\).
3. \((\text{i})+(\text{vi})+(\text{viii})\).
4. \((\text{iii})+(\text{vii})+(\text{ix})\).
5. \((\text{v})+(\text{vii})+(\text{ix})\).

**Proof.** If \(A/B\) is \(q\)-Galois then (i)–(ix) are all satisfied ([4, Theorems 3, 4 and 5] and [9, Theorem 6]). Conversely, if one of the conditions (1), (2) and (3) is satisfied and if \(T\) is in \(\mathcal{R}_{l.f}^{0}\), then \(T\cap H\) is in \(\mathcal{R}_{l.f}^{0}\) ([6, Lemma 1.6] and [5 Theorem 1.1]) and \(J(\mathfrak{G}(T, A/B), T)=J(\mathfrak{G}(T, A/B), T)\cap J(\tilde{V}|T, T)=J(\mathfrak{G}(T, A/B)|H\cap T, H\cap T)=J(\mathfrak{G}(H/B)|H\cap T, H\cap T)=B\), where \(\mathfrak{G}(H/B)\) means the Galois group of \(H/B\). Hence, \(A/B\) is \(q\)-Galois by [4, Theorems 3 and 4]. Finally, assume (4) or (5). Then, \(A/H\) is locally finite by Theorem 1 (d). If \(T\in \mathcal{R}_{l.f}^{0}\), then \(J(\mathfrak{G}(T, A/B), T)\subseteq J(\mathfrak{G}(T[H], A/B)|H\cap T, H\cap T)\subseteq J(\mathfrak{G}(H/B)|H\cap T, H\cap T)=B\). Hence, \(A/B\) is \(q\)-Galois by [4, Theorem 5].

Finally, we shall extend [7, Corollary 2] to simple ring extensions. Let

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\(^3\) As to notations, we follow [4] and [10].
Let \( \mathfrak{S} \) be a (multiplicative) sub-semigroup of \( \mathfrak{A} \). If \( \mathfrak{S}A_r \) and \( \mathfrak{S}A_l \) form subrings of \( \mathfrak{A} \) (or, \( A_r, \mathfrak{S} \subseteq \mathfrak{S}A_r \) and \( A_l, \mathfrak{S} \subseteq \mathfrak{S}A_l \)) and \( V_\mathfrak{R}(\mathfrak{S}) \cap A_r = B_r \) and \( V_\mathfrak{R}(\mathfrak{S}) \cap A_l = B_l \), then \( \mathfrak{S} \) is called a 

\textit{Galois semigroup of } \( A/B \).

**Lemma 3.** Let a simple ring \( A \) be \( B \cdot V \cdot A \)-irreducible, and \( \mathfrak{S} \) a Galois semigroup of \( A/B \) containing \( \overline{V} \). Let \( T \) be a \( B \cdot B \)-submodule of \( A \) possessing a linearly independent left \( B \)-basis.

(a) If \( T \) is left finite over \( B \) then \( (\mathfrak{S}|T)V_r \) possesses a linearly independent \( V_r \)-basis that forms at the same time a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \).

(b) In order that \( T \) be left finite over \( B \), it is necessary and sufficient that \( [(\mathfrak{S}|T)V_r|V_r] \) be finite.

**Proof.** By Corollary 2, \( B \) is regular and \( \mathfrak{S}A_r \) is dense in \( V_\mathfrak{R}(B_l) \).

(a) The proof will be completed in the same way as in [5, Lemma 1.2 (i)]. Let \( T = \{ g_{pq} ; p, q = 1, \ldots, u \} \) be a system of matrix units of \( V \) such that \( V_\mathfrak{R}(T) \) is a division ring. If \( g_r = g_{pp} \) then \( A = \oplus_i g_{pi}A \) and \( (g_{pp})^r \) induces a \( B \cdot A \)-isomorphism of \( g_{p}A \) onto \( g_qA \). Since \( A \) is homogeneously \( B \cdot A \)-completely reducible and \( [A|B_r \cdot A_r] = [V|V] = u \) (Lemma 2 (a)), \( g_{p}A \) is \( B \cdot A \)-irreducible, so that \( g_{p}A_r \) is \( B_r \cdot A_r \)-irreducible. Accordingly, \( (\sigma|T)(g_{p}A_r) \), being \( B_r \cdot A_r \)-homomorphic to \( (g_{p}A)_r \) for every \( \sigma \in \mathfrak{S} \), \( \text{Hom}_{B_r}(T, A) = \sum_{\sigma \in \mathfrak{S}} \sum_{p}(\sigma|T)(g_{p}A_r) = \oplus \{ (\sigma|T)|g_{p_i}A_r \} \), with some \( \sigma_i \in \mathfrak{S} \) and some \( g_{p_i} \), where each \( (\sigma|T)(g_{p_i}A_r) \) is \( B_r \cdot A_r \)-isomorphic to arbitrary fixed \( (g_{p}A)_r \). Recalling here that \( A = \oplus_i g_{p}A \), the last relation yields \( s = u \cdot [T:B] \), and so \( \mathfrak{S} = \sum_i (\sigma_i|T)(g_{p_i}V_r) = \oplus \{ (\sigma_i|T)(g_{p_i}V_r) \} \), \( \mathfrak{S} \cdot V_r \) possesses a linearly independent \( V_r \)-basis \( \{ \epsilon_1, \ldots, \epsilon_t \} \) and \( [\mathfrak{S} : V_r] = [T:B] \). Since \( (\mathfrak{S}|T)A_r = \mathfrak{S}A_r \) and \( [(\mathfrak{S}|T)A_r : A_r] = [T:B] \), the \( V_r \)-basis \( \{ \epsilon_1, \ldots, \epsilon_t \} \) is still a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \). Now, one will easily see that \( \mathfrak{S} = \text{Hom}_{B_r}(T, A) = (\mathfrak{S}|T)V_r \).

(b) In virtue of (a), it remains only to prove the sufficiency. If \( [(\mathfrak{S}|T)V_r|V_r] \) is finite then \( (\mathfrak{S}|T)V_r \) is finite over \( V_r \), and so \( (\mathfrak{S}|T)A_r = (\mathfrak{S}|T)V_r \cdot A_r \) is finite over \( A_r \), too. Hence, our assertion is a consequence of the density of \( \mathfrak{S}A_r \) in \( V_\mathfrak{R}(B_l) \).

**Proposition 1.** Assume that a simple ring \( A \) is \( B \cdot V \cdot A \)-irreducible and \( A \cdot B \cdot V \)-irreducible. Let \( \mathfrak{S} \) be a Galois semigroup of \( A/B \) containing \( \overline{V} \). Let \( T \) be a \( B \cdot B \)-submodule of \( A \) possessing a finite linearly independent left \( B \)-basis and a linearly independent right \( B \)-basis, and \( \{ \epsilon_1, \ldots, \epsilon_t \} \) a linearly independent \( V_r \)-basis of \( (\mathfrak{S}|T)V_r \) that forms at the same time a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \) (cf. Lemma 3 (a)). If \( V_A(\cup_i \{ T\epsilon_i \}) \cap V \) contains a unital division subring \( U \) such that \( [V:U]_r = [V:U]<\infty \), then \( [T:B]_r \leq [T:B]_l \).
Proof. By Corollary 2, B is regular, $\mathfrak{A}_{r}$ and $\mathfrak{A}_{l}$ are dense in $V_{\mathfrak{A}}(B_{l})$ and $V_{\mathfrak{A}}(B_{r})$, respectively. Hence, $\mathrm{Hom}_{B_{l}}(T, A) = (\mathfrak{A} | T) A_{r} = \bigoplus_{t=1}^{\infty} \epsilon_{t} A_{r}$ ($t = [T : B]_{l}$). Moreover, one will easily see that $\mathfrak{X} = \sum_{l=1}^{\infty} \epsilon_{l} V_{r} = \mathrm{Hom}_{B_{l}, B_{r}}(T, A) \supseteq (\mathfrak{A} | T)V_{l}$. If $\{v_{1}, \cdots, v_{m}\}$ is a linearly independent left U-basis of V, then by the assumption we have $\mathfrak{X} = \sum_{l=1}^{m} \epsilon_{l} (\sum_{k=1}^{\infty} U_{r} v_{kr}) = \sum_{l=1}^{m} \epsilon_{l} v_{kr} U_{l}$. Hence, $tm \geq [\mathfrak{X} : U_{l}]$, so that $[\mathfrak{X} | T)V_{l} | V_{l}]$ is finite. Accordingly, by the proposition symmetric to Lemma 3, it follows $\infty > [T : B]_{r} = [(\mathfrak{X} | T)V_{l} : V_{l}]$. Then, noting that $[\mathfrak{X} : U_{l}] = [\mathfrak{X} : V_{i}] \cdot m/|V_{l}V_{l} |$, we readily obtain $t \cdot |V_{l}V_{l}| > [\mathfrak{X} | V_{l}] \geq [(\mathfrak{X} | T)V_{l} | V_{l}] = [T : B]_{r} \cdot |V_{l}V_{l}|$. We have proved therefore $[T : B]_{l} \geq [T : B]_{r}$.

Now, we shall prove our last theorem.

**Theorem 3.** Let a simple ring $A$ be $B \cdot V \cdot A$-irreducible and $A \cdot B \cdot V$-irreducible, and $\mathfrak{A}$ a Galois semigroup of $A/B$.

(a) Let $T$ be a $B$-$B$-submodule of $A$ possessing a linearly independent left $B$-basis and a linearly independent right $B$-basis. If $A/H$ is left locally finite then $[T : B]_{l} = [T : B]_{r}$, provided we do not distinguish between two infinite dimensions.

(b) Let $V$ be contained in $B$. If $T$ is an intermediate ring of $A/B$ then $[T : B]_{l} = [T : B]_{r}$, provided we do not distinguish between two infinite dimensions.

Proof. One may remark first that $B$ and $H$ are regular by Corollary 2 and Lemma 2 (a), and assume that $\mathfrak{A}$ contains $\mathfrak{V}$.

(a) Since $A$ is $H \cdot V \cdot A$-irreducible and left locally finite over $H = V_{l}^{2}(H)$, $A/H$ is two-sided locally finite by Theorem 1 (d). Hence, by the symmetry of our assumption, it suffices to prove that if $[T : B]_{l} < \infty$ then $[T : B]_{r} \leq [T : B]_{l}$. Now, let $\{\epsilon_{1}, \cdots, \epsilon_{t}\}$ be a linearly independent $V_{r}$-basis of $(\mathfrak{A} | T) V_{r}$ that forms at the same time a linearly independent $A_{r}$-basis of $(\mathfrak{A} | T) A_{r}$ (Lemma 3). By Theorem 1 (d), there holds then $\infty > [H[E, \cup \{T_{d}\}] : H] = [V : V_{A}(E, \cup \{T_{d}\})] = [V : V_{A}(H[E, \cup \{T_{d}\})]$, where $E$ is a system of matrix units such that $V_{A}(E)$ is a division ring. Hence, $[T : B]_{r} \geq [T : B]_{l}$ by Proposition 1.

(b) Again by the symmetry of our assumption, it suffices to prove that if $[T : B]_{l} < \infty$ then $[T : B]_{r} \leq [T : B]_{l}$. Since $U = V_{A}(T) = V_{B}(T) \subseteq V_{A}(T \mathrm{Hom}_{A_{l}B_{r}}(T, A)) \subseteq V$ (field) and $[V : U] \subseteq [T : B]_{l} < \infty$ by Lemma 2, our assertion is again a consequence of Proposition 1.

As a direct consequence of Theorem 3 (a), we obtain the following, that contains evidently [7, Corollary 2].

**Corollary 3.** Let a simple ring $A$ be $B \cdot V \cdot A$-irreducible and $A \cdot B \cdot V$-irreducible, and $T$ an intermediate ring of $A/B$. If $J(\mathfrak{A}, A) = B$ and $A/H$...
is left locally finite then \([T : B]_{l} = [T : B]_{r}\), provided we do not distinguish between two infinite dimensions.

**Remark 2.** Theorem 3 (a) may be regarded as an extension of [1, Folgerung zu Satz VII, 2]. However, [1] contains considerable errors: The definition of \(e\text{Hom}_L(\mathcal{M}, \mathcal{N})\) is absurd, and Satz V, 1, Hilfssatz VI, 2 and Hilfssatz VI, 4 are open to doubt.

**References**


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