ON DIMENSIONS OF SIMPLE RING EXTENSIONS

By

Takasi NAGAHARA and Hisao TOMINAGA

Let $A$ be a division ring, and $B$ a division subring of $A$. If $T$ is an intermediate ring of $A/V_A^2(B)$ then $[T : V_A(B)]_r = [V_A(B) : V_A(T)]_r$, provided we do not distinguish between two infinite dimensions ([8, Lemma 2]). If $A/B$ is left locally finite then so is $A/V_A^2(B)$ ([8, Theorem 1]). Moreover, if $A/B$ is Galois and $A/V_A^2(B)$ is left locally finite then for any intermediate ring $B'$ of $A/B$ left finite over $B$ there holds $[B' : B]_r = [B' : B]$ ([7, Corollary 2]).

The purpose of the present paper is to extend those results stated above to simple ring extensions. As one will see later, our extension of [7, Corollary 2] is especially satisfactory (Theorem 3).

Throughout the present paper, we use the following conventions: $A$ will represent a ring with 1, $B$ a unital subring of $A$ (i.e. a subring of $A$ containing 1), $V$ the centralizer $V_A(B)$ of $B$ in $A$, and $H$ the double centralizer $V_A^2(B) = V_A(V)$ of $B$ in $A$. Moreover, $\mathfrak{A}$ and $\mathfrak{B}$ will denote the absolute endomorphism ring $\text{Hom}(A, A)$ of $A$ and the group of all $B$-ring automorphisms of $A$, respectively. If $X$ is a subset of $A$, then $X_l$, $X_r$, and $\overline{X}$ will mean the sets of all left multiplications effected by elements of $X$, of all the right multiplications effected by elements of $X$ and of all inner automorphisms effected by regular elements of $A$ contained in $X$, respectively.

The following easy lemma will play an essential role in our subsequent consideration.

**Lemma 1.** Let $A$ be a right Artinian ring, and $\mathfrak{B}$ a subring of $\mathfrak{A}$ such that $A$ is $V_\mathfrak{A}(\mathfrak{B})$-$\mathfrak{B}$-irreducible. If $\mathfrak{B}A_r = \mathfrak{B}$ and $A$ is $\mathfrak{B}$-unital (i.e. $x\mathfrak{B} \neq 0$ for every non-zero $x \in A$), then $V_\mathfrak{A}(\mathfrak{B})$ is an (Artinian) simple subring of $A_r$ and $V_\mathfrak{B}(\mathfrak{B})$ is the closure of $\mathfrak{B}$ (in the finite topology).

**Proof.** Noting that every $x\mathfrak{B}$ ($x \in A$) is a right $A$-submodule of $A$ and $A$ is $\mathfrak{B}$-unital, one will easily see that $A$ contains a minimal $\mathfrak{B}$-submodule $M$. Since $A = MV_\mathfrak{A}(\mathfrak{B}) = \sum_{\alpha \in V_\mathfrak{A}(\mathfrak{B})} M\alpha$ and each $M\alpha$ is either $\mathfrak{B}$-isomorphic to $M$ or 0, $A$ is homogeneously $\mathfrak{B}$-completely reducible. Hence, $V_\mathfrak{A}(\mathfrak{B})$ is simple, and $\mathfrak{B}$ is dense in $V_\mathfrak{B}(\mathfrak{B})$ ($\exists 1$) by [2, Theorem VI. 2.2]. Now, the rest of our assertion will be easily seen.

The converse of Lemma 1 will be rather familiar: Let $B$ be a direct
summand of the left $B$-module $A$. If a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_\mathfrak{A}(B_i)$ then it is known that $B_i = V_\mathfrak{B}(\mathfrak{B})$. In particular, if $\mathfrak{B}'A_r$ is dense in $V_\mathfrak{A}(B_i)$ then $J(\mathfrak{B}', A) = \{a \in A; \sigma a = a \text{ for every } \sigma \in \mathfrak{B}'\}$ coincides with $B$, where $\mathfrak{B}'$ is a group of $B$-ring automorphisms of $A$. Further, if $B$ is a simple ring and a subring $\mathfrak{B}$ of $\mathfrak{A}$ is dense in $V_\mathfrak{A}(B_i)$ then it turns out that $A$ is $V_\mathfrak{A}(\mathfrak{B})$-$\mathfrak{B}$-irreducible. Accordingly, combining the above with Lemma 1, we obtain the following:

**Corollary 1.** If $A$ is a right Artinian ring, then $B' \to V_\mathfrak{A}(B_i)$ and $\mathfrak{B}' \to (1) V_\mathfrak{A}(\mathfrak{B}')$ are mutually converse $1-1$ dual correspondences between unital simple subrings $B'$ of $A$ and closed intermediate rings $\mathfrak{B}'$ of $\mathfrak{A}/A_r$ such that $A$ is $V_\mathfrak{A}(\mathfrak{B}')$-$\mathfrak{B}'$-irreducible.

**Remark 1.** In Corollary 1, $\mathfrak{B}'$ can be characterized as a closed subring of $\mathfrak{A}$ such that $\mathfrak{B}'A_r = \mathfrak{B}'$ and that $A$ is $\mathfrak{B}'$-unital and $V_\mathfrak{A}(\mathfrak{B}')$-$\mathfrak{B}'$-irreducible (Lemma 1). Moreover, in case $A$ is a division ring, the $V_\mathfrak{A}(\mathfrak{B}')$-$\mathfrak{B}'$-irreducibility of $A$ is an easy consequence of the assumption that $\mathfrak{B}'A_r = \mathfrak{B}'$ and $A$ is $\mathfrak{B}'$-unital. Hence, Corollary 1 contains essentially [1, Satz V, 1].

The next lemma stated without proof is [3, Lemma 1], in particular, the first assertion is an immediate consequence of Lemma 1.

**Lemma 2.** Let $A$ be a simple ring, $W$ a unital subring of $V$, and let $A$ be $B \cdot W$-$A$-irreducible.

(a) $V$ and $V_\mathfrak{A}(W)$ are simple rings, and $A$ is homogeneously completely irreducible as $B$-$A$-module and $[A|B_i \cdot A_r] = [V|V]^2$.

(b) If $S$ is a unital simple subring of $B$ such that $[B:S] < \infty$, then $[V_\mathfrak{A}(S): V] \leq [B:S]$. If moreover $A/S$ is left locally finite, then $[V_\mathfrak{A}(S): V] \leq [B:S]$.

If $A$ is right Artinian and $B \cdot V$-$A$-irreducible, then $A$ is obviously a simple ring. In what follows, we shall often treat with a simple ring extension $A/B$ such that $A$ is $B \cdot V$-$A$-irreducible. Such an extension is, we believe, not so extraordinary. In fact, as was shown in [4] and [10], if $A/B$ is $q$-Galois and left locally finite then $A$ is $B \cdot V$-$A$-irreducible.

**Corollary 2.** Let a simple ring $A$ be $B \cdot V$-$A$-irreducible.

(a) If $\mathfrak{S}$ is a subset of $\mathfrak{A}$ containing $\mathfrak{V}$ such that $V_\mathfrak{A}(A_r[\mathfrak{S}]) = B_i$, then $B$ is regular and $A_r[\mathfrak{S}]$ is dense in $V_\mathfrak{A}(B_i)$.

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2) $[A|B_i \cdot A_r]$ means the length of the composition series of $A$ as $B_i \cdot A_r$-module, and $[V|V]$ does the capacity of the simple ring $V$ (= length of the composition series of $V$ as one-sided $V$-module).
(b) If \( \mathcal{S} \) is a subgroup of \( \mathfrak{G} \) containing \( \tilde{V} \) and \( J(\mathcal{S}, A) = B \), then \( B \) is regular, \( \mathcal{S} A_r \) is dense in \( V_A(B_i) \) and \( (\mathcal{S}|H)H_r \) is dense in \( \text{Hom}_{B_r}(H,H) \).

**Proof.** By Lemma 2 (a), \( V \) and \( H \) are simple rings.

(a) Since \( A_r[\mathcal{S}] \supseteq \tilde{V}A_r = V_I \cdot A_r \), \( A \) is \( V_A(\mathcal{S}) \cdot A_r \)-irreducible. Hence, the assertion is clear by Lemma 1.

(b) By the validity of (a), it suffices to prove the last assertion. Let \( h \) be an arbitrary non-zero element of \( H \). Then, \( (Bh)\mathcal{S}H_r = eH \) with some non-zero idempotent \( e \). Since \( A = (Bh)\mathcal{S}A_r = ((Bh)\mathcal{S}H_r)A_r = eA \), \( e \) has to be 1. Hence, \( H \) is \( B_r \cdot (\mathcal{S}|H)H_r \)-irreducible. Consequently, again by Lemma 1, \( (\mathcal{S}|H)H_r \) is dense in \( \text{Hom}_{B_r}(H,H) \).

Obviously, [8, Lemma 2] and [8, Theorem 1] are contained in the following theorem.

**Theorem 1.** Let a simple ring \( A \) be \( B \cdot V-A \)-irreducible.

(a) If \( T \) is an intermediate ring of \( A/H \) such that \( A \) is \( T-A \)-irreducible then \( [V:V_A(T)]_r = [T:H]_r \), provided we do not distinguish between two infinite dimensions (cf. Lemma 2 (a)).

(b) Let \( B \) be simple. If \( B' \) is an intermediate ring of \( A/B \) left finite over \( B \) such that \( A \) is \( B'-A \)-irreducible, then \( [V_2^2(\mathcal{S}B'):H]_r = [V : V_B(B')]_r < [V : V_B(B')]_l \) and \( V_2^2(B') = H[B'] \).

(c) Let \( B \) be simple. If \( B' \) is an intermediate ring of \( A/B \) right finite over \( B \) such that \( A \) is \( A-B' \)-irreducible, then \( [V_2^2(\mathcal{S}B'):H]_r = [V : V_B(B')]_r < [V : V_B(B')]_l \) and \( V_2^2(B') = H[B'] \).

(d) Let \( B \) be simple. If \( A/B \) is left (or right) locally finite, then \( A \) is \( h \)-Galois and (two-sided) locally finite over \( H \) and then \( A/A' \) is inner Galois and \( [V_2^2(\mathcal{S}B'):H]_r = [V : V_B(A')]_l = [V : V_B(A')]_r \) for every simple intermediate ring \( A' \) of \( A/H \) left (or right) finite over \( H \).

**Proof.** (a) Since \( V_I \cdot A_r \) is dense in \( \text{Hom}_{B_r}(A, A) \) by Lemma 1, one will easily see that \([(V_I[T]A_r:A_r]:A_r) = [T:H]_r \), provided we do not distinguish between two infinite dimensions (cf. [6, Lemma 1.4]). On the other hand, by [6, Lemma 1.4], \([(V_I[T]A_r:A_r):A_r] = [V : V_A(T)]_l \). Combining those, it follows at once our assertion.

(b) Obviously, \( H \subseteq H[B'] \subseteq V_2^2(B') \) and \( A \) is \( H[B']-A \)-irreducible. Since \( V_A(H[B']) = V_A(B') \) and \( \infty > [V : V_A(B')]_r \) by Lemma 2 (b), (a) implies that \( [V_2^2(B'):H]_r = [V : V_A(B')]_r = [H[B']:H]_r \).

(c) By Lemma 2 (b), we obtain \( \infty > [B':B]_r > [V : V_A(B')]_r > [V_2^2(B'):H]_r > [H[B']:H]_r > [V : V_A(B')]_r \), namely, \( [V : V_A(B')]_r = [H[B']:H]_r = [V_2^2(B'):H]_r \).

(d) By (b) (or (c)) and [3, Theorem 1], \( A/H \) is \( h \)-Galois and locally finite. Since \( \tilde{V} \cdot A_r = V_I \cdot A_r \) is dense in \( \text{Hom}_{B_r}(A, A) \) (Lemma 1), \( A/A' \) is inner Galois.
by [11, Proposition 4], and then \([A': H]_r = [A': H]_l = [V: V_A(A')]_r = [V: V_A(A')]_l\), by [3, Theorem 1] or [9, Theorem 8].

Now, let \(A/B\) be a left locally finite simple ring extension. We consider the following conditions:\(^3\):

(i) \(B\) is regular.

(ii) \(A\) is \(B\cdot V\cdot A\)-irreducible.

(iii) \(A\) is \(B\cdot V\cdot A\)-irreducible and \(\mathfrak{G}(A', A/B)|H = \mathfrak{G}(H, A/B)\) for every \(A'\in \mathcal{R}^0/H\) left finite over \(H\).

(iv) \(A\) is \(A\cdot B\cdot V\)-irreducible.

(v) \(A\) is \(A\cdot B\cdot V\)-irreducible and \(\mathfrak{G}(A', A/B)|H = \mathfrak{G}(H, A/B)\) for every \(A'\in \mathcal{R}^0/H\) left finite over \(H\).

(vi) \(\mathfrak{G}(B_1, A/B)|B_2 = \mathfrak{G}(B_2, A/B)\) for every \(B_1 \supseteq B_2\) in \(\mathcal{R}_{l.f}\).

(vii) \(H/B\) is Galois and \([V^2_A(T): H] = [V: V_A(T)]\), for every \(T\in \mathcal{R}^0_{l.f}\).

(viii) \((T\cap H)\mathfrak{G}(T, A/B)\subseteq H\) for every \(T\in \mathcal{R}^0_{l.f}\).

In [4, Theorems 3, 4 and 5], one of the present authors has given several useful conditions those which are equivalent to the condition that \(A/B\) be \(q\)-Galois. Now, we shall add other equivalent ones to those.

**Theorem 2.** Let a simple ring \(A\) be left locally finite over a simple ring \(B\). In order that \(A/B\) be \(q\)-Galois, it is necessary and sufficient that any of the following equivalent conditions be satisfied:

1. (i) + (vi) + (vii).
2. (iv) + (vi) + (vii).
3. (i) + (vi) + (viii).
4. (iii) + (vii) + (ix).
5. (v) + (vii) + (ix).

**Proof.** If \(A/B\) is \(q\)-Galois then (i)–(ix) are all satisfied ([4, Theorems 3, 4 and 5] and [9, Theorem 6]). Conversely, if one of the conditions (1), (2) and (3) is satisfied and if \(T\) is in \(\mathcal{R}^0_{l.f}\), then \(T\cap H\) is in \(\mathcal{R}_{l.f}\) ([6, Lemma 1.6] and [5 Theorem 1.1]) and \(J(\mathfrak{G}(T, A/B), T) = J(\mathfrak{G}(T, A/B), T) \cap J(\tilde{V}|T, T) = J(\mathfrak{G}(T, A/B)|H \cap T, H \cap T) = J(\mathfrak{G}(H/B)|H \cap T, H \cap T) = B\), where \(\mathfrak{G}(H/B)\) means the Galois group of \(H/B\). Hence, \(A/B\) is \(q\)-Galois by [4, Theorems 3 and 4]. Finally, assume (4) or (5). Then, \(A/H\) is locally finite by Theorem 1 (d). If \(T\in \mathcal{R}^0_{l.f}\), then \(J(\mathfrak{G}(T, A/B), T) \subseteq J(\mathfrak{G}(T[H], A/B)|H \cap T, H \cap T) \subseteq J(\mathfrak{G}(H/B)|H \cap T, H \cap T) = B\). Hence, \(A/B\) is \(q\)-Galois by [4, Theorem 5].

Finally, we shall extend [7, Corollary 2] to simple ring extensions. Let

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3) As to notations, we follow [4] and [10].
\( \mathfrak{S} \) be a (multiplicative) sub-semigroup of \( \mathfrak{A} \). If \( \mathfrak{S} A_r \) and \( \mathfrak{S} A_t \) form subrings of \( \mathfrak{A} \) (or, \( A_r \mathfrak{S} \subseteq \mathfrak{S} A_r \) and \( A_t \mathfrak{S} \subseteq \mathfrak{S} A_t \)) and \( V_r(\mathfrak{S}) \cap A_r = B_r \) and \( V_r(\mathfrak{S}) \cap A_t = B_t \), then \( \mathfrak{S} \) is called a Galois semigroup of \( A/B \).

**Lemma 3.** Let a simple ring \( A \) be \( B \cdot V \cdot A \)-irreducible, and \( \mathfrak{S} \) a Galois semigroup of \( A/B \) containing \( \mathfrak{V} \). Let \( T \) be a \( B \cdot B \)-submodule of \( A \) possessing a linearly independent left \( B \)-basis.

(a) If \( T \) is left finite over \( B \) then \( (\mathfrak{S}|T)V_r \) possesses a linearly independent \( V_r \)-basis that forms at the same time a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \).

(b) In order that \( T \) be left finite over \( B \), it is necessary and sufficient that \( [(\mathfrak{S}|T)V_r|V_r] \) be finite.

**Proof.** By Corollary 2, \( B \) is regular and \( \mathfrak{S} A_r \) is dense in \( V_r(\mathfrak{B}) \).

(a) The proof will be completed in the same way as in [5, Lemma 1.2 (i)]. Let \( l' = \{g_{pq} ; p, q = 1, \cdots, u \} \) be a system of matrix units of \( V \) such that \( V_\nu(l') \) is a division ring. If \( g_r = g_{pp} \) then \( A = \oplus_i^u g_{p} A \) and \( (g_{qp}) \) induces a \( B \cdot A \)-isomorphism of \( g_{p} A \) onto \( g_q A \). Since \( A \) is homogeneously \( B \cdot A \)-completely reducible and \( [A|B \cdot A_r] = [V|V] = u \) (Lemma 2 (a)), \( g_p A \) is \( B \cdot A \)-irreducible, so that \( (g_p A) \) is \( B_r \cdot A_r \)-irreducible. Accordingly, \( (\sigma|T)(g_{p} A) \), being \( B_r \cdot A_r \)-isomorphic to \( (g_p A) \), for every \( \sigma \in \mathfrak{S} \), \( \text{Hom}_{B_r}(T, A) = \sum_{\sigma \in \mathfrak{S}} \sum_{p}(\sigma|T)(g_{p} A) = \oplus_i^{|\mathfrak{S}|}(\sigma|T)(g_{p} A) \), with some \( \sigma \in \mathfrak{S} \) and some \( g_{p} \), where each \( (\sigma|T)(g_{p} A) \), is \( B_r \cdot A_r \)-isomorphic to arbitrary fixed \( (g_p A) \). Recalling here that \( A = \oplus_i^u g_{p} A \), the last relation yields \( s = u \cdot [T : B]_t \), and so \( \Sigma = \sum_{i} \sum_{p}(\sigma|T)(g_{p} V) = \oplus_i^{|\mathfrak{S}|}(\sigma|T)(g_{p} V) \), possesses a linearly independent \( V_r \)-basis \( \{\epsilon_1, \cdots, \epsilon_t \} \) and \( [\Sigma : V_r] = [T : B]_t \). Since \( (\mathfrak{S}|T)A_r = \Sigma A_r \) and \( [(\mathfrak{S}|T)A_r : A_r]_r = [T : B]_t \), the \( V_r \)-basis \( \{\epsilon_1, \cdots, \epsilon_r \} \) is still a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \). Now, one will easily see that \( \Sigma = \text{Hom}_{B_r}(T, A) = (\mathfrak{S}|T)V_r \).

(b) In virtue of (a), it remains only to prove the sufficiency. If \( [(\mathfrak{S}|T)V_r|V_r] \) is finite then \( (\mathfrak{S}|T)V_r \) is finite over \( V_r \), and so \( (\mathfrak{S}|T)A_r = ((\mathfrak{S}|T)V_r)A_r \) is finite over \( A_r \), too. Hence, our assertion is a consequence of the density of \( \mathfrak{S} A_r \) in \( V_r(\mathfrak{B}) \).

**Proposition 1.** Assume that a simple ring \( A \) is \( B \cdot V \cdot A \)-irreducible and \( A \cdot B \cdot V \)-irreducible. Let \( \mathfrak{S} \) be a Galois semigroup of \( A/B \) containing \( \mathfrak{V} \). Let \( T \) be a \( B \cdot B \)-submodule of \( A \) possessing a finite linearly independent left \( B \)-basis and a linearly independent right \( B \)-basis, and \( \{\epsilon_1, \cdots, \epsilon_r \} \) a linearly independent \( V_r \)-basis of \( (\mathfrak{S}|T)V_r \) that forms at the same time a linearly independent \( A_r \)-basis of \( (\mathfrak{S}|T)A_r \) (cf. Lemma 3 (a)). If \( V_r(A \cup (\cup_{i} T \epsilon_i)) \cap V \) contains a unital division subring \( U \) such that \( [V : U] = [V : U]_r < \infty \), then \( [T : B]_r \leq [T : B]_t \).
Proof. By Corollary 2, $B$ is regular, $\mathcal{G}_A$, and $\mathcal{G}_A$ are dense in $V_B(B)$ and $V_B(B)$, respectively. Hence, $\text{Hom}_{B_r}(T,A) = (\mathcal{G}|T)A_r = \oplus \{\varepsilon_i A_r \mid t = [T:B]_{l}\}$. Moreover, one will easily see that $\mathcal{X} = \oplus \{\varepsilon_i V_r = \text{Hom}_{B_r}(T,A) \supseteq (\mathcal{G}|T)V_l\}$. If $\{v_1, \cdots, v_m\}$ is a linearly independent left $U$-basis of $V$, then by the assumption we have $\mathcal{X} = \sum \{\mathcal{X}_i(\sum_{t=1}^m U_r v_{kr}) = \sum_{t,k} \varepsilon_i v_{kr} U_l\}$. Hence, $tm \geq [\mathcal{X}:U_l]_r$, so that $[(\mathcal{G}|T)V_l|V_l]$ is finite. Accordingly, by the proposition symmetric to Lemma 3, it follows $\infty \geq [T:B]_r = [\mathcal{G}|T)V_l|V_l]$. Then, noting that $[\mathcal{X}:U_l]_r = [\mathcal{X}_i|U_l] \cdot m/[V|V]$, we readily obtain $t \cdot [V|V] \geq [\mathcal{X}|V_l]_r \geq [(\mathcal{G}|T)V_l|V_l] = [T:B]_r \cdot [V|V]$. We have proved therefore $[T:B]_r \geq [T:B]_l$.

Now, we shall prove our last theorem.

**Theorem 3.** Let a simple ring $A$ be $B$-$V$-$A$-irreducible and $A$-$B$-$V$-irreducible, and $\mathcal{G}$ a Galois semigroup of $A/B$.

(a) Let $T$ be a $B$-$B$-submodule of $A$ possessing a linearly independent left $B$-basis and a linearly independent right $B$-basis. If $A/H$ is left locally finite then $[T:B]_l = [T:B]_r$, provided we do not distinguish between two infinite dimensions.

(b) Let $V$ be contained in $B$. If $T$ is an intermediate ring of $A/B$ then $[T:B]_l = [T:B]_r$, provided we do not distinguish between two infinite dimensions.

Proof. One may remark first that $B$ and $H$ are regular by Corollary 2 and Lemma 2 (a), and assume that $\mathcal{G}$ contains $\mathcal{V}$.

(a) Since $A$ is $H$-$V$-$A$-irreducible and left locally finite over $H = V_B(H)$, $A/H$ is two-sided locally finite by Theorem 1 (d). Hence, by the symmetry of our assumption, it suffices to prove that if $[T:B]_l < \infty$ then $[T:B]_l \leq [T:B]_l$. Now, let $\{e_1, \cdots, e_t\}$ be a linearly independent $V_r$-basis of $(\mathcal{G}|T)V_r$ that forms at the same time a linearly independent $A_r$-basis of $(\mathcal{G}|T)A_r$ (Lemma 3). By Theorem 1 (d), there holds then $\infty > [H[E, U \{T_{e_i}\}] : H] = [V : V_A(H[E, U \{T_{e_i}\}])]_r$, where $E$ is a system of matrix units such that $V_{A}(E)$ is a division ring. Hence, $[T:B]_l \geq [T:B]_l$ by Proposition 1.

(b) Again by the symmetry of our assumption, it suffices to prove that if $[T:B]_l < \infty$ then $[T:B]_l \leq [T:B]_l$. Since $U = V_A(T) = V_B(T) \subseteq V_A(T \text{Hom}_{A_r} (T, A)) \subseteq V$ (field) and $[V : U] \leq [T:B]_l < \infty$ by Lemma 2, our assertion is again a consequence of Proposition 1.

As a direct consequence of Theorem 3 (a), we obtain the following, that contains evidently [7, Corollary 2].

**Corollary 3.** Let a simple ring $A$ be $B$-$V$-$A$-irreducible and $A$-$B$-$V$-irreducible, and $T$ an intermediate ring of $A/B$. If $J(\mathcal{G}, A) = B$ and $A/H$
is left locally finite then \([T:B]_l = [T:B]_r\), provided we do not distinguish between two infinite dimensions.

**Remark 2.** Theorem 3 (a) may be regarded as an extension of [1, Folgerung zu Satz VII, 2]. However, [1] contains considerable errors: The definition of \(e\text{Hom}_{L}(M, N)\) is absurd, and Satz V, 1, Hilfssatz VI, 2 and Hilfssatz VI, 4 are open to doubt.

**References**


Okayama University
and
Hokkaido University

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