ON SOME PROPERTIES OF CERTAIN HYPERSURFACES
IN A K-SPACE

By

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Introduction. Recently Y. Tashiro [5] proved that an orientable hypersurface in an almost complex space has an almost contact structure and showed that the induced almost contact structure of the hypersurface in a Kählerian space is normal contact if and only if the second fundamental tensor of the hypersurface has the form $H_{ab} = q_{ab} + \mu \eta_a \eta_b$ [5].

The purpose of the present paper is to investigate some properties of a hypersurface with analogous conditions in a $K$-space. § 1 devoted to give the fundamental concepts of an almost Hermitian space, and we show some formulas concerning hypersurfaces in a $K$-space for the later use.

It is well-known that if the second fundamental tensor of a hypersurface in Euclidean space has the form $H_{ab} = q_{ab} + \mu \eta_a \eta_b$, then $\mu = 0$, that is, the hypersurface is totally umbilical [6], [7]. In § 2, we shall obtain the similar properties of such a hypersurface in the special $K$-space. In the last section we consider of the case that a hypersurface in a $K$-space admits the second fundamental tensor of more general form $H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b$, and we shall give properties of such a hypersurface in the special $K$-space.

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§ 1. Preliminaries. Let us consider a real $(2n + 2)$-dimensional almost Hermitian manifold $M^{2n+2}$ with local coordinate system $\{x^i\}$ and let $(F^i, g_{ij})$ be the almost Hermitian structure, that is, $F^i$ be the almost complex structure defined on $M^{2n+2}$ and $g_{ij}$ be the Riemannian metric tensor satisfying $g_{hk} = g_{ij} F^h_i F^j_k$. Then it follows that

$$F_{ij} = -F_{ji} \quad \text{and} \quad (F_{ij} = g_{ik} F^k_j)$$

A differentiable hypersurface $M^{2n+1}$ of $M^{2n+2}$ may be represented parametrically by the equation $X^i = X^i(u^a)^{2n+1}$. If we put

1) Numbers in brackets refer to the references at the end of the paper.
2) Throughout the present paper the Latin indices are supposed to run over the range $1, 2, \cdots, 2n+2$, and the Greek indices take the values $1, 2, \cdots, 2n+1$. 

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2. The second fundamental tensor of a hypersurface in a $K$-space admits the second fundamental tensor of more general form $H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b$, and we shall give properties of such a hypersurface in the special $K$-space.
$X^i = \frac{\partial x^i}{\partial u^\alpha}$,

$X^i$ span a tangent plane of $M^{2n+1}$ at each point, and the induced Riemannian metric tensor $g_{\alpha\beta}$ in $M^{2n+1}$ is given by

$$g_{\alpha\beta} = g_{ij}X^iX^j.$$  

Assuming that our hypersurface is orientable, we choose the unit normal vector $X^i$ to the hypersurface and put

$\varphi_{\beta}^\alpha = F_{j}^iX_i^\alpha X_{\beta}^j$,  
$\xi^\alpha = -F_{j}^iX_i^\alpha X^j$,  
$\eta_{\alpha} = F_{j}^iX_iX_a^j$,

where we put $X_i^\alpha = g^{\alpha\beta}q_{ij}X_{\beta}^j$ and $X_i = g_{ij}X^j$.

Then it is known that the quantities $\varphi_{\beta}^\alpha$, $\xi^\alpha$, $\eta_{\alpha}$ and $g_{\alpha\beta}$ satisfy the following conditions [3]:

$$\left\{\begin{array}{ll}
\xi^\alpha\eta_{\beta} = 1, & \text{rank } (\varphi_{\beta}^\alpha) = 2n, \\
\varphi_{\beta}^\alpha\xi^\beta = 0, & \varphi_{\beta}^\alpha\eta_{\beta} = 0, \\
\varphi_{\beta}^\alpha\varphi_{\gamma}^\beta = -\delta_{\gamma}^\alpha + \xi^\alpha\eta_{\gamma},
\end{array}\right.$$  

and

$$g_{\alpha\beta}\xi^\alpha = \eta_{\beta}, \quad g_{\alpha\beta}\varphi_{\gamma}^\alpha\varphi_{\delta}^\beta = g_{\gamma\delta} - \eta_{\gamma}\eta_{\delta}.$$  

Therefore we may consider the quantities $\varphi_{\beta}^\alpha$, $\xi^\alpha$, $\eta_{\alpha}$ and $g_{\alpha\beta}$ define an almost contact metric structure in $M^{2n+1}$. From (1.3) and (1.4) it follows that

$$\varphi_{\alpha\beta} = -\varphi_{\beta\alpha}. \quad (\varphi_{\alpha\beta} = g_{\alpha\gamma}\varphi_{\beta}^\gamma)$$

On making use of the Gauss equations we have from (1.2)

$$\left\{\begin{array}{l}
\varphi_{\beta\gamma} = F_{j,k}^i X_i^\alpha X_j^\beta X_k^\gamma + H^\gamma_{\beta\gamma} - H^\gamma_{\alpha\beta}, \\
\eta_{\gamma}^{\alpha\beta} = -F_{j,k}^i X_i^\gamma X_j^\alpha X_k^\beta + H^\gamma_{\beta\gamma},
\end{array}\right.$$  

where $H^\gamma_{\beta\gamma} = g^{\alpha\gamma}H_{\beta\gamma}$ and $H_{\beta\gamma}$ denotes the covariant component of the second fundamental tensor of $M^{2n+1}$.

Let $M^{2n+2}$ be a $K$-space, then we have the following condition [1]:

$$F_{\epsilon j,k} + F_{\epsilon k,j} = 0.$$  

If we put $F_{\epsilon j,k} X_i^\alpha X_j^\beta X_k^\gamma = A_{\alpha\beta\gamma}$, then by virtue of (1.1) and (1.7) we can find that the covariant tensor $A_{\alpha\beta\gamma}$ is skew-symmetric with respect to all indices, and it follows that if $A_{\alpha\beta\gamma} = 0$ holds good, then a $K$-space is necessarily a Kählerian space [10]. Moreover, after some calculations (1.6) become

3) In the present paper comma and semi-colon denotes covariant differentiation with respect to the Riemannian connection defined by $g_{ij}$ and its induced connection respectively.
(1.8) \[ \varphi_{a\beta;\gamma} = A_{a\beta\gamma} + H_{a\gamma}\eta^{\alpha} - H_{a\beta}\eta^{\alpha}, \]

(1.9) \[ \eta^{a}_{;\beta} = H\eta^{a}_{;\gamma} + A_{a\beta}\eta^{a\gamma}, \] [10].

Then for the Nijenhuis tensors of a hypersurface in a $K$-space we obtain the following expressions:

(1.10) \[ N_{a} = H_{\beta\gamma}\varphi^{\beta}_{a}, \]

(1.11) \[ N_{\beta\gamma} = 2A_{\alpha\beta\gamma}\eta^{\alpha} - 2A_{\alpha\delta}\eta^{\alpha}\varphi^{\delta}_{\gamma} - H_{a\gamma}\eta^{\alpha}\eta_{\beta} - H_{\alpha\beta}\eta^{\alpha}\eta_{\gamma}, \]

(1.12) \[ N_{\beta}^a = 2A_{\beta\gamma}^{a}\eta^{\gamma} + H_{\gamma}^{a}\eta^{\gamma}\eta_{\beta} - H_{\beta}^a - \varphi_{\beta}^{\gamma}\eta^{\gamma}_{;\tau}^{a}, \]

(1.13) \[ N_{\beta}^{a\gamma} = 2A_{a\beta}^{\delta}\varphi^{\delta}_{\gamma} - 2A_{\gamma\delta}^{a}\varphi^{\delta}_{\beta} + H_{a\gamma\delta}^{\delta} - H_{\delta}^{a}\eta^{\gamma}_{\beta} + \eta^{a}_{;\rho}^{\gamma}\eta^{\beta} - \eta^{a}_{;\gamma}^{\beta}\eta^{\gamma}_{\tau}. \]

§ 2. The hypersurface admitting the second fundamental tensor of the form $H_{a\beta} = g_{a\beta} + \mu\eta_{a}\eta_{\beta}$. In a $K$-space $M^{2n+2}$, the induced structure $\varphi_{ab}$ in a hypersurface $M^{2n+1}$ satisfies the relation

(2.1) \[ \varphi_{ab} = \eta_{a;\beta} + A_{ab}\gamma_{\beta}^{\gamma}, \]

if and only if the second fundamental tensor $H_{ab}$ of $M^{2n+1}$ has the form

(2.2) \[ H_{ab} = g_{ab} + \mu\eta_{a}\eta_{\beta}, \]

where $\mu$ is a scalar field in $M^{2n+1}$ [10].

Now, if we assume that the manifold $M^{2n+2}$ be a $K$-space with constant curvature, then the hypersurface satisfies the following Codazzi equation [2]:

(2.3) \[ H_{a\beta\gamma} - H_{\gamma\beta\alpha} = 0. \]

Substituting (2.2) into (2.3), we get

(2.4) \[ (\mu_{;\alpha})\eta_{a}\eta_{\beta} - (\mu_{;\beta})\eta_{\alpha}\eta_{\gamma} + \mu(\eta_{a;\beta}\eta_{\gamma} - \eta_{\gamma;\beta}\eta_{\alpha} + \eta_{\beta;\alpha}\eta_{\gamma} - \eta_{\gamma;\alpha}\eta_{\beta}) = 0. \]

Multiplying (2.4) by $\varphi^{a\gamma}\eta^{\beta}$ and summing for all indices, by virtue of (1.3) it follows that

(2.5) \[ \mu\cdot\varphi_{a\gamma}(\eta_{a;\gamma} - \eta_{\gamma;\alpha}) = 0. \]

By means of (2.1) and the skew-symmetric property of the tensor $A_{a\beta\gamma}$, we obtain $\mu = 0$. Thus we have the following theorem:

**Theorem 2.1.** If the second fundamental tensor $H_{ab}$ of a hypersurface in a $K$-space with constant curvature has the form

\[ H_{ab} = g_{ab} + \mu\eta_{a}\eta_{\beta}, \]

then $\mu = 0$ and the hypersurface is umbilical.

When a $K$-space is an Einstein space, we obtain the following theorem:

**Theorem 2.2.** If the second fundamental tensor $H_{ab}$ of a hypersurface
In an Einstein K-space is of the form

\[ H_{a\beta} = g_{a\beta} + \mu \eta_a \eta_\beta, \]

then \( \mu = \text{const.} \) and the mean curvature \( H \) of a hypersurface is constant.

**Proof.** Substituting (2.1) and (2.2) into the Codazzi equation [2]:

\[ (2.6) \quad H_{a\beta\gamma} - H_{\gamma \beta \alpha} = K_{ijkl} X_\gamma X_\alpha X_k X_l, \]

we obtain

\[ (\mu_{;\gamma}) \eta_a \eta_\beta - (\mu_{;\alpha}) \eta_\beta \eta_\gamma - \mu (2 \eta_\beta \varphi_{\alpha \gamma} + \eta_\alpha \varphi_{\beta \gamma} - \eta_\gamma \varphi_{\beta \alpha}) \]

\[ + \mu (A_{\delta \gamma} \eta \eta_\beta \varphi^\delta_\alpha + A_{\delta \gamma} \eta \eta_\alpha \varphi^\delta_\beta - A_{\delta \alpha} \eta \eta_\beta \varphi^\delta_\gamma - A_{\delta \alpha} \eta \eta_\gamma \varphi^\delta_\beta) \]

\[ = K_{ijkl} X_\gamma X_\alpha X_k X_l. \]

Since \( M^{2n+2} \) is an Einstein K-space, multiplying (2.7) by \( g^{a\beta} \) and summing for \( \alpha \) and \( \beta \), we get

\[ (\mu_{;\gamma}) \eta^a \eta_\gamma - (\mu_{;a}) \eta^{a} \eta_\gamma - \mu A_{\beta^\gamma \alpha} \eta^\alpha \eta_\gamma \varphi^\beta_\alpha = 0, \]

from which by multiplying (2.8) by \( \varphi^{\gamma \text{e}} \varphi_{*\lambda} \) and summing for \( \gamma \), we obtain

\[ (\mu_{;\gamma}) \eta^\gamma \eta_\lambda = \mu_{;\lambda}. \]

Moreover, differentiating (2.9) covariantly with respect to \( u^\alpha \) and multiplying by \( \varphi^{a\lambda} \) and summing for \( \lambda \), by virtue of (2.1) we have the following relation:

\[ (\mu_{;\gamma}) \eta^\gamma \eta_\lambda = 0. \]

Hence \( \mu \) is constant in \( M^{2n+1} \) from (2.9) and (2.10).

**§ 3. The hypersurface admitting the second fundamental tensor of the form** \( H_{a\beta} = \lambda g_{a\beta} + \mu \eta_a \eta_\beta \). In this section, we consider that a hypersurface of a K-space admits the second fundamental tensor \( H_{a\beta} \) of the form

\[ (3.1) \quad H_{a\beta} = \lambda g_{a\beta} + \mu \eta_a \eta_\beta, \]

where \( \lambda \) and \( \mu \) are scalar functions.

At the first, let us consider a relation of a hypersurface, which is equivalent to (3.1).

Substituting from (3.1) in the right hand member of (1.9), we have

\[ (3.2) \quad \eta_{a\beta} = \lambda \varphi_{a\beta} + A_{*\beta} \varphi^\beta_\alpha. \]

Next, we shall show that (3.2) is the sufficient condition that \( H_{a\beta} \) be of the form (3.1). Multiplying (3.2) by \( \eta^\beta \) and summing for \( \beta \), we get \( \eta_{a\beta} \eta^\beta = 0 \). From the last relation it follows that \( N_a = 0 \), then from (1.10) we obtain the relation

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for a suitable function $\rho$.

Moreover, by using (1.8) we have

$$(3.4)\quad \varphi_{\alpha \gamma ; \beta} + \varphi_{\beta \gamma ; \alpha} = 2H_{\alpha \beta} \eta_{\gamma} - H_{\beta \gamma} \eta_{\alpha} - H_{\alpha \gamma} \eta_{\beta}.$$

Multiplying (3.4) by $\eta^\gamma$ and summing for $\gamma$, by virtue of (3.3) it follows that

$$\frac{1}{2} (\varphi_{\alpha \gamma ; \beta} + \varphi_{\beta \gamma ; \alpha}) \eta^\gamma = H_{\alpha \beta} - \rho \eta_{\alpha} \eta_{\beta},$$

from which by virtue of (1.3) and (1.6)

$$(3.5)\quad H_{\alpha \beta} = \rho \eta_{\alpha} \eta_{\beta} + \frac{1}{2} (\varphi_{a \gamma ; \beta} + \varphi_{\beta \gamma ; \alpha}) \eta^\gamma.$$

Solving (3.2) for $\eta_{\alpha ; \beta}$ and substituting from the solution in the right hand member of (3.5), we obtain (3.1). Then we have the following theorem:

**Theorem 3.1.** The second fundamental tensor $H_{\alpha \beta}$ of a hypersurface in a K-space is of the form

$$H_{\alpha \beta} = \lambda g_{\alpha \beta} + \mu \eta_{\alpha} \eta_{\beta},$$

if and only if $\varphi_{a \beta}$ satisfies the relation

$$\eta_{\alpha ; \beta} = \lambda \varphi_{a \beta} + A_{a \beta} \eta \varphi^\alpha_{\beta}.$$

In particular, we have the following lemma in a constant curvature K-space:

**Lemma 3.2.** If the second fundamental tensor $H_{\alpha \beta}$ of a hypersurface in a K-space with constant curvature has the form

$$H_{\alpha \beta} = \lambda g_{\alpha \beta} + \mu \eta_{\alpha} \eta_{\beta},$$

then

$$\lambda \cdot \mu = 0.$$

**Proof.** Since $M^{2n+2}$ be a K-space with constant curvature, from (2.3) and (3.1), we get

$$(\lambda_{\gamma}) g_{\alpha \beta} - (\lambda_{\alpha} g_{\beta \gamma} + (\mu_{\gamma}) \eta_{\alpha} \eta_{\beta} - (\mu_{\alpha}) \eta_{\beta} \eta_{\gamma} + \mu (\eta_{\alpha \gamma} \eta_{\beta} + \eta_{\beta \gamma} \eta_{\alpha} - \eta_{\alpha \gamma} \eta_{\beta} - \eta_{\beta \gamma} \eta_{\alpha}) = 0 .$$

Multiplying this equation by $\varphi^{\alpha \gamma \beta}$ and summing for all indices, by virtue of (1.3) it follows that

$$(3.6)\quad \mu \cdot \varphi^{\alpha \gamma \beta} (\eta_{\alpha \gamma} \eta_{\beta} - \eta_{\beta \gamma} \eta_{\alpha}) = 0 .$$
Substituting (3.2) into (3.6), we have the conclusion.

**Theorem 3.3** If the second fundamental tensor $H_{ab}$ of a hypersurface in a $K$-space with constant curvature has the form

$$H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b ,$$

then the hypersurface must be reduced to one of the following three cases:

1. $H_{ab} = \mu \eta_a \eta_b$ for a suitable function $\mu$, i.e. rank $(H_{ab}) = 1$;
2. $H_{ab} = \lambda g_{ab}$ and $\lambda$ is constant, i.e. the hypersurface is umbilical;
3. $H_{ab} = 0$, i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.2, we must have one of the following three cases: $\lambda = 0, \mu = 0$ and $\lambda = \mu = 0$.

If $\lambda = 0$, then we have the case (1).
If $\mu = 0$, then the hypersurface is umbilical. Since in a hypersurface of constant curvature space the relation $H_{a;\gamma} - H_{\gamma;a} = 0$ is satisfied, from this relation and $H_{ab} = \lambda g_{ab}$, we see that $\lambda$ is constant. Then we have the case (2).
If $\lambda = \mu = 0$, then $H_{ab} = 0$, i.e. the hypersurface is totally geodesic.

When $M^{2n+2}$ be a space of constant curvature, from the Gauss equation [2]:

$$K_{a;\gamma\delta} = H_{a\gamma}H_{\delta\beta} - H_{a\beta}H_{\gamma\delta} + K_{ijkl}X_{\alpha}^{i}X_{\beta}^{j}X_{\gamma}^{k}X_{\delta}^{l} ,$$

we can easily obtain that the hypersurface is of constant curvature for above three cases. Thus we have the following corollary:

**Corollary 3.4.** If the second fundamental tensor $H_{ab}$ of a hypersurface in a $K$-space with constant curvature has the form

$$H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b ,$$

then the hypersurface is also of constant curvature.

Next, we assume that $M^{2n+2}$ be an Einstein $K$-space, then we have the following lemma:

**Lemma 3.5.** If the second fundamental tensor $H_{ab}$ of a hypersurface in an Einstein $K$-space has the form

$$H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b ,$$

then

$$(3.7) \quad \lambda(2n\lambda + \mu)_{;\gamma}\eta^\gamma = 0 .$$

**Proof.** Substituting (3.1) into (2.6) and multiplying by $g^{ab}$ and summing for $\alpha$ and $\beta$, we get

$$2n(\lambda_{;\gamma}) + (\mu_{;\gamma}) + (\mu_{;\alpha})\eta^\gamma \eta_\gamma + \mu(2\eta_{a;\gamma} \eta^a - \eta_{a;\gamma} \eta^a - \eta_{a;\gamma} \eta^a \eta_{a;\beta}) = K_{ijkl}X_{\gamma}^{i}X_{\delta}^{l} .$$
Since $M^{2n+2}$ is an Einstein $K$-space, we obtain by means of (3.2)
\[(2n\lambda + \mu)_{;r} + (\mu_{;a})\eta^a \eta_r - \mu \eta_r A_{ab} \eta^a \varphi^b = 0.\]
Multiplying the above equation by $\varphi^r \varphi_{ra}$ and summing for $r$, it follows that
\[(3.8) \quad (2n\lambda + \mu)_{;a} = (2n\lambda + \mu)_{;r} \eta^r \eta_a.\]
Differentiating (3.8) covariantly with respect to $u^a$ and multiplying by $\varphi^a$ and summing for $a$ and $\kappa$, we have the conclusion by means of (3.2).

**Theorem 3.6.** If the second fundamental tensor $H_{ab}$ of a hypersurface in an Einstein $K$-space has the form
\[H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b,\]
then the hypersurface must be reduced to one of the following four cases:

1. $H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b$, where $2n\lambda + \mu = \text{constant}$;
2. $H_{ab} = \lambda g_{ab}$ and $\lambda$ is constant, i.e. the hypersurface is umbilical;
3. $H_{ab} = \mu \eta_a \eta_b$ for a suitable function $\mu$, i.e. rank $(H_{ab}) = 1$;
4. $H_{ab} = 0$, i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.5, if $\lambda \neq 0$, then $(2n\lambda + \mu)_{;r} \eta^r = 0$. From this equation and (3.8), it follows that $2n\lambda + \mu = \text{constant}$. This is the case (1). Specially, if $\lambda \neq 0$ and $\mu = 0$, then $H_{ab} = \lambda g_{ab}$, where $\lambda$ is constant. Hence we have the case (2).

From (3.7) if $\lambda = 0$, then we have the case (3). Specially, if $\lambda = \mu = 0$, then $H_{ab} = 0$, that is, the hypersurface is totally geodesic.

**References**

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