ON SOME PROPERTIES OF CERTAIN HYPERSURFACES
IN A K-SPACE

By

Hidemaro KÔJYÔ

Introduction. Recently Y. Tashiro [5] proved that an orientable hypersurface in an almost complex space has an almost contact structure and showed that the induced almost contact structure of the hypersurface in a Kählerian space is normal contact if and only if the second fundamental tensor of the hypersurface has the form $H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta}$ [5].

The purpose of the present paper is to investigate some properties of a hypersurface with analogous conditions in a $K$-space. §1 devoted to give the fundamental concepts of an almost Hermitian space, and we show some formulas concerning hypersurfaces in a $K$-space for the later use.

It is well-known that if the second fundamental tensor of a hypersurface in Euclidean space has the form $H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta}$, then $\mu = 0$, that is, the hypersurface is totally umbilical [6], [7]. In §2, we shall obtain the similar properties of such a hypersurface in the special $K$-space. In the last section we consider of the case that a hypersurface in a $K$-space admits the second fundamental tensor of more general form $H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta}$, and we shall give properties of such a hypersurface in the special $K$-space.

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§1. Preliminaries. Let us consider a real $(2n+2)$-dimensional almost Hermitian manifold $M^{2n+2}$ with local coordinate system $\{x^{i}\}$ and let $(F_{ij}, g_{ij})$ be the almost Hermitian structure, that is, $F_{ij}$ be the almost complex structure defined on $M^{2n+2}$ and $g_{ij}$ be the Riemannian metric tensor satisfying $g_{hk} = g_{ij} F_{h}^{i} F_{k}^{j}$. Then it follows that

\begin{equation}
F_{ij} = -F_{ji}. \tag{1.1}
\end{equation}

A differentiable hypersurface $M^{2n+1}$ of $M^{2n+2}$ may be represented parametrically by the equation $X^{\ell} = X^{\ell}(\mu^{a})^{2n+1}$. If we put

1) Numbers is brackets refer to the references at the end of the paper.

2) Throughout the present paper the Latin indices are supposed to run over the range $1, 2, \ldots, 2n+2$, and the Greek indices take the values $1, 2, \ldots, 2n+1$. 

\[ F_{ij} = g_{ij} F^h_h \]
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\[ X^{i}_{a} = \frac{\partial x^{i}}{\partial u^{a}} \]

\( X^{i}_{a} \) span a tangent plane of \( M^{2n+1} \) at each point, and the induced Riemannian metric tensor \( g_{\alpha\beta} \) in \( M^{2n+1} \) is given by

\[ g_{\alpha\beta} = g_{ij} X^{i}_{a} X^{j}_{\beta} \]

Assuming that our hypersurface is orientable, we choose the unit normal vector \( X^{i} \) to the hypersurface and put

\[ \varphi_{\beta}^{a} = F_{j}^{i} X_{i}^{a} X_{\beta}^{j}, \quad \xi^{a} = -F_{j}^{i} X_{i}^{a} X^{j}, \quad \eta_{\alpha} = F_{j}^{i} X_{i}^{a} X^{j} \]

where we put \( X_{i}^{a} = g^{a\beta} q_{ij} X_{\beta}^{j} \) and \( X_{i} = g_{ij} X^{j} \).

Then it is known that the quantities \( \varphi_{\beta}^{a}, \xi^{a}, \eta_{\alpha} \) and \( g_{\alpha\beta} \) satisfy the following conditions [3]:

\[ \left\{ \begin{array}{l} \xi^{a} \eta_{a} = 1, \quad \text{rank} (\varphi_{\beta}^{a}) = 2n, \\ \varphi_{\beta}^{a} \xi^{a} = 0, \quad \varphi_{\beta} \eta_{a} = 0, \\ \varphi_{\beta} \varphi_{a} = -\delta_{a}^{\tau} + \xi_{\tau}, \end{array} \right. \]

and

\[ g_{a\beta} \xi^{a} = \eta_{\beta}, \quad g_{a\beta} \varphi_{\tau}^{a} \varphi_{\rho}^{a} = g_{\tau\rho} - \eta_{\tau} \eta_{\rho}. \]

Therefore we may consider the quantities \( \varphi_{\beta}^{a}, \xi^{a}, \eta_{\alpha} \) and \( g_{\alpha\beta} \) define an almost contact metric structure in \( M^{2n+1} \). From (1.3) and (1.4) it follows that

\[ \varphi_{a\beta} = -\varphi_{\beta a}. \quad (\varphi_{a\beta} = g_{a\gamma} \varphi_{\beta}^{\gamma}) \]

On making use of the Gauss equations we have from (1.2)

\[ \left\{ \begin{array}{l} \varphi_{\beta}^{a} = F_{j,k} X_{i}^{a} X_{\beta}^{j} + H_{j}^{a} \eta_{\beta} - H_{\beta}^{a} \varphi_{\tau}^{a}, \\ \eta_{\beta}^{i} = -F_{j,k} X_{i}^{a} X^{j} + H_{j}^{a} \varphi_{\tau}^{a}, \end{array} \right. \]

where \( H_{\beta}^{a} = g^{\sigma} H_{\beta}^{\sigma} \) and \( H_{\beta}^{a} \) denotes the covariant component of the second fundamental tensor of \( M^{2n+1} \).

Let \( M^{2n+2} \) be a K-space, then we have the following condition [1]:

\[ F_{ij,k} + F_{ik,j} = 0. \]

If we put \( F_{ij,k} X_{i}^{a} X^{j} = A_{a\beta} \), then by virtue of (1.1) and (1.7) we can find that the covariant tensor \( A_{a\beta} \) is skew-symmetric with respect to all indices, and it follows that if \( A_{a\beta} = 0 \) holds good, then a K-space is necessarily a Kählerian space [10]. Moreover, after some calculations (1.6) become

\[ F_{ij,k} + F_{ik,j} = 0. \]

3) In the present paper comma and semi-colon denotes covariant differentiation with respect to the Riemannian connection defined by \( g_{ij} \) and its induced connection respectively.
\( \varphi_{a;\beta} = A_{a\beta r} + H_{a\beta} \eta_{r} - H_{\beta r} \eta_{a} , \)

\( \eta^{a}_{;\beta} = H^{a}_{\beta \rho} \varphi^{a}_{\gamma} + A^{a}_{\delta \beta} \eta^{\delta} \varphi^{a}_{\gamma} , \) [10].

Then for the Nijenhuis tensors of a hypersurface in a \( K \)-space we obtain the following expressions:

\( N_{a} = H_{\beta \gamma} \varphi^{a}_{\beta} \eta^{\gamma} , \)

\( N_{\alpha \beta} = 2 A_{\alpha \beta \gamma} \eta^{\gamma} - 2 A_{\alpha \delta} \eta^{\gamma} \varphi^{\beta}_{\delta} \varphi^{\gamma}_{\delta} + H^{a}_{\gamma} \eta^{\alpha} \eta_{\beta} - H^{a}_{\alpha} \eta^{\beta} \eta_{\gamma} , \)

\( N_{\beta}^{\alpha} = 2 A^{\alpha}_{\beta \gamma} \eta^{\gamma} + H^{\alpha}_{\gamma} \eta^{\gamma} \eta_{\beta} - H^{\alpha}_{\beta} - \varphi^{\gamma}_{\beta} \eta^{\gamma}_{;\alpha} , \)

\( N^{a}_{\beta \gamma} = 2 A^{a}_{\beta \delta} \varphi^{\delta}_{\gamma} - 2 A^{a}_{\gamma \delta} \varphi^{\delta}_{\beta} + H^{a}_{\gamma} \eta_{\delta} \varphi^{a}_{\delta} + \eta^{a}_{;\rho} \eta_{\beta} - \eta^{a}_{;\gamma} \eta_{\beta} . \)

§ 2. The hypersurface admitting the second fundamental tensor of the form \( H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta} . \) In a \( K \)-space \( M^{2n+2} \), the induced structure \( \varphi_{a\beta} \) in a hypersurface \( M^{2n+1} \) satisfies the relation

\( \varphi_{a\beta} = \eta_{a;\beta} + A_{\gamma \delta \beta} \eta^{\delta} \varphi^{a}_{\gamma} , \)

if and only if the second fundamental tensor \( H_{a\beta} \) of \( M^{2n+1} \) has the form

\( H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta} , \)

where \( \mu \) is a scalar field in \( M^{2n+1} \) [10].

Now, if we assume that the manifold \( M^{2n+2} \) be a \( K \)-space with constant curvature, then the hypersurface satisfies the following Codazzi equation [2]:

\( H_{a\beta \gamma} - H_{\gamma \beta \alpha} = 0 . \)

Substituting (2.2) into (2.3), we get

\( (\mu_{\gamma}) \eta_{a \gamma \beta} - (\mu_{\alpha}) \eta_{\beta \alpha} \gamma_{\beta} + \mu (\eta_{a \gamma} \gamma_{\beta} - \eta_{\gamma \alpha} \gamma_{\beta} + \eta_{\beta \alpha} \gamma_{\gamma} - \eta_{\beta \gamma} \gamma_{\alpha}) = 0 . \)

Multiplying (2.4) by \( \varphi^{a\alpha} \gamma^{\beta} \) and summing for all indices, by virtue of (1.3) it follows that

\( \mu \cdot \varphi^{a\gamma} (\eta_{a \gamma} - \eta_{\gamma a}) = 0 . \)

By means of (2.1) and the skew-symmetric property of the tensor \( A_{a\beta \gamma} \), we obtain \( \mu = 0 \). Thus we have the following theorem:

**Theorem 2.1.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface in a \( K \)-space with constant curvature has the form

\( H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta} , \)

then \( \mu = 0 \) and the hypersurface is umbilical.

When a \( K \)-space is an Einstein space, we obtain the following theorem:

**Theorem 2.2.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface
in an Einstein K-space is of the form
\[ H_{a\beta} = g_{a\beta} + \mu \eta_{a} \eta_{\beta}, \]
then \( \mu = \text{const.} \) and the mean curvature \( H \) of a hypersurface is constant.

**Proof.** Substituting (2.1) and (2.2) into the Codazzi equation [2]:
\[ (2.6) \quad H_{a\beta;\gamma} - H_{\gamma\beta;\alpha} = K_{ijkl} X_{\gamma}^{i} X_{a}^{j} X_{\beta}^{k} X^{l}, \]
we obtain
\[ (2.7) \quad (\mu_{;\gamma}) \eta_{a} \eta_{\beta} - (\mu_{;\alpha}) \eta_{\beta} \eta_{\gamma} - \mu (2 \eta_{\beta} \varphi_{\alpha\gamma} + \eta_{\alpha} \varphi_{\beta\gamma} - \eta_{\gamma} \varphi_{\beta a}) + \mu (A_{\delta\gamma} \eta_{a} \eta_{\beta} \varphi_{\delta}^{*} + A_{\delta\gamma} \eta_{a} \eta_{\beta} \varphi_{\delta} - A_{\delta\alpha} \eta_{\beta} \varphi_{\gamma}^{*} - A_{\delta\alpha} \eta_{\gamma} \varphi_{\beta}^{*}) = K_{ijkl} X_{\gamma}^{i} X_{a}^{j} X_{\beta}^{k} X^{l}. \]

Since \( M^{2n+2} \) is an Einstein K-space, multiplying (2.7) by \( g^{a\beta} \) and summing for \( \alpha \) and \( \beta \), we get
\[ (2.8) \quad (\mu_{;\gamma}) - (\mu_{;\alpha}) \eta^{\alpha} \eta_{\gamma} - \mu A_{\beta^\alpha \gamma} \eta^{\beta} \varphi_{\alpha}^{*} \eta_{\gamma} = 0, \]
from which by multiplying (2.8) by \( \varphi^{*} \varphi_{\gamma}^{*} \) and summing for \( \gamma \), we obtain
\[ (2.9) \quad (\mu_{;\gamma}) \varphi^{*} \varphi_{\gamma} = \mu_{;\gamma}. \]
Moreover, differentiating (2.9) covariantly with respect to \( u^{*} \) and multiplying by \( \varphi_{\alpha}^{*} \) and summing for \( \alpha \), by virtue of (2.1) we have the following relation:
\[ (2.10) \quad (\mu_{;\gamma}) \varphi_{\gamma}^{*} \varphi_{\alpha} = 0. \]
Hence \( \mu \) is constant in \( M^{2n+1} \) from (2.9) and (2.10).

§ 3. The hypersurface admitting the second fundamental tensor of the form \( H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta} \). In this section, we consider that a hypersurface of a K-space admits the second fundamental tensor \( H_{a\beta} \) of the form
\[ (3.1) \quad H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta}, \]
where \( \lambda \) and \( \mu \) are scalar functions.

At the first, let us consider a relation of a hypersurface, which is equivalent to (3.1).

Substituting from (3.1) in the right hand member of (1.9), we have
\[ (3.2) \quad \eta_{a\beta} = \lambda \varphi_{a\beta} + A_{a\beta} \varphi_{\gamma}^{*} \varphi_{\gamma}. \]
Next, we shall show that (3.2) is the sufficient condition that \( H_{a\beta} \) be of the form (3.1). Multiplying (3.2) by \( \eta^{a} \) and summing for \( \beta \), we get \( \eta_{a\beta} \eta^{a} = 0. \)

From the last relation it follows that \( N_{a} = 0 \), then from (1.10) we obtain the relation...
(3.3) $H_{a\beta} = \rho \eta_{a}$

for a suitable function $\rho$.

Moreover, by using (1.8) we have

(3.4) $\varphi_{a;r;\beta} + \varphi_{r;i;\alpha} = 2H_{a;\beta} - H_{\alpha;\beta} - H_{a;r;\beta}$.

Multiplying (3.4) by $\eta_{r}$ and summing for $r$, by virtue of (3.3) it follows that

$$ \frac{1}{2} (\varphi_{a;r;\beta} + \varphi_{r;i;\alpha}) \eta_{r} = H_{a;\beta} - \rho \eta_{a} \eta_{\beta}, $$

from which by virtue of (1.3) and (1.6)

(3.5) $H_{a;\beta} = \rho \eta_{a} \eta_{\beta} + \frac{1}{2} (\varphi_{a;\gamma;\beta} + \varphi_{\beta;\gamma;\alpha})$.

Solving (3.2) for $\eta_{a;\beta}$ and substituting from the solution in the right hand member of (3.5), we obtain (3.1). Then we have the following theorem:

**Theorem 3.1.** The second fundamental tensor $H_{a;\beta}$ of a hypersurface in a K-space is of the form

$$ H_{a;\beta} = \lambda g_{a;\beta} + \mu \eta_{a} \eta_{\beta}, $$

if and only if $\varphi_{a;\beta}$ satisfies the relation

$$ \eta_{a;\beta} = \lambda \varphi_{a;\beta} + A_{a;\beta} \eta_{\beta} \varphi_{a}^{\beta}. $$

In particular, we have the following lemma in a constant curvature K-space:

**Lemma 3.2.** If the second fundamental tensor $H_{a;\beta}$ of a hypersurface in a K-space with constant curvature has the form

$$ H_{a;\beta} = \lambda g_{a;\beta} + \mu \eta_{a} \eta_{\beta}, $$

then

$$ \lambda \cdot \mu = 0. $$

**Proof.** Since $M^{2n+2}$ be a K-space with constant curvature, from (2.3) and (3.1), we get

$$ (\lambda_{\gamma}) g_{a;\beta} - (\mu_{\gamma}) g_{a;\beta} + (\mu_{\gamma}) \eta_{a} \eta_{\beta} - (\lambda_{\gamma}) \eta_{a} \eta_{\beta} $$

$$ + \mu (\eta_{a;\gamma;\beta} + \eta_{\gamma;\beta} - \eta_{\beta;\gamma} - \eta_{\gamma;\beta}) = 0. $$

Multiplying this equation by $\varphi^{\alpha;\gamma} \eta_{\beta}$ and summing for all indices, by virtue of (1.3) it follows that

(3.6) $\mu \cdot \varphi^{\alpha;\gamma} (\eta_{a;\gamma} - \eta_{\gamma;a}) = 0.$
Substituting (3.2) into (3.6), we have the conclusion.

**Theorem 3.3** If the second fundamental tensor $H_{a\beta}$ of a hypersurface in a K-space with constant curvature has the form

$$H_{a\beta} = \lambda g_{a\beta} + \mu \eta_a \eta_\beta,$$

then the hypersurface must be reduced to one of the following three cases:

1. $H_{a\beta} = \mu \eta_a \eta_\beta$ for a suitable function $\mu$, i.e. rank $(H_{a\beta}) = 1$;
2. $H_{a\beta} = \lambda g_{a\beta}$ and $\lambda$ is constant, i.e. the hypersurface is umbilical;
3. $H_{a\beta} = 0$, i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.2, we must have one of the following three cases: $\lambda=0$, $\mu=0$ and $\lambda=\mu=0$.

If $\lambda=0$, then we have the case (1).

If $\mu=0$, then the hypersurface is umbilical. Since in a hypersurface of constant curvature space the relation $H_{a\beta c}-H_{\beta c a}=0$ is satisfied, from this relation and $H_{a\beta}=\lambda g_{a\beta}$, we see that $\lambda$ is constant. Then we have the case (2).

If $\lambda=\mu=0$, then $H_{a\beta}=0$, i.e. the hypersurface is totally geodesic.

When $M^{2n+2}$ be a space of constant curvature, from the Gauss equation [2]:

$$K_{a\beta c d} = H_{a c} H_{\beta d} - H_{a d} H_{\beta c} + K_{ijkl} X_{\alpha}^{i} X_{\beta}^{j} X_{\gamma}^{k} X_{\delta}^{l},$$

we can easily obtain that the hypersurface is of constant curvature for above three cases. Thus we have the following corollary:

**Corollary 3.4.** If the second fundamental tensor $H_{a\beta}$ of a hypersurface in a K-space with constant curvature has the form

$$H_{a\beta} = \lambda g_{a\beta} + \mu \eta_a \eta_\beta,$$

then the hypersurface is also of constant curvature.

Next, we assume that $M^{2n+2}$ be an Einstein K-space, then we have the following lemma:

**Lemma 3.5.** If the second fundamental tensor $H_{a\beta}$ of a hypersurface in an Einstein K-space has the form

$$H_{a\beta} = \lambda g_{a\beta} + \mu \eta_a \eta_\beta,$$

then

$$\lambda(2n\lambda + \mu)_{;\gamma} \eta^\gamma = 0 .$$

**Proof.** Substituting (3.1) into (2.6) and multiplying by $g^{a\beta}$ and summing for $\alpha$ and $\beta$, we get

$$2n(\lambda_{;\gamma}) + (\mu_{;\gamma}) + (\mu_{,\gamma}) \eta_\gamma \eta_\delta + \mu(2\eta_{a\gamma} \eta^\gamma - \eta_{a\gamma} \eta^\gamma - \eta_a \eta^\gamma \eta_{\gamma\delta}) = K_{il} X_i^\gamma X_l^\gamma .$$
Since $M^{2n+2}$ is an Einstein $K$-space, we obtain by means of (3.2)
\[(2n\lambda + \mu)_{;\gamma} + (\mu_{;\alpha})\eta^\alpha \eta_\gamma - \mu \eta_\gamma A_{;\alpha} \eta^\alpha \varphi^\delta = 0\,.
\]
Multiplying the above equation by $\varphi^{\gamma \beta} \varphi_{\beta \iota}$ and summing for $\gamma$, it follows that
\[(3.8) \quad (2n\lambda + \mu)_{;\iota} = (2n\lambda + \mu)_{;\gamma} \eta^\gamma \eta_\iota\,.
\]
Differentiating (3.8) covariantly with respect to $u^\alpha$ and multiplying by $\varphi^a$ and summing for $\alpha$ and $\kappa$, we have the conclusion by means of (3.2).

**Theorem 3.6.** If the second fundamental tensor $H_{ab}$ of a hypersurface in an Einstein $K$-space has the form

\[H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b\,;
\]
then the hypersurface must be reduced to one of the following four cases:

1. $H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b$, where $2n\lambda + \mu =$ constant;
2. $H_{ab} = \lambda g_{ab}$ and $\lambda$ is constant, i.e. the hypersurface is umbilical;
3. $H_{ab} = \mu \eta_a \eta_b$ for a suitable function $\mu$, i.e. rank $(H_{ab}) = 1$;
4. $H_{ab} = 0$, i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.5, if $\lambda \neq 0$, then $(2n\lambda + \mu)_{;\gamma} \eta^\gamma = 0$. From this equation and (3.8), it follows that $2n\lambda + \mu =$ constant. This is the case (1). Specially, if $\lambda \neq 0$ and $\mu = 0$, then $H_{ab} = \lambda g_{ab}$, where $\lambda$ is constant. Hence we have the case (2).

From (3.7) if $\lambda = 0$, then we have the case (3). Specially, if $\lambda = \mu = 0$, then $H_{ab} = 0$, that is, the hypersurface is totally geodesic.

**References**


Department of Mathematics,
Hokkaido University

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