<table>
<thead>
<tr>
<th>Title</th>
<th>On Some Properties of Certain Hypersurfaces in a K-Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kôjyô, Hidemaro</td>
</tr>
<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University. Ser. 1 Mathematics, 19(3-4): 154-161</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1966</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/56079">http://hdl.handle.net/2115/56079</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
<tr>
<td>File Information</td>
<td>JFSH1U_19_N3-4_154-161.pdf</td>
</tr>
</tbody>
</table>

**Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP**
ON SOME PROPERTIES OF CERTAIN HYPERSURFACES
IN A K-SPACE

By

Hidemaro KÔJYÔ

Introduction. Recently Y. Tashiro [5] proved that an orientable hypersurface in an almost complex space has an almost contact structure and showed that the induced almost contact structure of the hypersurface in a Kählerian space is normal contact if and only if the second fundamental tensor of the hypersurface has the form \( H_{ab} = g_{ab} + \mu \eta_a \eta_b \) [5].

The purpose of the present paper is to investigate some properties of a hypersurface with analogous conditions in a K-space. §1 devoted to give the fundamental concepts of an almost Hermitian space, and we show some formulas concerning hypersurfaces in a K-space for the later use.

It is well-known that if the second fundamental tensor of a hypersurface in Euclidean space has the form \( H_{ab} = g_{ab} + \mu \eta_a \eta_b \), then \( \mu = 0 \), that is, the hypersurface is totally umbilical [6], [7]. In §2, we shall obtain the similar properties of such a hypersurface in the special K-space. In the last section we consider of the case that a hypersurface in a K-space admits the second fundamental tensor of more general form \( H_{ab} = \lambda g_{ab} + \mu \eta_a \eta_b \), and we shall give properties of such a hypersurface in the special K-space.

The author likes to express his sincere thanks to Dr. Y. Katsurada and Mr. T. Nagai who gave me many valuable suggestions and constant guidances.

§1. Preliminaries. Let us consider a real \((2n+2)\)-dimensional almost Hermitian manifold \( M^{2n+2} \) with local coordinate system \( \{x^i\} \) and let \( (F^i_j, g_{ij}) \) be the almost Hermitian structure, that is, \( F^i_j \) be the almost complex structure defined on \( M^{2n+2} \) and \( g_{ij} \) be the Riemannian metric tensor satisfying \( g_{hk} = g_{ij} F^i_h F^j_k \). Then it follows that

\[ F_{ij} = - F_{ji}, \quad (F_{ij} = g_{ik} F^k_j) \]

A differentiable hypersurface \( M^{2n+1} \) of \( M^{2n+2} \) may be represented parametrically by the equation \( X^i = X^i(\mu^a)^{2n} \). If we put

---

1) Numbers in brackets refer to the references at the end of the paper.

2) Throughout the present paper the Latin indices are supposed to run over the range 1, 2, ..., 2n+2, and the Greek indices take the values 1, 2, ..., 2n+1.
On Some Properties of Certain Hypersurfaces in a K-Space

\[ X_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}, \]

\( X_\alpha^i \) span a tangent plane of \( M^{2n+1} \) at each point, and the induced Riemannian metric tensor \( g_{\alpha\beta} \) in \( M^{2n+1} \) is given by

\[ g_{\alpha\beta} = g_{ij}X_\beta^i X_\beta^j. \]

Assuming that our hypersurface is orientable, we choose the unit normal vector \( X^i \) to the hypersurface and put

\[ \varphi_\beta^\alpha = F_{j}^{i}X_{i}^{a}X_{\beta}^{j}, \quad \xi^\alpha = -F_{j}^{i}X_{i}^{a}X^{j}, \quad \eta_\alpha = F_{j}^{i}X_{i}X_{a}^{j}, \]

where we put \( X_{i}^{a} = g^{a\beta}q_{ij}X_{\beta}^{j} \) and \( X_{i} = g_{ij}X^{j} \).

Then it is known that the quantities \( \varphi_\beta^\alpha, \xi^\alpha, \eta_\alpha \) and \( q_{a\beta} \) satisfy the following conditions [3]:

\begin{align*}
\xi^\alpha \eta_\alpha & = 1, & \text{rank } (\varphi_\beta^\alpha) & = 2n, \\
\varphi_\beta^\alpha \xi^\beta & = 0, & \varphi^\alpha \eta_\alpha & = 0, \\
\varphi^\alpha \varphi_\beta^\gamma & = -\delta_\gamma^\alpha + \xi^\alpha \eta_\gamma,
\end{align*}

and

\[ g_{\alpha\beta} \xi^\alpha = \eta_\beta, \quad g_{\alpha\beta} \varphi^\alpha \varphi_\beta^\gamma = g_{\gamma\delta} - \eta_\gamma \eta_\delta. \]

Therefore we may consider the quantities \( \varphi_\beta^\alpha, \xi^\alpha, \eta_\alpha \) and \( g_{\alpha\beta} \) define an almost contact metric structure in \( M^{2n+1} \). From (1.3) and (1.4) it follows that

\[ \varphi_{a\beta} = -\varphi_{\beta a}. \quad (\varphi_{a\beta} = g_{a\gamma} \varphi^\gamma \varphi_\beta^\gamma) \]

On making use of the Gauss equations we have from (1.2)

\begin{align*}
\varphi_{\beta i} &= F_{j,k}^i X_{i}^\alpha X_{\beta}^j X_{\gamma}^k + H^\gamma_{\beta} \eta_\gamma - H_\beta \eta_\alpha, \\
\eta^\alpha_{\beta i} &= -F_{j,k}^i X_{i}^\gamma X_{\beta}^j X_{\gamma}^k + H^\alpha_{\beta} \varphi_\beta^\gamma,
\end{align*}

where \( H^\alpha_{\beta} = g^{\alpha\gamma}H_{\beta\gamma} \) and \( H_{\beta\gamma} \) denotes the covariant component of the second fundamental tensor of \( M^{2n+1} \).

Let \( M^{2n+2} \) be a K-space, then we have the following condition [1]:

\[ F_{\epsilon j,k} + F_{\epsilon k,j} = 0. \]

If we put \( F_{\epsilon j,k} X_{\alpha}^i X_{\beta}^j X_{\gamma}^k = A_{\alpha\beta\gamma} \), then by virtue of (1.1) and (1.7) we can find that the covariant tensor \( A_{\alpha\beta\gamma} \) is skew-symmetric with respect to all indices, and it follows that if \( A_{\alpha\beta\gamma} = 0 \) holds good, then a K-space is necessarily a Kählerian space [10]. Moreover, after some calculations (1.6) become

3) In the present paper comma and semi-colon denotes covariant differentiation with respect to the Riemannian connection defined by \( g_{ij} \) and its induced connection respectively.
\[ \varphi_{a\beta;\gamma} = A_{a\beta\gamma} + H_{a\gamma} \eta^a - H_{a\beta} \eta^a, \]
\[ \eta^a;\beta = H^a_{\beta} \phi^a_\gamma + A_{a\beta} \eta^a \phi^a_\gamma, \]

Then for the Nijenhuis tensors of a hypersurface in a \( K \)-space we obtain the following expressions:

\[ N_a = H_{\beta\gamma} \phi^a_\beta \eta^\gamma, \]
\[ N_{\beta\gamma} = 2 A_{\alpha\beta\gamma} \eta^a - 2 A_{\alpha\delta\gamma} \eta^a \phi^\delta_\beta \phi^\gamma_\delta + H_{a\gamma} \eta^a \eta^\beta - H_{a\beta} \eta^a \eta^\gamma, \]
\[ N^a_\beta = 2 A^a_\beta \phi^a_\gamma - 2 A^a_\gamma \phi^a_\delta + H^\gamma_{\alpha\beta} \phi^\gamma_\delta - H^\delta_{\alpha\beta} \phi^\delta_\gamma + \eta^a_\alpha \phi^a_\beta - \eta^a_\beta \phi^a_\gamma, \]

**§ 2.** The hypersurface admitting the second fundamental tensor of the form \( H_{a\beta} = g_{a\beta} + \mu \eta_a \eta_\beta \). In a \( K \)-space \( M^{2n+2} \), the induced structure \( \varphi_{a\beta} \) in a hypersurface \( M^{2n+1} \) satisfies the relation

\[ \varphi_{a\beta} = \eta_{a\beta} + A_{\gamma\delta\beta} \eta^\gamma \phi^\delta_\alpha, \]

if and only if the second fundamental tensor \( H_{a\beta} \) of \( M^{2n+1} \) has the form

\[ H_{a\beta} = g_{a\beta} + \mu_\alpha \phi^\alpha_\beta, \]

where \( \mu \) is a scalar field in \( M^{2n+1} \) [10].

Now, if we assume that the manifold \( M^{2n+2} \) be a \( K \)-space with constant curvature, then the hypersurface satisfies the following Codazzi equation [2]:

\[ H_{a\beta;\gamma} - H_{\gamma\beta;\alpha} = 0. \]

Substituting (2.2) into (2.3), we get

\[ (\mu_\gamma \eta_{a\beta} - (\mu_\gamma \eta_{a\beta}) - \mu_\gamma \eta_{a\beta} + \mu_\gamma \eta_{a\beta} - \mu_\gamma \eta_{a\beta}) = 0. \]

Multiplying (2.4) by \( \phi^{a\gamma} \eta^\gamma \) and summing for all indices, by virtue of (1.3) it follows that

\[ \mu \cdot \phi^{a\gamma} (\eta_{a\gamma} - \eta_{\alpha\gamma}) = 0. \]

By means of (2.1) and the skew-symmetric property of the tensor \( A_{a\beta} \), we obtain \( \mu = 0. \) Thus we have the following theorem:

**Theorem 2.1.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface in a \( K \)-space with constant curvature has the form

\[ H_{a\beta} = g_{a\beta} + \mu \eta_a \eta_\beta, \]

then \( \mu = 0 \) and the hypersurface is umbilical.

When a \( K \)-space is an Einstein space, we obtain the following theorem:

**Theorem 2.2.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface
in an Einstein K-space is of the form

\[ H_{a\beta} = g_{a\beta} + \mu \eta_{a}\eta_{\beta}, \]

then \( \mu = \text{const.} \) and the mean curvature \( H \) of a hypersurface is constant.

**Proof.** Substituting (2.1) and (2.2) into the Codazzi equation [2]:

\[ H_{a\alpha\gamma} - H_{\alpha\gamma a} = K_{ijkl}X_{\gamma}^{i}X_{a}^{j}X_{\beta}^{k}X^{l}, \]

we obtain

\[ \begin{align*}
(\mu_{\alpha}) \eta_{a}\eta_{\beta} - (\mu_{\beta}) \eta_{a}\eta_{\gamma} - \mu (2\eta_{\beta} \varphi_{\alpha\gamma} + \eta_{\alpha} \varphi_{\beta\gamma} - \eta_{\gamma} \varphi_{\beta a}) \\
+ \mu (A_{\delta\gamma} \eta_{a} \varphi_{\beta}^{\delta} + A_{\delta\gamma} \eta_{\alpha} \varphi_{\beta}^{\delta} - A_{\delta\alpha} \eta_{a} \varphi_{\gamma}^{\delta} - A_{\delta\alpha} \eta_{\gamma} \varphi_{\beta}^{\delta}) \\
= K_{ijkl}X_{\gamma}^{i}X_{a}^{j}X_{\beta}^{k}X^{l}.
\end{align*} \]

Since \( M^{2n+2} \) is an Einstein K-space, multiplying (2.7) by \( g^{a\beta} \) and summing for \( \alpha \) and \( \beta \), we get

\[ (\mu_{\alpha}) - (\mu_{\beta}) \eta^{\alpha} \eta_{\beta} - \mu A_{\alpha\beta} \eta_{\alpha} \varphi_{\beta}^{\gamma} = 0, \]

from which by multiplying (2.8) by \( \varphi^{\gamma} \varphi_{\alpha} \) and summing for \( \gamma \), we obtain

\[ (\mu_{\alpha}) \eta^{\gamma} \eta_{\alpha} = \mu_{\alpha}. \]

Moreover, differentiating (2.9) covariantly with respect to \( u^{\alpha} \) and multiplying by \( \varphi^{\alpha\lambda} \) and summing for \( \lambda \), by virtue of (2.1) we have the following relation:

\[ (\mu_{\alpha}) \eta^{\gamma} \eta_{\lambda} = 0. \]

Hence \( \mu \) is constant in \( M^{2n+1} \) from (2.9) and (2.10).

**§ 3. The hypersurface admitting the second fundamental tensor of the form \( H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a}\eta_{\beta} \).** In this section, we consider that a hypersurface of a K-space admits the second fundamental tensor \( H_{a\beta} \) of the form

\[ H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a}\eta_{\beta}. \]

where \( \lambda \) and \( \mu \) are scalar functions.

At the first, let us consider a relation of a hypersurface, which is equivalent to (3.1).

Substituting from (3.1) in the right hand member of (1.9), we have

\[ \eta_{a\beta} = \lambda \varphi_{a\beta} + A_{a\beta} \varphi^{\gamma} \varphi_{\gamma}^{\beta}. \]

Next, we shall show that (3.2) is the sufficient condition that \( H_{a\beta} \) be of the form (3.1). Multiplying (3.2) by \( \eta^{\gamma} \) and summing for \( \beta \), we get \( \eta_{a\beta} \varphi^{\gamma} = 0. \)

From the last relation it follows that \( N_{a} = 0 \), then from (1.10) we obtain the relation
\[ H_{a\beta} = \rho \eta_{a} \]

for a suitable function \( \rho \).

Moreover, by using (1.8) we have

\[ \varphi_{a\beta;}^\gamma + \varphi_{\beta;\gamma}^a = 2H_{a\beta} \eta_{\gamma} - H_{\beta\gamma} \eta_{a} - H_{a\gamma} \eta_{\beta} . \]

Multiplying (3.4) by \( \eta^\gamma \) and summing for \( \gamma \), by virtue of (3.3) it follows that

\[ \frac{1}{2} (\varphi_{a\beta;}^\gamma + \varphi_{\beta;\gamma}^a) \eta_{\gamma} = H_{a\beta} - \rho \eta_{a} \eta_{\beta} , \]

from which by virtue of (1.3) and (1.6)

\[ H_{a\beta} = \rho \eta_{a} \eta_{\beta} + \frac{1}{2} (\varphi_{a\gamma;\beta}^\gamma + \varphi_{\beta;\gamma}^a) . \]

Solving (3.2) for \( \eta_{a;\beta} \) and substituting from the solution in the right hand member of (3.5), we obtain (3.1). Then we have the following theorem:

**Theorem 3.1.** The second fundamental tensor \( H_{a\beta} \) of a hypersurface in a K-space is of the form

\[ H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta} , \]

if and only if \( \varphi_{a\beta} \) satisfies the relation

\[ \eta_{a;\beta} = \lambda \varphi_{a\beta} + A_{\delta\beta} \eta \varphi_{\alpha}^\delta . \]

In particular, we have the following lemma in a constant curvature K-space:

**Lemma 3.2.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface in a K-space with constant curvature has the form

\[ H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta} , \]

then

\[ \lambda \cdot \mu = 0 . \]

**Proof.** Since \( M^{2n+2} \) be a K-space with constant curvature, from (2.3) and (3.1), we get

\[
\begin{aligned}
(\lambda_{t}) g_{a\beta} - (\lambda_{t}) g_{\beta t} + (\mu_{t}) \eta_{a} \eta_{\beta} - (\mu_{t}) \eta_{\beta} \eta_{t} \\
+ \mu (\eta_{a t} \eta_{\beta} + \eta_{t \beta} \eta_{a} - \eta_{t a} \eta_{\beta} - \eta_{a \beta} \eta_{t}) = 0 .
\end{aligned}
\]

Multiplying this equation by \( \varphi^{\alpha \gamma} \eta^\beta \) and summing for all indices, by virtue of (1.3) it follows that

\[ \mu \cdot \varphi^{\alpha \gamma} (\eta_{a;\gamma} - \eta_{t;\gamma}) = 0 . \]
Substituting (3.2) into (3.6), we have the conclusion.

**Theorem 3.3** If the second fundamental tensor \( H_{\alpha\beta} \) of a hypersurface in a \( K \)-space with constant curvature has the form

\[
H_{\alpha\beta} = \lambda g_{\alpha\beta} + \mu \eta_{\alpha} \eta_{\beta},
\]

then the hypersurface must be reduced to one of the following three cases:

1. \( H_{\alpha\beta} = \mu \eta_{\alpha} \eta_{\beta} \) for a suitable function \( \mu \), i.e. \( \text{rank } (H_{\alpha\beta}) = 1 \);
2. \( H_{\alpha\beta} = \lambda g_{\alpha\beta} \) and \( \lambda \) is constant, i.e. the hypersurface is umbilical;
3. \( H_{\alpha\beta} = 0 \), i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.2, we must have one of the following three cases; \( \lambda = 0 \), \( \mu = 0 \) and \( \lambda = \mu = 0 \).

If \( \lambda = 0 \), then we have the case (1).

If \( \mu = 0 \), then the hypersurface is umbilical. Since in a hypersurface of constant curvature space the relation \( H_{a\beta;\gamma} - H_{\gamma\beta;a} = 0 \) is satisfied, from this relation and \( H_{\alpha\beta} = \lambda g_{\alpha\beta} \), we see that \( \lambda \) is constant. Then we have the case (2).

If \( \lambda = \mu = 0 \), then \( H_{\alpha\beta} = 0 \), i.e. the hypersurface is totally geodesic.

When \( M^{2n+2} \) be a space of constant curvature, from the Gauss equation [2]:

\[
K_{a\beta\gamma\delta} = H_{a\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma} + K_{ijkl} X^{i}_{\alpha} X^{j}_{\beta} X^{k}_{\gamma} X^{l}_{\delta},
\]

we can easily obtain that the hypersurface is of constant curvature for above three cases. Thus we have the following corollary:

**Corollary 3.4.** If the second fundamental tensor \( H_{\alpha\beta} \) of a hypersurface in a \( K \)-space with constant curvature has the form

\[
H_{\alpha\beta} = \lambda g_{\alpha\beta} + \mu \eta_{\alpha} \eta_{\beta},
\]

then the hypersurface is also of constant curvature.

Next, we assume that \( M^{2n+2} \) be an Einstein \( K \)-space, then we have the following lemma:

**Lemma 3.5.** If the second fundamental tensor \( H_{a\beta} \) of a hypersurface in an Einstein \( K \)-space has the form

\[
H_{a\beta} = \lambda g_{a\beta} + \mu \eta_{a} \eta_{\beta},
\]

then

\[
\lambda(2n\lambda + \mu)_{;\gamma} \eta^{\gamma} = 0.
\]

**Proof.** Substituting (3.1) into (2.6) and multiplying by \( g^{a\beta} \) and summing for \( \alpha \) and \( \beta \), we get

\[
2n(\lambda_{;\gamma}) + (\mu_{;\gamma}) + (\mu_{a})\eta^{\gamma} \eta_{i} + \mu(2\eta_{a;\gamma} \eta^{a} - \eta_{i;\gamma} \eta^{a} - \eta_{i} g^{\alpha\beta} \eta_{\beta;i}) = K_{il} X^{i}_{\gamma} X^{l}.
\]
Since $M^{2n+2}$ is an Einstein $K$-space, we obtain by means of (3.2)
\[
(2n\lambda + \mu)_{;r} + (\mu_{;a})\eta^{a}\eta_{r} - \mu\eta_{r}A_{a\delta}\eta^{\delta} = 0.
\]
Multiplying the above equation by $\varphi^{g}\varphi_{p}$ and summing for $r$, it follows that
\[
(2n\lambda + \mu)_{;r} + (\mu_{;a})\eta^{a}\eta_{r} - \mu\eta_{r}A_{a\delta}\eta^{\delta} = 0.
\]
Differentiating (3.8) covariantly with respect to $u^{a}$ and multiplying by $\varphi^{a}$ and summing for $\alpha$ and $\kappa$, we have the conclusion by means of (3.2).

**Theorem 3.6.** If the second fundamental tensor $H_{a\beta}$ of a hypersurface in an Einstein $K$-space has the form
\[
H_{a\beta} = \lambda g_{a\beta} + \mu\eta_{a}\eta_{\beta},
\]
then the hypersurface must be reduced to one of the following four cases:

1. $H_{a\beta} = \lambda g_{a\beta} + \mu\eta_{a}\eta_{\beta}$, where $2n\lambda + \mu = \text{constant}$;
2. $H_{a\beta} = \lambda g_{a\beta}$ and $\lambda$ is constant, i.e. the hypersurface is umbilical;
3. $H_{a\beta} = \mu\eta_{a}\eta_{\beta}$ for a suitable function $\mu$, i.e. rank $(H_{a\beta}) = 1$;
4. $H_{a\beta} = 0$, i.e. the hypersurface is totally geodesic.

**Proof.** From the result of Lemma 3.5, if $\lambda \neq 0$, then $(2n\lambda + \mu)_{;r}\eta_{r} = 0$. From this equation and (3.8), it follows that $2n\lambda + \mu = \text{constant}$. This is the case (1). Specially, if $\lambda \neq 0$ and $\mu = 0$, then $H_{a\beta} = \lambda g_{a\beta}$, where $\lambda$ is constant. Hence we have the case (2).

From (3.7) if $\lambda = 0$, then we have the case (3). Specially, if $\lambda = \mu = 0$, then $H_{a\beta} = 0$, that is, the hypersurface is totally geodesic.

**References**


Department of Mathematics, Hokkaido University

(Received July 11, 1966)