ON ABELIAN EXTENSIONS OF SIMPLE RINGS

By

Kazuo KISHIMOTO*)

In [3], the author gave characterizations for a simple ring to have a cyclic extension, which generalizes the case of division ring extension treated by S. A. Amitsur in [1], while in [4], he studied as a generalized notion of a simple ring extension having a sigle commutative generator, a polynomial extension[1] of a simple ring as well as relationships between a cyclic extension and a Galois extension which is polynomial extension.

The present paper is a continuation of these earlier works to give, as main results, some necessary and sufficient conditions for a simple ring to have an abelian extension. §§ **1–2** are prerequisite preparations for the main purpose. In § **1**, we shall deal with an e-polynomial simple ring extension which is gneralized notion of a polynomial simple ring extension considered in [4]. It can be seen from the definition that the structure of each p^e-dimensional fundamental abelian extension of prime characteristic p and Kummer extension given in [Theorems 3,4 of [8] and Corollary 2 of [5]] are e-polynomial simple ring extensions. In § **2**, we shall construct two types of noncommutative) polynomial ring with several indeterminates over a simple ring. One is defined by means of derivations and the other is defined by means of endomorphisms both in the basic simple ring. Further, as residue class ring of above polynomial rings, two types of e-polynomial simple ring extensions will be defined, the former of which is concerned with p^e-dimensional fundamental abelian extension of prime characteristic p and the latter is concerned with a Kummer extension respectively.

Applying the results of the preceding sections, we shall give in § **3**, a necessary and sufficient condition for that a simple ring of prime characteristic p have a p^e-dimensional fundamental abelian extension in Theorem 3.1, and also a necessary and sufficient condition for a p^f-dimensional abelian extension T over S of prime characteristic p to be regularly embedded in a p^e-dimensional abelian extension R/S such that R/T is a fundamental abelian extension. Finally, **§4** is devoted to the Kummer extension case. Namely, we shall obtain a necessary and sufficient condition for a simple ring to have an n-dimensional

*) This work was done while the author was a Domestic fellow in the Mathematical Institute, Hokkaido University, on leave from Hokkaido University of Education.

1) Cf. [2].

abelian extension (whose center contains a primitive n-th root ζ of 1) in Theorem 4.1. Moreover, in Theorem 4.2, a necessary and sufficient condition for that a q-dimensional abelian extension T over S such that the center of T contains ζ and T is T_lS_r-irreducible can be regularly embedded in a qn-dimensional abelian extension R/S such that the center of R contains ζ will be given.

Throughout the present paper, we use the following conventions:

Let A be a ring and a an element of A.

$A^{\bullet}$: the multiplicative group consisting of the regular elements of A.

$\tilde{A}$: the set of all inner automorphisms determined by elements of $A^{\bullet}$.

I_a, $\tilde{a}$: the inner derivation $a_l - a_r$ determined by a, the inner automorphism $a_l a_r^{-1}$ determined by the regular element a respectively.

$J(\mathfrak{G}, A)$: the fixed-subring of A under $\mathfrak{G}$ where $\mathfrak{G}$ is an automorphism group of A.

$I_a|B$, $\mathfrak{G}|B$: the restriction of I_a to B, of $\mathfrak{G}$ to B where B is a subset of A.

$\mathfrak{G}(B)$: $\{\sigma \in \mathfrak{G};\ b\sigma = b$ for all $b \in B\}$.

$A \backslash B$: the set theoretical complement of B in A,

$LN_m(a;\rho)$ (resp. $RN_m(a;\rho)$) : $a\rho^{m-1} \cdot a\rho^{m-2} \cdots a\rho \cdot a$ (resp. $aa\rho \cdot a\rho^{m-1} a\rho^{m-1}$) where ρ is a (ring) homomorphism of A, by $N_m(a;\rho)$ we denote $LN_m(a;\rho)$ if it coincides with $RN_m(a;\rho)$.

$T_m(a;\rho)$: $\sum_{i=0}^{m-1} a\rho^i$.

$A[X;\rho, D]$: the ring of all polynomials $\{X^i a_i;\ a_i \in A\}$ which the multiplication is defined by $aX = X(a\rho) + aD$ for each $a \in A$ where D is a ρ-derivation in A. In particular, we set $A[X;D] = A[X;1,D]$, $A[X;\rho] = A[P;\rho,0]$ and $A[X] = A[X;1,0]$.

S, Z: a simple artinian ring, its center.

R, C: an extension simple artinian ring of S with common identity, its center.

V: $V_R(S)$, the centralizer of S in R.

$\oplus M_i$: a direct sum of modules M_i.

$\pi = \begin{pmatrix} 1, & 2, & \cdots, & e \\ \pi(1), & \pi(2), & \cdots, & \pi(e) \end{pmatrix}$: a permutation of e-letters.

The other notations and terminologies used in this paper, we follow [3].

§ 1. e-polynomial simple ring extension.

First of all, we shall begin our study with e-polynomial simple ring extensions over S.

R is called an *n-dimensional right e-polynomial simple ring extension over* S if R satisfies the following conditions:

(1) $[R:S]_r = n$,

(2) $R = \sum^{0\leq \nu_i \leq m_j-1} y_e^{\nu_e} y_{e-1}^{\nu_{e-1}} \cdots y_1^{\nu_1} S$ where $\prod_{i=1}^{e} m_i = n$, $m_i > 1$,

(3) $S(\pi(j)) = \sum_{0\leq \nu_{\pi(i)} \leq m_{\pi(i)}-1} y_{\pi(j)}^{\nu_{\pi(j)}} y_{\pi(j-1)}^{\nu_{\pi(j)}} \cdots y_{\pi(1)}^{\nu_{\pi(1)}} S$ is simple for each π and $1 \leqq j \leqq e$,

(4) $sy_i = y_i s_1^{(i)} + s_0^{(i)} (s_0^{(i)}, s_1^{(i)} \in S)$ for every $s \in S$, $y_i y_j = y_j y_i s_{ij} + t_{ij}$ for some $s_{ij}, t_{ij} \in S$.

(5) $y_i^{m_i} + \sum_{j=0}^{m_i-1} y_i^j u_j^{(i)} = 0$ $(u_j^{(i)} \in S)$.

Then, by $s \to s_1^{(i)}$, and $s \to s_0^{(i)}$, we can define a monomorphism ρ_i in S and ρ_i-derivation D_i in S. Further, s_{ij} is regular and $t_{ij} = -t_{ji} s_{ij}$ since $y_i y_j = y_j y_i s_{ij} + t_{ij} = y_i y_j s_{ji} s_{ij} + t_{ji} s_{ij} + t_{ij}$. Moreover, by (3) and (4), we can see that each $\rho_{\pi(j)}$ can be extended a monomorphism $P_{\pi(j)}$ of $S(\pi(j))$ by $y_{\pi(i)} P_{\pi(j)} = y_{\pi(i)} s_{\pi(i)\pi(j)}$ and, a tthe same time, $D_{\pi(j)}$ can be extended a $P_{\pi(j)}$-derivation in $S(\pi(j))$.

An n-dimensional right e-polynomial simple ring extension R over S is called *an n-dimensional e-polynomial simple ring extension over* S if $[R:S]_l = n$. Throughout in this section, we assume that R is an n-dimensional right e-polynomial simple ring extension over S, and in the rest, for the sake of simplicity, π will be omitted if there is no danger of misunderstanding.

Lemma 1.1. $[S(j):S(i)]_l = \prod_{p=i+1}^{j} \alpha_p$ *for each* π *and* $i<j$ *where* $\alpha_q = \sum_{j=1}^{m_q-1} ([S(q):S(q)P]_l)^j + 1$.

Proof. By (3) and (4), $S(j)$ is a right 1-polynomial simple ring extension over $S(j-1)$ $(j=1,2,\cdots,e$ and we set $S(0)=S)$. Hence $[S(j):S(j-1)]_l = \alpha_j$ by [4. Theorem 1.1]. Thus the assertion is clear from the fact that $[S(j):S(i)]_l = [S(j):S(j-1)]_l . [S(j-1):S(j-2)]_l \cdots [S(i+1):S(i)]_l$.

Corollary 1.1. *The following conditions are equivalent*:

(**a**) $[R:S]_l = [R:S]_r$.

(**b**) $[S(j):S(i)]_l = [S(j):S(i)]_r$ for each $i<j$ and $i=0,1,\cdots,e-1$.

(**c**) each ρ_j is an automorphism and it can be extended to an automorphism P_j in $S(j)$ for ech π.

Proof. (**a**)→(**b**) By Lemma 1.1, $[R:S]_l = \prod_{i=1}^{e} \alpha_i$ and $[S(i+1):S(i)]_l \geqq [S(i+1):S(i)]_r$. Hence we obtain (**b**).

(**b**)→(**c**) Since $[S(1):S]_l = [S(1):S]_r$, ρ_1 is an automorphism in S for each π. Next, $[S(j):S(j-1)]_l = [S(j):S(j-1)]_r$ shows that P_j is an automorphism in $S(j)$.

(**c**)→(**a**). This is a direct consquence of Lemma 1.1.

Corollary 1.2. *If* $D_i = 0$ *and* $t_{ij} = 0$ *for each* i, j, *then* $[R:S]_r = [R:S]_l$.

Proof. By [4. Corollary 1.2 (b)], $\rho_i = \bar{y}_i^{-1} | S$ is an automorphism in S. Further, $(y_j^{\nu_j} \cdot y_{j-1}^{\nu_{j-1}} \cdots y_1^{\nu_1} s) y_i = y_i (y_j^{\nu_j} \cdot y_{j-1}^{\nu_{j-1}} \cdots y_1^{\nu_1} s) P_i$ shows that $P_i | S(j) = \bar{y}_i^{-1} | S(j)$.

Hence we obtain $P_i|S(j)$ is an automorphism in $S(j)$. Thus the assertion is an immediate consequence of Corollary 1.1.

Proposition 1.1. (a) *If each* $\rho_i=1$, *and* $s_{ij}=1$ *then,*

$$[D_i, D_j]=D_iD_j-D_jD_i=I_{t_{ij}} \quad \textit{and}$$
$$t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=0.$$

(b) *If each* $D_i=0$ *and* $t_{ij}=0$ *then,*

$$\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=\tilde{s}_{ij} \quad \textit{and}$$
$$s_{ij}(s_{ik}\rho_j)s_{jk}=(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k).$$

Proof. (a) Since $\rho_i=1$, we have $D_i=I_{y_i}|S$. Then, $s(D_iD_j-D_jD_i)=s(y_iy_j-y_jy_i)-(y_iy_j-y_jy_i)s=st_{ij}-t_{ij}s$ $(s\in S)$ by the latter half condition of (4).

Next, $t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=[[y_i, y_j], y_k]+[[y_j, y_k], y_i]+[[y_k, y_i], y_j]=0$.

(b) Since $D_i=0$, we may assume that $\rho_i=\tilde{y}_i^{-1}|S$ is an automorphism in S [4. Corollary 1.2]. Further, $y_iy_j=y_jy_is_{ij}$ implies $s_{ij}=(y_jy_i)^{-1}y_iy_j$. Hence $s\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=y_i^{-1}y_j^{-1}y_iy_jsy_j^{-1}y_i^{-1}y_jy_i$ shows that $\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=\tilde{s}_{ij}$.

Next, $(y_iy_j)y_k=y_k(y_is_{ik}y_js_{jk})=y_k(y_iy_j(s_{ik}\rho_j)s_{jk})=y_k(y_jy_i(s_{ij}(s_{ik}\rho_j)s_{jk})$. On the other hand,

$$(y_iy_j)y_k=(y_jy_is_{ij})y_k=y_k(y_js_{jk}y_is_{ik}(s_{ij}\rho_k))=y_k(y_jy_i(s_{jk}\rho_i)\cdot s_{ik}(s_{ij}\rho_k))$$

Consequently, we have $s_{ij}(s_{sk}\rho_j)s_{jk}=(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k)$.

Proposition 1.2. *Let* $D_i=0$, $t_{ij}=0$ *and* $y_i^{m_i}\in S$ *for each* i. *Then* R *is* R_lS_r*-irreducible if and only if* V *is a division ring.*

Proof. Let V be a division ring. We show that $R=RxS$ for each $x\in R\backslash S$. Let $r=\sum y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots y_1^{\nu_1}s_{\nu_e\cdots\nu_1}$ $(s_{\nu_e\cdots\nu_1}\in S)$ be the shortest non zero relation relative to the representation by $\{y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots y_1^{\nu_1};\ 0\leqq\nu_i\leqq m_i-1\}$ in the element of RxS. Then we may assume that $r\in RxS\backslash S$ (otherwise, $r\in S$ implies $R=RrS=RxS$). Noting here $sr=\sum y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots y_1^{\nu_1}s\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}s_{\nu_e\cdots\nu_1}$ $(s\in S)$ is contained in RxS, we may set $r=y_e^{\alpha_e}y_{e-1}^{\alpha_{e-1}}\cdots y_1^{\alpha_1}+\sum y_e^{\mu_e}y_{e-1}^{\mu_{e-1}}\cdots y_1^{\mu_1}s_{\mu_e\cdots\mu_1}$ since $S\rho_e^{\alpha_e}\rho_{e-1}^{\alpha_{e-1}}\cdots\rho_1^{\alpha_1}sS=SsS=S$ for each $s(\neq 0)\in S$. Then $(y_e^{\alpha_e}y_{e-1}^{\alpha_{e-1}}\cdots y_1^{\alpha_1})^{-1}r=1+\sum y_e^{\mu_e-\alpha_e}y_{e-1}^{\mu_{e-1}-\alpha_{e-1}}\cdots y_1^{\mu_1-\alpha_1}t_{\mu_{e-\alpha_e}\cdots\mu_1-\alpha_1}\in RxS$ $(t_{\mu_{e-\alpha_e}\cdots\mu_1-\alpha_1}\in S)$. Since $y_i^{m_i}\in S$, we may assume that $r=1+\sum y_e^{\eta_e}y_{e-1}^{\eta_{e-1}}\cdots y_1^{\eta_1}s_{\eta_e\cdots\eta_1}$ $(s_{\eta_e\cdots\eta_1}\in S)$. Hence $sr-rs$ $(\in RxS)$ is zero for each $s\in S$. Consequently, $r\in V$ and hence $R=RrS=RxS$. The converse is evident.

Let A be a simple artinian ring over S such that $[A:S]_r$ is finite. A/S is called $\mathfrak{G}$*-regular* if A/S is Galois and there exists a Galois group $\mathfrak{G}$ such that $\#\mathfrak{G}=[A:S]_r$, while A is called *an abelian extension over* S *if* A/S is $\mathfrak{G}$-regular for some abelian F-group $\mathfrak{G}$.

Theorem 1.1. *Let* R *be an* n*-dimensional right* e*-polynomial simple ring extension over* S. *Assume* $n=p^e$, *the characteristic* $\chi(S)$ *of* S *is either zero or* $>p$, $\rho_i=1$ *and* $s_{ij}=1$ *for each* i,j. *Then,*

(a) $V=Z$ *implies* $\mathfrak{G}(R/S)=1$ *where* $\mathfrak{G}(R/S)$ *is the group of* S*-automor-*

phisms in R.

(**b**) *R/S is $\mathfrak{G}$-regular for some $\mathfrak{G}$ if and only if $R=S[C]$ and C/Z is a p^e-dimensional Galois extension.*

Proof. (**a**) Since $(sy_i)\sigma=s(y_i\sigma)$ for each $\sigma\in\mathfrak{G}(R/S)$ and $s\in S$, $y_i\sigma-y\in V=Z$. Hence $y_i\sigma=y_i+s_{i\sigma}$, $s_{i\sigma}\in Z$. Thus we may assume that $y_i\sigma=y_i+1$ (e.g. take $y_is_{i\sigma}^{-1}$). If $y_i^p=\sum_{j=0}^{p-1}y_i^js_j^{(i)}(s_j^{(i)}\in S)$, then $(y_i+1)^p=\sum_{k=0}^{p}\binom{p}{k}y_i^k=y_i^p+\sum_{k=0}^{p-1}\binom{p}{k}y_i^k=\sum_{k=0}^{a-1}y\left(\binom{p}{k}+s_k^{(i)}\right)=y_i^p\sigma=(\sum_{j=0}^{p-1}y_i^js_j^{(i)})\sigma=\sum_{j=0}^{p-1}(\sum_{k=0}^{j}\binom{j}{k}y_i^ks_j^{(i)}D_i^{j-k})$. From those, we can see that $p+s_{p-1}^{(i)}=s_{p-1}^{(i)}$, where it follows a contradiction $p=0$.

(**b**) Let R/S be $\mathfrak{G}$-regular for some $\mathfrak{G}$. If $\sigma(\neq 1)\in\mathfrak{G}$, then as is shown in the proof of (**a**), $y_i\sigma=y_i+v_{i\sigma}(v_{i\sigma}(\neq 0)\in V)$, and we may assume $y_i\sigma^k\neq y_i$ for each $0<k<p^j$ where p^j is the order of σ. Then $y_i\sigma^{p^j}=y_i+T_{p^j}(v_{i\sigma};\sigma)=y_i$ shows that $\#(\mathfrak{G}|V)=p^e$ and hence $V=C$. Thus $R=S\otimes_Z C$ and $J(\mathfrak{G}|C,C)=Z$ shows that C/Z is a p^e-dimensional Galois extension. The converse is evident.

The following is a slight generalization of Jacobson's Theorem.

Theorem 1.2. *Let R be an n-dimensional right e-polynomial simple ring over S. Assume $n=2^e$, $\chi(S)\neq 2$ and $t_{ij}=0$ for each i,j. If $[S:Z]$ is finite, the R/S is weakly Galois with respect to an abelian group $\mathfrak{G}$ such that $\#\mathfrak{G}=2^e$. In particular, if R is a division ring, then R/S is abelian.*

Proof. Since $S(1)/S$ is a right quadratic extension for each π, we may assume that $y_1^2\in S$, and $S(1)/S$ is a Galois extension with respect to τ_1 such that $y_1s\tau_1=-y_1s$ by [4. Corollary 4.1]. Let σ_i be an extension of τ_i such that $y_is\sigma_i=y_is\tau_i$ and $y_js\sigma_i=y_js$ if $i\neq j$. Then σ_i is an automorphism in R since $t_{ij}=0$, and $J(\sigma_i,R)=S[y_1,\cdots,\acute{y}_i,\cdots,y_e]$. If we set $\mathfrak{G}$, the group generated by $\sigma_1,\sigma_2,\cdots,\sigma_e$, then $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$ and $J(\mathfrak{G},R)=S$. If R is a division ring, then so is V, and hence the above fact shows that R/S is an abelian extension with respect to $\mathfrak{G}$.

§ 2. Construction of polynomial rings, e-polynomial simple ring extension.

Let (t_{ij}) be an $e\times e$ matrix with entries in S such that $t_{ij}=-t_{ji}$, $t_{ii}=0$ and let $D_1,D_2,\cdots,D_e$ be derivations in S satisfying $[D_i,D_j]=I_{t_{ij}}$ and $t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=0$. Further, by $S[\mathfrak{X}_{\pi(e)}]$ we denote $\{\sum X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1};\ s_{\nu_e\cdots\nu_1}\in S\}$, the set of all polynomials with e-indeterminates and the coefficients in S (the coefficients written on the right hand and X_i means $X_{\pi(i)}$). We define in $S[\mathfrak{X}_{\pi(e)}]$ the following multiplication rule and functions E_j, $j=1,2,\cdots,e$.

(1) $sX_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}=\sum_{\alpha_e=0}^{\nu_1}\sum_{\alpha_{e-1}=0}^{\nu_{e-1}}\cdots\sum_{\alpha_1=0}^{\nu_1}\binom{\nu_e}{\alpha_e}\cdots\binom{\nu_1}{\alpha_1}X_e^{\nu_e-\alpha_e}X_{e-1}^{\nu_{e-1}-\alpha_{e-1}}\cdots X_1^{\nu_1-\alpha_1}sD_e^{\alpha_e}D_{e-1}^{\alpha_{e-1}}\cdots D_1^{\alpha_1}$,

(2) $(X_i^{\nu_i}s)E_j=t_{ij}X_i^{\nu_i-1}s+X_it_{ij}X_i^{\nu_i-2}s+X_i^2t_{ij}X_i^{\nu_i-3}s+\cdots+X_i^{\nu_i-1}t_{ij}s+X_i^{\nu_i}(sD_j)$,

(3) $(X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s)E_j=(X_i^{\nu_i}E_j)(X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s)+X_i^{\nu_i}(X_{i-1}^{\nu_{i-1}}E_j).X_{i-2}^{\nu_{i-2}}\cdots X_1^{\nu_1}s+\cdots+X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_2^{\nu_2}(X_1^{\nu_1}E_j)s+X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}(sD_j)$,

(4) $(\sum_k Y_k)E_j=\sum_k(Y_kE_j)$ where $Y_k=X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1}$ for each s, $s_{\nu_e\cdots\nu_1}\in S$ and $i\leqq j$.

Proposition 2.1. *$S[\mathfrak{X}_{\pi(e)}]$ forms an associative ring by the following rule:*

$(\sum X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_i\cdots\nu_1})X_j=X_j(\sum X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_i\cdots\nu_1})+(\sum X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_i\cdots\nu_1})E_j$ *for evch $j\geqq i$.*

Moreover, when this is the case, $S[\mathfrak{X}_{\pi(e)}]$ is isomorphic to $S[\mathfrak{X}_{\pi'(e)}]$ for every π'.

Proof. We prove the assertion by induction on the number of indeterminates. $\{D_1, t_{11}=0\}$ satisfies the conditions stated above and $S[X_1, D_1]$ is an associative ring for each π. Hence we assume that $S[\mathfrak{X}_{\pi(e-1)}]=\{\sum X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}s_{\nu_{e-1}\cdots\nu_1};\ s_{\nu_{e-1}\cdots\nu_1}\in S\}$ is an associative ring for each π. Then it is clear that the map $\sum X_{\pi(e-1)}^{\nu_{e-1}}\cdots X_{\pi(1)}^{\nu_1}s_{\nu_{e-1}\cdots\nu_1}\to\sum X_{\pi'(i_{e-1})}^{\nu_{e-1}}\cdots X_{\pi'(i_1)}^{\nu_1}s_{\nu_{e-1}\cdots\nu_1}$ gives an isomorphism of $S[\mathfrak{X}_{\pi(e-1)}]$ to $S[\mathfrak{X}_{\pi'(e-1)}]$ where $\pi(j)=\pi(i_j)$. Now, the assertion is clear if we prove that E_e is a derivation in $S[\mathfrak{X}_{\pi(e-1)}]$, namely, $fE_eX_j^{\nu}+f\cdot(X_j^{\nu}E_e)=\left(\sum_{\alpha=0}^{\nu}\binom{\nu}{\alpha}X_j^{\nu-\alpha}fE_j^{\alpha}\right)E_e$ for each $f\in S[\mathfrak{X}_{\pi(j-1)}]$ and $j=1,2,\cdots,e-1$ (we set $S[\mathfrak{X}_{\pi(0)}]=S$) since $fX_j^{\nu}=\sum_{\alpha=0}^{\nu}\binom{\nu}{\alpha}X_j^{\nu-\alpha}fE_j^{\alpha}$. In $S[\mathfrak{X}_{\pi(e-1)}]$, $X_iX_j=X_jX_i+t_{ij}=X_jX_i-t_{ji}$ for each $j\geqq i$, and so we may assume that $E_i=I_{X_i}|S[\mathfrak{X}_{\pi(e-1)}]$, $E_j=I_{X_j}|S[\mathfrak{X}_{\pi(e-1)}]$ are derivations in $S[\mathfrak{X}_{\pi(e-1)}]$.

Firstly, we shall prove that $g[E_j, E_k]=gt_{jk}-t_{jk}g$ for each $g\in S[\mathfrak{X}_{\pi(e-1)}]$ and $j, k\leqq e-1$ by induction on the degree of g.

$X_is[E_j, E_k]=X_i[E_j, E_k]s+X_i(s[E_j, E_k])=(t_{ij}D_k-t_{ik}D_j)s+X_i(st_{jk}-t_{jk}s)=-t_{jk}D_is+X_i(st_{jk}-t_{jk}s)\ (s\in S)$ by the latter condition on $\{t_{ij}, D_k\}$. On the other hand, $X_ist_{jk}-t_{jk}X_is=X_i(st_{jk}-t_{jk}s)-t_{jk}D_is$. Therefore, we have $X_is[E_j, E_k]=X_ist_{jk}-t_{jk}X_is$. Suppose that $(\prod_{\alpha=e-1}^{1}X_\alpha^{\nu_\alpha}s)[E_j, E_k]=(\prod_{\alpha=e-1}^{1}X_\alpha^{\nu_\alpha}s)t_{jk}-t_{jk}(\prod_{\alpha=e-1}^{1}X_\alpha^{\nu_\alpha}s)$ for each $(\prod_{\alpha=e-1}^{1}X_\alpha^{\nu_\alpha}s)$ $(s\in S)$ such that $\sum_{\nu_\alpha}\nu=m$. Let $\sum_{\alpha=1}^{e-1}\mu_\alpha=m+1$. Then $(\prod_{\alpha=e-1}^{1}X_\alpha^{\mu_\alpha}s)[E_j, E_k]=(X_{e-1}^{\mu_{e-1}}\cdots X_1^{\mu_1-1}[E_j, E_k])X_1s+(X_{e-1}^{\mu_{e-1}}\cdots X_1^{\mu_1-1})\cdot(X_1s[E_j, E_k])=(X_{e-1}^{\mu_{e-1}}\cdots X_1^{\mu_1}t_{jk}-t_{jk}X_{e-1}^{\mu_{e-1}}\cdots X_1^{\mu_1})X_1s+(X_{e-1}^{\mu_{e-1}}\cdots X_1^{\mu_1})(X_1st_{jk}-t_{jk}X_1s)=(\prod_{\alpha=e-1}^{1}X_\alpha^{\mu_\alpha}s)t_{jk}-t_{jk}(\prod_{\alpha=e-1}^{1}X_\alpha^{\mu_\alpha}s)$.

Next, let $f\in S[X_{\pi(j-1)}]$. Then $fE_eX_j+f\cdot(X_jE_e)-(X_jE_ef+X_jfE_e+fE_jE_e)$ $=X_jfE_e+fE_eE_j+ft_{je}-(t_{je}f+X_jfE_e+fE_jE_e)=f[E_e,E_j]-ft_{ej}+t_{ej}f$. Since we may assume that f is contained in $S[\mathfrak{X}_{\pi'(e-1)}]$ which contains $S[\mathfrak{X}_{\pi(j-1)}]$ and X_e for a suitable π', $f[E_e,E_j]=ft_{ej}-t_{ej}f$. Thus we have $fE_eX_j+fX_jE_e=(X_jf+fE_j)E_e=(fX_j)E_e$.

Next, we shall show that $(X_j^qfX_j)E_e=(X_j^qfE_e)X_j+(X_j^qf)(X_jE_e)$ for each $f\in S[\mathfrak{X}_{\pi(j-1)}]\cdots\cdots\cdots(\alpha)$.

For, $(X_j^qfX_j)E_e=(X_j^{q+1}f+X_j^qfE_j)E_e$

$$\begin{aligned}&=X_j^qE_eX_jf+X_j^q(X_jfE_e)+X_j^qE_efE_j+X_j^qfE_jE_e\\&=X_j^qE_e(X_jf+fE_j)+X_j^q(X_jfE_e+fE_jE_e)\\&=X_j^qE_e(fX_j)+X_j^q(fX_jE_e)=X_j^qE_e(fX_j)+X_j^q(fE_eX_j+fX_jE_e)\\&=(X_j^qE_ef+X_j^qfE_e)X_j+X_j^qfX_jE_e=(X_j^qf)E_eX_j+(X_j^qf)(X_jE_e).\end{aligned}$$

Let us assume that $(fX_j^{\nu-1})E_e=fE_eX_j^{\nu-1}+fX_j^{\nu-1}E_e$. Then we can prove $(fX_j^\nu)E_e=fE_eX_j^\nu+fX_j^\nu E_e$ by use of (α).

For, $(fX_j^\nu)E_e=\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}X_j^{\nu-\mu-1}fE_jX_j)E_e=(\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}X_j^{\nu-\mu-1}fE_j^\mu)E_eX_j$ $+(\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}X_j^{\nu-\mu-1}fE_j^\mu)(X_jE_e)=(fX_j^{\nu-1})E_eX_j+(fX_j^{\nu-1})(X_jE_e)$ $=fE_eX^j+fX_j^{\nu-1}E_eX_j+(fX_j^{\nu-1})(X_jE_e)=fE_eX_j^\nu+fX_j^\nu E_e$.

Let $i=\pi'(\alpha_i)$ for each $i=1,2,\cdots,e$. Then the map $\sum X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1}\to$ $\sum X_{\pi'(\alpha_e)}^{\nu_e}X_{\pi'(\alpha_{e-1})}^{\nu_{e-1}}\cdots X_{\pi'(1)}^{\nu_1}s_{\nu_e\cdots\nu_1}$ gives an isomorphim, of $S[\mathfrak{X}_{\pi(e)}]$ onto $S[\mathfrak{X}_{\pi'(e)}]$.

By $S[\mathfrak{X}_{\pi(e)};\mathfrak{D}_{\pi(e)}]$ or sometimes, by $S[X_1,X_2,\cdots,X_e;D_1,D_2\cdots,D_e]$ we denote the polynomials ring obtained in Proposition 2.1.

Let (s_{ij}) be an $e\times e$ matrix with entries in $S^\cdot$ such that $s_{ij}=s_{ji}^{-1}$, $s_{ii}=1$, and let $\rho_1,\rho_2,\cdots,\rho_e$ be automorphisms in S satisfying $\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=\bar{s}_{ij}$ and $s_{ij}(s_{ik}\rho_j)s_{jk}=(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k)$. Further, we define the following multiplication rule and functions P_j, $j=1,2,\cdots,e$ in $S[\mathfrak{X}_{\pi(e)}]$.

(1) $sX_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}=X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}(s\rho_e^{\nu_e}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_1})$,

(2) $(X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s)P_j=(X_is_{ij})^{\nu_i}(X_{i-1}s_{i-1j})^{\nu_{i-1}}\cdots(X_1s_{1j})^{\nu_1}s\rho_j$,

(3) $(\sum_kY_k)P_j=\sum_k(Y_kP_j)$ where $Y_k=X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_2\cdots\nu_1}$,

for each $s,s_{\nu_e\cdots\nu_1}\in S$ and $j\geqq i$.

Proposition 2.2. *$S[\mathfrak{X}_{\pi(e)}]$ forms an associative ring by the following rule:*

$(\sum X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_i\cdots\nu_1})X_j=X_j(\sum X_i^{\nu_i}X_{i-1}^{\nu_{i-1}}\cdots X_1^{\nu_1}s_{\nu_i\cdots\nu_1})P_j$.

Moreover, when this is the case, $S[\mathfrak{X}_{\pi(e)}]$ is isomorphic to $S[\mathfrak{X}_{\pi'(e)}]$ for each π'.

Proof. We prove the assertion by induction on the number of indeterminates. $\{\rho_1,s_{11}=1\}$ satisfies the conditions stated above and $S[X_1;\rho_1]$ is an associative ring for each π. Hence we assume that $S[\mathfrak{X}_{\pi(e-1)}]$ is an associative

ring for each π. Then it is clear that the map $\sum X^{\nu_{e-1}}_{\pi(e-1)}\cdots X^{\nu_1}_{\pi(1)}s_{\nu_{e-1}\cdots\nu_1}\to\sum X^{\nu_{e-1}}_{\pi'(i_{e-1})}\cdots X^{\nu_1}_{\pi'(i_1)}s_{\nu_{e-1}\cdots\nu_1}$ gives an isomorphism of $S[\mathfrak{X}_{\pi(e-1)}]$ to $S[\mathfrak{X}_{\pi'(e-1)}]$ where $\pi(j)=\pi'(i_j)$. Now the assertion is clear if we prove that P_e is a homomorphism in $S[\mathfrak{X}_{\pi(e-1)}]$, namely, $fP_eX^\nu_jP_e=X^\nu_jP_efP^\nu_jP_e$ for each $f\in S[\mathfrak{X}_{\pi(j-1)}]$ and $j=1,2,\cdots,e-1$ (we set $S[\mathfrak{X}_{\pi(0)}]=S$) since $f\cdot X^\nu_j=X^\nu_jfP^\nu_j$. In $S[\mathfrak{X}_{\pi(e-1)}]$, each P_j $(j=1,2,\cdots,e-1)$ is a homomorphism since $(fg)X_j=f(gX_j)$ for each $f,g\in S[\mathfrak{X}_{\pi(j-1)}]$. Then $(\sum X^{\nu_i}_iX^{\nu_{i-1}}_{i-1}\cdots X^{\nu_1}_1s_{\nu_i\cdots\nu_1})P_j=0$ means $(\sum X^{\nu_i}_iX^{\nu_{i-1}}_{i-1}\cdots X^{\nu_1}_1u_{\nu_i\cdots\nu_1}s_{\nu_i\cdots\nu_1})=0$ where each $u_{\nu_i\cdots\nu_1}$ is a regular element of S. Hence each P_j is an automorphism in $S[\mathfrak{X}_{\pi(j)}]$. Firstly, we shall prove that $fP_k^{-1}P_j^{-1}\cdot P_kP_j=f\tilde{s}_{jk}$ for each $f\in S[\mathfrak{X}_{\pi(j-1)}]$ by induction on the degree of f.

For, $X_isP_k^{-1}P_j^{-1}P_kP_j=X_iP_k^{-1}P_j^{-1}P_kP_js\rho_k^{-1}\rho_j^{-1}\rho_k\rho_j$
$=X_is_{ij}(s_{ik}\rho_j)(s_{ji}\rho_j^{-1}\rho_k\rho_j)(s_{ki}\rho_k^{-1}\rho^{-1}_j\rho_k\rho_j)(s\rho_k^{-1}\rho_j^{-1}\rho_k\rho_j)$
$=X_i(s_{jk}\rho_is_{ik})(s_{ij}\rho_k)s_{kj})(s_{ji}\rho_j^{-1}\rho_k\rho_j)(s_{jk}s_{ki}s_{kj})(s\tilde{s}_{jk})$
$=X_i(s_{jk}\rho_is_{ik}(s_{ij}\rho_k)s_{kj})s_{jk}(s_{ji}\rho_k)s_{kj}(s_{jk}s_{ki}s_{kj})(s\tilde{s}_{jk})$
$=X_i(s_{jk}\rho_is_{kj})(s\tilde{s}_{jk})=(X_i\tilde{s}_{jk})(s\tilde{s}_{jk})$ by the latter conditions on $\{s_{ij},,P_k\}$.

Assume that $(X^{\nu_i}_iX^{\nu_{i-1}}_{i-1}\cdots X^{\nu_1}_1s)P_k^{-1}P_j^{-1}P_kP_j=(X^{\nu_i}_iX^{\nu_{i-1}}_{i-1}\cdots X^{\nu_1}_1s)\tilde{s}_{jk}$ for each $\sum_{j=1}^i\nu_j=m$. Let $\sum_{j=1}^i\mu_j=m+1$. Then $(X^{\mu_i}_iX^{\mu_{i-1}}_{i-1}\cdots X^{\mu_1}_1s)P_k^{-1}P_j^{-1}P_kP_j=(X^{\mu_i}_iX^{\mu_{i-1}}_{i-1}\cdots X^{\mu_1-1}_1)P_k^{-1}P_j^{-1}P_kP_j(X_1s)P_k^{-1}P_j^{-1}P_kP_j=(X^{\mu_i}_iX^{\mu_{i-1}}_{i-1}\cdots X^{\mu_1-1}_1)\tilde{s}_{jk}(X_1s)$. $\tilde{s}_{jk}=(X^{\mu_i}_iX^{\mu_{i-1}}_{i-1}\cdots X^{\mu_1}_1s)\tilde{s}_{jk}$.

Next, let $f\in S[\mathfrak{X}_{\pi(j-1)}]$. Then $fP_eX_jP_e-(X_jP_efP_jP_e)=X_j.fP_eP_js_{je}-X_js_{je}fP_jP_e=X_j(fP_eP_js_{je}-s_{je}fP_jP_e)$. Since we may assume that $fP_e^{-1}P_j^{-1}P_eP_j=f\tilde{s}_{je}$, we have $X_j(fP_eP_js_{je}-s_{je}fP_jP_e)=0$. Suppose that $fP^eX^{\nu-1}_jP_e=X^{\nu-1}_jP_efP^{\nu-1}_jP^{\nu-1}_jP_e$. Then,
$fP_eX^\nu_jP_e=fP_eX^{\nu-1}_jP_eX_jP_e=X^{\nu-1}_jP_efP^{\nu-1}_jP_eX_js_{je}=X^{\nu-1}_jP_eX_jfP^{\nu-1}_jP_eP_js_{je}$
$=X^{\nu-1}_jP_e(X_jP_es_{ej})fP^{\nu-1}_jP_eP_js_{je}=X^{\nu-1}_jP_eX_jP_efP^{\nu-1}_jP_eP_jP_j^{-1}P_e^{-1}P_jP_e$
$=X^\nu_jP_e\cdot fP^\nu_jP_e$. If $i=\pi'(\alpha_i)$ for each $i=1,2,\cdots,e$, the it is clear that $\sum X^{\nu_e}_eX^{\nu_{e-1}}_{e-1}\cdots X^{\nu_1}_1s_{\nu_e\cdots\nu_1}\to\sum X^{\nu_e}_{\pi'(e)}X^{\nu_{e-1}}_{\pi'(e-1)}\cdots X^{\pi_1}_{\pi'(1)}s_{\nu_e\cdots\nu_1}$ gives an isomorphism of $S[\mathfrak{X}_{\pi(e)}]$ to $S[\mathfrak{X}_{\pi'(e)}]$.

By $S[\mathfrak{X}_{\pi(e)};P_{\pi(e)}]$ or sometimes, by $S[X_1,X,\cdots,X_e;\rho_1,\rho_2,\cdots,\rho_e]$, we denote the polynomials ring obtained in Proposition 2.2.

Lemma 2.1. *Let $f_i(X_i)=X_i^{m_i}+\sum_{j=0}^{m_i-1}X_i^js_j^{(i)}$ $(s_j^{(i)}\in S)$ be a polynomial in $S[\mathfrak{X}_{\pi(e)}]$.*

(**a**) *If $f_i(X_i)\in S[\mathfrak{X}_{\pi(e)};D_{\pi(e)}]$, then $M=f_i(X_i)S[\mathfrak{X}_{\pi(e)};D_{\pi(i)}]$, a right ideal generated by $f_i(X_i)$ is a two-sided ideal if and only if $f_i(X_i)$ is central.*

(**b**) *If $f_i(X_i)\in S[\mathfrak{X}_{\pi(e)};P_{\pi(e)}]$, then M is a two-sided ideal if and only if $\{s_\nu^{(i)}\}$ satisfies the following conditions:*

(1) $s\rho_i^\nu s_\nu^{(i)}=s_\nu^{(i)}s\rho_i^{m_i}$,

(2) $RN_\nu(s_{ji};\rho_i)s_\nu^{(i)}=s_\nu^{(i)}\rho_jRN_{m_i}(s_{ji};\rho_i)$.

Proof. Since $f_i=f_i(X_i)$ is monic, $deg\, f_i g = deg\, f_i + deg\, g$ for each polynomial g. Hence M is a two-sided ideal if and only if $sf_i=f_i s'$ $(s'\in S)$ for every $s\in S$ and $X_j f_i = \begin{cases} f_i(X_j+f) & \text{in case (a).} \\ f_i(X_j t'+t) & \text{in case (b),} \end{cases}$ for some $t, t'\in S$.

(a) Let M be a two-sided ideal. Since $sf_i=X_i^{m_i}s+g_i$ $(deg\ g_i<m_i)$ and $f_i s'=X_i^{m_i}s'+g_i'$ $(deg\ g'<m)$, we can easily seen $s'=s$. Next, $X_j f_i=f_i X_j+f_i E_j$ $=f_i X_j+f_i t$ show that $f_i E_j=f_i t$, and hence $t=0$ and $f_i E_j=0$. Thus f_i is central. The converse is clear.

(b) $sf_i(X_i)=X_i^{m_i}s+\sum_{\nu=0}^{m_i-1}X_i^\nu s\rho_i^\nu s_\nu^{(i)}$ and $f_i(X_i)s'=X_i^{m_i}s'+\sum_{\nu=0}^{m_i-1}X_i^\nu s_\nu^{(i)}s'$, $=X^j f_i(X_i)=X_i^{m_i}(X_j P_i^{m_i})+\sum_{\nu=0}^{m_i-1}X_i^\nu(X_j P_i^\nu)s_\nu^{(i)}=X_i^{m_i}X_j RN_{m_i}(s_{ji};\rho_i)+\sum_{\nu=0}^{m_i-1}$ $X_i^\nu X_j RN_\nu(s_{ji};\rho_i)s_\nu^{(i)}$ and $f_i(X_i)(X_j t'+t)=X_i^{m_i}X_j t'+\sum_{\nu=0}^{m_i-1}X_i^\nu X_j s_\nu^{(i)}\rho_j t'+f_i(X_i)t$. These facts show that M is a two-sided ideal if and only if (1) and (2) are satisfied.

Lemma 2.2. *Let $f_i(X_i)$ be a polynomial given in Lemma 2.1, $i=1,2,\cdots,e$.*

(a) *If $f_i(X_i)\in S[\mathfrak{X}_{\pi(e)};\mathfrak{D}_{\pi(e)}]$ and $f_i(X_i)$ is central, then $M_k E_j\subseteqq M_k$ where $M_k=(f_1(X_1),f_2(X_2),\cdots,f_k(X_k))\,S[\mathfrak{X}_{\pi(e)};\mathfrak{D}_{\pi(e)}]$.*

(b) *If $f_i(X_i)$ satisfies the conditions of Lemma 2.1 (b), then $M_k P_j\subseteqq M_k$.*

Proof. (a) Since $E_j=I_{X_j}|S[\mathfrak{X}_{\pi(e)};\mathfrak{D}_{\pi(e)}]$, the assertion is clear.

(b) $(X_i^{m_i}+\sum_{\nu=0}^{m_i-1}X_i^\nu s_\nu^{(i)})P_j=X_i^{m_i}LN_{m_i}(s_{ij};\rho_i)+\sum_{\nu=0}^{m_i-1}X_i^\nu LN_\nu(s_{ij};\rho_i)s_\nu^{(i)}$. If we note that $(LN_\nu(s_{ij}:\rho))^{-1}=RN_\nu(s_{ji};\rho_i)$, the condition (2) of Lemma 2.1 (b) yields at once $M_k P_j\subseteqq M_k$.

By Lemma 2.2, we can see that each E_j(resp. P_j) induces a derivation (resp. homomorphism) in $S[\mathfrak{X}_{\pi(k)};\mathfrak{D}_{\pi(k)}]/M_k\cong S[x_1,x_2,\cdots,x_k]$ (resp. $S[X_{\pi(k)};P_{\pi(k)}]/M_k\cong S[x_1,x_2,\cdots,x_k]$) where x_i is the residue class of X_i modulo M_k. We denote by E_j (resp. P_j), these induced derivation (resp. homomorphism) again. Then $S[x_1,x_2,\cdots,x_k][X_{k+1};E_{k+1}]=\{\sum_\nu X_{k+1}^\nu f_\nu;f_\nu\in S[x_1,x_2,\cdots,x_k]\}$ (resp. $S[x_1,x_2,\cdots,x_k][X_{k+1};P_{k+1}]$) can be regarded as an associative ring by the rule $fX_{k+1}=X_{k+1}f+fE_{k+1}$ (resp. $fX_{k+1}=X_{k+1}fP_{k+1}$)) for each $f\in S[x_1,x_2,\cdots,x_k]$. Further, when this is the case, each polynomial $f_{k+1}(X_{k+1})$ given in Lemma 2.1 can be considered as a polynomial in $S[x_1,x_2,\cdots,x_k][X_{k+1};E_{k+1}]$ (resp. $S[x_1,x_2,\cdots,x_k][X_{k+1}:P_{k+1}]$) and $f_{k+1}(X_{k+1})S[x_1,x_2,\cdots,x_k][X_{k+1};E_{k+1}]$ is a two-sided ideal.

Now, let $S_k=S[\mathfrak{X}_{\pi(k)};\mathfrak{D}_{\pi(k)}]/f_k(X_k)S[\mathfrak{X}_{\pi(k)};\mathfrak{D}_{\pi(k)}]\cong S[w_k][\mathfrak{X}_{\pi(k+1)};\mathfrak{D}_{\pi(k+1)}]$ where w_k is the residue class of X_k modulo $f_k(X_k)S[\mathfrak{X}_{\pi(k)};\mathfrak{D}_{\pi(k)}]$, and let $S_i=S_{i+1}/f_i(X_i)S_{i+1}\cong S[w_k,w_{k-1},\cdots,w_i][\mathfrak{X}_{\pi(i-1)};\mathfrak{D}_{\pi(i-1)}]$, (more precisely, $S[w_k][\mathfrak{X}_{\pi(k-1)};\mathfrak{D}_{\pi(k-1)}]/f_{k-1}(X_{k-1})S[w_k][\mathfrak{X}_{\pi(k-1)};\mathfrak{D}_{\pi(k-1)}]\cong S[\bar{w}_k,w_{k-1}][\mathfrak{X}_{\pi(k-1)};\mathfrak{D}_{\pi(k-1)}]$ where $\bar{w}_k$ is the residue class of w_k modulo $f_{k-1}(X_{k-1})S[w_k][\mathfrak{X}_{\pi(k-1)};\mathfrak{D}_{\pi(k-1)}]$, but, by w_k we denote $\bar{w}_k$ again).

Lemma 2.3. (a) *S_1 is isomorphic to $S[\mathfrak{X}_{\pi(k)}; \mathfrak{D}_{\pi(k)}]/M_k$* (resp $S[\mathfrak{X}_{\pi(k)}; \mathfrak{P}_{\pi(k)}]/M_k$).

(b) *If $f_1(X_1)$ is w-irreducible in S_2 for each π, then $S[x_1, x_2, \cdots, x_k]$ is simple for each k and π.*

Proof. The following diagram is commutative where φ_i: $S_{i+1} \to S_{i+1}/f_i(X_i)S_{i+1}$, θ: $S[\mathfrak{X}_{\pi(k)}: \mathfrak{D}_{\pi(k)}] \to S[\mathfrak{X}_{\pi(k)}; \mathfrak{D}_{\pi(k)}]/M_k$, $\varphi = \varphi_k \varphi_{k-1} \cdots \varphi_1$, and $\Psi = (an$ *inverse image of* $\varphi) \cdot \theta$.

$$\begin{tikzcd}
S[\mathfrak{X}_{\pi(k)}; \mathfrak{D}_{\pi(k)}] \arrow[r, "1"] \arrow[d, "\varphi_k"] & S[\mathfrak{X}_{\pi(k)}; \mathfrak{D}_{\pi(k)}] \arrow[dd, "\theta"] \\
S_k \arrow[d] & \\
S_{k-1} \arrow[d, "\varphi_1"] & \\
S \arrow[r, "\varphi"] & S[x_1, x_2, \cdots, x_k]
\end{tikzcd}$$

For, $S_1 \ni \sum_\nu w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1} = 0$ $(s \in S)$, then each inverse image of $\sum w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1} (=0)$ of φ is contained in M_k. Hence $(\sum w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1})\Psi = 0$, and this means that Ψ is well defined. Next, let $(\sum w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1})\Psi = 0$. Then $(\sum w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1})\varphi^{-1} \subseteqq M$. Hence $\sum w_k^{\nu_k} w_{k-1}^{\nu_{k-1}} \cdots w_1^{\nu_1} s_{\nu_k \cdots \nu_1} = 0$.

(b) Since $S[w_2, w_3, \cdots, w_e] \supseteqq S[w_2, w_3, \cdots, w_k] \supseteqq S$ for each π, the w-irrducibility of $f_1(X_1)$ in S_{e-1} yields at once that of $f_1(X_1)$ in $S[w_2, w_3, \cdots, w_k][X_1; E_1]$. Hence $S[X_1; D_1]/f_1(X_1)s[X_1; D_1]$ is simple for each π. Thus repeating the procedures, we have the simplicity of $S[w_1, w_2, \cdots, w_k] \cong S[x_1, x_2, \cdots, x_k]$.

Let $f_i(X_i)$ $(i = 1, 2, \cdots, e)$ be a polynomial of $S[X_i; D_i]$ (resp. $S[X_i; P_i]$). Then $f_i(X_i)$ is called *s.w-irreducible* if *$f_i(X_i)$ is w-irreducible in* $S[X_1, X_2, \cdots, \check{X}_i, \cdots, X_k; E_1, E_2, \cdots, \check{E}_i, \cdots, E_k]/M$ (resp. $S[X_1, X_2, \cdots, \check{X}_i, \cdots, X_k; P_1, P_2, \cdots, \check{P}_i, \cdots, P_k]/M$) for each $1 \leqq k \leqq e$ where M is a two-sided ideal generated by $f_1(X_1), f_2(X_2), \cdots, f_i(\check{X}_i), \cdots, f_k(X_k)$. Moreover, $f_i(X_i)$ is called *s-irreducible if $f_i(X_i)$ is irreducible in* $S[X_1, X_2, \cdots, \check{X}_i, \cdots, X_k; E_1, E_2, \cdots, \check{E}_i, \cdots, E_k]/M$ (resp. $S[X_1, X_2, \cdots, \check{X}_i, \cdots, X_k; P_1, P_2, \cdots, \check{P}_i, \cdots, P_k]/M$).

In following we shall give necessary and sufficient conditions for a simple ring to have an e-polynomial simple ring extension of dimension $n = \prod_{i=1}^{e} m_i$, $m_i > 1$.

Proposition 2.3. (a) *In order that S have an $n = \prod_{i=1}^{e} m_i$ $(m > 1)$-dimensional e-polynomial simple ring extension such that $\rho_i = 1$ and $s_{ij} = 1$, it is necessary and sufficient that there exist derivations $D_1, D_2, \cdots, D_e$ in S, (t_{ij})*

an $e\times e$ *matrix with entries in* S *such that* $t_{ij}=-t_{ji}$, $t_{ii}=0$ *and* $(s_\nu^{(i)})$ $1\times m_i$ *matrices* $(i=1,2,\cdots,e)$ *satisfying*

(1) $[D_i, D_j]=I_{t_{ij}}$,

(2) $t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=0$,

(3) $\binom{m_i-k+h}{h}sD_i^h s_{m_i-k+h}^{(i)}=s_{m_i-k}^{(i)}s$,

(4) $s_{m_i-k}^{(i)}D_j=\sum_{h=0}^{k}\binom{m_i-k+h}{h}t_{ji}D_i^{h-1}s_{m_i-k+h}^{(i)}$,

(5) $f_i(X_i)=X_i^{m_i}+\sum_{\nu=0}^{m_i-1}X_i^\nu s_\nu^{(i)}$ *is s.w-irreducible.*

(**b**) *In order that* S *have an* $n=\prod_{i=1}^{e}m_i$ $(m_i>1)$*-dimensional e-polynomial simple ring extension such that* $D_i=0$ *and* $t_{ij}=0$, *it is necessary and sufficient that there exist automorphisms* $\rho_1,\rho_2,\cdots,\rho_e$ *in* S, (s_{ij}) *an* $e\times e$ *matrix with entries in* $S^\bullet$ *such that* $s_{ij}=s_{ji}^{-1}$, $s_{ii}=1$ *and* $(s_\nu^{(i)})$ $1\times m_i$ *matrices* $(i=1,2,\cdots,e)$ *satisfying*

(1) $\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=\tilde{\varepsilon}_{ij}$,

(2) $s_{ij}(s_{ik}\rho_j)s_{jk}=(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k)$,

(3) $s\rho_i^\nu s_\nu^{(i)}=s_\nu^{(i)}s\rho_i^{m_i}$,

(4) $RN_\nu(s_{ji};\rho_j)s_\nu^{(i)}=s_\nu^{(i)}\rho_j RN_{m_i}(s_{ji};\rho_i)$,

(5) $f_i(X_i)=X_i^{m_i}+\sum_{\nu=0}^{m_i-1}X_i^\nu s_\nu^{(i)}$ *is s.w-irreducible.*

Proof. By proposition 2.1, (1) and (2) are equivalent with the existence of a polynomial ring $S[\mathfrak{X}_e;\mathfrak{D}_e]$. Let $M=(f_1(X_1),f_2(X_2),\cdots,f_e(X_e)S[\mathfrak{X}_e;\mathfrak{D}_e]$. Then (3) and (4) imply the fact that each $f_i(X_i)$ is central. Hence M is a two-sided ideal. Now, by (5),$R^*\cong S[\mathfrak{X}_e;\mathfrak{D}_e]/M$ is an $n=\prod_{i=1}^{e}m_i$-dimensional right e-polynomial simple ring extension over S. Further, if we note that each $\rho_i=1$ and $s_{ij}=1$, $[R^*:S]_r=[R^*:S]_l$ by Corollary 1.1. Conversely, let $R=S[x_1,x_2,\cdots,x_e]$ be an $n=\prod_{i=1}^{e}m_i$ $(m_i>1)$-dimensional e-polynomial simple ring extension over S such that $sx_i=x_is+s_0^{(i)}$ $(s_0^{(i)}\in S)$ for every $s\in S$ and $x_ix_j=x_jx_i+t_{ij}$. Then, by Proposition 1,1 (a), each $D_i=I_{x_i}|S$, $t_{ij}=x_ix_j-x_jx_i$ satisfies (1) and (2). Let $x_i^{m_i}+\sum_{\nu=0}^{m_i-1}x_i^\nu s_\nu^{(i)}=0$. Then $s(x_i^{m_i}+\sum_{\nu=0}^{m_i-1}x_i^\nu s_\nu^{(i)})=(x_i^{m_i}+\sum_{\nu=0}^{m_i-1}x_i^\nu s_\nu^{(i)})s$ and $x_j(x_i^{m_i}+\sum_{\nu=0}^{m_i-1}x_i^\nu s_\nu^{(i)})=(x_i^{m_i}+\sum_{\nu=0}^{m_i-1}x_i^\nu s_\nu^{(i)})x_j$ imply (3) and (4) for each $1\times m_i$ matrix $(s_\nu^{(i)})$. Finally (5) is an immediate consequence of (3) of the definition of an e-polynomial simple ring extension and Lemma 2.3.

(**b**) Sufficiency is clear from the proof of (a). Conversely, let $R=S[x_1,x_2,\cdots,x_e]$ be an $n=\prod_{i=1}^{e}m_i$ $(m_i>1)$-dimensional e-polynomial simple ring extension over S with $sx_i=x_is\rho_i$ and $x_ix_j=x_jx_is_{ij}$, then each $\tilde{x}_i^{-1}|S=\rho_i$, $x_i^{-1}x_j^{-1}x_ix_j=s_{ij}$ satisfies conditions (1) and (2). The rest of the proof is same

as that of (a).

§ 3. Abelian extensions of prime characteristic p.

In this section, we shall deal with p^f-dimensional abelian extensions of a simple ring of prime characteristic p. A p^e-dimensional abelian extension R over S is called a p^e-dimensional fundamental abelian extension over S if $\chi(S)=p$ and there exists a Galois gronp $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, a direct product of σ_i, with $\#\sigma_i=p$ and $\mathfrak{G}$ is an F-group. Then, one of principal theorem of this section can be state

Theorem 3.1. *Let $\chi(S)=p$. In order that S have a p^e-dimensional fundmaental abelian extension, it is necessary and sufficient that there exist derivations $D_1, D_2, \cdots, D_e$ in S, (t_{ij}) an $e\times e$ matrix with entries in S such that $t_{ij}=-t_{ji}$, $t_{ii}=0$ and (s_i) $1\times e$ matrix with entries in S satisfying*

(1) $[D_i, D_j]=I_{t_{ij}}$,

(2) $t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=0$,

(3) $D_i^p-D_i=I_{s_i}$, $s_iD_i=0$,

(4) $t_{ij}D_j^{p-1}+t_{ji}+s_jD_i=0$,

(5) $X_{\pi(i)}^p-s_{\pi(i)}=X_{\pi(i)}^p-X_{\pi(i)}-s_{\pi(i)}$ *is s.w-irreducible.*

More precisely, if there exists $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying (1)–(5), *then $M=(X_1^p-s_1, X_2^p-s_2, \cdots, X_e^p-s_e)S[\mathfrak{X}_e;\mathfrak{D}_e]$ is a maximal ideal of $S[\mathfrak{X}_e;\mathfrak{D}_e]$, $R^*=S[y_1, y_2, \cdots, y_e]\cong S[\mathfrak{X}_e;\mathfrak{D}_e]/M$ is a p^e-dimensional fundamental abelian extension with respect to $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_e^*)$ such that $\#\sigma_i^*=p$ and $y_i\sigma_j^*=y_i+\delta_{ij}$ where each y_i is the residue class of X_i modulo M. Conversely, if R is a p^e-dimensional fundamental abelian extension over S with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, then we can find such $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying* (1)–(5) *that there holds an S-isomorphism φ^*: $R^*\cong R$ with $\varphi^*\sigma_i=\sigma_i^*\varphi^*$.*

Proof. Since $s(X_i^p-X_i-s_i)=X_i^ps-X_is+sD_i^p-sD_i-ss_i$, and $X_j(X_i^p-X_i-s_i)=X_i^pX_j+X_iX_j-sX_j+s_{ji}D_i^{p-1}-s_{ji}+s_iD_j$, (3) and (4) are equivalent with $X_i^p-s_i$ is central in $S[\mathfrak{X}_e;\mathfrak{D}_e]$. Thus, by Proposition 2.3 (a), $R^*=S[y_1, y_2, \cdots, y_e]\cong S[\mathfrak{X}_e;\mathfrak{D}_e]/M$ is a p^e-dimensional e-polynomial simple ring extension over S. Let Φ_i be the map of $S[\mathfrak{X}_e;\mathfrak{D}_e]$ defined by $\sum_\nu X_i^\nu f_\nu\to\sum_\nu(X_i+1)^\nu f_\nu$ ($f_\nu\in S[X_1, X_2, \cdots, \check{X}_i, \cdots, X_e;\ D_1, D_2, \cdots, \check{D}_i, \cdots, D_e]$). Then $(f\Phi_i)X_i\Phi_i=f^\bullet(X_i+1)=X_if+fE_i+f=(X_if+fE_i)\Phi_i$ shows that Φ_i is a ring homomorphism of $S[\mathfrak{X}_e:\mathfrak{D}_e]$, and moreover, $\Phi_i^p=1$ (since $\chi(S)=p$) means that Φ_i is an automorphism of order p. From the definition of Φ_i, $\Phi_i\Phi_j=\Phi_j\Phi_i$ is clear. Next, $(X_i^p-s_i)\Phi_j=(X_i+\delta_{ij})^p-(X_i+\delta_{ij})-s=X_i^p-s$. Hence each Φ_j induces an automorphism σ_j^* of order p in R^*. Let $(\sigma_j^*)\cap(\sigma_1^*, \sigma_2^*, \cdots, \sigma_{j-1}^*)\ni\sigma^*=\sigma_j^{*k}$. Then

$y_j\sigma^* = y_j + k = y_j\sigma_1^{*\alpha_1}\sigma_2^{*\alpha_2}\cdots\sigma_{j-1}^{*\alpha_{j-1}} = y_j$ show that $\sigma^* = 1$. Consequently, we have $\mathfrak{G}^*$, the group generated by $\sigma_1^*, \sigma_2^*, \cdots, \sigma_e^*$, is an abelian group of order p such that the direct product of $\sigma_1^*, \sigma_2^*, \cdots, \sigma_e^*$. Noting here $J(\sigma_i^*, R^*) = S[y_1, y_2, \cdots, \check{y}_i, \cdots, y_e]$, we have $J(\mathfrak{G}^*, R^*) = \bigcap_{i=1}^{e} J(\sigma_i^*, R^*) = S$. Thus R^*/S is a p^e-dimensional weakly Galois extension with respect to $\mathfrak{G}^*$. Thus in following, we shall show that $V^* = V_{R^*}(S)$ is simple. By [Lemma 2 of [6]], it suffices to prove that $C^*[v_1, v_2, \cdots, v_k]$ is a field where $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_k\} = \mathfrak{G}^*{}_{\cap}\tilde{V}^*$.

Since $\tilde{v}_i\sigma_j^* = \sigma_j^*\tilde{v}_i$, $v_i\sigma_j^* = c_{ij}v_i$ for some $c_{ij} \in C^* = V^{R^*}(R^*)$ (and $\tilde{v}_i\tilde{v}_j = \tilde{v}_j\tilde{v}_i$ since $v_i\tilde{v}_j = c \cdot v_i$ for some $c \in C^*$ implies $v_i = v_i\tilde{v}_j^p = c^p v_i$). Noting here that C^* is cyclic over $J(\sigma_j^* | C^*, C^*)$ and $N_p(c_{ij}; \sigma_j^*) = 1$, there exists an element $d_{ij} \in C^*$ such that $c_{ij}^{-1} = d_{ij}^{-1}(d_{ij}\sigma_j^*)$. Setting here $u_{ij} = v_i d_{ij}$, then u_{ij} is contained in $V_{S[s_1, y_2 \cdots \check{y}_j, \cdots, y_e]}(S) = V_j^*$. Hence $v_i \in C^*[V_j^*]$ for erch $j = 1, 2, \cdots, e$, and so, $v_i \in \bigcap_{j=1}^{e} C^*[V_j^*]$. Noting that $\bigcap_{j=1}^{e} C^* \otimes_{C^*\cap S} V^* = C^* \otimes_{C^*\cap S}(\bigcap_{j=1}^{e} V_j^*) = C^* \otimes_{C^*\cap S} Z$ (considered in $C^* \otimes_{C^*\cap S} V^*$) and $\bigcap_{j=1}^{e} C^*[V_j^*]$ is an epimorphic image of $\bigcap_{j=1}^{e} C^* \otimes_{C^*\cap S} V_j^*$, we have $v_i \in C^*[Z]$. Now we shall show that $C^*[Z]$ is a field. For, each $z \in Z$, $zE_k = zy_k - y_k z \in Z$ and hence, $z^p E_k = p(zE_k z^{p-1}) = 0$. Hence $z^p \in C^*{}_{\cap}S$. While, $C^* \otimes_{C^*\otimes S} Z$ is a commutative semi-simple artinian ring since $C^*/C^*{}_{\cap}S$ is a finite Galois extension field. Let $uv = 0$ for some $u, v \in C^* \otimes_{C^*\cap S} Z$. Then $0 = (uv)^p = u^p v^p$ yields at once $u^p = 0$ or $V^p = 0$ since $u^p, v^p \in C^*$, and hence $u = 0$ or $v = 0$. Thus $C^* \otimes_{C^*\cap S} Z \cong C^*[Z]$ is a field. Consequently, $C^*[v_i]$ is a field since $[C^*[v_i] : C^*] < \infty$. Repeating the same procedures, we have $C^*[v_1, v_2, \cdots, v_k] \subseteqq (C^*[Z])$ is a field.

Conversely, let R be a p^e-dimensinal fundamental abelian extension over S with respect to $\mathfrak{G} = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_e)$. Then $R = S[x_1, x_2, \cdots, x_e]$ with $x_i\sigma_j = x_i + \delta_{ij}$ [Corollary 2 of [5]]. Then $D_i = I_{x_i}|S$ is a derivation in S since $(sx_i - x_i s)\sigma_j = s(x_i + \delta_{ij}) - (x_i + \delta_{ij})s = sx_i - x_i s$. Further, $(x_i x_j - x_j x_i)\sigma_k = (x_i + \delta_{ik})$. $(x_j + \delta_{jk}) - (x_j + \delta_{jk})(x_i + \delta_{ik}) = x_i x_j - x_j x_i$. Hence, if we set $t_{ij} = x_i x_j - x_j x_i$, then $t_{ij} = -t_{ji}$, $t_{ii} = 0$ and $\{D_1, D_2, \cdots, D_e, (t_{ij})\}$ satisfies conditions (1) and (2). Thus there exists a polynomial ring $S[\mathfrak{X}_e; \mathfrak{D}_e]$. Next, $(x_i^p - x_i)\sigma_j = (x_i + \delta_{ij})^p - (x_i + \delta_{ij})$ shows that $s_i = x_i^p - x_i \in S$ and $X_i^p - s_i$ is central in $S[\mathfrak{X}_e; \mathfrak{D}_e]$. Hence conditions (3) and (4) are satisfied. Finally, if we note that V is a field [Theorem 5 of [5]], each intermediate ring of R/S is simple [Lemma 1.4 of [7]]. Hence $X_i^p - s_i$ is $s.w$-irreducible. Let φ^* be the map defined by $\sum x_e^{\nu_e} x_{e-1}^{\nu_{e-1}} \cdots x_1^{\nu_1} s_{\nu_e \cdots \nu_1} \to \sum y_e^{\nu_e} y_{e-1}^{\nu_{e-1}} \cdots y_1^{\nu_1} s_{\nu_e \cdots \nu_1}$ of $R = S[x_1, x_2, \cdots, x_e]$ *to* $R^* = S[y_1, y_2, \cdots, y_e]$, then it is clear that φ^* is an S-isomrophism of R to R^* satisfying $\varphi^*\sigma_i^* = \sigma_i\varphi^*$.

Corollary 3.1. Let $\chi(S)=p$. (**a**) *In order that S have a p^e-dimensional fundamental inner abelian extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying* (1)–(4) *of Theorem* 3.1 *and (z_i) a $1\times e$ matrix with entries in Z^* satisfying*

(6) $z_i D_j = z_i \delta_{ij}$.

(**b**) *In order that S have a p^e-dimensional fundamental outer abelian extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ji}), (s_i)\}$ satisfying* (1)–(5) *of Theorem* 3.1 *and*

(6′) $zD_j=0$ *for all* $j=1,2,\cdots,e$.

Proof. By Proposition 2.3 (a), there exists a p^e-dimensional e-polynomial simple ring extension $R^*=S[y_1, y_2, \cdots, y_e]$ over S where each y_i is the residue class of X_i modulo M. Let $\sum_{\nu=0}^{p-1} X_j^\nu f_\nu$ be a polynomial of $S[y_1, y_2, \cdots, y_{j-1}][X_j; E_j]$ $(f_\nu \in S[y_1, y_2, \cdots, y_{j-1}])$. Then, for each z_j satisfying (6), $z_j(\sum_{\nu=0}^{p-1} X_j^\nu f_\nu) = (\sum_{\nu=0}^{p-1}\sum_{i=0}^{\nu}\binom{\nu}{i} X_j^i z_j f_\nu) = \sum_{\nu=0}^{p-1}(\sum_{i=0}^{\nu}\binom{\nu}{i} X_j^i f_\nu z_j)$. Hence we have $(\sum_{\nu=0}^{p-1} X_j^\nu f_\nu) I_{x_j}^p = (p-1)!\, z_j^{p-1} f_{p-1}$ by induction on the degree of the polynomial. Thus each $X_i^p - s_i$ is $s.w$-irreducible. Consequently, R^* is a p^e-dimensional fundamental abelian extension of S. Next, $z_i D_j = z_i y_j - y_j z_i = z_i \delta_{ij}$ shows that $z_i y_j z_i^{-1} = y_j + \delta_{ij}$. Thus we may adapt $\tilde{z}_i$ as σ_i^* since $z_i^p \in C^*$.

Conversely, let R/S be a p^e-dimensional fundamental inner abelian extension over S with respect to $\mathfrak{G}=(\tilde{v}_1)\times(\tilde{v}_2)\times\cdots\times(\tilde{v}_e)$, $v_i \in V$. Since $V=C[Z]$ [Theorem 5 of [5]], we have $v_i = \sum_k c_k^{(i)} z_k$ where $\{c_k^{(i)}\}$ $(\subseteqq C)$ is a Z-basis for V. On the other hand, $v_i x_j v_i^{-1} = x_j + \delta_{ij}$ implies $v_i E_j = v_i x_j - x_j v_i = v_i \delta_{ij}$. Hence $v_i E_j = \sum_k c_k^{(i)}(z_k^{(i)} E_j) = v\delta_{ij}$ shows that the existence of $(z_1, z_2, \cdots, z_e)$ satisfying (6).

(**b**) Under the same notations of the proof of (a), if $\mathfrak{G}^*$ contains $\tau = \sigma_1^{*\alpha_1}\sigma_1^{*\alpha_2}\cdots\sigma_k^{*\alpha_k} = \tilde{v}$ $(v \in V^*)$, as was shown in the latter half of the proof of (a), there exists $z \in Z^*$ such that $zD_j = \alpha_j z$ $(j=1,2,\cdots,k)$, and this contradicts $ZD_j=0$ for all $j=1,2,\cdots,e$.

Conversely, if R is a p^e-dimensional outer Galois extension over S, then $Z \subseteqq C$. Hence $ZD_j=0$ since $CD_j=0$ for all $j=1,2,\cdots.e$.

R is called *a trivial extension over* S if $R=S[C]$. Then we can prove the following:

Corollary 3.2. *Let $\chi(S)=p$. Then the following conditions are equivalent.*

(**a**) *S has a p^e-dimensional fundamental trivial abelian extension.*

(**b**) *There exist elements $z_1, z_2, \cdots, z_e$ in Z^* such that the algbraic set associated with a two-sided ideol $M=(X_1^p - z_1, X_2^p - z, \cdots, X_e^p - z_e)Z[X_1, X_2, \cdots, X_e]$ is an algebraic variety.*

(**c**) *There exist elements $s_1, s_2, \cdots, s_e$ in S such that $X_i^p - s_i$ is $s.w$-irre-*

ducible in $S[X_1, X_2, \cdots, X_e]$.

Proof. (**a**)→(**b**) Let $R=S[C]=S\otimes_Z C$. Then $J(\mathfrak{G}|C, C)=Z$ means that C/Z is a p^e-dfmensional fundamental abelian extension with the (abelian) Galois group $\mathfrak{G}|C$. Hence (b) is clear.

(**a**)→(**c**) Let $R=S[C]$ amd $\{c_e^{\nu_e}c_{e-1}^{\nu_{e-1}}\cdots c_1^{\nu_1}\,;\ 0\leqq\nu_i\leqq p-1,\ c_i\in C\}$ be Z-basis such that $c_i^p-c_i\in Z$. Then $X_i^p-c_i$ is $s\cdot w$-irreducible in $S[X_1, X_2, \cdots, X_i]$.

(**b**)→(**a**) Since $Z'=Z[X_1, X_2, \cdots, X_e]/M$ is a p^e-dimensional fundamental abelian extension, $S\otimes_Z Z'$ is a requested one.

(**c**)→(**a**) Let $M=(X_1^p-s_1, X_2^p-s_2, \cdots, X_e^p-s_e)\,S[X_1, X_2, \cdots, X_e]$. Then $S[X_1, X_2, \cdots, X_e]/M$ is a requested one.

Let $X_i^p-s_i$ be s-irreducible. Then $S[\mathfrak{X}_e; \mathfrak{D}_e]/M$ is a division ring. Hence we obtain

Corollary 3.3. *Let S be a division ring of $\chi(S)=p$. In order that S have a p^e-dimensional fundamental abelian division ring extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying* (1)–(4) *of Theorem* 3.1 *and*

(5′) *$X_i^p-s_i$ is s-irreducible.*

Let R/S be a p^e-dimensional fundamental abelian extension with respect to $\mathfrak{G}$. Then R/S is called *a parallel extension* if $R=S[T]$ such that $T/T_\cap S$ is a p^e-dimensional fundamental abelian extension with respect to $\mathfrak{G}|T$. A p^e-dimensional fundamental parallel abelian extension is called *a d-parallel* (resp *an f-parallel*) *extension* if T can be a division ring (resp. *a field*).

Corollary 3.4. *Let $\chi(S)=p$.* (**a**) *In order that S have a p^e-dimensional fundamental parallel extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying* (1)–(5) *of Theorem* 3.1 *and*

(6) *there exists a simple subring $W(\ni 1)$ of S satisfying $WD_i\subseteqq W$, $\{\{t_{ij}\}, \{s_i\}\}\subseteqq W$ and $X_i^p-s_i$ is $s\cdot w$-irreducible.*

(**b**) *In order that S have a p^e-dimensional fundamental d-parallel abelian extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying* (1)–(5) *of Theorem* 3.1 *and*

(6′) *there exists a divison subring $W(\ni 1)$ of S satisfying $WD_i\subseteqq W$, $\{\{t_{ij}\}, \{s_i\}\}\subseteqq W$ and $X_i^p-s_i$ is s-irreducible in $W[\mathfrak{X}_{\pi(e)}; \mathfrak{D}_{\pi(e)}]$.*

(**c**) *In order that S have a p^e-dimensional fundamental f-parallel abelian extension, it is necessary and sufficient that there exist $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\{$ satifying* (1)–(5) *of Theorem* 3.1 *and*

(6″) *there exists a subfield $W(\ni 1)$ of S satisfying $X_i^p-s_i$ is s-irreducible in $W[X_1, X_2, \cdots, X_e]$.*

Proof. Let $R^*=S[y_1, y_2, \cdots, y_e]\cong S[\mathfrak{X}_e; \mathfrak{D}_e]/M$, and $T^*=\{\sum y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots$

$y_1^{\nu_1} w_{\nu_e \cdots \nu_1}$; $w_{\nu_e \cdots \nu_1} \in W\}$. Then $J(\mathfrak{G}^* | T^*, T^*) = T^* \cap S = W$ and $p^e = \# \mathfrak{G}^* = [T^* : W]$ and $R = S[T^*]$.

(**a**) *Since* $T^* \cong W[\mathfrak{X}_e; \mathfrak{D}_e]/M_W$, $M_W = (X_1^{\mathfrak{p}} - s_1, X_2^{\mathfrak{p}} - s_2, \cdots, X_e^{\mathfrak{p}} - s_e)\, W[\mathfrak{X}_e; \mathfrak{D}_e]$, T^* is a simple ring.

Conversely, if $R = S[T]$ is a p^e-dimeesional fundamental parallel abelian extension with respect to $\mathfrak{G}$, then $T/T \cap S$ is a p^e-dimensional fundamental abelian extension with respect to $\mathfrak{G}|T$. Hence there exists $\{t_1, t_2, \cdots, t_e\} \subseteqq T$ with $t_i \sigma_j = t_i + \delta_{ij}$. If we set $D_i = I_{t_i}|S$, $t_{ij} = t_i t_j - t_j t_i$, $s_i = t_i^{\mathfrak{p}} - t_i$ and $W = T \cap S$, then our assertion is easy by the proof of Theorem 3.1.

(**b**) Combining (a) with Corollary 3.3, we readily obtain (b).

(**c**) Combining (a) with Corollary 3.2, we readily obtain (c).

Now, in the following, we shall deal with a regularly embedding problem of an abelian extension.

Let R/S be $\mathfrak{G}$-regular, and an intermediate ring T of R/S be satisfy $J(\mathfrak{H}, T) = S$ and $[T : S] = \# \mathfrak{H}$ where $\mathfrak{H} \subseteqq \mathfrak{G}$. If $\mathfrak{H} = \mathfrak{G}|T$, T/S is said to be *regularly embedded in* R/S.

Let T/S be a p^e-dimensional abelian extension with respect to $\mathfrak{H} = (\tau_1) \times (\tau_2) \times \cdots \times (\tau_m)$, $\#(\tau_i) = p^{f_i}$. We shall give a necessary and sufficient condition for T/S can be regularly embedded in R/S where R/S is a p^{f+m+k}-dimensional abelian extension with respect to $\mathfrak{G} = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_m) \times (\sigma_{m+1}) \times \cdots \times (\sigma_{m+k})$ if $k \geqq 0$ and $\mathfrak{G} = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_{m+k}) \times (\tau_{m+k+1}) \times \cdots \times (\tau_m)$ if $-m < k < 0$.

Let s be an element of $S \subseteqq S[\mathfrak{X}_e; \mathfrak{D}_e]$, and let $\Delta_0^{(i)}(s) = 1$ and $\Delta_k^{(i)}(s) = \Delta_{k-1}^{(i)}(s) D_i + \Delta_{k-1}^{(i)}(s) s$. Then we have $(X_i + s)^n = \sum_{k=0}^{n} \binom{k}{n} X_i^k \Delta_{n-k}^{(i)}(s)$ [§1. I, (ii) of [4]].

Theorem 3.2. *Let* $\chi(S) = p$ *and* T/S *be a* p^f*-dimensional abelian extension with respect to* $\mathfrak{H} = (\tau_1) \times (\tau_2) \times \cdots \times (\tau_m)$, $\# \tau_i = p^{f_i}$. *In order that* T/S *can be regularly embedded to a* p_{f+m+k}*-dimensional abelian extension such that* $k \geqq -m$, *it is necessary and sufficient that there exist derivations* $D_1, D_2, \cdots, D_m, \cdots, D_{m+k}$ *in* T, (t_{ij}) *an* $(m+k) \times (m+k)$ *matrix with entries in* T, (t_i) *a* $1 \times (m+k)$ *matrix with entries in* T *satisfying the conditions* (1)–(5) *of Theorem* 3.1 *(in* T*) and* (b_{ij}) *an* $(m+k) \times (m+k)$ *matrix with entries in* T *satisfying*

(1) $D_i \tau_j - \tau_j D_i = \tau_j I_{b_{ij}}$,

(2) $T_{\# \tau_j}(b_{ij}; \tau_j) = \delta_{ij}$,

(3) $\Delta_p^{(i)}(b_{ij}) - b_{ij} = t_i(\tau_j - 1)$,

(4) $t_{ij}(\tau_k - 1) = b_{ik}(\tau_k^{-1} D_j \tau_k) - b_{jk} D_i$,

(5) $b_{ij}(\tau_k - 1) = b_{ik}(\tau_j - 1)$

More precisely, if there exists $\{D_1, \cdots, D_{m+k}, (t_{ij}), (t_i), (b_{ij})\}$ *satisfying the*

conditions, then $R^*=T[y_1, y_2, \cdots, y_m, y_{m+1}, \cdots, y_{m+k}]=T[\mathfrak{X}_{m+k}; \mathfrak{D}_{m+k}]/M$, $M=(X_1^{\mathfrak{p}}-t_1, X_2^{\mathfrak{p}}-t_2, \cdots, X_{m+k}^{\mathfrak{p}}-t_{m+k})T[\mathfrak{X}_{m+k}; \mathfrak{D}_{m+k}]$ *and* y_k *is the residue class of* X_k *modulo* M, *is a requested one with generating automorphisms* $\sigma_1^*, \sigma_2^*, \cdots, \sigma_m^*, \cdots, \sigma_{m+k}^*$ *such that* $(y_i t)\sigma_j^*=(y_i+b_{ij})(t\tau_j)$ $(t\in T)$ *and* $\mathfrak{G}^*|T=\mathfrak{H}$ *where* $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_m^*)\times\cdots\times(\sigma_{m+k}^*)$ *if* $k\geqq 0$ *and* $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_{m+k}^*)\times(\tau_{m+k+1})\times\cdots\times(\tau_m)$ *if* $k<0$. *Conversely, if* T/S *can be regularly embedded in a* p^{f+m+k}*-dimensional abelian extension with respect to* $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_{m+k})$ *then there exists such* $\{D_1, D_2, \cdots, D_{m+k}, (t_{ij}), (t_i), (b_{ij})\}$ *satisfying the conditions in Theorem* 3.1 *and* (1)–(5) *that there holds a* T*-isomorphism* $\varphi^*: R^*\cong R$ *with* $\varphi^*\sigma_i=\sigma_i^*\varphi^*$.

Proof. Firstly, we set $b_{ij}=\delta_{ij}$ if $k>0$, $j>m$ *or* $b_{ij}=0$, if $k<0$ and $j>m$. Then, clearly, they satisfy the conditions (1)–(5).

Let Φ_j $(j=1, 2, \cdots, m+k)$ be the mapping of $T[\mathfrak{X}_{m+k}; \mathfrak{D}_{m+k}]$ defined by $\sum X_{m+k}^{\nu_{m+k}}\cdots X_1^{\nu_1}t_{\nu_{m+k}\cdots\nu_1}\to\sum(X_{m+k}+b_{m+kj})^{\nu_{m+k}}\cdots(X_1+b_{1j})^{\nu_1}\cdot(t_{\nu_{m+k}\cdots\nu_1}\tau_j)$ (we set $\tau_j=1$ if $j>m$). Then we can prove that Φ_j is an automorphism by induction on the number of inderteminates. By (1), we have $(t\Phi_j)X_i\Phi_j=(t\tau_j)(X_i+b_{ij})=X_i(t\tau_j)+t\tau_j D_i+t\tau_j b_{ij}=X_i t\tau_j+tD_i\tau_j-(t\tau_j b_{ij}-b_{ij}t\tau_j)+t\tau_j b_{ij}=X_i t\tau_j+tD_i\tau_j+b_{ij}t\tau_j=(X_i t+tD_i)\Phi_j$. Hence Φ_j is a (ring) epimorphism of $T[X_i; D_i]$. Further $(\sum_{\nu=0}^m X_i^\nu t_\nu)\Phi_j=\sum_{\nu=0}^m(X_i+b_{ij})^\nu(t_\nu\tau_j)=X_i^m t_m\tau_j+X_i^{m-1}\left(\binom{m}{m-1}b_{ij}D_i t_m\tau_j+t_{m-1}\tau_j\right)+\cdots+(\Delta_m^{(i)}(b_{ij})t_m\tau_j+\cdots+t_0\tau_j)$ shows that Φ_j is an automorphism. Hence, we assume that Φ_j is an automorphism in $T[\mathfrak{X}_{m+k-1}; \mathfrak{D}_{m+k-1}]$. Thus it suffices to prove $(Y\Phi_j)(X_{m+k}\Phi_j)=(X_{m+k}\Phi_j)(Y\Phi_j)+(YE_{m+k})\Phi_j$ where $Y\in T[\mathfrak{X}_{m+k-1}; \mathfrak{D}_{m+k-1}]$. Now, $(Y\Phi_j)(X_{m+k}\Phi_j)=(Y\Phi_j)(X_{m+k}+b_{m+kj})=X_{m+k}(Y\Phi_j)+(Y\Phi_j)E_{m+k}+(Y\Phi_j)b_{m+kj}$ and $(X_{m+k}\Phi_j)(Y\Phi_j)+(YE_{m+k})\Phi_j=(X_{m+k}+b_{m+kj})(Y\Phi_j)+(YE_{m+k})\Phi_j$. Hence the assertion is clear if we show that $(Y\Phi_j)I_{b_{m+kj}}=Y(E_{m+k}\Phi_j-\Phi_j E_{m+k})$.

Now, $X_i(E_{m+k}\Phi_j-\Phi_j E_{m+k})=(t_{im+k})\tau_j-t_{im+k}-b_{ij}D_{m+k}=b_{ij}(\tau_j^{-1}D_{m+k}\tau_j)-b_{m+kj}D_i-b_{ij}D_{m+k}$ by (4) and $X_i\Phi_j I_{b_{m+kj}}=(X_i+b_{ij})b_{m+kj}-b_{m+kj}(X_i+b_{ij})=b_{ij}b_{m+kj}-b_{m+kj}b_{ij}-b_{m+kj}D_i=(b_{ij}\tau_j^{-1})\tau_j I_{b_{m+kj}}-b_{m+kj}D_i=(b_{ij}\tau_j^{-1})(D_{m+k}\tau_j-\tau_j D_{m+k})-b_{m+kj}D_i=b_{ij}(\tau_j^{-1}D_{m+k}\tau_j)-b_{ij}D_{m+k}-b_{m+kj}D_i$ by (1). Hence we have $X_i(E_{m+k}\Phi_j-\Phi_j E_{m+k})=(X_i\Phi_j)I_{b_{m+kj}}$.

Let us assume that $Y(E_{m+k}\Phi_j-\Phi_j E_{m+k})=(Y\Phi_j)I_{b_{m+kj}}$ for each $Y=X_{m+k-1}^{\nu_{m+k-1}}\cdots X_1^{\nu_1}t_{\nu_{m+k-1}\cdots\nu_1}\in T[\mathfrak{X}_{m+k-1}; \mathfrak{D}_{m+k-1}]$ such that $\deg Y=q$. Then, for each $Y=X_{m+k-1}^{\nu_{m+k-1}}\cdots X_i^{\nu_i+i}\cdots X_1^{\nu_1}t_{\nu_{n+k=1}\cdots\nu_1}$, $Y(E_{m+k}\Phi_j-\Phi_j E_{m+k})=Y_1(E_{m+k}\Phi_j-\Phi_j E_{m+k})(Y_2\Phi_j)+(Y_1\Phi_j)(Y_2(E_{m+k}\Phi_j-\Phi_j E_{m+k}))=(Y_1\Phi_j I_{b_{m+kj}})(Y_2\Phi_j)+(Y_1\Phi_j)(Y_2\Phi_j I_{b_{m+kj}})=(Y_1Y_2)\Phi_j I_{b_{m+kj}}$ where $Y_1=X_{m+k-1}^{\nu_{m+k-1}}\cdots X_i^{\nu_i}$, $Y_2=X_i X_{i-1}^{\nu_{i-1}}\ X_\nu^{\nu_1}t_{\nu_{m+k-1}\cdots\nu_1}$. Therefore, we can see Φ_j is an automorphism.

$X_i\Phi_j\Phi_k=X_i+b_{ik}+b_{ij}\tau_k$, $X_i\Phi_k\Phi_j=X_i+b_{ij}+b_{ik}\tau_j$ shows that $\Phi_j\Phi_k=\Phi_k\Phi_j$ if and only if there holds (5).

$X_i\Phi_j^{\#\tau_j}=X_i+T_{\#\tau_j}(b_{ij};\tau_j)$ shows that (2) is equivalent with $\#\Phi_j=p^{f_j+j}$ if $j\leqq m$ and $\#\Phi_j=p$ if $j>m$. Now $(X_i^p-t_i)\Phi_j=X_i^p+(\Delta_p^{(i)}(b_{ij})-b_{ij})-t_i\tau_j$ shows that (3) is equivalent with that $X_i^p-t_i$ is left invariant under Φ_j. Thus each Φ_j induces an automorphism σ_j^* of order p^{f_j+1} if $j\leqq m$, of order p if $j>m$ in $R^*=T[y_1,y_2,\cdots,y_{m+k}]\cong T[\mathfrak{X}_{m+k};\mathfrak{D}_{m+k}]/M$ where R^* is a p^{f+k+m}-dimensional simple ring extension over S.

Let $\sigma\in(\sigma_j^*)_\cap(\sigma_1^*,\sigma_2^*,\cdots,\sigma_{j-1}^*)$. If $\sigma=\sigma_j^{*\alpha_j}=\sigma_1^{*\alpha_1}\sigma_2^{*\alpha_2}\cdots\sigma_{j-1}^{*\alpha_{j-1}}$, we may assume that each α_i is a multiple of each $\#\tau_i$ since $1=\sigma|T=\tau_j^{\alpha_j}=\tau_1^{\alpha_1}\tau_2^{\alpha_2}\cdots\tau_{j-1}^{\alpha_{j-1}}$, Then $y_j\sigma=y_j+T_{\alpha_j}(b_{jj};\tau_j)=y_j+\alpha_j/\#\tau_j=y_j\sigma_1^{*\alpha_1}\sigma_2^{*\alpha_2}\cdots\sigma_{j-1}^{\alpha_{j-1}}=y_j$ yields at once $\sigma=1$. Hence $\mathfrak{G}^*$, the group generated by $\sigma_1^*,\sigma_2^*,\cdots,\sigma_{m+k}^*$, is coincides with the direct product of them. Finally, $S\subseteqq J(\mathfrak{G}^*,R^*)=J(\mathfrak{G}^*|T,T)=S$. Thus the rest, we shall show that $\mathfrak{G}^*$ is an F-group.

Firstly, we note that $C^*[Z]$ is a commutative semi-simple artinian ring as same reason as that of Theorem 3.1. Let $\{\tilde{v}_1,\cdots,\tilde{v}_m\}=\mathfrak{G}^*_\cap\tilde{V}^*$. Then $v_i\sigma_j^*=c_{ij}v_i$ for some $c_{ij}\in C^*$ such that $N_{\#\sigma_j^*}(c_{ij};\sigma_j^*)=1$ and $v_iv_j=v_jv_i$. Thus, we can see that $C^*[v_1,v_2,\cdots,v_m]\subseteqq C^*[Z]$. If $v=\sum c_\alpha w_\alpha$, $c_\alpha\in C^*$ and $w_\alpha=v_m^{\beta_m}v_{m-1}^{\beta_{m-1}}\cdots v_1^{\beta_1}$, is an arbitrary element of $C^*[v_1,v_2,\cdots,v_m]$, *then* $v^{p^a}\in C^*$ for some positive integer a. Thus $C^*[v_1,v_2,\cdots,v_m]$ is a field.

Conversely, let T/S can be regularly embedded in a p^{f+k+m}-dimensional abelian extension R/S with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_m)\times(\sigma_{m+1})\times\cdots\times(\sigma_{m+k})$ if $k\geqq 0$ and $\mathfrak{G}=(\sigma_1)\times\cdots\times(\sigma_{m+k})\times(\tau_{m+k+1})\times\cdots\times(\tau_m)$ if $-m<k<0$ such that $\#\sigma_j=(\#\tau_j)p$, $\sigma_j|T=\tau_j$ if $j\leqq m$ and $\#\sigma_j=p$, $\sigma_j|T=1$ if $j>m$. Since V is a field and $[R:T]=p^{m+k}$, R/T is a fundamental abelian extension with respect to $\mathfrak{G}(T)=(\sigma_1^{p^{f_1}})\times\cdots\times(\sigma_m^{p^{f_m}})\times(\sigma_{m+1})\times\cdots\times(\sigma_{m+k})$. Then $R=\bigoplus_{0\leqq\nu_i<p}x_e^{\nu_e}x_{e-1}^{\nu_{e-1}}\cdots x_1^{\nu_1}T$, $x_i\sigma_j^{p^{f_j}}-x_i=\delta_{ij}$ if $j\leqq m$ and $x_i\sigma_j-x_i=\delta_{ij}$ if $j>m$. Evidently $b_{ij}=x_i\sigma_j-x_i$ is contained in T and $T_{\#\tau_j}(b_{ij};\tau_j)=\sum_{k=0}^{\#\tau_j-1}(x_i\sigma_j-x_i)\sigma_j^k=\delta_{ij}$, further, $b_{ij}(\tau_k-1)=(x_i\sigma_j-x_i)(\sigma_k-1)=(x_i\sigma_j-x_i)\sigma_k-(x_i\sigma_j-x_i)=(x_i\sigma_k-x_i)\sigma_j-(x_i\sigma_k-x_i)=b_{ik}(\tau_j-1)$.

Now, we set $D_i=I_{x_i}$, $t_i=x_i^p-x_i$ and $t_{ij}=x_ix_j-x_jx_i$. Then one will easily check that $\{D_1,D_2,\cdots,D_{m+k},(t_i),(t_{ij})\}$ satisfies the conditions (1)–(5) of Theorem 3.1 in T.

$t(D_i\tau_j-\tau_jD_i)=(tx_i-x_it)\tau_j-(t\tau_jx_i-x_it\tau_j)=t\tau_j(x_i\sigma_j-x_i)-(x_i\sigma_j-x_i)t\tau_j$
$=t\tau_jI_{b_{ij}}$ $(t\in T)$.

$t_{ij}(\tau_k-1)=(x_ix_j-x_jx_i)\sigma_k-(x_ix_j-x_jx_i)=(x_ix_j)\sigma_k-x_i(x_j\sigma_k)-(x_jx_i)\sigma_k+x_j\sigma_kx_i-(x_j\sigma_k-x_j)x_i+x_i(x_j\sigma_k-x_j)=(x_i\sigma_k-x_i)x_j\sigma_k-x_j\sigma_k(x_i\sigma_k-x_i)-b_{jk}D_i$
$=b_{ik}\tau_k^{-1}D_j\tau_k-b_{jk}D_i$.

Finally, $(x_i^p-x_i)\sigma_j=x_i^p+\Delta_p^{(i)}(b_{ij})-x_i-b_{ij}$ shows that $\Delta_p^{(i)}(b_{ij})-b_{ij}=(x_i^p-x_i)(\sigma_j-1)=t_i(\tau_j-1)$.

The map $\varphi^*: \sum X_e^{\nu_e}\cdots X_1^{\nu_1}t_{\nu_e\cdots\nu_1}\to\sum y_e^{\nu_e}\cdots y_1^{\nu_1}t_{\nu_e\cdots\nu_1}$ $(t_{\nu_e\cdots\nu_1}\in T)$ is a T-isomorphism of R^* onto R satisfying $\varphi^*\sigma_i=\sigma_i^*\varphi^*$.

Combining Corollary 3.3 with Theorem 3.2, we can easily obtain

Corollary 3.5. *Let T/S be a p-dimensional abelian division ring extension of $\chi(S)=p$ with respect to $\mathfrak{H}=(\tau_1)\times(\tau_2)\times\cdots\times(\tau_m)$ such that $\#\tau_i=p^{f_i}$. In order that T/S can be regularly embedded in a p^{f+k+m}-dimensional abelian division ring extension such that $k\geqq -m$, it is necessary and sufficient that there exist derivations $D_1, D_2, \cdots, D_{m+k}$ in T, (t_{ij}) an $(m+k)\times(m+k)$ matrix with entries in T, (t_i) a $1\times(m+k)$ matrix with entries in T satisfying the conditions* (1)–(5′) *of Corollary* 3.3 *(in T) and (b_{ij}) an $(m+k)\times(m+k)$ matrix with entries in T satisfing the conditions* (1)–(5) *of Theorem* 3.2.

§ 4. Kummer extensions.

Throughout the present section, we assume that Z contains ζ, a primitive n-th root of 1. An n-dimensional abelian extension R over S is called *an n-dimensional Kummer extension* if $C\ni\zeta$ and there exists a Galois group $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, a direct product of σ_i with $\#\sigma_i=n_i$, $\prod_{i=1}^{e} n_i=n$ and $\mathfrak{G}$ is a DF-group.

One of principal theorem of this section is the following

Theorem 4.1. *In order that S have an n-dimensional Kummer extension, it is necessary and sufficient that there exist automorphisms $\rho_1, \rho_2, \cdots, \rho_e$ in S, (s_{ij}) an $e\times e$ matrix with entries in $S^\bullet$ such that $s_{ij}=s_{ji}^{-1}$, $s_{ii}=1$ and (s_i) a $1\times e$ matrix with entries in $S^\bullet$ satisfying*

(1) $\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=\tilde{s}_{ij}$,

(2) $s_{ij}(s_{ik}\rho_j)s_{jk}=(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k)$,

(3) $\rho_i^{n_i}=\tilde{s}_i^{-1}$ *such that* $\prod_{i=1}^{e} n_i=n$, $s_i\rho_i=s_i$ *and* $\zeta\rho_i=\zeta$,

(4) $RN_{n_i}(s_{ji};\rho_i)=(s_i^{-1}\rho_j)s_i$,

(5) $f(X_{\pi(i)})=X_{\pi(i)}^{n_{\pi(i)}}-s_{\pi(i)}$ *is s.w-irreducible in* $S[\mathfrak{X}_{\pi(e)};\mathfrak{P}_{\pi(e)}]$,

(6) *If $g(X)$ is a polynomial of $S[\mathfrak{X}_{\pi(e)};\mathfrak{P}_{\pi(e)}]\setminus M$, $M=(f(X_1), f(X_2),\cdots,f(X_e))$, then $\sum_i f_i(X)g(X)u_i=1$ modulo M for some $f_i(X)\in S[\mathfrak{X}_{\pi(e)};\mathfrak{P}_{\pi(e)}]$, $u_i\in S$.*

More precisely, if there exists $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying (1)–(6), *then $M=(f(X_1),f(X_2),\cdots,f(X_e))S[\mathfrak{X}_e;\mathfrak{P}_e]$ is maximal and $R^*=S[y_1,y_2,\cdots,y_e]\cong S[\mathfrak{X}_e;\mathfrak{P}_e]/M$, y_i is the residue class of X_i modulo* M, *is an n-dimensional Kummer extension with generating automorphisms $\sigma_1^*,\sigma_2^*,\cdots,\sigma_e^*$ such that*

$$x_i\sigma_j^*\begin{cases}=x_i\zeta^{n/n_i} & \text{if } i=j\\ =x_i & \text{if } i\neq j.\end{cases}$$

Conversely, if R is an n-dimensional Kummer extension over S with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, $\#\sigma_i=n_i$ and $\prod_{i=1}^{e} n_e=n$, then we can find such $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying (1)–(5) *and $S[\mathfrak{X}_e;\mathfrak{P}_e]$ satisfying* (6) *that there holds an S-isomorphism $\varphi^*: R^*\cong R$ with $\varphi^*\sigma_i=\sigma_i^*\varphi^*$ for each $i=1,2,\cdots,e$.*

Proof. By Proposition 2.2, there exists a polynomial ring $S[\mathfrak{X}_e;\mathfrak{P}_e]=\{\sum X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1};\ s_{\nu_e\cdots\nu_1}\in S\}$ such that $sX_i=X_i(s\rho_i)$, $X_iX_j=X_jX_is_{ij}$ ((1) and (2)).

Since $X_j(X_i^{n_i}s_i^{-1}-1)=X_i^{n_i}X_jRN_{n_i}(s_{ji};\rho_i)s_i^{-1}-X_j$, (4) means that $X_j(X_i^{n_i}s_i^{-1}-1)=(X_i^{n_i}s_i^{-1}-1)X_j$. Thus, with the first two conditions of (3), we can see that $M=(f(X_1),f(X_2),\cdots,f(X_e))S[\mathfrak{X}_e;\mathfrak{P}_e]$ is a two-sided ideal, and by (5), it is maximal.

Now, let Ψ_i $(i=1,2,\cdots,e)$ be the map of $S[\mathfrak{X}_e;\mathfrak{P}_e]$ defined by $\sum X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots X_i^{\nu_i}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1}\to\sum X_e^{\nu_e}X_{e-1}^{\nu_{e-1}}\cdots(X_i\zeta^{n/n_i})^{\nu_i}\cdots X_1^{\nu_1}s_{\nu_e\cdots\nu_1}$. Then Ψ_i is an automorphism. For, $f\Psi_iX_i\Psi_i=f\cdot X_i\zeta^{n/n_i}=X_i(fP_i)\zeta^{n/n_i}$ and $[X_i(fP_i)]\Psi_j=X_i\zeta^{n/n_i}fP_i=X_i(fP_i)\zeta^{n/n_i}$ (by the last condition of (3)) for each $f\in S[X_1,\cdots,\check{X}_i,\cdots,X_e;P_1,\cdots,\check{P}_i,\cdots,P_e]$. Hence Ψ_i is a ring epimorphism, and $\#\Psi_i=n_i$ shows that Ψ_i is an automorphism.

$(X_i^{n_i}-s_i)\Psi_i=(X_i\zeta^{n/n_i})^{n_i}-s_i=X_i^{n_i}-s_i$ implies that $M\Psi_i=M$. Thus each Ψ_i induces an automorphism σ_i^* of order n_i in $R^*=S[y_1,y_2,\cdots,y_e]$ where R^* is an n-dimensional simple ring over S which is isomorphic to $S[\mathfrak{X}_e;\mathfrak{P}_e]/M$ and y_i is the residue class of X_i modulo M.

Let $\sigma\in(\sigma_i^*)\cap(\sigma_1^*,\sigma_2^*,\cdots,\sigma_{i-1}^*)$. If $\sigma=\sigma_i^{*\alpha_i}=\sigma_1^{*\alpha_1}\sigma_2^{*\alpha_2}\cdots\sigma_{i-1}^{*\alpha_{i-1}}$, $x_i\sigma_i^*=x_i\zeta^{kn/n_i}=x_i\sigma_1^{*\alpha_1}\sigma_2^{*\alpha_2}\cdots\sigma_{i-1}^{*\alpha_{i-1}}=x_i$ shows that $\sigma=1$. Hence $\mathfrak{G}^*$, the group generated by $\sigma_1^*,\sigma_2^*,\cdots,\sigma_e^*$, coincides with a direct product of them since $\sigma_i^*\sigma_j^*=\sigma_j^*\sigma_i^*$ is clear. Thus we have $J(\mathfrak{G}^*,R^*)=\bigcap_{i=1}^{e}J(\sigma_i^*,R^*)=S$.

The condition (6) is equivalennt with that R^* is $R_l^*S_r$-irreducible. Hence, by Proposition 1.2, V^* is a division ring.

Conversely, if R is an n-dimensional Kummer extension over S with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, $\#\sigma_i=n_i$, then $R=\bigoplus_{0\leq\nu_i\leq n_i-1}x_e^{\nu_e}\cdots x_1^{\nu_1}S$ where each x_i is an element contained in $R^{\bullet}$ and $x_i\sigma_j\begin{cases}=x\zeta^{n/n_i} & \text{if } i=j\\ =x_i & \text{if } i\neq j\end{cases}$ [Theorem 4 of [8]].

We set $\rho_i=\tilde{x}_i^{-1}|S$, $s_i=x_i^{n_i}$, $s_{ij}=x_i^{-1}x_j^{-1}x_ix_j$. Then $(x_i^{-1}sx_i)\sigma_j=x_i^{-1}sx_i$, $x_i^{n_i}\sigma_j=x_i^{n_i}$ and $s_{ij}\sigma_k=s_{ij}$ show that ρ_i is an automorphism of S, s_i, $s_{ji}\in S$ respectively, and ρ_i satisfies (3).

Now $s\rho_j^{-1}\rho_i^{-1}\rho_j\rho_i=(x_jsx_j^{-1})\rho_i^{-1}\rho_j\rho_i=(x_ix_jsx_j^{-1}x_i^{-1})\rho_j\rho_i=x_i^{-1}x_j^{-1}x_ix_jsx_j^{-1}x_i^{-1}x_jx_i=s\tilde{s}_{ij}$ for each $s\in S$.

$(s_{ij})(s_{ik}\rho_j)s_{jk}=(x_i^{-1}x_j^{-1}x_ix_j)(x_j^{-1}x_i^{-1}x_k^{-1}x_ix_kx_j)(x_j^{-1}x_k^{-1}x_jx_k)=x_i^{-1}x_j^{-1}x_k^{-1}x_ix_jx_k$. On ther other hand, $(s_{jk}\rho_i)s_{ik}(s_{ij}\rho_k)=(x_i^{-1}x_j^{-1}x_k^{-1}x_jx_kx_i)(x_i^{-1}x_k^{-1}x_ix_k)$ $(x_k^{-1}x_i^{-1}x_j^{-1}x_ix_jx_k)=(x_i^{-1}x_j^{-1}x_k^{-1}x_jx_ix_k)(x_k^{-1}x_i^{-1}x_j^{-1}x_ix_jx_k)=x_i^{-1}x_j^{-1}x_k^{-1}x_ix_jx_k$. Thus there holds (2).

Next, $RN_{n_i}(s_{ji};\rho_i)=(s_{ji})(s_{ji}\rho_i)\cdots(s_{ji}\rho_i^{n_i-1})$
$=(s_{ji})(x_i^{-1}s_{ji}x_i)(x_i^{-2}s_{ji}x_i^2)\cdots(x_i^{1-n_i}s_{ji}x_i^{n_i-1})=(s_{ji}x_i^{-1})^{n_i-1}(s_{ji}x_i^{n_i-1})$
$=(x_j^{-1}x_i^{-1}x_j)^{n_i-1}(x_j^{-1}x_i^{-1}x_jx_i^{n_i})=(x_j^{-1}x_i^{1-n_i}x_j)(x_j^{-1}x_i^{-1}x_jx_i^{n_i})=x_j^{-1}x_i^{-n_i}x_jx_i^{n_i}=(s_i^{-1}\rho_j)s_i$.

Since $R=S[x_1,x_2,\cdots,x_e]\cong S[\mathfrak{X}_e;\mathfrak{P}_e]/M$, $M=(X_1^{n_1}-x_1^{n_1},X_2^{n_2}-x_2^{n_2},\cdots,X_e^{n_e}-x_e^{n_e})S[\mathfrak{X}_e;\mathfrak{P}_e]$, $X_iP_j=X_is_{ij}$, $S|P_j=\rho_j$, and V is a division ring, each $X_i^{n_i}-x_i^{n_i}$ is $s.w$-irreducible in $S[\mathfrak{X}_e;\mathfrak{P}_e]$ and $S[\mathfrak{X}_e;\mathfrak{P}_e]$ satisfies (6).

If we define the map φ^* of R^* to R by $\sum y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots y_1^{\nu_1}s_{\nu_e\cdots\nu_1}\to\sum x_e^{\nu_e}x_{e-1}^{\nu_{e-1}}\cdots x_1^{\nu_1}s_{\nu_e\cdots\nu_1}$, φ^* is an S-isomorphism satisfying $\sigma_i^*\varphi^*=\varphi^*\sigma_i$ for each $i=1,2,\cdots,e$.

Corollary 4.1. (**a**) *In order that S have an n-dimensional outer Kummer extension, it is necessary and sufficient that there exist $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying the conditions* (1)–(5) *of Theorem* 4.1 *and if $\rho_e^{\nu_e}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_1}=\tilde{s}$ for some $s\in S$, then,*

(7) $s\rho_i=RN_{\nu_1}(s_{i1};\rho_1)\{RN_{\nu_2}(s_{i_2};\rho_2)\}\rho_1^{\nu_1}\{RN_{\nu_3}(s_{i3};\rho_3)\}\rho_2^{\nu_2}\rho_1^{\nu_1}\cdots$
$RN_{\nu_e}(s_{ie};\rho_e)\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_1}s$.

(**b**) *In order that S have an n-dimensional inner Kummer extension it is necessary and sufficient that there exist $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying* (1)–(6) *of Theorem* 4.1 *and there exist a $1\times e$ matrix (t_i) with entries in T satisfying*

(7′) $\rho_e^{\nu_{ei}}\cdots\rho_1^{\nu_{1i}}=\tilde{t}_i$

$$t_i\rho_i=RN_{\nu_{1i}}(s_{i1};\rho_1)\{RN_{\nu_{2i}}(s_{i2};\rho_2)\}\rho_1^{\nu_{1i}}\{RN^{\nu_{3i}}(s_{i3};\rho_3)\}\rho_2^{\nu_{2i}}\rho_1^{\nu_{1i}}\cdots$$
$$\{RN_{\nu_{ei}}(s_{ie};\rho_e)\}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_{1i}}t_i\zeta^{n/n_i}$$
$$t_i\rho_j=RN_{\nu_{1i}}(s_{i1};\rho_1)\{RN_{\nu_{2i}}(s_{i2};\rho_2)\}\rho_1^{\nu_{1j}}\{RN_{\nu_{3i}}(s_{i3};\rho_3)\}\rho_2^{\nu_{2i}}\rho_1^{\nu_{1i}}\cdots$$
$$\{RN_{\nu_{ei}}(s_{ie};\rho_e)\}\rho_{e-1}^{\nu_{e-1}i}\cdots\rho_1^{\nu_{1i}}\ (i\neq j).$$

In particular.

(**c**) *In order that S have an n-dimensional inner Kummer extension such that $V=Z$, it is necessary and sufficient that there exist $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying* (1)–(4) *of Theoem* 4.1 *and there exist a $1\times n$ matrix (z_i) with entries in $Z^\bullet$ satisfying*

(7″) $z_i\rho_j\begin{cases}=z_i\zeta^{n/n_i} & \text{if } i=j\\ =z_i & \text{if } i\neq j.\end{cases}$

Proof. If there exists $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(s_i)\}$ satisfying the conditions (1)–(5) of Theorem 4.1, then there exists an n-dimensional simple ring extension $R^*=S[y_1,y_2,\cdots,y_e]$ over S such that $J(\mathfrak{G}^*,R^*)=S$.

(**a**) Let $v=\sum y_e^{\nu_e}y_{e-1}^{\nu_{e-1}}\cdots y_1^{\nu_1}s_{\nu_e\cdots\nu_1}$ be an element of V^*. Then $sv=vs$ ($s\in S$

yields at once $\sum y_e^{\nu_e}\cdots y_1^{\nu_1}s\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}s_{\nu_e\cdots\nu_1}=\sum y_e^{\nu_e}\cdots y_1^{\nu_1}s_{\nu_e\cdots\nu_1}s$. Hence each $y_e^{\nu_e}\cdots y_1^{\nu_1}s_{\nu_e\cdots\nu_1}$ is contained in V^* and $\rho_e^{\nu_e}\cdots\rho_1^{\nu_2}=\tilde{s}_{\nu_e\cdots\nu_1}$ since $S=Ss_{\nu_e\cdots\nu_1}S=S\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}s_{\nu_e\cdots\nu_1}S=s_{\nu_e\cdots\nu_1}S$. Thus $\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}=\tilde{s}$ if and only if $y_e^{\nu_e}\cdots y_1^{\nu_1}s$ is contained in V^*. Now, let $v=y_e^{\nu_e}\cdots y_1^{\nu_1}s$ be in V^*. Then $v\in C^*$ if and only if $y_iv=vy_i$ for each y_i, and, that is $RN_{\nu_1}(s_{i1};\rho_1)\{RN_{\nu_2}(s_{i2};\rho_2)\}\rho_1^{\nu_1}\{RN_{\nu_3}(s_{i3};\rho_3)\}\rho_2^{\nu_2}\rho_1^{\nu_1}\cdots\{RN_{\nu_e}(s_{ie};\rho_e)\}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_1}s=s\rho_i$.

(**b**) As was shown in the proof of (a), $\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}=\tilde{t}_i$ is equivalent with $v_i=y_e^{\nu_{ei}}\cdots y_1^{\nu_{1i}}t_i$ is contained in V^*. Hence if t_i satisfies the two conditions of (7′), then $v_iy_i=y_e^{\nu_{ei}}\cdots y_1^{\nu_{1i}}y_it_i\rho_i=y_e^{\nu_{ei}}\cdots y_1^{\nu_{1i}}y_i$. $RN_{\nu_{1i}}(s_{j1};\rho_1)\{RN_{\nu_{2i}}(s_{i1};\rho_2)\}\rho_1^{\nu_{1i}}\{RN_{\nu^2_3}(s_{i3};\rho_3)\}\rho_2^{\nu_{2i}}\rho_1^{\nu_{1i}}\cdots\{RN_{\nu_{ei}}(s_{ie};\rho_e)\}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_{1i}}\zeta^{n/n_i}=y_iv_i\zeta^{n/n_i}$ and $v_iy_j=y_e^{\nu_{ei}}\cdots y_1^{\nu_{1i}}y_jt_i\rho_j=y_e^{\nu_{ei}}\cdots y_1^{\nu_{1i}}y_jRN_{\nu_1}(s_{j1};\rho_1)\{RN(s_{j2};\rho_1)\}\rho_1^{\nu_{1i}}\{RN_{\nu_{3i}}(s_{j3};\rho_3)\}\rho_2^{\nu_{2i}}\rho_1^{\nu_{1i}}\cdots\{RN_{\nu_{ei}}(s_{je};\rho_e)\}\rho_{e-1}^{\nu_{e-1}}\cdots\rho_1^{\nu_{1i}}=y_jv_i$ for each $j\neq i$. Consequently, we have

$$v_iy_jv_i^{-1}\begin{cases}=y_j\zeta^{n/n_i} & \text{if } i=j\\ =y_j & \text{if } i\neq j.\end{cases}$$

Thus we may adapt $\tilde{v}_i$ as σ_i^*.

Conversely, if $R=S[x_1,x_2,\cdots,x_e]$ is an n-dimensional inner Kummer extension over S, then $\sigma_i=\tilde{v}_i=\widetilde{\sum x_e^{\nu_e}x_{e-1}^{\nu_{e-1}}\cdots x_1^{\nu_1}t^{(i)}_{\nu_e\cdots\nu_1}}$ for each $i=1,2,\cdots,e$. Hence each $t^{(i)}_{\nu_e\cdots\nu_1}$ satisfies the condition (7′).

(**c**) If there exists (z_i) satisfying (7″), then we can prove that each $f(X_i)=X_i^{n_i}-s_i$ is $s.w$-irreducible in $S[\mathfrak{X}_e;\mathfrak{P}_e]$. For let $g(X_i)=\sum_{\nu=0}^{k}X_i^{\nu}f_\nu$ be a proper left factor of $f(X_i)$ in $S[y_1,\cdots,y_{i-1}][X_i;P_i]$ where $f_\nu\in S[y_1,\cdots,y_{i-1}]$. Then $f_0\neq 0$. Since the constant term of $z_ig(X_i)-g(X_i)z_i\zeta^{kn/n_i}$ is $f_0z_i(1-\zeta^{kn/n_i})\neq 0$ and deg $(z_ig(X_i)-g(X_i)z_i\zeta^{kn/n_i})<k$, an easy induction will yield at once $(g(X_i))=S[y_1,\cdots,y_{i-1}][X_i;P_i]$.

Next, since $z_iy_jz_i^{-1}=y_jz_i\rho_iz_i^{-1}\begin{cases}=y_j\zeta^{n/n_i} & \text{if } i=j\\ =y_j & \text{if } i\neq j\end{cases}$, we may regard $(\tilde{z}_i)$ as (σ_i^*). Then $V^*=J(\mathfrak{G}^*|V^*,V^*)=Z$. The converse is almost evident.

Now as is shown in the proof of Corollary 4.1 (b), if $\rho_i=\tilde{s}_i$ $(s_i\in S)$ for each $i=1,2,\cdots,e$, then $y_is_i\in V^*$ where $R^*=S[y_1,\cdots,y_e]$. Hence we obtain $R^*=S\otimes_Z V^*=S[V^*]$. Using this fact we have

Corollary 4.2. *The followings are equivalent.*

(1) *S has an n-dimensional trivial Kummer extension.*

(2) *There exist elements $z_1,z_2,\cdots,z_e$ in Z such that the associated algebraic set of the ideal $(X_1^{n_1}-z_1,X_2^{n_2}-z_2,\cdots,X_e^{n_e}-z_e)$ in $Z[X_1,X_2,\cdots,X_e]$ is an algebraic variety.*

(3) *There exist inner autmorphisms $\rho_1,\rho_2,\cdots,\rho_e$ in S, (s_{ij}) an $e\times e$ matrix with entries in $S^\bullet$, (s_i) a $1\times e$ matrix with entries in $S^\bullet$ satisfying* (1)–(4) *of Theorem* 4.1 *and if $\rho_e^{\nu_e}\cdots\rho_1^{\nu_1}=\tilde{s}$ then s satisfies* (7′) *of Corollary* 4.1 (*a*).

Proof. The implication (1)→(2)→(3) is easy. (3)→(1) is an immediate consequence of the remark above mentioned and Corollary 4.1 (a).

The condition (6) of Theorem 4.1 was need only to see that the centralizer V^* is a division ring. Accordigly we have

Corollary 4.3. *Let S be a division ring. In order that S have an n-dimensional Kummer division ring extension, it is necessary and sufficient that there exist $\{\rho_1, \rho_2, \cdots, \rho_e, (s_{ij}), (s_i)\}$ satisfying* (1)–(4) *of Theorem* 4.1 *and*

(5′) *$f(X_i)=X_i^{n_i}-s_i$ is s-irreducible in $S[\mathfrak{X}_e; \mathfrak{P}_e]$.*

The following is corresponding to Corollary 3.4 and we can prove it as similar to that of Corollary 3.4.

Corollary 4.4. *Let S be a simple ring with automorphisms $\rho_1, \rho_2, \cdots, \rho_i$, (s_{ji}) and (s_i) satisfying the conditions* (1)–(6) *of Theorem* 4.1.

(**a**) *In order that S have an n-dimensional parallel Kummer extension, it is necessary and sufficient that there exists a simple subring W of S containing $\{s_{ji}\}$, $\{s_i\}$ such that $W\rho_i \subseteqq W$, $V_W(W) \ni \zeta$ and $X_i^{n_i}-s_i$ is s·.w-irreducible in $W[\mathfrak{X}_e; \mathfrak{P}_e]$ for each $i=1, 2, \cdots, e$.*

(**b**) *In order that S have an n-dimensional d-parallel Kumer extension, it is necessary and sufficient that there exist a division subring W of S containing $\{s_{ij}\}$, $\{s_i\}$ such that $W\rho_i \subseteqq W$, $V_W(W) \in \zeta$ and $X_i^{n_i}-s_i$ is s-irreducible in $W[\mathfrak{X}_e; \mathfrak{P}_e]$ for each $i=1, 2, \cdots, e$.*

(**c**) *In order that S have an n-dimensional f-parallel Kummer extension, it is necessary and sufficient that there exist a subfield W of S containing $\{s_{ij}\}$, $\{s_i\}$ such that $\rho_i | W=1$, $W \ni \zeta$ and $X_i^{n_i}-s_i$ is s-irreducible in $W[\mathfrak{X}_e; \mathfrak{P}_e]$ for each $i=1, 2, \cdots, e$.*

Finally, we shall deal with the regular embedding problem of Kummer case.

Lemma 4.1. *Let T be a q-dimensional simple ring extension over S such that T is T_lS_r-irreducible and $V_T(T) \ni \zeta$, and let R be an n-dimensional Kummer extension over T. Then R is R_lS_r-irreducible if and only if V is a division ring.*

Proof. Let $R=T[x, \cdots, x_e]$, $x_i^{n_i} \in T$, and let V be a division ring. We shall show that $R=RxS$ for each $x \in R \backslash S$. Let $y=\sum x_e^{\nu_e} x_{e-1}^{\nu_{e-1}} \cdots x_1^{\nu_1} s_{\nu_e \cdots \nu_1}$ be the element which is the shortest non zero relation contained in RxS. Then, without loss of generality, we may assume that $y \in R \backslash S$. Since T is T_lS_r-irreducible and each x_i is regular, we may assume that the constant term of y is 1. Thus $sy-ys=0$, thst is, $R=RyS=RxS$. The converse is clear.

In what follow, we assume that T is a q-dimensional abelian extension over S with respect to $\mathfrak{H}=(\tau_1) \times (\tau_2) \times \cdots \times (\tau_m)$, $\# \tau_i = q_i$ such that T is T_lS_r-irreducible and $V_T(T) \ni \zeta$.

Theorem 4.2. *In order that T can be regularly embedded in a qn-dimensional abelian extension R/S such that R/T is an n-dimensional Kummer extension such that V is a division ring, it is necessary and sufficient that there exist automorphisms $\rho_1, \rho_2, \cdots, \rho_{m+k}$ $(k>-m)$ in T, (s_{ij}) an $(m+k)\times(m+k)$ matrix with entries in $T^{\bullet}$, (t_i) a $1\times(m+k)$ matrix with entries in $T^{\bullet}$ satisfying the conditions* (1)–(5) *of Theorem* 4.1 *(in T) and (b_{ij}) an $(m+k)\times(m+k)$ matrix with entries in $T^{\bullet}$ satisfying*

(1) $\tau_j^{-1}\rho_i^{-1}\tau_j\rho_i=\tilde{b}_{ij}$,

(2) $\begin{cases} RN_{q_i}(b_{ii};\tau_i)=\zeta^{n/n_i} \\ RN_{q_j}(b_{ij};\tau_j)=1 \end{cases}$,

(3) $LN_{n_i}(b_{ij};\rho_i)=t_i^{-1}\cdot t_i\rho_j$,

(4) $s_{ij}(b_{ik}\rho_j)b_{jk}=(b_{jk}\rho_i)b_{ik}(s_{ij}\tau_k)$,

(5) $b_{ik}(b_{ij}\tau_k)=b_{ij}(b_{ik}\tau_j)$

(6) *If $g(X)$ is a polynomial in $T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]\setminus M$, $M=(X_1^{n_1}-t_1, X_2^{n_2}-t_2,\cdots,X_{m+k}^{n_{m+k}}-t_{m+k})$, then $\sum_i f_i(X)g(X)s_i=1$ modulo M for some $f_i(X)\in T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]$, $s\in S$.*

More precisely, if there exists $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(t_i),(b_{ij})\}$ satisfying the conditions, then $M=(X_1^{n_1}-t_1,\cdots,X_{m+k}^{n_{m+k}}-t_{m+k})T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]$ and $R^{**}=T[y_1,\cdots,y_m,y_{m+1},\cdots,y_{m+k}]\cong T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]/M$, y_i is the residue class of X_i modulo M, is a requested one with generating automorphisms $\sigma_1^*,\sigma_2^*,\cdots,\sigma_{m+k}^*$ such that $(y_it)\sigma_j^*=y_ib_{ij}t\tau_j$ $(t\in T)$ and $\mathfrak{G}^*|T=\mathfrak{H}$ where $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_m^*)\times(\sigma_{m+1}^*)\times\cdots\times(\sigma_{m+k}^*)$ if $k>0$ and $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_{m+k}^*)\times(\tau_{m+k+1})\times\cdots\times(\tau_m)$ if $k<0$. Conversely, if T/S be regularly embedded in a qn-dimensional abelian extension with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_{m+k})$ such that R/T is an n-dimensional Kummer extension, then there exists $\{\rho_1,\rho_2,\cdots,\rho_e,(s_{ij}),(t_i),(b_{ij})\}$ satisfying the conditions in Theorem 4.1 *(in T) and* (1)–(5) *that there holds a T-isomorphism $\varphi^*: R^*\cong R$ with $\varphi^*\sigma_i=\sigma_i^*\varphi^*$ for each $i=1,2,\cdots,m+k$.*

Proof. Firstly we set $b_{ij}\begin{cases} =\zeta^{n/n_i} & \text{if } k>0 \text{ and } i=j>m \\ =1 & \text{if } k>0,\ i\neq j \text{ and } i,j>m \\ & \text{or } k<0 \text{ and } j>m+k \end{cases}$

Then we can easily checked that they satisfy the conditions (1)–(5).

Let Ψ_j $(j=1,2,\cdots,m+k)$ be the mappings of $T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]$ defined by $\sum X_{m+k}^{\nu_{m+k}}\cdots X_1^{\nu_1}t_{\nu_{m+k}\cdots\nu_1}\to\sum(X_{m+k}b_{m+kj})^{\nu_{m+k}}\cdots(X_1b_{1j})^{\nu_1}(t_{\nu_{m+k}\cdots\nu_1}\tau_j)$ (we set $\tau_j=1$ if $j>m$). Then we can prove that Ψ_j is an automorphism by induction on the number of indeterminates.

$(t\Psi_j)X_i\Psi_j=(t\tau_j)(X_ib_{ij})=X_it\tau_i\rho_ib_{ij}$ and $(X_it\rho_i)\Psi_j=X_ib_{ij}+t\rho_i\tau_j$ and hence the condition (1) yields $(tX_i)\Psi_j=(X_it\rho_i)\Psi_j$, and further, $\#\Psi_j=n_jq_j$ (by (2)) shows that Ψ_j is an automorphism in $T[X_i;\rho_i]$. Hence we assume that the assertion is true for each $T[\mathfrak{X}_{m+k-1};\mathfrak{P}_{m+k-1}]$. Then it suffices to prove

$(Y\Psi_j)(X_{m+k}\Psi_j)=(X_{m+k}\Psi_j)(YP_{m+k}\Psi_j)$ where $Y\in T[\mathfrak{X}_{m+k-1};\mathfrak{P}_{m+k-1}]$. Since $(Y\Psi_j)(X_{m+k}\Psi_j)=Y\Psi_j(X_{m+k}b_{m+kj})=X_{m+k}(Y\Psi_jP_{m+k})b_{m+kj}$ and $(X_{m+k}\Psi_j)YP_{m+k}\Psi_j=X_{m+k}b_{m+kj}(YP_{m+k}\Psi_j)$, it suffices to prove $Y\Psi_jP_{m+k}=YP_{m+k}\Psi_j\tilde{b}_{m+kj}$. Now, $X_i\Psi_jP_{m+k}=(X_ib_{ij})P_{m+k}=X_is_{im+k}b_{ij}\rho_{m+k}$ and $X_iP_{m+k}\Psi_j\tilde{b}_{m+kj}=b_{m+kj}X_ib_{ij}(s_{im+k}\tau_j)b^{-1}_{m+kj}=X_ib_{m+kj}\rho_ib_{ij}(s_{im+k}\tau_j)b^{-1}_{m+kj}$. Hence the condition (4) shows that $X_i\Psi_jP_{m+k}=X_iP_{m+k}\Psi_j\tilde{b}_{m+kj}$. Thus we assume that $Y\Psi_jP_{m+k}=YP_{m+k}\Psi_j\tilde{b}_{m+kj}$ where $Y=X^{\nu_{m+k-1}}_{m+k-1}\cdots X^{\nu_i}_i\cdots X^{\nu_1}_1$. Then $(X^{\nu_{m+k-1}}_{m+k-1}\cdots X^{\nu_i+1}_i\cdots X^{\nu_1}_1)\Psi_jP_{m+k}=(Y_1\Psi_jP_{m+k})(Y_2\Psi_jP_{m+k})=Y_1P_{m+k}\Psi_j\tilde{b}_{m+kj}Y_2P_{m+k}\Psi_j\tilde{b}_{m+kj}=(X^{\nu_{m+k-1}}_{m+k-1}\cdots X^{\nu_i+1}_iX^{\nu_1}_1)P_{m+k}\Psi_j\tilde{b}_{m+kj}$ where $Y_1=X^{\nu_{m+k-1}}_{m+k-1}\cdots X^{\nu_i}_i$ and $Y_2=X_i\cdots X^{\nu_1}_1$. Consequently, we have $Y\Psi_jP_{m+k}=YP_{m+k}\Psi_j\tilde{b}_{m+kj}$ for each $Y\in T[\mathfrak{X}_{m+k-1};\mathfrak{P}_{m+k-1}]$.

$(X^{n_i}_it^{-1}_i-1)\Psi_j=X^{n_i}_iLN_{n_i}(b_{ij};\rho_i)t^{-1}_i\tau_j-1$ and the condition (3) show that $M\Psi_j=M$. Hence each Ψ_j induces an automorphism σ^*_j in $R^*=T[y_1,\cdots,y_{m+k}]\cong T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]/M$, y_i is the residue class of X_i modulo M, is a qn-dimensional simple ring extension over S. Moreover, by (2),

$$\#\sigma^*_j\begin{cases}=q_jn_j & \text{if } j\leqq m\\ =n_j & \text{if } j>m\end{cases}.$$

$y_i\sigma^*_j\sigma^*_k=y_ib_{ik}(b_{ij}\tau_k)=y_ib_{ij}(b_{ik}\tau_j)=y_i\sigma^*_k\sigma^*_j$ by condition (5) shows that $\sigma^*_j\sigma^*_k=\sigma^*_k\sigma^*_j$. Let $\sigma^*\in(\sigma^*_i)_\cap(\sigma^*_1,\sigma^*_2,\cdots,\sigma^*_{i-1})$. Then $\sigma^*=\sigma^{*\alpha_i}_i=\sigma^{*\alpha_1}_1\sigma^{*\alpha_2}_2\cdots\sigma^{*\alpha_{i-1}}_{i-1}$ and $\sigma^*|T=\tau^{\alpha_i}_i=\tau^{\alpha_1}_1\cdots\tau^{\alpha_{i-1}}_{i-1}$ show that each α_j is a mutiple of q_j, and hence $y_i\sigma^*=y_i\sigma^{*\alpha_1}_1\sigma^{*\alpha_2}_2\cdots\sigma^{*\alpha_{i-1}}_{i-1}=y_i$ by $RN_{q_j}(b_{ij};\tau_j)=1$. Consequently, we have $\sigma^*=1$. Thus, $\mathfrak{G}^*$, the group generated by $\sigma^*_1,\sigma^*_2,\cdots,\sigma^*_{m+k}$, is a direct product of each σ^*_i, and $J(\mathfrak{G}^*,R^*)=J(\mathfrak{G}^*|T,T)=J(\mathfrak{H},T)=S$.

Finally, if (6) is satisfied, then by Lemma 4.1, V^* is a division ring.

Conversely, let T/S can be regularly embedded in a qn-dimensional abelian extension R/S with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_m)\times\cdots\times(\sigma_{m+k})$ if $k\geqq 0$ and $\mathfrak{G}=(\sigma_1)\times\cdots\times(\sigma_{m+k})\times(\tau_{m+k+1})\times\cdots\times(\tau_m)$ if $-m<k<0$ such that $\#\sigma_j=(\#\tau_j)n_j$, $\sigma_j|T=\tau_j$ if $j\leqq m$ and $\#\sigma_j=n_j$, $\sigma_j|T=1$ if $j>m$.

Since V is a field and $[R:T]=n$, R/T is an n-dimensional Kummer extension with respect to $\mathfrak{G}(T)=(\sigma^{q_1}_1)\times(\sigma^{q_2}_2)\times\cdots\times(\sigma^{q_m}_m)\times(\sigma_{m+1})\times\cdots\times(\sigma_{m+k})$. Then $R=\bigoplus_{0\leqq\nu_i\leqq m_i-1}x^{\nu_{m+k}}_{m+k}\cdots x^{\nu_1}_1T$ where each $x_i\in R^\cdot$, $x_i\sigma^{q_j}_j\begin{cases}=x_i\zeta^{n/n_i} & \text{if } i=j\\ =x_i & \text{if } i\neq j\end{cases}$ for each $j\leqq m$ and $x_i\sigma_j\begin{cases}=x_i\zeta^{n/n_i} & \text{if } i=j\\ =x_i & \text{if } i\neq j\end{cases}$ for each $j>m$. Then $\rho_i=\tilde{x}^{-1}_i|T$, $t_i=x^{n_i}_i$, $s_{ij}=x^{-1}_ix^{-1}_jx_ix_j$ satisfy the conditions (1)–(5) of Theorem 4.1 (in T).

Now we set $b_{ij}=x^{-1}_i(x_i\sigma_j)$. Then $b_{ij}\sigma^{q_k}_k=x^{-1}_i\sigma^{q_k}_kx_i\sigma_j\sigma^{q_k}_k=\begin{cases}=x^{-1}_i\zeta^{-n/n_i}(x_i\sigma_j\zeta^{n/n_i})=x^{-1}_ix_i\sigma_j=b_{ij} & \text{if } i=j\\ =x^{-1}_i(x_i\sigma_j)=b_{ij} & \text{if } i\neq j\end{cases}$ for each $k\leqq m$ and $b_{ij}\sigma_k=(x^{-1}_i(x_i\sigma_j))\sigma_k=b_{ij}$ if $k>m$ show that $b_{ij}\in T$.

Next, $tb_{ij}=x^{-1}_i(x_i\sigma_j)t(x^{-1}_i\sigma_j)x_i=x^{-1}_i(x_it\tau^{-1}_jx^{-1}_i)\sigma_jx_i=x^{-1}_i(t\tau^{-1}_j\rho^{-1}_i)\cdot\sigma_jx_i=$

$x_i^{-1}(t\tau_j^{-1}\rho_i^{-1}\tau_j)x_i = t\tau_j^{-1}\rho_i^{-1}\tau_j\rho_i$.

$RN_{q_j}(b_{ij};\tau_j) = x_i^{-1}(x_i\sigma_j)(x_i^{-1}\sigma_j)(x_i\sigma_j^2)\cdots(x_i^{-1}\sigma_j^{q_j-1})(x_i\sigma_j^{q_j}) = x_i^{-1}x_i\sigma_j^{q_j}$

$\begin{cases} = \zeta^{n/n_i} & \text{if } i=j \\ = 1 & \text{if } i\neq j \end{cases}$ by the acts of $\sigma_j^{q_j}$ on x_i.

$LN_{n_i}(b_{ij};\rho_i) = (x_i^{1-n_i}x_i^{-1}(x_i\sigma_j)x_i^{n_i-1})(x_i^{2-n_i}x_i^{-1}(x_i^{-1}\sigma_j)x_i^{n_i-2})\cdots x_i^{-1}(x_i\sigma_j) = x_i^{-n_i}(x_i\sigma_j)^{n_i} = x_i^{-n_i}(x_i^{n_i}\sigma_j) = t_i^{-1}(t_i\sigma_j)$.

$s_{ij}(b_{ik}\rho_j)b_{jk} = (x_i^{-1}x_j^{-1}x_ix_j)(x_j^{-1}x_i^{-1}(x_i\sigma_k)x_j)x_j^{-1}(x_j\sigma_k) = x_i^{-1}x_j^{-1}(x_i\sigma_k)(x_j\sigma_k) = (x_i^{-1}(x_j^{-1}x_j\sigma_k)x_i)(x_i^{-1}(x_j^{-1}\sigma_k)(x_i\sigma_k)(x_i\sigma_k)) = b_{jk}\rho_i(x_i^{-1}(x_i\sigma_k)(x_i^{-1}\sigma_k)(x_j^{-1}x_ix_j)\sigma^k) = b_{jk}\rho_ib_{ik}(s_{ij}\tau_k)$.

$x_i(b_{ik}(b_{ij}\tau_k)) = x_i\sigma_j\sigma_k = x_i\sigma_k\sigma_j = x_ib_{ij}(b_{ik}\tau_j)$ shows that $b_{ik}(b_{ij}\tau_k) = b_{ij}(b_{ik}\tau_j)$.

Finally, since V is a division ring, T_lS_r-irreducibility of T yields at once the condition (6) by Lemma 4.1 in $T[\mathfrak{X}_{m+k};\mathfrak{P}_{m+k}]$ where $X_iP_j = X_is_{ij}$ and $S|P_j = \rho_j$.

The condition (6) of Theorem 4.2 was need only to see that the centralizer V^* is a division ring. Accordingly, we readily obtain the following

Corollary 4.5. *Let S be a division ring, and let T/S be a q-dimensional abelian division ring extension satisfying the condition of Theorem* 4.2. *In order that T/S can be regularly embedded in a qn-dimensional abelian division ring extension R/S such that R/T is an n-dimensional Kummer extension, it is necessary and sufficient that there exist $\{\rho_1, \rho_2, \cdots, \rho_e, (s_{ij}), (t_i), (b_{ij})\}$ satisfying* (1)–(5′) *of Corollary* 4.3 *(in T) and* (1)–(5) *of Theorem* 4.1.

References

[1] S. A. AMITSUR: Non-commutative cyclic fields, Duke Math. J., Vol. 21 (1954), 87–105.

[2] P. M. COHN: On a class of binomial extensions, Illinois J. of Math., Vol. 10 (1966), 418–424.

[3] K. KISHIMOTO: On cyclic extensions of simple rings, J. Fac Sci. Hokkaido Univ., Vol. 19 (1966), 74–85.

[4] ———: On polynomial extensions of simple rings, J. Fac. Sci. Hokkaido Univ., Vol 19 (1966), 169–180,

[5] K. KISHIMOTO, T. ONODERA and H. TOMINAGA: On normal basis theorem and the extension dimension, J. Fac. Sci. Hokkaido Univ., Vol. 18 (1964), 82–88.

[6] T. NAGAHARA, T. ONODERA and H. TOMINAGA: On normal basis theorem and strictly Galois extensions, Math. J. Okayama Univ., Vol. 8 (1958), 133–142.

[7] T. NAGAHARA and H. TOMINAGA: On Galois and locally Galois extensions of simple rings, Math. J. Okayama Univ., Vol. 10 (1961), 143–166.

[8] N. NOBUSAWA and H. TOMINAGA: Some remarks on strictly Galois extensions of simple rings, Math. J. Okayama Univ., Vol. 9 (1959), 13–17.

Department of Mathematics
Shinshû University, Matsumoto

(Received May 20, 1967)