ON ABELIAN EXTENSIONS OF SIMPLE RINGS

By

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In [3], the author gave characterizations for a simple ring to have a cyclic extension, which generalizes the case of division ring extension treated by S. A. Amitsur in [1], while in [4], he studied as a generalized notion of a simple ring extension having a single commutative generator, a polynomial extension\(^1\) of a simple ring as well as relationships between a cyclic extension and a Galois extension which is polynomial extension.

The present paper is a continuation of these earlier works to give, as main results, some necessary and sufficient conditions for a simple ring to have an abelian extension. §§ 1–2 are prerequisite preparations for the main purpose. In § 1, we shall deal with an \(e\)-polynomial simple ring extension which is generalized notion of a polynomial simple ring extension considered in [4]. It can be seen from the definition that the structure of each \(p^e\)-dimensional fundamental abelian extension of prime characteristic \(p\) and Kummer extension given in [Theorems 3, 4 of [8] and Corollary 2 of [5]] are \(e\)-polynomial simple ring extensions. In § 2, we shall construct two types of noncommutative) polynomial ring with several indeterminates over a simple ring. One is defined by means of derivations and the other is defined by means of endomorphisms both in the basic simple ring. Further, as residue class ring of above polynomial rings, two types of \(e\)-polynomial simple ring extensions will be defined, the former of which is concerned with \(p^e\)-dimensional fundamental abelian extension of prime characteristic \(p\) and the latter is concerned with a Kummer extension respectively.

Applying the results of the preceding sections, we shall give in § 3, a necessary and sufficient condition for that a simple ring of prime characteristic \(p\) have a \(p^e\)-dimensional fundamental abelian extension in Theorem 3.1, and also a necessary and sufficient condition for a \(p^e\)-dimensional abelian extension \(T\) over \(S\) of prime characteristic \(p\) to be regularly embedded in a \(p^e\)-dimensional abelian extension \(R/S\) such that \(R/T\) is a fundamental abelian extension. Finally, § 4 is devoted to the Kummer extension case. Namely, we shall obtain a necessary and sufficient condition for a simple ring to have an \(n\)-dimensional

\(^1\) Cf. [2].
abelian extension (whose center contains a primitive \( n \)-th root \( \zeta \) of 1) in Theorem 4.1. Moreover, in Theorem 4.2, a necessary and sufficient condition for that a \( q \)-dimensional abelian extension \( T' \) over \( S \) such that the center of \( T \) contains \( \zeta \) and \( T \) is \( T' S_r \)-irreducible can be regularly embedded in a \( qn \)-dimensional abelian extension \( R/S \) such that the center of \( R \) contains \( \zeta \) will be given.

Throughout the present paper, we use the following conventions:

Let \( A \) be a ring and \( a \) an element of \( A \).

\( A^* \): the multiplicative group consisting of the regular elements of \( A \).

\( \tilde{A} \): the set of all inner automorphisms determined by elements of \( A^* \).

\( I_a, \tilde{a} \): the inner derivation \( a_l-a_r \) determined by \( a \), the inner automorphism \( a_l a_r^{-1} \) determined by the regular element \( a \) respectively.

\( J(\mathfrak{G}, A) \): the fixed-subring of \( A \) under \( \mathfrak{G} \) where \( \mathfrak{G} \) is an automorphism group of \( A \).

\( I_a \mid B, \mathfrak{G} \mid B \): the restriction of \( I_a \) to \( B \), of \( \mathfrak{G} \) to \( B \) where \( B \) is a subset of \( A \).

\( \mathfrak{G}(B) \): \( \{ \sigma \in \mathfrak{G} ; b \sigma = b \ \text{for all} \ b \in B \} \).

\( A \backslash B \): the set theoretical complement of \( B \) in \( A \),

\( LN_m(a; \rho) \) (resp. \( RN_m(a; \rho) \)) : \( a \rho^{m-1} \cdot a \rho^{m-2} \cdots a \rho \cdot a \) (resp. \( a a \rho \cdot a \rho^{m-1} a \rho^{m-1} \)) where \( \rho \) is a (ring) homomorphism of \( A \), by \( N_m(a; \rho) \) we denote \( LN_m(a; \rho) \) if it coincides with \( RN_m(a; \rho) \).

\( T_m(a; \rho) : \sum_{i=0}^{m-1} a \rho^i \).

\( A[X; \rho, D] \): the ring of all polynomials \( \{ X^i a_i ; a_i \in A \} \) which the multiplication is defined by \( aX=X(a \rho)+aD \) for each \( a \in A \) where \( D \) is a \( \rho \)-derivation in \( A \). In particular, we set \( A[X;D]=A[X;1,D] \), \( A[X; \rho]=A[P; \rho, 0] \) and \( A[X]=A[X;1,0] \).

\( S, Z \): a simple artinian ring, its center.

\( R, C \): an extension simple artinian ring of \( S \) with common identity, its center.

\( V : V_R(S) \), the centralizer of \( S \) in \( R \).

\( \oplus M_i \): a direct sum of modules \( M_i \).

\( \pi = \left( 1, 2, \ldots, e \right) \): a permutation of \( e \)-letters.

The other notations and terminologies used in this paper, we follow [3].

\( \S 1. \ e \text{-polynomial simple ring extension.} \)

First of all, we shall begin our study with \( e \)-polynomial simple ring extensions over \( S \).

\( R \) is called an \( n \)-dimensional right \( e \)-polynomial simple ring extension over \( S \) if \( R \) satisfies the following conditions:
(1) \([R:S]_r = n\),
(2) \(R = \sum_{0 \leq m_{j} \leq m_{j-1}} y_{e_{j}}y_{e_{j-1}}\cdots y_{e_{1}}S\) where \(\prod_{t=1}^{e} m_{t} = n, m_{t} > 1\),
(3) \(S(\pi(j)) = \sum_{0 \leq m_{j} \leq m_{j-1}} y_{e_{j}}y_{e_{j-1}}\cdots y_{e_{1}}S\) is simple for each \(\pi\) and \(1 \leq j \leq e\),
(4) \(sy_{i} = y_{i}y_{s_{ij}} + t_{ij}\) for every \(s \in S\), \(y_{i}y_{j} = y_{j}y_{s_{ij}} + t_{ij}\) for some \(s_{s_{ij}}, t_{ij} \in S\).
(5) \(y_{m}^{m_{l}} + \sum_{j=0}^{m_{l}-1} y_{j}^{n_{j}}t_{j}^{(n_{j})} = 0\) \((t_{j}^{(n_{j})} \in S)\).

Then, by \(s \rightarrow s_{i}^{(n)}\), and \(s \rightarrow s_{0}^{(i)}\), we can define a monomorphism \(\rho_{i}^{e}\) in \(S\) and \(\rho_{r}\)-derivation \(D_{i}\) in \(S\). Further, \(s_{ij}\) is regular and \(t_{ij} = -t_{ji}s_{ij}\) since \(y_{i}y_{j} = y_{j}y_{s_{ij}} + t_{ij} = y_{i}y_{j}s_{ij} + t_{ji}s_{ij} + t_{ij}\). Moreover, by (3) and (4), we can see that each \(\rho_{i}(S)\) can be extended a monomorphism \(P_{i}(S)\) of \(S(\pi(j))\) by \(y_{n(i)}P_{n(i)} = y_{n(i)}S_{n(i)}(S_{n(i)})\) and, at the same time, \(D_{n(i)}\) can be extended a \(P_{n(i)}\)-derivation in \(S(\pi(j))\).

An \(n\)-dimensional right \(e\)-polynomial simple ring extension \(R\) over \(S\) is called an \(n\)-dimensional \(e\)-polynomial simple ring extension over \(S\) if \([R:S]_r = n\). Throughout this section, we assume that \(R\) is an \(n\)-dimensional right \(e\)-polynomial simple ring extension over \(S\), and in the rest, for the sake of simplicity, \(\pi\) will be omitted if there is no danger of misunderstanding.

**Lemma 1.1.** \([S(j):S(i)]_l = \Pi_{p=i+1}^{j} \alpha_{p}\) for each \(\pi\) and \(i < j\) where \(\alpha_{i} = \sum_{j=1}^{n} \Pi_{p=j+1}^{n} \alpha_{p}\).

**Proof.** By (3) and (4), \(S(j)\) is a right \(1\)-polynomial simple ring extension over \(S(j-1)\) \((j=1, 2, \cdots, e)\) and we set \(S(0) = S\). Hence \([S(j):S(j-1)]_r = \alpha_{j}\) by [4, Theorem 1.1]. Thus the assertion is clear from the fact that \([S(j):S(i)]_l = [S(j):S(j-1)]_l \cdot [S(j-1):S(j-2)]_l \cdots [S(i+1):S(i)]_l\).

**Corollary 1.1.** The following conditions are equivalent:
(a) \([R:S]_l = [R:S]_r\).
(b) \([S(j):S(i)]_l = [S(j):S(i)]_r\) for each \(i < j\) and \(i = 0, 1, \cdots, e-1\).
(c) Each \(\rho_{j}\) is an automorphism and it can be extended to an automorphism \(P_{j}\) in \(S(j)\) for each \(\pi\).

**Proof.** (a)\(\rightarrow\)(b) By Lemma 1.1, \([R:S]_l = \Pi_{i=0}^{n} \alpha_{i}\) and \([S(i+1):S(i)]_l \geq [S(i+1):S(i)]_r\). Hence we obtain (b).

(b)\(\rightarrow\)(c) Since \([S(1):S]_l = [S(1):S]_r\), \(\rho_{1}\) is an automorphism in \(S\) for each \(\pi\). Next, \([S(j):S(j-1)]_l = [S(j):S(j-1)]_r\) shows that \(P_{j}\) is an automorphism in \(S(j)\).

(c)\(\rightarrow\)(a). This is a direct consequence of Lemma 1.1.

**Corollary 1.2.** If \(D_{i} = 0\) and \(t_{ij} = 0\) for each \(i, j\), then \([R:S]_l = [R:S]_r\).

**Proof.** By [4, Corollary 1.2 (b)], \(\rho_{i} = \tilde{y}_{i}^{-1}\) is an automorphism in \(S\). Further, \((y_{j}^{j}y_{j-1}^{j-1}\cdots y_{i}^{j}S) y_{i} = y_{i}(y_{j}^{j}y_{j-1}^{j-1}\cdots y_{i}^{j}S) P_{i}\) shows that \(P_{i} | S(j) = \tilde{y}_{i}^{-1}| S(j)\).
Hence we obtain $P_i|S(j)$ is an automorphism in $S(j)$. Thus the assertion is an immediate consequence of Corollary 1.1.

**Proposition 1.1.** (a) If each $\rho_i=1$, and $s_{ij}=1$ then, 
$[D_i, D_j]=D_i D_j - D_j D_i = I_{s_{ij}}$ and 
$t_i D_k + t_k D_i + t_k t_i D_j = 0$.
(b) If each $D_i=0$ and $t_{ij}=0$ then, 
$\rho_j^{-1} \rho_i^{-1} \rho_j \rho_i = \bar{s}_{ij}$ and 
$s_{ij}(s_{ik} \rho_j) s_{jk} = (s_{jk} \rho_i) s_{ik} (s_{ij} \rho_k)$.

**Proof.** (a) Since $\rho_i=1$, we have $D_i = I_{y_{i}}|S$. Then, $s(D_i D_j - D_j D_i) = s(y_{i} y_{j} - y_{j} y_{i}) - (y_{i} y_{j} - y_{j} y_{i}) s = s t_{ij} - t_{ij}$ (for $s \in S$) by the latter half condition of (4).

Next, $t_i D_k + t_k D_i + t_k t_i D_j = [y_i, y_j] + [y_j, y_k] + [y_k, y_i] + [y_k, y_i]$, $y_j = 0$.
(b) Since $D_i=0$, we may assume that $\rho_i=\bar{y}_i^{-1} | S$ is an automorphism in $S$ [4. Corollary 1.2]. Further, $y_{i} y_{j} = y_{j} y_{i} s_{ij}$ implies $s_{ij} = (y_{j} y_{i})^{-1} y_{i} y_{j}$. Hence $\rho_j^{-1} \rho_i^{-1} \rho_j \rho_i = y_{i}^{-1} y_{j}^{-1} y_{j} y_{i} s_{ij} y_{i}^{-1} y_{j} y_{i}$ shows that $\rho_j^{-1} \rho_i^{-1} \rho_j \rho_i = \bar{s}_{ij}$.

Next, $(y_{i} y_{j}) y_{k} = y_{k} (y_{i} s_{jk} y_{k} s_{ik}) = y_{k} (y_{i} y_{j} (s_{jk} \rho_j) s_{ik}) = y_{k} (y_{i} y_{j} (s_{jk} \rho_j) s_{ik})$. On the other hand, 

$(y_{i} y_{j}) y_{k} = (y_{j} y_{i} s_{ij}) y_{k} = y_{k} (y_{i} y_{j} (s_{jk} \rho_j) s_{ik}) = y_{k} (y_{i} y_{j} (s_{jk} \rho_j) s_{ik})$.

Consequently, we have $s_{ij}(s_{ik} \rho_j) s_{jk} = (s_{jk} \rho_i) s_{ik} (s_{ij} \rho_k)$.

**Proposition 1.2.** Let $D_i=0$, $t_{ij}=0$ and $y_{i}^e \in S$ for each $i$. Then $R$ is $R_S r_{ij}$-irreducible if and only if $V$ is a division ring.

**Proof.** Let $V$ be a division ring. We show that $R=RxS$ for each $x \in R \setminus S$. Let $r=\sum y_{i}^e y_{i}^e \cdots y_{i}^e s_{e_{i-1}} (s_{e_{i-1}} \in S)$ be the shortest non zero relation relative to the representation by $\{ y_{i}^e y_{i}^e \cdots y_{i}^e ; 0 \leq e \leq m_i - 1 \}$ in the element of $RxS$. Then we may assume that $r \in RxS \setminus S$ (otherwise, $r \in S$ implies $R=Rr_S = RxS$). Noting here $s_{e_{i-1}} = \sum y_{i}^e y_{i}^e \cdots y_{i}^e s_{e_{i-1}} \in S$ is contained in $RxS$, we may set $r = y_{i}^e y_{i}^e \cdots y_{i}^e + \sum y_{i}^e y_{i}^e \cdots y_{i}^e s_{e_{i-1}}$ since $s_{e_{i-1}} \in S$. Hence $s_{e_{i-1}} \in S$ for each $s_{e_{i-1}} \in S$. Then $(y_{i}^e y_{i}^e \cdots y_{i}^e)^{-1} = 1 + \sum y_{i}^e y_{i}^e \cdots y_{i}^e s_{e_{i-1}}$. Hence $s_{e_{i-1}} \in S$ is zero for each $s_{e_{i-1}} \in S$. Consequently, $r \in V$ and hence $R=Rr_S = RxS$. The converse is evident.

Let $A$ be a simple Artinian ring over $S$ such that $[A : S]$ is finite. $A/S$ is called $\mathfrak{g}$-regular if $A/S$ is Galois and there exists a Galois group $\mathfrak{g}$ such that $\#\mathfrak{g} = [A : S]$, while $A$ is called an abelian extension over $S$ if $A/S$ is $\mathfrak{g}$-regular for some abelian $F$-group $\mathfrak{g}$.

**Theorem 1.1.** Let $R$ be an $n$-dimensional right $e$-polynomial simple ring extension over $S$. Assume $n=p^e$, the characteristic $\chi(S)$ of $S$ is either zero or $\chi(S)$, $\rho_i=1$ and $s_{ij}=1$ for each $i, j$. Then,

(a) $V=Z$ implies $\mathfrak{g}(R/S)=1$ where $\mathfrak{g}(R/S)$ is the group of $S$-automor-
phisms in $R$.
(b) $R/S$ is $\mathfrak{G}$-regular for some $\mathfrak{G}$ if and only if $R=S[C]$ and $C/Z$ is a $p^e$-dimensional Galois extension.

Proof. (a) Since $(sy_i)\sigma=s(y_\sigma)$ for each $\sigma\in\mathfrak{G}(R/S)$ and $s\in S$, $y_\sigma-y\in V=Z$. Hence $y_\sigma=y_t+s_{\mathfrak{G}}$, $s_{\mathfrak{G}}\in Z$. Thus we may assume that $y_\sigma=y_t+1$ (e.g. take $y_t^{-1}$). If $y_t^p=\sum_{j=0}^{p-1}y_t^j(s_t^j)\in S$, then $(y_t+1)^p=\sum_{k=0}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) y_t^k = y_t^p + \sum_{k=0}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) y_t^k + \sum_{k=0}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) y_t^k s_t^j D_t^{-k}.$
From these, we can see that $p+s_{p-1}^t=s_{p-1}^t$, where it follows a contradiction $p=0$.

(b) Let $R/S$ be $\mathfrak{G}$-regular for some $\mathfrak{G}$. If $\sigma(\neq 1)\in \mathfrak{G}$, then as is shown in the proof of (a), $y_\sigma=y_t+\tau_{\mathfrak{G}}(\tau_{\mathfrak{G}}(\neq 0)\in V)$, and we may assume $y_\sigma y_t^e=y_t$ for each $0<k<p^e$ where $p^e$ is the order of $\sigma$. Then $y_\sigma y_t^e=y_t+T_y(\tau_{\mathfrak{G}}; \sigma)=y_t$ shows that $\#(\mathfrak{G}|V)=p^e$ and hence $V=C$. Thus $R=S\otimes_\mathbb{Z}C$ and $J(\mathfrak{G}|C,C)=Z$ shows that $C/Z$ is a $p^e$-dimensional Galois extension. The converse is evident.

The following is a slight generalization of Jacobson's Theorem.

**Theorem 1.2.** Let $R$ be an $n$-dimensional right $e$-polynomial simple ring over $S$. Assume $n=2^e$, $\chi(S)\neq 2$ and $t_{ij}=0$ for each $i, j$. If $[S:Z]$ is finite, the $R/S$ is weakly Galois with respect to an abelian group $\mathfrak{G}$ such that $\#\mathfrak{G}=2^n$. In particular, if $R$ is a division ring, then $R/S$ is abelian.

Proof. Since $S(1)/S$ is a right quadratic extension for each $\pi$, we may assume that $y_t^e\in S$, and $S(1)/S$ is a Galois extension with respect to $\tau_1$ such that $y_{t\tau_1}=y_{t}$ by [4, Corollary 4.1]. Let $\sigma_1$ be an extension of $\tau_1$ such that $y_{\sigma_1+y_{t\tau_1}}=y_{\sigma_1+y_{t\tau_1}}$ if $i\neq j$. Then $\sigma_1$ is an automorphism in $R$ since $t_{ij}=0$, and $J(\sigma_1,R)=S[y_1, \ldots, y_d, \ldots, y_e]$. If we set $\mathfrak{G}$, the group generated by $\sigma_1$, $\sigma_2$, $\ldots$, $\sigma_e$, then $\#(\mathfrak{G}|C)=\sigma_1 \times \sigma_2 \times \cdots \times \sigma_e$ and $J(\mathfrak{G}, R)=S$. If $R$ is a division ring, then so is $V$, and hence the above fact shows that $R/S$ is an abelian extension with respect to $\mathfrak{G}$.


Let $(t_{ij})$ be an $e \times e$ matrix with entries in $S$ such that $t_{ij}=-t_{ji}$, $t_{ii}=0$ and let $D_1, D_2, \ldots, D_e$ be derivations in $S$ satisfying $[D_i, D_j]=I_{t_{ij}}$ and $t_{ij} D_k + t_{jk} D_i + t_{ki} D_j = 0$. Further, by $S[\mathbf{x}^{(e)}]$ we denote $\{\sum X_{e^1}^{\nu_{e^1}} \cdots X_{e^n}^{\nu_{e^n}}; s_{e^1 \cdots e^n} \in S\}$, the set of all polynomials with $e$-indeterminates and the coefficients in $S$ (the coefficients written on the right hand and $X_i$ means $X_{\sigma(i)}$). We define in $S[\mathbf{x}^{(e)}]$ the following multiplication rule and functions $E_j$, $j=1,2,\ldots,e$. 

Then each isomorphism clearly gives an isomorphism of $S[\mathfrak{X}_{\pi(e-1)}]$ to $S[\mathfrak{X}_{\pi(e)}]$ for every $\pi$. Now, the assertion is clear if we prove that $E_{\pi}$ is a derivation in $S[\mathfrak{X}_{\pi(e-1)}]$, namely, $fE_{e}X_{j}+f\cdot(X_{j}E_{e}) = \sum_{\nu=0}^{e-1} \sum_{\alpha=0}^{\nu} e-1 \mu_{e_{\alpha}}=m+1$

Firstly, we shall prove that $\nu_{e} \in S[\mathfrak{X}_{\pi(e-1)}]$ and $j, k \leq e-1$ by induction on the degree of $g$. $X_{e}[E_{j}, E_{k}] = X_{e}(E_{j}) E_{k} + X_{e}(s E_{e}) = \sum_{\alpha=0}^{\nu} X_{j}^{\nu_{\alpha}} f E_{j}$ In $S[\mathfrak{X}_{\pi(e-1)}]$, $X_{e}X_{j} = X_{j}X_{e} + t_{ij} = X_{e}X_{j} - t_{jk}$ for each $j \geq i$, and so we may assume that $E_{e} = I_{X_{e}}|S[\mathfrak{X}_{\pi(e-1)}]$, $E_{j} = I_{X_{j}}|S[\mathfrak{X}_{\pi(e-1)}]$ are derivations in $S[\mathfrak{X}_{\pi(e-1)}]$. Then suppose that $g[E_{j}, E_{k}] = g t_{jk} - t_{jk} g$ for each $g \in S[\mathfrak{X}_{\pi(e-1)}]$ and $j, k \leq e-1$ by induction on the degree of $g$. $X_{e}[E_{j}, E_{k}] = X_{e}(E_{j}) E_{k} + X_{e}(s E_{e}) = \sum_{\alpha=0}^{\nu} X_{j}^{\nu_{\alpha}} f E_{j}$ In $S[\mathfrak{X}_{\pi(e-1)}]$, $X_{e}X_{j} = X_{j}X_{e} + t_{ij} = X_{e}X_{j} - t_{jk}$ for each $j \geq i$, and so we may assume that $E_{e} = I_{X_{e}}|S[\mathfrak{X}_{\pi(e-1)}]$, $E_{j} = I_{X_{j}}|S[\mathfrak{X}_{\pi(e-1)}]$ are derivations in $S[\mathfrak{X}_{\pi(e-1)}]$.
Next, let $f \in S(X_{n(j-1)})$. Then $fE_{j}X_{j} + f \cdot (X_{j}E_{j}) = (X_{j}E_{j}f + X_{j}fE_{j} + fE_{j}E_{j}) = X_{j}fE_{j} + fE_{j}X_{j} + ft_{e_{j}} = (fE_{j} + fX_{j})E_{j} = f[E_{j}, E_{j}]$.

Since we may assume that $f$ is contained in $S(X_{n(j-1)})$ which contains $S(X_{n(j-1)})$ and $X_{j}$ for a suitable $n'$, $f[E_{j}, E_{j}] = ft_{e_{j}} = t_{e_{j}}f$. Thus we have $fE_{j}X_{j} + fX_{j}E_{j} = (X_{j}f + fE_{j})E_{j} = (fX_{j})E_{j}$.

Next, we shall show that $(X_{j}^{2}fX_{j})E_{j} = (X_{j}^{2}fE_{j})X_{j} + (X_{j}^{2}f)(X_{j}E_{j})$ for each $f \in S(X_{n(j-1)})$, $\cdots$, $(\alpha)$. For, $(X_{j}^{2}fX_{j})E_{j} = (X_{j}^{2}fX_{j})E_{j} = X_{j}^{2}E_{j}X_{j} + X_{j}^{2}(X_{j}fE_{j}) + X_{j}^{2}fE_{j}E_{j}$

$= X_{j}^{2}E_{j}X_{j} + X_{j}^{2}(X_{j}f + fE_{j}) + X_{j}^{2}fE_{j}E_{j}$

$= X_{j}^{2}E_{j}X_{j} + X_{j}^{2}(X_{j}f + fE_{j}) + X_{j}^{2}fE_{j}E_{j}$

$= (X_{j}^{2}fE_{j})X_{j} + (X_{j}^{2}f)(X_{j}E_{j})$.

Let us assume that $(fX_{j}^{2})E_{j} = fE_{j}X_{j} + fX_{j}E_{j}$. Then we can prove $(fX_{j}^{2})E_{j} = fE_{j}X_{j} + fX_{j}E_{j}$ by use of $(\alpha)$.

For, $(fX_{j}^{2})E_{j} = \sum_{\mu=0}^{\nu-1} \binom{\nu-1}{\mu} X_{j}^{\nu-1}fE_{j}(X_{j}^{\mu}E_{j}X_{j}) = (\sum_{\mu=0}^{\nu-1} \binom{\nu-1}{\mu} X_{j}^{\nu-1}fE_{j}E_{j})E_{j}X_{j}$

$= fE_{j}X_{j} + fX_{j}E_{j}X_{j} + (fX_{j}^{2})E_{j}$.

Let $i = n'(\alpha)$ for each $i = 1, 2, \cdots, e$. Then the map $\sum X_{e}^{\nu_{e}}X_{e^\nu_{e^{-1}}-\overline{1}}^{\nu_{e^{-1}}-\overline{1}}X_{1}^{\nu_{1}} \rightarrow \sum X_{e}^{\nu_{e}}X_{e^\nu_{e^{-1}}-\overline{1}}^{\nu_{e^{-1}}-\overline{1}}X_{1}^{\nu_{1}} \rightarrow$ gives an isomorphism, of $S(X_{n(e)})$ onto $S(X_{n(e)})$.

By $S(X_{n(e)}; D_{n(e)}$ or sometimes, by $S(X_{1}, X_{2}, \cdots, X_{e}; D_{1}, D_{2}, \cdots, D_{e}$) we denote the polynomials ring obtained in Proposition 2.1.

Let $(s_{ij})$ be an $e \times e$ matrix with entries in $S$ such that $s_{ij} = s_{ij}^{1}$, and let $\rho_{1}, \rho_{2}, \cdots, \rho_{e}$ be automorphisms in $S$ satisfying $\rho_{j}^{1} \rho_{e}^{1} \rho_{j} \rho_{e}^{1} = s_{ij}$ and $s_{ij}(s_{kj} \rho_{k}) = (s_{kj} \rho_{k}) s_{kj}$, for each $s, s_{ij} \in S$ and $j \geq i$.

**Proposition 2.2.** $S(X_{n(e)})$ forms an associative ring by the following rule:

$$(\sum X_{e}^{\nu_{e}}X_{e^\nu_{e^{-1}}-\overline{1}}^{\nu_{e^{-1}}-\overline{1}}X_{1}^{\nu_{1}} \rightarrow s_{ij} \rightarrow) X_{j} = X_{j}(\sum X_{e}^{\nu_{e}}X_{e^\nu_{e^{-1}}-\overline{1}}^{\nu_{e^{-1}}-\overline{1}}X_{1}^{\nu_{1}} \rightarrow s_{ij} \rightarrow) P_{j}.$$

Moreover, when this is the case, $S(X_{n(e)})$ is isomorphic to $S(X_{n(e)})$ for each $\pi'$.
ring for each $\pi$. Then it is clear that the map $\sum X_{j}^{p_{j}+1} \cdots X_{e-1}^{p_{e-1}} s_{e-1}^{p_{e-1} \cdots s_{e-1}} \rightarrow \sum X_{j}^{p_{j}+1} \cdots X_{e-1}^{p_{e-1}} s_{e-1}^{p_{e-1} \cdots s_{e-1}}$ gives an isomorphism of $S[X_{e-1};] \rightarrow S[\mathfrak{X}_{e-1}]$ where $\pi(j) = \pi'(j)$. Now the assertion is clear if we prove that $P_{e}$ is a homomorphism in $S[X_{e-1}]$, namely, $fP_{e}X_{j}P_{e} = X_{j}P_{e}fP_{e}P_{e}$ for each $f \in S[\mathfrak{X}_{e-1}]$ and $j = 1, 2, \cdots, e-1$ (we set $S[\mathfrak{X}_{e}] = S$) since $f \cdot X_{j} = X_{j}f$. In $S[X_{e-1}]$, each $P_{j}$ ($j = 1, 2, \cdots, e-1$) is a homomorphism since $(fg)X_{j} = f(gX_{j})$ for each $f, g \in S[\mathfrak{X}_{e-1}]$. Then $(\sum X_{i}^{p_{i}}X_{j}^{p_{j}+1} \cdots X_{i}^{p_{i}}s_{e-1}^{p_{e-1} \cdots s_{e-1}})P_{j} = 0$ means $(\sum X_{i}^{p_{i}}X_{j}^{p_{j}+1} \cdots X_{i}^{p_{i}}u_{e-1}^{p_{e-1} \cdots u_{e-1}})^{p_{j}} = 0$ where each $u_{e-1}^{p_{e-1} \cdots u_{e-1}}$ is a regular element of $S$. Hence each $P_{j}$ is an automorphism in $S[\mathfrak{X}_{e-1}]$. Firstly, we shall prove that $fP_{e}X_{j}^{p_{j}+1} \cdots P_{j}P_{j} = fX_{j}\tilde{s}_{jk}$ for each $f \in S[\mathfrak{X}_{e-1}]$ by induction on the degree of $f$.

For, $X_{j}s_{e}P_{e}X_{j}^{p_{e}+1} \cdots P_{j}P_{e} = X_{j}P_{e}X_{j}^{p_{j}+1} \cdots P_{e}P_{e}s_{e}^{p_{e}+1}P_{j}P_{e} = X_{j}P_{e}X_{j}^{p_{j}+1} \cdots P_{e}P_{j}P_{e} = X_{j}P_{e}X_{j}^{p_{j}+1} \cdots P_{e}P_{j}P_{e} = X_{j}P_{e}X_{j}^{p_{j}+1} \cdots P_{e}P_{j}P_{e}$ by the latter conditions on $\{s_{e}, P_{j}\}$.

Assume that $(X_{i}^{p_{i}}X_{i}^{p_{i}+1} \cdots X_{i}^{p_{i}}s)P_{k}^{-1}P_{j}^{-1}P_{k}P_{j} = (X_{i}^{p_{i}}X_{i}^{p_{i}+1} \cdots X_{i}^{p_{i}}s)s_{jk}$ for each $i = 1, 2, \cdots, e-1$. Let $\sum_{j=1}^{i} \mu_{j} = m + 1$. Then $(X_{i}^{p_{i}}X_{i}^{p_{i}+1} \cdots X_{i}^{p_{i}}s)P_{j}^{-1}P_{j}P_{j} = (X_{i}^{p_{i}}X_{i}^{p_{i}+1} \cdots X_{i}^{p_{i}}s)s_{jk}$ for each $i = 1, 2, \cdots, e-1$. $\tilde{\delta}_{jk} = (X_{i}^{p_{i}+1}X_{i}^{p_{i}+1} \cdots X_{i}^{p_{i}}s)\tilde{\delta}_{jk}$ gives an isomorphism of $S[X_{e-1};]$ to $S[\mathfrak{X}_{e-1}]$.

By $S[\mathfrak{X}_{e};]$, or sometimes, by $S[X_{1}, X_{2}, \cdots, X_{e}; \rho_{1}, \rho_{2}, \cdots, \rho_{e}]$, we denote the polynomials ring obtained in Proposition 2.2.

**Lemma 2.1.** Let $f_{e}(X_{e}) = X_{e}^{p_{e}} + \sum_{i=1}^{e} X_{i}s_{i}^{(e)}$ $(s_{i}^{(e)} \in S)$ be a polynomial in $S[\mathfrak{X}_{e}]$.

(a) If $f_{e}(X_{e}) \in S[\mathfrak{X}_{e}; D_{e}]$, then $M = f_{e}(X_{e})S[\mathfrak{X}_{e}; D_{e}]$, a right ideal generated by $f_{e}(X_{e})$ is a two-sided ideal if and only if $f_{e}(X_{e})$ is central.

(b) If $f_{e}(X_{e}) \in S[\mathfrak{X}_{e}; P_{e}]$, then $M$ is a two-sided ideal if and only if $\{s_{i}^{(e)}\}$ satisfies the following conditions:

1. $s_{i}^{(e)}\rho_{e}^{m_{i}} = s_{i}^{(e)}\rho_{e}^{m_{i}}$.
2. $RN_{e}(s_{i}^{(e)}; \rho_{e}) = s_{i}^{(e)}\rho_{e}RN_{e}(s_{i}^{(e)}; \rho_{e})$. 


Proof. Since $f_i = f_i(X_i)$ is monic, $\deg f_i g = \deg f_i + \deg g$ for each polynomial $g$. Hence $M$ is a two-sided ideal if and only if $sf_i = sf'_i$ ($s \in S$) for every $s \in S$ and $X_j f_i = \{ f_i(X_j + f) \}$ in case (a). $f_i(X_j t + t)$ in case (b), for some $t, t' \in S$.

(a) Let $M$ be a two-sided ideal. Since $sf_i = X_i s' + g_i (\deg g_i < m_i)$ and $f_i s' = X_i s'^{t'} + g'_i (\deg g' < m_i)$, we can easily see $s' = s$. Next, $X_j f_i = f_i X_j + f_i t'$ show that $f_i E_j = f_i t'$, and hence $t = 0$ and $f_i E_j = 0$. Thus $f_i$ is central. The converse is clear.

(b) $sf_i (X_i) = X_i s' + \sum_{\nu=0}^{m_i - 1} X_i^\nu s^{(i)}$ and $f_i (X_i) s' = X_i s'^t + \sum_{\nu=0}^{m_i - 1} X_i^\nu s^{(i)}$. Hence $X_i s'^t = X_i s'^t + \sum_{\nu=0}^{m_i - 1} X_i^\nu s^{(i)}$. These facts show that $M$ is a two-sided ideal if and only if (1) and (2) are satisfied.

Lemma 2.2. Let $f_i (X_i)$ be a polynomial given in Lemma 2.1, $i = 1, 2, \ldots, e$.

(a) If $f_i (X_i) \in S[\mathfrak{X}_{\pi(i)} ; \mathfrak{D}_{\pi(i)}]$ and $f_i (X_i)$ is central, then $M_k E_j \subseteq M_k$ where $M_k = (f_i(X_i), f_i(X_j), \ldots, f_i(X_k)) S[\mathfrak{X}_{\pi(e)} ; \mathfrak{D}_{\pi(e)}]$.

(b) If $f_i (X_i)$ satisfies the conditions of Lemma 2.1 (b), then $M_k P_j \subseteq M_k$.

Proof. (a) Since $E_j = I_x [S[\mathfrak{X}_{\pi(e)} ; \mathfrak{D}_{\pi(e)}]]$, the assertion is clear.

(b) $f_i (X_i)$ satisfies the conditions of Lemma 2.1 (b), then $M_k P_j \subseteq M_k$.

By Lemma 2.2, we can see that each $E_j$ (resp. $P_j$) induces a derivation (resp. homomorphism) in $S[\mathfrak{X}_{\pi(k)} ; \mathfrak{D}_{\pi(k)}]/M_k \cong S[x_1, x_2, \ldots, x_k]$ (resp. $S[\mathfrak{X}_{\pi(k)} ; \mathfrak{D}_{\pi(k)}]/M_k \cong S[x_1, x_2, \ldots, x_k]$) where $x_i$ is the residue class of $X_i$ modulo $M_k$. We denote by $E_j$ (resp. $P_j$), these induced derivation (resp. homomorphism) again. Then $S[x_1, x_2, \ldots, x_k] [X_{k+1} ; E_{k+1}] = \{ \sum_{i=1}^{k+1} f_i ; f_i \in S[x_1, x_2, \ldots, x_k] \}$ (resp. $S[x_1, x_2, \ldots, x_k] [X_{k+1} ; P_{k+1}] = \{ \sum_{i=1}^{k+1} f_i ; f_i \in S[x_1, x_2, \ldots, x_k] \}$) which can be regarded as an associative ring by the rule $fX_{k+1} = X_{k+1} f + fE_{k+1}$ (resp. $fX_{k+1} = X_{k+1} f P_{k+1}$) for each $f \in S[x_1, x_2, \ldots, x_k]$.

Further, when this is the case, each polynomial $f_{k+1} (X_{k+1})$ given in Lemma 2.1 is considered as a polynomial in $S[x_1, x_2, \ldots, x_k] [X_{k+1} ; E_{k+1}]$ (resp. $S[x_1, x_2, \ldots, x_k] [X_{k+1} ; P_{k+1}]$) and $f_{k+1} (X_{k+1}) S[x_1, x_2, \ldots, x_k] [X_{k+1} ; E_{k+1}]$ is a two-sided ideal.

Now, let $S_k = S[\mathfrak{X}_{\pi(k)} ; \mathfrak{D}_{\pi(k)}] / f_k (X_k) S[\mathfrak{X}_{\pi(k)} ; \mathfrak{D}_{\pi(k)}] \subseteq S[\omega_{k}] [\mathfrak{X}_{\pi(k+1)} ; \mathfrak{D}_{\pi(k+1)}]$ where $\omega_k$ is the residue class of $X_k$ modulo $f_k (X_k) S[\mathfrak{X}_{\pi(k)} ; \mathfrak{D}_{\pi(k)}]$ and $S_{k+1} = S_{k+1} / f_{k+1} (X_{k+1}) S[\mathfrak{X}_{\pi(k+1)} ; \mathfrak{D}_{\pi(k+1)}]$ (more precisely, $S[\omega_{k}] [\mathfrak{X}_{\pi(k+1)} ; \mathfrak{D}_{\pi(k+1)}] / f_{k+1} (X_{k+1}) S[\omega_{k}] [\mathfrak{X}_{\pi(k+1)} ; \mathfrak{D}_{\pi(k+1)}]$ where $\overline{\omega}_k$ is the residue class of $\omega_k$ modulo $f_{k+1} (X_{k+1}) S[\omega_k] [\mathfrak{X}_{\pi(k+1)} ; \mathfrak{D}_{\pi(k+1)}]$, but, by $\omega_k$ we denote $\overline{\omega}_k$ again).
Lemma 2.3. (a) $S_1$ is isomorphic to $S[X_n(k); D_n(k)]/M_k$ (resp $S[X_n(k); \mathfrak{P}_n(k)]/M_k$).

(b) If $f_i(X_i)$ is $w$-irreducible in $S_2$ for each $\pi$, then $S[x_1, x_2, \ldots, x_k]$ is simple for each $k$ and $\pi$.

Proof. The following diagram is commutative where $\phi_i: S_{e+1}\rightarrow S_{e+1}/f_{i}(X_{i})S_{e+1}$, $\theta: S[X_n(k); D_n(k)]\rightarrow S[X_n(k); D_n(k)]/M_k$, $\varphi=\varphi_k\varphi_{k-1}\cdots\varphi_1$, and $\Psi=(an$ inverse image of $\varphi)\cdot\theta$.

\[
\begin{array}{c}
S[X_n(k); D_n(k)] \xrightarrow{\varphi_k} S[X_n(k); D_n(k)] \\
\downarrow \phi_k \\
S_{k-1} \\
\downarrow \psi_1 \\
S \xrightarrow{\theta} S[x_1, x_2, \ldots, x_k]
\end{array}
\]

For, $S_1 \ni \sum w_{i}^{\nu}w_{i}^{\nu_{1}}\cdots w_{i}^{\nu_{\pi(k)}}s_{\nu_{1}\cdots\nu_{1}}=0$ $(s \in S)$, then each inverse image of $\sum w_{i}^{\nu}w_{i}^{\nu_{1}}\cdots w_{i}^{\nu_{\pi(k)}}s_{\nu_{1}\cdots\nu_{1}}=0$ of $\varphi$ is contained in $M_k$. Hence $(\sum w_{i}^{\nu}w_{i}^{\nu_{1}}\cdots w_{i}^{\nu_{\pi(k)}}s_{\nu_{1}\cdots\nu_{1}})\varphi^{-1} \subseteq M$. Hence $\sum w_{i}^{\nu}w_{i}^{\nu_{1}}\cdots w_{i}^{\nu_{\pi(k)}}s_{\nu_{1}\cdots\nu_{1}}=0$.

(b) Since $S[w_2, w_3, \ldots, w_k] \supseteq S[w_1, w_2, \ldots, w_k] \supseteq S$ for each $\pi$, the $w$-irreducibility of $f_i(X_i)$ in $S_{e-1}$ yields at once that of $f_i(X_i)$ in $S[w_2, w_3, \ldots, w_k] [X_i; E_i]$. Hence $S[X_i; D_i]/f_i(X_i)s[X_i; D_i]$ is simple for each $\pi$. Thus repeating the procedures, we have the simplicity of $S[w_1, w_2, \ldots, w_k] \cong S[x_1, x_2, \ldots, x_k]$.

Let $f_i(X_i)$ $(i=1, 2, \ldots, e)$ be a polynomial of $S[X_i; D_i]$ (resp $S[X_i; P_i]$). Then $f_i(X_i)$ is called $s.w$-irreducible if $f_i(X_i)$ is $w$-irreducible in $S[X_1, X_2, \ldots, \check{X}_i, \ldots, X_k; E_1, E_2, \ldots, \check{E}_i, \ldots, E_k]/M$ (resp. $S[X_1, X_2, \ldots, \check{X}_i, \ldots, X_k; P_1, P_2, \ldots, \check{P}_i, \ldots, P_k]/M$) for each $1 \leq k \leq e$ where $M$ is a two-sided ideal generated by $f_1(X_i), f_2(X_i), \ldots, f_i(X_i), \ldots, f_e(X_i)$. Moreover, $f_i(X_i)$ is called $s$-irreducible if $f_i(X_i)$ is irreducible in $S[X_1, X_2, \ldots, \check{X}_i, \ldots, X_k; E_1, E_2, \ldots, \check{E}_i, \ldots, E_k]/M$ (resp. $S[X_1, X_2, \ldots, \check{X}_i, \ldots, X_k; P_1, P_2, \ldots, \check{P}_i, \ldots, P_k]/M$).

In following we shall give necessary and sufficient conditions for a simple ring to have an $e$-polynomial simple ring extension of dimension $n=\prod_{\ell=1}^{e} m_{\ell}$, $m_{\ell}>1$.

Proposition 2.3. (a) In order that $S$ have an $n=\prod_{\ell=1}^{e} m_{\ell}$ $(m_{\ell}>1)$-dimensional $e$-polynomial simple ring extension such that $n_{\ell}=1$ and $s_{i\ell}=1$, it is necessary and sufficient that there exist derivations $D_1, D_2, \ldots, D_e$ in $S$, $(t_{ij})$
an $e \times e$ matrix with entries in $S$ such that $t_{ij} = -t_{ji}$, $t_{ii} = 0$ and $(s^{\ell}_{ij})$ a $1 \times m_{\ell}$ matrices $(i=1, 2, \cdots, e)$ satisfying

1. $[D_{i}, D_{j}] = I_{t_{ij}}$
2. $t_{ij}D_{k} + t_{jk}D_{i} + t_{kl}D_{j} = 0$
3. $(m_{\ell} - k + h)_{sD^{h}s_{m_{\ell}-k+h}^{(i)}} = s_{m_{\ell}-k}^{(i)}s$
4. $s_{m_{\ell}-k}^{(i)}D_{j} = \sum_{h=0}^{k} (m_{\ell} - k + h)_{sD^{h}s_{m_{\ell}-k+h}^{(i)}} = s_{m_{\ell}-k}^{(i)}s$
5. $f_{i}(X_{i}) = X_{i}^{m_{i}} + \Sigma_{\nu=0}^{m_{i}-1}X_{i}^{\nu}s_{\nu}^{(i)}$

Proof.

(b) In order that $S$ have an $n=I^{e}Tm_{i}i=1(m_{i}>1)$-dimensional $e$-polynomial simple ring extension such that $D_{i}=0$ and $t_{ij}=0$, it is necessary and sufficient that there exist automorphisms $\rho_{1}, \rho_{2}, \cdots, \rho_{e}$ in $S$, $(s_{ij})$ an $e \times e$ matrix with entries in $S^{*}$ such that $s_{ij} = s_{ji}^{-1}$, $s_{ii} = 1$ and $(s^{\ell}_{ij})$ matrices $(i=1, 2, \cdots, e)$ satisfying

1. $\rho_{j}^{-1}\rho_{i}^{-1}\rho_{j}\rho_{i} = t_{ij}^{(e)}$
2. $s_{ij}(s_{iA}\rho_{j})s_{jk} = (s_{jk}\rho_{i})s_{ik}(s_{ij}\rho_{k})$
3. $s\rho_{i}^{m_{\ell}}s_{\nu}^{(i)} = s_{\nu}^{(i)}s\rho_{i}^{m_{\ell}}$
4. $RN_{\nu}(s_{ji} ; \rho_{j})s_{\nu}^{(i)} = s_{\nu}^{(i)}\rho_{j}RN_{m_{i}}(s_{ji} ; \rho_{j})$
5. $f_{i}(X_{i}) = X_{i}^{m_{i}} + \Sigma_{\nu=0}^{m_{i}-1}X_{i}^{\nu}s_{\nu}^{(i)}$

is s.w.-irreducible.

(a) Let $R = S[x_{1}, x_{2}, \cdots, x_{e}]$ be an $n=\prod_{i=1}^{e}m_{i}(m_{i}>1)$-dimensional $e$-polynomial simple ring extension over $S$ with $sx_{i} = x_{i}s + s_{0}^{(i)}(s_{0}^{(i)} \in S)$ for every $s \in S$ and $x_{i}x_{j} = x_{j}x_{i} + t_{ij}$. Then, by Proposition 1, (a), each $D_{i}|_{x_{i}} = I_{t_{ij}}$, $t_{ij} = x_{i}x_{j} - x_{j}x_{i}$ satisfies (1) and (2). Let $x_{i}^{m_{i}} + \Sigma_{v=0}^{m_{i}-1}x_{i}^{v}s_{\nu}^{(i)} = 0$. Then $s(x_{i}^{m_{i}} + \Sigma_{v=0}^{m_{i}-1}x_{i}^{v}s_{\nu}^{(i)}) = (x_{i}^{m_{i}} + \Sigma_{v=0}^{m_{i}-1}x_{i}^{v}s_{\nu}^{(i)})s$ and $x_{j}(x_{i}^{m_{i}} + \Sigma_{v=0}^{m_{i}-1}x_{i}^{v}s_{\nu}^{(i)}) = (x_{i}^{m_{i}} + \Sigma_{v=0}^{m_{i}-1}x_{i}^{v}s_{\nu}^{(i)})x_{j}$ imply (3) and (4) for each $1 \times m_{i}$ matrix $(s^{\ell}_{ij})$. Finally (5) is an immediate consequence of (3) of the definition of an $e$-polynomial simple ring extension and Lemma 2.3.

(b) Sufficiency is clear from the proof of (a). Conversely, let $R = S[x_{1}, x_{2}, \cdots, x_{e}]$ be an $n=\prod_{i=1}^{e}m_{i}(m_{i}>1)$-dimensional $e$-polynomial simple ring extension over $S$ with $sx_{i} = x_{i}s\rho_{i}$ and $x_{i}x_{j} = x_{j}x_{i}s_{ij}$, then each $\tilde{x}_{i}^{-1}|S = \rho_{i}, x_{i}^{-1}x_{j}^{-1}x_{i}x_{j} = s_{ij}$ satisfies conditions (1) and (2). The rest of the proof is same.
§ 3. Abelian extensions of prime characteristic $p$.

In this section, we shall deal with $p^e$-dimensional abelian extensions of a simple ring of prime characteristic $p$. A $p^e$-dimensional abelian extension $R$ over $S$ is called a $p^e$-dimensional fundamental abelian extension over $S$ if $\chi(S)=p$ and there exists a Galois group $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, a direct product of $\sigma_i$, with $\#\sigma_i=p$ and $\mathfrak{G}$ is an $F$-group. Then, one of principal theorem of this section can be stated.

Theorem 3.1. Let $\chi(S)=p$. In order that $S$ have a $p^e$-dimensional fundamental abelian extension, it is necessary and sufficient that there exist derivations $D_1, D_2, \cdots, D_e$ in $S$, $(t_{ij})$ an $e\times e$ matrix with entries in $S$ such that $t_{ij}=-t_{ji}$, $t_{ii}=0$ and $(s_i)$ $1\times e$ matrix with entries in $S$ satisfying

1. $[D_1, D_2]=I_{t_{12}}$,
2. $t_{ij}D_k+t_{jk}D_i+t_{ki}D_j=0$,
3. $D_i^p-D_i=I_{t_{ii}}$, $s_iD_i=0$,
4. $t_{ij}D_j^{p-1}+t_{ji}+s_jD_i=0$,
5. $X_i^p-s_i=(X_i^p-t_{ij}D_j)^p-X_i^p-s$ is s.w.-irreducible.

More precisely, if there exists $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying (1)-(5), then $M=(X_1^p-s_1, X_2^p-s_2, \cdots, X_e^p-s_e)S[\mathfrak{X}_e; \mathfrak{D}_e]$ is a maximal ideal of $S[\mathfrak{X}_e; \mathfrak{D}_e]$, $R^*=S[y_1, y_2, \cdots, y_e]\cong S[\mathfrak{X}_e; \mathfrak{D}_e]/M$ is a $p^e$-dimensional fundamental abelian extension with respect to $\mathfrak{G}^*=(\sigma_1^*)\times(\sigma_2^*)\times\cdots\times(\sigma_e^*)$ such that $\#\sigma_i^*=p$ and $\mathfrak{G}^*=\mathfrak{G}$ is the residue class of $X_i$ modulo $M$. Conversely, if $R$ is a $p^e$-dimensional fundamental abelian extension over $S$ with respect to $\mathfrak{G}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$, then we can find such $\{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\}$ satisfying (1)-(5) that there holds an $S$-isomorphism $\varphi^*: R^*\cong R$ with $\varphi^*s_i=\sigma_i^*\varphi^*$.

Proof. Since $s(X_i^p-s_i)=(X_i^p)^p-s(X_i^p)+sD_i^{p-1}-sD_i+ss_i$, and $X_i(X_i^p-s_i)=X_i^p-s_i=\sum_{\nu}X_i^\nu f_{\nu}\rightarrow\sum_{\nu}(X_i+1)^\nu f_{\nu}(f_{\nu}\in S[\mathfrak{X}_e; \mathfrak{D}_e])$. Thus, by Proposition 2.3 (a), $R^*=S[y_1, y_2, \cdots, y_e]\cong S[\mathfrak{X}_e; \mathfrak{D}_e]/M$ is a $p^e$-dimensional $e$-polynomial simple ring extension over $S$. Let $\Phi_i$ be the map of $S[\mathfrak{X}_e; \mathfrak{D}_e]$ defined by $\sum_{\nu}X_i^\nu f_{\nu}\rightarrow\sum_{\nu}(X_i+1)^\nu f_{\nu}(f_{\nu}\in S[X_1, X_2, \cdots, X_e; D_1, D_2, \cdots, D_e])$. Then $(f\Phi_i)X_i\Phi_i=f^*(X_i+1)X_i$, $X_i\Phi_i=f^*(X_i+1)=X_i f+fE_i+f=(X_i f+fE_i)\Phi_i$ shows that $\Phi_i$ is a ring homomorphism of $S[\mathfrak{X}_e; \mathfrak{D}_e]$, and moreover, $\Phi_i^*=1$ (since $\chi(S)=p$) means that $\Phi_i$ is an automorphism of order $p$. From the definition of $\Phi_i$, $\Phi_i\Phi_j=\Phi_j\Phi_i$ is clear. Next, $(X_i^p-s_i)\Phi_j=(X_i^p+\delta_{ij})^p-(X_i^p+\delta_{ij})-s=X_i^p-s$. Hence each $\Phi_j$ induces an automorphism $\sigma_j^*$ of order $p$ in $R^*$. Let $(\sigma_j^*)\circ(\sigma_1^*, \sigma_2^*, \cdots, \sigma_{j-1}^*)=\sigma_j^*$. Then
$y_\sigma^\ast=y_j+k=y_\sigma^\ast y^\ast_1\sigma^\ast_2\cdots \sigma^\ast_j y^\ast_{j+1}=y_j$ show that $\sigma^\ast=1$. Consequently, we have $\Phi^\ast$, the group generated by $\sigma^\ast_1, \sigma^\ast_2, \cdots, \sigma^\ast_e$, is an abelian group of order $p$ such that the direct product of $\sigma^\ast_1, \sigma^\ast_2, \cdots, \sigma^\ast_e$. Noting here $J(\Phi^\ast, R^\ast)=S[y_1, y_2, \cdots, y_\epsilon, \cdots, y_e]$, we have $J(\Phi^\ast, R^\ast)=\bigcap_{i=1}^e J(\sigma^\ast_i, R^\ast)=S$. Thus $R^\ast/S$ is a $p^\ast$-dimensional weakly Galois extension with respect to $\Phi^\ast$. Thus in following, we shall show that $V^\ast=V_{R^\ast}(S)$ is simple. By [Lemma 2 of [6]], it suffices to prove that $C^\ast[v_1, v_2, \cdots, v_e]$ is a field where $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_e\}=\Phi^\ast_\vec{\sigma}V^\ast$.

Since $\tilde{v}_j\sigma^\ast_j=\sigma^\ast_j\tilde{v}_i$, $v_j\sigma^\ast_j=c_{ij}v_i$ for some $c_{ij}\in C^\ast=V^\ast(R^\ast)$ (and $\tilde{v}_j\tilde{v}_i=\tilde{v}_i\tilde{v}_j$, since $v_j\tilde{v}_j=c\cdot v_i$ for some $c \in C^\ast$ implies $v_i=v_i\tilde{v}_j=\sigma^\ast_j v_i$). Noting here that $C^\ast$ is cyclic over $J(\sigma^\ast_j|C^\ast, C^\ast)$ and $N_p(c_{ij}; \sigma^\ast_j)=1$, there exists an element $d_{ij}\in C^\ast$ such that $c_{ij}=d_{ij}(d_{ij}\sigma^\ast_j)$. Setting here $u_{ij}=v_id_{ij}$, then $u_{ij}$ is contained in $V_{\Phi^\ast(y_1, y_2, \cdots, y_e)}(S)=V^\ast_j$. Hence $v_i\in C^\ast[V^\ast_j]$ for each $j=1, 2, \cdots, e$, and so, $v_i\in \bigcap_{j=1}^e C^\ast[V^\ast_j]$. Noting that $\bigcap_{j=1}^e C^\ast\otimes_{C^\ast[V^\ast_j]} V^\ast C^\ast\otimes_{C^\ast[V^\ast_j]} Z$ (considered in $C^\ast\otimes_{C^\ast[V^\ast_j]} V^\ast$) and $\bigcap_{j=1}^e C^\ast[V^\ast_j]$ is an epimorphic image of $\bigcap_{j=1}^e C^\ast\otimes_{C^\ast[V^\ast_j]} V^\ast_j$, we have $v_i\in C^\ast[Z]$. Now we shall show that $C^\ast[Z]$ is a field.

For, each $z \in Z$, $zE_k=x_{ij}y_{ij}y_{ij}x_{ij}z \in Z$ and hence, $z^pE_k=p(zE_kx^{p-1})=0$. Hence $z^p \in C^\ast \otimes_{C^\ast[V^\ast]} Z$. While, $C^\ast\otimes_{C^\ast[V^\ast]} Z$ is a commutative semi-simple artinian ring since $C^\ast/C^\ast[V^\ast]$ is a finite Galois extension field. Let $uv=0$ for some $u, v \in C^\ast\otimes_{C^\ast[V^\ast]} Z$. Then $0=(uv)^p=uv^p$ yields at once $u^p=0$ or $v^p=0$ since $u^p, v^p \in C^\ast$, and hence $u=0$ or $v=0$. Thus $C^\ast\otimes_{C^\ast[V^\ast]} Z \cong C^\ast[Z]$ is a field. Consequently, $C^\ast[v_i]$ is a field since $[C^\ast[v_i]: C^\ast]<\infty$. Repeating the same procedures, we have $C^\ast[v_1, v_2, \cdots, v_k] \subseteq (C^\ast[Z])$ is a field.

Conversely, let $R$ be a $p^\ast$-dimensional fundamental abelian extension over $S$ with respect to $\Phi=(\sigma_1)(\sigma_2)\cdots(\sigma_e)$. Then $R=S[x_1, x_2, \cdots, x_e]$ with $x_j\sigma_j=x_j+\delta_{ij}$ [Corollary 2 of [5]]. Then $D_j=I_{\mathfrak{a}_j}|S$ is a derivation in $S$ since $(sx_j-x_js)\sigma_j=s(x_j+\delta_{ij})-(x_j+\delta_{ij})s=sx_j-x_js$. Further, $(x_jx_j-x_jx_i)\sigma_k=(x_j+\delta_{ik})$. $(x_j+\delta_{ik})-(x_j+\delta_{ik})(x_j+\delta_{ik})=x_jx_j-x_jx_j$. Hence, if we set $t_{ij}=x_jx_j-x_jx_i$, then $t_{ij}=-t_{ji}$, $t_{ii}=0$ and $\{D_1, D_2, \cdots, D_e, (t_{ij})\}$ satisfies conditions (1) and (2). Thus there exists a polynomial ring $S[\mathfrak{a}_j; \mathfrak{D}_e]$. Next, $(x_j^p-x_i)\sigma_j=(x_j+\delta_{ij})^p-(x_j+\delta_{ij})$ shows that $s_i=x_j^p-x_j \in S$ and $X_j^p-s_i$ is central in $S[\mathfrak{a}_j; \mathfrak{D}_e]$. Hence conditions (3) and (4) are satisfied. Finally, if we note that $V$ is a field [Theorem 5 of [5]], each intermediate ring of $R/S$ is simple [Lemma 1.4 of [7]]. Hence $X_j^p-s_i$ is $s.w.$-irreducible. Let $\varphi^\ast$ be the map defined by $\sum x_{ij}x_{ik}^p=\sum x_{ij}y_{ij}x_{ij}^p=\cdots=\sum y_{ij}x_{ij}^p=\cdots$. Then it is clear that $\varphi^\ast$ is an $S$-isomorphism of $R$ to $R^\ast$ satisfying $\varphi^\ast \sigma_1^\ast=\sigma_1 \varphi^\ast$. 
Corollary 3.1. Let \( \chi(S) = p \). (a) In order that \( S \) have a \( p \)-dimensional fundamental inner abelian extension, it is necessary and sufficient that there exist \( \{D_{1}, D_{2}, \ldots, D_{e}, (t_{ij}), (s_{i})\} \) satisfying (1)–(4) of Theorem 3.1 and \( (z_{i}) \) a \( 1 \times e \) matrix with entries in \( Z^{*} \) satisfying
\[
(6) \quad z_{i}D_{j} = z_{i}\delta_{ij}.
\]
(b) In order that \( S \) have a \( p \)-dimensional fundamental outer abelian extension, it is necessary and sufficient that there exist \( \{D_{1}, D_{2}, \ldots, D_{e}, (t_{ij}), (s_{i})\} \) satisfying (1)–(5) of Theorem 3.1 and
\[
(6') \quad zD_{j} = 0 \quad \text{for all} \quad j = 1, 2, \ldots, e.
\]

Proof. By Proposition 2.3 (a), there exists a \( p \)-dimensional \( e \)-polynomial simple ring extension \( R^{*} = S[y_{1}, y_{2}, \ldots, y_{e}] \) over \( S \) where each \( y_{i} \) is the residue class of \( X_{i} \) modulo \( M \). Let \( \sum_{i=0}^{p-1}X_{i}f_{i} \) be a polynomial of \( S[y_{1}, y_{2}, \ldots, y_{e-1}] \) \( [X_{j}; E_{j}] \) \( \left( f_{i} \in S[y_{1}, y_{2}, \ldots, y_{j-1}] \right) \). Then, for each \( z_{j} \) satisfying (6), \( z_{j}(\sum_{i=0}^{p-1}X_{i}f_{i}) \)
\[
= (\sum_{i=0}^{p-1}z_{j}^{i}X_{j}f_{i}) = \sum_{i=0}^{p-1}(\sum_{i=0}^{p-1}X_{j}^{i}f_{i}z_{j}),
\]
Hence we have \( (\sum_{i=0}^{p-1}X_{i}f_{i}) \)
\[
I_{x_{j}}^{p} = (p-1)!z_{j}^{-1}f_{p-1}
\]
by induction on the degree of the polynomial. Thus each \( X_{i} - s_{i} \) is s.w.-irreducible. Consequently, \( R^{*} \) is a \( p \)-dimensional fundamental abelian extension of \( S \). Next, \( z_{i}D_{j} = z_{i}y_{j} - y_{j}z_{j} = z_{i}\delta_{ij} \) shows that \( z_{i}y_{j}z_{j}^{-1} = y_{j} + \delta_{ij} \). Thus we may adapt \( z_{i} \) as \( \sigma_{i}^{*} \) since \( z_{i}^{*} \in C^{*} \).

Conversely, let \( R/S \) be a \( p \)-dimensional fundamental inner abelian extension over \( S \) with respect to \( \mathfrak{O} = (\tilde{v}_{1}) \times (\tilde{v}_{2}) \times \cdots \times (\tilde{v}_{e}) \), \( v_{i} \in V \). Since \( V = C[Z] \) [Theorem 5 of [5]], we have \( v_{i} = \sum_{k}c_{k}^{(i)}z_{k} \) where \( \{c_{k}^{(i)}\} \subseteqq (C) \) is a \( Z \)-basis for \( V \). On the other hand, \( v_{i}x_{j}v_{i}^{-1} = x_{j} + \delta_{ij} \) implies \( v_{i}E_{j} = v_{i}x_{j} - x_{j}v_{i} = v_{i}\delta_{ij} \). Hence
\[
v_{i}E_{j} = \sum_{k}c_{k}^{(i)}(z_{k}E_{j}) = v\delta_{ij},
\]
which shows that the existence of \( (z_{1}, z_{2}, \ldots, z_{e}) \) satisfying (6).

(b) Under the same notations of the proof of (a), if \( \mathfrak{O}^{*} \) contains \( \tau = \sigma_{1}^{*}s_{1}^{*}\sigma_{2}^{*} \cdots \sigma_{e}^{*}s_{e} = \tilde{v} \) \((v \in V^{*})\), as was shown in the latter half of the proof of (a), there exists \( z \in Z^{*} \) such that \( zD_{j} = \alpha_{j}x \) \((j = 1, 2, \ldots, k)\), and this contradicts \( ZD_{j} = 0 \) for all \( j = 1, 2, \ldots, e \).

Conversely, if \( R \) is a \( p \)-dimensional outer Galois extension over \( S \), then \( Z \subseteq C \). Hence \( ZD_{j} = 0 \) since \( CD_{j} = 0 \) for all \( j = 1, 2, \ldots, e \).

\( R \) is called a trivial extension over \( S \) if \( R = S[C] \). Then we can prove the following:

Corollary 3.2. Let \( \chi(S) = p \). Then the following conditions are equivalent.

(a) \( S \) has a \( p \)-dimensional fundamental trivial abelian extension.

(b) There exist elements \( z_{1}, z_{2}, \ldots, z_{e} \) in \( Z^{*} \) such that the algebraic set associated with a two-sided ideal \( M = (X_{1}^{p} - z_{1}, X_{2}^{p} - z_{2}, \ldots, X_{e}^{p} - z_{e})Z[X_{1}, X_{2}, \ldots, X_{e}] \) is an algebraic variety.

(c) There exist elements \( s_{1}, s_{2}, \ldots, s_{e} \) in \( S \) such that \( X_{i}^{p} - s_{i} \) is s.w.- irre-
ducible in \( S[X_1, X_2, \cdots, X_e] \).

**Proof.** (a)\( \rightarrow \) (b) Let \( R = S[C] = S \otimes Z C \). Then \( J(\mathfrak{G}|C, C) = Z \) means that 
\( C/Z \) is a \( p^e \)-dimensional fundamental abelian extension with the (abelian) Galois group \( \mathfrak{G}|C \). Hence (b) is clear.

(a)\( \rightarrow \) (c) Let \( R = S[C] \) and \( \{c_1^e, c_2^e, \cdots, c_i^e\} ; 0 \leq c_i \leq p - 1, c_i \in C \} \) be \( Z \)-basis such that \( c_i^e - c_i \in Z \). Then \( X_i^e - c_i \) is \( s \cdot w \)-irreducible in \( S[X_1, X_2, \cdots, X_e] \).

(b)\( \rightarrow \) (a) Since \( Z' = Z[X_1, X_2, \cdots, X_e]/M \) is a \( p^e \)-dimensional fundamental abelian extension, \( S \otimes Z' \) is a requested one.

(c)\( \rightarrow \) (a) Let \( M = (X_i^e - s_1, X_2^e - s_2, \cdots, X_i^e - s_i) S[X_1, X_2, \cdots, X_e] \). Then 
\( S[X_1, X_2, \cdots, X_e]/M \) is a requested one.

Let \( X_i^e - s_i \) be \( s \)-irreducible. Then \( S[X_i^e; D_i]/M \) is a division ring. Hence we obtain

**Corollary 3.3.** Let \( S \) be a division ring of \( \chi(S) = p \). In order that \( S \) have a \( p^e \)-dimensional fundamental abelian division ring extension, it is necessary and sufficient that there exist \( \{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\} \) satisfying (1)-(4) of Theorem 3.1 and

(5') \( X_i^e - s_i \) is \( s \)-irreducible.

Let \( R/S \) be a \( p^e \)-dimensional fundamental abelian extension with respect to \( \mathfrak{G} \). Then \( R/S \) is called a parallel extension if \( R = S[T] \) such that \( T/T \cap S \) is a \( p^e \)-dimensional fundamental abelian extension with respect to \( \mathfrak{G}|T \). A \( p^e \)-dimensional fundamental parallel abelian extension is called a \( d \)-parallel (resp an \( f \)-parallel) extension if \( T \) can be a division ring (resp a field).

**Corollary 3.4.** Let \( \chi(S) = p \). (a) In order that \( S \) have a \( p^e \)-dimensional fundamental parallel extension, it is necessary and sufficient that there exist \( \{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\} \) satisfying (1)-(5) of Theorem 3.1 and

(6) there exists a simple subring \( W(\exists 1) \) of \( S \) satisfying \( WD_i \subseteq W, \{t_{ij}\}, \{s_i\} \subseteq W \) and \( X_i^e - s_i \) is \( s \cdot w \)-irreducible.

(b) In order that \( S \) have a \( p^e \)-dimensional fundamental \( d \)-parallel abelian extension, it is necessary and sufficient that there exist \( \{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\} \) satisfying (1)-(5) of Theorem 3.1 and

(6') there exists a division subring \( W(\exists 1) \) of \( S \) satisfying \( WD_i \subseteq W, \{t_{ij}\}, \{s_i\} \subseteq W \) and \( X_i^e - s_i \) is \( s \)-irreducible in \( W[X_i^e; D_i] \).

(c) In order that \( S \) have a \( p^e \)-dimensional fundamental \( f \)-parallel abelian extension, it is necessary and sufficient that there exist \( \{D_1, D_2, \cdots, D_e, (t_{ij}), (s_i)\} \) satisfying (1)-(5) of Theorem 3.1 and

(6'') there exists a subfield \( W(\exists 1) \) of \( S \) satisfying \( X_i^e - s_i \) is \( s \)-irreducible in \( W[X_1, X_2, \cdots, X_e] \).

Proof. Let \( R^* = S[y_1, y_2, \cdots, y_e] \cong S[X_i^e; D_i]/M, \) and \( T^* = \{ \sum y_1^e y_2^e \cdots \)
Let $R/S$ be $\mathfrak{G}$-regular, and an intermediate ring $T$ of $R/S$ be satisfy $J(\mathfrak{G}, T)=S$ and $[T:S]=\#\mathfrak{G}$ where $\mathfrak{G}=\mathfrak{G}|T$. If $\mathfrak{G}=\mathfrak{G}|T$, $T/S$ is said to be regularly embedded in $R/S$.

Let $T/S$ be a $p^e$-dimensional abelian extension with respect to $\mathfrak{G}=\mathfrak{G}|T$, $T/S$ can be regularly embedded in $R/S$ where $R/S$ is a $p^{f+m+k}-dimensional$ abelian extension with respect to $\mathfrak{H}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_m)\times(\sigma_{m+1})\times\cdots\times(\sigma_{m+k})$ if $k\geq 0$ and $\mathfrak{H}=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_{m+k})\times(\tau_{m+k+1})\times\cdots\times(\tau_m)$ if $-m\leq k< 0$.

Let $s$ be an element of $S\subseteq S[\mathfrak{G}; \mathfrak{D}]$, and let $A_{(s)}(s)=1$ and $A_{(s)}(s)=A_{(s)}(s)D_t + A_{(s)}(s)s$. Then we have $(X_t+s)^n=\sum_{k=0}^{n(\begin{array}{l}k\end{array})-k}^{n}X_t^k\Delta(s)$.

**Theorem 3.2.** Let $\chi(S)=p$ and $T/S$ be a $p^e$-dimensional abelian extension with respect to $\mathfrak{G}=(\tau_1)\times(\tau_2)\times\cdots\times(\tau_m)$, $\#\mathfrak{G}=p^{f+e}$. In order that $T/S$ can be regularly embedded to a $p_{f+m+k}$-dimensional abelian extension such that $k\geq -m$, it is necessary and sufficient that there exist derivations $D_1, D_2, \cdots, D_{m+k}$ in $T$, $(t_{ij})$ a $1 \times (m+k) \times (m+k)$ matrix with entries in $T$, $(t_{ij})$ a $1 \times (m+k) \times (m+k)$ matrix with entries in $T$ satisfying the conditions (1)-(5) of Theorem 3.1 (in $T$) and $(b_{ij})$ an $(m+k) \times (m+k)$ matrix with entries in $T$ satisfying

\begin{enumerate}
  \item $D_t\tau_j - \tau_j D_t = \tau_j I_{b_{ij}},$
  \item $T_{\mathfrak{G}}(b_{ij}; \tau_j) = \delta_{ij},$
  \item $A_{(s)}(s) - b_{ij} = t_i(t_j-1),$
  \item $t_{ij} = b_{ik}(\tau_j I_{b_{ij}}) - b_{jk} D_t,$
  \item $b_{ij} \tau_j - 1 = b_{ik}(\tau_j - 1)$
\end{enumerate}

More precisely, if there exists $\{D_1, \cdots, D_{m+k}, (t_{ij}), (t_i), (b_{ij})\}$ satisfying the
conditions, then $R^* = T[y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_{m+k}] = T[x_{m+k} ; \mathfrak{D}_{m+k}] / M$, $M = (X^g_k - t_1, X^g_k - t_2, \ldots, X^g_k - t_m, X_{m+k} - t_{m+k}) T[x_{m+k} ; \mathfrak{D}_{m+k}]$ and $y_k$ is the residue class of $X_k$ modulo $M$, is a requested one with generating automorphisms $\sigma^*_1, \sigma^*_2, \ldots, \sigma^*_{m+k}$ such that $(y, t) \sigma^*_i = (y + b_{ij})(t \tau_j)$ $(t \in T)$ and $\mathfrak{S}^* | T = \mathfrak{S}$ where $\mathfrak{S}^* = (\sigma^*_1) \times (\sigma^*_2) \times \cdots \times (\sigma^*_{m+k})$ if $k \geq 0$ and $\mathfrak{S}^* = (\sigma^*_1) \times (\sigma^*_2) \times \cdots \times (\sigma^*_{m+k}) \times (\tau_{m+k+1}) \cdots \times (\tau_m)$ if $k < 0$. Conversely, if $T | S$ can be regularly embedded in a $| r = m + k |$-dimensional abelian extension with respect to $\mathfrak{S} = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_{m+k})$ then there exists such $\{D_1, D_2, \ldots, D_{m+k}, (t_{ij}), (t_i), (b_{ij})\}$ satisfying the conditions in Theorem 3.1 and (1)–(5) that there holds a $T$-isomorphism $\varphi^* : R^* \cong R$ with $\varphi^* \sigma_i = \sigma^*_i \varphi^*$.

**Proof.** Firstly, we set $b_{ij} = \delta_{ij}$ if $k > 0, j > m$ or $b_{ij} = 0$, if $k < 0$ and $j > m$. Then, clearly, they satisfy the conditions (1)–(5).

Let $\Phi_j$ $(j = 1, 2, \ldots, m + k)$ be the mapping of $T[x_{m+k}; \mathfrak{D}_{m+k}]$ defined by $\sum X_m t_m + \cdots X_{m+k} t_{m+k} \rightarrow \sum (X_m + b_{m+k}) t_m + \cdots (X_{m+k} + b_{m+k}) t_{m+k} \tau_j$ (we set $\tau_j = 1$ if $j > m$). Then we can prove that $\Phi_j$ is an automorphism by induction on the number of indeterminates. By (1), we have $(t \Phi_j) X_i \Phi_j = (t \tau_j)(X_i + b_{ij}) = X_i(t \tau_j) + t \tau_j D_i + t \tau_j b_{ij} = X_i t \tau_j + t D_i \tau_j - b_{ij} D_{m+k}$ if $m + k \geq 0$. Hence $\Phi_j$ is a (ring) epimorphism of $T[X_i ; D_i]$. Further $(\sum_{m+k} X_i t \Phi_j = \sum m \tau_j + X_{m+k} \tau_j)$ shows that $\Phi_j$ is an automorphism. Hence, we assume that $\Phi_j$ is an automorphism in $T[x_{m+k-1}; \mathfrak{D}_{m+k-1}]$. Thus it suffices to prove $(Y \Phi_j)(X_{m+k}) \Phi_j = (X_{m+k}) \Phi_j(Y \Phi_j)$ if $Y \in T[x_{m+k-1}; \mathfrak{D}_{m+k-1}]$. Now, $(Y \Phi_j)(x_{m+k}) \Phi_j = (Y \Phi_j)(x_{m+k} + b_{m+k}) = x_{m+k} (Y \Phi_j) + (Y \Phi_j) E_{m+k}$.

Let $X_i (E_{m+k} \Phi_j) = (t \Phi_j) X_i \Phi_j = (t \tau_j)(X_i + b_{ij}) = X_i t \tau_j + t D_i \tau_j - b_{ij} D_{m+k}$ if $m + k \geq 0$. Hence the assertion is clear if we show that $(Y \Phi_j) I_{b_{m+k} j} = Y (E_{m+k} \Phi_j - \Phi_j E_{m+k})$. Now, $X_i (E_{m+k} \Phi_j - \Phi_j E_{m+k}) = (t \tau_j) Y - \tau_j D_{m+k}$ by (4) and $X_i \Phi_j I_{b_{m+k} j} = X_i (t \tau_j) + t \tau_j b_{ij} = (b_{ij} D_{m+k} - b_{ij} D_{m+k} - b_{ij} D_{m+k})$ if $m + k \geq 0$. Hence we have $X_i (E_{m+k} \Phi_j - \Phi_j E_{m+k}) = (X_i \Phi_j) I_{b_{m+k} j}$.

Let us assume that $Y (E_{m+k} \Phi_j - \Phi_j E_{m+k}) = (Y \Phi_j) I_{b_{m+k} j} = (X_i \Phi_j) I_{b_{m+k} j} = (X_i \Phi_j) I_{b_{m+k} j}$ for each $X_i = X^{n_{m+k}}_{m+k-1}$, \ldots, $X_i t_{m+k} = X_{m+k} t_{m+k}$ such that $\deg Y = q$. Then, for each $X_i = X^{n_{m+k}}_{m+k-1}$, \ldots, $X_i t_{m+k}$, $X_i (E_{m+k} \Phi_j - \Phi_j E_{m+k}) = (Y \Phi_j) I_{b_{m+k} j}$.

Therefore, we can see $\Phi_j$ is an automorphism. $X_i \Phi_j K = X_i + b_{ik} + b_{ij} \tau_j$, $X_i \Phi_j \Phi_j = X_i + b_{ij} + b_{ij} \tau_j$ shows that $\Phi_j \Phi_j = \Phi_j \Phi_j$ if and only if there holds (5).
$X_0\Phi_j^x = X_t + T_{\Phi_j}(b_{ij}; \tau_j)$ shows that (2) is equivalent with $\#\Phi_j = p^{f+j}$ if $j \leq m$ and $\#\Phi_j = p$ if $j > m$. Now $(X_0^2 - t_i)\Phi_j = X_0^2 + (A_p^f)(b_{ij} - b_{ij} - t_i\tau_j)$ shows that (3) is equivalent with that $X_0^2 - t_i$ is left invariant under $\Phi_j$. Thus each $\Phi_j$ induces an automorphism $\sigma_j^*$ of order $p^{f+j+1}$ if $j \leq m$, of order $p$ if $j > m$ in $R*[y_1, y_2, \ldots, y_{m+k}] \cong T[\mathfrak{x}_{m+k}; \mathfrak{D}_{m+k}] / M$ where $R^*$ is a $p^{f+k+m}$-dimensional simple ring extension over $S$.

Let $\sigma = (\sigma_j^*)_{\cap}(\sigma_1^*, \sigma_2^*, \cdots, \sigma_j^*)$. If $\sigma = \sigma_j^* \sigma_j^*= \sigma_j^* \sigma_1^* \sigma_j^* \cdots \sigma_j^*$, we may assume that each $\alpha_j$ is a multiple of each $\tau_j$ since $1 = \sigma|T = \tau_j^f = \tau_j^p \tau_j^2 \cdots \tau_j^{p^m}$, Then $y\sigma = y_j + T_{\sigma_j}(b_{ij}; \tau_j) = y_j + \alpha_j / \#\tau_j = y_j \sigma_1^* \sigma_j^* \cdots \sigma_j^* = y_j$ yields at once $\sigma = 1$. Hence $\mathfrak{G}$, the group generated by $\sigma_1^*, \sigma_2^*, \cdots, \sigma_j^*$, is coincides with the direct product of them. Finally, $S \subseteq J(\mathfrak{G}, R^*) = J(\mathfrak{G}|T, T) = S$. Thus the rest, we shall show that $\mathfrak{G}$ is an F-group.

Firstly, we note that $C^*[Z]$ is a commutative semi-simple artinian ring as same reason as that of Theorem 3.1. Let $\{\bar{v}_1, \ldots, \bar{v}_m\} = \mathfrak{G} \cap \mathfrak{V}^*$. Then $v_i \sigma_j^* = c_i v_i$ for some $c_i \in C^*$ such that $N_{\mathfrak{G}}^*(c_{ij} ; \sigma_j^*) = 1$ and $v_i v_j = v_j v_i$. Thus, we can see that $C^*[v_1, v_2, \cdots, v_m] \cong C^*[Z]$. If $v = \sum c_i \alpha_i$, $c_i \in C^*$ and $\alpha_i = v_i^m v_i^{m-1} \cdots v_i^1$, is an arbitrary element of $C^*[v_1, v_2, \cdots, v_m]$, then $v^p \in C^*$ for some positive integer $a$. Thus $C^*[v_1, v_2, \cdots, v_m]$ is a field.

Conversely, let $T/S$ can be regularly embedded in a $p^{f+k+m}$-dimensional abelian extension $R/S$ with respect to $\mathfrak{G} = (\sigma_1) \times (\sigma_2) \times \cdots \times (\sigma_j) \times \cdots \times (\sigma_{m+k})$ if $k \geq 0$ and $\mathfrak{G} = (\sigma_1) \times \cdots \times (\sigma_{m+k}) \times (\tau_{m+k+1}) \times \cdots \times (\tau_m)$ if $-m < k < 0$ such that $\#\sigma_j = (\#\tau_j)^p$, $\sigma_j|T = \tau_j$ if $j \leq m$ and $\#\sigma_j = p$, $\sigma_j|T = 1$ if $j > m$. Since $V$ is a field and $[R : T] = 2^{m+k}$, $R/T$ is a fundamental abelian extension with respect to $\mathfrak{G}^* = (\sigma_1^*) \times \cdots \times (\sigma_{m+k}^*) \times (\tau_{m+k+1}) \times \cdots \times (\tau_m)$. Then $R = \bigoplus_{0 \leq x < p} x^\alpha x_{e^e} \cdots x_{e^1} T$, $x_\sigma \sigma_j^* - x_\sigma = \delta_{ij}$ if $j \leq m$ and $x_\sigma \sigma_j - x_\sigma = \delta_{ij}$ if $j > m$. Evidently $b_{ij} = x_\sigma \sigma_j - x_\sigma$ is contained in $T$ and $T_{\sigma_j}(b_{ij}; \tau_j) = \sum x_\sigma x_{e^e} \cdots x_{e^1} (x_\sigma \sigma_j - x_\sigma) \sigma_j^f = \delta_{ij}$, further, $b_{ij}(\tau_k - 1) = (x_\sigma \sigma_j - x_\sigma)(\sigma_j - 1) = (x_\sigma \sigma_j - x_\sigma) \sigma_j - (x_\sigma \sigma_j - x_\sigma) \sigma_j - (x_\sigma \sigma_j - x_\sigma) = b_{i k}(\tau_j - 1)$.

Now, we set $D_i = I_{x_i}$, $t_i = x_i - x_i$ and $t_{ij} = x_i x_j - x_j x_i$. Then one will easily check that $\{D_i, D_{ij}, D_m, \cdots, (t_i), (t_{ij})\}$ satisfies the conditions (1)–(5) of Theorem 3.1 in $T$.

$$t(D_x \sigma_j - \sigma_j D_x) = (t x_i - x_i t) \sigma_j - (t t_i x_i - x_i t_t) \sigma_j = t x_i \sigma_j - (x_i \sigma_j - x_i) t_t \sigma_j = t x_i I_{b_{ij}} (t \in T).$$

$$t_{ij}(\tau_k - 1) = (x_i x_j - x_j x_i) \sigma_k - (x_i x_j - x_j x_i) \sigma_k = (x_i x_j) \sigma_k - (x_i \sigma_j - x_j) x_i + x_i (x_i \sigma_j - x_j) = (x_i x_k - x_i) x_j \sigma_k - x_j \sigma_k (x_i \sigma_j - x_j) - b_{jk} D_i = b_{ik} \tau_j D_j \sigma_k - b_{jk} D_i.$$

Finally, $(x_i^p - x_i) \sigma_j = x_i^p + \Delta^f(b_{ij}) - x_i - b_{ij}$ shows that $\Delta^f(b_{ij}) - b_{ij} = (x_i^p - x_i)(\sigma_j - 1) = t_i (\tau_j - 1)$. 

K. Kishimoto
The map $\varphi^*: \sum x_{\nu}^{e_{\nu}} \cdots X_{1}^{e_{1}} t_{e_{\nu}^{-1}} \rightarrow \sum y_{\nu}^{e_{\nu}} \cdots y_{1}^{e_{1}} t_{e_{\nu}^{-1}} \in T$ is a $T$-isomorphism of $R^{*}$ onto $R$ satisfying $\varphi^* \sigma_i = \sigma_i^* \varphi^*$.

Combining Corollary 3.3 with Theorem 3.2, we can easily obtain

**Corollary 3.5.** Let $T/S$ be a $p$-dimensional abelian division ring extension of $\chi(S)=p$ with respect to $\Theta=\tau_1 \times \cdots \times \tau_m$ such that $\# \tau_i = p^{t_i}$.

In order that $T/S$ can be regularly embedded in a $p^{t_1 + \cdots + t_m}$-dimensional abelian division ring extension such that $k \geq -m$, it is necessary and sufficient that there exist derivations $D_1, D_2, \ldots, D_{n+k}$ in $T$, $(t_{ij})$ an $(m+k) \times (m+k)$ matrix with entries in $T$, $(t_i)$ a $1 \times (m+k)$ matrix with entries in $T$ satisfying the conditions (1)–(5)' of Corollary 3.3 (in $T$) and $(b_{ij})$ an $(m+k) \times (m+k)$ matrix with entries in $T$ satisfying the conditions (1)–(5) of Theorem 3.2.


Throughout the present section, we assume that $Z$ contains $\zeta$, a primitive $n$-th root of 1. An $n$-dimensional abelian extension $R$ over $S$ is called an $n$-dimensional Kummer extension if $C \ni \zeta$ and there exists a Galois group $G=\langle \sigma_1 \rangle \times \cdots \times \langle \sigma_e \rangle$, a direct product of $\sigma_i$ with $\# \sigma_i = \pi_i$, $\prod_i \pi_i = n$ and $G$ is a $DF$-group.

One of principal theorems of this section is the following

**Theorem 4.1.** In order that $S$ have an $n$-dimensional Kummer extension, it is necessary and sufficient that there exist automorphisms $\rho_1, \rho_2, \ldots, \rho_e$ in $S$, $(s_{ij})$ an $e \times e$ matrix with entries in $S^*$ such that $s_{ij}=s_{ji}^{-1}$, $s_{ii}=1$ and $(s_i)$ a $1 \times e$ matrix with entries in $S^*$ satisfying

1. $\rho_i^{-1}\rho_j \rho_i = \bar{s}_{ij}$,
2. $s_{ij}(s_{ik}) \rho_j = (s_{jk}) \rho_i (s_{ij} \rho_j)$,
3. $\rho_i^* = \bar{s}_{ii}^{-1}$ such that $\prod_{i=1}^{e} \pi_i = n$, $s_i \rho_i = s_i$ and $\zeta \rho_i = \zeta$,
4. $RN_{n_i}(\rho_i) = (s_i^{-1} \rho_i) s_i$,
5. $f(X_{\pi(e)}) = X_{\pi(e)}^{n_{\pi}} - s_{\pi(e)}$ is $s.w$-irreducible in $S[\mathfrak{X}_{\pi(e)}; \mathfrak{P}_{\pi(e)}]$,
6. If $g(X)$ is a polynomial of $S[\mathfrak{X}_{\pi(e)}; \mathfrak{P}_{\pi(e)}] \backslash M$, $M=(f(X_1), \ldots, f(X_e))$, then $\sum f_i(X)g(X)u_i = 1$ modulo $M$ for some $f_i(X) \in S[\mathfrak{X}_{\pi(e)}; \mathfrak{P}_{\pi(e)}]$, $u_i \in S$.

More precisely, if there exists $\{\rho_1, \rho_2, \ldots, \rho_e, (s_{ij}), (s_i)\}$ satisfying (1)–(6), then $M=(f(X_1), f(X_2), \ldots, f(X_e)) S[\mathfrak{X}; \mathfrak{P}_\pi]$ is maximal and $R^* = S[y_1, y_2, \ldots, y_e]$ $\equiv S[\mathfrak{X}; \mathfrak{P}_\pi]/M$, $y_i$ is the residue class of $X_i$ modulo $M$, is an $n$-dimensional Kummer extension with generating automorphisms $\sigma_1^*, \sigma_2^*, \ldots, \sigma_e^*$ such that $x_i \sigma_j^* \{ x_i^{n_i/n_j} \text{ if } i=j \}$.
Conversely, if $R$ is an $n$-dimensional Kummer extension over $S$ with respect to $\mathfrak{G}=(\sigma_{1})\times(\sigma_{2})\times\cdots\times(\sigma_{e}),$ $\#\sigma_{i}=n_{i}$ and $\prod_{i=1}^{e}n_{i}=n,$ then we can find such \(\{\rho_{1}, \rho_{2}, \cdots, \rho_{e}, (s_{ij}), (s_{i})\}\) satisfying (1)-(5) and $S[\mathfrak{x}_{e}; \Psi_{e}]$ satisfying (6) that there holds an $S$-isomorphism $\varphi^{*}: R^{*}\cong R$ with $\varphi^{*}\sigma_{i}=\sigma_{i}^{*}\varphi^{*}$ for each $i=1, 2, \cdots, e.$

Proof. By Proposition 2.2, there exists a polynomial ring $S[\mathfrak{x}_{e}; \Psi_{e}]$ = \{\(\sum X_{i}^{n_{i}}X_{i}^{n_{i}+1} \cdots X_{e}^{n_{i}+1}, s_{i} \in S\) \} such that $sX_{i}=X_{i}(s\rho_{i}), X_{i}X_{j}=X_{j}X_{i}s_{ij}$ (1) and (2)).

Since $X_{j}(X_{i}^{n_{i}}s_{i}^{-1}=1)=X_{i}^{n_{i}}X_{j}R\left(s_{j} ; \rho_{i}\right)s_{i}^{-1}=X_{j}$, (4) means that $X_{j}(X_{i}^{n_{i}}s_{i}^{-1}=1)=X_{i}^{n_{i}}s_{i}^{-1}=X_{j}.$ Thus, with the first two conditions of (3), we can see that $M=(f(X_{1}), f(X_{2}), \cdots, f(X_{e}))S[\mathfrak{x}_{e}; \Psi_{e}]$ is a two-sided ideal, and by (5), it is maximal.

Now, let $\Psi_{i}$ ($i=1, 2, \cdots, e$) be the map of $S[\mathfrak{x}_{e}; \Psi_{e}]$ defined by $\sum X_{i}X_{i+1} \cdots X_{e}X_{e-1}=X_{i}^{n_{i}}X_{i-1}^{n_{i}} \cdots X_{i-1}^{n_{i}}s_{i}^{-1}.$ Then $\Psi_{i}$ is an automorphism. For, $f\Psi_{i}X_{i}f\Psi_{i}=f \cdot X_{i}X_{i+1}^{n_{i}} \cdots X_{e}X_{e-1}^{n_{i}}s_{i}^{-1}X_{i-1}^{n_{i}} \cdots X_{i}^{n_{i}}s_{i}^{-1}=X_{i}^{n_{i}}s_{i}^{-1}=X_{j}.$ Thus, with the first two conditions of (3), we can see that $M=(f(X_{1}), f(X_{2}), \cdots, f(X_{e}))S[\mathfrak{x}_{e}; \Psi_{e}]$ is a two-sided ideal, and by (5), it is maximal.

Let $\sigma \in (\sigma_{i}^{*})(\sigma_{1}, \sigma_{2}, \cdots, \sigma_{i-1})$. If $\sigma=\sigma_{i}^{*}=\sigma_{1}^{*} \cdots \sigma_{i-1}^{*}$, then $x_{i}\sigma_{i}^{*}=x_{i}^{n_{i}}s_{i}^{-1}X_{i}^{n_{i}}s_{i}^{-1}=s_{i}$ implies that $M\Psi_{i}=M.$ Thus each $\Psi_{i}$ induces an automorphism $\sigma_{i}^{*}$ of order $n_{i}$ in $R^{*}=S[y_{1}, y_{2}, \cdots, y_{e}]$ where $R^{*}$ is an $n$-dimensional simple ring over $S$ which is isomorphic to $S[\mathfrak{x}_{e}; \Psi_{e}]/M$ and $y_{i}$ is the residue class of $X_{i}$ modulo $M.$

Let $\sigma \in (\sigma_{i}^{*})(\sigma_{1}, \sigma_{2}, \cdots, \sigma_{i-1}).$ If $\sigma=\sigma_{i}^{*}=\sigma_{1}^{*} \cdots \sigma_{i-1}^{*}$, then $x_{i}\sigma_{i}^{*}=x_{i}^{n_{i}}s_{i}^{-1}X_{i}^{n_{i}}s_{i}^{-1}=s_{i}$ shows that $\sigma=1$. Hence $\mathfrak{G}^{*}$, the group generated by $\sigma_{1}, \sigma_{2}, \cdots, \sigma_{e}$, coincides with a direct product of them since $\sigma_{i}^{*}\sigma_{j}^{*}=\sigma_{j}^{*}\sigma_{i}^{*}$ is clear.

Thus we have $J(\mathfrak{G}^{*}, R^{*})=\bigcap_{i=1}^{e}J(\sigma_{i}^{*}, R^{*})=S.$

The condition (6) is equivalent with that $R^{*}$ is $R^{*}S_{e}$-irreducible. Hence, by Proposition 1.2, $V^{*}$ is a division ring.

Conversely, if $R$ is an $n$-dimensional Kummer extension over $S$ with respect to $\mathfrak{G}=(\sigma_{1})\times(\sigma_{2})\times\cdots\times(\sigma_{e}),$ $\#\sigma_{i}=n_{i}$, then $R=\bigoplus x_{i}^{n_{i}}\cdots x_{i}^{n_{e}}S$ where each $x_{i}$ is an element contained in $R^{*}$ and $x_{i}\sigma_{j}^{*}=x_{i}$ if $i=j$ [Theorem 4 of [8]].

We set $\rho_{i}^{*}=\tilde{x}_{i}^{-1}|S,$ $s_{i}=x_{i}^{n_{i}},$ $s_{ij}=x_{i}^{n_{i}}x_{j}^{n_{j}}x_{i}x_{j}.$ Then $(x_{i}^{-1}s_{i})\sigma_{j}=x_{i}^{-1}s_{i}x_{j}$ and $s_{ij}\sigma_{k}=s_{ij}$ show that $\rho_{i}$ is an automorphism of $S,$ $s_{i},$ $s_{i} \in S$ respectively, and $\rho_{i}$ satisfies (3).

Now $s_{ij}^{*}\rho_{i}^{*}\rho_{j}^{*}=x_{j}^{n_{j}}x_{i}^{n_{i}}s_{ij}^{*}\rho_{i}^{*}=(x_{j}^{n_{j}}x_{i}^{n_{i}}x_{i}^{n_{j}}x_{j}^{n_{i}})\rho_{j}^{*}x_{i}^{-1}x_{j}^{-1}x_{j}x_{i}=s_{ij}^{*}$ for each $s \in S.$
4.1 Theorem.

Next, \( RN_{d_1}(s_{1i}; \rho_i) = (s_{1i}) \cdots (s_{1i}) = (s_{1i}) \cdots (s_{1i}) \)

\[= (s_{1i}) \cdots (s_{1i}) = (s_{1i}) \cdots (s_{1i}) \]

\[= (s_{1i}) \cdots (s_{1i}) = (s_{1i}) \cdots (s_{1i}) \]

Thus, there holds (2).

Since \( R = S[x_i, x_{2i}, \ldots, x_{en}] \) satisfies (1) of Theorem 4.1 and if \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \) for some \( s \in S \), then,

\[ \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \]

(7) \( s \rho_{i} = RN_{s_{1i}}(s_{1i}; \rho_{i}) \{RN_{s_{2i}}(s_{2i}; \rho_{i}) \rho_{i}^{*} \} \{RN_{s_{3i}}(s_{3i}; \rho_{i}) \rho_{i}^{*} \} \cdots \rho_{en}^{*} = \mathbf{1} \)

(8) \( s \rho_{i} = RN_{s_{1i}}(s_{1i}; \rho_{i}) \{RN_{s_{2i}}(s_{2i}; \rho_{i}) \rho_{i}^{*} \} \{RN_{s_{3i}}(s_{3i}; \rho_{i}) \rho_{i}^{*} \} \cdots \rho_{en}^{*} = \mathbf{1} \)

(b) In order that \( S \) have an \( n \)-dimensional outer Kummer extension, it is necessary and sufficient that there exist \( \{\rho_{1}, \rho_{2}, \ldots, \rho_{n}, (s_{ij}), (s_{i})\} \)

satisfying the conditions (1)–(5) of Theorem 4.1 and if \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \) for some \( s \in S \), then,

\[ \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \]

(7) \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \)

(8) \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \)

(c) In order that \( S \) have an \( n \)-dimensional inner Kummer extension such that \( V = Z \), it is necessary and sufficient that there exist \( \{\rho_{1}, \rho_{2}, \ldots, \rho_{e}, (s_{ij}), (s_{i})\} \)

satisfying (1)–(4) of Theorem 4.1 and there exist a \( 1 \times e \) matrix \( (t_{i}) \) with entries in \( T \) satisfying

(7') \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \)

(8') \( \rho_{(1)}^{*} \rho_{(2)}^{*} \cdots \rho_{en}^{*} = \mathbf{1} \)

In particular.

Proof. If there exists \( \{\rho_{1}, \rho_{2}, \ldots, \rho_{e}, (s_{ij}), (s_{i})\} \)

satisfying the conditions (1)–(5) of Theorem 4.1, then there exists an \( n \)-dimensional simple ring extension \( R^{*} = S[y_{1}, y_{2}, \ldots, y_{n}] \) over \( S \) such that \( J(\mathbb{S}, R^{*}) = S \).

(a) Let \( v = \sum y_{i}^{*} y_{i}^{*} \cdots y_{i}^{*} s_{en} \) be an element of \( V^{*} \). Then \( sv = vs \) \( (s \in S \)
yields at once $\sum y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1} = \sum y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$. Hence each $y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is contained in $V^*$ and $\rho_{e}^{e} \cdots \rho_{1}^{1} = \tilde{s}_{e}^{e} \cdots s_{1}^{1}$. Thus $\rho_{e}^{e} \cdots \rho_{1}^{1} = \tilde{s}$ if and only if $y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is contained in $V^*$. Now, let $v = y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ be in $V^*$. Then $v \in C^*$ if and only if $y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is contained in $V^*$. Hence $v_{e}^{e} \cdots v_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is in $V^*$.

(b) As was shown in the proof of (a), $\rho_{e}^{e} \cdots \rho_{1}^{1} = \tilde{e}$ is equivalent with $v_{e}^{e} \cdots v_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is contained in $V^*$. Hence if $t_{e}^{e} \cdots t_{1}^{1}$ satisfies the two conditions of $(7')$, then $v_{e} y_{e}^{e} \cdots v_{1}^{1} y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1} \rho_{e}^{e} \cdots \rho_{1}^{1} = \rho_{e}^{e} \cdots \rho_{1}^{1}$.

Conversely, if $R = S[x_{1}, x_{2}, \cdots, x_{e}]$ is an $n$-dimensional inner Kummer extension over $S$, then $\sigma_{e} = v_{e}^{e} \cdots v_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$ is contained in $V^*$. Hence $v_{e}^{e} \cdots v_{1}^{1} s_{e}^{e} \cdots s_{1}^{1} = \rho_{e}^{e} \cdots \rho_{1}^{1}$.

(c) If there exists $(z_{e})$ satisfying $(7'')$, then we can prove that each $f(X_{e}) = X_{e}^{e} - s_{e}$ is $s.w$-irreducible in $S[x_{e} ; \mathfrak{P}_{e}]$. For let $g(X_{e}) = \sum x_{e}^{e} X_{e}^{e}$ be a proper left factor of $f(X_{e})$ in $S[y_{1}, \cdots, y_{e-1}][X_{e} ; P_{e}]$ where $f_{e} \in S[y_{1}, \cdots, y_{e-1}]$. Then $f_{e} \neq 0$. Since the constant term of $z_{e} g(X_{e}) - g(X_{e}) z_{e}$ is a proper left factor of $f(X_{e})$ in $S[y_{1}, \cdots, y_{e-1}][X_{e} ; P_{e}]$, then $f_{e} \neq 0$.

Next, since $z_{e} y_{e}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1} = y_{j} y_{j}^{e} \cdots y_{1}^{1} s_{e}^{e} \cdots s_{1}^{1}$, we may regard $(\tilde{e})$ as $(\sigma_{e}^{*})$. Then $V^* = J_{S}^* \cap V^* = Z$. The converse is almost evident.

Now as was shown in the proof of Corollary 4.1 (b), if $\rho_{e}^{e} = \tilde{e} (s_{e} \in S)$ for each $i = 1, 2, \cdots, e$, then $y_{e} s_{e} \in V^*$ where $R^* = S[y_{1}, \cdots, y_{e}]$. Hence we obtain $R^* = S \otimes_{S} V^* = S[V^*]$. Using this fact we have

**Corollary 4.2.** The followings are equivalent.

1. $S$ has an $n$-dimensional trivial Kummer extension.
2. There exist elements $z_{1}, z_{2}, \cdots, z_{e}$ in $Z$ such that the associated algebraic set of the ideal $(X_{e}^{e} - z_{1}, X_{e}^{e} - z_{2}, \cdots, X_{e}^{e} - z_{e})$ in $Z[X_{1}, X_{2}, \cdots, X_{e}]$ is an algebraic variety.
3. There exist inner automorphisms $\rho_{1}, \rho_{2}, \cdots, \rho_{e}$ in $S$, $(s_{e})$ an $e \times e$ matrix with entries in $S^*$, $(s_{e})$ a $1 \times e$ matrix with entries in $S^*$ satisfying (1)-(4) of Theorem 4.1 and if $\rho_{e}^{e} \cdots \rho_{1}^{1} = \tilde{s}$ then $s$ satisfies $(7'')$ of Corollary 4.1 (a).
Proof. The implication (1)−(2)−(3) is easy. (3)−(1) is an immediate consequence of the remark above mentioned and Corollary 4.1 (a).

The condition (6) of Theorem 4.1 was need only to see that the centralizer $V^*$ is a division ring. Accordigly we have

Corollary 4.3. Let $S$ be a division ring. In order that $S$ have an $n$-dimensional Kummer division ring extension, it is necessary and sufficient that there exist $\{\rho_1, \rho_2, \cdots, \rho_e, (s_{ij}), (s_i)\}$ satisfying (1)−(4) of Theorem 4.1 and

\[ f(X) = X^{n_i} - s_i \]

is $s$-irreducible in $S[\mathfrak{x}_e; \mathfrak{y}_e]$. The following is corresponding to Corollary 3.4 and we can prove it as similar to that of Corollary 3.4.

Corollary 4.4. Let $S$ be a simple ring with automorphisms $\rho_1, \rho_2, \cdots, \rho_e, (s_{ij})$ and $(s_i)$ satisfying the conditions (1)−(6) of Theorem 4.1.

(a) In order that $S$ have an $n$-dimensional parallel Kummer extension, it is necessary and sufficient that there exists a simple subring $W$ of $S$ containing $\{s_{ij}\}, \{s_i\}$ such that $W\rho_i \subseteq W$, $V_w(W) \ni \zeta$ and $X_i^{n_i} - s_i$ is $s$-irreducible in $W[\mathfrak{x}_e; \mathfrak{y}_e]$ for each $i = 1, 2, \cdots, e$.

(b) In order that $S$ have an $n$-dimensional $d$-parallel Kummer extension, it is necessary and sufficient that there exist a division subring $W$ of $S$ containing $\{s_{ij}\}, \{s_i\}$ such that $W\rho_i \subseteq W$, $V_w(W) \in \zeta$ and $X_i^{n_i} - s_i$ is $s$-irreducible in $W[\mathfrak{x}_e; \mathfrak{y}_e]$ for each $i = 1, 2, \cdots, e$.

(c) In order that $S$ have an $n$-dimensional $f$-parallel Kummer extension, it is necessary and sufficient that there exist a subfield $W$ of $S$ containing $\{s_{ij}\}, \{s_i\}$ such that $\rho_i | W = 1$, $W \ni \zeta$ and $X_i^{n_i} - s_i$ is $s$-irreducible in $W[\mathfrak{x}_e; \mathfrak{y}_e]$ for each $i = 1, 2, \cdots, e$.

Finally, we shall deal with the regular embedding problem of Kummer case.

Lemma 4.1. Let $T$ be a $q$-dimensional simple ring extension over $S$ such that $T$ is $T_vS,\text{-irreducible}$ and $V_T(T) \ni \zeta$, and let $R$ be an $n$-dimensional Kummer extension over $T$. Then $R$ is $R_vS,\text{-irreducible}$ if and only if $V$ is a division ring.

Proof. Let $R = T[x, \cdots, x_v], x_i^{n_i} \in T$, and let $V$ be a division ring. We shall show that $R = RxS$ for each $x \in R \setminus S$. Let $y = \sum_1^7 x_i^{n_i} s_{ij} x_j^{n_j} s_{ij} x_i^{n_i} s_{ij}$, be the element which is the shortest non zero relation contained in $RxS$. Then, without loss of generality, we may assume that $y \in R \setminus S$. Since $T$ is $T_vS,\text{-irreducible}$ and each $x_i$ is regular, we may assume that the constant term of $y$ is 1. Thus $sy - ys = 0$, that is, $R = RyS = RxS$. The converse is clear.

In what follow, we assume that $T$ is a $q$-dimensional abelian extension over $S$ with respect to $\mathfrak{D} = (\tau_1) \times (\tau_2) \times \cdots \times (\tau_m)$, $\# \tau_i = q_i$ such that $T$ is $T_vS,\text{-irreducible}$ and $V_T(T) \ni \zeta$. 
Theorem 4.2. In order that $T$ can be regularly embedded in a $q$-dimensional abelian extension $R/S$ such that $R/T$ is an $n$-dimensional Kummer extension such that $V$ is a division ring, it is necessary and sufficient that there exist automorphisms $\rho_1, \rho_2, \ldots, \rho_{m+k}$ $(k > -m)$ in $T$, $(s_{ij})$ an $(m+k) \times (m+k)$ matrix with entries in $T^*$, $(t_i)$ a $1 \times (m+k)$ matrix with entries in $T^*$ satisfying the conditions (1)–(5) of Theorem 4.1 (in $T$) and $(b_{ij})$ an $(m+k) \times (m+k)$ matrix with entries in $T^*$ satisfying

1. $\tau_j \rho_i \tau_j \rho_i = b_{ij}$,
2. $\{R N_{s_i}(b_{ij} ; \tau_j) = \xi^{n/m};
3. LN_{s_i}(b_{ij} ; \rho_i) = t_i^{-1} \cdot t_i \rho_i$,
4. $s_{ij}(b_{ik} \rho_i) b_{jk} = (b_{ik} \rho_i) b_{ik} (s_{ij} \tau_j)$,
5. $b_{ik}(b_{ij} \tau_j) = b_{ij}(b_{ik} \tau_j)$

(6) If $g(x)$ is a polynomial in $T[X_{m+k} ; \mathfrak{P}_{m+k}] \setminus M$, $M = (X_{1}^{n_1} - t_1, \ldots, X_{m+k}^{n_{m+k}} - t_{m+k})$, then $\sum \sigma(x) g(x) s_i = 1$ modulo $M$ for some $f_i(x) \in T[X_{m+k} ; \mathfrak{P}_{m+k}]$, $\sigma \in S$.

More precisely, if there exists $\{\rho_1, \rho_2, \ldots, \rho_e, (s_{ij}), (t_i), (b_{ij})\}$ satisfying the conditions, then $M = (X_{1}^{n_1} - t_1, \ldots, X_{m+k}^{n_{m+k}} - t_{m+k}) T[X_{m+k} ; \mathfrak{P}_{m+k}]$ and $R^* = T[y_1, \ldots, y_m, x_1, \ldots, x_{m+k}] \cong T[X_{m+k} ; \mathfrak{P}_{m+k}] / M$, $y_i$ is the residue class of $X_i$ modulo $M$, is a requested one with generating automorphisms $\sigma_1^*, \sigma_2^*, \ldots, \sigma_{m+k}^*$ such that $(y_i t) \sigma^*_i = y_i b_{ij} t \rho_i$ $(t \in T)$ and $G^* \cap T = G$ where $G^* = \{\sigma_1^* \times \sigma_2^* \times \cdots \times (\sigma_{m+k}^*) \times (\sigma_{m+k+1}^*) \times \cdots \times (\sigma_1^*)\times \cdots \times (\sigma_{m+k}^*) \times (\tau_{m+k+1}) \times \cdots \times (\tau_m)$ if $k < 0$. Conversely, if $T/S$ be regularly embedded in a $q$-dimensional abelian extension with respect to $G = \{\sigma_1 \times \sigma_2 \times \cdots \times (\sigma_{m+k})\}$ such that $R/T$ is an $n$-dimensional Kummer extension, then there exists $\{\rho_1, \rho_2, \ldots, \rho_e, (s_{ij}), (t_i), (b_{ij})\}$ satisfying the conditions in Theorem 4.1 (in $T$) and (1)–(5) that there holds a $T$-isomorphism $\phi^* : R^* \cong R$ with $\phi \sigma_i = \sigma_i^* \phi^*$ for each $i = 1, 2, \ldots, m+k$.

Proof. Firstly we set $b_{ij} \sigma^*_i = \xi^{n/m}$ if $k > 0$ and $i = j > m$

Then we can easily checked that they satisfy the conditions (1)–(5).

Let $\psi_j$ $(j = 1, 2, \ldots, m+k)$ be the mappings of $T[X_{m+k} ; \mathfrak{P}_{m+k}]$ defined by

$$\sum X_{m+k}^r \cdots X_{m+k}^r \psi_j = \sum (X_{m+k}^r b_{m+k} \cdots X_{m+k}^r \psi_j) \psi_j (t_{m+k} \cdots t_j)$$

(we set $t_j = 1$ if $j > m$). Then we can prove that $\psi_j$ is an automorphism by induction on the number of indeterminates.

$(\psi_j) X_i \psi_j = (\tau_j) (X_i \psi_j) = X_i t \tau_j \rho_i t \rho_i$ and $(X_i t \rho_i) \psi_j = X_i b_{ij} t + t \rho_i \tau_j$ and hence the condition (1) yields $(t X_i) \psi_j = (X_i t \rho_i) \psi_j$, and further, $\# \psi_j = n \sigma_j$ (by (2)) shows that $\psi_j$ is an automorphism in $T[X_i ; \rho_i]$. Hence we assume that the assertion is true for each $T[X_{m+k-1} ; \mathfrak{P}_{m+k-1}]$. Then it suffices to prove
Now, since each $\mathfrak{G}=(\sigma_{1})x(\sigma_{2})x\cdots x(\sigma_{m})x\cdots x(\sigma_{m+k})$ and each $\sigma_{j}|T=1$, we have that $\sigma_{j}^{*}$ is a product of $q_{j}$, and hence $\mathfrak{G}^{*}$ is an automorphism of $\mathfrak{G}$ such that $\mathfrak{G}^{*}=(\sigma_{1})^{*q_{1}}x(\sigma_{2})^{*q_{2}}x\cdots x(\sigma_{m})^{*q_{m}}x\cdots x(\sigma_{m+k})^{*q_{m+k}}$. Consequently, we have $\mathfrak{G}^{*}|T=\mathfrak{G}$.

Finally, if (6) is satisfied, then by Lemma 4.1, $V^{*}$ is a division ring.

Conversely, let $T/S$ be a regular embedded in a $qn$-dimensional abelian extension $R/S$ with respect to $\mathfrak{G}=(\sigma_{1})x(\sigma_{2})x\cdots x(\sigma_{m})x\cdots x(\sigma_{m+k})$ if $k\geq 0$ and $\mathfrak{G}=(\sigma_{1})\times\cdots\times(\sigma_{m+k})$ if $-m<k<0$ such that $\mathfrak{G}^{*}|T=T^{*}$. Then $\mathfrak{G}^{*}|T=T^{*}$.

Finally, if (6) is satisfied, then by Lemma 4.1, $V^{*}$ is a division ring.

Conversely, let $T/S$ be a regular embedded in a $qn$-dimensional abelian extension $R/S$ with respect to $\mathfrak{G}=(\sigma_{1})x(\sigma_{2})x\cdots x(\sigma_{m})x\cdots x(\sigma_{m+k})$ if $k\geq 0$ and $\mathfrak{G}=(\sigma_{1})\times\cdots\times(\sigma_{m+k})$ if $-m<k<0$ such that $\mathfrak{G}^{*}|T=T^{*}$. Then $\mathfrak{G}^{*}|T=T^{*}$.

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Finally, if (6) is satisfied, then by Lemma 4.1, $V^{*}$ is a division ring.

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Finally, if (6) is satisfied, then by Lemma 4.1, $V^{*}$ is a division ring.

Conversely, let $T/S$ be a regular embedded in a $qn$-dimensional abelian extension $R/S$ with respect to $\mathfrak{G}=(\sigma_{1})x(\sigma_{2})x\cdots x(\sigma_{m})x\cdots x(\sigma_{m+k})$ if $k\geq 0$ and $\mathfrak{G}=(\sigma_{1})\times\cdots\times(\sigma_{m+k})$ if $-m<k<0$ such that $\mathfrak{G}^{*}|T=T^{*}$.
$x_i^{-1}(t \tau_j^{-1} \rho_j \tau_j) x_i = t \tau_j^{-1} \rho_j \tau_j \rho_i.
\begin{align*}
RN_{i_k}(b_{ij} ; \tau_j) &= x_i^{-1} (x_i \sigma_j) (x_i^{-1} \sigma_j) (x_i \sigma_j^2) \cdots (x_i^{-1} \sigma_j^{q_j-1}) (x_i \sigma_j^q) = x_i^{-1} x_i \sigma_j^q, \\
\{1, 1\} &\text{ if } i = j, \\
\{1, 0\} &\text{ if } i \neq j.
\end{align*}
\begin{align*}
\text{LN}_{i_k}(b_{ij} ; \rho_i) &= (x_i^{-n_i} x_i^{-1} (x_i \sigma_j) x_i^{n_i-1}) (x_i^{-1} \sigma_j) x_i^{-1} (x_i \sigma_j) x_i^{-1} (x_i \sigma_j^2) \cdots x_i^{-1} (x_i \sigma_j^{q_j}) = x_i^{-n_i} (x_i \sigma_j)^{q_j} = x_i^{-n_i} (x_i \sigma_j)^{q_j} = t_i^{q_j} (t_i \sigma_j).
\end{align*}
\begin{align*}
s_{ij}(b_{ik} \rho_j) b_{jk} &= (x_i^{-1} x_j^{-1} x_i x_j) (x_i^{-1} x_j^{-1} x_i \sigma_k) (x_i x_j) = x_i^{-1} x_j^{-1} (x_i \sigma_k) (x_i x_j) = (x_i^{-1} (x_i x_j \sigma_k) x_i) (x_i^{-1} (x_i \sigma_k) x_i) (x_i \sigma_k) (x_i \sigma_k) = b_{jk} \rho_i (x_i^{-1} (x_i \sigma_k) x_i^{-1} \sigma_k) (x_i x_j) = b_{jk} \rho_i (x_i x_j) = b_{jk} (s_{ij} \tau_k).
\end{align*}
\begin{align*}
x_i (b_{ik} (b_{jk} \tau_k)) &= x_i x_j \sigma_k = x_i x_j \sigma_j = x_i b_{ij} (b_{ik} \tau_k) \text{ shows that } b_{ik} (b_{ij} \tau_k) = b_{ij} (b_{ik} \tau_k).
\end{align*}
Finally, since $V$ is a division ring, $T/S$, irreducibility of $T$ yields at once the condition (6) by Lemma 4.1 in $T[\mathfrak{F}_{m+k}, \mathfrak{P}_{m+k}]$ where $X_i P_J = X_i s_{ij}$ and $S|P_J = \rho_j$.

The condition (6) of Theorem 4.2 was need only to see that the centralizer $V^*$ is a division ring. Accordingly, we readily obtain the following.

**Corollary 4.5.** Let $S$ be a division ring, and let $T/S$ be a $q$-dimensional abelian division ring extension satisfying the condition of Theorem 4.2. In order that $T/S$ can be regularly embedded in a $qn$-dimensional abelian division ring extension $R/S$ such that $R/T$ is an $n$-dimensional Kummer extension, it is necessary and sufficient that there exist $\{\rho_1, \rho_2, \cdots, \rho_n, s_{ij}, (t_i), (b_{ij})\}$ satisfying (1)–(5) of Corollary 4.3 (in $T$) and (1)–(5) of Theorem 4.1.

References


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